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Dynamics of actions of automorphisms of discrete groups G on Sub_G and applications to lattices in Lie groups

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Abstract. For a locally compact Hausdorff group G and the compact space Sub_G of closed subgroups of G endowed with the Chabauty topology, we study the dynamics of actions of automorphisms of G on Sub_G in terms of distality and expansivity. We prove that an infinite discrete group G, which is either polycyclic or a lattice in a connected Lie group, does not admit any automorphism which acts expansively on Sub^c_G, the space of cyclic subgroups of G, while only the finite order automorphisms of G act distally on Sub^c_G. For an automorphism T of a connected Lie group G which keeps a lattice Γ invariant, we compare the behaviour of the actions of T on Sub_G and Sub_Γ in terms of distality. Under certain necessary conditions on the Lie group G, we show that T acts distally on Sub_G if and only if it acts distally on Sub_Γ. We also obtain certain results about the structure of lattices in a connected Lie group.

1. Introduction

A homeomorphism T of a (Hausdorff) topological space X is said to be *distal* if, for every pair of elements $x, y \in X$ with $x \neq y$, the closure of $\{(T^n(x), T^n(y)) \mid n \in \mathbb{Z}\}$ in $X \times X$ does not intersect the diagonal $\{(a,a) \mid a \in X\}$. If X is compact and metrizable with a metric d, then T is distal if and only if given $x, y \in X$ with $x \neq y$, $\inf\{d(T^n(x), T^n(y)) \mid n \in \mathbb{Z}\} > 0$. Distal maps on compact spaces were introduced by David Hilbert to study the dynamics of non-ergodic maps and studied by many mathematicians in different contexts; see Ellis [9], Furstenberg [10], Moore [20], Raja and Shah [28, 29], Shah [31], Shah and Yadav [33–35], and the references cited therein.

For a metrizable topological space X with a metric d, a homeomorphism T of X is said to be *expansive* if there exists $\varepsilon > 0$ satisfying the following: if $x, y \in X$ with $x \neq y$, then $d(T^n(x), T^n(y)) > \varepsilon$ for some $n \in \mathbb{Z}$. Here, ε is said to be an expansive constant for T. The notion of expansivity was introduced by Utz [36] and studied by many in different contexts (see Bryant [6], Schmidt [30], Choudhuri and Raja [8], Glöckner and Raja [12], Shah [32], and the references cited therein). It is known that on any compact metric space, the expansivity of a homeomorphism is independent of the metric [38], and the class of distal homeomorphisms and that of expansive homeomorphisms are disjoint from each other [6].

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Here, we study the dynamics of the actions of automorphisms of certain locally compact groups G on the compact space of closed subgroups of G in terms of distality and expansivity. Let G be a locally compact (Hausdorff) topological group and let Sub_G denote the set of all closed subgroups of G equipped with the Chabauty topology [7]. Then Sub_G is compact and Hausdorff. It is metrizable if G is locally compact and second countable (see [11] and [3, Section 1, Chapter E] for more details). Let $\mathrm{Aut}(G)$ denote the group of all automorphisms of G. There is a natural action of $\mathrm{Aut}(G)$ on Sub_G; namely, $(T, H) \mapsto T(H)$, $T \in \mathrm{Aut}(G)$, $H \in \mathrm{Sub}_G$. For each $T \in \mathrm{Aut}(G)$, the map $H \mapsto T(H)$ defines a homeomorphism of Sub_G [14, Proposition 2.1], and the corresponding map from $\mathrm{Aut}(G) \to \mathrm{Homeo}(\mathrm{Sub}_G)$ is a group homomorphism.

For a locally compact second countable group G, we say that $T \in \mathrm{Aut}(G)$ acts distally (resp. expansively) on Sub_G if the homeomorphism of Sub_G corresponding to T is distal (resp. expansive). The distality of such an action was first studied by Shah and Yadav [35] for connected Lie groups and the expansivity of the action was first studied by Prajapati and Shah [25] for locally compact groups.

Let Sub^a_G (resp. Sub^c_G) denote the space of all closed abelian (resp. discrete cyclic) subgroups of G. They are invariant under the action of $\mathrm{Aut}(G)$, Sub^a_G is always closed in Sub_G, while Sub^c_G is closed for many discrete groups G [23]. In particular, this also holds for any discrete polycyclic group G. Here we will show that Sub^c_G is closed for any lattice G in a connected Lie group (see Corollary 3.5). We focus on studying the distality and expansivity of the actions of automorphisms of a discrete group G on Sub^c_G when G is polycyclic or a lattice in a connected Lie group.

For a lattice Γ in a connected Lie group G, the set Sub^c_Γ is usually much smaller than Sub^a_G. In fact, Sub^c_Γ is countable since Γ is countable, as the latter is a discrete subgroup of a connected Lie group. But Sub^a_G is uncountable if G is noncompact. For a connected Lie group G without any nontrivial compact connected central subgroup, only those automorphisms contained in compact subgroups of $\mathrm{Aut}(G)$ act distally on Sub^a_G [35, Theorem 4.1]. We would like to know if there exists an automorphism which is not contained in a compact subgroup of $\mathrm{Aut}(G)$, keeps a lattice Γ-invariant, and acts distally on Sub^c_Γ. This does not happen for a large class of Lie groups G as shown by Theorem 4.7. More specifically, we may ask if Γ admits any infinite order automorphism which acts distally on Sub^c_Γ. The answer is negative as illustrated by the following result which is a part of Theorem 4.6.

Theorem 1.1. *Let Γ be a lattice in a connected Lie group and let $T \in \mathrm{Aut}(\Gamma)$. Then T acts distally on Sub^c_Γ if and only if $T^n = \mathrm{Id}$ for some $n \in \mathbb{N}$.*

We compare the behaviour of distality of the T-actions on Sub^c_Γ and Sub_G, when $T \in \mathrm{Aut}(G)$ and $T(\Gamma) = \Gamma$, in Theorem 4.7 which unifies and generalises the results obtained by Palit and Shah in [23]. We also construct counter examples to show that Theorem 4.7 is the best possible result in this direction.

It was shown in [25] for an almost connected locally compact metrizable group G that if $T \in \mathrm{Aut}(G)$ acts expansively on Sub_G, then G is finite. In particular, a nontrivial connected Lie group G does not admit any automorphism which acts expansively on Sub_G. Now the question arises whether there exists an automorphism of G which keeps a lattice Γ invariant and acts expansively on Sub_Γ. The answer is again negative if G is noncompact, and equivalently if Γ is infinite.

Theorem 1.2. *A lattice Γ in a connected noncompact Lie group does not admit any automorphism which acts expansively on* Sub^c_Γ.

We also prove similar results as above for discrete (infinite) polycyclic groups (see Theorems 4.5 and 5.2)

For many groups G, the (compact) spaces Sub_G, Sub^a_G and the closure of Sub^c_G have been identified (see e.g. Baik and Clavier [1, 2], Bridson, de la Harpe, and Kleptsyn [5], and Pourezza and Hubbard [24]). For the 3-dimensional Heisenberg group $\mathbb{H}$, the structure of the space of lattices in $\mathbb{H}$ and the action of $\mathrm{Aut}(\mathbb{H})$ on certain subspaces of $\mathrm{Sub}_\mathbb{H}$ have also been studied in [5]. Since the homeomorphisms of Sub_G arising from the action of $\mathrm{Aut}(G)$ form a large subclass of $\mathrm{Homeo}(\mathrm{Sub}_G)$, it is important to study the dynamics of such homeomorphisms of Sub_G.

Some results about distal actions are proven for automorphisms belonging to the class (NC) which was introduced in [35]. For a locally compact metrizable group G, an automorphism $T \in \mathrm{Aut}(G)$ is said to belong to (NC) if for every nontrivial closed cyclic subgroup A of G, $T^{n_k}(A) \not\to \{e\}$ in Sub_G for any sequence $\{n_k\} \subset \mathbb{Z}$. The class (NC) of automorphisms is studied in details in [35] for connected Lie groups, and in [23] for lattices in certain connected Lie groups. The class (NC) is larger than the set of those which act distally on Sub^a_G or the closure of Sub^c_G as illustrated by [23, Example 3.11] and Example 4.8. However, for many groups G, it turns out to be the same as the set of those which act distally on Sub_G; see [35, Theorem 4.1], [23, Corollary 3.9 and Theorem 3.16], and also Theorem 4.6.

We prove some results on the structure of lattices in connected Lie groups which are useful for the proofs of the main results about distal and expansive actions. It is known that any closed subgroup H of a connected Lie group admits a unique maximal solvable normal subgroup (say) H_{rad} [26, Corollary 8.6]. We show for a lattice Γ in a connected Lie group G that Γ_{rad} is polycyclic and $\Gamma/\Gamma_{\mathrm{rad}}$ is either finite or it admits a subgroup of finite index which is a lattice in a connected semisimple Lie group without compact factors and with finite center (see Proposition 3.3, see also Proposition 3.2). Using known results for semisimple and nilpotent groups and the Borel density theorem, we show that if the radical of G is simply connected and nilpotent and a Levi subgroup of G is either trivial or has no compact factors, then no nontrivial automorphism of G acts trivially on a lattice Γ; in particular, the centraliser of Γ in G is the center of G (see Proposition 3.6). We also prove an elementary but crucial lemma about the structure of Sub_G for a class of countable discrete groups G with the property that the set of roots of g in G is finite for every $g \in G$; the lemma also shows that such a G does not admit any automorphism that acts expansively on Sub^c_G (which is closed) unless G is finite (see Lemma 5.1).

In Section 2, we state some basic results and properties of distal and expansive actions. We also describe the topology of Sub_G for locally compact groups G. In Section 3, we prove some results about the structure of lattices in Lie groups. For the action of an automorphism of a discrete group G on Sub_G, where G is either polycyclic or a lattice in a connected Lie group, we explore the distality of this action in Section 4, while Section 5 deals with the study of the expansivity of this action.

We will assume that all our topological groups are locally compact Hausdorff and second countable. For a locally compact Hausdorff group G, it is second countable if and only if it is first countable (metrizable) and σ-compact. We will use results from [23, 25, 35], where it is assumed that both G and Sub_G are metrizable, for which it is sufficient to assume that G is second countable. A compact Hausdorff first countable group is second countable. If G is a countable discrete group or a closed subgroup of a Lie group, then G is second countable.

For a topological group G with the identity e, and a subgroup $H \subset G$, let H^0 denote the connected component of the identity e in H, $[H, H]$ the commutator subgroup of H, $Z(H)$ the center of H, and $Z_G(H)$ the centraliser of H in G. An element $x \in G$ is said to be a torsion element if $x^n = e$ for some $n \in \mathbb{N}$. A group G is said to be *torsion-free* if it does not have any nontrivial torsion element. For any $x \in G$, by convention, $x^0 = e$. Similarly, $T^0 = \mathrm{Id}$, the identity map, for any bijective map T of a space X. For $x \in G$, let R_x denote the set of roots of x in G; i.e., $R_x = \{y \in G \mid y^n = x \text{ for some } n \in \mathbb{N}\}$. Note that if $G = \mathbb{Z}^d$, or more generally, if G is a finitely generated nilpotent group, then R_x is finite for every $x \in G$ [15, Example 3.1.12, Theorems 3.1.13 and 3.1.17].

For a connected Lie group G, let $\mathcal{G}$ denote the Lie algebra of G and let $\exp : \mathcal{G} \to G$ be the exponential map. For any $T \in \mathrm{Aut}(G)$, there exists a unique Lie algebra automorphism $\mathrm{d}\,T : \mathcal{G} \to \mathcal{G}$ which satisfies $\exp(\mathrm{d}\,T(v)) = T(\exp(v))$, $v \in \mathcal{G}$. Recall that $\mathrm{Ad} : G \to \mathrm{GL}(\mathcal{G})$, the adjoint representation of G on $\mathcal{G}$, is defined as $\mathrm{Ad}(g) = \mathrm{d}(\mathrm{inn}(g))$, $g \in G$, where $\mathrm{inn}(g)$ denotes the inner automorphism of G by g; i.e., $\mathrm{inn}(g)(x) = gxg^{-1}$, $x \in G$. Note that $\mathrm{Ad}(G)$ is a connected Lie subgroup of $\mathrm{GL}(\mathcal{G})$. The radical (resp. nilradical) of G is the maximal connected solvable (resp. nilpotent) normal subgroup of G and, G is said to be semisimple if its radical is trivial. A connected Lie group G is said to be reductive if its Lie algebra $\mathcal{G}$ is reductive; equivalently, $\mathrm{Ad}(G)$ is semisimple. Note that G is reductive if and only if its radical is central in G and, equivalently, if G is an almost direct product of a connected semisimple Lie group and $Z(G)$. A connected Lie group is said to be linear if it is isomorphic to a subgroup of $\mathrm{GL}(n, \mathbb{R})$ for some $n \in \mathbb{N}$. We will use certain results about the structure of linear groups, Lie groups, and Lie algebras which are standard and can be found in any basic textbook on Lie groups (see e.g. [16, 37]).

2. Preliminaries

For a (metrizable) topological space X, let $\mathrm{Homeo}(X)$ denote the space of all homeomorphisms of X. We first state some known properties of distal and expansive actions for a compact space X. Let $T \in \mathrm{Homeo}(X)$. Then T^n is distal (resp. expansive) for some $n \in \mathbb{Z} \setminus \{0\}$ if and only if T^n is so for all $n \in \mathbb{Z} \setminus \{0\}$. If $Y \subset X$ is a nonempty T-invariant subspace and if T is distal (resp. expansive), then $T|_Y$ is so. If $S \in \mathrm{Homeo}(X)$, then T is distal (resp. expansive) if and only if STS^{-1} is so. An expansive homeomorphism of a compact metric space X has only finitely many fixed points and, hence, the set of its periodic points is countable. If a topological space is discrete, then any homeomorphism is distal as well as expansive. The identity map of a space is distal by definition, but it need not be expansive (for example, if the metric space is non-discrete).

Given a locally compact (Hausdorff) group G, the Chabauty topology on Sub_G was introduced by Chabauty [7]. A sub-basis of the Chabauty topology on Sub_G is given by the sets of the following form $O_1(K) = \{A \in \mathrm{Sub}_G \mid A \cap K = \emptyset\}$ and $O_2(U) = \{A \in \mathrm{Sub}_G \mid A \cap U \neq \emptyset\}$, where K (resp. U) is a compact (resp. an open) subset of G.

