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Stable loops and almost transverse surfaces

Michael Landry

Abstract. We use veering triangulations to study homology classes on the boundary of the cone over a fibered face of a compact fibered hyperbolic three-manifold. This allows us to give a hands-on proof of an extension of Mosher's transverse surface theorem to the setting of manifolds with boundary. We also show that the cone over a fibered face is dual to the cone generated by the homology classes of a canonical finite collection of curves called *minimal stable loops* living in the associated veering triangulation.

1. Introduction

In this paper, we use veering triangulations to study the suspension flows of pseudo-Anosov homeomorphisms on compact orientable surfaces and their relation to the unit ball of the Thurston norm. We call these flows *circular pseudo-Anosov flows*. The term *circular* indicates that the flow is a suspension flow, and is standard. However, pseudo-Anosov flows as defined by Mosher in [19] live only in closed three-manifolds, making the terminology here slightly nonstandard. We use the term "compact hyperbolic three-manifold" throughout to refer to compact three-manifolds whose interiors admit complete hyperbolic metrics of finite volume.

1.1. Main results and outline

Let M be a compact hyperbolic three-manifold with fibered face $\sigma \subset H_2(M, \partial M; \mathbb{R})$. Each integral point in the interior of $\operatorname{cone}(\sigma) := \mathbb{R}_{\geq 0} \cdot \sigma$ corresponds to a fibration $M \to S^1$ (the nonprimitive integral points correspond to fibrations with disconnected fibers). There is a circular pseudo-Anosov flow φ called the *suspension flow of* σ, unique up to reparameterization and conjugation by homeomorphisms of M isotopic to the identity, which collates the fibers and monodromies of all fibrations of M corresponding to σ. More precisely, each fiber of a fibration $M \to S^1$ is isotopic to a cross section of φ, and the monodromy of the fibration is given by the first return map of φ on that cross section. This theorem was proven for closed M by Fried in [7, Theorem 14.11]; a proof in the case when $\partial M \neq \varnothing$ has yet to appear in the literature and so we include one in Appendix A.

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In addition to capturing data about the integral points of $\operatorname{int}(\operatorname{cone}(\sigma))$, the suspension flow φ detects the integral points in $\partial \operatorname{cone}(\sigma)$. In this direction, our first result is a generalization of Mosher's transverse surface theorem [18, Theorem 1.4] to the setting of manifolds with boundary.

Theorem 3.5 (Almost transverse surfaces). *Let M be a compact hyperbolic three-manifold, with a fibered face σ of $B_x(M)$ and associated suspension flow φ. Let $\alpha \in H_2(M, \partial M)$ be an integral homology class. Then $\alpha \in \operatorname{cone}(\sigma)$ if and only if α is represented by a surface almost transverse to φ.*

In the theorem statement, the symbol $B_x(M)$ denotes the unit ball of the Thurston norm. A surface is *almost transverse* to φ if it is transverse to a closely related flow $\varphi^\sharp$ obtained from φ by a process called *dynamically blowing up* singular orbits, which amounts to replacing some number of singular orbits of φ by mapping tori of homeomorphisms of finite trees. The precise definition of almost transversality is found in Section 3.2. Note that transverse implies almost transverse, so in the case $\alpha \in \operatorname{int}(\operatorname{cone}(\sigma))$ the forward direction of Theorem 3.5 amounts to the theorem of Fried mentioned above; the case of most interest here is when $\alpha \in \partial \operatorname{cone}(\sigma)$. We remark that this generalization of the transverse surface theorem to the setting of manifolds with boundary has already proven useful: in [2] the authors use Theorem 3.5 to prove results about the asymptotic translation length of pseudo-Anosov mapping classes acting on the curve graph of a surface.

The main tool in the proof of Theorem 3.5 is an object τ, canonically associated to φ, called the *veering triangulation* of σ. The veering triangulation τ is an ideal triangulation of a cusped hyperbolic three-manifold M' obtained from M by deleting finitely many closed curves from $\operatorname{int}(M)$. We find it fruitful in this paper to view τ as sitting in M as an ideal triangulation of a subspace.

Loosely speaking, the strategy of our proof of Theorem 3.5 is to arrange a surface to lie in a regular neighborhood of the 2-skeleton of the veering triangulation away from the singular orbits of φ, and then use our knowledge of how the veering triangulation sits in relation to φ in order to appropriately blow up the flow near the singular orbits. Mosher, without the machinery of veering triangulations available to him, performed a deep analysis of the dynamics of the lift of φ to the cyclic cover of M associated to the surface. Including this dynamical analysis, his complete proof of the theorem spans [16–18]. It is an advertisement for the power of veering triangulations that once their combinatorics are understood, the proof of Theorem 3.5 is fairly simple.

The construction of τ depends on φ, and so one might expect the combinatorics of τ to encode information about φ. The main result of Section 4, Theorem 4.9, gives an example of this. Before stating the result, we state a few facts about φ. The suspension flow φ has the following property: a cohomology class $u \in H^1(M;\mathbb{R})$ is Lefschetz dual to a class in $\operatorname{cone}(\sigma) := \mathbb{R}_{\geq 0} \cdot \sigma$ if and only if u is nonnegative on $\mathcal{C}_\varphi$, the *cone of homology directions* of φ. Hence computing $\mathcal{C}_\varphi$ is equivalent to computing $\operatorname{cone}(\sigma)$. For a flow F on M, $\mathcal{C}_F$ is the smallest closed cone containing the projective accumulation points of homology classes of nearly closed orbits of F. Since φ is a circular pseudo-Anosov flow,

C_φ has a more convenient characterization as the smallest closed cone containing the homology classes of the closed orbits of φ. In fact, Fried showed that $\mathcal{C}_\varphi$ is generated by the homology classes of the simple loops in the directed graph associated to any Markov partition for φ [7, 8].

While a Markov partition is a noncanonical object, the following result gives a canonical finite family of curves whose homology classes generate $\mathcal{C}_\varphi$. The family is easily defined in terms of the structure of τ and has an interpretation in terms of φ.

Theorem 4.9 (Stable loops). *Let M be a compact hyperbolic three-manifold with fibered face σ. Let τ and φ be the associated veering triangulation and circular pseudo-Anosov flow, respectively. Then $\mathcal{C}_\varphi$ is the smallest convex cone containing the homology classes of the minimal stable loops of τ.*

The *stable loops* of τ are a family of closed curves carried by the so-called *stable train track of τ*, which lies in the 2-skeleton of τ and records the combinatorics of the intersection of the stable foliation of φ with the 2-skeleton. A *minimal stable loop* is a stable loop traversing each switch of the stable train track at most once. Precise definitions are given in Section 4.3 (for the stable train track) and Section 4.5 (for stable loops).

To interpret Theorem 4.9, we recall a theorem of Cooper, Long, and Reid: an immersed surface S transverse to a circular pseudo-Anosov flow in a closed three-manifold is a virtual fiber of M if and only if the singular foliation on S induced by its intersection with the stable foliation of the flow has no closed leaves [4, Theorem 1.2]. As a consequence, the induced foliation on an embedded surface transverse to φ and representing a homology class in $\partial\,\text{cone}(\sigma)$ must have a closed leaf. This closed leaf represents a nontrivial element of the fundamental group of the corresponding leaf of the stable foliation of φ, and is therefore freely homotopic to a multiple of a closed orbit γ of φ whose homology class has 0 intersection with the homology class of the surface. It follows that $[\gamma]$ lies in the boundary of $\mathcal{C}_\varphi$.

In our context, we would like to understand the homology classes generating $\mathcal{C}_\varphi$. One of the things we show in the course of proving Theorem 4.9 is that any integral class in $\partial\,\text{cone}(\sigma)$ is represented by a surface almost transverse to φ whose intersection with the stable foliation of φ contains a closed leaf which can be homotoped to a stable loop.

By choosing an integral class lying in an appropriate top-dimensional face of $\text{cone}(\sigma)$, we can produce a stable loop whose homology class lies in any given 1-dimensional face of $\mathcal{C}_\varphi$, giving the result.

In other words, the stable loops needed to generate $\mathcal{C}_\varphi$ arise naturally from closed leaves of induced singular foliations on surfaces transverse and almost transverse to φ.

To briefly outline the paper: in Section 2, we provide background on the Thurston norm, circular pseudo-Anosov flows, and branched surfaces. We then develop some of the combinatorial structure of veering triangulations. In Section 3, we review the notion of dynamic blowups and prove Theorem 3.5. In Section 4, we show that a property called *infinite flippability* distinguishes fibers carried by the 2-skeleton of a veering triangulation, and use this fact to prove Theorem 4.9.

We include two appendices. In Appendix A, mentioned above, we prove that Fried's results from [7] concerning cross sections and the duality of $\mathcal{C}_\varphi$ and $\mathrm{cone}(\sigma)$ hold for compact hyperbolic three-manifolds with boundary. In Appendix B, we explain how to use Theorem 3.5 to show the results of [13] hold for manifolds with boundary, extending that paper's partial answer of a question of Oertel from [20]. In particular, we obtain the following corollary.

Corollary B.4. *Let L be a fibered hyperbolic link with at most three components. Let M_L be the exterior of L in S^3. Any fibered face of $B_x(M_L)$ is spanned by a taut branched surface.*

1.2. Context, references

A veering triangulation is a special type of *taut ideal triangulation*. Taut ideal triangulations were introduced by Lackenby in [12] as combinatorial analogues of taut foliations, where he uses them to give an alternative proof of Gabai's theorem that the singular genus of a knot is equal to its genus. He states that "one of the principal limitations of taut ideal triangulations is that they do not occur in closed three-manifolds," and asks the following question:

Question (Lackenby). Is there a version of taut ideal triangulations for closed three-manifolds?

While we do not claim a comprehensive answer to Lackenby's question, an undercurrent of this paper (and [13]) is that for fibered hyperbolic three-manifolds, possibly closed, a veering triangulation of a dense open submanifold is a useful version of a taut ideal triangulation.

Veering triangulations are introduced by Agol in [1]. There is a canonical veering triangulation associated to any fibered face of a hyperbolic three-manifold, and Guéritaud showed [11] that it can be built directly from the suspension flow. If taut ideal triangulations are combinatorial analogues of taut foliations, Guéritaud's construction allows us to view veering triangulations as combinatorializations of pseudo-Anosov flows.

2. Preliminaries

In this paper, all manifolds are orientable and all homology and cohomology groups have coefficients in $\mathbb{R}$.

2.1. The Thurston norm, fibered faces, relative Euler class

We review some facts about the Thurston norm, which can be found in [22]. Let M be a compact, irreducible, boundary irreducible, atoroidal, anannular three-manifold (∂M may be empty). If S is a connected surface embedded in M, define

$$\chi_-(S) = \max\{0, -\chi(S)\},$$

where χ denotes Euler characteristic. If S is disconnected, let $\chi_-(S) = \sum_i \chi_-(S_i)$, where the sum is taken over the connected components of S. For any integral homology class $\alpha \in H_2(M, \partial M)$, we can find an embedded surface representing α. Define

$$x(\alpha) = \min \left\{\chi_-(S) \mid S \text{ is an embedded surface representing } \alpha\right\}.$$

Then x extends by linearity and continuity from the integer lattice to a vector space norm on $H_2(M, \partial M)$ called the *Thurston norm* [22]. We mention that Thurston defined x more generally to be a seminorm on $H_2(M, \partial M)$ for any compact orientable M. However, in this paper x will always be a norm, since the manifolds we consider will not admit essential surfaces of nonnegative Euler characteristic.

The unit ball of x is denoted by $B_x(M)$. As a consequence of x taking integer values on the integer lattice, $B_x(M)$ is a finite-sided polyhedron with rational vertices. Our convention in this paper is that a *face* of $B_x(M)$ is a *closed* cell of the polyhedron.

We say that an embedded surface S is *taut* if it is incompressible and realizes the minimal χ_- in $[S]$. If $\Sigma \subset M$ is the fiber of a fibration $\Sigma \hookrightarrow M \to S^1$, then Σ is taut, any taut surface representing $[\Sigma]$ is isotopic to Σ, and $[\Sigma]$ lies in int(cone(σ)) for some top-dimensional face σ of $B_x(M)$. Moreover, any other integral class representing a class in int(cone(σ)) is represented by the fiber of some fibration of M over S^1. Such a top-dimensional face σ is called a *fibered face*.

Let ξ be an oriented plane field on M which is transverse to ∂M. If we fix an outward pointing section of $\xi|_{\partial M}$, this determines a *relative Euler class* $e_\xi \in H^2(M, \partial M)$. For a relative 2-cycle S, $e_\xi([S])$ is the first obstruction to finding a nonvanishing section of $\xi|_S$ agreeing with the outward pointing section on $\partial S \subset \partial M$. A reference on relative Euler class is [21].

If φ is a flow on M tangent to ∂M, let $T\varphi$ be the oriented line field determined by the tangent vectors to orbits of φ. Let ξ_φ be the oriented plane field which is the quotient bundle of TM by $T\varphi$. We can think of ξ_φ as a subbundle of TM by choosing a Riemannian metric and identifying ξ_φ with the orthogonal complement of $T\varphi$. For notational simplicity, we define $e_\varphi = e_{\xi_\varphi}$, the relative Euler class of ξ_φ.

Fix a fibration $Y \hookrightarrow M \to S^1$, which allows us to express $M \cong (Y \times [0,1])/(y,1) \sim (g(y), 0)$ for some homeomorphism g of Y. Let TY be the tangent plane field to the foliation of M by $(Y \times \{t\})$'s. Let φ be the *suspension flow* of g, which moves points in M along lines (y,t) for fixed y, gluing by g at the boundary of $Y \times [0,1]$. We have $\xi_\varphi \cong TY$ and hence $e_\varphi = e_{TY}$. For some fibered face σ, we have $[Y] \in$ int(cone(σ)). We have $x([Y]) = -\chi(Y) = -e_{TY}([Y])$, i.e., x and e_{TY} agree on $[Y]$. In fact, more is true: cone(σ) is exactly the subset of $H_2(M, \partial M)$ on which $-e_{TY}$ and x agree.

It can be fruitful to think of a properly embedded surface S in M as representing both a homology class in $H_2(M, \partial M)$ and a cohomology class in $H^1(M)$ mapping homology classes of closed curves to their intersection number with S. As such we will sometimes think of x as a norm on $H^1(M)$ via Lefschetz duality. The image in $H^1(M)$ of a face σ of $B_x(M)$ will be denoted by σ_{LD}, and in general the subscript LD, when attached to an object, will denote the Lefschetz dual of that object.

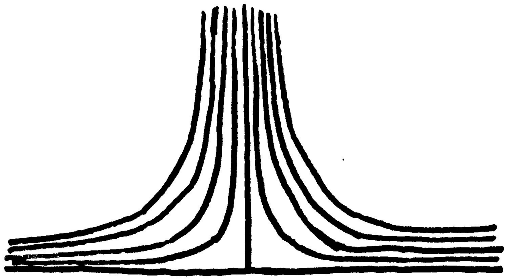

Figure 1. A boundary singularity of an invariant singular foliation of a generic pseudo-Anosov map has three separatrices.

