A description of $\mathrm{Aut}(dV_n)$ and $\mathrm{Out}(dV_n)$ using transducers

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Abstract. The groups dV_n are an infinite family of groups, first introduced by C. Martínez-Pérez, F. Matucci and B. E. A. Nucinkis, which includes both the Higman–Thompson groups V_n $(= 1V_n)$ and the Brin–Thompson groups nV $(= nV_2)$. A description of the groups $\mathrm{Aut}(G_{n,r})$ (including the groups $G_{n,1} = V_n$) has previously been given by C. Bleak, P. Cameron, Y. Maissel, A. Navas, and F. Olukoya. Their description uses the transducer representations of homeomorphisms of Cantor space introduced in a paper of R. I. Grigorchuk, V. V. Nekrashevich, and V. I. Sushchanskii, together with a theorem of M. Rubin. We generalise the transducers of the latter paper and make use of these transducers to give a description of $\mathrm{Aut}(dV_n)$ which extends the description of $\mathrm{Aut}(1V_n)$ given in the former paper. We make use of this description to show that $\mathrm{Out}(dV_2) \cong \mathrm{Out}(V_2) \wr S_d$, and more generally give a natural embedding of $\mathrm{Out}(dV_n)$ into $\mathrm{Out}(G_{n,n-1}) \wr S_d$.

1. Introduction

In Matthew G. Brin's 2004 paper [5], he introduces the family of simple groups dV, which serve as d-dimensional analogues to Thompson's group V. The present paper is concerned with finding a "nice" way to represent the automorphism groups of the groups dV. To do this we follow a similar path to that by Collin Bleak, Peter Cameron, Yonah Maissel, Andrés Navas, and Feyishayo Olukoya [2], but via a more category theoretic perspective. This enables us to prove a conjecture made by Nathan Barker in 2012:

Theorem 1.1. *For all $d \geq 1$, we have $\mathrm{Out}(dV) \cong \mathrm{Out}(V) \wr S_d$ (using the standard action of S_d on d points).*

We view the transducers of [8] as a category in their own right, and then identify subcategories of transducers which are more appropriate for representing homeomorphisms of "n-dimensional" Cantor spaces. Similarly to [2], we then employ Rubin's theorem [14] to represent the elements of $\mathrm{Aut}(dV)$ as homeomorphisms, which in turn are represented with transducers. We also extend the description of $\mathrm{Out}(V)$ given in [2] to $\mathrm{Out}(dV)$.

From this perspective we are able to represent the automorphisms of the encompassing family of groups dV_n, first introduced in the paper [11] of Martínez-Pérez, Matucci, and

Nucinkis. We also describe the outer automorphisms of these groups with transducers and give the following theorem extending the one given for dV:

Theorem 1.2. *For all $d \geq 1$ and $n \geq 2$ we have*

$$\text{Out}(dV_n) \cong \left\{ \mathbf{T} \in \text{Out}(G_{n,n-1})^d : \prod_{i<d} (\mathbf{T}\boldsymbol{\pi}_i)\overline{\text{sig}} = 1 \right\} \rtimes S_d,$$

where the action of S_d is the standard permutation of coordinates, $\boldsymbol{\pi}_i$ is the i-th projection map and $\overline{\text{sig}}$ is the homomorphism of [12, Definition 7.6].

The outer automorphisms of Thompson groups have a history in the literature. In [4, 7], Brin and Fernando Guzmán study the automorphisms of F and T type groups. As previously mentioned, the authors of [2] gave a means of describing $\text{Out}(G_{n,r})$ with transducers, in particular the way the groups $\text{Out}(1V_n)$ are viewed in this paper is theirs. More recently Feyishayo Olukoya has used transducer based methods to study the outer automorphisms of the groups $T_{n,r}$ in [12].

The family of groups dV has also been extensively studied in the literature. In [5], it is proved that the groups dV are all infinite, simple, and finitely generated. In [6], Brin goes on to give an explicit finite presentation for $2V$ with 8 generators and 70 relations. The paper [3] of Collin Bleak and Daniel Lanoue uses Rubin's theorem to show that dV and nV are non-isomorphic for $d \neq n$.

In [9], Johanna Hennig and Francesco Matucci show that in general dV can be finitely presented with $2d + 4$ generators and $10d^2 + 10d + 10$ relations. More recently, Martyn Quick [13] has built much smaller presentations for dV, using only 2 generators as well as $2d^2 + 3d + 13$ relations.

It is shown in [2] that $\text{Aut}(G_{n,r})$ embeds in the rational group $\mathcal{R}$ of finite transducers as defined in [8]. In [1], it is shown that there is a natural topological conjugacy embedding $2V$ into $\mathcal{R}$ as well (which can be naturally generalised to dV). It is therefore natural to ask if this conjugacy sends $\text{Aut}(dV)$ to a subgroup of $\mathcal{R}$. In this paper we give examples to demonstrate that this fails for all $d \geq 2$ (see Section 5).

Mark V. Lawson and Alina Vdovina have constructed many additional Thompson-like groups using the notion of "k-monoids" in [10]. Their Proposition 3.9 suggests that the methods of this document are likely only compatible with groups corresponding to the k-monoids which are finite products of finite rank free monoids.

2. Preliminaries

We will compose functions from left to right and we will always index from 0. For $n \in \mathbb{N}$, we will use the notations X_n and $\bar{n}$ to denote the set $\{0, 1, \ldots, n-1\}$. We use the former when thinking of this set as an alphabet, and the latter when thinking of an initial segment of $\mathbb{N}$. We denote the free monoid of all finite words over a finite alphabet X by X^*. That

is,

$$X^* := \bigcup_{n \in \mathbb{N}} X^n.$$

This notably includes the empty word which we denote by ε. If $w \in X^*$, then we define $|w|$ to be the length of w as a word (the value of n such that $w \in X^n$). We also follow the standard convention of identifying a letter with a word of length 1. If X is an alphabet, then the set $X^* \cup X^\omega$ is naturally partially ordered by

$$x \leq y \text{ if and only if } x \text{ is a "prefix" of } y.$$

We also extend this partial order to the sets $(X^* \cup X^\omega)^d$ via the product order.

Definition 2.1. We will denote all projection morphisms by $\pi_0, \pi_1, \dots$ in all categories with products (the specific morphism used will be determined by the context).

If $x \in (X_n^*)^d$, $y \in (X_n^* \cup X_n^\omega)^d$ and $x \leq y$, then we define $y - x$ to be the unique $z \in (X_n^* \cup X_n^\omega)^d$ such that $xz = y$ (using coordinate-wise concatenation).

Definition 2.2. If $n \geq 2$, then we define $\mathbb{C}_n := X_n^\omega$ to be the usual Cantor space with the product topology. Moreover, if $w \in (X_n^*)^d$, then we define

$$w\mathbb{C}_n^d := \{x \in \mathbb{C}_n^d : w \leq x\}.$$

Note that these sets are clopen, and the collection of all such sets is a basis for $\mathbb{C}_n^d$. Such basic open sets will be referred to as *cones*.

Definition 2.3. If X is a topological space, then we denote the homeomorphism group of X by $H(X)$.

We can now give the definition of the groups dV_n which we will use throughout the paper.

Definition 2.4. Suppose that F_1, F_2 are finite subsets of $(X_n^*)^d$, such that

$$\{w\mathbb{C}_n^d : w \in F_1\} \quad \text{and} \quad \{w\mathbb{C}_n^d : w \in F_2\}$$

are partitions of $\mathbb{C}_n^d$, and $\phi : F_1 \to F_2$ is a bijection. We call such sets F_1, F_2 *complete prefix codes* for $\mathbb{C}_n^d$. We then define the *prefix exchange map* $f_\phi : \mathbb{C}_n^d \to \mathbb{C}_n^d$ as follows: If $w \in F_1$ and $x \in w\mathbb{C}_n^d$, then

$$(x)f_\phi = (w\phi)(x - w).$$

Such prefix exchange maps are always homeomorphisms and the set of all such maps under composition forms the group dV_n (or just dV if $n = 2$).

