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On the Bieri–Neumann–Strebel–Renz invariants of the weak commutativity construction $\mathfrak{X}(G)$

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Abstract. For a finitely generated group G, we calculate the Bieri–Neumann–Strebel–Renz invariant $\Sigma^1(\mathfrak{X}(G))$ for the weak commutativity construction $\mathfrak{X}(G)$. Identifying $S(\mathfrak{X}(G))$ with $S(\mathfrak{X}(G)/W(G))$, we show $\Sigma^2(\mathfrak{X}(G),\mathbb{Z}) \subseteq \Sigma^2(\mathfrak{X}(G)/W(G),\mathbb{Z})$ and $\Sigma^2(\mathfrak{X}(G)) \subseteq \Sigma^2(\mathfrak{X}(G)/W(G))$, that are equalities when $W(G)$ is finitely generated, and we explicitly calculate $\Sigma^2(\mathfrak{X}(G)/W(G),\mathbb{Z})$ and $\Sigma^2(\mathfrak{X}(G)/W(G))$ in terms of the Σ-invariants of G. We calculate completely the Σ-invariants in dimensions 1 and 2 of the group $\nu(G)$ and show that if G is finitely generated group with finitely presented commutator subgroup then the non-abelian tensor square $G \otimes G$ is finitely presented.

1. Introduction

In this paper, we consider the Bieri–Neumann–Strebel–Renz Σ-invariants of the weak commutativity construction $\mathfrak{X}(G)$. By definition, $\Sigma^m(G,\mathbb{Z})$ and $\Sigma^m(G)$ are subsets of the character sphere $S(G) = \mathrm{Hom}(G,\mathbb{R})/\sim$, where $\chi_1 \sim \chi_2$ if $\chi_1 \in \mathbb{R}_{>0}\chi_2$. The importance of the Σ-invariants is that they control which subgroups of G that contain the commutator have homological type FP_m or homotopical type F_m. The first Σ-invariant was defined by Bieri and Strebel in [9], where it was used to classify all finitely presented metabelain groups. In the case of metabelian groups, $\Sigma^1(G)$ has a strong connection with the valuation theory from commutative algebra that was used by Bieri and Groves to prove that the complement of $\Sigma^1(G)$ in the character sphere $S(G)$ is a spherical rational polyhedron [6]. In [7], Bieri, Neumann and Strebel defined the invariant $\Sigma^1(G)$ for any finitely generated group G, and for 3-manifold groups, they linked $\Sigma^1(G)$ with the Thurston norm [39].

Though in general it is difficult to calculate the Σ-invariants, they are known for several classes of groups though sometimes in low dimensions or in specific cases. The case of the Thompson group F was considered by Bieri, Geoghegan and Kochloukova [5] with a geometric proof given in [41], and the case of generalized Thompson groups $\mathrm{F}_{n,\infty}$ by Kochloukova [24] and by Zaremsky [42]. The case of free-by-cyclic group was studied by Funke and Kielak [18, 22], Cashen and Levitt [13], and the case of Poincaré duality group of dimension 3 by Kielak [22]. In [16], Dowdall, Kapovich and Leininger consider links between dynamical properties of the expanding action of $\mathbb{Z}$ on a free finite rank free

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group and Σ^1. Though the case of right-angled Artin group was completely resolved by Meier, Meinert and Van Wyk [31], the case of general Artin groups is widely open. Still, there are some results on $\Sigma^1(G)$ for specific Artin groups by Almeida [1], Almeida and Kochloukova [2]. In [14], Almeida and Lima calculated $\Sigma^1(G)$ for Artin group of finite type (i.e., spherical type). The case of combinatorial wreath product was considered by Mendonça [15].

Let G be a group and $\overline{G}$ an isomorphic copy of the group G. The group $\mathfrak{X}(G)$ was defined by Sidki in [38] by the presentation

$$\mathfrak{X}(G) = \langle G, \overline{G} \mid [g, \overline{g}] = 1 \text{ for } g \in G \rangle,$$

where $\overline{g}$ is the image of g in $\overline{G}$. In [38, Theorem C], Sidki proved that if $G \in \mathcal{P}$, then $\mathfrak{X}(G) \in \mathcal{P}$ when $\mathcal{P}$ is one of the following classes of groups: finite π-groups, where π is a set of primes; finite nilpotent groups; solvable groups and perfect groups. Later the following classes of groups $\mathcal{P}$ were added to the above list: finitely generated nilpotent groups by Gupta, Rocco and Sidki [20], virtually polycyclic groups by Lima and Oliveira [30], soluble groups of homological type FP_∞ by Kochloukova and Sidki [28], finitely presented groups by Kochloukova and Bridson [12], finitely generated virtually nilpotent groups [11], finitely generated Engel groups [11]. In this paper, we add to the above list the class of finitely generated groups for which $\Sigma^1(G)^c = S(G) \setminus \Sigma^1(G)$ is a rationally defined spherical polyhedron. By definition, a subset of $S(G)$ is a rationally defined spherical polyhedron if it is a finite union of finite intersection of closed rationally defined semispheres in $S(G)$, where rationality means that the semisphere is defined by a rational vector.

The group $\mathfrak{X}(G)$ has a special normal abelian subgroup $W(G)$ such that $\mathfrak{X}(G)/W(G)$ is a subdirect product of $G \times G \times G$ that maps surjectively on pairs. Our first result calculates $\Sigma^1(\mathfrak{X}(G))$ for any finitely generated group G.

Theorem A. *Let G be a finitely generated group and $\chi\colon \mathfrak{X}(G) \to \mathbb{R}$ be a character. Consider the homomorphisms $\chi_1, \chi_2\colon G \to \mathbb{R}$, where $\chi_1(g) = \chi(g)$ and $\chi_2(g) = \chi(\overline{g})$. Then $[\chi] \in \Sigma^1(\mathfrak{X}(G))$ if and only if one of the following holds:*

(1) $\chi_1 \neq 0$, $\chi_2 \neq 0$ *and* $\chi_1 \neq \chi_2$;

(2) $\chi_1 = 0$, $[\chi_2] \in \Sigma^1(G)$;

(3) $\chi_2 = 0$, $[\chi_1] \in \Sigma^1(G)$;

(4) $\chi_1 = \chi_2 \neq 0$ *and* $[\chi_1] \in \Sigma^1(G)$.

In particular, identifying $S(\mathfrak{X}(G))$ with $S(\mathfrak{X}(G)/W(G))$ via the epimorphism $\mathfrak{X}(G) \to \mathfrak{X}(G)/W(G)$, we have the equality $\Sigma^1(\mathfrak{X}(G)) = \Sigma^1(\mathfrak{X}(G)/W(G))$.

Theorem A implies immediately the following corollary.

Corollary B1. *Let G be a finitely generated group. Then $\Sigma^1(G)^c$ is a rationally defined spherical polyhedron if and only if $\Sigma^1(\mathfrak{X}(G))^c$ is a rationally defined spherical polyhedron.*

Let $\pi_1\colon \mathfrak{X}(G) \to G$ be the epimorphism that sends $\overline{G}$ to 1 and is identity on G. Let $\pi_2\colon \mathfrak{X}(G) \to G$ be the epimorphism that sends G to 1 and sends $\overline{g}$ to g for every $g \in G$. Finally, set the epimorphism $\pi_3\colon \mathfrak{X}(G) \to G$ that sends both g and $\overline{g}$ to g for every $g \in G$.

Corollary B2. *Let G be a finitely generated group and N be a subgroup of $\mathfrak{X}(G)$ that contains the commutator subgroup $\mathfrak{X}(G)'$. Then N is finitely generated if and only if $\pi_1(N)$, $\pi_2(N)$ and $\pi_3(N)$ are all finitely generated. In particular, $\mathfrak{X}(G)'$ is finitely generated if and only if G' is finitely generated.*

Limit groups were defined by Sela and independently studied by Kharlampovich and Myasnikov, who referred to them as fully residually free groups. Limit groups were discovered during the development of the theory that led to the solution of the Tarski problem on the elementary theory of non-abelian free groups of finite rank in [21] and [37]. In [23], Kochloukova showed that for a non-abelian limit group G we have $\Sigma^1(G) = \emptyset$.

Corollary C. *Let G be a non-abelian limit group. Then*

$$\Sigma^1(\mathfrak{X}(G)) = \big\{[\chi] \in S(\mathfrak{X}(G)) \mid \chi_1 \neq 0, \chi_2 \neq 0, \chi_1 \neq \chi_2\big\}.$$

Now, we continue with the study of the Σ-invariants focusing on $\Sigma^2(\mathfrak{X}(G), \mathbb{Z})$ and $\Sigma^2(\mathfrak{X}(G))$. Our first result in this direction is for non-abelian limit groups G.

Proposition D. *Let G be a finitely generated group. Suppose that $\Sigma^1(G) = \emptyset$. Then $\Sigma^2(\mathfrak{X}(G), \mathbb{Z}) = \emptyset$. In particular, this holds for non-abelian limit groups G.*

Recall that the group $\mathfrak{X}(G)$ has a special normal abelian subgroup $W(G)$ such that $\mathfrak{X}(G)/W(G)$ is a subdirect product of $G \times G \times G$ that maps surjectively on pairs. This together with the result of Bridson, Howie, Miller, Short [10, Theorem A] implies that $\mathfrak{X}(G)/W(G)$ is finitely presented whenever G is finitely presented. A homological version of this result was proved in [28, Theorem D] using Σ-theory, i.e., $\mathfrak{X}(G)/W(G)$ is FP_2 whenever G is FP_2. Recently Bridson and Kochloukova generalised this by showing in [12] that $\mathfrak{X}(G)$ is finitely presented (resp. FP_2) if and only if G is finitely presented (resp. FP_2). This does not generalise to homological type FP_3, since for finitely generated free non-cyclic group G the group $\mathfrak{X}(G)$ is not of type FP_3 [12].

In a recent work, Kochloukova and Lima [26] studied the Σ-invariants of subdirect products of non-abelian limit groups; in particular, this applies for $\mathfrak{X}(G)/W(G)$ when G is a non-abelian limit group. In the following theorem, the groups are not presumed limit groups and different techniques from [26] are applied.

Theorem E1. *Let G be of type FP_2 and $\chi\colon K = \mathfrak{X}(G)/W(G) \to \mathbb{R}$ be a character. Let $\chi_1, \chi_2\colon G \to \mathbb{R}$ be characters defined by $\chi_1(g) = \chi(g)$ and $\chi_2(g) = \chi(\overline{g})$. Then $[\chi] \in \Sigma^2(K, \mathbb{Z})$ if and only if one of the following cases holds:*

(1) $\chi_1 = 0$, $[\chi_2] \in \Sigma^2(G, \mathbb{Z})$;

(2) $\chi_2 = 0$, $[\chi_1] \in \Sigma^2(G, \mathbb{Z})$;

(3) $\chi_1 = \chi_2 \neq 0$ *and* $[\chi_1] \in \Sigma^2(G, \mathbb{Z})$;

(4) $\chi_1 \neq 0$, $\chi_2 \neq 0$, $\chi_1 \neq \chi_2$ *and one of the following holds:*

(4.1) $\{[\chi_1], [\chi_2]\} \subseteq \Sigma^1(G)$;

(4.2) $\{[\chi_1], [\chi_1 - \chi_2]\} \subseteq \Sigma^1(G)$;

(4.3) $\{[\chi_2], [\chi_2 - \chi_1]\} \subseteq \Sigma^1(G)$.

The proof of Theorem E1 uses substantially Theorem 2.7. Since Theorem 2.7 has a homotopical version, we have a homotopical version of Theorem E1.

Theorem E2. *Let G be a finitely presented group and $\chi\colon K = \mathfrak{X}(G)/W(G) \to \mathbb{R}$ be a character. Let $\chi_1, \chi_2\colon G \to \mathbb{R}$ be characters defined by $\chi_1(g) = \chi(g)$ and $\chi_2(g) = \chi(\overline{g})$. Then $[\chi] \in \Sigma^2(K)$ if and only if one of the following conditions holds:*

(1) $\chi_1 = 0$, $[\chi_2] \in \Sigma^2(G)$;

(2) $\chi_2 = 0$, $[\chi_1] \in \Sigma^2(G)$;

(3) $\chi_1 = \chi_2 \neq 0$ *and* $[\chi_1] \in \Sigma^2(G)$;

(4) $\chi_1 \neq 0$, $\chi_2 \neq 0$, $\chi_1 \neq \chi_2$ *and one of the following holds:*

(4.1) $\{[\chi_1], [\chi_2]\} \subseteq \Sigma^1(G)$;

(4.2) $\{[\chi_1], [\chi_1 - \chi_2]\} \subseteq \Sigma^1(G)$;

(4.3) $\{[\chi_2], [\chi_2 - \chi_1]\} \subseteq \Sigma^1(G)$.

In general, little is known for $W(G)$. In [28], Kochloukova and Sidki proved using homological methods that if G is FP_2 and G'/G'' is finitely generated, then $W(G)$ is finitely generated. In Theorem F1, we get partial information on $\Sigma^2(\mathfrak{X}(G), \mathbb{Z})$.

Theorem F1. *Suppose G is a group of type FP_2. For a character $\chi\colon \mathfrak{X}(G) \to \mathbb{R}$, define $\widehat{\chi}\colon \mathfrak{X}(G)/W \to \mathbb{R}$ to be the character induced by χ, where $W = W(G)$.*

(a) *If $[\chi] \in \Sigma^2(\mathfrak{X}(G), \mathbb{Z})$, then $[\widehat{\chi}] \in \Sigma^2(\mathfrak{X}(G)/W, \mathbb{Z})$.*

(b) *Suppose W is finitely generated and $[\widehat{\chi}] \in \Sigma^2(\mathfrak{X}(G)/W, \mathbb{Z})$. Then*

$$[\chi] \in \Sigma^2(\mathfrak{X}(G), \mathbb{Z}).$$

In particular, this holds when the abelianization of the commutator group $G' = [G, G]$ is finitely generated.

Theorem F1 has the following homotopical version.

Theorem F2. *Suppose G is a finitely presented group. For a character $\chi\colon \mathfrak{X}(G) \to \mathbb{R}$ define $\widehat{\chi}\colon \mathfrak{X}(G)/W \to \mathbb{R}$ to be the character induced by χ, where $W = W(G)$.*

(a) *If $[\chi] \in \Sigma^2(\mathfrak{X}(G))$, then $[\widehat{\chi}] \in \Sigma^2(\mathfrak{X}(G)/W)$.*

(b) *Suppose W is finitely generated and $[\widehat{\chi}] \in \Sigma^2(\mathfrak{X}(G)/W)$. Then $[\chi] \in \Sigma^2(\mathfrak{X}(G))$. In particular, this holds when the abelianization of the commutator group $G' = [G, G]$ is finitely generated.*

The next result follows from Theorems A, E1, E2, F1 and F2. Though it is an easy corollary of those theorems, it looks more symmetric than the theorems it derives from. For a group H and a subgroup N of H, by definition $S(H,N) = \{[\chi] \in S(H) \mid \chi(N) = 0\}$.

