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Iterated Minkowski sums, horoballs and north-south dynamics

Jeremias Epperlein and Tom Meyerovitch

Abstract. Given a finite generating set A for a group Γ, we study the map $W \mapsto WA$ as a topological dynamical system – a continuous self-map of the compact metrizable space of subsets of Γ. If the set A generates Γ as a semigroup and contains the identity, there are precisely two fixed points, one of which is attracting. This supports the initial impression that the dynamics of this map is rather trivial. Indeed, at least when $\Gamma = \mathbb{Z}^d$ and $A \subseteq \mathbb{Z}^d$ is a finite positively generating set containing the identity, the natural invertible extension of the map $W \mapsto W + A$ is always topologically conjugate to the unique "north-south" dynamics on the Cantor set. In contrast to this, we show that various natural "geometric" properties of the finitely generated group (Γ, A) can be recovered from the dynamics of this map, in particular, the growth type and amenability of Γ. When $\Gamma = \mathbb{Z}^d$, we show that the volume of the convex hull of the generating set A is also an invariant of topological conjugacy. Our study introduces, utilizes and develops a certain convexity structure on subsets of the group Γ, related to a new concept which we call the sheltered hull of a set. We also relate this study to the structure of horoballs in finitely generated groups, focusing on the abelian case.

1. Introduction

In this paper we study the topological dynamical system associated to a finitely generated group via the "Minkowski product". We denote the collection of subsets of a countably infinite group Γ by $\mathcal{P}(\Gamma)$. This space is naturally identifiable with the space $\{0,1\}^\Gamma$, thus naturally equipped with a topology turning it into a topological Cantor set.

Any finite subset $A \subseteq \Gamma$ defines a continuous map $\varphi_A : \mathcal{P}(\Gamma) \to \mathcal{P}(\Gamma)$ given by $\varphi_A(W) := WA$, where

$$WA := \{wa : w \in W, a \in A\}.$$

The set WA is often called *the Minkowski product* of W and A.

Most instances in the literature deal with the abelian case, where the group operation is usually denoted by $+$ and the set

$$W + A := \{w + a : w \in W, a \in A\}$$

is called the *Minkowski sum*.

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Our point of view here is to study $(\mathcal{P}(\Gamma), \varphi_A)$ as a dynamical system. We ask how dynamical properties of φ_A relate to algebraic properties of Γ and more specifically to geometric properties of the Cayley graph of Γ with respect to the set A. We focus on the case where A is *positively generating*, i.e. $\Gamma = \bigcup_{n=1}^{\infty} A^n$ and $1_\Gamma \in A$. We call such a pair (Γ, A) as above simply a *finitely generated group*.

In this case $\lim_{n\to\infty} \varphi_A^n(W) = \Gamma$ for any non-empty set $W \subset \Gamma$, so Γ and $\emptyset$ are the only fixed point for φ_A, and the forward orbit of any set in $\mathcal{P}(\Gamma) \setminus \{\emptyset\}$ converges to the fixed point Γ. Thus, the only ergodic φ_A-invariant probability measures on $\mathcal{P}(\Gamma)$ are the delta measures δ_Γ and $\delta_\emptyset$. So from the ergodic theory viewpoint, the maps φ_A are completely trivial. Nevertheless, from the point of view of topological dynamics it turns out that the system $(\mathcal{P}(\Gamma), \varphi_A)$ encodes non-trivial properties of the finitely generated group (Γ, A).

Call a property P of finitely generated groups *dynamically recognizable* (among a family $\mathcal{G}$ of finitely generated groups) if for any pair of finitely generated groups (Γ_1, A_1) and (Γ_2, A_2) (in the family $\mathcal{G}$) such that $(\mathcal{P}(\Gamma_1), \varphi_{A_1}) \cong (\mathcal{P}(\Gamma_2), \varphi_{A_2})$, (Γ_1, A_1) has the property P if and only if (Γ_2, A_2) has the property P. This approach is analogous to the study of group properties only depending on the Cayley graph in geometric group theory.

We show that the following properties are dynamically recognizable:

- Growth type (polynomial, exponential, ...), see Corollary 4.9.
- Rank and volume of the convex hull of the generating set, among $\{\mathbb{Z}^d : d \in \mathbb{N}\}$, see Corollary 4.10.
- The exponential growth rate among e.g. free groups, Corollary 4.12.
- Amenability, see Corollary 5.8.