2.2. Circular pseudo-Anosov flows

Let F be a flow on M which is tangent to ∂M. By *flow* we mean a continuous action of $\mathbb{R}$ on M with C^1 orbits. We call the orbits *flow lines*. Recall that a *cross section* to F is a fiber of a fibration $M \to S^1$ whose fibers are transverse to F. Flows which admit cross sections are called *circular*.

Let $g\colon Y \to Y$ be a pseudo-Anosov map on a compact surface Y which is maximally blown up on ∂Y in the sense that the singularities on ∂Y of its invariant foliations (see [23, §3]) each have three *separatrices* in the sense of [5, Chapter 5, §5.1] (see Figure 1). We call g a *generic* pseudo-Anosov map. Up to conjugacy by a homeomorphism of Y isotopic to the identity, g is unique within its isotopy class. We call the suspension flow φ of a generic pseudo-Anosov map a *circular pseudo-Anosov flow*. In contrast, one can also define pseudo-Anosov flows which are not circular, essentially by requiring neighborhoods of closed orbits be modeled on closed orbits of circular pseudo-Anosov flows (see [3, Definition 6.41]). We will not use these flows, however—we mention them simply to provide context.

The suspension of the stable and unstable foliations of g give two codimension-1 foliations in M preserved by φ called the *stable* and *unstable* foliations of φ. These foliations are transverse to one another in $\mathrm{int}(M)$. The closed orbits corresponding to the singular points of g lying in $\mathrm{int}(Y)$ are called *singular orbits*. The closed orbits lying on ∂M are called ∂-*singular orbits*. See Figure 2.

If Z is a cross section to a flow φ, the *first return map* of Z is the map sending $z \in Z$ to the first point in its forward orbit under φ lying in Z. If φ is a circular pseudo-Anosov flow, the first return map of any cross section to φ will be pseudo-Anosov (this was proved in [7, Lemma 14.2] for closed manifolds, but the proof in general is essentially the same).

2.3. Review of the veering triangulation

A *taut tetrahedron* is an ideal tetrahedron with the following edge and face decorations. Four edges are labeled 0 and two are labeled π. Two faces are cooriented outwards and two are cooriented inwards, and faces of opposite coorientation meet only along edges

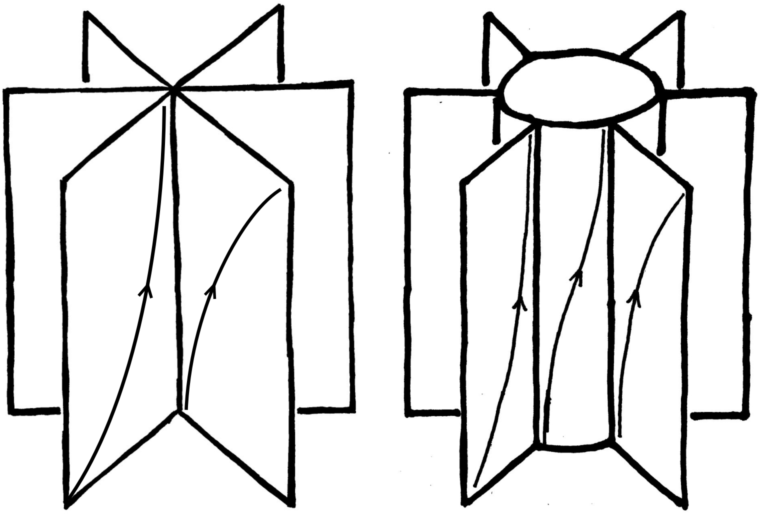

Figure 2. The behavior of circular pseudo-Anosov flow near a singular orbit (left) and a boundary component of the ambient three-manifold (right).

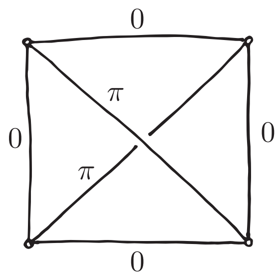

Figure 3. A taut tetrahedron. The labels on edges indicate interior angles, and our convention is that the coorientation of each face points out of the page.

labeled 0. The edge labels should be thought of as the interior angles of the corresponding edges; see Figure 3. We define the *top* (resp., *bottom*) of a taut tetrahedron t to be the union of the two faces whose coorientations point out of (resp., into) t. The *top* (resp., *bottom*) π*-edge* of t will be the edge labeled π lying in the top (resp., bottom) of t.

Definition 2.1 ([12]). A *taut ideal triangulation* of a three-manifold is an ideal triangulation by taut tetrahedra such that

(1) when two faces are identified their coorientations agree, and

(2) the sum of interior angles around a single edge is 2π.

The 0 and π angle labels tell us how to "pinch" a taut ideal triangulation along its edges so that the 2-skeleton has a well-defined tangent space at every point. We will always assume that the 2-skeleton of a taut ideal triangulation is embedded in such a way.

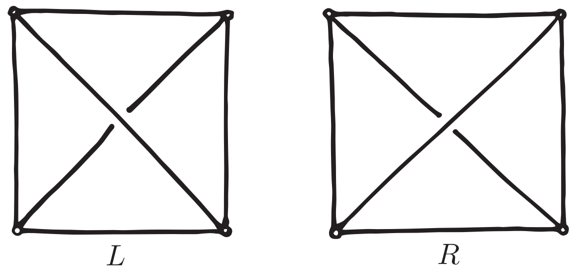

Figure 4. When the edges which are bottommost in the page are distinguished, the taut tetrahedron on the left is of type L and the taut tetrahedron on the right is of type R.

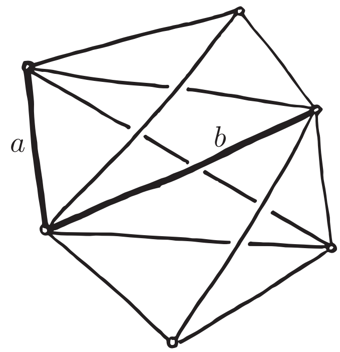

Figure 5. Part of a veering triangulation. The edge a is left veering, while b is right veering.

Note that up to orientation-preserving combinatorial equivalence, there are two types of taut tetrahedron with a distinguished 0-edge. We call these L and R and they are shown in Figure 4.

Definition 2.2. Let e be an edge of a taut ideal triangulation $\triangle$ such that no two interior angles of π are adjacent around e. If e has the property that all tetrahedra for which e is a 0-edge are of type R when e is distinguished, we say that e is *right veering*. Symmetrically, if they are all of type L we say e is *left veering*. If every edge of $\triangle$ is either right or left veering, $\triangle$ is *veering*; see Figure 5.

A consequence of the above condition that no two interior π angles be adjacent is that each edge of a veering triangulation has degree ≥ 4.

2.4. The veering triangulation of a fibered face

In this section, we move somewhat delicately between compact manifolds and their interiors. The reason for this is that we wish to work in a compact manifold and study its second homology rel boundary, but veering triangulations live most naturally in cusped manifolds.

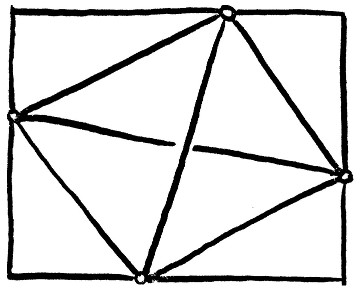

Figure 6. A maximal rectangle and the image of the associated taut tetrahedron under flattening.

Veering triangulations are introduced by Agol in [1], where he canonically associates a veering triangulation to a pseudo-Anosov surface homeomorphism. Let $g\colon Y \to Y$ be a pseudo-Anosov map on a compact surface Y with associated stable and unstable measured foliations $\mathcal{F}^s$ and $\mathcal{F}^u$. Let $Y' = \operatorname{int} Y \setminus \{\text{singularities of } \mathcal{F}^s, \mathcal{F}^u\}$ and $g' = g|_{Y'}$, and let M_g and $M_{g'}$ be the respective mapping tori of g and g'. Agol constructs an ideal veering triangulation of $M_{g'}$ from a sequence of Whitehead moves between ideal triangulations of Y' which are dual to a periodic splitting sequence of measured train tracks carrying the stable lamination of g'. Each Whitehead move corresponds to gluing a taut tetrahedron to Y', and the resulting taut ideal triangulation of the $\mathbb{Z}$-cover associated to Y' descends to a taut ideal triangulation of $M_{g'}$. We call a taut ideal triangulation of this type *layered on* Y'. Agol shows that up to combinatorial equivalence there is only one veering triangulation of $M_{g'}$ which is layered on Y, and we call this the *veering triangulation of* g.

In [11], Guéritaud provides an alternative construction of the veering triangulation of g which we summarize now; a nice account is also given in [15].

Let $\widetilde{Y}'$ be the universal cover of Y', and let $\widehat{Y}'$ be the space obtained by attaching a point to each lift of an end of Y' to $\widetilde{Y}'$. The measured foliation $\mathcal{F}^s|_{Y'}$ lifts to $\widetilde{Y}'$ and gives rise to a measured foliation on $\widehat{Y}'$ with singularities at points of $\widehat{Y}' \setminus \widetilde{Y}'$. We call this the *vertical foliation*, and the analogous measured foliation coming from $\mathcal{F}^u|_{Y'}$ is called the *horizontal foliation*. In pictures, we will arrange the vertical and horizontal foliations so they are actually vertical and horizontal in the page.

The transverse measures on the vertical and horizontal foliations give $\widehat{Y}'$ a singular flat structure. A *singularity-free rectangle* is a subset of $\widehat{Y}'$ which can be identified with $[0,1] \times [0,1]$ such that for all $t \in [0,1]$, $\{t\} \times [0,1]$ (resp., $[0,1] \times \{t\}$) is a leaf of the vertical (resp., horizontal) foliation.

We consider the family of singularity-free rectangles which are maximal with respect to inclusion. Any such maximal rectangle has one singularity in the interior of each edge.

Each maximal rectangle R defines a map $f_R\colon t_R \to R$ of a taut tetrahedron into R which "flattens" t_R and has the following properties: the pullback of the orientation on R induces the correct coorientation on each triangle in t_R, the top (resp., bottom) π-edge of t_R is mapped to a segment connecting the singularities on the horizontal (resp., vertical) edges of R, and $f(e)$ is a geodesic in the singular flat structure of $\widehat{Y}'$; see Figure 6.

We can build a complex from $\bigcup t_{R_i}$, where the union is taken over all maximal rectangles, by making all the identifications of the following type: let R_1 and R_2 be maximal rectangles, and suppose that Δ_1 and Δ_2 are faces of t_{R_1} and t_{R_2} such that $f_{R_1}(\Delta_1) = f_{R_2}(\Delta_2)$. Then we identify Δ_1 and Δ_2. The resulting complex is a taut ideal triangulation of $\widetilde{Y'} \times \mathbb{R}$, and one checks that it is veering. Guéritaud shows that it descends to a layered veering triangulation of $M_{g'}$, which must be the one constructed by Agol.

While the veering triangulation is canonically associated to $g\colon Y \to Y$, it is in fact "even more canonical" than that. To elaborate, we need the following result.

Theorem 2.3 (Fried). *Let φ be a circular pseudo-Anosov flow on a compact three-manifold M. Let S be a cross section of φ and let σ be the fibered face of $B_x(M)$ such that $[S] \in \operatorname{cone}(\sigma)$. An integral class $\alpha \in H_2(M, \partial M)$ lies in $\operatorname{int}(\operatorname{cone}(\sigma))$ if and only if α is represented by a cross section to φ. Up to reparameterization and conjugation by homeomorphisms of M isotopic to the identity, φ is the only such circular pseudo-Anosov flow.*

Thus we we will speak of *the suspension flow of a fibered face*. Theorem 2.3 says that all the monodromies of fibrations coming from σ are realized as first return maps of the suspension flow. Theorem 2.3 is proven by Fried in [7, Theorem 14.11] in the case where M is closed. The proof of this more general result, when the fiber possibly has boundary, does not seem to exist in the literature. For the reader's convenience, we provide a proof in Appendix A (Theorem A.7).

Set $M = M_g$. Let σ be the face of $B_x(M_g)$ such that $[Y] \in \operatorname{cone}(\sigma)$, let φ be the associated suspension flow, and let $\varphi' = \varphi|_{M_{g'}}$.

The lift $\widetilde{\varphi'}$ of φ' to the universal cover $\widetilde{M_{g'}}$ of $M_{g'}$ is *product covered*, i.e., conjugate to the unit-speed flow in the z direction on $\mathbb{R}^3$. Consequently, the quotient of $\widetilde{M_{g'}}$ by the flowing action of $\mathbb{R}$, which we call the *flowspace of* φ' and denote by $\operatorname{flowspace}(\varphi')$, is homeomorphic to $\mathbb{R}^2$. In addition, $\operatorname{flowspace}(\varphi')$ has two transverse (unmeasured) foliations which are the quotients of the lifts of the (2-dimensional) stable and unstable foliations of φ' to $\widetilde{M_{g'}}$.

Let Z be another fiber of M over S^1 such that $[Z] \in \operatorname{cone}(\sigma)$, with monodromy $h\colon Z \to Z$. Let Z', $\widetilde{Z'}$, and $\widehat{Z'}$ be obtained from Z and h in the same way that Y', $\widetilde{Y'}$, and $\widehat{Y'}$ were obtained from Y and g.

By Theorem 2.3, both $\widetilde{Y'}$ and $\widetilde{Z'}$ and their vertical and horizontal foliations can be identified with $\operatorname{flowspace}(\varphi')$ and its foliations by forgetting measures. Hence we can identify $\widehat{Y'}$ and $\widehat{Z'}$ together with their vertical and horizontal foliations. The maximal rectangles from Guéritaud's construction depend only on the vertical and horizontal foliations, and not on their transverse measures. The geodesics defining the edges of a tetrahedron do depend on the measures (and hence on g and h), but for either pair of measures, the geodesics will be transverse to both the vertical and horizontal foliations. We see then that the triangles of the veering triangulation of

$$\widetilde{Y'} \times \mathbb{R} \cong \widetilde{Z'} \times \mathbb{R} \cong \widetilde{M_{g'}}$$

are well defined up to isotopy. It follows that the veering triangulations of g and h are the same up to isotopy in $M_{g'} = M_{h'} = \mathrm{int}(M) \setminus \{\text{singular orbits of } \varphi\}$.

Synthesizing the above discussion, we have shown the following.

Theorem 2.4 (Agol). *Let M be a compact hyperbolic three-manifold, and suppose that Y and Z are fibers of fibrations $M \to S^1$ with monodromies g and h such that $[Y], [Z] \in \mathrm{cone}(\sigma)$ for some fibered face of $B_x(M)$. Then the veering triangulations of g and h are combinatorially equivalent.*

It therefore makes sense to speak of *the veering triangulation of a fibered face* of $B_x(M)$.

2.5. Some notation

We now fix some notation which will hold for the remainder of the paper.

Let M be a compact hyperbolic three-manifold, and let φ be a circular pseudo-Anosov flow on M. Let c be the union of the singular orbits $c_1, \ldots, c_n$ of φ and let U_i be a small regular neighborhood of c_i. Let V be a small regular neighborhood of ∂M and put $U = V \cup (\bigcup_i U_i)$ and $\mathring{M} = M \setminus U$. Let σ be the (closed) fibered face of $B_x(M)$ determined by φ, with associated veering triangulation τ.