Any closed subgroup of $\mathbb{R}$ is either a discrete group generated by a real number or the whole group $\mathbb{R}$, and $\mathrm{Sub}_{\mathbb{R}}$ is homeomorphic to $[0, \infty]$ with a compact topology. Any proper closed subgroup of $\mathbb{Z}$ is of the form $n\mathbb{Z}$ for some $n \in \mathbb{N} \cup \{0\}$, and $\mathrm{Sub}_{\mathbb{Z}}$ is homeomorphic to $\{\frac{1}{n} \mid n \in \mathbb{N}\} \cup \{0\}$. The space $\mathrm{Sub}_{\mathbb{R}^2}$ is homeomorphic to $\mathbb{S}^4$ [24]. Note that the space $\mathrm{Sub}_{\mathbb{R}^n}$ is simply connected for all $n \in \mathbb{N}$ [19, Theorem 1.3].

We now state a criterion for convergence of sequences in Sub_G when it is metrizable [3].

For a locally compact second countable group G, a sequence $\{H_n\}_{n\in\mathbb{N}}$ in Sub_G converges to H in Sub_G if and only if the following conditions hold:

(I) for any $h \in H$, there exists a sequence $\{h_n\}$ with $h_n \in H_n$, $n \in \mathbb{N}$, such that $h_n \to h$;

(II) for any unbounded sequence $\{n_k\} \subset \mathbb{N}$, if $\{h_{n_k}\}_{k\in\mathbb{N}}$ is such that $h_{n_k} \in H_{n_k}$, $k \in \mathbb{N}$, and $h_{n_k} \to h$, then $h \in H$.

In case G is discrete, we know from [23, Lemma 3.2] that $H_n \to H$ in Sub_G if and only if $H = \bigcup_{n=1}^{\infty} \bigcap_{k=n}^{\infty} H_k$. In particular, (for such a group G) if $H_n \to H$, then $h \in H$ if and only if $h \in H_n$ for all large n. We will use these criteria for convergence for discrete groups frequently. Also, for a discrete group G, if each H_n is cyclic and $H_n \to H$, then H is an increasing union of cyclic groups; in particular, one can replace H_n by $H_n' = \bigcap_{k=n}^{\infty} H_k$ and assume that $H_n \subset H_{n+1}$.

It is easy to see that Sub_G^a, the set of all closed abelian subgroups of G, is closed in Sub_G, but the same need not be true for Sub_G^c, the set of discrete cyclic subgroups; e.g. $G = \mathbb{R}$. Even if G is discrete, Sub_G^c need not be closed, e.g. G is the group consisting of all roots of unity in the unit circle endowed with the discrete topology, and for a prime p, the groups H_n of p^nth roots of unity are cyclic, $n \in \mathbb{N}$, but $H_n \to \bigcup_{n\in\mathbb{N}} H_n$, which is not cyclic. From now on, for a group G, when we say that Sub_G^c is closed, we mean that Sub_G^c is closed in Sub_G. In [23], for a discrete group G, various conditions for Sub_G^c to be closed are discussed. We now state and prove an elementary lemma which gives one more useful condition involving quotient groups. Note that a discrete group is second countable if and only if it is countable.

Lemma 2.1. *Let G be a discrete countable group and let H be any normal subgroup of G. If Sub^c_H and $\mathrm{Sub}^c_{G/H}$ are closed, then Sub^c_G is closed. In particular, if H has finite index in G and Sub^c_H is closed, then Sub^c_G is closed.*

Proof. If G is finite, then Sub_G is finite and discrete and hence Sub^c_G is closed. Suppose that G is not finite. If $H = \{e\}$ or $H = G$, then the assertions are obvious.

Now suppose that H is a proper subgroup of G. Let $\psi : G \to G/H$ be the natural projection. Suppose that Sub^c_H and $\mathrm{Sub}^c_{G/H}$ are closed. Let G_{x_n} be the cyclic group generated by $x_n \in G$, $n \in \mathbb{N}$, such that $G_{x_n} \to L$ for some $L \in \mathrm{Sub}_G$. By [23, Lemma 3.2], $L = \bigcup_n G_n$, where $G_n = \bigcap_{k=n}^{\infty} G_{x_k}$. In particular, L is an increasing union of cyclic groups G_n. Then $\psi(L) = \bigcup_n \psi(G_n)$, an increasing union of cyclic groups, and hence $\psi(G_n) \to \psi(L)$ in $\mathrm{Sub}_{G/H}$. As $\mathrm{Sub}^c_{G/H}$ is closed, $\psi(L)$ is cyclic, and hence $\psi(L) = \psi(G_n)$ for all large n. Therefore, $LH = G_n H$, and hence $L = G_n(L \cap H)$ for all large n. Since $L \cap H = \bigcup_n (G_n \cap H)$, $G_n \cap H \to L \cap H$ in Sub_H. As each $G_n \cap H$ is cyclic and Sub^c_H is closed, we have that $L \cap H$ is cyclic, and hence that $G_n \cap H = L \cap H$ for all large n. Therefore, we get that $L = G_n$ for all large n, and L is cyclic. This implies that Sub^c_G is closed.

If G/H is finite, then so is $\mathrm{Sub}_{G/H}$. Therefore, the second statement follows easily from the first. ∎

For a locally compact (Hausdorff) group G, recall that there is a natural group action of $\mathrm{Aut}(G)$, the group of automorphisms of G, on Sub_G as follows:

$$\mathrm{Aut}(G) \times \mathrm{Sub}_G \to \mathrm{Sub}_G; \quad (T, H) \mapsto T(H), \quad T \in \mathrm{Aut}(G),\ H \in \mathrm{Sub}_G.$$

The map $H \mapsto T(H)$ is a homeomorphism of Sub_G for each $T \in \mathrm{Aut}(G)$ [14, Proposition 2.1], and the corresponding map from $\mathrm{Aut}(G)$ to $\mathrm{Homeo}(\mathrm{Sub}_G)$ is a homomorphism.

If T is an automorphism of G which is second countable, and H is a closed normal T-invariant subgroup, then that T acts distally (resp. expansively) on Sub_G implies that T acts distally (resp. expansively) on both Sub_H and $\mathrm{Sub}_{G/H}$ (see [35, Lemma 3.1] and [25, Lemma 2.3]). However, the converse is not true as illustrated by [35, Example 3.2] and [25, Example 4.1]. In fact, Example 3.2 in [35] shows that for $G = \mathbb{R}^2$ and a subgroup $H = \mathbb{R}$, $T|_H = \mathrm{Id}$ and T acts trivially on G/H, but T does not act distally on Sub_G. However, the following elementary but useful lemma shows that in such a case T acts trivially on $G/Z(H)$, and in particular that $T = \mathrm{Id}$ if $Z(H) = \{e\}$. We include a short proof for the sake of completeness.

Lemma 2.2. *Let G be a locally compact (Hausdorff) group and let H be a closed normal subgroup. Let $T \in \mathrm{Aut}(G)$ be such that $T|_H = \mathrm{Id}$. Then T acts trivially on $G/Z_G(H)$. In particular, if T acts trivially on G/H, then T acts trivially on $G/Z(H)$.*

Proof. Let $g \in G$ and $h \in H$. As H is normal and $T|_H = \mathrm{Id}$, we have that

$$ghg^{-1} = T(ghg^{-1}) = T(g)hT(g^{-1}).$$

Therefore, $g^{-1}T(g)$ centralises every $h \in H$. This implies that $T(g) \in gZ_G(H)$. Moreover, if T acts trivially on G/H, then $T(g) \in g(H \cap Z_G(H)) = gZ(H)$, $g \in G$; i.e., T acts trivially on $G/Z(H)$. ■

3. Structure and properties of lattices in connected Lie groups

In this section, we discuss the structure and some properties of lattices in connected Lie groups and prove some useful results. Recall that a discrete subgroup Γ of a locally compact group G is a lattice in G if G/Γ carries a finite G-invariant measure. It is shown by Mostow [22] that any lattice in a connected solvable Lie group G is co-compact in G. If G is a simply connected nilpotent Lie group which admits a lattice, then any automorphism of the lattice extends to a unique automorphism of G (see Theorem 2.11 and Corollary 1 following it in [26]). In general, a lattice need not be co-compact and an automorphism of a lattice need not extend (uniquely) to an automorphism of the ambient group. If a connected Lie group admits a lattice, then it is unimodular. However, there are connected unimodular Lie groups which do not admit a lattice (see e.g. [26, Theorem 2.12 and Remark 2.14]). Any lattice in a connected Lie group is finitely generated (see Remark 3.4). We now define polycyclic groups, many of which arise as lattices in solvable Lie groups.

A group G is *polycyclic* if it admits a finite sequence of subgroups $G = G_0 \supset G_1 \supset \cdots \supset G_k = \{e\}$ such that each G_{i+1} is normal in G_i and G_i/G_{i+1} is cyclic, $0 \le i \le k-1$. Moreover, if G_i/G_{i+1} is infinite, $0 \le i \le k-1$, then G is said to be *strongly polycyclic*.

Polycyclic groups are finitely generated and solvable. Every infinite polycyclic group admits a subgroup of finite index which is strongly polycyclic. Also, every subgroup of a polycyclic (resp. strongly polycyclic) group is polycyclic (resp. strongly polycyclic) and hence it is finitely generated. It is easy to see that a group is polycyclic if and only if it is solvable and every subgroup of it is finitely generated. In particular, a finitely generated nilpotent group is polycyclic since all its subgroups are finitely generated. It follows from Corollary 3.9 of [26] that any discrete subgroup of a connected solvable Lie group is polycyclic. The following lemma extends this to any discrete solvable subgroup of a connected Lie group. The lemma is essentially known for subgroups of a connected solvable Lie group [26] (see also [13]) and the statement (1) below is proven in [17], while (2) and (3) may be known in general but we give a short proof for the sake of completeness.

Lemma 3.1. *Let H be a closed solvable subgroup of a connected Lie group. Then the following hold:*

(1) *H is compactly generated;*

(2) *H/H^0 is polycyclic;*

(3) *H admits a normal subgroup L of finite index such that $[L, L]$ is nilpotent.*

Proof. Let G be a connected Lie group containing H. As observed above, (1) is already known [17, Main Theorem]. As H is compactly generated, H/H^0 is a discrete finitely generated solvable group, and every subgroup of H/H^0 is also finitely generated since it is a quotient of a closed solvable subgroup of G. Therefore, H/H^0 is polycyclic. Therefore, (2) holds. Now we prove (3).

Suppose that G is a closed linear Lie group; i.e., G is a closed subgroup of $\mathrm{GL}(n,\mathbb{R})$ for some $n \in \mathbb{N}$. Let $\widetilde{H}$ be the Zariski closure of H in $\mathrm{GL}(n,\mathbb{R})$. Then $\widetilde{H}$ is solvable and it has finitely many connected components. Hence $L = H \cap (\widetilde{H})^0$ is a normal subgroup of finite index in H and it is contained in $(\widetilde{H})^0$. Then

$$[L, L] \subset \big[(\widetilde{H})^0, (\widetilde{H})^0\big]$$

which is nilpotent [26, Proposition 3.11]. That is, (3) holds in this case.

Let G be any connected Lie group. If H is central in G, then (3) holds trivially. Now suppose that H is not central in G. Let $\pi : G \to \mathrm{GL}(\mathcal{G})$ be defined as $\pi(g) = \mathrm{Ad}(g)$, $g \in G$, where $\mathcal{G}$ is the Lie algebra of G, which is a finite dimensional real vector space. Then $\ker \pi = Z(G)$, the center of G, and $\overline{\pi(H)}$ is a closed solvable subgroup of $\overline{\pi(G)}$, which is a closed connected linear Lie group. As shown above, $\overline{\pi(H)}$ has a normal subgroup (say) L' of finite index such that $[L', L']$ is nilpotent. Sine L' is also open in $\overline{\pi(H)}$, $\pi(H^0) \subset \overline{\pi(H)}^0 \subset L'$, and $L'\pi(H)$ is open in $\overline{\pi(H)}$; and hence it is equal to $\overline{\pi(H)}$. As $(L'\pi(H))/L'$ is finite, $L'' := L' \cap \pi(H)$ is a normal subgroup of finite index in $\pi(H)$. Let $L = \pi^{-1}(L'') \cap H$. Then $H^0 \subset L$ and L is an open normal subgroup of finite index in H. Moreover, $[L, L]$ is nilpotent as $[\pi(L), \pi(L)]$ is so and $\ker \pi = Z(G)$. Thus, (3) holds. ∎

Any closed subgroup H of a connected Lie group admits a unique maximal solvable normal subgroup [26, Corollary 8.6], and we denote it by H_{rad}. Note that H_{rad} is closed and characteristic in H. In particular, a lattice Γ in a connected Lie group G admits a unique maximal solvable normal subgroup Γ_{rad}. Moreover, Γ_{rad} is polycyclic by Lemma 3.1. The following proposition about certain properties of lattices in connected Lie groups will be useful. If G is a connected solvable Lie group, then the statements (b) and (c) of the proposition are easy to show. The statement (d) below may be known.

Proposition 3.2. *Let Γ be a lattice in a connected Lie group G. Then the following statements hold.*

(a) *The unique maximal solvable normal subgroup Γ_{rad} of Γ is polycyclic.*

(b) *There exists a unique maximal nilpotent normal subgroup Γ_{nil} in Γ. If Γ_{nil} is finite, then Γ_{rad} is finite.*

(c) *If Γ_{rad} is finite, then the following hold: the radical of G is compact and central in G. The group G is either compact and abelian, or G is reductive and it is an almost direct product of a compact group and a semisimple group with finite center. Moreover, G admits a finite central subgroup F such that G/F is linear.*

(d) *If G is semisimple and has no compact factors, then $\Gamma_{\mathrm{rad}} = Z(\Gamma) \subset Z(G)$ and it is a subgroup of finite index in $Z(G)$.*

Proof. (a) This follows from Lemma 3.1 (2).

(b) Any nilpotent normal subgroup of Γ is contained in $\Gamma_{\rm rad}$. As $\Gamma_{\rm rad}$ is polycyclic, by [26, Corollary 2 to Lemma 4.7], $\Gamma_{\rm rad}$ has a unique maximal nilpotent normal subgroup, which is also the unique maximal nilpotent normal subgroup of Γ.