From the geometric point of view, the dynamical invariants underlying the above results are related to a certain convexity structure we introduce on subsets of an arbitrary finitely generated group (Γ, A), and to a new concept which we call the *sheltered hull*. This is introduced in Section 3. The convexity structure given by the sheltered hull is related to the notion of a *horoball*, which we recall and discuss in Section 2.

The dynamical system $(\mathcal{P}(\Gamma), \varphi_A)$ only depends on the directed Cayley graph $\mathrm{Cayley}(\Gamma, A)$. One can easily generalize the map φ_A to general countable, locally finite, directed graphs (see Section 2), and for many results we do not actually need that this graph is a Cayley graph. Hence this is the level of generality we assume if it does not add further complications.

The dynamically recognizable properties above, captured by the sheltered hull, are in some sense based on "quantifying non-invertibility" of the map φ_A in a manner which is invariant under topological conjugacy. It is thus natural to ask if there are dynamically recognizable properties that can be "detected disregarding non-invertibility". Thus, in Section 7 we consider "the invertible analog" of φ_A: Namely, the natural extension of the restriction of φ_A to its eventual image (for brevity we refer to this as "the natural extension of φ_A"). This is a homeomorphism with "north-south dynamics", which we discuss and recall in Section 6. In Section 10 we show that, at least for $\Gamma = \mathbb{Z}^d$, the natural extensions are all topologically conjugate to one another.

The first part of the paper, up to Section 7, deals with results about general finitely generated groups (in some cases these are general results about locally finite graphs). In the second part of the paper, from Section 8 onwards, we specialize to the case $\Gamma = \mathbb{Z}^d$. In Section 8 we discuss the structure of horoballs in $\mathbb{Z}^d$ with respect to a generating set A and show that up to translation these are in a natural bijection with the faces of the convex hull of A (viewed as a subset of $\mathbb{R}^d$). En route we recall some old results about the structure of iterated Minkowski sums in $\mathbb{Z}^d$. In Section 9, still working with $\Gamma = \mathbb{Z}^d$, we consider the space of horoballs (and its closure) as a topological space and a $\mathbb{Z}^d$-dynamical system. In particular, we observe that its homeomorphism type is uniquely determined by the rank d (Corollary 9.7). We also provide an explicitly checkable characterization of the topological conjugacy class of the associated dynamical system (Theorem 9.8). Our proof in Section 10 that the natural extension of $(\mathcal{P}(\mathbb{Z}^d), \varphi_A)$ is perfect, thus topologically conjugate to the unique north-south dynamics on the Cantor set, is contrasted in Section 11, where we show that the topological structure of the eventual image is sensitive to the specific generating set $A \subset \mathbb{Z}^d$, already when $d = 2$. In Section 12 we discuss the problem of when $(\mathcal{P}(\Gamma_1), \varphi_{A_1})$ factors onto $(\mathcal{P}(\Gamma_2), \varphi_{A_2})$, a problem which for the most part we have not been able to resolve.

Currently we do not have a reasonable necessary and sufficient condition for the existence of a conjugacy between $(\mathcal{P}(\Gamma_1), \varphi_{A_1})$ and $(\mathcal{P}(\Gamma_2), \varphi_{A_2})$, even in the case $\Gamma_1 = \Gamma_2 = \mathbb{Z}^d$. This remains an open problem for future work.

2. Horoballs in directed graphs and finitely generated groups

2.1. Quasi-metrics on directed graphs

Let $G = (V(G), E(G))$ be a countably infinite, locally finite directed graph. We assume throughout that G is strongly connected, meaning that for every $v, w \in V(G)$ there is a directed path in G from v to w.

Let $\mathcal{P}(V(G))$ denote the collection of subsets of $V(G)$, which we identify with $\{0,1\}^{V(G)}$, equipped with the product topology. Consider the continuous self-map

$$\varphi_G : \mathcal{P}(V(G)) \to \mathcal{P}(V(G)),$$
$$\varphi_G(W) := W \cup \big\{v \in V(G) : \exists w \in W \text{ s.t. } (w, v) \in E(G)\big\}.$$

Via the identification of subsets of $V(G)$ with their characteristic functions this can be rewritten as follows:

$$\mathbb{1}_{\varphi_G(W)}(w) = \max\big\{\mathbb{1}_W(v) : (v, w) \in E(G) \text{ or } w = v\big\}.$$

