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Finitely generated groups acting uniformly properly on hyperbolic space

Robert Kropholler and Vladimir Vankov

Abstract. We construct an uncountable sequence of groups acting uniformly properly on hyperbolic spaces. We show that only countably many of these groups can be virtually torsion-free. This gives new examples of groups acting uniformly properly on hyperbolic spaces that are not virtually torsion-free and cannot be subgroups of hyperbolic groups.

1. Introduction

We say that a group G acts *properly* on a metric space X if for all $r > 0$ and $x \in X$ there exists N such that $|\{g \in G \mid d(x, gx) \leq r\}| \leq N$. A group is said to be *hyperbolic* if it acts properly and cocompactly by isometries on a hyperbolic metric space. It is currently not known whether hyperbolic groups which are not virtually torsion-free exist. There have been many interesting generalisations of this notion, for instance: acylindrically hyperbolic groups [14] or relatively hyperbolic groups [6]. In this paper, we will study the class of groups with uniformly proper actions studied in [4].

Definition 1.1. Let G be a group acting on a metric space X. We say that the action is *uniformly proper* if for every $r > 0$ there exists N such that for all $x \in X$,

$$\left|\left\{g \in G \mid d(x, gx) \leq r\right\}\right| \leq N.$$

One should note that N in this definition only depends on r and not on x. We insist on this, as otherwise all groups admit a proper action on a hyperbolic space, namely their combinatorial horoball [7].

The class of groups acting uniformly properly on hyperbolic spaces includes all subgroups of hyperbolic groups. In [4], the question of whether these two classes coincide is asked. We show that in fact there are uncountably many finitely generated groups acting on hyperbolic spaces, while there are only countably many finitely generated subgroups of hyperbolic groups. We also show that many of the examples created are not virtually torsion-free.

Theorem 1.2. *There exist uncountably many finitely generated groups H_W acting uniformly properly on hyperbolic spaces. Moreover, at most countably many of the groups obtained are virtually torsion-free.*

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The construction starts by making a subgroup H of a hyperbolic group G that is finitely generated but not finitely presented. The presentation for H has relators contained in a set V. By considering a subset $W \subset V$, we obtain a group H_W by replacing relators in W with their p-th powers.

The algebraic structure of the groups constructed is similar to that of subgroups of hyperbolic groups. Thus it would be of interest to obtain a characterisation of when these groups embed in hyperbolic groups.

For example, if the set of relators for which p-th powers are taken is chosen in a periodic way, then the group H_W is a subgroup of a hyperbolic group. In this instance, the hyperbolic group acts properly cocompactly on a CAT(0) cube complex and so is virtually special [1]. Hence, H_W is virtually torsion-free in this case.

One should compare this to [15], where a similar construction is applied to generalised Bestvina–Brady groups. There it is shown that if H_W is virtually torsion-free, then the set of relators for which p-th powers are taken has to be periodic. Therefore, we may conjecture that this is the case here as well.

Conjecture 1.3. *The group H_W is virtually torsion-free if and only if W is a periodic subset of V.*

This is the case if and only if H_W embeds in a hyperbolic group.

A similar conjecture regarding generalised Bestvina–Brady groups, in a non-hyperbolic context, appears in [11]. However, it is interesting to note that it is the converse which is open in that setting.

In Section 2, we construct a non-positively curved cube complex X whose fundamental group G is hyperbolic. In Section 3, we give a Morse function on X and find a subgroup of G that is finitely generated but not finitely presented. In Section 4, we take branched covers of the cube complex X to obtain uncountably many isomorphism classes of groups acting uniformly properly on hyperbolic spaces. Finally, in Section 5, we provide a criterion for when these groups are virtually torsion-free and show that this criterion can only be satisfied by countably many of the groups constructed in Section 4.

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2. A construction of a hyperbolic group

We begin by constructing one hyperbolic group with a subgroup that is finitely generated but not finitely presented. We will then take branched covers of a cube complex to obtain our sequence of groups with the desired properties.