The homology long exact sequence associated to the triple $(M, U, \partial M)$ contains the sequence

$$H_2(U, \partial M) \xrightarrow{0} H_2(M, \partial M) \to H_2(M, U).$$

By excision, $H_2(M, U) \cong H_2(\mathring{M}, \partial\mathring{M})$. Hence there is an injective map

$$P\colon H_2(M, \partial M) \to H_2(\mathring{M}, \partial\mathring{M}).$$

At the level of chains, the map corresponds to sending a relative 2-chain S to $S \setminus U$. We call P the *puncturing map*, and if $\alpha \in H_2(M, \partial M)$ we often write $\mathring{\alpha}$ to mean $P(\alpha)$.

2.6. Veering triangulations, branched surfaces, and boundaries of fibered faces

Recall that a *branched surface* in M is an embedded 2-complex transverse to ∂M which is locally modeled on the quotient of a stack of disks $D_1, \ldots, D_n$ such that for $i = 1, \ldots, n-1$, D_i is glued to D_{i+1} along the closure of a component of the complement of a smooth arc through D_i. The quotient is given a smooth structure such that the inclusion of each D_i is smooth (see Figure 7). Branched surfaces were introduced in [6], but the definition here is a slightly more general one appearing in [20].

Let $B \subset M$ be a branched surface. The union of points in B with no neighborhood homeomorphic to a disk is called the *branching locus* of B, and the components of the complement of the branching locus are called *sectors*. Let $N(B)$ be the closure of a regular neighborhood of B. Then $N(B)$ can be foliated by closed intervals transverse to B, and we call this the *normal foliation* of $N(B)$. If the normal foliation is oriented, we say that B is an *oriented* branched surface. In this paper, all branched surfaces will be oriented. If

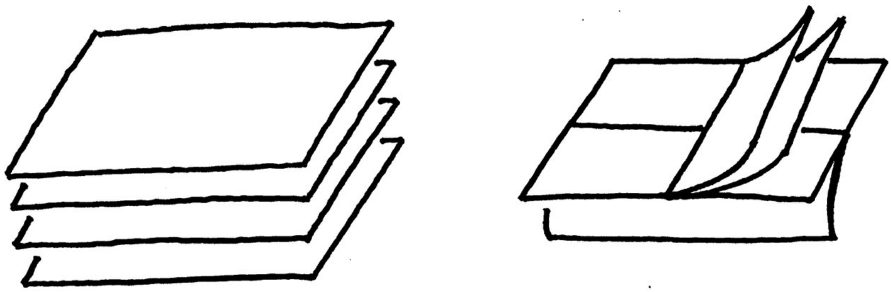

Figure 7. A portion of a branched surface, obtained as the quotient of a stack of disks.

$S \subset N(B)$ is a cooriented surface properly embedded in M which is positively transverse to the normal foliation in the sense that its coorientation is compatible with orientation of the normal foliation, we say that S is *carried by* B. A surface carried by B gives a system of nonnegative integer weights on the sectors of B which is compatible with natural linear equations along the branching locus of B. Any system of nonnegative real weights w satisfying these linear equations gives a 2-cycle and thus determines a homology class $[w] \in H_2(M, \partial M)$. We say that $[w]$ is *carried by* B. The collection of homology classes carried by B is clearly a convex cone.

Put $\mathring{\tau} = \tau \cap \mathring{M}$. Then the 2-skeleton of $\mathring{\tau}$ has the structure of a cooriented branched surface. We call this branched surface $B_{\mathring{\tau}}$.

Corollary 2.5 (Agol). *Let σ be the closed face of $B_x(M)$ determined by φ, and let $\mathring{\sigma}$ be the face of $B_x(\mathring{M})$ such that $P(\sigma) \subset$* cone$(\mathring{\sigma})$. *The cone of classes in $H_2(\mathring{M}, \partial\mathring{M})$ carried by $B_{\mathring{\tau}}$ is equal to* cone$(\mathring{\sigma})$.

Proof. Let Σ be a fiber of M such that $[\Sigma] \in$ cone$(\mathring{\sigma})$. By Theorem 2.4, τ can be built as a layered triangulation on int(Σ). It follows that $[\Sigma]$ is carried by $B_{\mathring{\tau}}$. Therefore, any integral class in int(cone$(\mathring{\sigma})$) is carried, so every rational class in int(cone$(\mathring{\sigma})$) is carried.

We can find a closed oriented transversal through each point of $B_{\mathring{\tau}}$, so $B_{\mathring{\tau}}$ is a *homology branched surface* in the sense of [20]; it follows that the cone of classes it carries is closed (see [13, Lemma 2.1] for details). Since each class in cone$(\mathring{\sigma})$ is approximable by a sequence of rational classes in the interior of the cone, the proof is finished. ∎

We say a homology branched surface B is *taut* if any surface carried by B is taut. We record the following useful lemma.

Lemma 2.6. *$B_{\mathring{\tau}}$ is taut.*

Proof. This is [12, Theorem 3], stated in the special case when the taut ideal triangulation is veering. ∎

2.7. Relating $\partial\mathring{\tau}$ and φ

The complex $\mathring{\tau}$ is a decomposition of $\mathring{M}$ into truncated taut tetrahedra; see Figure 8. Let t be a taut tetrahedron of τ, and let t' be the truncation of t obtained by deleting $t \cap U$. The

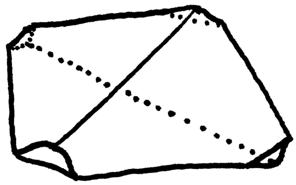

Figure 8. A truncated taut tetrahedron. The coorientation is such that the solid π-edge is on top.

Figure 9. Upward (left) and downward (right) flat triangles.

top and *bottom of* t', and *top* and *bottom* π*-edges of* t' are defined to be the restrictions to t' of the corresponding parts of t. Each of the four faces of t' corresponding to ideal vertices of t is a triangle with two interior angles of 0 and one of π, which we call a *flat triangle*. Because the faces of t are cooriented, a flat triangle inherits a coorientation on its edges. We say a flat triangle which is cooriented outwards (resp., inwards) at its π vertex is an *upward* (resp., *downward*) flat triangle, as in Figure 9. The upwards flat triangles of t' correspond to the ideal vertices connected by the top π-edge of t'.

The flat triangles of $\mathring{\tau}$ give a triangulation of ∂M. Some of the combinatorics of this triangulation are described in [10, 11, 13]. The terms ladder, ladderpole, and rung, defined below, first appeared in [10]. Some of these ideas are illustrated in Figure 10.

The union u of all upward flat triangles is a collection of annuli. The triangulation restricted to each annulus component of u has the property that each edge of a flat triangle either traverses the annulus or lies on the boundary of the annulus. The same holds for the union d of all downward flat triangles. A component of u is called an *upward ladder* and a component of d is called a *downward ladder*. We call the boundary components of ladders *ladderpoles*.

The 1-skeleton of this triangulation by flat triangles of $\partial\mathring{M}$ is a cooriented train track, which we call $\partial\mathring{\tau}$. We define a notion of left and right on each branch e of $\partial\mathring{\tau}$: orient ∂M inwards, and map some neighborhood of e homeomorphically to $\mathbb{R}^2$ so that e is identified with $[-1,1]\times\{0\}$ and the pushforward of e's coorientation points in the positive y direction. Define the *left* (resp., *right*) *switch of* e to be the preimage of -1 (resp., 1). We orient each branch of $\partial\mathring{\tau}$ from right to left, and this consistently determines an orientation on $\partial\mathring{\tau}$. If an oriented curve is carried by $\partial\mathring{\tau}$ such that orientations agree, we say that γ is *positively carried* by $\partial\mathring{\tau}$. By our choice of orientation if S is a surface carried by $B_{\mathring{\tau}}$, then its boundary, given the orientation induced by an outward-pointing vector field on $\partial\mathring{M}$, is positively carried by $\partial\mathring{\tau}$.

We call branches of $\partial\mathring{\tau}$ contained in ladderpoles *ladderpole branches*, and branches that traverse ladders *rungs*. We define the *left* (resp., *right*) *ladderpole* of a ladder to be the ladderpole containing the left (resp., right) boundary switches of its rungs. A closed

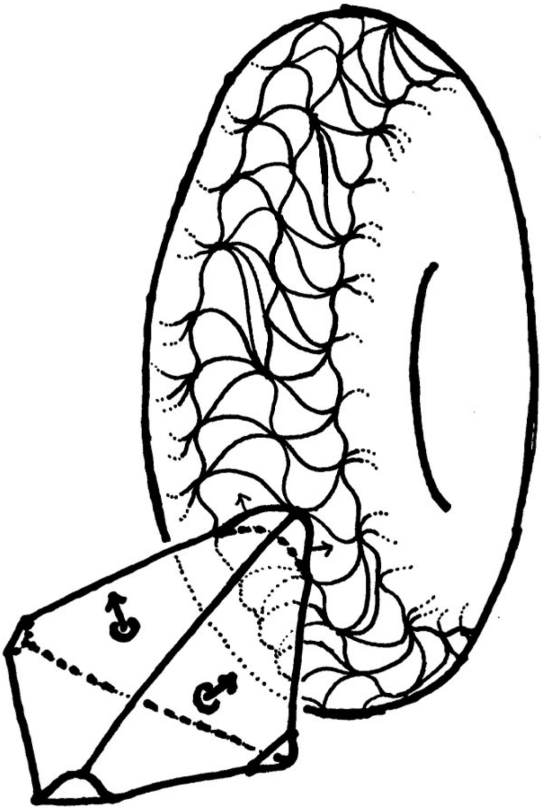

Figure 10. A truncated taut tetrahedron meeting a component of $\partial\mathring{M}$ in an upward flat triangle. An upward and downward ladder are shown. The top π-edge of the truncated taut tetrahedron must be left veering since it corresponds to a vertex in the right ladderpole of an upward ladder.

oriented curve positively carried by $\partial\mathring{\tau}$ and traversing only ladderpole branches is called a *ladderpole curve*.

Note that a switch of $\partial\mathring{\tau}$ corresponds to an edge of τ. The combinatorics of the flat and veering triangulations are related by the following lemma, the proof of which is elementary.

Lemma 2.7. *Let v be a switch of $\partial\mathring{\tau}$ corresponding to an edge e of τ. If v lies in the left ladderpole of an upward ladder, then e is right veering. If v lies in the right ladderpole of an upward ladder, then e is left veering.*

Consider a singular orbit c_i of φ and fix a cross section Y of φ. Let p be a point of $c_i \cap Y$. If ℓ is a separatrix of the stable foliation of the first return map at p, let L denote the orbit of ℓ under φ. In the path topology, L is the quotient of a half-closed, half open annulus A by a map which wraps ∂A around c_i some finite number of times. The flow lines

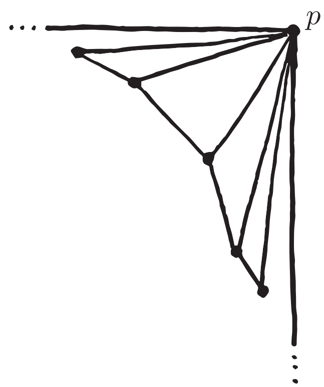

Figure 11. Four triangles of $\widehat{T}$, lying in a single quadrant bounded by vertical and horizontal leaves which meet at the singular point p of $\widehat{Y'}$.

of $\varphi|_L$ converge in forward time to c_i. Since L is dense in M, the intersection $L \cap \mathrm{cl}(U_i)$ has many components; we will call the component containing c_i a *stable flow prong* of c_i. An *unstable flow prong* of c_i is defined symmetrically with an unstable separatrix.

Lemma 2.8. *A stable flow prong of c_i intersects ∂U_i in the interior of an upward ladder. Symmetrically, an unstable flow prong of c_i intersects ∂U_i in the interior of a downward ladder.*

Proof. Fix a fiber Y of M with $[Y] \in \mathrm{cone}(\sigma)$, and let $\widetilde{Y'}$ and $\widehat{Y'}$ be as in Section 2.4. Let

$$M' = \mathrm{int}(M) \setminus \{\text{singular orbits of } \varphi\}.$$

Pick a singular orbit c_i of φ and consider the left ladderpole ℓ of an upward ladder of U_i. The vertices of ℓ define a family T of triangles of τ. Let $T' = T \cap U_i$, let $\widetilde{T'}$ be a component of the lift of T' to $\widetilde{M'}$, the universal cover of M', let $\widetilde{T}$ be the union of triangles in the veering triangulation of $\widetilde{M'}$ intersecting $\widetilde{T'}$, and let $\widehat{T}$ be the closure in $\widehat{Y'}$ of the projection of $\widetilde{T}$. This is a bi-infinite sequence of triangles, each sharing an edge with the next, and all sharing a single vertex at a singular point p of $\widehat{Y'}$.

The vertical and horizontal foliations define infinitely many quadrants each meeting p in a corner of angle $\frac{\pi}{2}$, which are each bounded by one vertical and one horizontal leaf. We claim that $\widehat{T}$ lives in only one of these quadrants, a situation depicted in Figure 11.

In $\widehat{Y'}$, the edges of positive slope are right veering while the edges of negative slope are left veering; this follows immediately from Figure 4 and the fact that the edge connecting the horizontal edges of a maximal rectangle lies above the edge connecting the vertical edges. Thus every edge in $\widehat{T}$ meeting p lies in a quadrant with horizontal left boundary leaf and vertical right boundary leaf (where our notion of left and right is determined looking at p from inside $\widetilde{Y'}$).

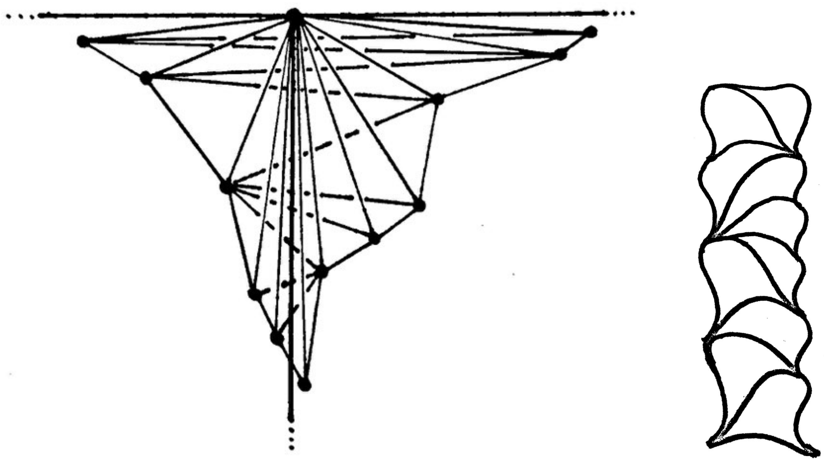

Figure 12. These 10 taut tetrahedra (left) define this part of an upward ladder (right) on the torus coming from the top, central singularity on the left.

Fix one of the triangles of $\widehat{T}$, defined by edges e_1 and e_2 meeting p. The two edges lie in the interior of a singularity-free rectangle with vertical and horizontal sides containing p in its boundary. Such a rectangle lies in the union of two adjacent quadrants of p, only one of which can have horizontal left boundary and vertical right boundary. Therefore, e_1 and e_2 lie in the same quadrant, and all the edges of $\widehat{T}$ lie in the same quadrant by induction.