Corollary G. *The following statements hold:*

(a) *Suppose that G is a finitely generated group. Then we have the disjoint union*
$$\Sigma^1(\mathfrak{X}(G))^c = V_1 \cup V_2 \cup V_3,$$
where
$$V_i = \Sigma^1(\mathfrak{X}(G))^c \cap S(\mathfrak{X}(G), \mathrm{Ker}(\pi_i))$$
and the epimorphism $\pi_i \colon \mathfrak{X}(G) \to G$ induces a bijection $\pi_i^* \colon \Sigma^1(G)^c \to V_i$.

(b) *Suppose that G is a finitely presented group. Then*
$$W_1 \cup W_2 \cup W_3 \cup (V_1 + V_2) \cup (V_2 + V_3) \cup (V_1 + V_3) \subseteq \Sigma^2(\mathfrak{X}(G))^c,$$
where
$$W_i = \Sigma^2(\mathfrak{X}(G))^c \cap S(\mathfrak{X}(G), \mathrm{Ker}(\pi_i)),$$
and the epimorphism $\pi_i \colon \mathfrak{X}(G) \to G$ induces a bijection $\pi_i^* \colon \Sigma^2(G)^c \to W_i$.

(c) *Suppose that G is of homological type* FP_2. *Then*
$$M_1 \cup M_2 \cup M_3 \cup (V_1 + V_2) \cup (V_2 + V_3) \cup (V_1 + V_3) \subseteq \Sigma^2(\mathfrak{X}(G), \mathbb{Z})^c,$$
where
$$M_i = \Sigma^2(\mathfrak{X}(G), \mathbb{Z})^c \cap S(\mathfrak{X}(G), \mathrm{Ker}(\pi_i))$$
and the epimorphism $\pi_i \colon \mathfrak{X}(G) \to G$ induces a bijection $\pi_i^* \colon \Sigma^2(G, \mathbb{Z})^c \to M_i$.

As a corollary of Theorems F1 and F2, we obtain the following result.

Corollary H. *Let G be a group of type* FP_2 *(resp. finitely presented). For a subgroup N of $\mathfrak{X}(G)$ that contains the commutator $\mathfrak{X}(G)'$ and such that N is of type* FP_2 *(resp. finitely presented), we have that $N/W(G)$ is of type* FP_2 *(resp. finitely presented) too. Furthermore, for $m \geq 1$, the commutator $\mathfrak{X}(G)'$ is* FP_m *(resp. finitely presented) if and only if the commutator G' is* FP_m *(resp. finitely presented). When this happens, $W(G)$ is finitely generated.*

Inspired by Theorems F1 and F2, we suggest the following conjecture.

Conjecture I. *Let G be a group of type* FP_2 *(resp. finitely presented). Then, identifying $S(\mathfrak{X}(G))$ with $S(\mathfrak{X}(G)/W(G))$ via the epimorphism $\mathfrak{X}(G) \to \mathfrak{X}(G)/W(G)$, we have the equality*
$$\Sigma^2(\mathfrak{X}(G), \mathbb{Z}) = \Sigma^2(\mathfrak{X}(G)/W(G), \mathbb{Z}) \quad (\textit{resp. } \Sigma^2(\mathfrak{X}(G)) = \Sigma^2(\mathfrak{X}(G)/W(G))).$$

Note that by Theorems F1 and F2, we have that Conjecture I holds when G'/G'' is finitely generated. Note that Theorem E1 implies that for a non-abelian limit group G we have $\Sigma^2(\mathfrak{X}(G)/W(G), \mathbb{Z}) = \emptyset$, so Conjecture I holds for non-abelian limit groups. Another way to state Conjecture I is that in Corollary G in parts (b) and (c) the inclusions are equalities.

In Section 8, we consider the non-abelian tensor square $G \otimes G$ of a group G and the construction $\nu(G)$. In [35], Rocco defined for an arbitrary group G the group $\nu(G)$. In [17], Ellis and Leonard studied a similar construction. The construction $\nu(G)$ is strongly related to the construction $\mathfrak{X}(G)$ in the following way: there is a central subgroup Δ of $\nu(G)$ such that

$$\nu(G)/\Delta \simeq \mathfrak{X}(G)/R(G),$$

where $R(G)$ is a normal subgroup of $\mathfrak{X}(G)$ such that $W(G)/R(G) \simeq H_2(G, \mathbb{Z})$. By [35], the non-abelian tensor square $G \otimes G$ is isomorphic to the subgroup $[G, \overline{G}]$ of $\nu(G)$. Using properties of $\mathfrak{X}(G)/W$, we show the following result.

Proposition J. *Let G be a finitely generated group such that the commutator subgroup G' is finitely presented (resp. is* FP_2*). Then the non-abelian tensor square $G \otimes G$ is finitely presented (resp. is* FP_2*).*

In Section 8, we determine the invariants $\Sigma^1(\nu(G))$, $\Sigma^2(\nu(G), \mathbb{Z})$ and $\Sigma^2(\nu(G))$ and identify them with the corresponding invariants of $\mathfrak{X}(G)/W$.

2. Preliminaries on the Σ-invariants

In [40], Wall defined a group G to be of homotopical type F_n if there is a classifying space $K(G, 1)$ with finite n-skeleton. The homotopical type F_2 coincides with finite presentability (in terms of generators and relations). A homological version of this property, called FP_n, was defined by Bieri in [3]. A group G is of homological type FP_n if the trivial $\mathbb{Z}G$-module $\mathbb{Z}$ has a projective resolution with all modules finitely generated in dimension $\leq n$.

Higher-dimensional homological invariants $\Sigma^n(G, A)$ for a $\mathbb{Z}G$-module A were defined by Bieri and Renz in [8], where they showed that $\Sigma^n(G, \mathbb{Z})$ controls which subgroups of G that contain the commutator are of homological type FP_n. In [34], Renz defined the higher-dimensional homotopical invariant $\Sigma^n(G)$ for groups G of homotopical type F_n and, similarly to the homological case $\Sigma^n(G)$, controls the homotopical finiteness properties of the subgroups of G above the commutator. In all cases the Σ-invariants are open subsets of the character sphere $S(G)$. For a group G of type F_n we have $\Sigma^n(G) = \Sigma^n(G, \mathbb{Z}) \cap \Sigma^2(G)$. The description of the Σ-invariants of right-angled Artin groups by Meier, Meinert and Van Wyk show that the inclusion $\Sigma^n(G) \subseteq \Sigma^n(G, \mathbb{Z})$ is not necessary an equality for $n \geq 2$ [31]. For the homotopical invariants, we note that $\Sigma^1(G) = \Sigma^1(G, \mathbb{Z})$ and in general $\Sigma^n(G) \subseteq \Sigma^n(G, \mathbb{Z})$.

By definition, a character $\chi\colon G \to \mathbb{R}$ is a non-zero homomorphism and $\Sigma^n(G, \mathbb{Z})$ is a subset of the character sphere $S(G)$. The character sphere $S(G)$ is the set of equivalence classes $[\chi]$ of characters $\chi\colon G \to \mathbb{R}$, where two characters χ_1 and χ_2 are equivalent if one is obtained from the other by multiplication with any positive real number. For a fixed character $\chi\colon G \to \mathbb{R}$, define

$$G_\chi = \{g \in G \mid \chi(g) \geq 0\}.$$

Recall that for an associative ring R and R-module A, we say that A is of type FP_n over R if A has a projective resolution over R where all projectives in dimension up to n are finitely generated, i.e., there is an exact complex

$$\mathcal{P}\colon \cdots \to P_i \to P_{i-1} \to \cdots \to P_0 \to A \to 0,$$

where each P_j is a projective R-module and for $i \leq n$ we have that P_i is finitely generated.

Let D be an integral domain. By definition for a (left) DG-module A

$$\Sigma^n_D(G, A) = \big\{[\chi] \in S(G) \mid A \text{ is of type } \mathrm{FP}_n \text{ as a } DG_\chi\text{-module}\big\}.$$

When A is the trivial (left) DG-module D, denote by $\Sigma^n(G, D)$ the invariant $\Sigma^n_D(G, D)$.

Note that if $\Sigma^n(G, \mathbb{Z})$ is a non-empty set, then G is FP_n, in particular, is finitely generated. Later we will need the description of $\Sigma^1(G)$ given by the Cayley graph of a finitely generated group G. Let X be a finite generating set of G. Consider the Cayley graph Γ of G associated with the generating set X, i.e., the set of vertices is $V(\Gamma) = G$ and the set of edges is $E(\Gamma) = X \times G$ with the edge $e = (x, g)$ having beginning g and end gx. The group G acts on Γ via left multiplication on $V(\Gamma)$, and $he = (x, hg)$ for any $h \in G$. The letter x is called the label of the edge e, we write (x^{-1}, gx) for the inverse of e and call x^{-1} the label of e^{-1}. For a fixed character $\chi\colon G \to \mathbb{R}$, we write Γ_χ for the (full) subgraph of Γ spanned by the vertices in G_χ. By definition,

$$\Sigma^1(G) = \big\{[\chi] \in S(G) \mid \Gamma_\chi \text{ is a connected graph}\big\}.$$

Suppose now that G is finitely presented with a finite presentation $\langle X | R\rangle$. Gluing to the Cayley graph Γ at every vertex 2-cells that spell out the relations of R we get the Cayley complex $\mathcal{C}$ associated to the above finite presentation. For a fixed character $\chi\colon G \to \mathbb{R}$, we denote by $\mathcal{C}_\chi$ the (full) subcomplex of $\mathcal{C}$ spanned by the vertices in G_χ. By definition,

$$\Sigma^2(G) = \big\{[\chi] \in S(G) \mid \text{there is a finite presentation for which } \mathcal{C}_\chi \text{ is 1-connected}\big\}.$$

The first result is folklore, it is an obvious corollary of the fact that $\Sigma^1(G, \mathbb{Z}) = \Sigma^1(G)$ and tensoring is a right exact functor.

Lemma 2.1. *Let $\pi\colon G_1 \to G_2$ be an epimorphism of finitely generated groups, $\mu_2\colon G_2 \to \mathbb{R}$ be a character (i.e., non-zero homomorphism) and $\mu_1 = \mu_2 \circ \pi$. Suppose that $[\mu_1] \in \Sigma^1(G_1)$. Then $[\mu_2] \in \Sigma^1(G_2)$.*

We warn the reader that the previous lemma does not hold for Σ^2.

Theorem 2.2 ([31, Theorem 9.3]). *Let H be a subgroup of G, M be a DG-module and $\xi\colon G \to \mathbb{R}$ be a character. If $[G : H] < \infty$, then*

$$[\xi|_H] \in \Sigma^n_D(H, A) \;\Leftrightarrow\; [\xi] \in \Sigma^n_D(G, A).$$

In particular, if $n = 0$, then

A is a finitely generated DG_ξ-module $\Leftrightarrow$ A is a finitely generated $DH_{\xi|_H}$-module.

In [4], Bieri and Geoghegan proved a formula for the homological invariants $\Sigma^n(-, F)$ for a direct product of groups, where F is the trivial module and is a field. If F is substituted with the trivial module $\mathbb{Z}$, the result is wrong in both homological and homotopical settings provided the dimension is sufficiently high, see [31] and [36].

Theorem 2.3 (Direct product formula [4, Theorem 1.3 and Proposition 5.2]). *Let $n \geq 0$ be an integer, G_1 and G_2 be finitely generated groups and F be a field. Then,*

$$\Sigma^n(G_1 \times G_2, F)^c = \bigcup_{p=0}^{n} \Sigma^p(G_1, F)^c * \Sigma^{n-p}(G_2, F)^c,$$

where $*$ denotes the join of subsets of the character sphere $S(G_1 \times G_2)$ and ^{c} denotes the set-theoretic complement of subsets of a suitable character sphere.

The above theorem means that if $\mu\colon G_1 \times G_2 \to \mathbb{R}$ is a character with $\mu_1 = \mu\,|_{G_1}$ and $\mu_2 = \mu\,|_{G_2}$, then $[\mu] \in \Sigma^n(G_1 \times G_2, F)^c = S(G_1 \times G_2, F) \setminus \Sigma^n(G_1 \times G_2, F)$ precisely when one of the following conditions holds:

(1) $\mu_1 \neq 0$, $\mu_2 \neq 0$ and $[\mu_1] \in \Sigma^p(G_1, F)^c = S(G_1) \setminus \Sigma^p(G_1, F)$, $[\mu_2] \in \Sigma^{n-p}(G_2, F)^c = S(G_2) \setminus \Sigma^{n-p}(G_2, F)$ for some $0 \leq p \leq n$; or

(2) one of the characters μ_1, μ_2 is trivial and for the non-trivial one, say μ_i, we have $[\mu_i] \in \Sigma^n(G_i, F)^c = S(G_i) \setminus \Sigma^n(G_i, F)$.

Though Theorem 2.3 does not hold in general when F is not a field, it holds in small dimensions $1 \leq n \leq 2$.

Theorem 2.4 ([19]). *Let $1 \leq n \leq 2$ be an integer, G_1, G_2 be finitely generated groups. Then,*

$$\Sigma^n(G_1 \times G_2, \mathbb{Z})^c = \bigcup_{p=0}^{n} \Sigma^p(G_1, \mathbb{Z})^c * \Sigma^{n-p}(G_2, \mathbb{Z})^c,$$

where $*$ denotes the join of subsets of the character sphere $S(G_1 \times G_2)$ and ^{c} denotes the set-theoretic complement of subsets of a suitable character sphere.

Theorem 2.5 ([8,34]). *Let G be a group of type F_n (resp. FP_n) and N be a subgroup of G that contains the commutator subgroup G'. Then N is of type F_n (resp. FP_n) if and only if*

$$S(G, N) = \big\{[\chi] \in S(G) \mid \chi(N) = 0\big\} \subseteq \Sigma^n(G) \quad (\textit{resp. } \Sigma^n(G, \mathbb{Z})).$$

Bieri and Renz proved the homological version of Theorem 2.5 in [8]. The homotopical version for $m = 2$ was proved by Renz in [34], and the general homotopical case $m \geq 3$ follows from the formula $\Sigma^m(G) = \Sigma^m(G, \mathbb{Z}) \cap \Sigma^2(G)$.

The following theorem can be traced back to several papers: Gehrke results in [19]; the Meier, Meinert and Van Wyk description of the Σ-invariants for right-angled Artin groups [31] or the Meinert result on the Σ-invariants for direct products of virtually free groups [32].