Under the assumption that the graph G is strongly connected, there are precisely two fixed points for φ_G, namely $V(G)$ and $\emptyset$. Also, for any $W \in \mathcal{P}(V(G)) \setminus \{\emptyset\}$, we have $\lim_{n\to\infty} \varphi_G^n(W) = V(G)$. For $v, w \in V(G)$ define

$$d(w, v) = \min\big\{n \in \mathbb{N}_0 : v \in \varphi_G^n(\{w\})\big\}.$$

The function $d : V(G) \times V(G) \to \mathbb{N}_0$ is a *quasi-metric*, meaning it satisfies the axioms of a metric, apart from symmetry. For $w, v \in V(G)$, $d(w, v)$ is the minimal number of edges in a directed path from w to v. For background on quasi-metrics, sometimes also called "non-symmetric metrics", see for instance [11, 30]. In the case where G is an undirected graph (which we think of as a directed graph with edges going both ways), d is called the graph metric. The reason we allow for directed graphs is to handle non-symmetric generating sets in finitely generated groups. We can also express φ_G^n directly in terms of the quasi-metric as $\varphi_G^n(W) = \{v \in V(G) : \exists w \in W \text{ s.t. } d(w, v) \leq n\}$. Many notions in metric geometry still make sense in our setting:

We define a *geodesic ray* in a graph G as a sequence of vertices $(\gamma_n)_{n \in \mathbb{N}_0}$ in $V(G)$ such that the shortest directed path from γ_i to γ_j for $i > j$ has length $i - j$. In particular, this means that $(\gamma_{n+1}, \gamma_n)_{n \in \mathbb{N}_0} \in E(G)$ for every n.

For a vertex $v \in V$, we refer to $\varphi_G^n(\{v\})$ as *the ball of radius n centered at v*. In the case where the graph G is undirected, this is actually a ball with respect to the graph metric. In general, for $W \subseteq V(G)$ one can think of $\varphi_G^n(W)$ as 'the set of elements accessible from W in n steps'.

The space of balls is clearly invariant under φ_G but it is not a closed subset of $\mathcal{P}(V(G))$. The new elements arising in the closure are called horoballs.

Definition 2.1. Let G be a locally finite graph. A limit point in $\mathcal{P}(V(G))$ of a sequence of balls $(\varphi_G^{n_i}(\{v_i\}))_{i \in \mathbb{N}}$ with radii $(n_i)_{i \in \mathbb{N}}$ tending to infinity is called a *horoball* in G if it is non-empty and has non-empty complement. Let $\mathrm{Hor}(G)$ be the set of all horoballs in G.

The definition of horoballs is due to Gromov [10] and comes from hyperbolic geometry. It makes sense in any quasi-metric space. Gromov's definition is slightly different from ours, but we will see later in this section that they are equivalent. For the definition in terms of limits of balls, see for instance [17]. See also [1], where horoballs "tangent to a base point" are called "cones" (in the context of finitely generated groups, where the base point is the identity element of the group).

2.2. Busemann balls and Gromov's horofunction boundary

Busemann balls are horoballs coming from geodesic rays:

Definition 2.2. A *Busemann ball* in G is a set $H \in \mathcal{P}(V(G)) \setminus \{\emptyset, V(G)\}$ of the form $\bigcup_{r=0}^{\infty} \varphi_G^r(\{\gamma_r\})$, where $(\gamma_n)_{n \in \mathbb{N}_0} \in V(G)^{\mathbb{N}_0}$ is a geodesic ray. Note that this is an increasing union, so H is indeed a limit of balls.

There is a slightly more classical approach to horoballs via Gromov's horofunction boundary [10], which we now recall (restricting to the case where the underlying quasi-metric space is a locally finite directed graph). See [26] and references therein for background and further details. Fix a base vertex v_0 in $V(G)$ and consider the space $C(V(G), \mathbb{Z})$ of integer-valued continuous functions on $V(G)$ with the topology of pointwise convergence (in a more general setting one needs to consider real-valued functions, with the topology of uniform convergence on compact sets). We have an embedding $\varrho : V(G) \to C(V(G), \mathbb{Z})$ via $w \mapsto \varrho_w$ with $\varrho_w(v) = d(w, v) - d(w, v_0)$. The elements of $C(V(G), \mathbb{Z})$ which are in the closure of $\varrho(V(G))$ but not in $\varrho(V(G))$ are called *horofunctions*. By definition, for every $x \in V(G)$, the sublevel sets of the function ϱ_x are balls around x. As in [26], horoballs are sometimes defined as sublevel sets of horofunctions. The following proposition verifies that in our setting sublevel sets of horofunctions are exactly accumulation points of balls (one implication follows directly from [26, Proposition 2.8]).