The following construction can be found in [12]. We use a sizeable graph constructed in the appendix of [8]; for a figure of this graph see [5, Figure 6.1 on p. 102].

Let Γ be the graph with vertex set $A \sqcup B$, where $A = A^- \sqcup A^+$ and $B = B^- \sqcup B^+$, with $A^-, A^+, B^-, B^+ = \mathbb{Z}/9\mathbb{Z}$. There is an edge a to b if any of the following hold:

(1) if $a \in A^+, b \in B^+$, then $a = b$ or $a = b + 1$;

(2) if $a \in A^+, b \in B^-$, then $a = b$ or $a = b - 2$;

(3) if $a \in A^-, b \in B^+$, then $a = b$ or $a = b + 2$;

(4) if $a \in A^-, b \in B^-$, then $a = b + 1$ or $a = b + 2$.

Proposition 2.1. Γ *has no embedded loops of length* < 5.

The full subgraph of Γ *spanned by* $A^s \sqcup B^t$ *is a loop of length* 18*, for all choices of* s, t.

Proof. These conditions correspond to modularity conditions mod 9. For full details, see the appendix in [8]. ∎

Let Λ_A be the graph with two vertices a^+, a^- and $|A|$ edges each running from a^- to a^+. Define Λ_B similarly. The squares in $\Lambda_A \times \Lambda_B$ are in one-to-one correspondence with $A \times B$.

Let $X_\Gamma \subset \Lambda_A \times \Lambda_B$ be the cubical subcomplex containing $(\Lambda_A \times \Lambda_B)^{(1)}$ and those squares (a, b) such that (a, b) is an edge of Γ.

Proposition 2.2. *The link of any vertex of* X_Γ *is* Γ.

Proof. There are four vertices in X_Γ but the definition permits a symmetry taking any vertex to any other. Thus we will focus on the case of the vertex $v = (a^+, b^+)$. Let $L = \mathrm{Lk}(v, X_\Gamma)$.

Since $(\Lambda_A \times \Lambda_B)^{(1)} \subset X_\Gamma$, we see that $V(L) = A \sqcup B$. There is an edge in L from the vertex a to the vertex b exactly when there is a square at v with edges a, b. We can see that this is exactly the case when (a, b) is an edge of Γ. ∎

Since Γ is triangle-free, we deduce that X_Γ is a non-positively curved cube complex. We can now apply a theorem of Moussong [13].

Theorem 2.3. *Let* $\tilde{X}$ *be a* 2*-dimensional* CAT(0) *curved cube complex. Suppose that the link of each vertex does not contain an embedded loop of length* 4*. Then* $\tilde{X}$ *is hyperbolic; in fact, it supports a* CAT(-1) *metric.*

We already know that Γ has no cycles of length less than 5, therefore we can conclude the following corollary:

Corollary 2.4. $\pi_1(X_\Gamma)$ *is hyperbolic.*

3. The first example of a finitely generated, not finitely presented subgroup

To find the first example of a subgroup that is finitely generated but not finitely presented, we will use Morse theory. For details on Morse theory, see [2].

Throughout, the circle S^1 will be triangulated with one vertex and one edge, which we will give an arbitrary orientation.

Give each edge of Λ_A an orientation by orienting towards a^+ if $a \in A^+$ and towards a^- if $a \in A^-$. Define an orientation on Λ_B similarly.

Define $f_A \colon \Lambda_A \to S^1$ by mapping each edge linearly via its orientation. Define f_B similarly. We can now define a map $f \colon \Lambda_A \times \Lambda_B \to S^1 = \mathbb{R}/\mathbb{Z}$ by

$$(x, y) \mapsto f_A(x) + f_B(y).$$

We can restrict to X_Γ in order to define a function which we also denote $f \colon X_\Gamma \to S^1$.