Applying this analysis to each ladderpole yields the result. ∎

For a picture illustrating the situation in Lemma 2.8; see Figure 12.

3. Almost transverse surfaces

3.1. Dynamic blowups

We now describe the process of dynamically blowing up a singular periodic orbit γ of a pseudo-Anosov flow φ, which can be thought of as replacing a singular orbit by the suspension of a homeomorphism of a tree. For more details, the reader can consult [19, §3.5].

Let $q \in \mathbb{N}$, $q \geq 3$. Define a *pseudo-Anosov star* to be a directed tree T embedded in the plane with $2q$ edges meeting at a central vertex v, such that the orientations of edges around v alternate between inward and outward with respect to v. We say that a directed tree $T^\sharp$ is a *dynamic blowup* of T if the closed neighborhood of each vertex of $T^\sharp$ is a pseudo-Anosov star, and there exists a cellular map $\pi\colon T^\sharp \to T$ preserving edge orientations such that π is injective on the complement of $\pi^{-1}(v)$. See Figure 13 for two examples.

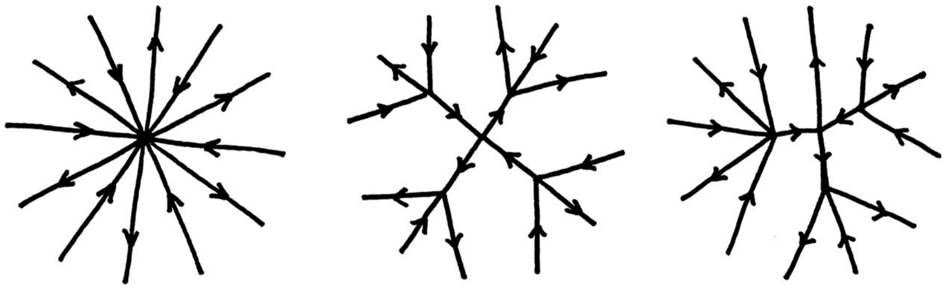

Figure 13. A pseudo-Anosov star (left) and two possible dynamic blowups.

Let γ be a singular periodic orbit of φ meeting q stable and q unstable flow prongs, and suppose that φ rotates the flow prongs by $2\pi \cdot \frac{p}{q}$ traveling once around γ, where $\gcd(p, q) = 1$.

The intersection of the flow prongs of γ with a local cross section of φ gives a pseudo-Anosov star T with $2q$ edges, with each edge oriented according to whether points in that flow prong spiral towards or away from γ in forward time. Let $T^\sharp$ be a dynamic blowup of T that is invariant under rotation by $2\pi \cdot \frac{p}{q}$, and let G be the preimage of the central vertex of T under the collapsing map $\pi\colon T^\sharp \to T$.

There is a flow $\varphi^\sharp$ on M which replaces γ by the suspension of a homeomorphism $h\colon G \to G$ with the following properties. Each edge E of G is mapped by h to its image under rotation by $2\pi \cdot \frac{p}{q}$, and $h^q(E)$ fixes vertices and moves interior points in the direction E inherits from $T^\sharp$.

The orbit of G under $\varphi^\sharp$ is a complex of annuli A invariant under $\varphi^\sharp$. A point interior to an annulus of A spirals away from one boundary circle and towards the other in forward time, and the boundaries of annuli are closed orbits of $\varphi^\sharp$. The flows $\varphi^\sharp$ and φ are semi-conjugate via a map $K\colon M \to M$ collapsing A to γ which is injective on the complement of A.

We say that $\varphi^\sharp$ is obtained from φ by *dynamically blowing up* γ. A flow obtained from φ by dynamically blowing up some collection of singular orbits is a *dynamic blowup* of φ.

The effect of dynamically blowing up a singular orbit is to pull apart its flow prongs, otherwise leaving the dynamics of the flow unchanged; see Figure 14.

3.2. Statement of transverse surface theorem, and Mosher's approach

Definition 3.1. Let ψ be a circular pseudo-Anosov flow on a three-manifold N. We say that an oriented surface S embedded in N is *almost transverse to* ψ if there is a dynamic blowup $\psi^\sharp$ of ψ such that S is transverse to $\psi^\sharp$ and the orientation of $TS \oplus T\psi^\sharp$ agrees with that of the tangent bundle of N at every point in N.

Transverse surface theorem (Mosher). *Let N be a closed hyperbolic three-manifold with fibered face F and associated suspension flow ψ. An integral class $\alpha \in H_2(N)$ lies in* $\mathrm{cone}(F)$ *if and only if it is represented by a surface which is almost transverse to ψ.*

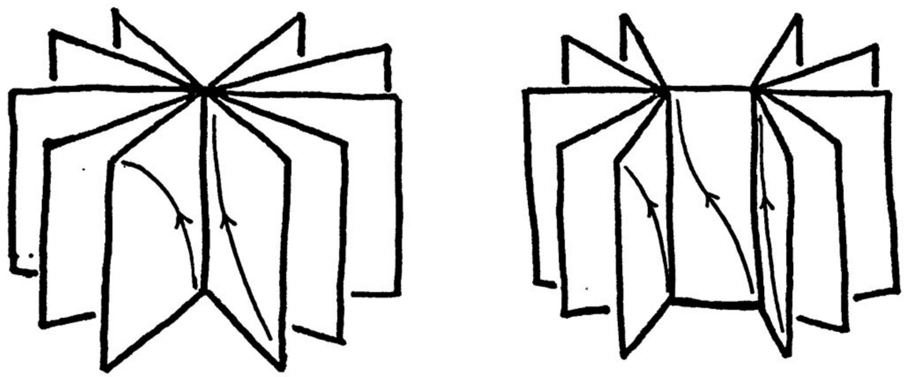

Figure 14. A dynamic blowup pulls apart the flow prongs of a singular orbit.

Let S be a surface which is almost transverse to ψ, and thus transverse to some dynamic blowup $\psi^\sharp$ of ψ. Then S is taut and $[S] \in \operatorname{cone}(F)$. This is true even when N has boundary, as we will now show.

Note that since ψ is a topologically transitive flow, meaning it has a dense orbit, $\psi^\sharp$ is as well. This follows from the fact that the collapsing map $K\colon M \to M$ semiconjugating $\varphi^\sharp$ and φ is injective on the complement of complex of annuli introduced by the dynamic blowup.

Consider a point $s \in S'$ for some component S' of S. There is an open neighborhood $\varepsilon(s)$ of s which is homeomorphic to $(-1,1)^3$ such that the restricted flow lines of $\psi^\sharp$ correspond to the vertical lines $\{a\} \times \{b\} \times (-1,1)$ for $-1 < a, b < 1$, and $S' \cap \varepsilon(s)$ corresponds to $(-1,1)^2 \times \{0\}$. Let o be a dense orbit of $\psi^\sharp$. We can take a segment of o with endpoints near each other in $\varepsilon(s)$ and attach endpoints with a short path to obtain a closed curve $\gamma(o)$ positively intersecting S', so S' is homologically nontrivial. In particular, it follows that S has no sphere or disk components.

We record a lemma:

Lemma 3.2. *Let $\varphi^\sharp$ be a dynamic blowup of φ. Then the tangent vector fields of φ and $\varphi^\sharp$ are homotopic, and consequently $e_\varphi = e_{\varphi^\sharp}$.*

Next, note that the restriction of $\xi_{\psi^\sharp}$ to S is homotopic to TS, so $\langle e_{\psi^\sharp}, [S]\rangle = \chi([S])$. Here $\langle \cdot, \cdot \rangle$ denotes the pairing of H^2 with H_2. Hence

$$\begin{aligned} x\big([S]\big) &\le -\chi(S) \\ &= \big\langle -e_{\psi^\sharp}, [S]\big\rangle \\ &= \big\langle -e_{\psi}, [S]\big\rangle \\ &\le x\big([S]\big). \end{aligned}$$

The first inequality holds because S has no sphere or disk components. The final inequality follows from the fact that x is a supremum of finitely many linear functionals on $H_2(N)$ which include $\langle -e_\varphi, \cdot \rangle$. The fact that $x([S]) = -\chi(S)$ means that S is taut, while $\langle -e_\varphi, [S]\rangle$ means that x and $\langle -e_\varphi, \cdot \rangle$ agree on $[S]$ so $[S] \in \mathrm{cone}(\sigma)$.

In light of the above discussion, to prove the transverse surface theorem it suffices to produce an almost transverse representative of any integral class in $\mathrm{cone}(F)$, and since any such class in the interior of $\mathrm{cone}(F)$ is represented by a cross section, it suffices to produce an almost transverse representative for any integral class in $\partial\,\mathrm{cone}(F)$.

Mosher's proof of the transverse surface theorem spans [16–18]; we give a brief summary here. Given an integral class $\alpha \in H_2(N)$ lying in $\partial\,\mathrm{cone}(\sigma)$, we consider its Poincaré dual $u \in H^1(N)$. Associated to u is an infinite cyclic covering space $N_{\mathbb{Z}}$. Mosher shows that there is a way to dynamically blow up a collection of singular orbits of ψ to get a dynamic blowup $\psi^\sharp$ that lifts to a flow $\widetilde{\psi}^\sharp$ on $M_{\mathbb{Z}}$ with nice dynamics. More specifically, he defines a natural partial order $\leq$ on the set of chain components of $\widetilde{\psi}^\sharp$ and shows that $\widetilde{\psi}^\sharp$ has finitely many chain components up to the deck action of $\mathbb{Z}$. He constructs a strongly connected directed graph Γ with vertices the deck orbits of chain components of $\widetilde{\psi}^\sharp$, and edges determined by $\leq$. He shows that flow isotopy classes of surfaces transverse to $\psi^\sharp$ and compatible with u are in bijection with positive cocycles on Γ representing a cohomology class $v \in H^1(\Gamma)$ which is determined by u. Finally, he proves the existence of such a cocycle.

3.3. A veering proof of the transverse surface theorem

The proof of the transverse surface theorem which we present in this section depends on the combinatorial Lemma 3.4, the statement of which requires some definitions.

A *pseudo-Anosov tree* T is a directed tree embedded in the plane such that the closed neighborhood of each non-leaf vertex is a pseudo-Anosov star. We can embed T in a disk Δ such that its leaves lie in $\partial\Delta$.

Let d be the closure of a component of $\Delta \setminus T$. Then d is homeomorphic to a closed disk and ∂d is a union of edges in T and one closed interval I in $\partial\Delta$. The boundary of I is composed of two leaves of T. Each leaf can be assigned a $+$ or $-$ depending on whether the corresponding edge of T points into or away from the leaf, respectively. We endow I with the orientation pointing from $-$ to $+$.

Lemma 3.3. *Let A, B be complementary regions of T which are incident along a single vertex of T, and the orientations on* $\mathrm{cl}(A) \cap \partial\Delta$ *and* $\mathrm{cl}(B) \cap \partial\Delta$ *are opposite. Then there is a unique dynamic blowup $T^\sharp$ of T with one more edge than T such that A and B are incident along an edge of $T^\sharp$.*

A picture makes this obvious; in lieu of a proof, see Figure 15 for a diagram of this dynamic blowup (for simplicity, we are abusing notation slightly by identifying A and B with the corresponding complementary regions of $T^\sharp$).

If we choose an orientation for $\partial\Delta$, then any point p lying in a component I of $\partial\Delta \setminus T$ can be given a sign according to whether the orientation of Δ agrees with the orientation of the component of $\partial\Delta \setminus T$ containing p.

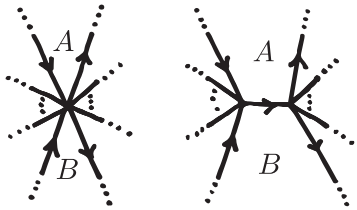

Figure 15. An illustration of Lemma 3.3.

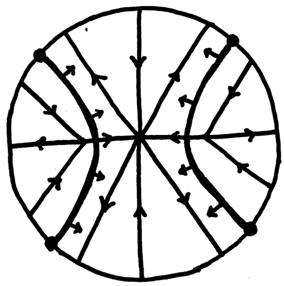

Figure 16. In this picture, we see a pseudo-Anosov tree T and an even family of size 4 being π-symmetrically filled in over T by two cooriented line segments.

An *even family* E for a pseudo-Anosov tree T is a finite subset of $\partial\Delta \setminus T$ which represents 0 in $H_0(\partial\Delta)$ when each $p \in E$ is given a sign as in the previous paragraph and E is viewed as a 0-chain. We say that an even family can be *filled in over* $\boldsymbol{T}$ if there exists a family L of disjoint cooriented line segments with $L \cap \partial\Delta = \partial L$ such that

- $\partial L = E$ as chains, i.e., the coorientations at each point in ∂L agree with the orientation of each segment of $\partial\Delta \setminus T$, and
- for each segment ℓ of L, ℓ intersects T only transversely in the interior of edges such that the coorientation of ℓ agrees with the orientation of the intersected edge.

If T and E above are symmetric under rotation of Δ by an angle of θ, and L can be chosen to respect this symmetry, we say that E can be *θ-symmetrically filled in over T*. Figure 16 shows an example.

Lemma 3.4. *Let $\ast$ and E be a pseudo-Anosov star and an even family for $\ast$, respectively, that are symmetric under rotation by θ. There exists a dynamic blowup $\ast^\sharp$ of $\ast$ such that E can be θ-symmetrically filled in over $\ast^\sharp$.*

Note that the lemma statement includes the case $\theta = 0$.

Proof. Choose a pair of points p_1, q_1 in E of opposite sign which are circularly adjacent and let $p_1, \dots, p_n$ and $q_1, \dots, q_n$ be all their images, without repeats, under rotation of Δ by θ. Let P_i, Q_i be the components of $\Delta \setminus S$ corresponding to p_i, q_i, respectively. If P_i and Q_i are incident along an edge of $\ast$, then there is a family of cooriented line segments

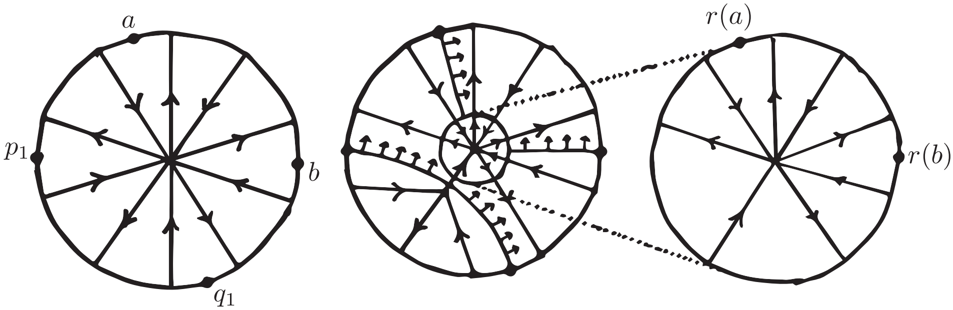

Figure 17. A diagram of the proof of Lemma 3.4 when $\ast$ is the pseudo-Anosov star on the left with an even family of size 4, and $\theta = 0$. In the notation of the proof, $E = \{p_1, q_1, a, b\}$ and $E_1 = \{r(a), r(b)\}$.

filling in $\{p_i, q_i\}_{i=1}^n$ over $\ast$. Otherwise, P_i and Q_i are incident at the vertex of $\ast$ and determine a dynamic blowup of $\ast$ as in Lemma 3.3. Since the pairs (p_i, q_i) are unlinked in $\partial\Delta$, we may perform all n of these dynamic blowups in concert to obtain a dynamic blowup $\ast^\sharp$ of $\ast$ such that $\{p_i, q_i\}_{i=1}^n$ can be θ-symmetrically filled in over $\ast^\sharp$.