Theorem 2.6 ([19, 31, 32]). *Let* F_2 *be the free group on* 2 *generators. If* $\chi\colon \mathrm{F}_2^s = \mathrm{F}_2 \times \cdots \times \mathrm{F}_2 \to \mathbb{R}$ *is a character whose restriction on precisely* n *copies of* F_2 *is non-zero, then* $[\chi] \in \Sigma^{n-1}(\mathrm{F}_2^s) \setminus \Sigma^n(\mathrm{F}_2^s)$.

The next result was recently obtained by Kochloukova and Mendonça. It should be viewed as a monoidal version of Theorem 2.5. It is surprising it was not discovered earlier, as Theorem 2.5 is quite well known and the proof of Theorem 2.7 in [27] is based on ideas from the proof of Theorem 2.5 but is slightly more technical.

Theorem 2.7 ([27]). *The following assertions hold:*

(a) *Let* $[H, H] \subseteq K \subseteq H$ *be groups such that* H *and* K *are of type* FP_n. *Let* $\chi\colon K \to \mathbb{R}$ *be a character such that* $\chi([H, H]) = 0$. *Then* $[\chi] \in \Sigma^n(K, \mathbb{Z})$ *if and only if* $[\mu] \in \Sigma^n(H, \mathbb{Z})$ *for every character* $\mu\colon H \to \mathbb{R}$ *that extends* χ.

(b) *Let* $[H, H] \subseteq K \subseteq H$ *be groups such that* H *and* K *are finitely presented. Let* $\chi\colon K \to \mathbb{R}$ *be a character such that* $\chi([H, H]) = 0$. *Then* $[\chi] \in \Sigma^2(K)$ *if and only if* $[\mu] \in \Sigma^2(H)$ *for every character* $\mu\colon H \to \mathbb{R}$ *that extends* χ.

3. Preliminaries on subdirect products and limit groups

The class of limit groups contains all finite rank free groups and the orientable surface groups. It coincides with the class of the fully residually free groups G, i.e., for every finite subset X of G there is free group F and a homomorphism $\varphi\colon G \to F$ whose restriction on X is injective. Limit groups are of type FP_∞, finitely presented and of finite cohomological dimension.

A subgroup $G \subseteq G_1 \times \cdots \times G_m$ is a subdirect product if the projection map $p_i\colon G \to G_i$ is surjective for all $1 \leq i \leq m$. Denote by $p_{i_1,\dots,i_n}\colon G \to G_{i_1} \times \cdots \times G_{i_n}$ the projection map that sends $(g_1, \dots, g_m)$ to $(g_{i_1}, \dots, g_{i_n})$.

Theorem 3.1 ([23]). *Let* $G \subseteq G_1 \times \cdots \times G_m$ *be a subdirect product of non-abelian limit groups* $G_1, \dots, G_m$ *such that* $G \cap G_i \neq 1$ *for every* $1 \leq i \leq m$ *and* G *be of type* FP_n *for some* $n \leq m$. *Then* $p_{i_1,\dots,i_n}(G)$ *has finite index in* $G_{i_1} \times \cdots \times G_{i_n}$ *for every* $1 \leq i_1 < \cdots < i_n \leq m$.

Theorem 3.2 ([23]). *Let* G *be a non-abelian limit group. Then* $\Sigma^1(G) = \emptyset$.

The following conjecture was defined by Kuckuck in [29].

Conjecture 3.3 (The virtual surjection conjecture [29]). *Let $G \subseteq G_1 \times \cdots \times G_m$ be a subdirect product of groups $G_1, \ldots, G_m$ such that $G \cap G_i \neq 1$ for every $1 \leq i \leq m$ and each G_i is of homotopical type F_n for some $n \leq m$. Suppose that $p_{i_1,\ldots,i_n}(G)$ has finite index in $G_{i_1} \times \cdots \times G_{i_n}$ for every $1 \leq i_1 < \cdots < i_n \leq m$. Then G is of type F_n.*

The motivation behind the virtual surjection conjecture is that it holds for $n = 2$ [10]; that particular case was established by Bridson, Howie, Miller and Short as a corollary of the 1-2-3 theorem. Furthermore, the virtual surjection conjecture holds for any n when G contains $G'_1 \times \cdots \times G'_m$ [29]. A homological version of the virtual surjection conjecture was suggested in [25] and proved for $n = 2$.

Theorem 3.4 ([25]). *Let $G \subseteq G_1 \times \cdots \times G_m$ be a subdirect product of groups $G_1, \ldots, G_m$ such that $G \cap G_i \neq 1$ for every $1 \leq i \leq m$ and each G_i is of homological type FP_2. Suppose that $p_{i_1,i_2}(G)$ has finite index in $G_{i_1} \times G_{i_2}$ for every $1 \leq i_1 < i_2 \leq m$. Then G is of type FP_2.*

4. Preliminaries on $\mathfrak{X}(G)$

Recall that
$$\mathfrak{X}(G) = \langle G, \overline{G} \mid [g, \overline{g}] = 1 \text{ for } g \in G\rangle,$$
where $\overline{G}$ is an isomorphic copy of the group G and $\overline{g}$ is the image of $g \in G$ in $\overline{G}$. In [38], Sidki defined the normal subgroup
$$L = L(G) = \langle\{\overline{g}^{-1} g \mid g \in G\}\rangle$$
of $\mathfrak{X}(G)$. Note that
$$\mathfrak{X}(G) = L \rtimes G.$$
In [30], Lima and Oliveira showed that the abelianization L/L' is finitely generated whenever G is finitely generated. In [12], Bridson and Kochloukova generalized this by showing that when G is finitely generated, L is finitely generated.

Another important normal subgroup of $\mathfrak{X}(G)$ is
$$D = D(G) = [G, \overline{G}].$$

There are the following canonical epimorphisms of groups:
$$\mathfrak{X}(G) \to \mathfrak{X}(G)/D \simeq G \times \overline{G} \simeq G \times G \quad \text{and} \quad \mathfrak{X}(G) \to \mathfrak{X}(G)/L \simeq G.$$

The diagonal map of these two epimorphisms induces a map
$$\rho\colon \mathfrak{X}(G) \to G \times G \times G$$

with kernel

$$W = W(G) = L(G) \cap D(G).$$

This map, after some permutation of the factors G in $G \times G \times G$, can be explicitly given by

$$\rho(g) = (g, g, 1) \quad \text{and} \quad \rho(\overline{g}) = (1, g, g).$$

Note that

$$\mathrm{Im}(\rho) = \left\{(g_1, g_2, g_3) \mid g_1 g_2^{-1} g_3 \in G'\right\}$$

is a subdirect product of $G \times G \times G$ that maps surjectively on pairs and contains the commutator subgroup $G' \times G' \times G'$. The defining relations of $\mathfrak{X}(G)$ easily imply that

$$[L, D] = 1,$$

and this property is crucial to develop the structure theory for $\mathfrak{X}(G)$. For example, it implies that $W(G)$ is an abelian group. Furthermore, $W(G)$ can be viewed as $\mathfrak{X}(G)$-module via conjugation with DL acting trivially. Thus $W(G)$ is a $\mathfrak{X}(G)/DL$-module and $\mathfrak{X}(G)/DL \simeq G/G'$ is abelian.

The following result was proved by Kochloukova and Sidki in [28] using homological techniques. Note that every finitely presented group is FP_2.

Theorem 4.1 ([28]). *If G is of homological type FP_2 and G'/G'' is finitely generated, then $W(G)$ is finitely generated.*

The question whether $\mathfrak{X}(G)$ is FP_n when G is FP_n was resolved by Bridson and Kochloukova in [12] with affirmative answer for $n = 2$ and negative for $n \geq 3$.

Theorem 4.2 ([12]). *If G is finitely presented (resp. FP_2), then $\mathfrak{X}(G)$ is finitely presented (resp. FP_2). But if G is FP_n for some $n \geq 3$, then $\mathfrak{X}(G)$ is not necessary FP_n, since for F a free non-cyclic group of finite rank $\mathfrak{X}(F)$ is not FP_3.*

5. The main results for $\Sigma^1(\mathfrak{X}(G))$

Throughout this section G is a *finitely generated group*,

$$\chi\colon \mathfrak{X}(G) \to \mathbb{R}$$

is a character and

$$\chi_0\colon \mathfrak{X}(G)/D \to \mathbb{R}$$

is the character induced by χ. Note that

$$\mathfrak{X}(G)/D \simeq G \times \overline{G} \simeq G \times G.$$

We write

$$\chi_0 = (\chi_1, \chi_2)\colon G \times G \to \mathbb{R}$$

and note that $\chi_1(g) = \chi(g)$ and $\chi_2(g) = \chi(\overline{g}_2)$.

Lemma 5.1. *Suppose that G is a finitely generated group and $[\chi] \in \Sigma^1(\mathfrak{X}(G))$. Then $[\chi_0] \in \Sigma^1(\mathfrak{X}(G)/D)$ and one of the following holds:*

(1) $\chi_1 \neq 0$, $\chi_2 \neq 0$;

(2) $\chi_1 = 0$, $[\chi_2] \in \Sigma^1(G)$;

(3) $\chi_2 = 0$, $[\chi_1] \in \Sigma^1(G)$.

Furthermore, if $\chi_1 = \chi_2 \neq 0$, then $[\chi_1] \in \Sigma^1(G)$.

Proof. The fact that $[\chi_0] \in \Sigma^1(\mathfrak{X}(G)/D)$ follows immediately by Lemma 2.1. Note that $\Sigma^1(\mathfrak{X}(G)/D) \simeq G \times G$. By Theorem 2.4 and the fact that $\Sigma^1(-, \mathbb{Z}) = \Sigma^1(-)$, we have

$$\begin{aligned}\Sigma^1(G \times G) = \big\{[(\chi_1, \chi_2)] \in S(G \times G) \mid \chi_1 \neq 0, \chi_2 \neq 0 \text{ or } \chi_1 = 0, [\chi_2] \in \Sigma^1(G) \\ \text{or } \chi_2 = 0, [\chi_1] \in \Sigma^1(G)\big\}. \end{aligned} \tag{5.1}$$

If $\chi_1 = \chi_2$, then $\chi(L) = 0$ and χ induces a character $\widehat{\chi}: \mathfrak{X}(G)/L \simeq G \to \mathbb{R}$. Note that $\widehat{\chi}$ can be identified with χ_1 and by Lemma 2.1 $[\widehat{\chi}] \in \Sigma^1(G)$. ∎

Lemma 5.2. *Suppose that G is a finitely generated group, $[\chi_0] = [(\chi_1, \chi_2)] \in \Sigma^1(G \times G)$ and $\chi_1 \neq \chi_2$. Then $[\chi] \in \Sigma^1(\mathfrak{X}(G))$.*

Proof. Note that since $\chi_1 \neq \chi_2$ we have $\chi(L) \neq 0$. From the very beginning, we can fix an element $a \in L$ such that $\chi(a) \geq 1$ and include it in a fixed finite generating set Y of $\mathfrak{X}(G)$. Let Γ be the Cayley graph of $\mathfrak{X}(G)$ with respect to Y. Let $\widehat{Y}$ be the image of Y in $\mathfrak{X}(G)/D$ and let $\widehat{\Gamma}$ be the Cayley graph of $\mathfrak{X}(G)/D$ with respect to the finite generating set $\widehat{Y}$. By definition, Γ_χ is the subgraph of Γ spanned by $\mathfrak{X}(G)_\chi = \{h \in \mathfrak{X}(G) \mid \chi(h) \geq 0\}$ and $\widehat{\Gamma}_{\chi_0}$ is the subgraph of $\widehat{\Gamma}$ spanned by $(\mathfrak{X}(G)/D)_{\chi_0} = \{h \in \mathfrak{X}(G)/D \mid \chi_0(h) \geq 0\}$.

Let $g \in \mathfrak{X}(G)_\chi$ and write $\widehat{g}$ for the image of g in $\mathfrak{X}(G)/D$. Since $[\chi_0] \in \Sigma^1(\mathfrak{X}(G)/D)$, we deduce that there is a path $\widehat{\gamma}$ in $\widehat{\Gamma}_{\chi_0}$ that starts at $1_{\mathfrak{X}(G)/D}$ and finishes at $\widehat{g}$. Then we can lift the path $\widehat{\gamma}$ to a path γ in Γ_χ that starts at $1_{\mathfrak{X}(G)}$, i.e., under the canonical epimorphism $\pi: F(Y) \to F(\widehat{Y})$ (where $F(Y)$ and $F(\widehat{Y})$ are the free groups with basis Y and $\widehat{Y}$ respectively) the label $l(\gamma)$ of γ is sent to the label $l(\widehat{\gamma})$. Then the path γ finishes at an element of $\mathfrak{X}(G)$ that is mapped under the canonical epimorphism $\mathfrak{X}(G) \to \mathfrak{X}(G)/D$ to $\widehat{g}$, i.e., the final point is tg for some $t \in D$.

Suppose there is a path γ_0 in Γ_χ that starts at $1_{\mathfrak{X}(G)}$ and finishes at t. Then the composition path $\gamma_0^{-1}\gamma$ is a path in Γ_χ that starts at t and finishes at tg. Finally, since $\chi(t) \in \chi(D) = 0$ we deduce that $t^{-1}.(\gamma_0^{-1}\gamma)$ is a path in Γ_χ that starts at $1_{\mathfrak{X}(G)}$ and finishes at g. Thus $[\chi] \in \Sigma^1(\mathfrak{X}(G))$ as required.

Finally, we construct the path γ_0. First, we start with any path $\widetilde{\gamma}$ in Γ that starts at $1_{\mathfrak{X}(G)}$ and finishes at t. Note that for m sufficiently large positive integer, we have that $\widetilde{\gamma}$ is inside $\Gamma_{\chi \geq -m}$. Recall that $a \in L \cap Y$ is an element such that $\chi(a) \geq 1$. Let δ_m be the path in Γ_χ that starts at $1_{\mathfrak{X}(G)}$ and has label $a^m = a \dots a$, and let δ_{-m} be the path in Γ that starts at $1_{\mathfrak{X}(G)}$ and has label $a^{-m} = a^{-1} \dots a^{-1}$. Then the path $a^m.\widetilde{\gamma}$ is inside Γ_χ, starts at a^m and finishes at $a^m t$. And since $\chi(t) = 0$, the path $a^m t.\delta_{-m}$ is inside Γ_χ, starts at $a^m t$ and finishes at $a^m t a^{-m}$. Note that $t \in D$ and $a \in L$. Since $[D, L] = 1$, we deduce that $a^m t a^{-m} = t$, hence the concatenation $\gamma_0 = \delta_m(a^m.\widetilde{\gamma})(a^m t.\delta_{-m})$ is a path inside Γ_χ that starts at $1_{\mathfrak{X}(G)}$ and finishes at t. ■

Lemma 5.3. *Suppose that H is a finitely generated group. Let N be a finitely generated normal subgroup of H and $\chi\colon H \to \mathbb{R}$ be a character such that $\chi(N) = 0$. Let $\widetilde{\chi}\colon H/N \to \mathbb{R}$ be the character induced by χ. Assume that $[\widetilde{\chi}] \in \Sigma^1(H/N)$. Then $[\chi] \in \Sigma^1(H)$.*

In particular, for a finitely generated group G, $H = \mathfrak{X}(G)$ and $N = L$, if $\chi_1 = \chi_2 \neq 0$ and $[\chi_1] \in \Sigma^1(G)$, then $[\chi] \in \Sigma^1(\mathfrak{X}(G))$.