Lifting this function to $\widetilde{X_\Gamma}$ gives us a Morse function $f \colon \widetilde{X_\Gamma} \to \mathbb{R}$.

Since the local geometry at each vertex of $\widetilde{X_\Gamma}$ is determined by that of X_Γ, we will abuse notation and label the vertices in $\widetilde{X_\Gamma}$ by the vertex they map to in X_Γ.

Proposition 3.1. *The descending link of (a^s, b^t) is the full subgraph of Γ spanned by $A^s \sqcup B^t$. The ascending link of (a^s, b^t) is the full subgraph of Γ spanned by $A^{-s} \sqcup B^{-t}$. Hence, both the ascending and descending links are cycles of length* 18.

Proof. The descending link of v is the full subgraph of Γ spanned by the vertices corresponding to edges oriented towards v. In the case of (a^s, b^t), these are exactly the edges coming from $A^s \sqcup B^t$. Thus we obtain the desired result.

The ascending link is computed similarly, with the signs reversed. ∎

Theorem 3.2. *Let $X = X_\Gamma$ and f be the complex and Morse function from above. Let H be the kernel of f_*. Then $\pi_1(X)$ is a hyperbolic group and H is finitely generated and not finitely presented.*

Proof. By Corollary 2.4, we know that $\pi_1(X)$ is a hyperbolic group. By Proposition 3.1, we have that the ascending and descending links are copies of S^1. Thus they are connected graphs and so H is finitely generated by [2, Theorem 4.1]. To see that H is not finitely presented, we can use [3, Theorem 4.7], which shows that it is not of type FP_2. ∎

It will be useful in the next sections to consider a presentation for H. Let Z be the cyclic cover of X_Γ corresponding to H. There is a map $h \colon Z \to \mathbb{R}$ which is the lift of f to Z. Let Z_0 denote the preimage of 0. This is a graph and the inclusion $Z_0 \to Z$ gives a surjection on fundamental groups. From [2], we know that as we expand Z_0 to $h^{-1}([-t, t])$, the homotopy changes by coning off the ascending and descending links of vertices. Since each ascending or descending link is a copy of S^1, we can see that each vertex v with $h(v) \neq 0$ gives one relation in the presentation of H.

Let v be a vertex of Z such that $h(v) > 0$. Thus we obtain a relation in H from v which corresponds to coning off the descending link of v. We can think about the descending link of v as a subset of Z in the $\frac{1}{4}$-neighbourhood of v. Denote this loop by γ_v. We can homotope γ_v to Z_0 in order to obtain the relation r_v corresponding to v.

Later, it will be useful to know that γ_v is not null-homotopic in $Z \smallsetminus \{v\}$. We will require the following graph theoretic lemma.

Lemma 3.3. *Let $\mathcal{G}$ be a countable graph and $\mathcal{A}$ a connected subgraph. Then there is a retraction $\mathcal{G} \to A$.*

Proof. Since any space retracts onto its connected components, we can assume that $\mathcal{G}$ is connected. Order the edges $e_1, e_2, \ldots$ of $\mathcal{G} \smallsetminus \mathcal{A}$. Let $\mathcal{A}_0 = \mathcal{A}$ and $\mathcal{A}_{i+1} = \mathcal{A}_i \cup e_i$. Thus $\mathcal{G}$ is the direct limit of the $\mathcal{A}_i$. Since $\mathcal{G}$ is connected, we can choose the ordering above such that e_n intersects $\mathcal{A}_{n-1}$. Thus, $\mathcal{A}_n$ is connected for all n.

We will define a retraction $r_n \colon \mathcal{A}_n \to \mathcal{A}_{n-1}$ as follows. If e_n has exactly 1 endpoint v in $\mathcal{A}_{n-1}$, then define r_n by sending e_n to v and the identity on $\mathcal{A}_{n-1}$.

If e_n has both endpoints v, w in $\mathcal{A}_{n-1}$, then we define r_n by sending e_n to a path from v to w in $\mathcal{A}_{n-1}$ and the identity on $\mathcal{A}_{n-1}$.