Let L be the family of cooriented line segments filling in $\{p_i, q_i\}_{i=1}^n$. Let $E' = E \setminus \{p_i, q_i\}_{i=1}^n$. By construction, E' is contained in a single component Δ' of $\Delta \setminus L$, and $S^\sharp \cap \Delta'$ has a single vertex v which is preserved under rotation by θ. There exists a closed disk $\Delta_1 \subset \Delta'$ centered around v which is also preserved under rotation by θ. We can connect each point e in E' by a cooriented line segment to its image $r(e)$ under a retraction $r\colon \Delta \to \Delta_1$ such that the union of these segments is invariant under rotation by θ. Let $E_1 = \{d(e) \mid e \in E'\}$, and $\ast_1 = \ast^\sharp \cap \Delta_1$. A picture of this situation is shown in Figure 17.

We see that E_1 is an even family for $\ast_1$ which is smaller than E. Iterating this procedure, we eventually θ-symmetrically fill in E over a dynamic blowup of $\ast$. ∎

Now we are equipped to prove the transverse surface theorem. As noted in Section 1, we actually prove a generalization to compact manifolds which might have boundary.

Theorem 3.5 (Almost transverse surfaces). *Let M be a compact hyperbolic three-manifold, with a fibered face σ of $B_x(M)$ and associated circular pseudo-Anosov suspension flow φ. Let $\alpha \in H_2(M, \partial M)$ be an integral homology class. Then $\alpha \in$ cone(σ) if and only if α is represented by a surface almost transverse to φ.*

Proof. By the discussion following the statement of the transverse surface theorem in Section 3.2, the homology class of any surface almost transverse to φ lies in cone(σ). Hence we need only produce a representative of α transverse to some dynamic blowup of φ. Since any integral class in int(cone(σ)) is represented by a cross section, we assume that $\alpha \in \partial\,$cone(σ).

Our strategy will be to take a nice representative of $\mathring{\alpha}$ which is transverse to φ and complete it over U by gluing in disks and annuli which are also transverse to φ. Where

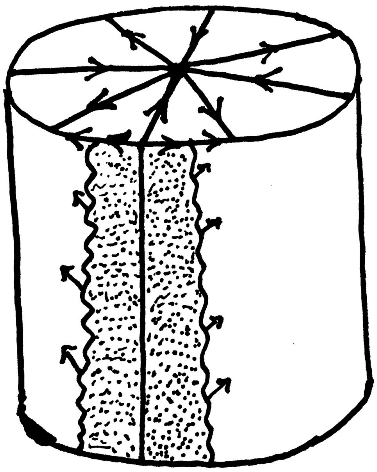

Figure 18. A portion of U_i is shown, illustrating implications of Lemma 2.8. The top disk is Δ. The shaded region denotes an upward ladder. Its boundary ladderpoles inherit a coorientation from $B_{\mathring{\tau}}$ agreeing with the orientations of the intervals of $\partial\Delta \setminus S$. The vertical line inside the ladder is the intersection of a stable flow prong with ∂U_i.

necessary, we dynamically blow up some singular orbits of φ and glue in annuli which are transverse to the blown up flow.

By Corollary 2.5, $\mathring{\alpha}$ has some representative $\mathring{A}$ which is carried by $B_{\mathring{\tau}}$, so we can assume that $\mathring{A}$ lies in $N(B_{\mathring{\tau}})$ transverse to the normal foliation. Since $B_{\mathring{\tau}}$ is transverse to φ, so is $\mathring{A}$.

First, to each boundary component of $\mathring{A}$ lying in ∂V (see Section 2.5 for notation), we glue an annulus which extends that component of $\partial\mathring{A}$ to ∂M maintaining transversality to φ.

Next, let c_i be a singular orbit of φ whose flow prongs φ rotates by θ. The proof of [13, Lemma 3.3] goes through in our setting, allowing us to conclude that $\mathring{A} \cap \partial U_i$ is either

(a) empty,

(b) a collection of meridians of $\mathrm{cl}(U_i)$ or

(c) a collection of ladderpole curves which is nulhomologous in $H_1(\partial(U_i))$.

In case (a), we have nothing to do. In case (b), the curves can be capped off by meridional disks of $\mathrm{cl}(U_i)$ transverse to φ.

In case (c), we consider a meridional disk Δ of $\mathrm{cl}(U_i)$ which is transverse to φ. The intersection of the flow prongs of c_i with Δ gives a pseudo-Anosov star $\ast$. Further, we claim that $\Delta \cap \mathring{A}$ is an even family for $\ast$.

By Lemma 2.8, each interval component of $\partial\Delta \setminus \ast$ intersects a single ladderpole, and the orientation of the interval agrees with the coorientation the ladderpole inherits from $B_{\mathring{\tau}}$; see Figure 18.

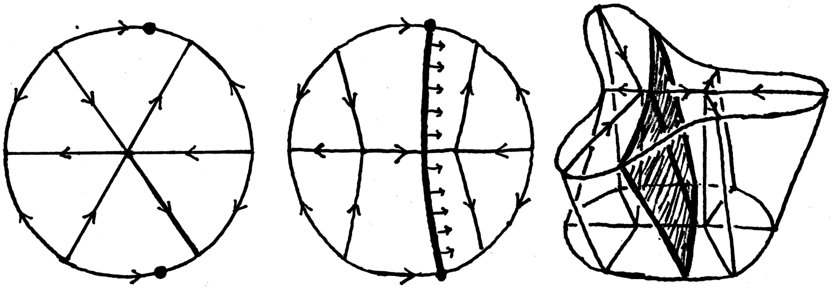

Figure 19. In this example, c_i is a 3-pronged singular orbit, $\theta = 0$, and $\mathring{A}$ intersects ∂U_i in two curves which gives an even family E of size 2 (left). We dynamically blow up S to $S^\sharp$ and fill in E over $S^\sharp$ by L, which in this case is a single cooriented line segment (center). We can model $\varphi^\sharp|_{U_i}$ as the vertical flow restricted to the distorted cyclinder shown on the right, with the top and bottom identified by a homeomorphism. Then we see L gives rise to an annulus which is transverse to $\varphi^\sharp$.

Because $\partial\mathring{A} \cap \partial U_i$ is a collection of ladderpole curves nulhomologous in $H_1(\partial U_i)$, it consists of equal numbers of left and right ladderpole curves of upward ladders. It follows that $E = \partial\mathring{A} \cap \Delta$ is an even family for $\ast$.

By Lemma 3.4, there exists a dynamic blowup $\ast^\sharp$ of $\ast$ such that E can be θ-symmetrically filled in by a collection of cooriented line segments L over $\ast^\sharp$. The tree $\ast^\sharp$ determines a dynamic blowup $\varphi_i^\sharp$ of φ. We can suspend L to a family of annuli with boundary $\partial\mathring{A}$ that are transverse to $\varphi^\sharp$ (see Figure 19 and caption).

By gluing these annuli to $\mathring{A}$, we eliminate all boundary components of $\mathring{A}$ meeting ∂U_i. The coorientations agree along $\partial\mathring{A}$ by construction.

By repeating this procedure at every singular orbit of φ, we obtain a surface A which is transverse to a dynamic blowup of φ. The image of $[A]$ under the puncturing map P is evidently $\mathring{\alpha}$. Since P is injective, $[A] = \alpha$. ■

4. Homology directions and τ

Our notation for this section is the same as defined at the beginning of Section 2.5: φ is a circular pseudo-Anosov flow on a compact three-manifold M, and σ is the associated fibered face with veering triangulation τ. By M' we mean $\mathrm{int}(M) \setminus \{\text{singular orbits of } \varphi\}$.

4.1. Convex polyhedral cones

We recall some facts about convex polyhedral cones (for a reference see, e.g., [9, §1.2]). Let A be a subset of a finite-dimensional real vector space V. Define the *dual* of A to be

$$A^\vee = \{u \in V^* \mid u(a) \geq 0 \; \forall a \in A\}.$$

A *convex polyhedral cone* in V is the collection of all linear combinations, with nonnegative coefficients, of finitely many vectors. If C is a convex polyhedral cone in V, then $C^\vee$ is a convex polyhedral cone in V^*. We have the relation $C^{\vee\vee} = C$.

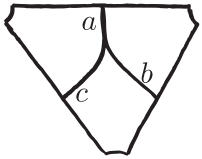

Figure 20. With labels as shown, a is a mixed branch of $\mathrm{St}(\tau)$ and b and c are small.

A *face* of C is defined to be the intersection of C with the kernel of an element in $C^\vee$. The *dimension* of a face of a convex polyhedral cone is the dimension of the vector subspace generated by points in the face. A top-dimensional proper face of a convex polyhedral cone is called a *facet* of the cone. If F is a face of C, define

$$F^* = \{u \in C^\vee \mid u(v) = 0 \ \forall v \in F\}.$$

Then F^* is a face of $C^\vee$ with dimension$(F^*) =$ dimension$(V) -$ dimension(F), and $F \mapsto F^*$ defines a bijection between the faces of C and the faces of $C^\vee$. We indulge in some foreshadowing by remarking that, in particular, $F \mapsto F^*$ restricts to a bijection between the one-dimensional faces of C and the facets of $C^\vee$.

We can identify $H_1(M)$ with $H^1(M)^*$ via the universal coefficients theorem, so if C is a convex polyhedral cone in $H^1(M)$, we will view $C^\vee$ as living in $H_1(M)$ and vice versa.

4.2. Flipping

Recall that $B_{\mathring{\tau}}$ is the 2-skeleton of $\mathring{\tau}$ viewed as a branched surface. Let $\triangle$ be a truncated taut tetrahedron of $\mathring{\tau}$. If S is carried with positive weights on both of the sectors of $B_{\mathring{\tau}}$ corresponding to the bottom of $\triangle$, then S may be isotoped upwards through $\triangle$ to a new surface carried by $B_{\mathring{\tau}}$ such that the uppermost (with respect to the orientation of the normal foliation of $N(B_{\mathring{\tau}})$) portion of S which was carried by the bottom of $\triangle$ is now carried by the top of $\triangle$. Outside of a neighborhood of $\triangle$, this isotopy is the identity. We call this isotopy an *upward flip*. If S_1 is the image of S under a single upward flip of S, we say that S_1 is an *upward flip of* S.

4.3. A train track on τ

Let T be a train track embedded in the 2-skeleton of τ with a single trivalent switch lying in the interior of each ideal triangle such that each edge of τ intersects T in a single point. Since each edge of τ has degree at least 4, this point is a switch of T with valence ≥ 4. Note that T is nonstandard in two ways: it is embedded in the 2-skeleton of τ rather than a surface, and it is not trivalent.

Let t be a triangle of τ, and let s be the switch of T interior to t. The interior of t is divided into three disks by $T \cap t$, one of which has a cusp at s. The branch which is disjoint from the cusped region is called a *mixed branch* of T. A branch which is not large is called a *small branch* of T; see Figure 20.

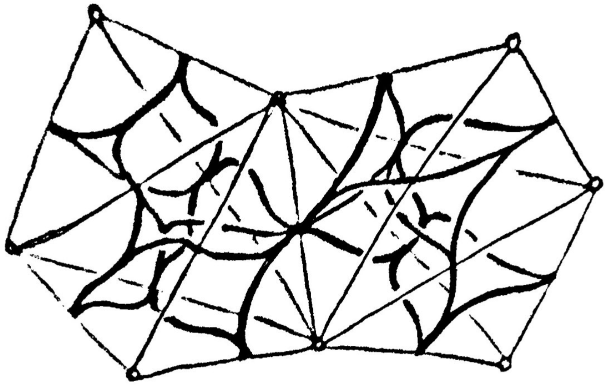

Figure 21. A portion of a veering triangulation and its stable train track.

We now define a particular train track in the 2-skeleton of τ which we call the *stable train track of* τ and denote it by $\mathrm{St}(\tau)$. For each triangle t of τ, we place a trivalent switch in the interior of t, and connect the branch which is large at the switch to the unique edge of t which is the bottom π-edge of the taut tetrahedron which t bounds below. We connect the two small branches to the other two edges of t. Gluing so that $\mathrm{St}(\tau)$ intersects each edge of τ in a point yields our desired train track. This is precisely the train track we would get from Agol's construction of τ by fixing a fibration, building τ as a layered triangulation on the fiber by looking at dual triangulations to a periodic maximal splitting sequence of the monodromy's stable train track, and recording the switches of stable train tracks on the relevant ideal triangles. We will also view $\mathrm{St}(\tau)$ as living in $B_{\mathring{\tau}}$. Figure 21 shows a portion of a veering triangulation with its stable train track.

Any surface S carried by $B_{\mathring{\tau}}$ naturally inherits a trivalent train track from $\mathrm{St}(\tau)$ which we call $\mathrm{St}(S)$. There is a natural cellular map $\mathrm{St}(S) \to \mathrm{St}(\tau)$, which allows us to identify each branch of $\mathrm{St}(S)$ with two branches of $\mathrm{St}(\tau)$. This map need be neither surjective nor injective. A branch of $\mathrm{St}(S)$ composed of two mixed branches of $\mathrm{St}(\tau)$ is called a *large branch* of $\mathrm{St}(S)$. This agrees with the standard notion of a large branch of a trivalent train track in a surface (see, e.g., [14, §2.3]).

By construction, we have the following.

Observation 4.1. Let S be a surface carried by $B_{\mathring{\tau}}$. There exists an upward flip of S if and only if $\mathrm{St}(S)$ contains a large branch.

Any curve carried by $\mathrm{St}(S)$ corresponds to a curve carried by $\mathrm{St}(\tau)$ under the map $\mathrm{St}(S) \to \mathrm{St}(\tau)$, and we will abuse terminology slightly by considering a curve carried by $\mathrm{St}(S)$ to also be carried by $\mathrm{St}(\tau)$.

4.4. Infinite flippability

A finite or infinite sequence $\{S_i\}$ of surfaces carried by $B_{\mathring{\tau}}$ such that each successive element is an upward flip of the previous element is called a *flipping sequence*. Any element of an infinite flipping sequence is called *infinitely flippable*.

If A is a surface carried by a branched surface with positive weights on each sector, we say that $[A]$ is *fully carried*.

We use these new words in a mathematical sentence:

Observation 4.2. Let $\mathring{A}$ be a surface carried by $B_{\mathring{\tau}}$ which is a fiber of $\mathring{M}$. Because τ can be built as a layered triangulation on the extension of $\mathring{A}$ to M', $\mathring{A}$ is infinitely flippable and some positive integer multiple of $[\mathring{A}]$ is fully carried by $B_{\mathring{\tau}}$.