Remark 2.5. There is a complete prefix code for $\mathbb{C}_n^d$ of size m if and only if $m \in (1 + (n-1)\mathbb{N})$.

3. Generalizing the transducers of Grigorchuk, Nekrashevich, and Sushchanskii

A transducer, as introduced by Grigorchuk, Nekrashevich, and Sushchanskii (which we shorten to GNS) in [8], can be thought of as a way of assigning each letter of an alphabet, a transformation of a "state set", together with a word to write for each state. These are then extended to all words in the input alphabet via the universal property of the free monoid. With reading elements of a monoid in mind, the following is a natural generalisation of their transducer definition:

Definition 3.1. We say that $T := (Q_T, D_T, R_T, \pi_T, \lambda_T)$ is a transducer if

(1) Q_T is a set (called the set of states),

(2) D_T is a semigroup (called the domain semigroup),

(3) R_T is a semigroup (called the range semigroup),

(4) $\pi_T : Q_T \times D_T \to Q_T$ is a (right) action of D_T on the set Q_T (called the transition function),

(5) $\lambda_T : Q_T \times D_T \to R_T$ is a function with the property that for all $q \in Q_T$ and $s, t \in D_T$ we have

$$(q, st)\lambda_T = (q, s)\lambda_T((q, s)\pi_T, t)\lambda_T \quad \text{(called the output function)}.$$

We will often refer to the domain semigroup and range semigroup of a transducer as simply its domain and range.

Note that a one state transducer is equivalent to a semigroup homomorphism.

Definition 3.2. Let A, B be transducers. We say that ϕ is a transducer homomorphism from A to B (written $\phi : A \to B$), if ϕ is a 3-tuple (ϕ_Q, ϕ_D, ϕ_R) with the following properties:

(1) $\phi_R : R_A \to R_B$ is a semigroup homomorphism,

(2) $\phi_D : D_A \to D_B$ is a semigroup homomorphism,

(3) $\phi_Q : Q_A \to Q_B$ is a function, such that for all $q \in Q_A$ and $s \in D_A$ we have

$$(q, s)\pi_A\phi_Q = (q\phi_Q, s\phi_D)\pi_B \quad \text{and} \quad (q, s)\lambda_A\phi_R = (q\phi_Q, s\phi_D)\lambda_B.$$

If furthermore, the maps ϕ_D, ϕ_R are identity maps, then we say that ϕ is *strong*.

Remark 3.3. We now have two categories which have the class of transducers as their object class. The first has transducer homomorphisms (using coordinate-wise composition) as its class of morphisms and the second has only the class of strong transducer homomorphisms.

Definition 3.4. A transducer homomorphism ϕ is called a *quotient* map if each of ϕ_Q, ϕ_D, ϕ_R is surjective.

We say that transducers A and B are isomorphic (denoted $A \cong B$) if they are isomorphic in the category of transducers and transducer homomorphisms. Similarly, we say that A and B are strongly isomorphic (denoted $A \cong_S B$) if they are isomorphic in the category of transducers and strong transducer homomorphisms.

The next definition gives us a means of minimizing our transducers which coincides with the GNS notion of combining equivalent states.

Definition 3.5. If T is a transducer with R_T cancellative, then we define its *minimal transducer* M_T to be $(Q_T / \sim_{M_T}, D_T, R_T, \pi_{M_T}, \lambda_{M_T})$ where $\sim_{M_T}, \pi_{M_T}$ and λ_{M_T} are defined as follows:

   (1) $\sim_{M_T}$ is the equivalence relation

$$\{(p, q) \in Q_T^2 : (p, s)\lambda_T = (q, s)\lambda_T \text{ for all } s \in D_T\},$$

   (2) if $q \in Q_T, s \in D_T$, then $([q]_{\sim_{M_T}}, s)\pi_{M_T} = [(q, s)\pi_T]_{\sim_{M_T}}$,
   (3) if $q \in Q_T, s \in D_T$, then $([q]_{\sim_{M_T}}, s)\lambda_{M_T} = (q, s)\lambda_T$.

It is routine to verify that this is well defined (as R_T is cancellative) and the natural strong quotient candidate $q_T : T \to M_T$, with $(p)q_{T\,Q} = [p]_{\sim_{M_T}}$ is a strong quotient map.

Lemma 3.6. *If A is a transducer with R_A cancellative, then all strong quotient maps $\phi : A \to B$ are left divisors of q_A.*

Proof. We define $\psi : B \to M_A$ by having ψ_D, ψ_R be the identity maps and defining ψ_Q by

$$((q)\phi_Q)\psi_Q := (q)q_{A\,Q}.$$

If we can show that ψ is a well-defined transducer homomorphism, then the result follows.

Note that all the maps $\phi_D, \phi_R, \psi_D, \psi_R, q_{A\,D}$ and $q_{A\,R}$ are identity maps so we can ignore them for the purposes of this proof. We first show that ψ is well defined. Suppose that $q_0, q_1 \in Q_A$ satisfy $(q_0)\phi_Q = (q_1)\phi_Q$. We need to show that $q_0 \sim_{M_T} q_1$. Let $s \in D_A$ be arbitrary, then

$$(q_0, s)\lambda_A = ((q_0)\phi_Q, s)\lambda_B = ((q_1)\phi_Q, s)\lambda_B = (q_1, s)\lambda_A.$$

We next show that ψ is a homomorphism. Using the fact that ϕ_Q is surjective, let $(p, s) = ((q)\phi_Q, s) \in Q_B \times D_B$ be arbitrary. We need only verify that ψ satisfies condition (3) from the definition of a transducer homomorphism. We have

$$\begin{aligned}
(p, s)\pi_B\psi_Q &= ((q)\phi_Q, s)\pi_B\psi_Q && \text{by the definition of } p \\
&= (q, s)\pi_A\phi_Q\psi_Q && \text{because } \phi \text{ is a homomorphism} \\
&= ((q)q_{A\,Q}, s)\pi_{M_A} && \text{because } q_A \text{ is a homomorphism} \\
&= (((q)\phi_Q)\psi_Q, s)\pi_{M_A} && \text{by the definition of } \psi \\
&= ((p)\psi_Q, s)\pi_{M_A} && \text{by the definition of } p,
\end{aligned}$$

and similarly $(p, s)\lambda_B = ((p)\psi_Q, s)\lambda_{M_A}$ as required. ∎

Definition 3.7. If T is a transducer, and we restrict Q_T, D_T, and R_T to sets which are (together) closed under the transition and output functions, then we obtain another transducer. We call such a transducer a *subtransducer* of T.

Definition 3.8. If $d \in \mathbb{N}$ and $n \in \mathbb{N}\backslash\{0, 1\}$, then we define a (d, n)-transducer to be a transducer T with $(X_n^*)^d$ as its domain and range, and such that the transition function is a monoid action (so you don't transition when reading the identity).

If T is a transducer, $q \in Q_T$, and $w \in D_T$, then we will view the maps $(q, \cdot)\pi_T$ and $(q, \cdot)\lambda_T$ as reading w through a path in T from q, ending at the state $(q, w)\pi_T$, and writing $(q, w)\lambda_T$ along the way (similarly to GNS transducers). Note that unlike GNS transducers, there is not always a "best" way of splitting up this path into minimal steps, for example $(0, 0) \in (X_2^*)^2$ could naturally be decomposed as either $(0, \varepsilon)(\varepsilon, 0)$ or $(\varepsilon, 0)(0, \varepsilon)$.