Proof. Let Γ be the Cayley graph of H with respect to a fixed finite generating set Y and $\widehat{Y}$ be the image of Y in H/N. Let $\widehat{\Gamma}$ be the Cayley graph of H/N with respect to the generating set $\widehat{Y}$.

Fix $g \in H_\chi$ and consider $\widehat{g}$ the image of g in H/N. Since $[\widetilde{\chi}] \in \Sigma^1(H/N)$, we deduce that there is a path $\widehat{\gamma}$ in $\widehat{\Gamma}_{\widetilde{\chi}}$ that starts at $1_{H/N}$ and finishes at $\widehat{g}$. Then we can lift the path $\widehat{\gamma}$ to a path γ in Γ_χ that starts at 1_H. Note that the path γ finishes at an element of H of the type tg for some $t \in N$.

Suppose there is a path γ_0 in Γ_χ that starts at 1_H and finishes at t. Then $\gamma_0^{-1}\gamma$ is a path in Γ_χ with beginning t and end tg. Finally, since $\chi(t) \in \chi(N) = 0$, we get that $t^{-1}(\gamma_0^{-1}\gamma)$ is a path in Γ_χ with beginning 1_H and end g. Thus $[\chi] \in \Sigma^1(H)$ as required.

Finally, we construct the path γ_0. Consider a finite generating set Y_1 of N; we can choose Y such that $Y_1 \subseteq Y$. Then we can link the elements 1_H and t with a path γ_0 in $\Gamma_{\chi=0}$ whose label is a word on $Y_1^{\pm 1}$, where $\Gamma_{\chi=0}$ is the subgraph of Γ generated by $\mathrm{Ker}(\chi)$.

Finally, for the case $H = \mathfrak{X}(G)$, $N = L$ observe that by [12, Proposition 2.3], if G is finitely generated, then L is finitely generated. ■

Lemmas 5.1, 5.2 and 5.3 imply the following corollary.

Corollary 5.4. *Let G be a finitely generated group and $\chi\colon \mathfrak{X}(G) \to \mathbb{R}$ be a character. Consider the homomorphisms $\chi_1, \chi_2\colon G \to \mathbb{R}$, where $\chi_1(g) = \chi(g)$ and $\chi_2(g) = \chi(\overline{g})$. Then $[\chi] \in \Sigma^1(\mathfrak{X}(G))$ if and only if one of the following holds:*

(1) $\chi_1 \neq 0$, $\chi_2 \neq 0$ *and* $\chi_1 \neq \chi_2$;

(2) $\chi_1 = 0$, $[\chi_2] \in \Sigma^1(G)$;

(3) $\chi_2 = 0$, $[\chi_1] \in \Sigma^1(G)$;

(4) $\chi_1 = \chi_2 \neq 0$ *and* $[\chi_1] \in \Sigma^1(G)$.

Denote by

$$\pi\colon \mathfrak{X}(G) \to \mathfrak{X}(G)/W$$

the canonical projection.

Lemma 5.5. *Suppose that G is a finitely generated group. Let*

$$\widehat{\chi}\colon \mathfrak{X}(G)/W \to \mathbb{R}$$

be a character, $\chi_1, \chi_2 \colon G \to \mathbb{R}$ be the characters defined by $\chi_1(g) = \widehat{\chi}\pi(g)$ and $\chi_2(g) = \widehat{\chi}\pi(\bar{g})$. Then $[\widehat{\chi}] \in \Sigma^1(\mathfrak{X}(G)/W)$ if and only if one of the following conditions holds:

(1) $\chi_1 \neq 0$, $\chi_2 \neq 0$ *and* $\chi_1 \neq \chi_2$*;*

(2) $\chi_1 = 0$, $[\chi_2] \in \Sigma^1(G)$*;*

(3) $\chi_2 = 0$, $[\chi_1] \in \Sigma^1(G)$*;*

(4) $\chi_1 = \chi_2 \neq 0$ *and* $[\chi_1] \in \Sigma^1(G)$.

Proof. Let

$$\chi = \widehat{\chi}\pi \colon \mathfrak{X}(G) \to \mathbb{R}.$$

We claim that $[\widehat{\chi}] \in \Sigma^1(\mathfrak{X}(G)/W)$ if and only if $[\chi] \in \Sigma^1(\mathfrak{X}(G))$. Recall that $\Sigma^1(\mathfrak{X}(G))$ is described in Corollary 5.4. If $[\chi] \in \Sigma^1(\mathfrak{X}(G))$, by Lemma 2.1 $[\widehat{\chi}] \in \Sigma^1(\mathfrak{X}(G)/W)$.

Suppose now that $[\widehat{\chi}] \in \Sigma^1(\mathfrak{X}(G)/W)$. Then since $W \subseteq D$, the group $\mathfrak{X}(G)/D$ is a quotient of $\mathfrak{X}(G)/W$, and by Lemma 2.1 for the character

$$(\chi_1, \chi_2) \colon \mathfrak{X}(G)/D \to \mathbb{R}$$

we have $[(\chi_1, \chi_2)] \in \Sigma^1(\mathfrak{X}(G)/D)$. By (5.1) either $\chi_1 = 0$, $[\chi_2] \in \Sigma^1(G)$ or $\chi_2 = 0$, $[\chi_1] \in \Sigma^1(G)$ or $\chi_1 \neq 0$, $\chi_2 \neq 0$.

Suppose that $\chi_1 = \chi_2$. Note that for the epimorphism $\pi_0 \colon \mathfrak{X}(G)/W \to \mathfrak{X}(G)/L$ we have that $\widehat{\chi} = \chi_1\pi_0$, where we have identified $\mathfrak{X}(G)/L$ with G. Then by Lemma 2.1, since $[\widehat{\chi}] \in \Sigma^1(\mathfrak{X}(G)/W)$, we deduce that $[\chi_1] \in \Sigma^1(G)$. ∎

Proof of Theorem A. It follows immediately from Corollary 5.4 and Lemma 5.5. ∎

Proof of Corollary B2. Suppose that $\pi_1(N)$, $\pi_2(N)$ and $\pi_3(N)$ are all finitely generated. Let $\chi \colon \mathfrak{X}(G) \to \mathbb{R}$ be a character such that $\chi(N) = 0$. We aim to show that $[\chi] \in \Sigma^1(\mathfrak{X}(G))$. Then by Theorem 2.5 we will obtain that N is finitely generated as required.

(1) Suppose that $\chi_1 = 0$. Then $\chi_2 \neq 0$. Since $\chi(N) = 0$, we have $\chi_2(\pi_2(N)) = 0$. By Theorem 2.5, the fact that $\pi_2(N)$ is finitely generated implies that $[\chi_2] \in \Sigma^1(G)$.

(2) Suppose that $\chi_2 = 0$. Then $\chi_1 \neq 0$. Since $\chi(N) = 0$, we have $\chi_1(\pi_1(N)) = 0$. By Theorem 2.5, the fact that $\pi_1(N)$ is finitely generated implies that $[\chi_1] \in \Sigma^1(G)$.

(3) Suppose that $\chi_1 = \chi_2 \neq 0$. Since $\chi(N) = 0$, we have $\chi_1(\pi_3(N)) = 0$. By Theorem 2.5, the fact that $\pi_3(N)$ is finitely generated implies that $[\chi_1] \in \Sigma^1(G)$.

(4) The final case is $\chi_1 \neq 0$, $\chi_2 \neq 0$ and $\chi_1 \neq \chi_2$.

Then by Theorem A in all four cases $[\chi] \in \Sigma^1(\mathfrak{X}(G))$ as required.

Finally, apply the above for $N = \mathfrak{X}(G)'$ to deduce that $\mathfrak{X}(G)'$ is finitely generated if and only if G' is finitely generated. This completes the proof of Corollary B2. ∎

Proof of Corollary C. Note that by Theorem 3.2, for a non-abelian limit group G we have $\Sigma^1(G) = \emptyset$. Then by Theorem A

$$\Sigma^1(\mathfrak{X}(G)) = \{[\chi] \in S(\mathfrak{X}(G)) \mid \chi_1 \neq 0,\ \chi_2 \neq 0,\ \chi_1 \neq \chi_2\}.$$ ∎

6. Some results on $\Sigma^2(\mathfrak{X}(G), \mathbb{Z})$ and $\Sigma^2(\mathfrak{X}(G))$

In this section, we prove results that do not require Theorem 2.7. Note that if G is FP_2 then by [12] $\mathfrak{X}(G)$ is FP_2 too. The last condition is necessary for $\Sigma^2(\mathfrak{X}(G), \mathbb{Z}) \neq \emptyset$, but as we will see from the results in this section it is not sufficient, i.e., there are groups G of type FP_2 such that $\Sigma^2(\mathfrak{X}(G), \mathbb{Z}) = \emptyset$.

Lemma 6.1. *Let H be a group of type* FP_2*, N a normal subgroup of H, $[\chi] \in \Sigma^2(H, \mathbb{Z})$ such that $\chi(N) = 0$ and $\widetilde{\chi}\colon H/N \to \mathbb{R}$ be the character induced by χ. Suppose further that $N/[N, N]$ is finitely generated as a left $\mathbb{Z}(H/N)_{\widetilde{\chi}}$-module, where H/N acts on $N/[N, N]$ via conjugation. Then $[\widetilde{\chi}] \in \Sigma^2(H/N, \mathbb{Z})$.*

Proof. Since $[\chi] \in \Sigma^2(H, \mathbb{Z})$, there is a free resolution

$$\mathcal{P}\colon \cdots \to P_2 \to P_1 \to P_0 = \mathbb{Z}H_\chi \to \mathbb{Z} \to 0$$

of the trivial left $\mathbb{Z}H_\chi$-module $\mathbb{Z}$, where P_1 and P_2 are finitely generated as $\mathbb{Z}H_\chi$-modules. Consider the complex of free $\mathbb{Z}(H/N)_{\widetilde{\chi}}$-modules

$$\mathcal{R} = \mathbb{Z} \otimes_{\mathbb{Z}N} \mathcal{P}\colon \cdots \to R_2 \xrightarrow{d_2} R_1 \xrightarrow{d_1} R_0 = \mathbb{Z}(H/N)_{\widetilde{\chi}} \to \mathbb{Z} \to 0.$$

Note that $\mathcal{R}$ is in general not exact and

$$H_0(\mathcal{R}) = 0 \quad \text{and} \quad H_1(\mathcal{R}) = H_1(N, \mathbb{Z}) = N/[N, N],$$

where the last follows from the fact that $\mathcal{P}$ can be viewed as a free resolution of $\mathbb{Z}N$-modules. Since R_2 is finitely generated as a $\mathbb{Z}(H/N)_{\widetilde{\chi}}$-module, we conclude that $\mathrm{Im}(d_2)$ is finitely generated as a $\mathbb{Z}(H/N)_{\widetilde{\chi}}$-module. This together with the fact that

$$N/[N, N] = H_1(\mathcal{R}) = \mathrm{Ker}(d_1)/\mathrm{Im}(d_2)$$

is finitely generated as a $\mathbb{Z}(H/N)_{\widetilde{\chi}}$-module implies that $\mathrm{Ker}(d_1)$ is finitely generated as a $\mathbb{Z}(H/N)_{\widetilde{\chi}}$-module. Hence $\mathbb{Z}$ is FP_2 as $\mathbb{Z}(H/N)_{\widetilde{\chi}}$-module, i.e., $[\widetilde{\chi}] \in \Sigma^2(H/N, \mathbb{Z})$. ∎

Corollary 6.2. *Let N be a normal subgroup of $\mathfrak{X}(G)$, $[\chi] \in \Sigma^2(\mathfrak{X}(G), \mathbb{Z})$ such that $\chi(N) = 0$ and $\widetilde{\chi}\colon \mathfrak{X}(G)/N \to \mathbb{R}$ be the character induced by χ. Suppose further that $N/[N, N]$ is finitely generated as a left $\mathbb{Z}(\mathfrak{X}(G)/N)_{\widetilde{\chi}}$-module, where $\mathfrak{X}(G)/N$ acts on $N/[N, N]$ via conjugation. Then $[\widetilde{\chi}] \in \Sigma^2(\mathfrak{X}(G)/N, \mathbb{Z})$.*

Proposition 6.3. *Let $[\chi] \in \Sigma^2(\mathfrak{X}(G), \mathbb{Z})$, $\chi_0 = (\chi_1, \chi_2)\colon \mathfrak{X}(G)/D \simeq G \times G \to \mathbb{R}$ and $\widehat{\chi}\colon \mathfrak{X}(G)/W \to \mathbb{R}$ be the characters induced by χ. Then the following conditions hold:*

1. *if $\chi_1 \neq \chi_2$ then $[\chi_0] \in \Sigma^2(\mathfrak{X}(G)/D, \mathbb{Z})$;*
2. *if $\chi_1 \neq \chi_2$ then $[\widehat{\chi}] \in \Sigma^2(\mathfrak{X}(G)/W, \mathbb{Z})$;*
3. *if $\chi_1 = \chi_2$ then $\chi(L) = 0$ and for the character $\widetilde{\chi}\colon \mathfrak{X}(G)/L \to \mathbb{R}$ induced by χ we have $[\widetilde{\chi}] \in \Sigma^2(\mathfrak{X}(G)/L, \mathbb{Z})$. Identifying $\mathfrak{X}(G)/L$ with G and $\widetilde{\chi}$ with χ_1 we get $[\chi_1] \in \Sigma^2(G, \mathbb{Z})$.*

Proof. Note that the condition $\chi_1 \neq \chi_2$ is equivalent to $\chi(L) \neq 0$. Since $\Sigma^2(\mathfrak{X}(G), \mathbb{Z}) \neq \emptyset$, we deduce that $\mathfrak{X}(G)$ is FP_2, hence its retract G is FP_2.