The cone of homology directions of a flow F, denoted by $\mathcal{C}_F$, is the smallest closed cone containing the projective accumulation points of nearly closed orbits of F. Since in our case φ is a circular pseudo-Anosov flow, there is a more convenient characterization of $\mathcal{C}_\varphi$ as the smallest closed, convex cone containing the homology classes of the closed orbits of φ (see the proof of Lemma A.3). As we noted in the introduction, Fried showed that it suffices to take the smallest convex cone containing a certain *finite* collection of closed orbits. It follows that $\mathcal{C}_\varphi$ is a rational convex polyhedral cone.

Let τ be the veering triangulation of g. Define a *τ-transversal* to be an oriented curve in M' which is *positively transverse* to the 2-skeleton of τ, i.e., intersects the 2-skeleton only transversely and agrees with the coorientation of τ. Let $\mathcal{C}_\tau \subset H_1(M)$ be the smallest closed cone containing the homology class of each closed τ-transversal.

Proposition 4.3. *Let ℓ be a closed τ-transversal. Then $\ell \in \mathcal{C}_\varphi \setminus \{0\}$. Moreover, $\mathcal{C}_\varphi = \mathcal{C}_\tau$.*

Remark 4.4. There are several cones at play in the following proof. As an aid to the reader, we provide the following summary:

Cone	Habitat	Description
$\mathrm{cone}(\sigma)$	$H_2(M, \partial M)$	cone over the fibered face
$\mathrm{cone}(\sigma_{\mathrm{LD}})$	$H^1(M)$	Lefschetz dual of $\mathrm{cone}(\sigma)$
$\mathcal{C}_\varphi$	$H_1(M)$	Cone of homology directions of φ
$\mathcal{C}_\tau$	$H_1(M)$	Cone generated by closed τ-transversals

By the work of Fried and also Theorem A.7 for the case with boundary, we have $\mathrm{cone}(\sigma_{\mathrm{LD}}) = (\mathcal{C}_\varphi)^\vee$.

Proof. Since both cones are closed, to show that $\mathcal{C}_\varphi = \mathcal{C}_\tau$ it suffices to show that the homology class of every closed orbit lies in $\mathcal{C}_\tau$, and that the homology class of every closed τ-transversal lies in $\mathcal{C}_\varphi$.

Suppose that o is a closed orbit of φ. If o lies interior to $\mathrm{int}(M) \setminus c$, then o is already a closed τ-transversal. Otherwise, o is a singular or ∂-singular orbit and can be isotoped onto a closed τ-transversal lying in the interior of a ladder in $\partial \mathring{M}$. Hence $\mathcal{C}_\varphi \subset \mathcal{C}_\tau$.

Now suppose that ℓ is a closed τ-transversal. We can isotope ℓ into $\mathring{M}$ such that ℓ is positively transverse to $B_{\mathring{\tau}}$. Let $\beta \in \mathrm{int}(\mathrm{cone}(\sigma))$ be an integral class, and let $\mathring{B}$ be a representative of $\mathring{\beta}$ carried by $B_{\mathring{\tau}}$. Since $\mathring{B}$ is a fiber of $\mathring{M}$, by Observation 4.2 there exists a surface $n\mathring{B}$ (topologically $n\mathring{B}$ is n parallel copies of $\mathring{B}$ for some positive integer n)

representing $n\mathring{\beta}$ which is fully carried by $B_{\mathring{\tau}}$. We can cap off the boundary components of $n\mathring{B}$ to obtain a surface nB in M representing $n\beta$ whose intersection with $\mathring{M}$ is $n\mathring{B}$. Since ℓ is positively transverse to $B_{\mathring{\tau}}$, it has positive intersection with nB. Letting $\beta_{\mathrm{LD}} \in H^1(M)$ denote the Lefschetz dual of β, we see that $\beta_{\mathrm{LD}}([\ell]) > 0$. Viewing $[\ell]$ as a linear functional on $H^1(M)$, we see that $[\ell]$ is strictly positive on $\mathrm{int}(\sigma_{\mathrm{LD}})$, so $[\ell]$ is nonnegative on $\mathrm{cone}(\sigma_{\mathrm{LD}})$, whence $\ell \in \mathrm{cone}(\sigma_{\mathrm{LD}})^\vee \setminus \{0\}$.

As noted in the above remark, we have $\mathrm{cone}(\sigma_{\mathrm{LD}}) = (\mathcal{C}_\varphi)^\vee$. Hence $\mathrm{cone}(\sigma_{\mathrm{LD}})^\vee = \mathcal{C}_\varphi$, so $[\ell] \in \mathcal{C}_\varphi \setminus \{0\}$. ∎

We now introduce some ideas and notation that will be used in the following proof. If R is a union of 2-cells and 3-cells of $\mathring{\tau}$, we say that a 2-cell f in R is *outward (inward) pointing* if $f \in \partial R$ and the 3-cell above (below) f is not in R. Note that a 2-cell can be both inward and outward pointing, for example if R is a single 2-cell. Recall that $N(B_{\mathring{\tau}})$ denotes a regular neighborhood of $B_{\mathring{\tau}}$, foliated by intervals. Let

$$\mathrm{coll}\colon N(B_\tau) \to B_\tau$$

be the map collapsing the intervals.

Proposition 4.5. *A surface carried by $B_{\mathring{\tau}}$ is a fiber of $\mathring{M}$ if and only if it is infinitely flippable.*

Proof. The forward implication follows immediately from Observation 4.2.

For the other direction, we will show that if a flipping sequence is such that there is a 3-cell of $\mathring{\tau}$ which is not swept across by a flip in the sequence, then the sequence is finite. Therefore, if S is infinitely flippable, some integer multiple of $[S]$ will be fully carried by $B_{\mathring{\tau}}$. Any fully carried homology class has positive intersection with any closed transversal to τ and thus is represented by a fiber by Proposition 4.3 and Theorem A.7. Hence S is a taut by Lemma 2.6. Any taut surface homologous to a fiber is isotopic to a fiber by [22, Theorem 4], so S will be a fiber.

Let $\{S_i\}$ be a flipping sequence. We assume that S_1 is connected, as otherwise we can apply the following reasoning to the flipping sequence associated to each component of S_1. Let R_n be the union of $\bigcup_{i=1}^n \mathrm{coll}(S_i)$ with all 3-cells swept across by the flipping sequence $\{S_i\}_{i=1}^n$. Since $\mathring{\tau}$ is a finite complex, the chain $R_1 \subset R_2 \subset R_3 \subset \cdots$ stabilizes at some R_N. Let $R = R_N$ and suppose that $R \subsetneq M$. We claim that there is some 3-cell t of $\mathring{\tau}$ whose bottom π-edge e lies in R such that t does not lie in R. This follows from the fact that each R_i possesses an equal number of inward and outward pointing 2-cells, so if $R \subsetneq M$, then R must have an outward pointing 2-cell.

Let S_k be a surface such that $\mathrm{coll}(S_k)$ contains e. Any surface carried by $B_{\mathring{\tau}}$ inherits an ideal triangulation of its interior in the obvious way, and a flipping sequence gives a sequence of diagonal exchanges between ideal triangulations of a reference copy of the surface. Let Σ be a reference copy of S_k. Then $\{S_i\}_{i \geq k}$ gives a sequence of ideal triangulations of Σ related by diagonal exchanges, and e is an edge of each triangulation.

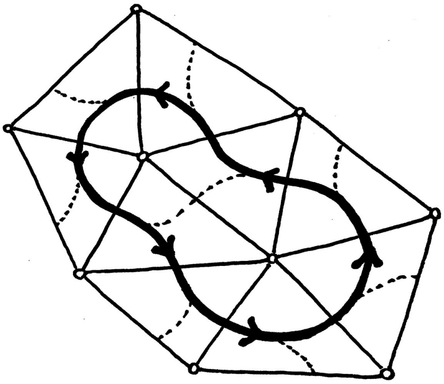

Figure 22. A stable loop.

We say that an edge in this sequence of diagonal exchanges is *adjacent* to a particular diagonal exchange if it is a boundary edge of the ideal quadrilateral whose diagonal is exchanged. Because each edge of $\mathring{\tau}$ is incident to only finitely many tetrahedra, each edge in this sequence of ideal triangulations of σ can be adjacent to only finitely many diagonal exchanges before it either disappears or remains forever. Since e is present in each triangulation, there exists $j \geq k$ and edges e', e'' which form a triangle with e such that the triangle (e, e', e'') is present in S_i for $i \geq j$. Since e' is adjacent to only finitely many diagonal exchanges, it is also eventually incident to a triangle (e', e''', e'''') that is fixed by the sequence of diagonal exchanges, and similarly for e''. Each triangulation has the same number of triangles, so continuing in this way we eventually cover Σ by triangles which are fixed. Therefore, the sequence is finite. ∎

Remark 4.6. We did not use the veering structure in the proof of the reverse direction, so it holds for general taut ideal triangulations.

4.5. Stable loops

Let λ be a closed curve carried by $\mathrm{St}(\tau)$. If λ has the property that it traverses alternately small and mixed branches of $\mathrm{St}(\tau)$, we call λ a *stable loop*. If λ additionally has the property that it traverses each switch of $\mathrm{St}(\tau)$ at most once, then we say that λ is a *minimal stable loop*. Since τ consists of finitely many ideal tetrahedra, $\mathrm{St}(\tau)$ has finitely many switches and thus finitely many minimal stable loops. We endow each stable loop λ with an orientation such that at a switch in the interior of a 2-cell, λ passes from a mixed branch to a small branch (see Figure 22).

Note that for any veering triangulation ρ, $\mathrm{St}(\rho)$ has stable loops. It is easily checkable that for any 2-cell $\triangle$ of $\mathring{\rho}$, $\mathrm{St}(\rho) \cap \triangle$ determines the left or right veeringness of two out of the three edges of $\triangle$ not lying in the boundary of the ambient three-manifold, as shown in Figure 23. The small branch of $\mathrm{St}(\rho)$ incident to a right veering edge is called a *right small branch*, and a *left small branch* is defined symmetrically. To produce a stable loop

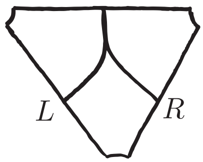

Figure 23. The intersection of the stable traintrack of a veering triangulation restricted to any 2-cell determines the left or right veeringness of two edges as shown, where the coorientation is pointing out of the page.

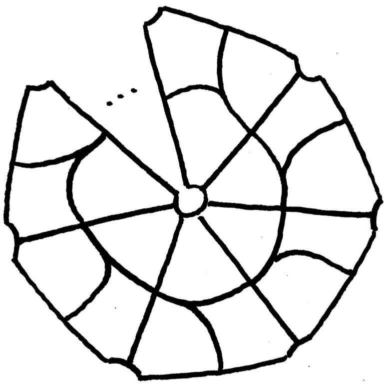

Figure 24. The 2-cells incident to a ladderpole give rise to a stable loop.

we can choose, for example, the left ladderpole L of an upward ladder and look at the collection Λ of all 2-cells of $\mathring{\rho}$ meeting L. The edges of $\mathring{\rho}$ corresponding to vertices in L are all right veering by Lemma 2.7, so $\mathrm{St}(\rho) \cap \Lambda$ is as shown in Figure 24, and carries a stable loop. We claim that this stable loop is also minimal, or equivalently that the edges of L are in bijection with the 2-cells of Λ. This is true: for each 2-cell $\triangle$ meeting L, $\triangle$ is incident to L along the *unique* truncated ideal vertex bounded by the edges meeting the large branch and right small branch of $\mathrm{St}(\rho) \cap \triangle$.

Observe that for any stable loop λ carried by $\mathrm{St}(\tau)$, we have $[\lambda] \in \mathcal{C}_\tau$. This is because the condition that λ traverses alternately mixed and small branches of $\mathrm{St}(\tau)$ implies that λ can be perturbed to a closed τ-transversal, as shown in Figure 25. Hence we have the following lemma.

Lemma 4.7. *Let λ be a stable loop. Then $[\lambda] \in \mathcal{C}_\varphi$.*

Let S be a surface carried by $\mathring{\tau}$ which is not a fiber of a fibration $\mathring{M} \to S^1$. By Proposition 4.5, any flipping sequence starting with S is finite. Therefore, S is isotopic to a surface S' which is carried by $B_{\mathring{\tau}}$ such that S' has no upward flips. We call such a surface *unflippable*.

Proposition 4.8. *Let S be a surface carried by $B_{\mathring{\tau}}$ which is unflippable. Then* $\mathrm{St}(S)$ *carries a stable loop of τ.*

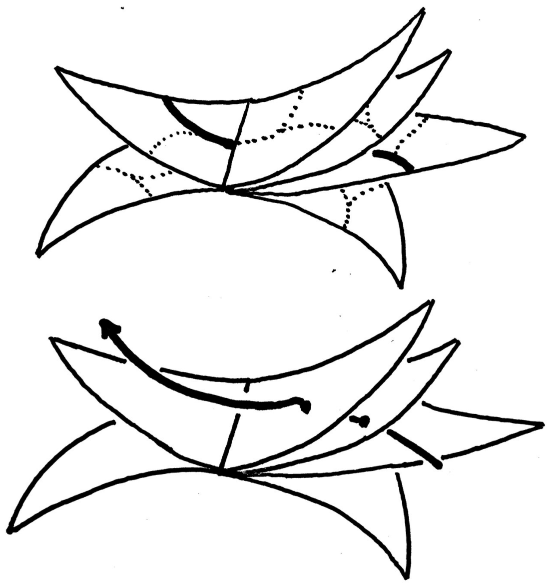

Figure 25. A stable loop may be perturbed to a closed τ-transversal.

Proof. The unflippability of S is equivalent to $\mathrm{St}(S)$ having no large branches. We define a curve carried by $\mathrm{St}(S)$ as follows: start at any switch of $\mathrm{St}(S)$, and travel along its large half-branch. When arriving at the next switch, exit along that switch's large half-branch, and so on. Since $\mathrm{St}(S)$ has finitely many branches, eventually the path will return to a branch it has previously visited, at which point we obtain a closed curve carried by $\mathrm{St}(S)$. By construction, it alternates between mixed and small branches of $\mathrm{St}(\tau)$. ■

We now prove the main result of this section. Recall that the subscript LD attached to an object denotes its image under Lefschetz duality.

Theorem 4.9 (Stable loops). *Let M be a compact hyperbolic three-manifold with fibered face σ. Let τ and φ be the associated veering triangulation and circular pseudo-Anosov flow, respectively. Then $\mathcal{C}_\varphi$ is the smallest convex cone containing the homology classes of the minimal stable loops of τ.*

Proof. By Lemma 4.7, the cone generated by the minimal stable loops lies in $\mathcal{C}_\varphi$. Hence it suffices to show that every 1-dimensional face of $\mathcal{C}_\varphi$ is generated by the homology class of a minimal stable loop.

Suppose first that $\mathrm{dimension}(H_2(M, \partial M)) = 1$. Then $H_1(M) = \mathbb{R}$, and $\mathcal{C}_\varphi$ is a ray. Let λ be a minimal stable loop of τ. Since $[\lambda] \in \mathcal{C}_\varphi \setminus \{0\}$ by Proposition 4.3, we have proved the claim in this case.

Now suppose that $\mathrm{dimension}(H_2(M, \partial M)) > 1$. Consider a 1-dimensional face Φ of $\mathcal{C}_\varphi = (\mathrm{cone}(\sigma_{\mathrm{LD}}))^\vee$, which can be characterized as $\Phi = (F_{\mathrm{LD}})^*$ for some facet F of $\mathrm{cone}(\sigma)$.