Proposition 2.3. *Let F be a horofunction on G. For every $r \in \mathbb{Z}$ the sublevel set $H_{F,R} := \{v \in V(G) : F(v) \leq R\}$ is either a horoball or equal to $V(G)$. Conversely, if H is a horoball, then there is $R \in \mathbb{Z}$ and a horofunction F such that $H = H_{F,R}$.*

Proof. Let $(\varrho_{w_k})_{k\in\mathbb{N}}$ be a sequence of functions as above converging to a horofunction F. Then

$$\begin{aligned} B_k &:= \left\{v \in V(G) : \varrho_{w_k}(v) \leq R\right\} \\ &= \left\{v \in V(G) : d(w_k, v) \leq d(w_k, v_0) + R\right\} \\ &= \varphi_G^{d(w_k,v_0)+R}(\{w_k\}) \end{aligned}$$

is a sequence of balls converging to $H_{F,R}$.

The set $H_{F,R}$ is non-empty because for non-negative R every B_k contains v_0 and for negative R it has non-empty intersection with $\{v \in V(G) : d(v, v_0) = |R|\}$. Thus, $H_{F,R}$ is either a horoball or equal to $V(G)$.

Conversely, let $(\varphi_G^{r_k}(\{w_k\}))_{k\in\mathbb{N}}$ be a sequence of balls converging to the horoball H. We will show that $r_k - d(w_k, v_0)$ is bounded. Let u be a point on the 'boundary of H', namely $u \in \varphi_G(H) \setminus H$. For sufficiently large k we have $d(w_k, u) = r_k + 1$ and hence (keeping in mind that d is not necessarily symmetric),

$$|r_k - d(w_k, v_0)| = |d(w_k, u) - 1 - d(w_k, v_0)| \leq d(v_0, u) + 1 + d(u, v_0).$$

Therefore we can find a subsequence of balls $\varphi_G^{r_k}(\{w_k\})$ converging to H with $r_k - d(w_k, v_0)$ constant equal to $R \in \mathbb{Z}$.

Next we will show that the corresponding functions ϱ_{w_k} are uniformly bounded. Namely for $v \in V(G)$ we have $|\varrho_{w_k}(v)| = |d(w_k, v) - d(w_k, v_0)| \leq d(v, v_0) + d(v_0, v)$ by the triangle inequality. Therefore we can select a subsequence of $(w_k)_{k\in\mathbb{N}}$ such that ϱ_{w_k} converges to a horofunction F. But then $H_{F,R}$ is the limit of the sequence of sets

$$\begin{aligned} \left\{v \in V(G) : \varrho_{w_k}(v) \leq R = r_k - d(w_k, v_0)\right\} &= \left\{v \in V(G) : d(w_k, v) \leq r_k\right\} \\ &= \varphi_G^{r_k}(\{w_k\}) \end{aligned}$$

and hence $H_{F,R} = H$. ∎

Thus, for fixed $R \in \mathbb{Z}$, the map $F \mapsto H_{F,R}$ from the horofunction boundary to the space of horoballs (and possibly the whole vertex set) is surjective. It is natural to ask the following.

Question 2.4. When is the map $F \mapsto H_{F,R}$ from the horofunction boundary to the space of horoballs (and possibly the whole vertex set) injective for fixed $R \in \mathbb{Z}$?

If we take the limit along a geodesic ray γ starting in $x_0 \in V(G)$, the functions ϱ_{γ_k} converge to a special kind of horofunction, called a *Busemann function*, see [18, 25]. The sublevel sets of Busemann functions are precisely the Busemann balls. There are known examples of Cayley graphs with horofunctions which are not Busemann functions, see [24,25,28]. In response to a question raised in a preliminary version of this work, Salo [19] provided various examples of Cayley graphs with non-Busemann horoballs and showed that connectedness of every horoball is equivalent to "almost-convexity" as introduced by Cannon [3]. Furthermore, Salo shows that the lamplighter group has horoballs which are not even coarsely connected. The following observation shows that horoballs that are minimal with respect to inclusion are Busemann.