Now suppose that $\Gamma_{\rm nil}$ is finite. We show that $\Gamma_{\rm rad}$ is also finite. If possible suppose that $\Gamma_{\rm rad}$ is infinite. Since $\Gamma_{\rm rad}$ is polycyclic, it has a normal subgroup Γ' of finite index which is strongly polycyclic and $[\Gamma', \Gamma']$ is nilpotent (see [26, Lemma 4.6 and Corollary 4.11]). Since $[\Gamma', \Gamma'] \subset \Gamma_{\rm nil}$ which is finite, and the former is torsion-free, we get that $[\Gamma', \Gamma']$ is trivial and hence Γ' is abelian. Therefore, $\Gamma' \subset \Gamma_{\rm nil}$ is finite. Since Γ' is torsion-free, it is trivial. This implies that $\Gamma_{\rm rad}$ is finite.

(c) Now suppose that $\Gamma_{\rm rad}$ is finite. We first show that the radical R of G is compact. Suppose that G is solvable. Then $G = R$, $\Gamma = \Gamma_{\rm rad}$, and Γ is finite. By [26, Theorem 3.1], G is compact. Then G is abelian [18, Lemma 2.2]. In this case, G itself is linear [16, Chapter XVIII, Theorem 3.2]. Now suppose that G is not solvable. Let $G = SR$ be a Levi decomposition, where S is a Levi subgroup of G. Note that S is a connected semisimple Lie subgroup.

Let K be the maximal compact normal subgroup of G. Suppose that K^0 is trivial. Then we show that S does not admit any nontrivial compact factor which centralises R. If possible, suppose that S has a nontrivial compact factor (say) C which centralises R. As C is normal in S and it centralises R, it is normal in G. This implies that $C \subset K^0$, which leads to a contradiction. Therefore, S does not have any compact factor centralising the radical R. By [26, Corollary 8.28], $\Gamma \cap R$ is a lattice in R and it is normal in Γ. Therefore, $\Gamma \cap R \subset \Gamma_{\rm rad}$, and hence it is finite. This implies that R is compact since $\Gamma \cap R$ is co-compact in R [26, Theorem 3.1].

Suppose that K^0 is nontrivial. As K is compact and normal in G, $\Gamma \cap K$ is a finite normal subgroup of Γ. Let $\Delta := Z_\Gamma(\Gamma \cap K)$. Then Δ is a normal subgroup of finite index in Γ, and hence it is a lattice in G. Moreover, $\Delta \cap K$ is finite and central in Δ. From (a), we get that Δ admits a unique maximal solvable normal subgroup $\Delta_{\rm rad}$, which is characteristic in Δ, and hence normal in Γ. Therefore, $\Delta_{\rm rad} \subset \Gamma_{\rm rad}$ and hence it is finite. Let $\pi : G \to G/K$ be the natural projection. Then $R' := \pi(R)$ is the radical of $\pi(G)$. Since $\pi(G)$ has no nontrivial compact normal subgroup, $\pi(\Delta) \cap R'$ is a lattice in R' and it is normal in $\pi(\Delta)$. As $\ker \pi \cap \Delta$ is central in Δ, we get that $\Delta \cap \pi^{-1}(R') = \Delta \cap KR$ is a solvable normal subgroup of Δ, and hence it is finite. Now $\pi(\Delta) \cap R'$ is finite, and being a lattice in R', it is cocompact in R'. Therefore, R' is compact. Since $\pi(G)$ has no nontrivial compact normal subgroups, $R' = \pi(R)$ is trivial, and hence $R \subset K$ and it is compact.

Now R is compact, and by [18, Lemma 2.2], R is abelian. As R is normal in G, and the latter is connected, by [18, Theorem 4], R is central in G. For the Lie algebra $\mathcal{G}$ of G and $\mathrm{Ad} : G \to \mathrm{GL}(\mathcal{G})$, we have that $\mathrm{Ad}(G) = \mathrm{Ad}(S)$ is semisimple and G is reductive. Also, G is an almost direct product of S and R, and $S \cap R$ is central in G. Let $S = K'S'$, where K' is a product of all compact factors of S (K' is trivial if S has no compact factors), and S' is a connected semisimple Lie group without compact factors. Then $K'R \subset K^0$. Since $K \cap S'$ is normal in S', it is closed in S' [27]. Therefore, $K \cap S'$ is compact, and hence it is finite and central in S' (as S' has no compact factors). In particular, $K'R = K^0$ and $G = KS' = K^0S'$, an almost direct product.

Now we show that the center of $\pi(S')$ is finite, where $\pi : G \to G/K$ is as above. This would also imply that $Z(S')$ is finite and also that $Z(S)$ is finite. We know that $\Delta \cap K \subset Z(\Delta)$ and $\pi(\Delta)$ is a lattice in $\pi(G) = \pi(S')$. Therefore,

$$\pi(\Delta)_{\mathrm{rad}} = \pi(\Delta_{\mathrm{rad}}) \subset \pi(\Gamma_{\mathrm{rad}})$$

is finite, and hence $Z(\pi(\Delta))$ is finite. Since $\pi(S')$ is a semisimple group without compact factors, by [26, Corollary 5.18], $Z(\pi(\Delta)) \subset Z(\pi(S'))$. Moreover, by [26, Theorem 5.17], $Z(\pi(S'))\pi(\Delta)$ is discrete, and hence $Z(\pi(\Delta))$ is a subgroup of finite index in $Z(\pi(S'))$. This implies that $Z(\pi(S'))$ is finite. Here, $\ker \pi \cap S' = K \cap S'$, and as noted above, it is finite and central in S'. This, together with the fact that $\pi(Z(S')) \subset Z(\pi(S'))$ is finite, implies that $Z(S')$ is finite. Therefore, $Z(S)$ is finite. For $\mathrm{Ad} : G \to \mathrm{GL}(\mathscr{G})$ as above, $\mathrm{Ad}(S)$ is closed in $\mathrm{GL}(\mathscr{G})$ and it is isomorphic to $S/(S \cap Z(G))$. Let $F = S \cap Z(G)$. Then F is finite and S/F, being isomorphic to $\mathrm{Ad}(S)$, is linear. Now the radical of G/F is compact and abelian, and the Levi subgroup of G/F is linear. Hence by [16, Chapter XVIII, Theorems 3.2 and 4.2], G/F is linear.

(d) Let G be a semisimple group without compact factors. As observed above, $Z(\Gamma) \subset Z(G)$ and it is a subgroup of finite index in $Z(G)$ [26, Theorem 5.17 and Corollary 5.18]. Let $\psi : G \to G/Z(G)$ be the natural projection, where $Z(G)$, the center of G, is discrete. Then for some $n \in \mathbb{N}$, $\psi(G)$ is a closed subgroup of $\mathrm{GL}(n, \mathbb{R})$ which is almost algebraic (i.e., $\psi(G)$ is a subgroup of finite index in an algebraic subgroup of $\mathrm{GL}(n, \mathbb{R})$), and $\psi(G)$ has trivial center. By [26, Theorem 5.17], $Z(G)\Gamma$ is closed, and hence $\psi(\Gamma)$ is a lattice in $\psi(G)$. Let L be the Zariski closure of $\psi(\Gamma_{\mathrm{rad}})$ in $\mathrm{GL}(n, \mathbb{R})$. Then L has finitely many connected components and $L^0 \subset \psi(G)$ as $\psi(G)$ is almost algebraic. Moreover, $\psi(\Gamma)$ normalises L, and hence it normalises L^0. As G has no compact factors, $\psi(\Gamma)$ is Zariski dense in $\psi(G)$. Hence, the preceding assertion implies that $\psi(G)$ normalises both L and L^0. Now L^0 is a connected solvable normal subgroup of $\psi(G)$ and the latter is semisimple. Therefore, L^0 is trivial, and hence L as well as $L \cap \psi(G)$ are finite. Since $L \cap \psi(G)$ is also normal in $\psi(G)$, it is central in $\psi(G)$. Therefore, $L \cap \psi(G)$ is trivial, hence we get that $\Gamma_{\mathrm{rad}} \subset Z(G)$, and that $\Gamma_{\mathrm{rad}} = Z(\Gamma)$ is a subgroup of finite index in $Z(G)$. ■

The following proposition about certain aspects of the structure of lattices in connected Lie groups will be very useful in proving the main results about expansivity and distality.

Proposition 3.3. *Let Γ be a lattice in a connected Lie group G and let Γ_{rad} be the unique maximal solvable normal subgroup of Γ. Then the following hold.*

(1) *Either $\Gamma/\Gamma_{\mathrm{rad}}$ is finite or Γ admits a normal subgroup Λ of finite index such that $\Gamma_{\mathrm{rad}} \subset \Lambda$ and $\Lambda/\Gamma_{\mathrm{rad}}$ is a lattice in a connected semisimple Lie group G' with finite center and without compact factors, where G' is a quotient group of G.*

(2) *$\Gamma/\Gamma_{\mathrm{rad}}$ admits a torsion-free subgroup Γ' of finite index with the property that the set R_g of roots of g in Γ' is finite for all $g \in \Gamma'$.*

Proof. (1) *Step 1*. Suppose that G is a connected semisimple Lie group without compact factors. By Proposition 3.2 (d), $\Gamma_{\mathrm{rad}} = Z(\Gamma) \subset Z(G)$ and it is a subgroup of finite index in $Z(G)$. Then G/Γ_{rad} is a connected semisimple Lie group without compact factors. Moreover, it has finite center, since $Z(G/\Gamma_{\mathrm{rad}}) = Z(G)/\Gamma_{\mathrm{rad}}$ (the latter statement follows from the fact that the center of any connected semisimple Lie group is discrete). Let $G' = G/\Gamma_{\mathrm{rad}}$. Then $\Gamma/\Gamma_{\mathrm{rad}}$ is a lattice in G'.

Step 2. We now note a useful general statement: for any closed normal subgroup H of G and a natural projection $\rho : G \to G/H$, if $\rho(\Gamma)$ is a lattice in $\rho(G)$ and $\Gamma \cap H$ is solvable, then $\rho(\Gamma_{\mathrm{rad}}) = \rho(\Gamma)_{\mathrm{rad}}$. One way inclusion $\rho(\Gamma_{\mathrm{rad}}) \subset \rho(\Gamma)_{\mathrm{rad}}$ is obvious. The equality follows from the facts that $\Gamma \cap H$, the kernel of $\rho|_\Gamma$, is a solvable normal subgroup of Γ, which in turn implies that the inverse image of any solvable normal subgroup of $\rho(\Gamma)$ under $\rho|_\Gamma$ is solvable and normal in Γ.

Step 3. Let G be any connected Lie group. If G is solvable, then so is Γ, and hence $\Gamma = \Gamma_{\mathrm{rad}}$ and (1) holds. Let K be the unique maximal compact connected normal subgroup of G and let $\Gamma_1 = Z_\Gamma(\Gamma \cap K)$. Then Γ_1 is a normal subgroup of finite index in Γ and, $\Gamma_1 \cap K$ is central in Γ_1. Hence $\Gamma_1 \cap K$ is contained in Γ_{rad}. Let $\Gamma_2 = \Gamma_1\Gamma_{\mathrm{rad}}$ and let $F = \Gamma_2 \cap K$. Then Γ_2 is a normal subgroup of finite index in Γ. We claim that F is solvable.

Let $\chi : \Gamma_2 \to \Gamma_2/\Gamma_1$ be the natural projection. Then $\chi(\Gamma_2) = \chi(\Gamma_{\mathrm{rad}})$ is solvable. Since $\ker \chi \cap F = \Gamma_1 \cap K$ is abelian, it follows that F is solvable. Replacing Γ_1 by Γ_2 we may assume that $\Gamma_{\mathrm{rad}} \subset \Gamma_1$ and $\Gamma_1 \cap K$ is a finite solvable group which is normal in Γ, and contained in Γ_{rad}. Moreover, $(\Gamma_1)_{\mathrm{rad}}$, being characteristic in Γ_1, is normal in Γ. Therefore, $(\Gamma_1)_{\mathrm{rad}} = \Gamma_{\mathrm{rad}}$.

Let $\pi : G \to G/K$ be the natural projection. Since $\ker \pi \cap \Gamma_1 = K \cap \Gamma_1$ is solvable, it follows from Step 2 that $\pi(\Gamma_{\mathrm{rad}}) = \pi(\Gamma)_{\mathrm{rad}}$. Observe that $\pi(R)$ is the radical of G/K. Since G/K has no nontrivial compact normal subgroup, it follows from [26, Corollary 8.28] that $\pi(\Gamma_1) \cap \pi(R)$ is a lattice in $\pi(R)$. Since $\pi(\Gamma_1) \cap \pi(R)$ is solvable and normal in $\pi(\Gamma)$, we get that $\pi(\Gamma_1) \cap \pi(R) \subset \pi(\Gamma_{\mathrm{rad}})$ and $\pi^{-1}(\pi(\Gamma_1) \cap \pi(R)) \cap \Gamma_1 \subset \Gamma_{\mathrm{rad}}$.

Step 4. Let $\pi_1 : G \to G/KR$ be the natural projection, where R is the radical of G. Since G/K has no nontrivial compact normal subgroup, arguing as in the proof of Proposition 3.2 (c), we get from [26, Corollary 8.28] that $\pi_1(\Gamma_1)$ is a lattice in G/KR which is semisimple. Let K' be the product of all compact (simple) factors of G/KR (we choose K' to be trivial if G/KR has no compact factors). Then K' is normal in $\pi_1(G)$. Moreover, $\pi_1(\Gamma) \cap K'$ is finite. Arguing as in Step 3, we get that $\pi_1(\Gamma)$, and hence Γ has a normal subgroup (say) Λ_1 of finite index such that $\pi_1(\Lambda_1) \cap K'$ is central in $\pi_1(\Lambda_1)$. Let $\Lambda = \Lambda_1 \cap \Gamma_1$. Then Λ is normal in Γ as well as in Γ_1 and Γ/Λ is finite. As $\pi_1(\Gamma_{\mathrm{rad}}) \subset \pi_1(\Gamma)_{\mathrm{rad}}$, arguing as in Step 3, we get that $\pi_1(\Lambda\Gamma_{\mathrm{rad}}) \cap K'$ is a solvable normal subgroup of $\pi_1(\Gamma)$.