Let $\alpha \in H_2(M, \partial M)$ be a primitive integral class lying in the relative interior of F. By Corollary 2.5, Proposition 4.5, and Proposition 4.8, there exists an unflippable surface $\mathring{S}$ representing $\mathring{\alpha}$ and carried by $B_{\mathring{\tau}}$ such that $\mathrm{St}(\mathring{S})$ carries a stable loop λ.

As in the proof of Proposition 4.3, we can cap off $\mathring{S}$ to obtain a surface S in M representing α with $S \cap \mathring{M} = \mathring{S}$. Since λ may be isotoped off of S, $\alpha_{\mathrm{LD}} \in \ker([\lambda])$, viewing $[\lambda]$ as a linear functional on $H^1(M)$. It follows that $[\lambda] \in \Phi \setminus \{0\}$, so Φ is generated by $[\lambda]$. Because F has codimension 1, $(F_{\mathrm{LD}})^*$ has dimension 1 and is thus generated by $[\lambda]$.

Since $\mathring{S}$ is embedded in $N(B_{\mathring{\tau}})$, λ never traverses a switch of $\mathrm{St}(\tau)$ in two directions. This is clear for switches of $\mathrm{St}(\tau)$ lying in the interiors of 2-cells, and for a switch lying on an edge of $\mathring{\tau}$; such behavior would force $\mathring{S}$ to be non-embedded.

Any curve carried by a train track which traverses each switch in at most one direction, and also never traverses the same branch twice, must never traverse the same vertex twice. Therefore, by cutting and pasting λ along any edges of $\mathrm{St}(\tau)$ carrying λ with weight > 1, we see that λ is homologous to a union of minimal stable loops $\lambda_1, \ldots, \lambda_n$:

$$[\lambda] = [\lambda_1] + \cdots + [\lambda_n].$$

Since $[\lambda]$ lies in a one-dimensional face of $\mathcal{C}_\varphi$, we conclude that for each n there is some positive integer n_i such that $\lambda = n_i[\lambda_i]$. It follows that Φ is generated by the homology class of a minimal stable loop. ∎

Remark 4.10. We could just as easily have defined *unstable loops* using a symmetrically defined *unstable train track* of τ, and flipped surfaces downward in the arguments above to achieve the corresponding results.

Combining Proposition 4.3 and Theorem 4.9, we have the following immediate corollary, which is not obvious from the definitions.

Corollary 4.11. *The cone $\mathcal{C}_\tau$ is the smallest convex cone containing the homology classes of the stable loops of τ.*

A. The suspension flow is canonical when our manifold has boundary

The purpose of this appendix is to record a generalization of Fried's theory, relating circular pseudo-Anosov flows on closed three-manifolds to fibered faces, that does not currently exist in the literature. The generalization here is to the case of circular pseudo-Anosov flows on compact three-manifolds, possibly with boundary.

Theorem A.7. *Let φ be a circular pseudo-Anosov flow on a compact three-manifold M with cross section Y. Let σ be the fibered face of $B_x(M)$ such that $[Y] \in \mathrm{cone}(\sigma)$. Let $\alpha \in H_2(M, \partial M)$ be an integral class. The following are equivalent:*

(1) *α lies in* $\mathrm{int}(\mathrm{cone}(\sigma))$,

(2) *the Lefschetz dual of α is positive on the homology directions of φ,*

(3) *α is represented by a cross section to φ.*

Moreover, φ is the unique circular pseudo-Anosov flow admitting cross sections representing classes in $\operatorname{cone}(\sigma)$ *up to reparameterization and conjugation by homeomorphisms of M isotopic to the identity.*

Homology directions are essentially projectivized homology classes of nearly-closed orbits of φ. Their precise definition is given below, in Section A.1.

Some experts may have verified for themselves that this generalization works. However, the results in this paper depend on it, so we include a proof. Our proof attempts to follow the arguments of Fried, making modifications when necessary to deal with boundary components.

A.1. Homology directions and cross sections

We begin by recalling some definitions and a result from [8]. Let M be a compact smooth manifold, and let D_M be the quotient of $H_1(M)$ by positive scalar multiplication, endowed with the topology of the disjoint union of a sphere and an isolated point corresponding to 0. We denote the quotient map $H_1(M) \to D_M$ by π. Let φ be a C^1 flow on M which is tangent to ∂M. Let $\varphi_t(a)$ denote the image of $a \in M$ under the time t map of φ.

A *closing sequence based at $m \in M$* is a sequence of points $(m_k, t_k) \in M \times \mathbb{R}$ with $m_k \to m \in M$, $\varphi_{t_k}(m_k) \to m$, and $t_k \not\to 0$. For sufficiently large k, the points m_k and $\varphi_{t_k}(m_k)$ lie in a small ball B around M. We can define a closed curve γ_k based at m by traveling along a short path in B from m to m_k, flowing to $\varphi_{t_k}(m_k)$, and returning to m by a short path in B. The γ_k are well defined up to isotopy.

Since D_M is compact, $\pi([\gamma_k])$ must have accumulation points. Any such accumulation point δ is called a *homology direction for φ*. We call (m_k, t_k) a *closing sequence for δ* if $\pi([\gamma_k]) \to \delta$. The set of homology directions for φ is denoted by D_φ. Let $\delta \in D_M$ and $\alpha \in H^1(M)$. While $\alpha(\delta)$ is not well defined unless we choose a norm on $H_1(M)$ and identify D_M with the vectors of length 0 and 1, we can say whether α is positive, negative, or zero on δ.

We define $\mathcal{C}_\varphi$, the *cone of homology directions* of φ, by $\mathcal{C}_\varphi := \pi^{-1}(D_\varphi \cup \{0\})$.

Fried gives a useful criterion for when an integral cohomology class in $H^1(M)$ is *compatible with a cross section*, i.e., has a cross section of φ representing its Lefschetz dual.

Theorem A.1 (Fried). *Let $\alpha \in H^1(M)$ be an integral class. Then α is compatible with a cross section to φ if and only if $\alpha(\delta) > 0$ for all $\delta \in D_\varphi$.*

We relate two useful observations of Fried. The first is that if (m_k, t_k) is a closing sequence for $\delta \in D_\varphi$ based at m and t_k has a bounded subsequence, then m lies on a periodic orbit o_m and $\delta = \pi([o_m])$. Thus $(m_k = m,\ t_k = kp)$, where p is the period of m, is also a closing sequence for δ. The second observation is that if φ admits a cross section Y, then each $\delta \in D_\varphi$ admits a closing sequence (m_k, t_k) with $m, m_k, \varphi_{t_l}(m_k) \in Y$. This can be seen by flowing each point in a closing sequence for δ until it meets Y for the first time. We record these observations in a lemma.

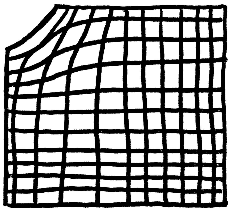

Figure 26. A pentagon. The diagonal edge on the upper left is a ∂-edge.

Lemma A.2 (Fried). *Suppose that φ admits a cross section Y, and let $\delta \in D_\varphi$. Then δ admits a closing sequence (m_k, t_k) based at m with $m, m_k, \varphi_{t_k}(m_k) \in Y$ and $t_k \to \infty$.*

A.2. Proving Theorem A.7

Throughout this section, our notation mimics that in Section 2.4. We consider a compact hyperbolic three-manifold M and a circular pseudo-Anosov flow φ on M admitting a cross section Y with first return map g. We assume that φ is parameterized so that for all $z \in Z$, we have $\varphi_1(z) = g(z)$.

As in [5, Exposé 10, §10.5], we can find a Markov partition $\mathcal{M}$ for $g\colon Y \to Y$, where the definition of Markov partition is altered slightly to account for the fact that Y may have boundary (Markov "rectangles" touching ∂Y are actually pentagons). We will call the elements of $\mathcal{M}$ *shapes*, and the elements of $\mathcal{M}$ which touch ∂Y *pentagons*. By the construction in [5] we can assume that an edge of a pentagon meeting ∂Y in a single point is contained in a separatrix of the stable or unstable foliation of g. Each pentagon has a single edge entirely contained in ∂Y called a ∂-edge; see Figure 26.

Let G be the directed graph associated to $\mathcal{M}$, whose vertices are labeled by the elements of $\mathcal{M}$ and whose edges are $(r_i, r_j) \in \mathcal{M} \times \mathcal{M}$, where g stretches r_i over r_j. By a *cycle* or *path* in G, we mean a directed cycle or directed path, respectively. As in the case of closed surfaces, G is a strongly connected directed graph, meaning that for any vertices r_i, r_j of G, there exists a path from r_i to r_j.

Let $\mathcal{O}_\varphi$ be the set of closed orbits of φ, and let L_G be the collection of cycles in G. Let $p_1 \in \mathcal{M}$ be a pentagon with ∂-edge e. The image of e under g^i is a ∂-edge of some pentagon p_i, and there exists some finite n for which $(p_1, \ldots, p_n)$ is a cycle. Let $\partial L_G \subset L_G$ be the collection of all such cycles.

We now define a surjection $\omega\colon L_G \to \mathcal{O}_\varphi$. Let $\ell \in \partial L_G$, and pick one of its vertices labeled by a pentagon $p \in \mathcal{M}$. Let $e = p \cap \partial Y$. If we give ∂Y the orientation induced by an outward-pointing vector field, we induce an orientation on e. Let e_+ be the positive endpoint of e with respect to this orientation. Since the other edge of p containing e_+ lies in a separatrix of the stable or unstable foliation of g, the orbit $o(e_+)$ of e_+ under φ is a ∂-singular orbit. Let $\omega(\ell) = o(e_+)$; this is well defined (i.e., it does not depend on the initial choice of p). Next, any cycle $\ell \in L_G \setminus \partial L_G$ determines a unique closed orbit of φ, just as in the case when Y is a closed surface. We let $\omega(\ell)$ be this closed orbit.

We say that a path $(r_1, \dots, r_n)$ in G is *simple* if $r_i \neq r_j$ for $1 \le i,\ j \le n,\ i \neq j$. Similarly, a cycle $(r_1, \dots, r_n)$, $r_1 = r_n$ in G is *simple* if $r_i \neq r_j$ for $2 \le i,\ j \le n-1$, $i \neq j$. Let $S_G \subset L_G$ be the set of simple cycles in G. Since $\mathcal{M}$ is finite, S_G is finite.

We now define a finite set $B \subset H_1(M)$. Let $B_1 = \{[\omega(s)] \in H_1(M) \mid s \in S_G\}$. For every simple path $p = (r_1, \dots, r_n) \in G$ which is *almost closed*, i.e., starts and ends on vertices labeled by shapes in $\mathcal{M}$ meeting along an edge or vertex, let $o(p)$ be an orbit segment starting in r_1, passing sequentially through the r_i's, and ending in r_n. Let $\varepsilon(p)$ be a segment in Y connecting the endpoints of $o(p)$ and supported in $r_1 \cup r_2$. Let $B_2 = \{[o(p) * \varepsilon(p)] \mid p \text{ is an almost closed simple path in } G\}$, where $*$ is concatenation of paths. Let $B = B_1 \cup B_2$. The salient feature of B that will be used below is that it is finite, and hence bounded.

Lemma A.3. *The cone C_φ of homology directions of φ is a finite-sided rational convex polyhedral cone.*

Proof. First, we claim that $\mathcal{C}_\varphi$ is generated by the set of homology classes of orbits of φ, $\{[o] \mid o \in \mathcal{O}_\varphi\}$.

Let $\delta \in D_\varphi$. By Lemma A.2, δ admits a closing sequence (m_k, t_k) based at $m \in Y$ with $m_k \in Y$, $t_k \in \mathbb{Z}_+$ for all k, and $t_k \to \infty$.

Consider the curves γ_k, which for sufficiently large k we can express as

$$\gamma_k = o_k * \varepsilon_k,$$

where o_k is the curve $\varphi_t(m_k)$, $0 \le t \le t_k$ and ε_k is a short curve in Y from $\varphi_{t_k}(m_k)$ to m_k supported in the union of at most two shapes of $\mathcal{M}$. We can lift o_k to a path $r(o_k) = (r_1, \dots, r_{n(k)})$ in G. For sufficiently large k, $r(o_k)$ is not simple. Let

$$\ell(o_k) = (r_a, r_{a+1}, \dots, r_{b-1}, r_b)$$

be the longest cycle subpath of $r(o_k)$. Then $(r_1, \dots, r_{a-1}, r_a, r_{b+1}, \dots, r_{n(k)})$ is simple and corresponds to some orbit segment q_k with endpoints in rectangles which are either equal or intersect along an edge or vertex. Let s_k be a segment connecting the endpoints of $q(k)$ supported in $r_1 \cup r_{n(k)}$. We have

$$[\gamma_k] = [q_k * s_k] + \big[\omega(\ell_k)\big],$$

and $[q_k * s_k] \in B$. Letting $k \to \infty$, the intersection of $[\gamma_k]$ with $[Y]$ approaches infinity so $\{[\gamma_k]\}$ is unbounded. Since B is bounded, we conclude that

$$\lim_{k\to\infty} \pi\big([\gamma_k]\big) = \lim_{k\to\infty} \pi\big(\omega(\ell_k)\big),$$

so δ is projectively approximated by homology classes of closed orbits. Since the homology class of each closed orbit lies in $\mathcal{C}_\varphi$, we have that $\mathcal{C}_\varphi$ is the smallest closed cone containing the homology classes of closed orbits of φ as claimed.

Next we will show that $\mathcal{C}_\varphi$ is convex. Let λ_1, λ_2 be two closed orbits of φ that respectively pass through rectangles r_1 and r_2. Let ℓ_i, $i = 1, 2$, be cycles in G such that $\omega(\ell_i) = \lambda_i$. Let $\nu_{1,2}$ (resp., $\nu_{2,1}$) be a path in G from r_1 to r_2 (resp., r_2 to r_1). Letting

$$\gamma_n = \omega\big((\ell_1)^n * \nu_{1,2} * (\ell_2)^n * \nu_{2,1}\big),$$

we have

$$[\gamma_n] = n[\lambda_1] + n[\lambda_2] + \big[\omega(\nu_{1,2} * \nu_{2,1})\big].$$

Therefore,

$$\pi\big([\lambda_1] + [\lambda_2]\big) = \lim_{n\to\infty} \pi\big([\gamma_n]\big) \in D_\varphi,$$

so $[\lambda_1] + [\lambda_2] \in \mathcal{C}_\varphi$ and $\mathcal{C}_\varphi$ is convex.