By composing these retractions, we obtain a retraction $\mathcal{A}_n \to \mathcal{A}$ for each n. Passing to the direct limit, we get a retraction $\mathcal{G} \to \mathcal{A}$ as desired. ∎

Lemma 3.4. *Let γ_v be the loop described above. Then γ_v is not homotopically trivial in $Z \smallsetminus \{v\}$.*

Proof. Let $\widetilde{Z}$ be the universal cover of Z and ρ the associated covering map. We obtain another covering $q \colon \widetilde{Z} \smallsetminus \rho^{-1}(v) \to Z \smallsetminus \{v\}$. Note that $\widetilde{Z}$ is also the universal cover of X_Γ, so is a CAT(0) cube complex. Since γ_v is a null-homotopic loop in Z, we can see that it lifts to a loop l in $\widetilde{Z}$. This lift lies in the $\frac{1}{4}$-neighbourhood of a lift w of v.

There is a retraction $\widetilde{Z} \smallsetminus \{w\} \to \mathrm{Lk}(w)$. By Lemma 3.3, there is a further retraction $\mathrm{Lk}(w) \to l$. Hence, we can see that l is not null-homotopic in $\widetilde{Z} \smallsetminus \{w\}$. Since q_* is an injection and $q_*(l) = \gamma_v$, we get that γ_v is not null-homotopic in $Z \smallsetminus \{v\}$. ∎

4. Branched covers and uncountably many groups

Throughout this section, let p be a prime number. Denote by $\Gamma(\lambda)$ the full subgraph of Γ spanned by λ.

We will now take branched covers of Z to obtain the uncountable family of groups. To do the branching, we mimic section 21 of [10]. There, a branched cover is obtained for any regular cover of L. In our case, we can follow a similar procedure. However, we are unable to appeal to the embedding results used in [10].

We will focus on the vertices which map to (a^+, b^+). At these vertices, the ascending link is given by $\Gamma(A^- \sqcup B^-)$ and the descending link is given by $\Gamma(A^+ \sqcup B^+)$. Let Λ be the subgraph of Γ containing both of these graphs and one edge joining them. By Lemma 3.3, there is a retraction $\Gamma \to \Lambda$ and a further map $\Lambda \to S^1$ mapping both loops homeomorphically to S^1.

We can now pull back the p-fold cover of S^1 to Λ. Let $\overline{\Gamma}$ be the corresponding cover of Γ and p the covering map. Thus $\overline{\Gamma}$ is a normal covering space with group of deck transformations $\mathbb{Z}/p\mathbb{Z}$. By construction, the preimages of $\Gamma(A^+ \sqcup B^+)$ and $\Gamma(A^- \sqcup B^-)$ are connected graphs isomorphic to S^1. In fact, these are p-fold covers of $\Gamma(A^+ \sqcup B^+)$ and $\Gamma(A^- \sqcup B^-)$, respectively.

Recall that we have a Morse function $h: Z \to \mathbb{R}$. Let V be the set of vertices in Z that map to (a^+, b^+) in X and $h(v) > 0$. The set V is countably infinite and there is a natural bijection with $\mathbb{N}$. Note that we are aiming to construct groups which are not virtually torsion-free. Hence guided by Conjecture 1.3, we consider only positive branching points with respect to the Morse function to eliminate any possibility of $\mathbb{Z}$-periodicity in our set W. Thus the set V considered here is only a half-ray of the more general case, where it would naturally biject with $\mathbb{Z}$. As a consequence, this construction is also slightly simpler.

For each $W \subset V$, let Z_W be the space obtained from Z by removing all vertices $w \in W$ and adding a disk to each loop in $\mathrm{Lk}(w)$ in the image of $\overline{\Gamma}$. Let T_w be the complex obtained from $\mathrm{Lk}(w)$ by attaching these disks. Let $H_W = \pi_1(Z_W)$.