If T is a (d, n)-transducer, then (like GNS transducers) we can naturally extend this idea to "infinite words", which in this case means elements of $(X_n^\omega)^d$, by reading arbitrarily long finite prefixes of an element and taking the limit of the elements written.

Definition 3.9. If T is a (d, n)-transducer and $q \in Q_T$, then we define $f_{T,q} : (X_n^\omega)^d \to (X_n^\omega \cup X_n^*)^d$ to be the map which maps $w \in (X_n^\omega)^d$ to the word written when w is read in T from the state q.

Note that if A is a (d, n)-transducer, $q \in Q_A$ and $\phi : A \to B$ is a strong transducer homomorphism, then $f_{A,q} = f_{B,(q)\phi_Q}$. In particular, this is true of the homomorphism q_A.

Definition 3.10. Similarly to GNS we say that a (d, n)-transducer T is *degenerate* if there exist $q \in Q_T$, $i \in \bar{d}$ and $x \in \mathbb{C}_n^d$ such that $(x)f_{T,q}\pi_i$ is finite.

We will often use the following fact without comment:

Remark 3.11. If T is a non-degenerate (d, n)-transducer and $q \in Q_T$, then for all $m \in \mathbb{N}$ there is $k \in \mathbb{N}$ such that reading an element of $(X_n^k)^d$ always writes a word whose length is at least m in every coordinate.

There are two important ways by which we combine our transducers, there is "composition" as was done in GNS, and taking products in the categorical sense.

Definition 3.12. If A and B are transducers, such that the range of A is contained in the domain of B, then we define their *composite* by

$$AB = (Q_{AB}, D_{AB}, R_{AB}, \pi_{AB}, \lambda_{AB}),$$

where

(1) $Q_{AB} := Q_A \times Q_B$, $D_{AB} := D_A$, $R_{AB} := R_B$,

(2) $((a, b), s)\pi_{AB} = ((a, s)\pi_A, (b, (a, s)\lambda_A)\pi_B)$,

(3) $((a, b), s)\lambda_{A,B} = (b, (a, s)\lambda_A)\lambda_B$.

As was the case in GNS, this definition is constructed so that whenever A, B are non-degenerate (d, n)-transducers, and $(p, q) \in A \times B$, we obtain

$$f_{A,p} f_{B,q} = f_{AB,(p,q)}.$$

Definition 3.13. If $(A)_{i \in I}$ are transducers, then we define $\prod_{i \in I} A_i := P$ where

$$Q_P := \prod_{i \in I} Q_{A_i}, \quad D_P := \prod_{i \in I} D_{A_i}, \quad R_P := \prod_{i \in I} R_{A_i},$$

and for all $(p_i)_{i \in I} \in Q_P$ and $(s_i)_{i \in I} \in D_P$ we have

$$((p_i)_{i \in I}, (s_i)_{i \in I})\pi_P = ((p_i, s_i)\pi_{A_i})_{i \in I},$$
$$((p_i)_{i \in I}, (s_i)_{i \in I})\lambda_P = ((p_i, s_i)\lambda_{A_i})_{i \in I}.$$

For $i \in I$ we then define $\pi_i : P \to A_i$ to be the transducer homomorphism (π_i, π_i, π_i). One can verify that this is a product in the category theoretic sense (using transducer homomorphisms but not strong transducer homomorphisms).

The following definition gives us, for each homeomorphism h of $\mathbb{C}_n^d$, a transducer M_h representing it. From the definition, one can see that this transducer has no inaccessible states, has complete response and has no distinct but equivalent states. So in particular when $d = 1$, the transducer M_h is the minimal transducer representing h as described by GNS.

Definition 3.14. If $h \in H(\mathbb{C}_n^d)$, then we define T_h to be the (d, n)-transducer with

(1) $Q_{T_h} := (X_n^*)^d$,

(2) $(s, t)\pi_{T_h} = st$,

(3) $((s, t)\lambda_{T_h})\pi_i$ is $b - a$, where b is the longest common prefix of the words in the set $((st\mathbb{C}_n^d)h)\pi_i$ and a is the longest common prefix of the words in the set $((s\mathbb{C}_n^d)h)\pi_i$.

(As h is a homeomorphism, the set $(st\mathbb{C}_n^d)h$ is always open and thus $(s, t)\lambda_{T_h}$ is always an element of $(X_n^*)^d$.) Moreover, as was the case in GNS, if $q = 1_{(X_n^*)^d}$ then $f_{T_h,q} = h$. We also define $M_h := M_{T_h}$.

Remark 3.15. If $h \in H(\mathbb{C}_n^d)$ and $q \in Q_{M_h}$, then $f_{M_h,q}$ is injective with clopen image.

The proof of the following theorem is analogous to the proof of the analogous theorem in GNS (the above construction deals with the homeomorphism case).

Theorem 3.16. *A function $h : \mathbb{C}_n^d \to \mathbb{C}_n^d$ is continuous if and only if there are a non-degenerate (d, n)-transducer T and $q \in Q_T$ such that $h = f_{T,q}$.*

4. Generalizing the synchronizing homeomorphisms of Bleak, Cameron, Maissel, Navas, and Olukoya

As was the case in [2] when analysing $\text{Aut}(G_{n,r})$, we now want to restrict to the transducers which give us the automorphisms we want. We thus extend the notion of synchronization given there.

Definition 4.1. We say that a (d, n)-transducer T is *synchronizing at level k* if for all $q_1, q_2 \in Q_T$ and $w \in (X_n^*)^d$ with $\min(\{|w\boldsymbol{\pi}_i| : i \in \bar{d}\}) \geq k$, we have $(q_1, w)\pi_T = (q_2, w)\pi_T$. We say that T is *synchronizing* if there is a level at which it is synchronizing. The *synchronizing length* of a synchronizing transducer T is

$$\min(\{k \in \mathbb{N} : T \text{ is synchronizing at level } k\}).$$

In this case we define the function

$$\mathfrak{s}_T := \{(w, q) \in (X_n^*)^d \times Q_T : \text{for all } p \in Q_T \text{ we have } (w, p)\pi_T = q\}.$$

So $\mathfrak{s}_T$ is basically π_T restricted to the part of its domain where the input state is not needed. The image of $\mathfrak{s}_T$, denoted $\text{Core}(T)$, is called the *core* of T.

It is useful to think of the core of a synchronizing (d, n)-transducer as the place reached when a sufficient amount of information has been read in each coordinate. In particular, if a word is read from any core state of a synchronizing (d, n)-transducer T, then you stay in the core, thus $\text{Core}(T)$ is a (synchronizing) subtransducer of T (when given the restrictions of the transition and output functions of T).

As $\text{Core}(T) = ((X_n^k)^d)\mathfrak{s}_T$ (where k is the synchronizing length of T), it follows that $\text{Core}(T)$ is always finite.

The following proposition is routine to verify, and shows that our transducer framework describes dV_n in a manner analogous to the way in which the transducers of GNS describe V_n.

Proposition 4.2. *If $h \in H(\mathbb{C}_n^d)$, then $h \in dV_n$ if and only if M_h is a synchronizing transducer whose core consists of a single "identity" state.*

Unlike for V_n, the transducers for elements of dV_n can sometimes be infinite. For example the transducer representing the baker's map of $2V$ is infinite (as can be seen in Figure 1) as it can't write anything until something is read in the first coordinate.

We will now introduce the monoids $d\mathcal{S}_{n,1}$, $\widetilde{d\mathcal{O}_{n,1}}$, $d\mathcal{B}_{n,1}$ and $d\mathcal{O}_{n,1}$ which generalise the monoids $\mathcal{S}_{n,1}$, $\widetilde{\mathcal{O}_{n,1}}$, $\mathcal{B}_{n,1}$ and $\mathcal{O}_{n,1}$ of [2].