It remains to show that $\mathcal{C}_\varphi$ is finite-sided and rational. To do this, it suffices to show that $\mathcal{C}_\varphi$ is the convex cone generated by $S_\varphi = \{[\omega(\ell)] \mid \ell \in S_G\}$. It is clear that $S_\varphi \subset \mathcal{C}_\varphi$. On the other hand, let $o \in \mathcal{O}_\varphi$. There exists some cycle ℓ in G such that $\omega(\ell) = o$, and ℓ is a concatenation of simple cycles $\ell = s_1 * \cdots * s_n$. By cutting and pasting, we see that

$$[o] = \big[\omega(s_1)\big] + \cdots + \big[\omega(s_n)\big].$$

Hence $\mathcal{O}_\varphi$ is contained in the cone generated by S_φ, so $\mathcal{C}_\varphi$ is also. ∎

Let F be a circular flow on M. Following Fried, we define two sets in $H^1(M)$:

$$\mathcal{C}_\mathbb{R}(F) = \big\{u \in H^1(M) \mid u(D_F) > 0\big\},$$

and

$$\mathcal{C}_\mathbb{Z}(F) = \big\{u \in \mathcal{C}_\mathbb{R}(F) \mid u \text{ is an integral point}\big\}.$$

We can think of $\mathcal{C}_\mathbb{R}(F)$ as the set of linear functionals on $H_1(M)$ which are positive on $\pi^{-1}(D_F)$. Since this is an open condition, $\mathcal{C}_\mathbb{R}(F)$ is an open cone, and it is also clearly convex.

Proposition A.4. *Let F and F' be two circular pseudo-Anosov flows on M. Then $\mathcal{C}_\mathbb{R}(F)$ and $\mathcal{C}_\mathbb{R}(F')$ are either disjoint or equal.*

Proof. Suppose that $\mathcal{C}_\mathbb{R}(F) \cap \mathcal{C}_\mathbb{R}(F')$ is nonempty. We will show that $D_F = D_{F'}$ and hence $\mathcal{C}_\mathbb{R}(F) = \mathcal{C}_\mathbb{R}(F')$.

The intersection is open, so we can find a primitive class $u \in \mathcal{C}_\mathbb{Z}(F) \cap \mathcal{C}_\mathbb{Z}(F')$. By Theorem A.1, there are fibrations $f, f' \colon M \to S^1$ whose fibers are transverse to F and F', respectively, and are homologous. Let Z and Z' be fibers of f and f', respectively. By [22, Theorem 4], Z' is isotopic to Z. By the isotopy extension theorem, the isotopy extends to an ambient isotopy of M.

Let F'' be the image of F' under this isotopy; Z is a cross section of F and F''. We reparametrize F and F'' so that the first return maps $\rho, \rho'' \colon Z \to Z$ of F and F'' are given by flowing for time 1 along the respective flows.

The maps ρ and ρ'' are both pseudo-Anosov representatives of the same isotopy class, so they are strictly conjugate. This means that there exists a map $g: Z \to Z$ which is isotopic to the identity such that

$$\rho \circ h = h \circ \rho''.$$

This isotopy extends to an ambient isotopy of M. Let F''' denote the image of F'' under this ambient isotopy. By construction, the first return map of F''' on Z is ρ.

Now we need a lemma.

Lemma A.5. *Let $p \in \partial Z$ be the boundary point of a leaf ℓ_p of the stable or unstable foliation of ρ. Then $F_t(p)$ and $F_t'''(p)$, $0 \leq t \leq 1$, are homotopic in ∂M rel endpoints.*

Proof of Lemma A.5. We first cut M open along Z. The result is a manifold with boundary that we can identify with $Z \times [0, 1]$, such that F is identified with the vertical flow. Let $\pi_Z: Z \times [0, 1] \to Z \times \{0\}$ be the projection. We identify $Z \times \{0\}$ with Z.

Consider the homotopy $g_t: Z \times [0, 1] \to Z$ given by $g_t(z) = \pi_Z(F_t'''(z))$. We claim that $g_t(p)$, $0 \leq t \leq 1$, is not an essential loop in ∂Z.

Let $\widetilde{Z}$ be the universal cover of Z, and let $\tilde{g}_t$ be the unique homotopy of $\mathrm{id}_{\widetilde{Z}}$ that covers g_t. We see that g_t preserves each component of the union of lines in $\partial\widetilde{Z}$ covering ∂Z. Hence it fixes the ends of $\widetilde{Z}$.

Let $\tilde{\ell}_p$ be a lift of ℓ_p to $\widetilde{Z}$. It is a ray $[0, \infty) \to \widetilde{Z}$ with its endpoint on a lift $\widetilde{\partial_\ell}$ of a component of ∂Z and the other end exiting an end of $\widetilde{Z}$.

Suppose that $g_t(p)$, $0 \leq t \leq 1$, is essential in ∂Z. Then $\tilde{g}_1$ carries $\tilde{\ell}_p$ to a separate lift $\widetilde{\ell_p}'$ of ℓ_p with its endpoint on $\widetilde{\partial_\ell}$. Since $\widetilde{g_1}$ fixes the ends of $\widetilde{Z}$, $\widetilde{\ell_p}$ and $\widetilde{\ell_p}'$ must exit the same end. We conclude that $\widetilde{\ell_p} = \widetilde{\ell_p}'$, a contradiction.

It follows that $F_t'''(p)$, $0 \leq t \leq 1$, can be homotoped rel endpoints to a vertical arc in $\partial Z \times [0, 1]$, proving the claim. ∎

With our lemma in hand, we can finish proving Proposition A.4. We define a map conjugating F and F''', which we will show is isotopic to the identity. For $z \in Z$, $t \in [0, 1]$ let

$$C\big(F_t(z)\big) = F_t'''(z).$$

As the first return maps of F and F''' to Z are both equal to ρ, C is well defined.

Fix a basepoint $z_o \in \partial Z$ lying in a ∂-singular orbit o of F and let ζ_o be a curve which starts at z_o, travels along o for time 1, and returns to z_o via a path in Z. If G is a set of generators of $\pi_1(Z, z_o)$, then $G \cup \{\zeta\}$ generates $\pi_1(M, z_o)$. Since C restricted to Z is the identity, $C_*: \pi_1(M, z_o) \to \pi_1(M, z_o)$ fixes each element of G. Since C_* also fixes $[\zeta_o]$ by Lemma A.5, we see that C_* is the identity map. Since M is a $K(\pi_1(M), 1)$ space, C must be homotopic to the identity. In fact, C is isotopic to the identity by a theorem of Waldhausen [24, Theorem 7.1] which states that any homeomorphism of a compact, irreducible, boundary irreducible, Haken three-manifold which is homotopic to the identity is isotopic to the identity.

Conjugating a flow by a homeomorphism isotopic to the identity does not change its set of homology directions. Hence $D_F = D_{F'}$, so $C_{\mathbb{R}}(F) = \mathcal{C}_{\mathbb{R}}(F')$ as desired. ∎

Let σ_{LD} denote the image of σ under the Lefschetz duality isomorphism

$$H_2(M, \partial M) \cong H^1(M).$$

Proposition A.6. $\mathcal{C}_{\mathbb{R}}(\varphi) = \mathrm{int}(\mathrm{cone}(\sigma_{\mathrm{LD}}))$.

Proof. Let α be Lefschetz dual to a class in $\mathcal{C}_{\mathbb{Z}}(\varphi)$. By Theorem A.1, α is represented by a cross section Σ to φ. By [22, Theorem 5], α lies interior to the cone over *some* top-dimensional face of $B_x(M)$. We show that this face is in fact σ.

As a leaf of a taut foliation, Σ is taut. Since Σ is a cross section to φ, the tangent plane field $T\Sigma$ of the fibration $\Sigma \hookrightarrow M \to S^1$ is homotopic to ξ_φ (recall from Section 2.1 that ξ_φ is the quotient of T_M by T_φ, the tangent line bundle to the 1-dimensional foliation by flowlines of φ). The same is true for TY, so the relative Euler classes of the two plane fields are equal. Let e_Y denote this Euler class. We have

$$x(\alpha) = -\chi(\Sigma) = -e_Y(\alpha),$$

so α lies in the portion of $H_2(M, \partial M)$, where x agrees with $-e_Y$. By the discussion in Section 2.1, this is $\mathrm{cone}(\sigma)$.

It follows that

$$\mathcal{C}_{\mathbb{Z}}(\varphi) \in \mathrm{int}\left(\mathrm{cone}(\sigma_{\mathrm{LD}})\right),$$

so every rational point in $\mathcal{C}_{\mathbb{R}}(\varphi)$ lies in $\mathrm{int}(\mathrm{cone}(\sigma_{\mathrm{LD}}))$. Since $\mathcal{C}_{\mathbb{R}}(\varphi)$ and $\mathrm{int}(\mathrm{cone}(\sigma_{\mathrm{LD}}))$ are both open, $\mathcal{C}_{\mathbb{R}}(\varphi) \subset \mathrm{int}(\mathrm{cone}(\sigma_{\mathrm{LD}}))$.

Now suppose that

$$\mathcal{C}_{\mathbb{R}}(\varphi) \subsetneq \mathrm{int}\left(\mathrm{cone}(\sigma_{\mathrm{LD}})\right).$$

We have $\mathcal{C}_{\mathbb{R}}(\varphi) = \mathrm{int}(\mathcal{C}_\varphi^\vee)$. By Lemma A.3, $\mathcal{C}_\varphi$ is a rational convex polyhedral cone, so $\mathcal{C}_{\mathbb{R}}(\varphi)$ is the interior of a rational convex polyhedral cone. Hence there is an integral cohomology class $v \in \mathrm{int}(\mathrm{cone}(\sigma_{\mathrm{LD}})) \cap \partial\mathcal{C}_{\mathbb{R}}(\varphi)$.

The Lefschetz dual of v is represented by a cross section to another circular pseudo-Anosov flow φ'. We must have $\mathcal{C}_{\mathbb{R}}(\varphi) \cap \mathcal{C}_{\mathbb{R}}(\varphi') \neq \varnothing$, but the cones cannot be equal because $v \notin \mathcal{C}_{\mathbb{R}}(\varphi)$. This contradicts Proposition A.4, so we conclude that

$$\mathcal{C}_{\mathbb{R}}(\varphi) = \mathrm{int}\left(\mathrm{cone}(\sigma_{\mathrm{LD}})\right).$$ ∎

We remark that the proof of the inclusion $\mathcal{C}_{\mathbb{R}}(\varphi) \subset \mathrm{int}(\mathrm{cone}(\sigma_{\mathrm{LD}}))$ did not require φ to be pseudo-Anosov, so the corresponding statement is still true if we replace M by any compact three-manifold and φ by any circular flow.

We conclude this section by observing that we have proven Theorem A.7.

Theorem A.7. *Let φ be a circular pseudo-Anosov flow on a compact three-manifold M with cross section Y. Let σ be the fibered face of $B_x(M)$ such that $[Y] \in \mathrm{cone}(\sigma)$. Let $\alpha \in H_2(M, \partial M)$ be an integral class. The following are equivalent:*

(1) α *lies in* $\operatorname{int}(\operatorname{cone}(\sigma))$,

(2) *the Lefschetz dual of* α *is positive on the homology directions of* φ,

(3) α *is represented by a cross section to* φ.

Moreover, φ *is the unique circular pseudo-Anosov flow admitting cross sections representing classes in* $\operatorname{cone}(\sigma)$ *up to reparameterization and conjugation by homeomorphisms of* M *isotopic to the identity.*

Proof of Theorem A.7. (1)$\Leftrightarrow$(2): This is a restatement of Proposition A.6.

(2)$\Leftrightarrow$(3): This is a restatement of Theorem A.1.

The truth of the last claim can be seen from the proof of Proposition A.4. Recall that we showed that if F^1, F^2 are circular pseudo-Anosov flows admitting homologous cross sections, then they are conjugate by a homeomorphism of M isotopic to the identity. ■

B. Face-spanning taut homology branched surfaces in manifolds with boundary

Let M be a three-manifold such that x is a norm on $H_2(M, \partial M)$, and let B be a taut branched surface in M. The cone of homology classes carried by B is contained in $\operatorname{cone}(F)$ for some face F of $B_x(M)$ (this is because one can see, via cutting and pasting surfaces carried by B, that x is linear on the cone of carried classes). If this cone of carried classes is *equal* to $\operatorname{cone}(F)$, we say that B *spans* F. In [20], Ulrich Oertel asked when a face of the Thurston norm ball is spanned by a single taut homology branched surface. Recall that *taut* means every surface carried by B is taut, and that a *homology branched surface* has a closed oriented transversal through every point.

In [13, Theorem 3.9], we gave a sufficient criterion for a fibered face of a closed hyperbolic three-manifold to admit a spanning taut homology branched surface via a construction using veering triangulations. In this appendix, we describe why that criterion is also sufficient in the broader setting of this paper, i.e., when the compact hyperbolic three-manifold in question possibly has boundary.

The general result is the following.

Theorem B.1. *Let* σ *be a fibered face of a compact hyperbolic three-manifold, and let* φ *be the suspension flow of* σ. *If each singular orbit of* φ *witnesses at most two ladderpole boundary classes of* σ, *then there exists a taut branched surface* B_σ *spanning* σ.

A *ladderpole vertex class* is a primitive integral class α lying in a 1-dimensional face of $\operatorname{cone}(\sigma)$ such that $\mathring{\alpha}$ is represented by a surface $\mathring{A}$ carried by $B_{\mathring{\tau}}$ and for some U_i, $\partial\mathring{A} \cap U_i$ is a collection of ladderpole curves. Note that here we make no requirements on the boundary components of $\mathring{A}$ which lie in V.

The technical lemma that allows us to prove Theorem B.1 is the following, which was proven in [13] only for closed hyperbolic three-manifolds.

Lemma B.2. *Let σ, φ be as above. Let $\alpha \in \operatorname{cone}(\sigma)$ be an integral class. Then*

$$x(\alpha) = x(\mathring{\alpha}) - i(\alpha, c),$$

where c is the union of the singular orbits of φ.

Proof. Let $\mathring{A}$ be a surface carried by $B_{\mathring{\tau}}$ and representing $\mathring{\alpha}$. By our proof of Theorem 3.5, there exists a surface A which is almost transverse to φ and represents α with $\chi_-(A) = \chi_-(\mathring{A}) - i(\alpha, c)$. Since A is almost transverse to φ, A is taut. Since $B_{\mathring{\tau}}$ is a taut branched surface, $\mathring{A}$ is taut. Therefore, $x(\alpha) = x(\mathring{\alpha}) - i(\alpha, c)$. ∎

The proof above represents a significant shortening of the proof of the corresponding lemma in [13, Lemma 3.6]. The ingredients that make this possible are (a) we now know that the transverse surface theorem holds when our manifold has boundary and (b) we can assume that our almost transverse surface representative of α lies in a neighborhood of $B_{\mathring{\tau}}$ away from the singular orbits, and is simple in a neighborhood of the singular orbits.

With Lemma B.2 proven, the proof of Theorem B.1 proceeds exactly as in [13].

We once again observe that the condition on ladderpole vertex classes is satisfied when dimension$(H_2(M, \partial M)) \leq 3$, so we have the following corollary.

Corollary B.3. *Let σ be a fibered face of a compact hyperbolic three-manifold M such that the dimension of $H_2(M, \partial M)$ is at most three. Any fibered face of $B_x(M)$ is spanned by a taut homology branched surface.*

Finally, as a special case of the above we observe that the result holds for exteriors of links with ≤ 3 components.

Corollary B.4. *Let L be a fibered hyperbolic link with at most three components. Let M_L be the exterior of L in S^3. Any fibered face of $B_x(M_L)$ is spanned by a taut homology branched surface.*

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Michael Landry
Department of Mathematics and Statistics, Washington University in St. Louis,
St. Louis, MO 63130, USA; mlandry@wustl.edu