(1) By Corollary 6.2 applied for $N = D$, it remains to prove that $D/[D, D]$ is finitely generated as a $\mathbb{Z}(\mathfrak{X}(G)/D)_{\chi_0}$-module. The fact that G is FP_2 implies that $\mathfrak{X}(G)/D \simeq G \times G$ is FP_2, and so any relation module of $G \times G$ (with respect to a finite generating set of $G \times G$) is finitely generated as a $G \times G$-module. Hence any quotient of a relation module of $G \times G$ (with respect to a finite generating set of $G \times G$) is finitely generated as a $G \times G$-module; in particular, $D/[D, D]$ is finitely generated as a $\mathfrak{X}(G)$-module (via conjugation). Since $\chi(L) \neq 0$, we have that $\mathfrak{X}(G) = \mathfrak{X}(G)_\chi L$. This combined with the fact that L and D act trivially (via conjugation) on $D/[D, D]$ implies that $D/[D, D]$ is finitely generated as a $\mathbb{Z}(\mathfrak{X}(G)_\chi/D)$-module. Finally, note that $\mathfrak{X}(G)_\chi/D = (\mathfrak{X}(G)/D)_{\chi_0}$.

(2) Note that $\mathfrak{X}(G)/W$ is a subdirect product of $G \times G \times G$ that maps surjectively on pairs. Thus since G is FP_2, we can apply Theorem 3.4 to deduce that $\mathfrak{X}(G)/W$ is FP_2, hence $W/[W, W] = W$ is finitely generated as a $\mathfrak{X}(G)$-module via conjugation. Since $\chi(L) \neq 0$, we can use $\mathfrak{X}(G) = \mathfrak{X}(G)_\chi L$ and the fact that L acts trivially on W via conjugation to deduce that W is finitely generated as a $\mathbb{Z}(\mathfrak{X}(G)_\chi/W)$-module. Finally, note that $\mathfrak{X}(G)_\chi/W = (\mathfrak{X}(G)/W)_{\widetilde{\chi}}$.

(3) Suppose now that $\chi(L) = 0$. This is equivalent to $\chi_1 = \chi_2$. Consider the decomposition $\mathfrak{X}(G) = L \rtimes G$. Then the character χ induces a character $\widetilde{\chi}\colon \mathfrak{X}(G)/L \to \mathbb{R}$ that, after identifying $\mathfrak{X}(G)/L$ with G, is the character χ_1. By Corollary 6.2, to show that $[\chi_1] \in \Sigma^2(G, \mathbb{Z})$, it suffices to show that $L/[L, L]$ is finitely generated as a $\mathbb{Z}(\mathfrak{X}(G)/L)_{\widetilde{\chi}}$-module. As observed before, Bridson and Kochloukova showed in [12] that L is finitely generated whenever G is finitely generated. The fact that $L/[L, L]$ is finitely generated for a finitely generated group G was proved earlier by Lima and Oliveira in [30]. ■

Lemma 6.4. *Let H be a finitely presented group, N a normal subgroup of H, $[\chi] \in \Sigma^2(H)$ such that $\chi(N) = 0$ and $\widetilde{\chi}\colon H/N \to \mathbb{R}$ be the character induced by χ. Suppose further that N is finitely generated as a left $H_{\widetilde{\chi}}$-group, where H acts on N via conjugation. Then $[\widetilde{\chi}] \in \Sigma^2(H/N)$.*

Proof. Since $[\chi] \in \Sigma^2(H)$, there is a finite presentation $H = \langle X \mid R \rangle$ such that for the Cayley complex Γ associated to this presentation and its subcomplex Γ_χ spanned by the vertices $H_\chi = \{h \in H \mid \chi(h) \geq 0\}$ we have that Γ_χ is 1-connected. The free left H-action on Γ induces an N-action on Γ_χ and thus we have a covering map

$$p\colon \Gamma_\chi \to \Gamma_\chi/N.$$

Since $\pi_1(\Gamma_\chi) = 1$, we have that

$$N \simeq \pi_1(\Gamma_\chi/N).$$

Since N is finitely generated as a left $H_{\widetilde{\chi}}$-group, there are elements $b_1, \ldots, b_m \in N$ such that $N = \langle {}^{H_\chi}b_1, \ldots, {}^{H_\chi}b_m \rangle$.

Consider the finite presentation $H/N = \langle X \mid R, b_1, \ldots, b_m \rangle$. The Cayley complex $\widehat{\Gamma}$ associated to this presentation is obtained from Γ changing vertex set H to H/N and gluing at each vertex extra 2-cells whose boundaries are closed paths $\gamma_1, \ldots, \gamma_m$ with labels that correspond to $b_1, \ldots, b_m$. Then there is a non-positive real number d such that $\gamma_1, \ldots, \gamma_m$ are closed paths homotopic to a point in $\widehat{\Gamma}_{\widetilde{\chi} \geq d}$, where $\widehat{\Gamma}_{\widetilde{\chi} \geq d}$ is the subcomplex of $\widehat{\Gamma}$ spanned by the vertices in $\{g \in H/N \mid \widetilde{\chi}(g) \geq d\}$. Thus $\widehat{\Gamma}_{\widetilde{\chi} \geq 0}$ is $\widehat{\Gamma}_{\widetilde{\chi}}$.

The fact that $\pi_1(\Gamma_\chi/N) \simeq N = \langle {}^{H_\chi} b_1, \ldots, {}^{H_\chi} b_m \rangle$ implies that the inclusion of spaces $\widehat{\Gamma}_{\widetilde{\chi}} \subseteq \widehat{\Gamma}_{\widetilde{\chi} \geq d}$ induces the trivial map $\pi_1(\widehat{\Gamma}_{\widetilde{\chi}}) \to \pi_1(\widehat{\Gamma}_{\widetilde{\chi} \geq d})$. This is one of the definitions of Σ^2, hence $[\widetilde{\chi}] \in \Sigma^2(H/N)$.

Alternatively, we can assume from the very beginning that the fixed generating set X contains a finite fixed subset of H. In particular, we can assume that X contains the set $\{b_1, \ldots, b_m\}$. This guarantees that $\widehat{\Gamma}_{\widetilde{\chi}}$ is 1-connected. ∎

Corollary 6.5. *Let G be a finitely presented group and let N be a normal subgroup of $\mathfrak{X}(G)$, $[\chi] \in \Sigma^2(\mathfrak{X}(G))$ such that $\chi(N) = 0$ and $\widetilde{\chi}\colon \mathfrak{X}(G)/N \to \mathbb{R}$ be the character induced by χ. Suppose further that N is finitely generated as a left $\mathfrak{X}(G)_{\widetilde{\chi}}$-group, where $\mathfrak{X}(G)$ acts (on the left) on N via conjugation. Then $[\widetilde{\chi}] \in \Sigma^2(\mathfrak{X}(G)/N)$.*

Proposition 6.6. *Let G be a finitely presented group, $[\chi] \in \Sigma^2(\mathfrak{X}(G))$,*

$$\chi_0 = (\chi_1, \chi_2)\colon \mathfrak{X}(G)/D \simeq G \times G \to \mathbb{R}$$

and $\widehat{\chi}\colon \mathfrak{X}(G)/W \to \mathbb{R}$ be the characters induced by χ. Then the following conditions hold:

(1) *if $\chi_1 \neq \chi_2$ then $[\chi_0] \in \Sigma^2(\mathfrak{X}(G)/D)$;*

(2) *if $\chi_1 \neq \chi_2$ then $[\widehat{\chi}] \in \Sigma^2(\mathfrak{X}(G)/W)$;*

(3) *if $\chi_1 = \chi_2$ then $\chi(L) = 0$, and for the character $\widetilde{\chi}\colon \mathfrak{X}(G)/L \to \mathbb{R}$ induced by χ we have $[\widetilde{\chi}] \in \Sigma^2(\mathfrak{X}(G)/L)$. Identifying $\mathfrak{X}(G)/L$ with G and $\widetilde{\chi}$ with χ_1, we get $[\chi_1] \in \Sigma^2(G)$.*

Proof. By Theorem 4.2, since G is finitely presented, $\mathfrak{X}(G)$ is finitely presented.

(1) By Corollary 6.2 applied for $N = D$, it remains to prove that D is finitely generated as a $\mathfrak{X}(G)_{\chi_0}$-group where $\mathfrak{X}(G)_{\chi_0}$ acts (on the left) via conjugation. The fact that G is finitely presented implies that $\mathfrak{X}(G)/D \simeq G \times G$ is finitely presented. Hence D is finitely generated as a normal subgroup of $\mathfrak{X}(G)$, i.e., is finitely generated as a $\mathfrak{X}(G)$-group, where $\mathfrak{X}(G)$ acts (on the left) via conjugation. Since $\chi(L) \neq 0$, we have that $\mathfrak{X}(G) = \mathfrak{X}(G)_\chi L$. This combined with the fact that $[L, D] = 1$ implies that L acts trivially on D via conjugation, hence the $\mathfrak{X}(G)$ action on D via conjugation factors through an action of $\mathfrak{X}(G)_\chi$.

(2) Note that $\mathfrak{X}(G)/W$ is a subdirect product of $G \times G \times G$ that maps surjectively on pairs. Since by [10] the virtual surjection conjecture holds for $n = 2$, we deduce that $\mathfrak{X}(G)/W$ is finitely presented, hence W is finitely generated as a normal subgroup of $\mathfrak{X}(G)$. Since W is abelian, this is equivalent to W being finitely generated as a left $\mathbb{Z}\mathfrak{X}(G)$-module via conjugation. Since $\chi(L) \neq 0$, we can use $\mathfrak{X}(G) = \mathfrak{X}(G)_\chi L$ and the

fact that L acts trivially on W via conjugation (since $W = L \cap D$ and $[L, D] = 1$) to deduce that W is finitely generated as a left $\mathbb{Z}\mathfrak{X}(G)_\chi$-module.

(3) Suppose now that $\chi(L) = 0$. This is equivalent to $\chi_1 = \chi_2$. Consider the decomposition $\mathfrak{X}(G) = L \rtimes G$. Then the character χ induces a character $\widetilde{\chi}\colon \mathfrak{X}(G)/L \to \mathbb{R}$ that after identifying $\mathfrak{X}(G)/L$ with G is the character χ_1. By Corollary 6.5, to show that $[\chi_1] \in \Sigma^2(G)$, it suffices to show that L is finitely generated as a left $\mathfrak{X}(G)_{\widetilde{\chi}}$-group. As observed before, Bridson and Kochloukova showed in [12] that L is finitely generated as a group whenever G is finitely generated. ∎

Proof of Proposition D. Suppose that $[\chi] \in \Sigma^2(\mathfrak{X}(G), \mathbb{Z})$. Then, by Proposition 6.3, for the induced character

$$\chi_0 = (\chi_1, \chi_2)\colon \mathfrak{X}(G)/D \simeq G \times G \to \mathbb{R}$$

either $[\chi_0] \in \Sigma^2(G \times G, \mathbb{Z})$ or $\chi_1 = \chi_2$ and $[\chi_1] \in \Sigma^2(G, \mathbb{Z})$. Since

$$\Sigma^2(G, \mathbb{Z}) \subseteq \Sigma^1(G, \mathbb{Z}) = \Sigma^1(G)$$

we have that $\Sigma^2(G, \mathbb{Z})$ is empty if $\Sigma^1(G)$ is empty. Note that by Theorem 2.4 (i.e., the direct product formula holds in dimension 2), $\Sigma^2(G \times G, \mathbb{Z})$ is empty if $\Sigma^1(G)$ is empty, a contradiction with the existence of χ. ∎

Theorem 6.7 ([33, Corollary 4.2]). *Let H be a finitely presented group, N a normal subgroup of H that is finitely presented, $\mu\colon H/N \to \mathbb{R}$ a character and $\pi\colon H \to H/N$ be the canonical epimorphism. Then $[\mu] \in \Sigma^2(H/N)$ if and only if $[\mu\pi] \in \Sigma^2(H)$.*

We will need the homological version of the above result.

Theorem 6.8. *Suppose N is a normal subgroup of H such that both N and H are* FP_2*, $\chi\colon H \to \mathbb{R}$ is a character such that $\chi(N) = 0$, $\widetilde{\chi}\colon H/N \to \mathbb{R}$ is the character induced by χ and $[\widetilde{\chi}] \in \Sigma^2(H/N, \mathbb{Z})$. Then $[\chi] \in \Sigma^2(H, \mathbb{Z})$.*

Proof. Consider the short exact sequence of groups

$$1 \to N \to H \to H/N \to 1$$

and the induced short exact sequence of monoids

$$1 \to N \to H_\chi \to (H/N)_{\widetilde{\chi}} \to 1.$$

This induces a LHS spectral sequence

$$E^2_{p,q} = H_p((H/N)_{\widetilde{\chi}}, H_q(N, V)) = \mathrm{Tor}_p^{\mathbb{Z}(H/N)_{\widetilde{\chi}}}(\mathbb{Z}, \mathrm{Tor}_q^{\mathbb{Z}N}(\mathbb{Z}, V))$$

that converges to $H_{p+q}(H_\chi, V) = \mathrm{Tor}_{p+q}^{\mathbb{Z}H_\chi}(\mathbb{Z}, V)$ for a fixed $\mathbb{Z}H_\chi$-module V. We set $V = \prod \mathbb{Z}H_\chi$.

Note that $[\widetilde{\chi}] \in \Sigma^2(H/N, \mathbb{Z})$ is equivalent to $\mathrm{Tor}_i^{(H/N)_{\widetilde{\chi}}}(\mathbb{Z}, -)$ commutes with direct products for $i = 0$ and $i = 1$. Note that this follows from Bieri's criterion [3, p. 12, Theorem 1.3, iiia)']. By the same argument, the condition $[\chi] \in \Sigma^2(H, \mathbb{Z})$ is equivalent to $\mathrm{Tor}_i^{\mathbb{Z}H_\chi}(\mathbb{Z}, \prod \mathbb{Z}H_\chi) = 0$ for $i = 1$ and $\mathrm{Tor}_0^{\mathbb{Z}H_\chi}(\mathbb{Z}, \prod \mathbb{Z}H_\chi) \simeq \prod \mathrm{Tor}_0^{\mathbb{Z}H_\chi}(\mathbb{Z}, \mathbb{Z}H_\chi)$.