Definition 4.3. We say an element $f \in H(\mathbb{C}_n^d)$ is *synchronizing* if M_f is synchronizing. We define $d\mathcal{S}_{n,1}$ to be the set of synchronizing elements of $H(\mathbb{C}_n^d)$.

Remark 4.4. If A and B are synchronizing, non-degenerate transducers, then so is their composite AB. This works as you can synchronize the first coordinate using the synchron-
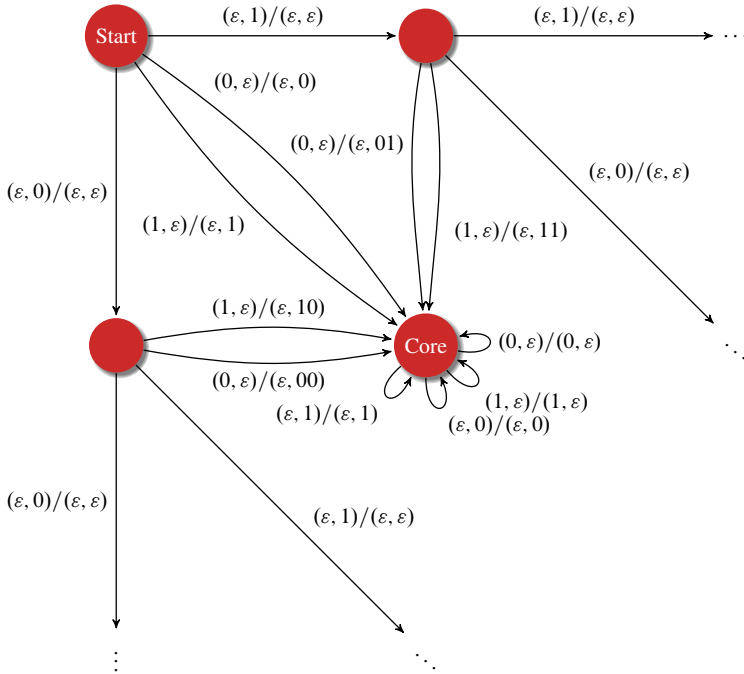
Figure 1. Part of a minimal transducer with $(X_2^*)^2$ as domain and range. This represents the baker's map in $2V$ (this transducer is infinite and every state has an edge with the core as its target).

izing property of A, and once the first coordinate is in the finite non-degenerate core of A, one can read enough so that the output of A synchronizes B as well.

Corollary 4.5. *If $f, g \in d\mathcal{S}_{n,1}$ then* $\mathrm{Core}(M_f M_g)$ *is a subtransducer of the composite transducer* $\mathrm{Core}(M_f)\,\mathrm{Core}(M_g)$.

Corollary 4.6. *The set $d\mathcal{S}_{n,1}$ is always a monoid.*

Proof. It follows from Remark 4.4 that, if $f, g \in d\mathcal{S}_{n,1}$, then $M_f M_g$ is synchronizing.

We will now essentially minimise $M_f M_g$ in the GNS fashion and obtain M_{fg}. Let

$$q_f := (1_{(X_n^*)^d})q_{T_f\,Q} \quad \text{and} \quad q_g := (1_{(X_n^*)^d})q_{T_g\,Q}.$$

We have

$$fg = f_{M_f,q_f} f_{M_g,q_g} = f_{M_f M_g,(q_f,q_g)}.$$

We then define A to be the transducer with the same states, domain, range and transition function as $M_f M_g$ but with $(q,w)\lambda_A = b - s$, where b is the longest common prefix of the set $(w\mathbb{C}_n^d)f_{M_f M_g,q}$ and s is the longest common prefix of the set $(\mathbb{C}_n^d)f_{M_f M_g,q}$ (it

follows from Remark 3.15 that b and s are finite). Let A' be the subtransducer of A consisting of the states that are accessible from (q_f, q_g) (the image of $((q_f, q_g), \cdot)\pi_A$).

It follows from the definition that A' is a strong quotient of the transducer T_{fg}. Thus M_{fg} is a strong quotient of A'. As A has the same transitions as $M_f M_g$, it follows that A is synchronizing. Moreover, since A' is a subtransducer of A, we get that A' is synchronizing and thus M_{fg} is also synchronizing (as a strong quotient of A'). ∎

Corollary 4.7. *The binary operation*

$$[\mathrm{Core}(M_f)]_{\cong_S}[\mathrm{Core}(M_g)]_{\cong_S} = [\mathrm{Core}(M_{fg})]_{\cong_S}$$

on the set $\widetilde{d\,\mathcal{O}_{n,1}} := \{[\mathrm{Core}(M_f)]_{\cong_S} : f \in d\,\mathcal{S}_{n,1}\}$ *is well defined. Moreover, this set with this operation is a quotient of the monoid* $d\,\mathcal{S}_{n,1}$.

Proof. This is well defined as the strong isomorphism type $\mathrm{Core}(M_{fg})$ can be found by removing incomplete response from $\mathrm{Core}(M_f)\,\mathrm{Core}(M_g)$, combining equivalent states and passing to the core (in the same manner as the proof of Corollary 4.6). ∎

Definition 4.8. We define $d\,\mathcal{B}_{n,1}$ to be the group of units of $d\,\mathcal{S}_{n,1}$, and

$$d\,\mathcal{O}_{n,1} := \big\{[\mathrm{Core}(M_f)]_{\cong_S} : f \in d\,\mathcal{B}_{n,1}\big\}.$$

Lemma 4.9. *The map* $f \mapsto [\mathrm{Core}(M_f)]_{\cong_S}$ *is a surjective group homomorphism from* $d\,\mathcal{B}_{n,1}$ *to* $d\,\mathcal{O}_{n,1}$ *with kernel* $d\,V_n$.

Proof. This map is a homomorphism by the definition of multiplication in $\widetilde{d\,\mathcal{O}_{n,1}}$, it is surjective by the definition of $d\,\mathcal{O}_{n,1}$ and thus as $d\,\mathcal{B}_{n,1}$ is a group, $d\,\mathcal{O}_{n,1}$ is also. The identity of $d\,\mathcal{O}_{n,1}$ is the image of the identity map, and is thus the single state "identity" transducer. From Proposition 4.2, we get that $d\,V_n$ is the kernel. ∎

We have now introduced the monoids we need. We now begin showing that $d\,\mathcal{B}_{n,1}$ coincides with the normalizer of $d\,V_n$ in $H(\mathbb{C}_n^d)$ (the case with $d = 1$ was done in [2]).

Lemma 4.10. *Let* $h \in N_{H(\mathbb{C}_n^d)}(d\,V_n)$ *and* $s, t \in (X_n^*)^d \setminus \{1_{(X_n^*)^d}\}$. *Let* $q_h := (1_{(X_n^*)^d})q_{T_h Q}$. *There exists* $K_{h,s,t} \in \mathbb{N}$ *such that for all* $a \in (X_n^*)^d$ *with*

$$\min\big(\{|a\pi_i| : i \in \overline{d}\}\big) \geq K_{h,s,t},$$

we have $(q_h, sa)\pi_{M_h} = (q_h, ta)\pi_{M_h}$.

Proof. For all $x \in (X_n^*)^d$, let $q_x := (q_h, x)\pi_{M_h}$. Let $f \in d\,V_n$ be such that f replaces the prefix s with the prefix t. By the choice of h, there is some $g \in nV$ such that $h^{-1}fh = g$ and so $fh = hg$.