To each W there is also an associated CAT(0) cube complex as follows. Let $\widetilde{Z_W}$ be the universal cover of Z_W. Let $\mathcal{D}$ be the collection of disks added to $Z \setminus W$ to obtain Z_W. Let $\widetilde{\mathcal{D}}$ be the preimages of the added disks in $\widetilde{Z_W}$. There is a covering map $\widetilde{Z_W} \setminus \widetilde{\mathcal{D}} \to Z \setminus W$. By lifting the metric and taking a completion, we obtain a CAT(0) cube complex Y_W. We can view the group H_W as the group of deck transformations for the branched cover $Y_W \to Z$. Thus there is an action of H_W on Y_W which is free away from vertices.

Lemma 4.1. *The groups H_W admit a uniformly proper action on a hyperbolic space.*

Proof. The group H_W acts on the cube complex Y_W. Since the link of each vertex in Y_W is a cover of Γ, we see that the link of each vertex contains no cycles of length 4. Hence Theorem 2.3 shows that Y_W is a hyperbolic metric space.

Let B_W be the barycentric subdivision of Y_W. Let C_W be the subgraph of B_W spanned by the vertices which are not vertices of Y_W. The graph C_W is quasi-isometric to Y_W and hence hyperbolic. The action of H_W on Y_W preserves C_W and acts freely on this subgraph.

There are two types of vertex in C_W. The vertices that are barycenters of faces have valence 4. The vertices that are barycenters of edges have valence equal to the number of two-cells containing the edge. This is equal to the size of the link of a vertex in Γ. Thus, these vertices also have valence 4.

The action of H_W on C_W is also by deck transformations, hence for uniform properness it suffices to consider the action on vertices. Since the valence of every vertex in C_W is 4, balls of radius r grow less than 4^r. Because the action is free, the number of group elements that send a vertex no more than distance r away is then also bounded by 4^r. Thus the action is uniformly proper. ∎

To finish this section, we study a presentation for H_W. In the original complex Z, we had one relation for each vertex v. As shown in Lemma 3.4, the words r_v do not represent the trivial element of $\pi_1(Z \setminus \{v\})$. Since in the descending link we are gluing a disk to γ_v^p if $v \in W$, we can see that a set of relators for H_w is $\{r_v \mid v \notin W\} \cup \{r_v^p \mid v \in W\}$.

The following lemma shows that r_v does not represent the trivial element of H_W if $v \in W$. This will be useful for showing that these groups are not virtually torsion-free.

Lemma 4.2. *If $v \in W$, then the loop γ_v is not null-homotopic in Z_W.*

Also, r_v does not represent the trivial word in H_W.

Proof. As before, let $\widetilde{Z_W}$ be the universal cover of Z_W. Let Y_W be the cube complex obtained by removing $\widetilde{\mathcal{D}}$ from $\widetilde{Z_W}$ and taking a completion. Suppose that γ_v is trivial in Z_W. Then γ_v lifts to $\widetilde{Z_W}$ in the neighbourhood of a vertex w. There is a retraction $Y_W \smallsetminus \{w\} \to \mathrm{Lk}(w, Y_W)$.

Let D be a disk in $\widetilde{\mathcal{D}}$ not based at w. Since Y_W is contractible, we can see that ∂D is a null-homotopic loop in Y_W. Thus under the retraction, the boundary of each disk D is sent to a trivial loop in $\mathrm{Lk}(v, Y_W)$.

We can now obtain a retraction $\widetilde{Z_W} \to L_w$, where L_w is the space obtained from $\mathrm{Lk}(w, Y_W)$ by attaching those disks in $\widetilde{\mathcal{D}}$ which are based at w. Since this is a retraction, it gives a surjection on fundamental groups. Thus we can see that L_w is simply connected.