(1) Note that

$$E^2_{1,0} = \mathrm{Tor}_1^{\mathbb{Z}(H/N)_{\widetilde{\chi}}}\Big(\mathbb{Z}, \mathrm{Tor}_0^{\mathbb{Z}N}\Big(\mathbb{Z}, \prod \mathbb{Z}H_\chi\Big)\Big).$$

Since N is finitely generated, we deduce that $\mathrm{Tor}_0^{\mathbb{Z}N}(\mathbb{Z}, -)$ commutes with direct products. Thus

$$\begin{aligned} H_0\Big(N, \prod \mathbb{Z}H_\chi\Big) &= \mathrm{Tor}_0^{\mathbb{Z}N}\Big(\mathbb{Z}, \prod \mathbb{Z}H_\chi\Big) \simeq \prod \mathrm{Tor}_0^{\mathbb{Z}N}(\mathbb{Z}, \mathbb{Z}H_\chi) \\ &\simeq \prod (\mathbb{Z}H_\chi/N) \simeq \prod \mathbb{Z}(H/N)_{\widetilde{\chi}}. \end{aligned}$$

Since $[\widetilde{\chi}] \in \Sigma^2((H/N)_{\widetilde{\chi}}, \mathbb{Z})$, we deduce that

$$\mathrm{Tor}_1^{\mathbb{Z}(H/N)_{\widetilde{\chi}}}\Big(\mathbb{Z}, \prod \mathbb{Z}(H/N)_{\widetilde{\chi}}\Big) \simeq \prod \mathrm{Tor}_1^{\mathbb{Z}(H/N)_{\widetilde{\chi}}}(\mathbb{Z}, \mathbb{Z}(H/N)_{\widetilde{\chi}}) = \prod 0 = 0.$$

Combining the above equalities we deduce that

$$E^2_{1,0} = 0, \quad \text{hence } E^\infty_{1,0} = 0.$$

Now

$$E^2_{0,1} = \mathrm{Tor}_0^{\mathbb{Z}(H/N)_{\widetilde{\chi}}}\Big(\mathbb{Z}, \mathrm{Tor}_1^{\mathbb{Z}N}\Big(\mathbb{Z}, \prod \mathbb{Z}H_\chi\Big)\Big).$$

Note that N is FP_2, hence $\mathrm{Tor}_1^{\mathbb{Z}N}(\mathbb{Z}, -)$ commutes with direct products. Thus

$$\mathrm{Tor}_1^{\mathbb{Z}N}\Big(\mathbb{Z}, \prod \mathbb{Z}H_\chi\Big) \simeq \prod \mathrm{Tor}_1^{\mathbb{Z}N}(\mathbb{Z}, \mathbb{Z}H_\chi) \simeq \prod 0 = 0,$$

where we have used that $\mathbb{Z}H_\chi$ is a free $\mathbb{Z}N$-module and so $\mathrm{Tor}_1^{\mathbb{Z}N}(\mathbb{Z}, \mathbb{Z}H_\chi) = 0$. Hence

$$E^2_{0,1} = \mathrm{Tor}_0^{\mathbb{Z}(H/N)_{\widetilde{\chi}}}(\mathbb{Z}, 0) = 0.$$

Then $E^\infty_{0,1} = E^\infty_{1,0} = 0$. Finally, the convergence of the spectral sequence gives a short exact sequence of abelian groups

$$0 \to E^\infty_{0,1} \to H_1\Big(H_\chi, \prod \mathbb{Z}H_\chi\Big) \to E^\infty_{1,0} \to 0,$$

hence

$$H_1\Big(H_\chi, \prod \mathbb{Z}H_\chi\Big) = 0.$$

(2) It remains to show that $\mathrm{Tor}_0^{\mathbb{Z}H_\chi}(\mathbb{Z}, \prod \mathbb{Z}H_\chi) \simeq \prod \mathrm{Tor}_0^{\mathbb{Z}H_\chi}(\mathbb{Z}, \mathbb{Z}H_\chi)$. This is equivalent to $[\chi] \in \Sigma^1(H)$ and follows from Lemma 5.3. ∎

Corollary 6.9. *Suppose N is a normal subgroup of $\mathfrak{X}(G)$ such that N and G are* FP_2*, $\chi\colon \mathfrak{X}(G) \to \mathbb{R}$ is a character such that $\chi(N) = 0$, $\widetilde{\chi}\colon \mathfrak{X}(G)/N \to \mathbb{R}$ is the character induced by χ and $[\widetilde{\chi}] \in \Sigma^2(\mathfrak{X}(G)/N, \mathbb{Z})$. Then $[\chi] \in \Sigma^2(\mathfrak{X}(G), \mathbb{Z})$.*

Corollary 6.10. *Suppose $\chi\colon \mathfrak{X}(G) \to \mathbb{R}$ is a character such that $\chi(L) = 0$ and $\chi_1 = \chi\,|_G$. Suppose that L is* FP_2 *and that $[\chi_1] \in \Sigma^2(G, \mathbb{Z})$. Then $[\chi] \in \Sigma^2(\mathfrak{X}(G), \mathbb{Z})$.*

Proof. Since $\Sigma^2(G, \mathbb{Z}) \neq \emptyset$, we have that G is FP_2, hence by Theorem 4.2 $\mathfrak{X}(G)$ is FP_2. We apply Corollary 6.9 for $N = L$ and identify $\mathfrak{X}(G)/L$ with G. Note that the character $\widehat{\chi}\colon \mathfrak{X}(G)/L \to \mathbb{R}$ under the identification of $\mathfrak{X}(G)/L$ with G is identified with χ_1. ∎

7. Proofs of Theorems E1, E2, F1 and F2, and Corollaries G and H

Let $H = G \times G \times G$ and $K = \mathrm{Im}(\rho)$. Recall that

$$\mathrm{Im}(\rho) = \{(g_1, g_2, g_3) \mid g_1 g_2^{-1} g_3 \in G'\},$$

hence

$$[\mathrm{Im}(\rho), \mathrm{Im}(\rho)] = [H, H].$$

Proof of Theorem E1. Suppose that

$$\mu = (\mu_1, \mu_2, \mu_3)\colon H = G \times G \times G \to \mathbb{R}$$

is a character extending χ. By Theorem 2.7, $[\chi] \in \Sigma^2(K, \mathbb{Z})$ if and only if for every character μ extending χ we have $[\mu] \in \Sigma^2(H, \mathbb{Z})$. Note that in dimension 2 the Σ direct product formula holds, see Theorem 2.4, hence $[\mu] \in \Sigma^2(H, \mathbb{Z})$ precisely if one of the following cases holds for the characters μ_1, μ_2, μ_3:

(a) Two characters from $\{\mu_1, \mu_2, \mu_3\}$ are 0, and the third corresponds to an element of $\Sigma^2(G, \mathbb{Z})$.

(b) One character from $\{\mu_1, \mu_2, \mu_3\}$ is 0, and the other two are non-zero and at least one corresponds to an element of $\Sigma^1(G)$.

(c) The three characters μ_1, μ_2, μ_3 are non-zero.

Note that, since μ is an extension of χ, we have

$$\chi((g_1, g_1 g_3, g_3)) = \mu_1(g_1) + \mu_2(g_1 g_3) + \mu_3(g_3).$$

Hence

$$\chi_1(g_1) = \chi((g_1, g_1, 1)) = \mu_1(g_1) + \mu_2(g_1), \quad \text{i.e., } \chi_1 = \mu_1 + \mu_2 \tag{7.1}$$

and

$$\chi_2(g_3) = \chi((1, g_3, g_3)) = \mu_2(g_3) + \mu_3(g_3), \quad \text{i.e., } \chi_2 = \mu_2 + \mu_3. \tag{7.2}$$

Then $\mu_1 = \chi_1 - \mu_2$ and $\mu_3 = \chi_2 - \mu_2$. Suppose that for each μ that extends χ we have $[\mu] \in \Sigma^2(H, \mathbb{Z})$. If $\mu_1 \neq 0$, $\mu_2 \neq 0$ and $\mu_3 \neq 0$, we are in case (c) and $[\mu] \in \Sigma^2(H, \mathbb{Z})$. There are several more cases to consider.

(a1) Assume that $\mu_1 = 0 = \mu_2$. Then by (7.1) and (7.2), $\mu_3 = \chi_2$, $\chi_1 = 0$ and $[\mu] \in \Sigma^2(H, \mathbb{Z})$ is equivalent to $[\mu_3] \in \Sigma^2(G, \mathbb{Z})$. Thus

$$\chi_1 = 0, \quad [\chi_2] \in \Sigma^2(G, \mathbb{Z}).$$

(a2) Assume that $\mu_1 = 0 = \mu_3$. Then by (7.1) and (7.2), $\chi_1 = \mu_2 = \chi_2$ and $[\mu] \in \Sigma^2(H, \mathbb{Z})$ is equivalent to $[\mu_2] \in \Sigma^2(G, \mathbb{Z})$. Thus

$$\chi_1 = \chi_2, \quad [\chi_1] \in \Sigma^2(G, \mathbb{Z}).$$

(a3) Assume that $\mu_2 = 0 = \mu_3$. Then by (7.1) and (7.2) $\chi_2 = 0$, $\mu_1 = \chi_1$ and $[\mu] \in \Sigma^2(H, \mathbb{Z})$ is equivalent to $[\mu_1] \in \Sigma^2(G, \mathbb{Z})$. Thus

$$\chi_2 = 0, \quad [\chi_1] \in \Sigma^2(G, \mathbb{Z}).$$

(b1) Assume that $\mu_1 = 0, \mu_2 \neq 0, \mu_3 \neq 0$. Then by (7.1) and (7.2), $\mu_2 = \chi_1$, $\mu_3 = \chi_2 - \mu_2 = \chi_2 - \chi_1$. Then $[\mu] \in \Sigma^2(H, \mathbb{Z})$ is equivalent to the condition that both μ_2 and μ_3 are non-zero and at least one represents an element from $\Sigma^1(G)$, i.e., $\chi_1 \neq 0$, $\chi_2 \neq \chi_1$ and at least one of the elements of $\{[\chi_1], [\chi_2 - \chi_1]\}$ belongs to $\Sigma^1(G)$.

(b2) Assume that $\mu_3 = 0, \mu_2 \neq 0, \mu_1 \neq 0$. Then by (7.1) and (7.2), $\mu_2 = \chi_2$, $\mu_1 = \chi_1 - \mu_2 = \chi_1 - \chi_2$. Then $[\mu] \in \Sigma^2(H, \mathbb{Z})$ is equivalent to the condition that both μ_1 and μ_2 are non-zero and at least one represents an element from $\Sigma^1(G)$, i.e., $\chi_2 \neq 0$, $\chi_1 \neq \chi_2$ and at least one of the elements of $\{[\chi_2], [\chi_1 - \chi_2]$ belongs to $\Sigma^1(G)$.

(b3) Assume that $\mu_2 = 0, \mu_1 \neq 0, \mu_3 \neq 0$. Then by (7.1) and (7.2), $\mu_1 = \chi_1$, $\mu_3 = \chi_2$. Then $[\mu] \in \Sigma^2(H, \mathbb{Z})$ is equivalent to the condition that both μ_1 and μ_3 are non-zero and at least one represents an element from $\Sigma^1(G)$, i.e., $\chi_2 \neq 0$, $\chi_1 \neq 0$ and at least one of the elements of$\{[\chi_1], [\chi_2]\}$ belongs to $\Sigma^1(G)$.

As a corollary of (b1), (b2) and (b3), if $\chi_1 \neq 0$, $\chi_2 \neq 0$ and $\chi_1 \neq \chi_2$, one of the following conditions should hold:

$$\{[\chi_1], [\chi_2]\} \subseteq \Sigma^1(G) \text{ or } \{[\chi_1], [\chi_1 - \chi_2]\} \subseteq \Sigma^1(G) \text{ or } \{[\chi_2], [\chi_2 - \chi_1]\} \subseteq \Sigma^1(G).$$

This completes the proof of Theorem E1. ∎

We note that the proof of Theorem E2 is similar to the proof of Theorem E1, since we can use the homotopical part of Theorem 2.7. We can obtain the proof of Theorem E2 from the proof of Theorem E1 by substituting $\Sigma^2(G, \mathbb{Z})$ with $\Sigma^2(G)$.

Proof of Theorem F1. (a) Assume now that $[\chi] \in \Sigma^2(\mathfrak{X}(G), \mathbb{Z})$. By part 2 of Proposition 6.3, if $\chi(L) \neq 0$, i.e., $\chi_1 \neq \chi_2$, we deduce that $[\hat{\chi}] \in \Sigma^2(\mathfrak{X}(G)/W, \mathbb{Z})$.

By Proposition 6.3, when $\chi_1 = \chi_2$ we have $[\chi_1] \in \Sigma^2(G, \mathbb{Z})$. Then by Theorem E1, $[\hat{\chi}] \in \Sigma^2(\mathfrak{X}(G)/W, \mathbb{Z})$.

(b) Note that by Theorem 4.1, if G is FP_2 and the abelianization of the commutator group $[G, G]$ is finitely generated, then $W(G)$ is finitely generated as abelian group.

Note that by Corollary 6.9 for $N = W$ when W is finitely presented (in our case it is a finitely generated abelian group) and $[\hat{\chi}] \in \Sigma^2(\mathfrak{X}(G)/W, \mathbb{Z})$, we can deduce that $[\chi] \in \Sigma^2(\mathfrak{X}(G), \mathbb{Z})$. ∎

Proof of Theorem F2. (a) Assume now that $[\chi] \in \Sigma^2(\mathfrak{X}(G))$. By part 2 of Proposition 6.6, if $\chi(L) \neq 0$, i.e., $\chi_1 \neq \chi_2$, we deduce that $[\hat{\chi}] \in \Sigma^2(\mathfrak{X}(G)/W)$.

By Proposition 6.6, when $\chi_1 = \chi_2$ we have $[\chi_1] \in \Sigma^2(G)$. Then, by Theorem E2, $[\hat{\chi}] \in \Sigma^2(\mathfrak{X}(G)/W)$.

(b) Note that by Corollary 6.9 for $N = W$ when W is finitely presented and $[\hat{\chi}] \in \Sigma^2(\mathfrak{X}(G)/W)$, we can deduce that $[\chi] \in \Sigma^2(\mathfrak{X}(G))$. ∎

Proof of Corollary G. (a) By Theorem A, $[\chi] \in \Sigma^1(\mathfrak{X}(G))^c$ if and only if one of the following conditions holds:

(1) $\chi_2 = 0, [\chi_1] \in \Sigma^1(G)^c$;

(2) $\chi_1 = 0, [\chi_2] \in \Sigma^1(G)^c$;

(3) $\chi_1 = \chi_2 \neq 0$ and $[\chi_1] \in \Sigma^1(G)^c$,

where $\chi_1, \chi_2 \colon G \to \mathbb{R}$ are characters defined by $\chi_1(g) = \chi(g)$ and $\chi_2(g) = \chi(\bar{g})$.