Proposition 2.5. *Let H be a horoball and $v \in H$. There is a Busemann ball B such that $v \in B \subseteq H$.*

Proof. Let $\varphi_G^{r_k}(\{w_k\})$ be a sequence of balls converging to H. For each k, let $\gamma^k = (\gamma^k_{\ell_k}, \gamma^k_{\ell_k - 1}, \dots, \gamma^k_0)$ be a shortest path in G starting in $w_k = \gamma^k_{\ell_k}$ and ending in $v = \gamma^k_0$. Because G is locally finite, and in particular has finite in-degrees, we can assume that for every $\ell \in \mathbb{N}_0$ the sequence $(\gamma^k_\ell)_{k \in \mathbb{N}_0}$ stabilizes, hence we can assume that the sequences γ^k converge to a geodesic ray $\gamma = (\gamma_k)_{k \in \mathbb{N}_0}$ with $\gamma_0 = v$. The Busemann ball corresponding to the geodesic ray γ is contained in H and contains v. ∎

Since unions of horoballs will play the role of the eventual image for the dynamical systems $(\mathcal{P}(V(G)), \varphi_G)$, we mention the following conclusion of Proposition 2.5.

Corollary 2.6. *Every set which is a union of horoballs is also a union of Busemann balls.*

As we will see in Section 8, in every Cayley graph of $\mathbb{Z}^n$ there are only finitely many horoballs up to translation. It is thus natural to ask:

Question 2.7. Which Cayley graphs allow for finitely many horoballs up to translation?

Tointon and Yadin [22, Conjecture 1.3] ask if groups of polynomial growth admit finitely many horofunctions, and recall an observation of Karlsson that an affirmative solution would yield an alternate proof of Gromov's theorem on groups of polynomial growth.

2.3. Iterated Minkowski products, positively generating sets and Cayley graphs

Recall that a subset A in a group Γ is called *positively generating* if $\Gamma = \bigcup_{n=1}^{\infty} A^n$. This means that A is a generating set for Γ considered as a semigroup. Throughout the paper when we consider a finitely generated group (Γ, A), we will assume A is a finite positively generating set which contains the identity, although we will not necessarily assume A is symmetric. The Cayley graph Cayley(Γ, A) for the finitely generated group (Γ, A) has vertex set Γ and edges of the form (g, ga) with $g \in \Gamma$ and $a \in A$. In this case the map $\varphi_{\text{Cayley}(\Gamma,A)}$ coincides with the Minkowski product maps φ_A given by

$$\varphi_A(W) := WA.$$

Note that φ_A is a cellular automaton over the group Γ, in the sense that it is a continuous map that commutes with the Γ action of translation from the left.

Definition 2.8. For a finitely generated group (Γ, A) an A-*horoball* is a non-empty subset of Γ that is the limit of balls $\{g_n A^{r_n}\}$ that is neither a ball itself nor the whole set Γ. We abbreviate Hor(Cayley(Γ, A)) by Hor(Γ, A).

Example 2.9. Consider $\Gamma = \mathbb{Z}^2$ with the generating set $A = \{-1, 0, 1\}^2$. Then horoballs are either translated vertical or horizontal halfspaces or translated quadrants. As we will show, up to translation there are therefore eight of them, corresponding to the 4 edges and 4 vertices of the convex hull of A, which is a square.

We note that $\overline{\text{Hor}(\Gamma, A)} = \text{Hor}(\Gamma, A) \cup \{\emptyset, \Gamma\}$ is a closed Γ-invariant subset of $\mathcal{P}(\Gamma)$. It is natural to wonder what properties of the group Γ or of the generating set A can be extracted from the topology of $\overline{\text{Hor}(\Gamma, A)}$ or from the dynamics of the Γ action on $\overline{\text{Hor}(\Gamma, A)}$.

Question 2.10. Let A_1, A_2 be two positively generating sets for Γ. Is it the case that $\overline{\text{Hor}(\Gamma, A_1)}$ and $\overline{\text{Hor}(\Gamma, A_2)}$ are homeomorphic?

We will later provide an affirmative answer in the particular case $\Gamma = \mathbb{Z}^d$ (see Corollary 9.7 below).