We know from the latter part of Step 3 that $\pi^{-1}(\pi(\Gamma_1) \cap \pi(R)) \cap \Gamma_1 \subset \Gamma_{\mathrm{rad}}$; i.e., $((\Gamma_1 \cap RK)K) \cap \Gamma_1 = (\Gamma_1 \cap KR)(K \cap \Gamma_1) \subset \Gamma_{\mathrm{rad}}$. Therefore, $\Gamma_1 \cap KR \subset \Gamma_{\mathrm{rad}}$. This implies in particular that $\ker \pi_1 \cap \Lambda\Gamma_{\mathrm{rad}} \subset KR \cap \Gamma_1 \subset \Gamma_{\mathrm{rad}}$ and $\ker \pi_1 \cap \Lambda\Gamma_{\mathrm{rad}}$ is a solvable normal subgroup in Γ as well as in Γ_1. Replacing Λ by $\Lambda\Gamma_{\mathrm{rad}}$ and arguing as in

Step 3, we get that $\Lambda \cap K$ (resp. $\pi_1(\Lambda) \cap K'$) is a solvable normal subgroup of both Γ and $\Gamma \cap K$ (resp. $\pi_1(\Gamma)$ and $\pi_1(\Gamma) \cap K'$).

Step 5. Note that Λ is a lattice in G, it is normal in Γ and $\Gamma_{\mathrm{rad}} \subset \Lambda$, and hence $\Gamma_{\mathrm{rad}} = \Lambda_{\mathrm{rad}}$. As observed above, $\Lambda \cap \ker \pi_1 = \Lambda \cap KR$ is solvable, hence it follows from Step 2 that $\pi_1(\Lambda_{\mathrm{rad}}) = \pi_1(\Lambda)_{\mathrm{rad}}$. Also, $\pi_1(\Lambda) \cap K'$, being solvable and normal in $\pi_1(\Gamma)$, is contained in $\pi_1(\Lambda_{\mathrm{rad}})$. Therefore, by Step 2, we have that $\pi_1^{-1}(K') \cap \Lambda \subset \Lambda_{\mathrm{rad}}$. Let $L = \pi_1^{-1}(K')$. Then L is a closed normal subgroup of G, $KR \subset L$, L/R is compact. Moreover, either $G = L$ or G/L is a connected semisimple Lie group without compact factors. We also have that $L \cap \Lambda \subset \Lambda_{\mathrm{rad}}$ which is solvable.

If $G = L$, then Λ is solvable, $\Lambda = \Gamma_{\mathrm{rad}}$ and $\Gamma/\Gamma_{\mathrm{rad}}$ is finite and (1) holds in this case. Now suppose $G \neq L$. Let $\pi_2 : G \to G/L$ be the natural projection. Since $\pi_2(G) = \pi_1(G)/K'$ and $\pi_1(\Lambda)$ is a lattice in $\pi_1(G)$, we get that both $\pi_2(\Gamma)$ and $\pi_2(\Lambda)$ are lattices in $\pi_2(G)$. By Step 2, we have that $\pi_2(\Lambda_{\mathrm{rad}}) = \pi_2(\Lambda)_{\mathrm{rad}}$.

As shown in Step 1 above (see also the proof of Proposition 3.2 (d)), $\pi_2(\Lambda)_{\mathrm{rad}} = Z(\pi_2(\Lambda)) \subset Z(\pi_2(G))$. Therefore, $\pi_2(\Lambda_{\mathrm{rad}}) = Z(\pi_2(\Lambda))$ which is central in $\pi_2(G)$. Let $M = \pi_2^{-1}(Z(\pi_2(\Lambda)))$. Then $M = \Lambda_{\mathrm{rad}} \ker \pi_2 = \Lambda_{\mathrm{rad}} L$. This, together with the fact that $L \cap \Lambda$ is solvable, implies that $M \cap \Lambda = \Lambda_{\mathrm{rad}}(L \cap \Lambda) = \Lambda_{\mathrm{rad}}$. Now G/M is isomorphic to $\pi_2(G)/\pi_2(\Lambda_{\mathrm{rad}})$, which is a connected semisimple group with finite center and without compact factors. Moreover, $(\Lambda M)/M$ is a lattice in G/M and it is isomorphic to $\Lambda/\Lambda_{\mathrm{rad}}$. As $\Lambda_{\mathrm{rad}} = \Gamma_{\mathrm{rad}}$, we get that $\Lambda/\Gamma_{\mathrm{rad}}$ is also a lattice in G/M. Let $G' = G/M$. Then (1) holds.

(2) If $\Gamma/\Gamma_{\mathrm{rad}}$ is finite, then we can take Γ' to be trivial and the assertion follows immediately. Now suppose that $\Gamma/\Gamma_{\mathrm{rad}}$ is infinite. From (1), there exists a normal subgroup Λ of finite index in Γ such that $\Lambda_{\mathrm{rad}} = \Gamma_{\mathrm{rad}}$ and $\Lambda/\Gamma_{\mathrm{rad}}$ is a lattice in a connected semisimple Lie group G' with finite center and without compact factors. Let $\Lambda' = \Lambda/\Gamma_{\mathrm{rad}}$ and let $\psi : G' \to G'/Z(G')$ be the natural projection, where $Z(G')$ is the center of G'. As $Z(G')$ is finite, $\psi(\Lambda')$ is a lattice in $\psi(G')$. Since $G'/Z(G')$ is a semisimple linear Lie group with trivial center, $\psi(\Lambda')$ is finitely generated [21, Section 4.7, Chapter 4]. We get from Selberg's lemma that $\psi(\Lambda')$ admits a torsion-free subgroup (say) Λ'' of finite index. Since $(\Lambda')_{\mathrm{rad}} = \{e\}$, we have that $\Lambda' \cap Z(G') = \{e\}$, and hence $\psi^{-1}(\Lambda'') \cap \Lambda'$ is a torsion-free subgroup of finite index in Λ'. This, together with [23, Lemma 3.13], implies that $\Lambda' = \Lambda/\Gamma_{\mathrm{rad}}$ admits a torsion-free subgroup (say) Γ' of finite index such that the set R_g of roots of g in Γ' is finite for every $g \in \Gamma'$. Since Λ is a subgroup of finite index in Γ, we have that Γ' is a subgroup of finite index in $\Gamma/\Gamma_{\mathrm{rad}}$. ∎

Remark 3.4. It is well known that any lattice Γ in a connected Lie group is finitely generated [26, Remarks 6.58]; though the proof is difficult to find except in the case of linear semisimple groups [21, Section 4.7, Chapter 4] or that of solvable Lie groups [26, Corollary 3.9]. Using Proposition 3.3 (1), one can show that Γ is finitely generated as follows: since Γ_{rad} is polycyclic, and hence finitely generated, it is enough to show that $\Gamma/\Gamma_{\mathrm{rad}}$ is finitely generated. By Proposition 3.3 (1), replacing Γ by a subgroup of finite index, we get that $\Gamma/\Gamma_{\mathrm{rad}}$ is either finite or a lattice in a connected semisimple Lie group

G' with finite center $Z(G')$. In the first case, it is obvious. In the second case, as in the proof of Proposition 3.3 (2) above, we get that the image of $\Gamma/\Gamma_{\mathrm{rad}}$ is a lattice in the connected semisimple linear group $G'/Z(G')$, and hence the image is finitely generated. Since $Z(G')$ is finite, It follows that $\Gamma/\Gamma_{\mathrm{rad}}$ is finitely generated.

For a lattice Γ in a connected Lie group G, it is shown in [23] that Sub^c_Γ is closed in Sub_Γ if G is either solvable or semisimple. Here, we generalise this to lattices in any connected Lie group.

Corollary 3.5. *Let G be a connected Lie group and let Γ be a lattice in G. Then Sub^c_Γ is closed.*

Proof. By Lemma 3.1 (2), Γ_{rad} is polycyclic, every subgroup of it is finitely generated and, by [23, Lemma 3.3], $\mathrm{Sub}^c_{\Gamma_{\mathrm{rad}}}$ is closed. From Proposition 3.3 (1), we have that Γ has a normal subgroup Λ of finite index such that $\Gamma_{\mathrm{rad}} \subset \Lambda$ and $\Lambda/\Gamma_{\mathrm{rad}}$ is either finite or it is a lattice in a connected semisimple Lie group. In the first case, $\mathrm{Sub}^c_{\Lambda/\Gamma_{\mathrm{rad}}}$ is finite, and in the second case, it is closed by [23, Lemma 3.14]. As Γ/Λ is finite, so is $\mathrm{Sub}_{\Gamma/\Lambda}$. Now by Lemma 2.1, Sub^c_Γ is closed. ∎

The following useful proposition holds for simply connected nilpotent groups by [26, Theorem 2.11]. The second statement is known for connected semisimple Lie groups without compact factors (see [26, Corollary 5.18]); the first statement should also be known in this case as it follows from the Borel density theorem. The proposition generalises these two special cases.

Proposition 3.6. *Let G be a connected Lie group and let Γ be a lattice G. Suppose that the radical of G is simply connected and nilpotent and a Levi subgroup of G is either trivial or has no compact factors. Then the following hold:*

(1) *if $\tau \in \mathrm{Aut}(G)$ is such that $\tau|_\Gamma = \mathrm{Id}$, then $\tau = \mathrm{Id}$;*

(2) *the centraliser of Γ in G is the center of G.*

Proof. For any $x \in Z_G(\Gamma)$, the centraliser of Γ in G, the inner automorphism $\mathrm{inn}(x)$ of G acts trivially on Γ and (1) implies that $\mathrm{inn}(x) = \mathrm{Id}$, and hence $x \in Z(G)$, the center of G; i.e., (2) holds. Now we prove (1). Let $\tau \in \mathrm{Aut}(G)$ be such that $\tau|_\Gamma = \mathrm{Id}$. We want to show that $\tau = \mathrm{Id}$.

Suppose that G is nilpotent. As G is simply connected, any automorphism of Γ extends uniquely to an automorphism of G [26, Theorem 2.11]. Therefore, $\tau = \mathrm{Id}$ and (1) holds in this case.

Now suppose that G is semisimple. Then G has no compact factors. Recall that $\mathrm{Aut}(G)$ is identified with a subgroup of $\mathrm{GL}(\mathcal{G})$, under the map $T \mapsto \mathrm{d}\,T$, $T \in \mathrm{Aut}(G)$, and the topology inherited by it as a subspace of $\mathrm{GL}(\mathcal{G})$ coincides with the compact-open topology and it is a Lie group. Note that $\mathcal{G}$ is a real vector space of dimension (say) n. Let $\mathrm{M}(\mathcal{G})$ be the space of all linear maps on $\mathcal{G}$. Then $\mathrm{M}(\mathcal{G})$ is a real vector space of dimension n^2 and $\mathrm{GL}(\mathcal{G}) \subset \mathrm{M}(\mathcal{G})$. Let $\rho : G \to \mathrm{GL}(\mathrm{M}(\mathcal{G}))$ be defined as given by

$\rho(g)(w) = \mathrm{Ad}(g)w\,\mathrm{Ad}(g^{-1})$, $g \in G$, $w \in \mathrm{M}(\mathcal{G})$. Since $\mathrm{Ad} : G \to \mathrm{Ad}(G) \subset \mathrm{GL}(\mathcal{G})$ is a continuous homomorphism, and the conjugation action of $\mathrm{GL}(\mathcal{G})$ on $\mathrm{M}(\mathcal{G})$ is a continuous linear group action, we get that ρ is a continuous representation of G on $\mathrm{M}(\mathcal{G})$. Now $\rho(g)(\mathrm{d}\,\tau) = \mathrm{Ad}(g\tau(g^{-1}))\,\mathrm{d}\,\tau$, $g \in G$. Then $\rho(g)(\mathrm{d}\,\tau) = \mathrm{d}\,\tau$ for all $g \in \Gamma$. Since Γ is a lattice in G which is semisimple and has no compact factors, we get by the Borel density theorem that $\rho(g)(\mathrm{d}\,\tau) = \mathrm{d}\,\tau$ for all $g \in G$. This implies that $\mathrm{Ad}(g\tau(g^{-1})) = \mathrm{Id}$, and hence that $g\tau(g^{-1}) \in Z(G)$ for all $g \in G$. Since G is connected, $\tau \in \mathrm{Aut}(G)$, and $Z(G)$ is discrete, it follows that $\tau(g) = g$ for $g \in G$.

Now suppose that G is neither nilpotent nor semisimple. We have a Levi decomposition $G = SN$, where S is a (semisimple) Levi subgroup without compact factors and N is a simply connected nilpotent normal closed subgroup. Let $\pi : G \to G/N$ be the natural projection, where G/N is a connected semisimple Lie group isomorphic to $\pi(S)$. Then $\pi(\Gamma)$ is a lattice in $\pi(G)$ [26, Corollary 8.27 or 8.28]. As τ keeps N invariant, we have an automorphism $\overline{\tau}$ of $\pi(G)$ corresponding to τ, which acts trivially on $\pi(\Gamma)$. As $\pi(G) = \pi(S)$ has no compact factors, we get from above that $\overline{\tau} = \mathrm{Id}$; i.e., τ acts trivially on G/N. We also have that $\Gamma \cap N$ is a lattice in N [26, Corollary 8.28]. As $\tau|_\Gamma = \mathrm{Id}$ and $\Gamma \cap N$ is a lattice in N, τ acts trivially on N. Now from Lemma 2.2, we have that τ also acts trivially on $G/Z(N)$, where $Z(N)$ is the center of N.

Since S is a Levi subgroup, so is $\tau(S)$. We also have that $\tau(S) \subset SZ(N)$. We now show that $SZ(N)$ is closed. Note that G/N is isomorphic to $S/(S \cap N)$ and $S \cap N$ is a central subgroup of S, and hence $S \cap N$ has a subgroup of finite index which is central in G. As N is simply connected and nilpotent, $Z(G) \cap N = Z(G)^0 \subset Z(N)$. Moreover, $N/Z(G)^0$ is also simply connected. Hence we get that $S \cap N \subset Z(G)^0 \subset Z(N)$. Now it follows easily that $SZ(N)$ is closed. As S and $\tau(S)$ are also Levi subgroups of $SZ(N)$, we have that $\tau(S) = aSa^{-1}$ for some $a \in Z(N)$ [16]. Therefore, $\mathrm{inn}(a)^{-1} \circ \tau(S) = S$. Let $s \in S$ be fixed. Now $s^{-1}a^{-1}\tau(s)a \in S$. Since $\tau(s) \in sZ(N)$ and $Z(N)$ is normal in G, we get that $s^{-1}a^{-1}\tau(s)a \in Z(N)$. Let $A = \{s^{-1}a^{-1}\tau(s)a \mid s \in S\}$. Then A is connected, $e \in A$, and $A \subset S \cap Z(N)$; the latter is a discrete (central) subgroup of S. This implies that $A = \{e\}$. Therefore, $\tau(s) = asa^{-1}$ for all $s \in S$. Since $a \in Z(N)$, $G = SN$, and τ acts trivially on N, we get that $\tau(g) = aga^{-1}$ for all $g \in G$, i.e., $\tau = \mathrm{inn}(a)$, where $a \in Z(N)$.