By the definition of V_i, we have that $[\chi] \in V_i$ if and only if χ satisfies the i-th condition. Hence

$$\Sigma^1(G)^c = V_1 \cup V_2 \cup V_3.$$

(b) Identifying $S(\mathfrak{X}(G))$ with $S(\mathfrak{X}(G)/W(G))$ via the projection map

$$\mathfrak{X}(G) \to \mathfrak{X}(G)/W(G)$$

and by Theorem F2, we have $\Sigma^2(\mathfrak{X}(G)) \subseteq \Sigma^2(\mathfrak{X}(G)/W(G))$. Hence

$$\Sigma^2(\mathfrak{X}(G)/W(G))^c \subseteq \Sigma^2(\mathfrak{X}(G))^c.$$

By Theorem E2, $[\chi] \in \Sigma^2(\mathfrak{X}(G)/W(G))^c$ if and only if one of the following conditions holds:

(1) $\chi_2 = 0, [\chi_1] \in \Sigma^2(G)^c$;

(2) $\chi_1 = 0, [\chi_2] \in \Sigma^2(G)^c$;

(3) $\chi_1 = \chi_2 \neq 0$ and $[\chi_1] \in \Sigma^2(G)^c$;

(4) $\chi_1 \neq 0, \chi_2 \neq 0, \chi_1 \neq \chi_2$ and one of the following holds:

(4a) $\{[\chi_1], [\chi_2]\} \subseteq \Sigma^1(G)^c$;

(4b) $\{[\chi_1], [\chi_2 - \chi_1]\} \subseteq \Sigma^1(G)^c$;

(4c) $\{[\chi_2], [\chi_1 - \chi_2]\} \subseteq \Sigma^1(G)^c$.

By the definition of W_i, we have that $[\chi] \in W_i$ if and only if χ satisfies the i-th condition for $1 \leq i \leq 3$.

In case (4a), $[\chi] = [(\chi_1, \chi_2)] = [(\chi_1, 0)] + [(0, \chi_2)]$ is a typical element of $V_1 + V_2$.

In case (4b), $[\chi] = [(\chi_1, \chi_2)] = [(0, \chi_2 - \chi_1)] + [(\chi_1, \chi_1)]$ is a typical element of $V_2 + V_3$.

In case (4c), $[\chi] = [(\chi_1, \chi_2)] = [(\chi_2, \chi_2)] + [(\chi_1 - \chi_2, 0)]$ is a typical element of $V_3 + V_1$.

(c) The proof is the obvious homological modification of (b). ■

Proof of Corollary H. Let $\mu: \mathfrak{X}(G)/W(G) \to \mathbb{R}$ be a character that vanishes on $N/W(G)$. We define a character $\chi: \mathfrak{X}(G) \to \mathbb{R}$ as the composition of the canonical projection $\mathfrak{X}(G) \to \mathfrak{X}(G)/W(G)$ with μ. Thus $\mu = \widehat{\chi}$ and since $\chi(N) = 0$ and N is FP_2 (resp. finitely presented) by Theorem 2.5, $[\chi] \in \Sigma^2(\mathfrak{X}(G), \mathbb{Z})$ (resp. $[\chi] \in \Sigma^2(\mathfrak{X}(G))$). Then by Theorem F1, $[\widehat{\chi}] \in \Sigma^2(\mathfrak{X}(G)/W(G), \mathbb{Z})$ (resp. by Theorem F2, $[\widehat{\chi}] \in \Sigma^2(\mathfrak{X}(G)/W(G))$), hence by Theorem 2.5 again $N/W(G)$ is FP_2 (resp. finitely presented).

Let us consider the case $N = \mathfrak{X}(G)'$. Note that $\mathfrak{X}(G) = L \rtimes G$, hence $\mathfrak{X}(G)' = \langle L', [L, G]\rangle \rtimes G'$. Suppose that $\mathfrak{X}(G)'$ is FP_m for some $m \geq 1$ (resp. finitely presented). Since the property FP_m (resp. finitely presented) passes to retracts, we deduce that G' is FP_m (resp. finitely presented).

For the converse, assume that G' is FP_m for some $m \geq 1$ (resp. finitely presented). Since FP_1 is equivalent to finite generation, we have that G' is finitely generated and by Theorem 4.1 $W(G)$ is finitely generated. Then

$$\mathfrak{X}(G)'/W = \mathrm{Im}(\rho)' = G' \times G' \times G' \quad \text{is } \mathrm{FP}_m \text{ (resp. finitely presented).}$$

Since W is abelian and finitely generated, it is of type FP_m and finitely presented. This implies that $\mathfrak{X}(G)'$ is FP_m (resp. finitely presented) as claimed. ■

Remark 7.1. Though it is tempting to study the structure of $\Sigma^n(\mathfrak{X}(G), \mathbb{Z})$ and $\Sigma^n(\mathfrak{X}(G))$ for $n \geq 3$, there are some structural problems. Firstly, we do not have a criterion to know when $\mathfrak{X}(G)$ is of type FP_3 and by [12], even for nice groups such as finite rank free non-cyclic groups G, the group $\mathfrak{X}(G)$ is not FP_3. Secondly, if we want to find a higher-dimensional version of Theorems E1 and E2, it is natural to apply Theorem 2.7 for $H = G \times G \times G$ and $K = \mathrm{Im}(\rho) \simeq \mathfrak{X}(G)/W(G)$. But though there is a direct product Σ-formula that holds for the homological Σ-invariants with coefficients in a field [4], a similar formula does not hold for $\Sigma^n(-, \mathbb{Z})$ for $n \geq 4$ [36] or for $\Sigma^n(-)$ for $n \geq 3$ [31].

8. On the finite presentability of the non-abelian tensor square and on the Σ-invariants of $\nu(G)$

Let G be a group. In [35], Rocco defined a group given by the following presentation

$$\nu(G) = \left\langle G, \overline{G} \mid [g_1, \overline{g}_2]^{g_3} = [g_1^{g_3}, \overline{g_2^{g_3}}] = [g_1, \overline{g}_2]^{\overline{g}_3} \right\rangle,$$

where $\overline{G}$ is an isomorphic copy of G. By [35], for the non-abelian tensor square $G \otimes G$, we have an isomorphism

$$G \otimes G \simeq [G, \overline{G}],$$

where $[G, \overline{G}]$ is the subgroup of $\nu(G)$ generated by $\{[g_1, \overline{g}_2] \,|g_1 \in G, \overline{g}_2 \in \overline{G}\}$. Furthermore, by [35], there is a subgroup $\Delta \subseteq \nu(G)' \cap Z(\nu(G))$ such that

$$\nu(G)/\Delta \simeq \mathfrak{X}(G)/R, \tag{8.1}$$

with an isomorphism that is an identity on $G \cup \overline{G}$, R is a special normal subgroup of $\mathfrak{X}(G)$ that is contained in $W = W(G)$ and $W/R \simeq H_2(G, \mathbb{Z})$. Thus Δ is a quotient of $H_2(\nu(G)/\Delta, \mathbb{Z}) \simeq H_2(\mathfrak{X}(G)/R, \mathbb{Z})$.

Lemma 8.1. *Let G be a group of type* FP_2. *Then Δ is finitely generated and there is a normal subgroup W_0 in $\nu(G)$ such that W_0 is finitely generated nilpotent of class at most* 2 *and*

$$\nu(G)/W_0 \simeq \mathfrak{X}(G)/W. \tag{8.2}$$

Proof. Let W_0 be the normal subgroup of $\nu(G)$ such that W_0/Δ is the preimage of W/R in $\nu(G)/\Delta$ under isomorphism (8.1). Then

$$W_0/\Delta \simeq W/R \simeq H_2(G, \mathbb{Z})$$

and Δ is a quotient of $H_2(\mathfrak{X}(G)/R, \mathbb{Z})$. Since G is FP_2, then $H_2(G, \mathbb{Z})$ is finitely generated, hence $W/R \simeq H_2(G, \mathbb{Z})$ is a finitely generated abelian group, hence it is finitely presented and so is FP_2. Furthermore, when G is FP_2, by [28, Theorem D], $\mathfrak{X}(G)/W$ is FP_2. Since the property FP_2 is extension closed [3, Exercise, p. 23], we deduce that $\mathfrak{X}(G)/R$ is FP_2, hence $H_2(\mathfrak{X}(G)/R, \mathbb{Z})$ is finitely generated. Then its quotient Δ is a finitely generated central subgroup of $\nu(G)$. Then W_0 is nilpotent of class at most 2 and both Δ and $W_0/\Delta \simeq H_2(G, \mathbb{Z})$ are finitely generated. Then we can deduce that W_0 is finitely generated. ∎

Proposition 8.2. *Let G be a group. We identify $S(\nu(G))$ with $S(\nu(G)/W_0)$ via the canonical projection map $\nu(G) \to \nu(G)/W_0$ and we identify $S(\nu(G)/W_0)$ with $S(\mathfrak{X}(G)/W)$ via isomorphism* (8.2). *Then*

(a) $\Sigma^2(\nu(G)) = \Sigma^2(\mathfrak{X}(G)/W)$ *if G is finitely presented;*

(b) $\Sigma^2(\nu(G), \mathbb{Z}) = \Sigma^2(\mathfrak{X}(G)/W, \mathbb{Z})$ *if G is* FP_2.

Remark. Recall that in Theorems E1 and E2, we have calculated both $\Sigma^2(\mathfrak{X}(G)/W, \mathbb{Z})$ and $\Sigma^2(\mathfrak{X}(G)/W)$.

Proof. (a), (b) In both cases G is FP_2 and, by Lemma 8.1, W_0 is finitely generated nilpotent of class at most 2. Hence W_0 is finitely presented. By construction $W_0 \subseteq \nu(G)'$ and, identifying $\nu(G)/W_0$ with $\mathfrak{X}(G)/W$, we have $\Sigma^2(\nu(G)/W_0) = \Sigma^2(\mathfrak{X}(G)/W)$ and $\Sigma^2(\nu(G)/W_0, \mathbb{Z}) = \Sigma^2(\mathfrak{X}(G)/W, \mathbb{Z})$ whenever the invariants are defined.

If G is finitely presented, by Theorem 6.7, $\Sigma^2(\nu(G)) = \Sigma^2(\nu(G)/W_0)$. If G is FP_2, then by Lemma 6.1 and Theorem 6.8, $\Sigma^2(\nu(G), \mathbb{Z}) = \Sigma^2(\nu(G)/W_0, \mathbb{Z})$. We observe that here we can apply Lemma 6.1 since $W_0/[W_0, W_0]$ is finitely generated. ■

If we want to calculate $\Sigma^1(\nu(G))$ only under the assumption that G is finitely generated, we cannot assume that W_0 is finitely generated since G in general is not of type FP_2.

But we can follow the ideas from the proofs from Section 5. To do so, we need to define two groups in $\nu(G)$ that would play the roles of D and L from $\mathfrak{X}(G)$. Here we set $L_\nu(G)$ as the subgroup of $\nu(G)$ generated by $\{g\bar{g}^{-1} \mid g \in G\}$ and

$$D_\nu(G) = [G, \bar{G}] \quad \text{in } \nu(G).$$

Note that both are normal subgroups in $\nu(G)$ and that the defining relations of $\nu(G)$ imply that

$$[D_\nu(G), L_\nu(G)] = 1. \tag{8.3}$$

Note that $\nu(G)/D_\nu(G) \simeq G \times \bar{G} \simeq \mathfrak{X}(G)/D$.

Lemma 8.3. *Let G be a finitely generated group. Then $L_\nu(G)$ is finitely generated.*

Proof. (1) We will prove first that the abelianization of $L_\nu(G)$ is finitely generated. By [38, Theorem 2.1.1], the group

$$\mathcal{E}(G) = \big\langle G, \bar{G} \mid [\{\langle g\bar{g}^{-1} \mid g \in G\}\rangle, \langle\{g\bar{g}^{-1}\rangle \mid g \in G\}] = 1\big\rangle$$

is isomorphic to $\mathrm{Aug}(\mathbb{Z}G) \rtimes G$. Here $\mathrm{Aug}(\mathbb{Z}G)$ is the augmentation ideal of $\mathbb{Z}G$. We write an element of $\mathrm{Aug}(\mathbb{Z}G) \rtimes G$ as (λ, g), where $\lambda \in \mathrm{Aug}(\mathbb{Z}G)$ and $g \in G$. The product is given by the formula $(\lambda_1, g_1)(\lambda_2, g_2) = (\lambda_1 + g_1\lambda_2, g_1g_2)$, and the isomorphism

$$\theta\colon \mathcal{E}(G) \to \mathrm{Aug}(\mathbb{Z}G) \rtimes G$$

sends $g \in G$ to $(0, g)$ and $\bar{g} \in \bar{G}$ to $(g - 1, g)$. Note that θ induces an isomorphism

$$\nu(G)/L_\nu(G)' \simeq (\mathrm{Aug}(\mathbb{Z}G)/I) \rtimes G,$$

where I is a left ideal of $\mathrm{Aug}(\mathbb{Z}G)$ and the above isomorphism restricted to the abelianization of $L_\nu(G)$ is

$$L_\nu(G)/L_\nu(G)' \simeq \mathrm{Aug}(\mathbb{Z}G)/I.$$

By the definition of θ,

$$\begin{aligned}\theta([g_1, \bar{g}_2]) &= \theta(g_1^{-1})\theta(\bar{g}_2^{-1})\theta(g_1)\theta(\bar{g}_2) = (0, g_1^{-1})(g_2^{-1} - 1, g_2^{-1})(0, g_1)(g_2 - 1, g_2)\\ &= (\alpha(g_1, g_2), [g_1, g_2]),\end{aligned}$$

where

$$\begin{aligned}\alpha(g_1, g_2) &= g_1^{-1}(g_2^{-1} - 1) + g_1^{-1}g_2^{-1}g_1(g_2 - 1)\\ &= g_1^{-1}g_2^{-1} + [g_1, g_2] - g_1^{-1} - g_1^{-1}g_2^{-1}g_1.\end{aligned}$$

The relation $[g_1,\bar{g}_2]^{g_3}=[g_1^{g_3},\overline{g_2^{g_3}}]$ in $\nu(G)$, where $g_1,g_2\in G$, $\bar{g}_2\in\bar{G}$, implies

$$\begin{aligned}(g_3^{-1}\alpha(g_1,g_2),[g_1^{g_3},g_2^{g_3}]) &= (0,g_3^{-1})(\alpha(g_1,g_2),[g_1,g_2])(0,g_3)\\ &= \theta(g_3^{-1})\theta([g_1,\bar{g}_2])\theta(g_3)=\theta([g_1,\bar{g}_2]^{g_3})=\theta([g_1^{g_3},\overline{g_2^{g_3}}])\\ &= (\alpha(g_1^{g_3},g_2^{g_3}),[g_1^{g_3},g_2^{g_3}]) \quad \text{in } (\mathrm{Aug}(\mathbb{Z}G)/I)\rtimes G.\end{aligned}$$