Let $q_f := (1_{(X_n^*)^d})q_{T_f Q}$ and $q_g := (1_{(X_n^*)^d})q_{T_g Q}$. It follows that

$$f_{M_f M_h, (q_f, q_h)} = fh = hg = f_{M_h M_g, (q_h, q_g)}.$$

Let I_f, I_g be the core states of M_f and M_g, respectively (which don't do anything). Note that

$$((q_f, q_h), s)\pi_{M_f M_h} = (I_f, q_t),$$
$$((q_h, q_g), s)\pi_{M_h M_g} = (q_s, (q_g, (q_h, s)\lambda_{M_h})\pi_{M_g}).$$

Let $K \in \mathbb{N}$ be such that for all $w \in (X_n^*)^d$ with $\min(\{|w\boldsymbol{\pi}_i| : i \in \bar{d}\}) \geq K$, we have that

$$\min\big(\{|((q_h, s)\pi_{M_h}, w)\lambda_{M_h}\boldsymbol{\pi}_i| : i \in \bar{d}\}\big)$$

is at least the synchronizing length of M_g.

Let $a \in (X_n^*)^d$ be arbitrary such that $\min(\{|w\boldsymbol{\pi}_i| : i \in \bar{d}\}) \geq K$. We have

$$((q_f, q_h), sa)\pi_{M_f M_h} = (I_f, q_{ta}),$$
$$((q_h, q_g), sa)\pi_{M_h M_g} = (q_{sa}, I_g).$$

Thus, for all $v \in (X_n^\omega)^d$ we have

$$
\begin{aligned}
((q_f, q_h), sa)\lambda_{M_f M_h}(v) f_{M_h, q_{ta}} &= ((q_f, q_h), sa)\lambda_{M_f M_h}(v) f_{M_f M_h, (I_f, q_{ta})} \\
&= (sav)fh \\
&= (sav)hg \\
&= ((q_h, q_g), sa)\lambda_{M_h M_g}(v) f_{M_h M_g, (q_{sa}, I_g)} \\
&= ((q_h, q_g), sa)\lambda_{M_h M_g}(v) f_{M_h, q_{sa}}.
\end{aligned}
$$

It follows that $((q_f, q_h), sa)\lambda_{M_f M_h}$ and $((q_h, q_g), sa)\lambda_{M_h M_g}$ are comparable in each coordinate.

If there was a coordinate in which $((q_f, q_h), sa)\lambda_{M_f M_h}$ and $((q_h, q_g), sa)\lambda_{M_h M_g}$ differed, it would follow that either the map $f_{M_h, q_{ta}}$ or $f_{M_h, q_{sa}}$ has its image contained in a proper cone. This is impossible as M_h by definition has no incomplete response. Thus

$$((q_f, q_h), sa)\lambda_{M_f M_h} = ((q_h, q_g), sa)\lambda_{M_h M_g}.$$

We deduce from the equality

$$((q_f, q_h), sa)\lambda_{M_f M_h}(v) f_{M_h, q_{ta}} = ((q_h, q_g), sa)\lambda_{M_h M_g}(v) f_{M_h, q_{sa}}$$

that $f_{M_h, q_{ta}} = f_{M_h, q_{sa}}$. As M_h is minimal, it follows that $q_{ta} = q_{sa}$ as required. ∎

Lemma 4.11. *The group $N_{H(\mathbb{C}_n^d)}(dV_n)$ is contained in $d\mathcal{B}_{n,1}$.*

Proof. The proof of this is essentially the same as [2, Corollary 6.17]. The idea is as follows: We need only show containment in $d\mathcal{S}_{n,1}$ because $N_{H(\mathbb{C}_n^d)}(dV_n)$ is a group with the same identity as $d\mathcal{S}_{n,1}$. Thus we need only show the synchronizing condition. So it suffices to show that, for all $h \in N_{H(\mathbb{C}_n^d)}(dV_n)$, there is a $K \in \mathbb{N}$ such that the state reached

by reading an arbitrary word from $(1_{(X_n^*)^d})q_{T_h Q}$ is determined by the last K letters of the word (in every coordinate). We do this by collapsing an arbitrary given input word from the front by repeated applications of Lemma 4.10 using s with size 1 and t with size 2 (where size means the sum of the lengths of the coordinates). ∎

We now recall the theorem of Rubin which connects our arguments to automorphism groups:

Theorem 4.12 (Rubin's theorem [14]). *Let G be a group of homeomorphisms of a perfect, locally compact, Hausdorff topological space X. For $U \subseteq X$ let*

$$G_U := \{g \in G : (x)g = x \text{ for all } x \in X \setminus U\}.$$

Suppose further that for all $x \in X$ and U a neighbourhood of x, we have $(x)G_U$ is somewhere dense. If $\phi : G \to G$ is a group isomorphism, then there is a $\psi_\phi \in H(X)$ such that $(g)\phi = \phi_\phi^{-1} g \psi_\phi$ for all $g \in G$.

In [2], it is shown that Rubin's theorem allows us to naturally embed $\mathrm{Aut}(G_{n,r})$ into $H(\mathbb{C}_{n,r})$. This same argument also applies to dV_n, and in fact to any group with an action satisfying the hypothesis of Rubin's theorem.

Corollary 4.13. *The groups $\mathrm{Aut}(dV_n)$ and $N_{H(\mathbb{C}_n^d)}(dV_n)$ are isomorphic.*

Theorem 4.14. *The groups $\mathrm{Aut}(dV_n)$ and $d\mathcal{B}_{n,1}$ are isomorphic.*

Proof. We have $\mathrm{Aut}(dV_n) \cong N_{H(\mathbb{C}_n^d)}(dV_n)$ by Corollary 4.13, and we have

$$N_{H(\mathbb{C}_n^d)}(dV_n) \subseteq d\mathcal{B}_{n,1} \subseteq N_{H(\mathbb{C}_n^d)}(dV_n)$$

by Lemma 4.11 and Lemma 4.9. ∎

Corollary 4.15. *The groups $\mathrm{Out}(dV_n)$ and $d\mathcal{O}_{n,1}$ are isomorphic.*

Proof. This follows from Theorem 4.14 and Lemma 4.9. ∎

Corollary 4.16. *For $d, m \in \mathbb{N} \setminus \{0\}$ and $n \in \mathbb{N} \setminus \{0, 1\}$ the group $\mathrm{Aut}(dV_n)^m$ embeds in the group $\mathrm{Aut}((md)V_n)$.*

Proof. If $(d\mathcal{B}_n)^m$ acts on $(\mathbb{C}_n^d)^m \cong \mathbb{C}_n^{dm}$ in the natural fashion, then these homeomorphisms are contained in the group $(md)\mathcal{B}_n$. ∎

Corollary 4.17. *The group $\mathrm{Aut}(dV_n)$ is countably infinite.*

Proof. The group dV_n is countably infinite and the group $d\mathcal{O}_{n,1}$ is countable (it consists of the isomorphism classes of transducers with finitely many states and finitely generated domains). ∎
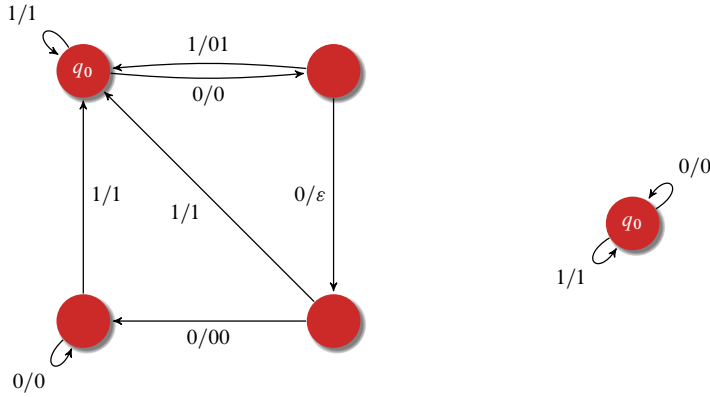
Figure 2. Two transducers with domain and range X_2^*.