Note that $Z(N)$ is isomorphic to $\mathbb{R}^d$ for some $d \in \mathbb{N}$ and it is normal in G. There is a natural action of G on $Z(N)$ by conjugation, which factors through $\pi(G) = \pi(S)$. Moreover, if $g \in \Gamma$, then $\tau(g) = g = aga^{-1}$, and hence $gag^{-1} = a$. Since $a \in Z(N)$, the action of $\pi(\Gamma)$ on $Z(N)$ fixes a. As $\pi(\Gamma)$ is a lattice in $\pi(S)$ which is semisimple and has no compact factors, we get by the Borel density theorem that the action of $\pi(G)$ on $Z(N)$ fixes a. This implies that $gag^{-1} = a$ for all $g \in G$; i.e., $\tau = \mathrm{Id}$ and (1) holds. ∎

The following useful lemma generalises Lemma 3.15 and Theorem 3.16 ((5)⇒(4)) in [23] which are for connected semisimple Lie groups. Note that the proof of (1) (resp. (2)) below has similar arguments as in [23] but uses Proposition 3.6 (2) instead of Corollary 5.18 in [26] (resp. (1) below instead of Lemma 3.15 in [23]). We give a proof for the sake of completeness.

Lemma 3.7. *Let G be a connected Lie group. Suppose that the radical of G is nilpotent and the maximal compact connected normal subgroup of a Levi subgroup of G is contained in the maximal compact normal subgroup of G. Let Γ be a lattice in G and let Γ' be a subgroup of finite index in Γ. Then*

(1) $Z(\Gamma) \cap \Gamma'$ *is a subgroup of finite index in* $Z(\Gamma')$*;*

(2) *If* $T \in \mathrm{Aut}(\Gamma)$ *is such that* $T|_{\Gamma'} = \mathrm{Id}$*, then* $T^n = \mathrm{Id}$ *for some* $n \in \mathbb{N}$.

Proof. The conditions imply that G has a Levi decomposition $G = SN$, where N is the nilradical of G and S is either trivial or its maximal compact connected normal subgroup is contained in the maximal compact connected normal subgroup (say) K of G. Note that Γ', being a subgroup of finite index in the lattice Γ, is also a lattice in G.

(1) If S is either trivial or it has no compact factors, and N is simply connected and nilpotent, we have from Proposition 3.6 (2) that $Z_G(\Gamma') \subset Z(G)$, hence $Z(\Gamma') \subset Z(\Gamma)$, and hence $Z(\Gamma) \cap \Gamma' = Z(\Gamma')$.

Let $\psi : G \to G/K$ be the natural projection, where K is the maximal compact connected normal subgroup of G. Then $G/K = \psi(S)\psi(N)$, where $\psi(N)$ is nilpotent, and it is simply connected, as the largest compact normal subgroup C of N is contained in K and N/C is simply connected. Here, $\psi(S)$ is either trivial or it is a semisimple Levi subgroup, and the condition on S implies that $\psi(S)$ has no compact factors. As $\psi(\Gamma)$ and $\psi(\Gamma')$ are lattices in $\psi(G)$, we get as above that $Z(\psi(\Gamma)) \cap \psi(\Gamma') = Z(\psi(\Gamma'))$. Therefore, $\psi(Z(\Gamma')) \subset Z(\psi(\Gamma')) \subset Z(\psi(\Gamma))$. Let $x \in Z(\Gamma')$ and $g \in \Gamma$ be fixed. Then $\psi(x)\psi(g)\psi(x^{-1}) = \psi(g)$, and hence $x_g = xgx^{-1}g^{-1} \in K$.

We first assume that Γ' is normal in Γ. Then $Z(\Gamma')$, and hence $Z(\Gamma') \cap K$ is normal in Γ and $x_g \in Z(\Gamma') \cap K$. Here, $Z(\Gamma') \cap K$ is finite. Let n be the order of $Z(\Gamma') \cap K$. As $x \in Z(\Gamma')$ and it commutes with x_g, we get that $(x_g)^n = x^n g x^{-n} g^{-1} = e$. Since this holds for all $g \in \Gamma$ and $x \in Z(\Gamma')$, we have that $x^n \in Z(\Gamma)$ for all $x \in Z(\Gamma')$. As $Z(\Gamma')$ is compactly generated and abelian, and $Z(\Gamma) \cap \Gamma' \subset Z(\Gamma')$, it follows that $Z(\Gamma) \cap \Gamma'$ has finite index in $Z(\Gamma')$.

Now suppose that Γ' is not normal in Γ. Then Γ' has a subgroup (say) Γ'' of finite index which is normal in Γ. Now from above, we have that $Z(\Gamma'')/(Z(\Gamma) \cap \Gamma'')$ and $Z(\Gamma'')/(Z(\Gamma') \cap \Gamma'')$ are finite. As $Z(\Gamma')/(Z(\Gamma') \cap \Gamma'')$ is also finite, it is easy to see that $Z(\Gamma')/(Z(\Gamma) \cap \Gamma')$ is finite.

(2) Let $T \in \mathrm{Aut}(\Gamma)$ be such that $T|_{\Gamma'} = \mathrm{Id}$. Passing to a subgroup of finite index of Γ', we may assume that Γ' is normal in G. Replacing T by T^l for some $l \in \mathbb{N}$, we may assume that T^l acts trivially on Γ/Γ'. By Lemma 2.2, T^l acts trivially on $\Gamma/Z(\Gamma')$. From (1), we have that $Z(\Gamma) \cap \Gamma'$ is a subgroup of finite index (say) m in $Z(\Gamma')$. Let $x \in \Gamma$. Then $T^l(x) = xy$ for some $y \in Z(\Gamma')$, and $T^{lm}(x) = xy^m$, where $y^m \in Z(\Gamma) \cap \Gamma'$. Therefore, $xy^m = y^m x$. Let k be the index of Γ' in Γ. Then

$$T^{lm}(x^k) = x^k y^{km} = x^k.$$

Therefore, $y^{km} = e$. Now $T^{klm}(x) = xy^{km} = x$. Since this holds for all $x \in \Gamma$, we have that $T^n = \mathrm{Id}$ for $n = klm$. ■

Now we illustrate by several examples that the conditions in Proposition 3.6 and Lemma 3.7 are necessary. Observe that Lemma 3.7 holds for lattices in semisimple Lie groups as well as for those in nilpotent Lie groups. Example 3.11 in [23] shows that there is a simply connected solvable Lie group G which admits two lattices $\Gamma_1 \subset \Gamma_2$, such that Γ_1 is abelian while $Z(\Gamma_2) = Z(G)$ has infinite index in Γ_1. Example 4.8 shows that for a nontrivial compact subgroup K of $\mathrm{GL}(d, \mathbb{R})$, $d \geq 3$, the group $G = K \ltimes \mathbb{R}^d$ with a lattice $\Gamma \subset \mathbb{R}^d$ admits a certain element $z \in \mathbb{R}^d$, which is not centralised by K. Since K could be chosen to be abelian, and hence G could be solvable. Both the above examples illustrate that the condition that the radical is nilpotent is necessary in both. Since K in Example 4.8 could also be chosen to be semisimple, but it is not normal in G, the condition that a semisimple Levi subgroup has no compact factors is necessary in Proposition 3.6, and the condition that the maximal compact subgroup of a Levi subgroup of G is contained in the maximal compact normal subgroup of G is necessary in Lemma 3.7. One can also take $G = K \times H$, where K is any nontrivial compact connected non-abelian Lie group (which has a compact semisimple Levi subgroup) and H is a connected semisimple Lie group (e.g. $H = \mathrm{SL}(n, \mathbb{R})$ for some $n \geq 2$) or H is isomorphic to $\mathbb{R}^d$, for some $d \geq 3$, then any lattice Γ in H is a lattice in G and $K \subset Z_G(\Gamma)$ but $K \not\subset Z(G)$. Hence the condition that a Levi factor of G has no compact factors is necessary in Proposition 3.6. If $G = \mathbb{H}/D$, where $\mathbb{H}$ is the Heisenberg group of 3×3 strictly upper triangular real (unipotent) matrices and $D = \mathrm{GL}(3, \mathbb{Z}) \cap Z(\mathbb{H})$ is a discrete central subgroup of $\mathbb{H}$ which is isomorphic to $\mathbb{Z}$, then $Z(G) = Z(\mathbb{H})/D$ is compact. Let $\xi : \mathbb{H} \to G = \mathbb{H}/D$ be the natural projection and let $\Gamma = \mathrm{GL}(3, \mathbb{Z}) \cap \mathbb{H}$. Then $\Gamma \cap Z(\mathbb{H}) = D$ and $\xi(\Gamma)$ is a lattice in G. Now $\xi(\Gamma)$ is abelian and it is easy to show that $Z_G(\xi(\Gamma)) = \xi(\Gamma) \times Z(G) \neq Z(G)$. Thus the condition that the (nil)radical is simply connected in Proposition 3.6 is necessary.

4. Distal actions of automorphisms of lattices Γ in Lie groups on Sub_Γ

In this section, for a lattice Γ in a connected Lie group, we discuss and characterise automorphisms of Γ which are in class (NC) and also those which act distally on Sub^c_Γ. For a certain class of connected Lie groups G, we characterise those automorphisms of G which keep a lattice Γ invariant and act distally on Sub^c_Γ. We first state and prove some useful elementary results for countable discrete groups. Note that any finitely generated abelian group has a unique maximal finite subgroup. We denote by G_x the (cyclic) group generated by x in a group.

Recall our assumption that all our groups here are second countable, which enables us to give a metric on Sub_G. Any discrete group is second countable if and only if it is countable. Any discrete subgroup of a connected Lie group is countable. Any finitely generated group and, in particular, any polycyclic group is countable.

Lemma 4.1. *Let G be a discrete countable group and let $T \in \mathrm{Aut}(G)$ be such that $T \in$ (NC). Let H be a normal subgroup of G such that $T|_H = \mathrm{Id}$. Let x be a nontrivial* *torsion element in G such that $T(x) \in xH$. Then there exist $l, n \in \mathbb{N}$ such that $x^l \neq e$ and $T^n(x^l) = x^l$.*

Moreover, if $Z(H)$ is finitely generated, then for the order m of the unique maximal finite group of $Z(H)$, the following holds: for every nontrivial torsion element $x \in G$ with $T(x) \in xH$, there exists $l \in \mathbb{N}$, which depends on x, such that $T^m(x^l) = x^l \neq e$.

Proof. Let $x \in G$ be a nontrivial torsion element and let $n_x \in \mathbb{N} \setminus \{1\}$ be the smallest number such that $x^{n_x} = e$. Then $T(x^j) = x^j y_j$ for some $y_j \in H$, $1 \leq j < n_x$. If $y_j = e$ for some j with $1 \leq j < n_x$, then $T(x^j) = x^j \neq e$ and the first assertion holds for $l = j$ and $n = 1$. Suppose that $y_j \neq e$ for all j, $1 \leq j < n_x$. Now we show that y_l has finite order for some l such that $1 \leq l < n_x$. Since $T \in (\mathrm{NC})$ and Sub_G is compact, $T^{n_k}(G_x) \to A \neq \{e\}$ for some unbounded monotone sequence $\{n_k\} \subset \mathbb{N}$. Let $a \in A$ be such that $a \neq e$. Since G is discrete and $x^{n_x} = e$, passing to a subsequence of $\{n_k\}$, we get that for all k, $T^{n_k}(x^l) = a$, for some fixed l such that $1 \leq l < n_x$. Therefore, $x^l y_l^{n_k} = a$, and hence $y_l^{n_k} = x^{-l} a$ for all k. This implies that y_l has finite order (say) n. Since $T(x^l) = x^l y_l$, we get that $T^n(x^l) = x^l y_l^n = x^l$.

Now suppose that $Z(H)$ is finitely generated. To prove the last statement, we may assume that $G = \{g \mid T(g) \in gH\}$. By Lemma 2.2, we get that T acts trivially on $G/Z(H)$. Let m be the order of the unique maximal finite subgroup (say) F of $Z(H)$. We get as above that for a torsion element $x \in G$, there exists $l \in \mathbb{N}$, which depends on x, such that $x^l \neq e$ and $T(x^l) = x^l y_l \in x^l F$. Therefore, $T^m(x^l) = x^l y_l^m = x^l \neq e$. ■

The following corollary will be used often and it is an easy consequence of Lemma 4.1. We give a short proof for the sake of completeness.

Corollary 4.2. *Let G be a discrete countable group and let H be a normal subgroup of finite index in G such that $Z(H)$ is finitely generated. Let $T \in \mathrm{Aut}(G)$ be such that $T \in (\mathrm{NC})$ and $T|_H = \mathrm{Id}$. Then there exists $m \in \mathbb{N}$ such that $G' = \{g \in G \mid T^m(g) = g\}$ is a subgroup of finite index in G, $H \subset G'$, T^m acts trivially on $G/Z(H)$, and the following holds: for every nontrivial element $x \in G$, $G_x \cap G' \neq \{e\}$. Moreover, if $x \notin G'$, then $\{T^n(x)\}_{n \in \mathbb{N}}$ is infinite (unbounded).*

Proof. If $G = H$ or if G is finite, then $T^m = \mathrm{Id}$ for some $m \in \mathbb{N}$ and the assertions follow trivially for $G' = G$. Suppose that $G \neq H$ and G is infinite. Since G/H is finite, there exists $k \in \mathbb{N}$ such that T^k acts trivially on H and on G/H. By Lemma 2.2, T^k acts trivially on $G/Z(H)$. Let F denote the unique maximal finite subgroup of $Z(H)$ and let d be the order of F. Let $m = kd$ and let $G' = \{g \in G \mid T^m(g) = g\}$. Since $T|_H = \mathrm{Id}$, we have that $H \subset G'$ and G' is a subgroup of finite index in G. Also, T^m acts trivially on $G/Z(H)$. For every element x of infinite order, $G_x \cap G' \neq \{e\}$. Since $T \in (\mathrm{NC})$, we have that $T^k \in (\mathrm{NC})$. Applying Lemma 4.1 for T^k instead of T and H as above, we get that for every nontrivial torsion element x, there exists $l \in \mathbb{N}$ which depends on x, such that $T^m(x^l) = x^l \neq e$; i.e., $G_x \cap G' \neq \{e\}$.