Then

$$g_3^{-1}\alpha(g_1,g_2)=\alpha(g_1^{g_3},g_2^{g_3})=g_3^{-1}\alpha(g_1,g_2)g_3 \quad \text{in } \mathrm{Aug}(\mathbb{Z}G)/I$$

and so

$$\alpha(g_1,g_2)(g_3-1)\in I. \tag{8.4}$$

Note that

$$\theta(g_3\bar{g}_3^{-1})=(0,g_3)(g_3^{-1}-1,g_3^{-1})=(1-g_3,1)$$

and

$$\begin{aligned}\theta([g_1,\bar{g}_2]^{g_3\bar{g}_3^{-1}}) &= \theta(g_3\bar{g}_3^{-1})^{-1}\theta([g_1,\bar{g}_2])\theta(g_3\bar{g}_3^{-1})\\ &= (g_3-1,1)(\alpha(g_1,g_2),[g_1,g_2])(1-g_3,1)\\ &= (g_3-1+\alpha(g_1,g_2)+[g_1,g_2](1-g_3),[g_1,g_2]).\end{aligned}$$

The relation $[g_1,\bar{g}_2]^{g_3\bar{g}_3^{-1}}=[g_1,\bar{g}_2]$ in $\nu(G)$ implies

$$g_3-1+\alpha(g_1,g_2)+[g_1,g_2](1-g_3)=\alpha(g_1,g_2) \quad \text{in } \mathrm{Aug}(\mathbb{Z}G)/I,$$

hence

$$([g_1,g_2]-1)(g_3-1)=0 \quad \text{in } \mathrm{Aug}(\mathbb{Z}G)/I. \tag{8.5}$$

Since

$$\begin{aligned}\alpha(g_1,g_2) &= g_1^{-1}g_2^{-1}+1-g_1^{-1}-g_2^{-1}\\ &= (g_1^{-1}-1)(g_2^{-1}-1) \quad \text{in } \mathrm{Aug}(\mathbb{Z}G)/\mathrm{Aug}(\mathbb{Z}G'),\end{aligned}$$

by (8.4) and (8.5), we get

$$0=\alpha(g_1,g_2)(g_3-1)=(g_1^{-1}-1)(g_2^{-1}-1)(g_3-1) \quad \text{in } \mathrm{Aug}(\mathbb{Z}G)/I.$$

Thus for any $t_1,t_2,t_3\in G$, we have

$$(t_1-1)(t_2-1)(t_3-1)\in I. \tag{8.6}$$

Note that if X is a finite generating set of G then $\mathrm{Aug}(\mathbb{Z}G)$ as a $\mathbb{Z}$-module is generated by $\bigcup_{k\geq 1}Y_k$, where $Y_k=\{(x_{i_1}-1)\dots(x_{i_k}-1)\mid x_{i_1},\dots,x_{i_k}\in X\cup X^{-1}\}$. Using (8.6), we obtain that $\mathrm{Aug}(\mathbb{Z}G)/I$ as a $\mathbb{Z}$-module is generated by the finite set $Y_1\cup Y_2$.

(2) Finally, we prove that $L_\nu(G)$ is finitely generated. Since we have a central extension $1\to\Delta\to L_\nu(G)\to L_\nu(G)/\Delta\to 1$, by [12, Lemma 2.2], $L_\nu(G)$ is finitely generated if $L_\nu(G)/\Delta$ is finitely generated and the abelianization of $L_\nu(G)$ is finitely generated. Now isomorphism (8.1) induces an isomorphism $L_\nu(G)/\Delta\simeq L/R$ and, since L is finitely generated [12, Proposition 2.3], $L_\nu(G)/\Delta$ is finitely generated too. ∎

Proposition 8.4. *Let G be a finitely generated group. Via the canonical projection map $\nu(G) \to \nu(G)/W_0$ we identify $S(\nu(G))$ with $S(\nu(G)/W_0)$ and we identify $S(\nu(G)/W_0)$ with $S(\mathfrak{X}(G)/W)$ via isomorphism (8.2). Then*

$$\Sigma^1(\nu(G), \mathbb{Z}) = \Sigma^1(\mathfrak{X}(G)/W, \mathbb{Z}).$$

Remark. We recall that $\Sigma^1(\mathfrak{X}(G)/W, \mathbb{Z})$ was calculated in Section 5.

Proof. I. By (8.3), the same argument from Lemma 5.2 applies with $\mathfrak{X}(G)$ substituted by $\nu(G)$, i.e., if $\chi\colon \nu(G) \to \mathbb{R}$ is a character such that for $\chi_0 = (\chi_1, \chi_2)\colon G \times G \to \mathbb{R}$ defined by $\chi_1(g) = \chi(g)$, $\chi_2(g) = \chi(\overline{g})$ we have that $\chi_1 \neq \chi_2$ and $[\chi_0] \in \Sigma^1(G \times G)$, then $[\chi] \in \Sigma^1(\nu(G))$. Recall that $[\chi_0] \in \Sigma^1(G \times G)$ is equivalent to one of the following: 1) $\chi_1 = 0$, $[\chi_2] \in \Sigma^1(G)$; 2) $\chi_2 = 0$, $[\chi_1] \in \Sigma^1(G)$; 3) $\chi_1 \neq 0$, $\chi_2 \neq 0$.

II. In Lemma 8.3 we proved that $L_\nu(G)$ is a finitely generated subgroup of $\nu(G)$. Then by Lemma 5.3 applied for $H = \nu(G)$ and $N = L_\nu(G)$, we deduce that if $\chi_1 = \chi_2 \neq 0$ and $[\chi_1] \in \Sigma^1(G)$, then $[\chi] \in \Sigma^1(\nu(G))$.

I. and II. together with the description of $\Sigma^1(\mathfrak{X}(G)/W)$ in Section 5 imply that $\Sigma^1(\mathfrak{X}(G)/W) \subseteq \Sigma^1(\nu(G))$.

On the other hand, since $W_0 \subseteq \nu(G)'$, we can apply Lemma 2.1 and deduce

$$\Sigma^1(\nu(G)) \subseteq \Sigma^1(\nu(G)/W_0) = \Sigma^1(\mathfrak{X}(G)/W),$$

where the last equality follows from isomorphism (8.2). ∎

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References

[1] K. Almeida, The BNS-invariant for Artin groups of circuit rank 2. *J. Group Theory* **21** (2018), no. 2, 189–228 Zbl 1425.20019 MR 3770010

[2] K. Almeida and D. H. Kochloukova, The Σ^1-invariant for Artin groups of circuit rank 1. *Forum Math.* **27** (2015), no. 5, 2901–2925 Zbl 1362.20031 MR 3393382

[3] R. Bieri, *Homological dimension of discrete groups*. 2nd edn., Queen Mary College Mathematics Notes, Queen Mary College, London, 1981 Zbl 0357.20027 MR 715779

[4] R. Bieri and R. Geoghegan, Sigma invariants of direct products of groups. *Groups Geom. Dyn.* **4** (2010), no. 2, 251–261 Zbl 1214.20047 MR 2595091

[5] R. Bieri, R. Geoghegan, and D. H. Kochloukova, The sigma invariants of Thompson's group F. *Groups Geom. Dyn.* **4** (2010), no. 2, 263–273 Zbl 1214.20048 MR 2595092

[6] R. Bieri and J. R. J. Groves, The geometry of the set of characters induced by valuations. *J. Reine Angew. Math.* **347** (1984), 168–195 Zbl 0526.13003 MR 733052

[7] R. Bieri, W. D. Neumann, and R. Strebel, A geometric invariant of discrete groups. *Invent. Math.* **90** (1987), no. 3, 451–477 Zbl 0642.57002 MR 914846

[8] R. Bieri and B. Renz, Valuations on free resolutions and higher geometric invariants of groups. *Comment. Math. Helv.* **63** (1988), no. 3, 464–497 Zbl 0654.20029 MR 960770

[9] R. Bieri and R. Strebel, Valuations and finitely presented metabelian groups. *Proc. London Math. Soc. (3)* **41** (1980), no. 3, 439–464 Zbl 0448.20029 MR 591649

[10] M. R. Bridson, J. Howie, C. F. Miller, III, and H. Short, On the finite presentation of subdirect products and the nature of residually free groups. *Amer. J. Math.* **135** (2013), no. 4, 891–933 Zbl 1290.20024 MR 3086064

[11] M. R. Bridson and D. H. Kochloukova, Weak commutativity, virtually nilpotent groups, and Dehn functions. 2022, arXiv:2202.03796

[12] M. R. Bridson and D. H. Kochloukova, Weak commutativity and finiteness properties of groups. *Bull. Lond. Math. Soc.* **51** (2019), no. 1, 168–180 Zbl 1445.20028 MR 3919568

[13] C. H. Cashen and G. Levitt, Mapping tori of free group automorphisms, and the Bieri–Neumann–Strebel invariant of graphs of groups. *J. Group Theory* **19** (2016), no. 2, 191–216 Zbl 1368.20048 MR 3466593

[14] K. E. de Almeida and F. F. Lima, Finite graph product closure for a conjecture on the BNS-invariant of Artin groups. *Arch. Math. (Basel)* **116** (2021), no. 2, 131–139 Zbl 07322643 MR 4218608

[15] L. A. de Mendonça, The weak commutativity construction for Lie algebras. *J. Algebra* **529** (2019), 145–173 Zbl 1465.17018 MR 3937764

[16] S. Dowdall, I. Kapovich, and C. J. Leininger, McMullen polynomials and Lipschitz flows for free-by-cyclic groups. *J. Eur. Math. Soc. (JEMS)* **19** (2017), no. 11, 3253–3353 Zbl 1433.20012 MR 3713041

[17] G. Ellis and F. Leonard, Computing Schur multipliers and tensor products of finite groups. *Proc. Roy. Irish Acad. Sect. A* **95** (1995), no. 2, 137–147 Zbl 0863.20010 MR 1660373

[18] F. Funke and D. Kielak, Alexander and Thurston norms, and the Bieri–Neumann–Strebel invariants for free-by-cyclic groups. *Geom. Topol.* **22** (2018), no. 5, 2647–2696 Zbl 1486.20056 MR 3811767

[19] R. Gehrke, The higher geometric invariants for groups with sufficient commutativity. *Comm. Algebra* **26** (1998), no. 4, 1097–1115 Zbl 0905.20024 MR 1612192

[20] N. Gupta, N. Rocco, and S. Sidki, Diagonal embeddings of nilpotent groups. *Illinois J. Math.* **30** (1986), no. 2, 274–283 Zbl 0592.20047 MR 840126

[21] O. Kharlampovich and A. Myasnikov, Elementary theory of free non-abelian groups. *J. Algebra* **302** (2006), no. 2, 451–552 Zbl 1110.03020 MR 2293770

[22] D. Kielak, The Bieri–Neumann–Strebel invariants via Newton polytopes. *Invent. Math.* **219** (2020), no. 3, 1009–1068 Zbl 1480.20101 MR 4055183

[23] D. H. Kochloukova, On subdirect products of type FP_m of limit groups. *J. Group Theory* **13** (2010), no. 1, 1–19 Zbl 1245.20065 MR 2604842

[24] D. H. Kochloukova, On the Σ^2-invariants of the generalised R. Thompson groups of type F. *J. Algebra* **371** (2012), 430–456 Zbl 1279.20068 MR 2975404

[25] D. H. Kochloukova and F. F. Lima, Homological finiteness properties of fibre products. *Q. J. Math.* **69** (2018), no. 3, 835–854 Zbl 1443.20080 MR 3859210

[26] D. H. Kochloukova and F. F. Lima, On the Bieri–Neumann–Strebel–Renz invariants of residually free groups. *Proc. Edinb. Math. Soc. (2)* **63** (2020), no. 3, 807–829 Zbl 07261166 MR 4163872

[27] D. H. Kochloukova and L. Mendonça, On the Bieri–Neumann–Strebel–Renz Σ-invariants of the Bestvina–Brady groups. *Forum Math.* **34** (2022), no. 3, 605–626 Zbl 07524160 MR 4415959

[28] D. H. Kochloukova and S. Sidki, On weak commutativity in groups. *J. Algebra* **471** (2017), 319–347 Zbl 1401.20057 MR 3569188

[29] B. Kuckuck, Subdirect products of groups and the n-$(n+1)$-$(n+2)$ conjecture. *Q. J. Math.* **65** (2014), no. 4, 1293–1318 Zbl 1353.20037 MR 3285772

[30] B. C. R. Lima and R. N. Oliveira, Weak commutativity between two isomorphic polycyclic groups. *J. Group Theory* **19** (2016), no. 2, 239–248 Zbl 1339.20028 MR 3466596

[31] J. Meier, H. Meinert, and L. Van Wyk, Higher generation subgroup sets and the Σ-invariants of graph groups. *Comment. Math. Helv.* **73** (1998), no. 1, 22–44 Zbl 1037.20037 MR 1610579

[32] H. Meinert, The geometric invariants of direct products of virtually free groups. *Comment. Math. Helv.* **69** (1994), no. 1, 39–48 Zbl 0834.20039 MR 1259605

[33] H. Meinert, Actions on 2-complexes and the homotopical invariant Σ^2 of a group. *J. Pure Appl. Algebra* **119** (1997), no. 3, 297–317 Zbl 0879.57010 MR 1453518

[34] B. Renz, *Geometrische Invarianten und Endlichkeitseigenschaften von Gruppen*. Ph.D. thesis, Johann Wolfgang Goethe-Universität Frankfurt am Main, 1988 Zbl 0746.20021

[35] N. R. Rocco, On a construction related to the nonabelian tensor square of a group. *Bol. Soc. Brasil. Mat. (N.S.)* **22** (1991), no. 1, 63–79 Zbl 0791.20020 MR 1159385

[36] D. Schütz, On the direct product conjecture for sigma invariants. *Bull. Lond. Math. Soc.* **40** (2008), no. 4, 675–684 Zbl 1188.20059 MR 2441140

[37] Z. Sela, Diophantine geometry over groups. VI. The elementary theory of a free group. *Geom. Funct. Anal.* **16** (2006), no. 3, 707–730 Zbl 1118.20035 MR 2238945

[38] S. Sidki, On weak permutability between groups. *J. Algebra* **63** (1980), no. 1, 186–225 Zbl 0442.20014 MR 568571

[39] W. P. Thurston, A norm for the homology of 3-manifolds. *Mem. Amer. Math. Soc.* **59** (1986), no. 339, 99–130 Zbl 0585.57006 MR 823443

[40] C. T. C. Wall, Finiteness conditions for CW-complexes. *Ann. of Math. (2)* **81** (1965), 56–69 Zbl 0152.21902 MR 171284

[41] S. Witzel and M. C. B. Zaremsky, The Σ-invariants of Thompson's group F via Morse theory. In *Topological methods in group theory*, pp. 173–193, London Math. Soc. Lecture Note Ser. 451, Cambridge University Press, Cambridge, 2018 Zbl 1430.20052 MR 3889108

[42] M. C. B. Zaremsky, On the Σ-invariants of generalized Thompson groups and Houghton groups. *Int. Math. Res. Not. IMRN* **2017** (2017), no. 19, 5861–5896 Zbl 1404.20035 MR 3741884

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