5. Rationality and representations

Unfortunately, unlike with $\mathcal{B}_{2,1}$, representing elements of $2\mathcal{B}_{2,1}$ with transducers can sometimes result in a transducer which has infinitely many states. In Figure 1 we see that the baker's map when represented by a transducer, in the way described in this paper, has infinitely many states.

However, if we want our pictures to be finite, then we can consider the submonoid of $(X_n^*)^d$ consisting of those elements $w \in (X_n^*)^d$ such that $|w\boldsymbol{\pi}_i|$ is the same for all i. As elements of $d\mathcal{B}_{n,1}$ are synchronizing, it follows that if we restrict the domain of a minimal transducer representing an element of $d\mathcal{B}_{n,1}$ to this submonoid, then we will only need finitely many states to represent it. In Figure 3 we see the baker's map represented in this fashion. The main problem with this representation is that composing functions represented with these transducers is much harder (due to the fact that the transducers have distinct domains and ranges).

Figure 2 displays two elements of $\mathcal{B}_{2,1}$, where the q_0 are the initial states. These transducers are particularly nice as they are finite. The first one acts by swapping the strings "0" and "00" wherever it sees them, and the second one is the identity. These transducers are special in that they are each equal to their own cores. Thus, these pictures also represent elements of $\mathcal{O}_{2,1}$ if we ignore the choice of initial state. In Figure 4 we see the categorical product of the transducers in Figure 2, which represents an element of $2\mathcal{B}_{2,1}$ (and also an element of $2\mathcal{O}_{2,1}$).

In [2], it is shown that $\mathcal{B}_{2,1}$ consists of rational homeomorphisms, and in [1, Theorem 5.2] it is shown that $2V$ naturally embeds in $\mathcal{R}_4$ via conjugation by a homeomorphism between $\mathbb{C}_4$ and $\mathbb{C}_2^2$. The map of [1, Theorem 5.2] acts by converting a pair of words $(x_0 x_1 \ldots, y_0 y_1 \ldots)$ to a word $(x_0, y_0)(x_1, y_1) \ldots$ over the alphabet
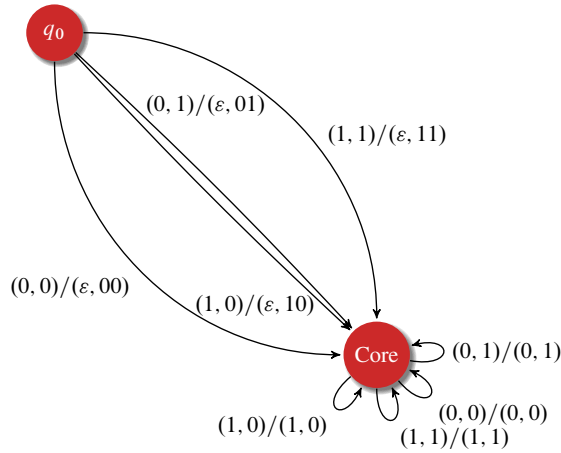
$$\{(0,0), (0,1), (1,0), (1,1)\}.$$

Figure 3. The subtransducer of the transducer in Figure 1 with $\{w \in (\{0,1\}^*)^2 : |w\pi_0| = |w\pi_1|\}$ as domain and no longer accessible states removed.
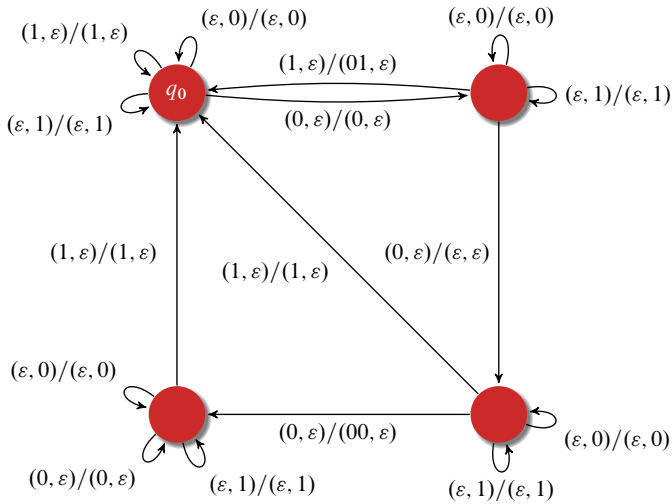

Figure 4. The transducer obtained by taking the categorical product of the transducers in Figure 2.

It is natural to ask if the same map gives an embedding of $2\mathcal{B}_{2,1}$ into $\mathcal{R}$. However it is routine to verify that if we conjugate the homeomorphism defined by the transducer of Figure 4, then the resulting map is not rational. This happens because this transducer is a product of a transducer which never resizes words, with a transducer which does this arbitrarily amounts (for example by reading the words $(01)^n$). So it seems that there is no good way to describe the groups $\mathrm{Aut}(d\,V_n)$ using only finite transducers.

6. A closer look at the groups $d\mathcal{O}_{n,1}$

We now want to pin down what the core transducers representing $d\mathcal{O}_{n,1}$ look like. We are going to end up with a semidirect product, so we will deal with the acting part of the product first. Before that we introduce a notation which we will use repeatedly throughout this section.

Definition 6.1. For $d \geq 1$, $n \geq 2$ and $S \subseteq \bar{d}$, we define

$$F_{d,n,S} := \{w \in (X_n^*)^d : w\boldsymbol{\pi}_i = \varepsilon \text{ for all } i \in \bar{d}\setminus S\}.$$

That is, $F_{d,n,S}$ is the submonoid of $(X_n^*)^d$, consisting of those elements only allowed to be non-trivial in the coordinates in S.

Lemma 6.2. *Let T be a transducer representing an element of $\widetilde{d\mathcal{O}_{n,1}}$. If $i \in \bar{d}$, then there is a unique $(i)\psi_T \in \bar{d}$, such that for all $q \in Q_T$ and $l \in F_{d,n,\{i\}}$, we have $(q,l)\lambda_T \in F_{d,n,\{(i)\psi_T\}}$.*

Proof. We start by showing the existence of $(i)\psi_T$. First note that for all $q \in Q_T$, the map $\underline{f}_{T,q}$ is necessarily injective (Remark 3.15). Suppose for a contradiction that there are $i \in \bar{d}$, $l_0, l_1 \in F_{d,n,\{i\}}$, $q_0, q_1 \in Q_T$ and $\alpha, \beta \in \bar{d}$ such that $\alpha \neq \beta$ and

$$|(q_0,l_0)\lambda_T\boldsymbol{\pi}_\alpha| > 0, \quad |(q_1,l_1)\lambda_T\boldsymbol{\pi}_\beta| > 0.$$

We may assume without loss of generality that $\alpha = 0$ and $\beta = 1$.

For all $j \in \bar{d}\setminus\{0,1\}$ let $q_j \in Q_T$, $i_j \in \bar{d}$ and $l_j \in F_{d,n,\{i_j\}}$ be such that $|(q_j,l_j)\lambda_T\boldsymbol{\pi}_j| > 0$ (note these must exist as T is non-degenerate and the sets $F_{d,n,\{i_j\}}$ for $i_j \in \bar{d}$ generate $(X_n^*)^d$). Also let $i_0 = i_1 = i$.

It is now the case that if we read a word in coordinate i_j, it is possible to write in coordinate j (if we are in the correct state). Moreover, $i_0 = i_1$, so there is a coordinate b such that we can write words in any given coordinate without reading from coordinate b. We will use this observation to contradict injectivity.

For each state q_j we choose some $w_j \in (q_j)\mathfrak{s}_T^{-1}$. If we read $w_j l_j$ from anywhere, we will write non-trivially into the coordinate j. Consider w_j as $w_{j,b}w_j'$ where $w_{j,b} \in F_{d,n,\{b\}}$ and $w_j' \in F_{d,n,\bar{d}\setminus\{b\}}$.