For $x \in G$, we have $T^k(x) = xy$ for some $y \in Z(H)$ and $T^{kn}(x) = xy^n$, $n \in \mathbb{N}$. Suppose that $\{T^n(x)\}_{n \in \mathbb{N}}$ is finite. Then so is $\{T^{kn}(x)\}_{n \in \mathbb{N}}$, hence y has finite order and

$y \in F$. Therefore, $T^m(x) = xy^d = x$, and hence $x \in G'$. This shows that if $x \in G \setminus G'$, then $\{T^n(x)\}_{n\in\mathbb{N}}$ is infinite. ∎

The following two lemmas will be useful for proving Theorems 4.5 and 4.6.

Lemma 4.3. *Let G be a discrete countable group and let $T \in \mathrm{Aut}(G)$. Suppose that H is a normal subgroup of finite index in G such that $T|_H = \mathrm{Id}$, the center $Z(H)$ of H is finitely generated, and Sub^c_H is closed. Then Sub^c_G is closed and the following holds: T acts distally on Sub^c_G if and only if $T^m = \mathrm{Id}$ for some $m \in \mathbb{N}$.*

Proof. Since Sub^c_H is closed and G/H is finite, it follows from Lemma 2.1 that Sub^c_G is closed. If $T^m = \mathrm{Id}$, then it acts distally on Sub_G, and hence so does T. Now suppose that T acts distally on Sub^c_G. Then $T \in$ (NC) and by Corollary 4.2, there exists $m \in \mathbb{N}$ such that $M = \{g \in G \mid T^m(g) = g\}$ has finite index in G, $H \subset M$, T^m acts trivially on $G/Z(H)$, and the following holds: for every nontrivial element $x \in G$, $G_x \cap M \neq \{e\}$, and if $x \notin M$, then $\{T^n(x)\}_{n\in\mathbb{N}}$ is infinite.

We show that $T^m = \mathrm{Id}$. If possible, suppose that $x \in G$ is such that $T^m(x) \neq x$. Then $x \notin M$. Let $l \in \mathbb{N}$ be the smallest integer such that $x^l \in M$. Then $x^l \neq e$ and, from our assumption, $l \neq 1$ and l is less than or equal to the index of M in G.

Let $T_1 = T^m$. We know from above that $T_1(x) \in xZ(H)$ for all $x \in G$ and $T_1(x) \neq x$. Therefore, $G_x \neq G_x \cap M \neq \{e\}$. There exists an unbounded monotone sequence $\{j_k\} \subset \mathbb{N}$ such that $T_1^{j_k}(G_x) \to L$ (say) in Sub^c_G. Then $G_x \cap M \subset L \cap M \subset L$. Note that $G_x \cap M$ is T_1-invariant. Moreover, if $g \in L \cap M$, then $g \in T_1^n(G_x)$ for some $n = j_k$, and hence $g \in G_x$ as $T_1(g) = g$. That is, $L \cap M = G_x \cap M$ and it is cyclic. As $T_1 = T^m$ acts distally on Sub^c_G, we have that $L \cap M \neq L$. Let $a \in L$ be such that $a \notin M$. Since G is discrete, replacing a by a^{-1} if necessary, we get that there exists a sequence $\{n_k\} \subset \mathbb{N}$ such that $T_1^{j_k}(x^{n_k}) = a$ for all large k. Since $x^l \in M$, we have that $T_1^{j_k}(x^{ln_k}) = x^{ln_k} = a^l$ for large k. Therefore, we have that either $\{n_k\}$ is an eventually constant sequence or x has finite order. In either case, passing to a subsequence, we can choose $n_k = n_0$ for all k.

Now we have that $a = T_1^{j_k}(x^{n_0})$ for all k. Note that $n_0 = il + i_0$ for some $i \in \{0\} \cup \mathbb{N}$ and for some fixed i_0 with $0 \le i_0 < l$. Let $k \in \mathbb{N}$ be fixed. Then $a = T_1^{j_k}(x^{i_0})x^{il} \in x^{i_0}M$ as $x^{il} \in M$ and $T_1(x) \in xM$. Here, $i_0 \neq 0$ as $a \notin M$. Since T_1 acts trivially on $G/Z(H)$, $T_1(x^{i_0}) = x^{i_0}y$ for some $y \in Z(H)$. Therefore,

$$a = T_1^{j_k}(x^{n_0}) = x^{il}T_1^{j_k}(x^{i_0}) = x^{il}x^{i_0}y^{j_k} = x^{n_0}y^{j_k}.$$

Hence $y^{j_k} = x^{-n_0}a$ for all $k \in \mathbb{N}$. Since $\{j_k\}$ is unbounded, we get that y has finite order, and hence $y \in F$, where F is the unique maximal finite subgroup of $Z(H)$ with order d (say). This implies that $T^{md}(x^{i_0}) = T_1^d(x^{i_0}) = x^{i_0}y^d = x^{i_0}$ and that $\{T^n(x^{i_0})\}_{n\in\mathbb{N}}$ is finite. This leads to a contradiction as $0 < i_0 < l$ and $x^{i_0} \notin M$. Therefore, $T^m(x) = x$ for all $x \in G$; i.e., $T^m = \mathrm{Id}$. ∎

Lemma 4.4. *Let G be a discrete group and let $T \in \mathrm{Aut}(G)$ be such that $T \in$ (NC). Let H be a closed normal strongly polycyclic subgroup such that $T|_H = \mathrm{Id}$ and $T(x) \in xH$ for all $x \in G$. Then G admits a normal subgroup G' of finite index such that $T|_{G'} = \mathrm{Id}$.*

Proof. The assertion follows trivially if $H = G$. Suppose that $H \neq G$. By Lemma 2.2, $T(x) \in xZ(H)$ for the center $Z(H)$ of H, which is normal in G. If $Z(H)$ is trivial, then $T = \mathrm{Id}$. Now suppose that $Z(H)$ is nontrivial. Since H is strongly polycyclic, so is $Z(H)$. Replacing H by $Z(H)$, we may, without loss of any generality, assume that H is compactly generated, abelian, and torsion-free. Therefore, $H = \mathbb{Z}^d$ for some $d \in \mathbb{N}$.

Now we have a natural homomorphism $\varrho : G \to \mathrm{GL}(d, \mathbb{Z})$, as H is normal in G, which is defined as

$$\varrho(x) = \mathrm{inn}(x)|_H, \quad x \in G,$$

where $\mathrm{inn}(x)$ is the inner automorphism by x in G. Note that $\ker\varrho$ is a normal subgroup of G which contains H. Let $x \in \ker\varrho$. We show that $T(x) = x$. If possible, suppose that $T(x) = xy$ and $y \neq e$, $y \in H$. Then $xy = yx$ and $T(x^m) = x^m y^m \neq x^m$ for all $m \in \mathbb{Z} \setminus \{0\}$ as H is torsion-free. By Lemma 3.12 of [35], this leads to a contraction as $T \in$ (NC). Therefore, $T(x) = x$ for all $x \in \ker\varrho$. If $\ker\varrho$ is a subgroup of finite index in G, then we can choose $G' = \ker\varrho$.

Now suppose that $G/\ker\varrho$ is infinite. As $\mathrm{GL}(n, \mathbb{Z})$ is finitely generated, by [4, Corollary 17.7], $\mathrm{GL}(n, \mathbb{Z})$ has a subgroup (say) M of finite index which is net, i.e., for every $g \in M$, the multiplicative group generated by eigenvalues of g in $\mathbb{C} \setminus \{0\}$ is torsion-free. Replacing M by a subgroup of finite index, we may assume that M is normal in $\mathrm{GL}(n, \mathbb{Z})$. Let $G'' = \varrho(G) \cap M$. Then G'' is a normal torsion-free subgroup of finite index in $\varrho(G)$ and it is net. Now let $G' = \varrho^{-1}(G'')$. It is a normal subgroup of finite index in G and it contains H. We show that $T|_{G'} = \mathrm{Id}$.

As noted above, $T \in$ (NC) and T acts trivially on both G/H and H. If possible, suppose that $x \in G'$ is such that $T(x) = xy$ for some $y \in H$, $y \neq e$. If $xy = yx$, then arguing as above using [35, Lemma 3.12], we arrive at a contraction as $T \in$ (NC). Now suppose that $xy \neq yx$. By [35, Lemma 3.12], we get that $T(x^l) = x^l$, for some $l \in \mathbb{N} \setminus \{1\}$. This implies that $(xy)^l = x^l$. For $a := \sum_{i=1}^{l} x^i y x^{-i}$ in $Z(H) = \mathbb{Z}^d$, which is an additive group, we get that $a = 0$. Then, $a = x^{-1}ax$, and hence $x^l y x^{-l} = y$. Therefore, $\varrho(x)$ has an eigenvalue which is a nontrivial root of unity. This leads to a contradiction as $x \in G'$ and $\varrho(G')$ is net. Hence $T|_{G'} = \mathrm{Id}$. ∎

Every polycyclic group contains a unique maximal nilpotent normal subgroup. The following theorem about distality for polycyclic groups generalises Theorem 3.10 of [23] as lattices in a connected solvable Lie group are polycyclic. More generally, Theorem 4.5 holds for any discrete solvable subgroup of a connected Lie group due to Lemma 3.1, and it will be useful in proving Theorem 4.6 for lattices in a connected Lie group. Note that the class of polycyclic groups is strictly larger than that of lattices in connected solvable Lie groups (see [26, Examples 4.29–4.33]). Example 3.11 in [23] illustrates that not all the statements in the theorem are equivalent.

Theorem 4.5. *Let G be a discrete polycyclic group and let $T \in \mathrm{Aut}(G)$. Let G_{nil} be the unique maximal nilpotent normal subgroup of G. Then Sub^c_G is closed and* (1)*,* (2) *are equivalent as well as* (3)*–*(6) *are equivalent.*

(1) $T \in$ (NC).

(2) *There exist $n \in \mathbb{N}$ and a subgroup G' of finite index in G containing $G_{\rm nil}$ such that $T^n|_{G'} = \mathrm{Id}$, the identity map on G', and $G' \cap G_x \neq \{e\}$ for every nontrivial element x in G.*

(3) *T acts distally on* Sub^c_G.

(4) *T acts distally on* Sub^a_G.

(5) *T acts distally on* Sub_G.

(6) $T^n = \mathrm{Id}$ *for some $n \in \mathbb{N}$.*

If G is nilpotent, then (1)–(6) *are equivalent.*

Proof. Since G is polycyclic, every subgroup of G is finitely generated, by [23, Lemma 3.3], Sub^c_G is closed. If G is finite, then so is Sub_G and (1)–(6) hold trivially as $T^n = \mathrm{Id}$ for some $n \in \mathbb{N}$. Now suppose that G is infinite. Suppose that (1) holds. We know that $G_{\rm nil}$ is characteristic in G, and hence it is T-invariant. Since $G_{\rm nil}$ is finitely generated and nilpotent, the set R_g of roots of $g \in G$ is finite for every $g \in G$ [15, Theorems 3.1.13 and 3.1.17]. As $T|_{G_{\rm nil}} \in$ (NC), by [23, Proposition 3.8], $T^{k_1}|_{G_{\rm nil}} = \mathrm{Id}$ for some $k_1 \in \mathbb{N}$. If $G = G_{\rm nil}$, then (2) holds for $G' = G$ and $n = k_1$.

Suppose that $G/G_{\rm nil}$ is nontrivial and finite. Since $T^{k_1} \in$ (NC) and $Z(G_{\rm nil})$ is finitely generated, applying Corollary 4.2 for T^{k_1}, we get that (2) holds for some subgroup G' containing $G_{\rm nil}$ and some $n \in \mathbb{N}$.

Now suppose that $G/G_{\rm nil}$ is infinite. Then we can choose a strongly polycyclic subgroup (say) L of finite index which is T-invariant and normal in G. Then $L_{\rm nil}$ is T-invariant and normal in G and $L_{\rm nil} \subset G_{\rm nil}$. By [26, Corollary 4.11], $L/L_{\rm nil}$ admits an abelian subgroup of finite index which is finitely generated and infinite. Now we can choose a T-invariant subgroup (say) L' of finite index in L such that $L_{\rm nil} \subset L'$ and $L'/L_{\rm nil}$ is torsion-free.

Let $\overline{T} : L'/L_{\rm nil} \to L'/L_{\rm nil}$ be the automorphism corresponding to $T|_{L'}$. By [23, Lemma 3.5], $\overline{T} \in$ (NC). Since $L'/L_{\rm nil}$ is finitely generated and abelian, arguing as above, we get that $\overline{T}^{k_2}$ acts trivially on $L'/L_{\rm nil}$ for some $k_2 \in \mathbb{N}$. Let $k = \mathrm{lcm}(k_1, k_2)$. Then T^k acts trivially on $L'/L_{\rm nil}$ and also on $L_{\rm nil}$ which is contained in $G_{\rm nil}$. As $L_{\rm nil}$ is strongly polycyclic, by Lemma 4.4, we get that L' has a normal subgroup (say) L'' of finite index such that $T^k|_{L''} = \mathrm{Id}$. Now we can replace L'' by a subgroup of finite index and assume that L'' is normal in G. Let $G'' = L''G_{\rm nil}$. Then $T^k|_{G''} = \mathrm{Id}$ and it is a normal subgroup of finite index in G. As $T^k \in$ (NC) and $Z(G'')$ is finitely generated, we get by Corollary 4.2 that (2) holds for some subgroup G' such that $G_{\rm nil} \subset G'' \subset G'$ and some $n \in \mathbb{N}$.