We also know that L_w is a cover of T_v, with $\pi_1(T_v)$ being $\mathbb{Z}/p\mathbb{Z}$, generated by γ_v. Thus we can see that γ_v does not lift to a loop in L_w and hence does not lift to a loop in $\widetilde{Z_W}$. Thus γ_v is not null-homotopic in Z_W. Since γ_v represents the word r_v in H_W, we see that r_v does not represent the identity in H_W. ■

Using the invariant defined in [9], based on that of [10], we can see that there are uncountably many isomorphism classes of groups H_W. Recall the following definition.

Definition 4.3. Let $R \subset F(A)$ be a set of words in the free group on a finite set A. Let G be the group generated by A. Define $\mathcal{R}(G, A, R) = \{w \in R \mid r(A) = 1 \text{ in } G\}$.

In [9], it is shown that for a fixed set R and fixed group G, there are only countably many possibilities for $\mathcal{R}(G, A, R)$. Let A be the generating set for H_W coming from $\pi_1(Z_0)$. Let $R = \{r_v \mid v \in Z\}$. From Lemma 4.2, we can see that $\mathcal{R}(H_W, A, R) = \{r_v \mid v \notin W\}$. Thus we obtain the following,

Proposition 4.4. *The groups H_W form an uncountable family of groups.*

This allows us to obtain the first part of Theorem 1.2 as follows:

Theorem 4.5. *There are groups of the form H_W which cannot be subgroups of hyperbolic groups.*

As such, there are finitely generated groups acting uniformly properly on hyperbolic spaces which cannot be subgroups of hyperbolic groups.

Proof. Since hyperbolic groups are finitely presented, there are only countably many of them. In a given hyperbolic group G, there are only countably many finitely generated subgroups. Thus there are only countably many finitely generated subgroups of hyperbolic groups. Since there are uncountably many groups of the form H_W, we obtain examples of finitely generated groups which act uniformly properly on hyperbolic spaces which are not subgroups of hyperbolic groups. ■

5. Virtually torsion-free criterion

We will now prove that only countably many of the groups H_W are virtually torsion-free. Denote by $o(g)$ the order of a group element.

Let r_v be the sequence of words coming from γ_v as v runs over the vertices of Z. We know that there is a generating set A such that H_W has the presentation $\langle A \mid R\rangle$, where $R = \{r_v \mid v \notin W\} \cup \{r_v^p \mid v \in W\}$. By Lemma 4.2, we know that r_v is not trivial if $v \in W$. Thus it has order p in this case.

Let G be a group and $\phi\colon H_W \to G$ a homomorphism. If $v \notin W$, then $o(\phi(r_v)) = 1$. If $v \in W$, then $o(\phi(r_v)) \mid p$ and so is either 1 or p. Let O be the subset of V consisting of those r_v such that $o(\phi(r_v)) = p$.

Now suppose that H_W is virtually torsion-free. Thus H_W contains a finite index subgroup F_W which is torsion-free. By considering the action of H_W on the cosets of F_W, we obtain a homomorphism $\phi\colon H_W \to S_n$, where $n = |H_W : F_W|$.

Since F_W is torsion-free, we can see that if $v \in W$, then $r_v \notin F_W$. Hence, we obtain $o(\psi(r_v)) = p$ for all $v \in W$. Thus, $O = W$ in this case.

The homomorphism ϕ is determined by a map $A \to S_n$. There are only finitely many such maps for a fixed n. And thus only countably many such maps as n varies.

The set O is determined by the map $A \to S_n$. Thus there can only be countably many sets O picked out by this process. There are uncountably many groups H_W, with one for each $W \subset V$. Thus only countably many of them can be torsion-free.

This completes the proof of Theorem 1.2.

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Robert Kropholler
Faculty of Mathematics and Computer Science, University of Münster, Einsteinstraße 62, 48149 Münster, Germany; robertkropholler@gmail.com

Vladimir Vankov
School of Mathematics, University of Bristol, Fry Building, Bristol BS8 1UG, UK; vlad.vankov@gmail.com