Consider the elements

$$s_m := w_{0,b}w_{1,b}\ldots w_{d-1,b}(w_0'l_0w_1'l_1\ldots w_{d-1}'l_{d-1})^m \in (X_n^*)^d.$$

As the $w_{j,b}$ type elements will commute with all other kind of elements in the product defining of s_m, by commuting the words so that $w_{j,b}(w_0'l_0w_1'l_1\ldots w_{d-1}'l_{d-1})^m$ is a part of the product defining s_m, it follows that for all $q \in Q_T$ we have

$$|(q,s_m)\lambda_T\boldsymbol{\pi}_j| \geq m$$

for all j.

Thus all elements of $(X_n^\omega)^d$ which have all the s_m as prefixes have the same image under $f_{T,q}$. This is a contradiction as there are infinitely many such elements and $f_{T,q}$ is injective.

It remains to show the uniqueness of $(i)\psi_T$. The only way $(i)\psi_T$ could be non-unique is if T never writes anything when reading from coordinate i. In this case it follows from the existence of the other $(j)\psi_T$, that there is some coordinate into which T never writes, which is impossible as T is non-degenerate. ∎

Definition 6.3. If T is a transducer representing an element of $\widetilde{d\mathcal{O}_{n,1}}$, then we define $\psi_T : \bar{d} \to \bar{d}$ to be the map which was shown to be well defined in Lemma 6.2.

Theorem 6.4. *The group $d\mathcal{O}_{n,1}$ is isomorphic to $d\mathcal{K}_{n,1} \rtimes S_d$, where S_d acts by permuting the coordinates of $\mathbb{C}_n^d$ and $d\mathcal{K}_{n,1} = \{[T]_{\cong_S} \in d\mathcal{O}_{n,1} : \psi_T = \mathrm{id}\}$.*

Proof. Note that the map $[T]_{\cong_S} \to \psi_T$ is a monoid homomorphism to the full transformation monoid on d points. As $d\mathcal{O}_{n,1}$ is a group, it follows that ψ_T is always a permutation for $[T]_\cong \in d\mathcal{O}_{n,1}$. To see that the map is onto the symmetric group and the extension splits, note that for an arbitrary $f \in S_d$ the map

$$(p_0, p_1, \ldots, p_{d-1}) \xrightarrow{h} (p_{(0)f}, p_{(1)f}, \ldots, p_{(d-1)f})$$

is an element of $d\mathcal{B}_{n,1}$. Moreover, $[\mathrm{Core}(M_h)]_{\cong_S}$ maps to f under the homomorphism. This proves the theorem. ∎

We next need to understand the group $d\mathcal{K}_{n,1}$. To this end, we recall the groups $\mathcal{O}_{n,n-1}$ of [2]. These are groups of synchronizing core $(1, n)$-transducers, which are isomorphic to the outer automorphism groups of $G_{n,n-1}$.

In [2], it was shown that $\mathcal{O}_{n,n-1}$ contains $\mathcal{O}_{n,j}$ for all j, and that a $(1, n)$-transducer represents an element of $\mathcal{O}_{n,n-1}$ if and only if it is minimal (in the sense of GNS), synchronizing, it is its own core, all its states are injective, all its states have clopen image and it is invertible. In particular, $1\mathcal{O}_{n,1} = \mathcal{O}_{n,1}$ is a subgroup of $\mathcal{O}_{n,n-1}$.

Theorem 6.5. *If $[T]_{\cong_S} \in d\mathcal{K}_{n,1}$, then there are $T_0, T_1, \ldots, T_{d-1} \in \mathcal{O}_{n,n-1}$ such that*

$$T \cong_S \prod_{i \in \bar{d}} T_i.$$

Proof. For each $i \in \bar{d}$, let

$$\sim_i := \left\{ (p, q) \in Q_T^2 : \text{there is } w \in F_{d,n,\{i\}} \text{ with } (p, w)\pi_T = q \right\}.$$

One can check that each $\sim_i$ is an equivalence relation. If $q \in Q_T$, $i \in \bar{d}$, we restrict the domain and range of T to $F_{d,n,\{i\}}$ and restrict the state set of T to $[q]_{\sim_i}$, then we obtain a subtransducer $S_{q,i}$ of T.

Moreover, for all $w \in F_{d,n,\bar{d}\setminus\{i\}}$, one can check that the map $q \mapsto (q, w)\pi_T$ is a strong transducer isomorphism from $S_{q,i}$ to $S_{(q,w)\pi_T, i}$.

We will show that $S_{q,i} \in \mathcal{O}_{n,n-1}$ (if we make the natural identification between $F_{d,n,\{i\}}$ and X_n^*). It suffices to check that the conditions given in [2] are satisfied. The transducer $S_{q,i}$ has no inaccessible states by the definition of $\sim_i$, it has complete response because T has complete response, it is synchronizing and is its own core because T is and it has injective state functions because T does (Remark 3.15). By Remark 3.15, each state function $f_{T,q}$ of T has clopen image. As the image of a state function $f_{S_{q,i},p}$ of $S_{q,i}$ is a projection of the image of the corresponding state function $f_{T,p}$ of T (because ψ_T is well defined), it follows that this image of $f_{T,p}$ is compact and open (hence clopen). It remains to show that $S_{q,i}$ has no distinct but equivalent states. As T has no distinct but equivalent states, it suffices to show that if p_0, p_1 are equivalent states in $S_{q,i}$, then they are equivalent in T. Let $j \in \bar{d}$, and $w \in F_{d,n,\{j\}}$. It suffices to show that $(p_0, w)\lambda_T = (p_1, w)\lambda_T$. If $j = i$, then this follows by the assumption on p_0, p_1. Otherwise let $s \in F_{d,n,\{i\}}$ be such that $(p_0, s)\pi_T = p_1$. Then

$$(p_0, w)\lambda_T \pi_j = (p_0, ws)\lambda_T \pi_j = (p_0, sw)\lambda_T \pi_j$$
$$= ((p_0, s)\lambda_T (p_1, w)\lambda_T)\pi_j = (p_1, w)\lambda_T \pi_j$$

as required. So we can conclude that $S_{q,i} \in \mathcal{O}_{n,n-1}$ (if we make the natural identification between $F_{d,n,\{i\}}$ and X_n^*).

For each $i \in \bar{d}$, let $S_{q,i}$ be isomorphic to $T_i \in \mathcal{O}_{n,n-1}$ via an isomorphism which uses π_i as the domain and range isomorphisms (recall that q has no effect on the isomorphism type). Moreover, let $\phi_{q,i} : S_{q,i} \to T_i$ be the unique such transducer isomorphism (this is unique as the image of an arbitrary state is determined by any one of its synchronizing words). Let $\phi_i : T \to T_i$ be the transducer homomorphism with

$$\phi_{i\,D} = \phi_{i\,T} = \pi_i, \quad \phi_{i\,Q} = \bigcup_{q \in Q_T} \phi_{q,i\,Q}.$$

We then define $\phi : T \to \prod_{i \in \bar{d}} T_i$ to be the unique transducer homomorphism such that for all $i \in \bar{d}$ we have

$$\phi \pi_i = \phi_i.$$

We need to show that ϕ is a strong isomorphism. We have $\phi_D = \phi_R = \mathrm{id}$ by definition, so we need only show that ϕ_Q is a bijection. It must be surjective as its target is synchronizing, and so each state is the image of the state reached in T by reading one of its synchronizing words. Injectivity is also immediate as T was assumed to be minimal and hence has no proper strong quotients. ∎

It is routine to check that the multiplication in $d\mathcal{K}_{n,1}$ is also compatible with the multiplication in $\mathcal{O}_{n,n-1}$, so we can make the following definition:

Definition 6.6. We define an embedding $\alpha : d\mathcal{K}_{n,1} \to \mathcal{O}_{n,n-1}^d$ by

$$\prod_{i \in \bar{d}} ([T]_{\cong_S})\alpha \pi_i \cong_S T.$$

Corollary 6.7. *The group* $\mathrm{Out}(d\,V_n)$ *embeds in the group* $\mathcal{O}_{n,n-1} \wr S_d$.