Now suppose that (2) holds. If G is nilpotent, then $G = G_{\rm nil} = G'$ and hence (2)$\Rightarrow$(1). Suppose that G is not nilpotent. Let G' and n be as in (2). Then G' is a subgroup of finite index (say) m and $T^n|_{G'} = \mathrm{Id}$. We show that $T^n \in$ (NC). For $x \in G$, by (2), there exists $l \in \mathbb{N}$, such that $e \neq x^l \in G' \cap G_x$. Hence, $T^n(G_{x^l}) = G_{x^l} \neq \{e\}$ and $T^{nm_j}(G_x) \not\to \{e\}$ for any sequence $\{m_j\} \subset \mathbb{Z}$. Therefore, $T^n \in$ (NC), and hence $T \in$ (NC) and (1) holds.

We know that (6)⇒(5)⇒(4)⇒(3). Now we show that (3)⇒(6). Suppose that (3) holds. That is, T acts distally on Sub^c_G. Then $T \in$ (NC) and hence (2) holds. Let G' be a subgroup of finite index in G as in (2). In particular, $T^n|_{G'} = \mathrm{Id}$ for some $n \in \mathbb{N}$. We may replace G' by a normal subgroup of finite index and assume that it is normal in G. As $Z(G')$ is finitely generated, by Lemma 4.3, we get that $T^n = \mathrm{Id}$ for some $n \in \mathbb{N}$, and hence (6) holds. Therefore, (3)–(6) are equivalent.

If G is nilpotent, then $G = G_{\mathrm{nil}}$ and 2 ⇔ 6, and hence (1)–(6) are equivalent. ∎

Theorems 4.6 and 4.7 together generalise Corollary 3.9, Theorem 3.10, and Theorem 3.16 of [23] which are for lattices in simply connected nilpotent, simply connected solvable, and connected semisimple Lie groups, respectively. We will illustrate by constructing several counter examples that these theorems are the best possible results in this direction. Note that for any lattice Γ in a connected Lie group, Sub^c_Γ is closed in Sub_Γ by Corollary 3.5.

Theorem 4.6. *Let Γ be a lattice in connected Lie group G. Let $T \in \mathrm{Aut}(\Gamma)$. Then* (1)–(2) *are equivalent and* (3)–(6) *are equivalent.*

(1) $T \in$ (NC).

(2) *There exist $n \in \mathbb{N}$ and a subgroup Γ' of finite index in Γ such that $\Gamma_{\mathrm{nil}} \subset \Gamma'$, $T^n|_{\Gamma'} = \mathrm{Id}$, the identity map on Γ' and for every nontrivial element $x \in \Gamma$, $G_x \cap \Gamma' \neq \{e\}$.*

(3) *T acts distally on Sub^c_Γ.*

(4) *T acts distally on Sub^a_Γ.*

(5) *T acts distally on Sub_Γ.*

(6) *$T^n = \mathrm{Id}$ for some $n \in \mathbb{N}$.*

If the radical of G is nilpotent and the maximal compact connected normal subgroup of a Levi subgroup of G is contained in the maximal compact normal subgroup of G, then (1)–(6) *are equivalent.*

Proof. If G is semisimple, then the assertions in the theorem follow from [23, Theorem 3.16]. If G, or more generally, Γ is nilpotent, then $\Gamma = \Gamma_{\mathrm{nil}}$ and (1)–(6) are equivalent by Theorem 4.5 as Γ is polycyclic in this case.

Suppose that G is compact. Equivalently, Γ is finite, hence $T^n = \mathrm{Id}$ for some $n \in \mathbb{N}$, and (1)–(6) are equivalent.

We now assume that G is not compact (equivalently, Γ is not finite). We know that (6)⇒(5)⇒(4)⇒(3). Suppose that (3) holds. Note that (3)⇒(1). Suppose that (1)⇒(2). Let Γ' be a subgroup of finite index in Γ such that $T^n|_{\Gamma'} = \mathrm{Id}$ for some $n \in \mathbb{N}$. We may replace Γ' by a subgroup of finite index and assume that Γ' is normal in Γ. Since $Z(\Gamma')$ is finitely generated, we get from Lemma 4.3 that $T^n = \mathrm{Id}$ for some $n \in \mathbb{N}$. Therefore, (3)–(6) are equivalent. Now it is enough to prove that (1)–(2) are equivalent. Note that (2)⇒(1) follows easily as in the proof of Theorem 4.5 ((2)⇒(1)). Now we show that (1)⇒(2).

Suppose that Γ is solvable. Then it is polycyclic by Lemma 3.1 (2), and it follows from Theorem 4.5 that (1)⇒(2).

Now suppose that $\Gamma \neq \Gamma_{\rm rad}$. Then G is a (noncompact) Lie group which is not solvable. Suppose that (1) holds. We know that $T(\Gamma_{\rm rad}) = \Gamma_{\rm rad}$ and that $\Gamma_{\rm rad}$ is polycyclic by Proposition 3.2. Note that $\Gamma_{\rm nil} \subset \Gamma_{\rm rad}$ and $\Gamma_{\rm nil}$ is also the unique maximal nilpotent normal subgroup of $\Gamma_{\rm rad}$. By Theorem 4.5, $\Gamma_{\rm rad}$ has a subgroup (say) Δ of finite index such that $\Gamma_{\rm nil} \subset \Delta$ and $T^{n_1}|_\Delta = \mathrm{Id}$ for some $n_1 \in \mathbb{N}$. Replacing Δ by a subgroup of finite index containing $\Gamma_{\rm nil}$, we may assume that Δ is normal in $\Gamma_{\rm rad}$. Suppose that $\Gamma_{\rm rad}$ is a subgroup of finite index in Γ. Then we can replace Δ again by a subgroup of finite index containing $\Gamma_{\rm nil}$ so that it is normal in Γ. As $T^{n_1} \in$ (NC) and $Z(\Delta)$ is finitely generated, we get from Corollary 4.2 that (2) holds for some Γ' with $\Gamma_{\rm nil} \subset \Delta \subset \Gamma'$, and some $n \in \mathbb{N}$.

Now suppose that $\Gamma/\Gamma_{\rm rad}$ is infinite. By Proposition 3.3, Γ has a subgroup (say) Λ of finite index such that $\Gamma_{\rm rad} = \Lambda_{\rm rad}$, $\Lambda/\Gamma_{\rm rad}$ is torsion-free, and the set R_g of roots of g in $\Lambda/\Gamma_{\rm rad}$ is finite for every $g \in \Lambda/\Gamma_{\rm rad}$. We may replace Λ by a subgroup of finite index and assume that Λ is normal in Γ and $T(\Lambda) = \Lambda$.

Suppose that $T \in$ (NC). Then $T|_\Lambda \in$ (NC). Let $\eta : \Gamma \to \Gamma/\Gamma_{\rm rad}$ be the natural projection and let $\overline{T}$ be the automorphism of $\Gamma/\Gamma_{\rm rad}$ corresponding to T. Since Γ is discrete, $\mathrm{Sub}_{\Gamma/\Gamma_{\rm rad}}$ is metrizable, and since $\eta(\Lambda)$ is torsion-free, we get by [23, Lemma 3.5] that $\overline{T}|_{\eta(\Lambda)} \in$ (NC). As the set R_g of roots of g in $\eta(\Lambda)$ is finite for every $g \in \eta(\Lambda)$, we get from [23, Proposition 3.8] that for some $n_2 \in \mathbb{N}$, $\overline{T}^{n_2}|_{\eta(\Lambda)} = \mathrm{Id}$; i.e., T^{n_2} acts trivially on $\Lambda/\Gamma_{\rm rad}$.

Suppose that $\Gamma_{\rm rad}$ is a finite group of order (say) n'. Replacing n_2 by its multiple in $\mathbb{N}$, we may assume that T^{n_2} acts trivially on $\Gamma_{\rm rad}$. As T^{n_2} acts trivially on $\Lambda/\Gamma_{\rm rad}$, replacing n_2 by $n_2 n'$, we get that $T^{n_2}|_\Lambda = \mathrm{Id}$. By Lemma 3.1, $Z(\Lambda)$ is finitely generated, and since $T^{n_2} \in$ (NC), by Corollary 4.2 we get that (2) holds for some subgroup Γ' containing Λ, and some $n \in \mathbb{N}$. We note here that $\Gamma_{\rm nil} \subset \Gamma_{\rm rad} \subset \Lambda \subset \Gamma'$.

Now suppose that $\Gamma_{\rm rad}$ is infinite. Note that T^{n_2} acts trivially on $\Lambda/\Gamma_{\rm rad}$. We also have from above that $\Gamma_{\rm rad}$ has a normal subgroup Δ of finite index such that $T^{n_1}|_\Delta = \mathrm{Id}$, where $n_1 \in \mathbb{N}$ and $\Gamma_{\rm nil} \subset \Delta$. Here, Δ is infinite since $\Gamma_{\rm rad}$ is so. Passing to a subgroup of finite index in Δ, we may assume that Δ is strongly polycyclic. Let m be the index of Δ in $\Gamma_{\rm rad}$. Let L_m be the subgroup generated by $\{x^m \mid x \in \Gamma_{\rm rad}\}$. Then L_m has finite index in $\Gamma_{\rm rad}$ [26, Lemma 4.4]. It is easy to see that $L_m \subset \Delta$ and it is characteristic in $\Gamma_{\rm rad}$. Replacing Δ by L_m, we may assume that Δ is T-invariant and normal in Γ. Note that $\Gamma_{\rm nil}$ may not be contained in Δ now but $(\Gamma_{\rm nil}\Delta)/\Delta$ is finite, $\Delta_{\rm nil} \subset \Gamma_{\rm nil}$, and $T^{n_1}|_{\Gamma_{\rm nil}} = \mathrm{Id}$. We may also replace n_2 by its multiple in $\mathbb{N}$ and assume that T^{n_2} acts trivially on $\Gamma_{\rm rad}/\Delta$.

Let $n_3 = \mathrm{lcm}(n_1, n_2)$ and let $T_1 = T^{n_3 m}$, where m is the index of Δ in $\Gamma_{\rm rad}$. Then T_1 acts trivially on Λ/Δ and on Δ. Since $T_1 \in$ (NC), we get by Lemma 4.4 that there is a normal subgroup (say) G'' of finite index in Λ such that $T_1|_{G''} = \mathrm{Id}$. Note that $T_1|_{\Gamma_{\rm nil}} = \mathrm{Id}$. Let $H = \Gamma_{\rm nil} G''$. As $Z(H)$ is finitely generated, we get from Corollary 4.2 that (2) holds for some subgroup Γ' containing $\Gamma_{\rm nil}$ and some $n \in \mathbb{N}$.

As for the last assertion, if G is as in Lemma 3.7, it follows from the lemma that (2)$\Rightarrow$(6), and hence (1)–(6) are equivalent. ■

For an automorphism τ of a connected Lie group G which keeps a lattice Γ invariant, we compare the distality of the τ-actions on Sub_G and Sub_Γ in the following theorem which generalises Corollary 3.9 and Theorem 3.16 of [23]. From now on till the end of the section, any of the statements (1) to (6) refers to the respective statement in Theorem 4.6.

Theorem 4.7. *Let G be a connected Lie group and let Γ be a lattice in G. Let $\tau \in \mathrm{Aut}(G)$ be such that $\tau(\Gamma) = \Gamma$ and let $T = \tau|_\Gamma$. Suppose that the radical of G is simply connected and nilpotent and the maximal compact connected normal subgroup of a Levi subgroup of G is normal in G. Then* (1)–(6) *of Theorem* 4.6 *are equivalent and they are equivalent to each of the following statements* (7)–(10)*:*

(7) $\tau \in (\mathrm{NC})$*;*

(8) τ *acts distally on* Sub^a_G*;*

(9) τ *acts distally on* Sub_G*;*

(10) τ *is contained in a compact subgroup of* $\mathrm{Aut}(G)$.

Moreover, if a Levi subgroup of G has no compact factors, then each of the statements (1)–(10) *are equivalent to the following:*

(11) $\tau^n = \mathrm{Id}$ *for some* $n \in \mathbb{N}$.

Proof. Let G, Γ, τ, and T be as in the hypothesis. By Theorem 4.6, (1)–(6) are equivalent.

We first assume that G is compact. Then the condition on the radical implies that the compact group G is either trivial or semisimple, and hence $\mathrm{Aut}(G)$ is also compact. Therefore, (1)–(10) are equivalent by [23, Theorem 3.16], and the additional condition that a Levi subgroup of G has no compact factors implies that G is trivial in this case, hence $\tau = \mathrm{Id}$ and (1)–(11) are equivalent.

We now assume that G is not compact (equivalently, Γ is not finite). Let K be the maximal compact connected normal subgroup of G and let N be the nilradical of G. As N is simply connected, $K \cap N = \{e\}$. Then G has no nontrivial compact connected central subgroup. By [35, Theorem 4.1], we have that (7)–(10) are equivalent. Also, (8)$\Rightarrow$(4), and hence (6) also holds since (1)–(6) are equivalent. Now suppose that (6) holds; i.e., $\tau^n|_\Gamma = T^n = \mathrm{Id}$ for some $n \in \mathbb{N}$.

By the conditions on the structure of G, we have a Levi decomposition $G = SN$, where N is simply connected and nilpotent and the Levi subgroup S is either trivial or it is semisimple. Moreover, the maximal compact connected normal subgroup of S is contained in K; in fact, it is equal to K, since $K \cap N = \{e\}$. Now if K is trivial or, equivalently, S has no compact factors, then by Proposition 3.6, $\tau^n = \mathrm{Id}$; i.e., (11) holds and hence (1)–(11) are equivalent.

Suppose K is nontrivial. We show that (10) holds. Let $\psi : G \to G/K$ be the natural projection. Since $\tau(K) = K$, we have the corresponding action of τ on G/K. Then $\psi(G)$ has no nontrivial compact connected normal subgroup, $\psi(\Gamma)$ is a lattice in $\psi(G)$, and we get from (6) that τ^n acts trivially on $\psi(\Gamma)$ for some $n \in \mathbb{N}$. Now $\psi(G) = \psi(S)\psi(N)$, where the Levi subgroup $\psi(S)$ is either trivial or it has no compact factors and $\psi(N)$ is simply connected as $K \cap N = \{e\}$. By Proposition 3.6, τ^n acts trivially on $\psi(G)$. This implies that, for any $x \in G$, $\tau^n(x) \in xK$, and if $x \in N$, $x^{-1}\tau^n(x) \in K \cap N = \{e\}$.