Proof. Follows from Definition 6.6 (Theorem 6.5), Theorem 6.4 and Corollary 4.15. ∎

We now have a connection between $d\,\mathcal{O}_{n,1}$ and $\mathcal{O}_{n,n-1}$. To pin this down precisely we recall the map $\overline{\mathrm{sig}} : \mathcal{O}_{n,n-1} \to (\mathbb{Z}/(n-1)\mathbb{Z}, \times)$ of [12, Definition 7.6].

This group homomorphism takes an $\mathcal{O}_{n,n-1}$ transducer to the unique element $(m + (n-1)\mathbb{Z}) \in \mathbb{Z}/(n-1)\mathbb{Z}$, such that when a cone is read through the transducer, the transducer writes m disjoint cones. We can naturally use this map to define a new homomorphism from $\mathcal{O}_{n,n-1}^d$.

Definition 6.8. We define the homomorphism $\overline{\mathrm{sig}_d} : \mathcal{O}_{n,n-1}^d \to (\mathbb{Z}/(n-1)\mathbb{Z}, \times)$ by

$$(\mathbf{T})\overline{\mathrm{sig}_d} = \prod_{i \in \bar{d}} ((\mathbf{T})\pi_i)\overline{\mathrm{sig}}.$$

Moreover, we observe that this definition functions as one might expect:

Lemma 6.9. *If* $\mathbf{T} \in \mathcal{O}_{n,n-1}^d$, *then* $(\mathbf{T})\overline{\mathrm{sig}_d}$ *is the unique element* $(m + (n-1)\mathbb{Z})$ *of* $\mathbb{Z}/(n-1)\mathbb{Z}$, *such that* m *disjoint cones are written when a cone is read through* $\prod_{i \in \bar{d}} (\mathbf{T}\pi_i)$.

Proof. This is well defined by Remark 2.5. The result follows from the observations that a (d, n) cone is the same as a product of d $(1, n)$ cones, and the set of words writable from a state in a product of transducers is the product of the sets of words which can be written in each coordinate. ∎

In [12, Proposition 7.7] and [2, Theorem 9.5], the signature map has been used to identify the groups $\mathcal{O}_{n,r}$ inside of $\mathcal{O}_{n,n-1}$. We can now do the same with $d\,\mathcal{K}_{n,1}$ via a similar argument.

Lemma 6.10. *If* $\mathbf{T} \in (\mathcal{O}_{n,n-1})^d$, *then* $P := \prod_{i \in \bar{d}} \mathbf{T}\pi_i$ *is strongly isomorphic to a transducer representing an element of* $d\,\mathcal{K}_{n,1}$ *if and only if* $\mathbf{T} \in \ker(\overline{\mathrm{sig}_d})$.

Proof. ($\Rightarrow$) Suppose that $f \in d\,\mathcal{B}_{n,1}$ is such that the transducer P is strongly isomorphic to $\mathrm{Core}(M_f)$. If U is clopen in $\mathbb{C}_n^d$, then let $\mathrm{count}(U)$ be the smallest number of cones in a decomposition of U into cones. Let k be the synchronizing length of M_f. It follows that

$$\begin{aligned}
1 + (n-1)\mathbb{Z} &= \mathrm{count}((\mathbb{C}_n^d)f) + (n-1)\mathbb{Z} \\
&= \sum_{w \in (X_n^k)^d} \mathrm{count}((\mathbb{C}_n^d)f_{M_f,(w)\mathfrak{s}_{M_f}}) + (n-1)\mathbb{Z} \\
&= \sum_{w \in (X_n^k)^d} (\mathbf{T})\overline{\mathrm{sig}_d} + (n-1)\mathbb{Z} \\
&= n^{kd}(\mathbf{T})\overline{\mathrm{sig}_d} + (n-1)\mathbb{Z} \\
&= (\mathbf{T})\overline{\mathrm{sig}_d} + (n-1)\mathbb{Z}.
\end{aligned}$$

The result follows.

($\Longleftarrow$) Let $q \in Q_P$ be arbitrary. Then let $\bigcup_{a \in A} a\mathbb{C}_n^d$ be a decomposition of img($f_{P,q}$) into disjoint cones. We have that $|A| \in 1 + (n-1)\mathbb{Z}$. Let $k \in \mathbb{N}$ be greater than $|A|$, and such that for all $w \in (X_n^k)^d$ and $i \in \bar{d}$, we have

$$|(q,w)\lambda_P \boldsymbol{\pi}_i| \geq \max\{|a\boldsymbol{\pi}_j| : a \in A, j \in \bar{d}\}.$$

For all $w \in (X_n^k)^d$ let $a_w \in A$ be such that a_w is a prefix of $(q,w)\lambda_P$. For all $a \in A$ we now have that

$$\{((q,w)\lambda_P - a_w) \, \text{img}(f_{P,(q,w)\pi_T}) : w \in (X_n^k)^d \text{ has } a_w = a\}$$

is a partition of $\mathbb{C}_n^d$.

As $|A| \in 1 + (n-1)\mathbb{N}$, there is a complete prefix code B of size $|A|$. Let $\phi : A \to B$ be a bijection. It follows that

$$\{(a_w)\phi((q,w)\lambda_P - a_w) \, \text{img}(f_{P,(q,w)\pi_P}) : w \in (X_n^k)^d\}$$

is a partition of $\mathbb{C}_n^d$. We now define an element $f \in H(\mathbb{C}_n^d)$ as follows:
If $w \in (X_n^k)^d$ and $\bar{x} \in \mathbb{C}_n^d$ then

$$(w\bar{x})f = (a_w)\phi((q,w)\lambda_P - a_w)(\bar{x})f_{P,(q,w)\pi_P}.$$

It is routine to verify that M_f is synchronizing and has core strongly isomorphic to P. Thus $f \in d\mathcal{S}_{n,1}$, and P represents an element of $d\widetilde{\mathcal{O}_{n,1}}$. As ker($\overline{\text{sig}_d}$) is a group, we have $\mathbf{T}^{-1} \in \text{ker}(\overline{\text{sig}_d})$, and so by the same argument

$$P' := \prod_{i \in \bar{d}} (\mathbf{T}\boldsymbol{\pi}_i)^{-1}$$

also represents an element of $d\widetilde{\mathcal{O}_{n,1}}$. Thus P is $d\mathcal{O}_{n,1}$. As P is a product of $(1,n)$-transducers, we also have $\psi_P = \text{id}$, so the result follows. $\blacksquare$

We now have all the tools to prove Theorem 1.2 from the introduction (the statement is a bit simpler now that we have defined $\overline{\text{sig}_d}$).

Theorem 6.11. *For all $d \geq 1$ and $n \geq 2$ we have* Out(dV_n) $\cong$ ker($\overline{\text{sig}_d}$) $\rtimes S_d$, *where the action of S_d is the standard permutation of coordinates.*

Proof. By Corollary 4.15 we have Out(dV_n) $\cong d\mathcal{O}_{n,1}$. Thus the result follows from Theorem 6.4 and Lemma 6.10. $\blacksquare$

Corollary 6.12. *For all $d \geq 1$ we have* Out(dV) $\cong$ Out(V) $\wr S_d$ *(using the standard action of S_d on d points).*

Proof. This follows from the previous theorem together with the observation that $(\mathbb{Z}/(2-1)\mathbb{Z}, \times)$ is the trivial group. $\blacksquare$

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