Therefore, $\tau^n|_N = \mathrm{Id}$. As $K \subset S$, it follows that $\tau^n(S) = S$. Here, $S = S'K = KS'$, where S' is either trivial or it is a connected semisimple Lie group without compact factors and $\tau^n(S') = S'$. Suppose that S' is nontrivial. Let $F = \{x^{-1}\tau^n(x) \mid s \in S'\}$. Then $e \in F \subset S' \cap K$ and F is connected as S' is connected. Since $S' \cap K$ is finite, we get that F is trivial, and hence $\tau^n(x) = x$ for all $x \in S'$. Therefore, τ^n acts trivially on $S'N$, which is a co-compact normal subgroup of G.

Since K is a compact connected semisimple Lie group, its automorphism group contains the group of inner automorphisms of K as a subgroup of finite index, and hence it is compact. Moreover, as K is normal and N is simply connected, elements of K centralise N, and also $S'N$. Note that $G = KN$ (resp. $G = KS'N$) and τ^n acts trivially on N (resp. $S'N$). Since $\tau(K) = K$, $\tau^n|_N = \mathrm{Id}$, and $\tau^n|_{S'} = \mathrm{Id}$, replacing n by its multiple in $\mathbb{N}$, we have that τ^n is an inner automorphism of G by an element of K. Therefore, τ generates a relatively compact group in $\mathrm{Aut}(G)$. That is, (6)⇒(10), and hence (1)–(10) are equivalent. ■

Now we illustrate by examples that Theorems 4.6 and 4.7 are the best possible results for lattices in a connected Lie group. Example 3.11 in [23] shows that a simply connected solvable Lie group G can admit a lattice Γ and an automorphism $\tau \in \mathrm{Aut}(G)$ such that τ keeps Γ invariant, $\tau|_\Gamma \in$ (NC) but τ does not act distally on Sub^c_Γ. This illustrates that neither (1)–(6) nor (1)–(10) above are equivalent in general.

If G is a compact connected semisimple Lie group, then the trivial subgroup is a lattice which is invariant under any automorphism of G and $\mathrm{Aut}(G)$ is a nontrivial compact group which contains elements of infinite order. Hence for such a G, (10) above holds, but (11) cannot hold in general. If $G = \mathbb{T}^d$, $d \geq 2$, then its lattices are finite and any automorphism of G keeps each finite group $G_n = \{g \in G \mid g^n = e\}$ invariant for any $n \in \mathbb{N}$. Note that $\mathrm{Aut}(G)$ is isomorphic to $\mathrm{GL}(d, \mathbb{Z})$ and, by Selberg's lemma, admits a subgroup of finite index which is torsion-free. Hence for such a G, (6) above holds for any lattice, but (10) or (11) cannot hold in general.

Now we give an example of a class of groups $G = K \ltimes \mathbb{R}^d$, $d \geq 3$, where K is any nontrivial compact connected subgroup of $\mathrm{GL}(d, \mathbb{R})$ and G admits an automorphism τ and a lattice Γ such that $\tau|_\Gamma = \mathrm{Id}$ but τ does not generate a relatively compact group in $\mathrm{Aut}(G)$, hence (1)–(6) hold in this case, but none of (7)–(11) holds. The group G as above has compact or trivial Levi subgroups, and it is solvable if K is abelian. Example 4.8 together with the examples mentioned above illustrate that the conditions in Theorem 4.7 that the radical is simply connected and nilpotent and the maximal compact connected normal subgroup of a Levi subgroup is normal in the whole group are necessary for the equivalence of (1)–(10).

Example 4.8. Let $G = K \ltimes \mathbb{R}^d$, for some $d \geq 3$, and let K be any nontrivial compact connected subgroup of $\mathrm{GL}(d, \mathbb{R})$, where the group operation is given by $(h, x)(k, y) = (hk, k^{-1}(x) + y)$, $h, k \in K$, $x, y \in \mathbb{R}^d$. Let $\Gamma = \mathbb{Z}^d \subset \mathbb{R}^d$. Since K is compact, Γ is a lattice in G. Choose $k \in K$ and $z \in \mathbb{R}^d$ such that $k(z) \neq z$. Let $\tau = \mathrm{inn}(z)$, the inner automorphism of G by z; i.e., $\tau(g) = zgz^{-1}$ for all $g \in G$. Then τ acts trivially on Γ. Now we prove that the closed subgroup generated by τ is noncompact in $\mathrm{Aut}(G)$. It is enough to show that $\{\tau^n(k) \mid n \in \mathbb{N}\}$ is unbounded. Note that $\tau^n(k) = (k, k^{-1}(nz) - nz) = (k, n(k^{-1}(z) - z)) = (k, ny)$, where $y = k^{-1}(z) - z \in \mathbb{R}^d$. Since $k(z) \neq z$, $y \neq 0$. Therefore, $\{ny \mid n \in \mathbb{N}\}$ is unbounded, and hence $\{\tau^n(k) \mid n \in \mathbb{N}\}$ is unbounded. Note that K could be chosen to be abelian or semisimple and it is not normal in G. Here (1)–(6) hold but none of the (7)–(11) in Theorem 4.7 holds.

5. Expansive actions of automorphisms of lattices Γ in Lie groups on Sub_Γ

In this section, we study expansive actions of automorphisms of G on Sub^c_G, the space of discrete cyclic subgroups of G, for a certain class of discrete groups G which include discrete polycyclic groups. We also show that a lattice Γ in a connected noncompact Lie group does not admit any automorphism which acts expansively on Sub^c_Γ.

For a countable discrete group G with the property that the set R_g of roots of g in G is finite for every $g \in G$, Sub^c_G is closed in Sub_G [23, Lemma 3.4]. For such a group G, it is shown in [23, Proposition 3.8] that only finite order automorphisms act distally on Sub^c_G, in case G is finitely generated. Here, we study the expansivity of actions of automorphisms of G on Sub^c_G in the following.

Lemma 5.1. *Let G be a discrete countable group with the property that the set R_g of roots of g in G is finite for all $g \in G$. Then the complement of any neighbourhood of $\{e\}$ in Sub^c_G is finite. Moreover, G does not admit any automorphism which acts expansively on Sub^c_G unless G is finite.*

Proof. If G is finite, then Sub_G is finite, and the first assertion holds trivially and any automorphism of G acts expansively on G. Now suppose that G is infinite. Let G_x be the cyclic group generated by $x \in G$. If possible, suppose that there are infinitely many elements outside some open neighbourhood U of $\{e\}$ in Sub^c_G; namely, $G_{x_n} \notin U$, $x_n \neq x_m$ for all $m, n \in \mathbb{N}$. Note that Sub^c_G, being closed in Sub_G, is compact. Passing to a subsequence if necessary, we get that $G_{x_n} \to G_x$ for some $x \in G$, as $n \to \infty$. Then $G_x \notin U$, and hence $x \neq e$. As G is discrete, there exists $n_0 \in \mathbb{N}$ such that $x \in G_{x_n}$ for all $n \geq n_0$. It follows that $x = x_n^{m_n}$ for some $m_n \in \mathbb{Z}$ and for all $n \geq n_0$. Replacing x by x^{-1} if necessary, we may assume that $m_n \in \mathbb{N}$ for infinitely many n. This leads to a contradiction as x_n's are distinct and R_x is finite. Hence, given any neighbourhood U of $\{e\}$ in Sub^c_G, $\mathrm{Sub}^c_G \setminus U$ is finite.

Let $T \in \mathrm{Aut}(G)$. If possible, suppose that T acts expansively on Sub^c_G with an expansive constant $\varepsilon > 0$. Let $U = \{H \in \mathrm{Sub}^c_G \mid d(H, \{e\}) < \varepsilon\}$, where d is the metric on Sub_G. Since G is infinite and R_e is finite, there exists $x \in G$ which generates an infinite cyclic group. For every $k \in \mathbb{N}$, there exists $n_k \in \mathbb{Z}$ such that $d(T^{n_k}(G_{x^k}), \{e\}) > \varepsilon$. Since $\mathrm{Sub}^c_G \setminus U$ is finite, we have that $T^{n_k}(G_{x^k}) = T^{n_l}(G_{x^l})$ for infinitely many k and l with $k \neq l$. Here, $n_k \neq n_l$ if $l \neq k$, as x has infinite order. Let $y = x^k$ for some fixed k. Then

$y = (T^{n_l - n_k}(x))^l$ for infinitely many l. Since R_y is finite, we get that $\{T^{n_l - n_k}(x)\}_{l \in \mathbb{N}}$ is finite. As $n_i \neq n_j$, if $i, j \in \mathbb{N}$ and $i \neq j$, there exists $m \in \mathbb{N}$ such that $T^m(x) = x$. Now $T^m(G_{x^k}) = G_{x^k}$ for all $k \in \mathbb{N}$. That is, T^m has infinitely many fixed points in Sub^c_G. This leads to a contradiction, due to [38, Theorem 5.26]. Therefore, T is not expansive. ■

Note that Lemma 5.1, in particular, implies that any discrete finitely generated infinite nilpotent group G does not admit any automorphism that acts expansively on Sub^c_G, as the set of roots of g is finite for every $g \in G$ [15]. Such groups G form a proper subclass of (discrete) polycyclic groups. If G is any discrete polycyclic group, then every subgroup of it is finitely generated, and by [23, Lemma 3.3], Sub^c_G is closed in Sub_G. The following theorem shows that such a G does not admit any automorphism which acts expansively on G, unless G is finite. The theorem will be useful in the proof of Theorem 1.2. As noted before, the class of polycyclic groups is strictly larger than that of lattices in connected solvable Lie groups.

Theorem 5.2. *Let G be an infinite discrete polycyclic group and let $T \in \mathrm{Aut}(G)$. Then the T-action on Sub^c_G is not expansive. In particular, this holds when G is any discrete solvable subgroup of a connected Lie group.*

Proof. Let G' be a strongly polycyclic subgroup of finite index in G. Passing to a subgroup of finite index, we may assume that G' is T-invariant. Now we may replace G by G' and assume that G is strongly polycyclic. Let $G(0) = G$ and let $G(n+1) = [G(n), G(n)]$ be the commutator subgroup of $G(n)$, $n \in \mathbb{N} \cup \{0\}$. Each $G(n)$ is a characteristic subgroup of G. Since G is solvable, there exists $k \in \mathbb{N} \cup \{0\}$ such that $G(k) \neq \{e\}$ and $G(k+1) = \{e\}$. Here, $G(k)$ is an infinite strongly polycyclic abelian T-invariant group. Therefore, replacing G by $G(k)$ if necessary, we may assume that G is abelian. Now G is isomorphic to $\mathbb{Z}^n$ for some $n \in \mathbb{N}$. It is easy to see that the set R_g of roots of g is finite in $G = \mathbb{Z}^n$ (see also [15, Example 3.1.12]). By Lemma 5.1, the T-action on Sub^c_G is not expansive.

Every discrete solvable subgroup of a connected Lie group is polycyclic by Lemma 3.1 (2), hence the second assertion follows from the first. ■

Recall that for a locally compact group G, Sub^a_G is the set of all closed abelian subgroups of G. It is closed in Sub_G. The following result is already known for all connected Lie groups [25, Theorem 3.1]. Combining it with Theorem 5.2, we get the following generalisation.

Corollary 5.3. *Let G be an infinite Lie group and let $T \in \mathrm{Aut}(G)$. If G/G^0 is polycyclic, then the T-action on Sub^a_G is not expansive. In particular, if G is an infinite closed subgroup of a connected Lie group H such that G is either solvable or normal in H, then the T-action on Sub^a_G is not expansive.*

Proof. Note that each of the conditions on G implies that G is second countable. The connected component G^0 of the identity e in G is a closed T-invariant subgroup of G and it is a Lie group. If $G^0 \neq \{e\}$, by [25, Theorem 3.1], the T-action on $\mathrm{Sub}^a_{G^0}$, and hence on

Sub_G^a, is not expansive. If $G^0 = \{e\}$, then G is discrete. If G/G^0 is polycyclic, i.e., G is polycyclic, by Lemma 5.1, the T-action on Sub_G^c and, hence, on Sub_G^a is not expansive.

Suppose that G is a closed subgroup of a connected Lie group H. If G is solvable, then by Lemma 3.1 (2), G/G^0 is polycyclic. If G is normal in H, then so is G^0, and G/G^0, being a discrete normal subgroup of the connected Lie group H/G^0, is central in H/G^0, and hence it is polycyclic. Therefore, in either case, the second assertion follows from the first. ∎

Note that a lattice Γ in a compact Lie group is finite and hence all its automorphisms act expansively on Sub_Γ. Now we are ready to prove one of the main results about expansivity; namely, Theorem 1.2, which states that a lattice Γ in a connected noncompact Lie group does not admit any automorphism which acts expansively on Sub_Γ^c.

Proof of Theorem 1.2. Let Γ be a lattice in a connected noncompact Lie group G and let $T \in \mathrm{Aut}(\Gamma)$. By Proposition 3.2 (a), the largest solvable normal subgroup Γ_{rad} of Γ is polycyclic. Note that Γ_{rad} is characteristic in Γ, and hence it is T-invariant.

If possible, suppose that T acts expansively on Sub_Γ^c. Then $T|_{\Gamma_{\mathrm{rad}}}$ acts expansively on $\mathrm{Sub}_{\Gamma_{\mathrm{rad}}}^c$. By Theorem 5.2, Γ_{rad} is finite. By Proposition 3.3 (2), Γ has a subgroup of finite index Λ containing Γ_{rad} such that the set R_g of roots of g in $\Lambda/\Gamma_{\mathrm{rad}}$ is finite, for every $g \in \Lambda/\Gamma_{\mathrm{rad}}$. This, together with the fact that Γ_{rad} is finite, implies that the set R_g of roots of g in Λ is finite for every $g \in \Lambda$. Passing to a subgroup of finite index if necessary, we may assume that Λ is T-invariant. Since T acts expansively on Sub_Γ^c and, hence, on Sub_Λ^c, we get from Lemma 5.1 that Λ is finite, and hence Γ is also finite. This leads to a contradiction as G is noncompact [26, Remark 5.2 (2) and Lemma 5.4]. Hence, T does not act expansively on Sub_Γ^c. ∎

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