3. The sheltered hull

From now on let G be a locally finite strongly connected directed graph. Our next goal is to introduce a convexity structure on the vertices of G. We will use this convexity structure in later sections to extract an invariant of topological conjugacy. This new convexity structure might also be of independent interest from the point of view of geometric group theory.

Definition 3.1. Given $W \subseteq V(G)$ and $r > 0$ we define the r-*sheltered hull* of W to be

$$S_G^r(W) := \{v \in V(G) : \varphi_G^r(\{v\}) \subseteq \varphi_G^r(W)\}.$$

The G-*sheltered hull* of W is then given by

$$S_G(W) := \bigcup_{r=1}^{\infty} S_G^r(W).$$

We call a set $W \subseteq V(G)$ *G-sheltered* if it agrees with its G-sheltered hull. When G is clear from the context, we omit it and write "sheltered" instead of G-sheltered.

Remark 3.2. In 'mathematical morphology', a subfield of image analysis, the operation $W \mapsto S_G^r(W)$ is called *closing*, see e.g. [20].

Remark 3.3. The definition of the sheltered hull naturally extends to general metric and quasi-metric spaces. In Euclidean space, the interior of the sheltered hull of a set agrees with the interior of the convex hull.

We denote by $\overline{G}$ the graph with the same vertex set as G and the directions of edges in G reversed, that is, $E(\overline{G}) = \{(w, v) : (v, w) \in E(G)\}$.

There is an interesting characterization of the complement of $S_G^r(W)$ in terms of r-balls of $\overline{G}$:

Proposition 3.4. *A vertex $v \in V(G)$ is contained in the complement of $S_G^r(W)$ if and only if it is covered by an r-ball in $\overline{G}$ which is disjoint from W. More precisely,*

$$V(G) \setminus S_G^r(W) = \bigcup \left\{ \varphi_{\overline{G}}^r(\{u\}) : u \in V(G),\ \varphi_{\overline{G}}^r(\{u\}) \cap W = \emptyset \right\}.$$

Proof. Suppose $u \in V(G)$ and $\varphi_{\overline{G}}^r(\{u\}) \cap W = \emptyset$ or, in other words, $u \notin \varphi_G^r(W)$. Let $v \in \varphi_{\overline{G}}^r(\{u\})$. Then $u \in \varphi_G^r(\{v\})$ and therefore $\varphi_G^r(\{v\}) \not\subseteq \varphi_G^r(W)$. Thus $v \notin S_G^r(W)$.

Now let $v \in V(G) \setminus S_G^r(W)$. Then $\varphi_G^r(\{v\})$ contains at least one element u not contained in $\varphi_G^r(W)$. This implies $\varphi_{\overline{G}}^r(\{u\}) \cap W = \emptyset$. ∎

In analogy to Proposition 3.4 we can characterize the complement of $S_G(W)$ as the union of all horoballs in $\overline{G}$ disjoint from W. See Figure 1 for an illustration.

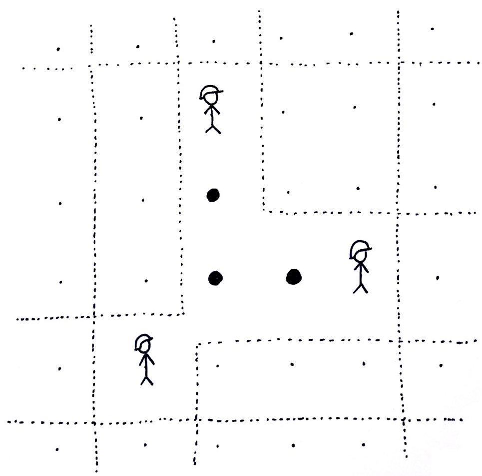


Figure 1. The sheltered hull of three elements (depicted by little guards providing shelter from the evil horoballs) of $\mathbb{Z}^2$ with generator $\{-1, 0, 1\}^2$ and the horoballs covering the complement.

Proposition 3.5. *For every non-empty subset of $W \subseteq \Gamma$ we have*

$$V(G) \setminus S_G(W) = \bigcup \{H \in \mathrm{Hor}(\overline{G}) : H \cap W = \emptyset\}.$$

Proof. Let $v \in V(G) \setminus S_G(W)$. By Proposition 3.4 for every $r \in \mathbb{N}$ there is a vertex $u_r \in V(G)$ such that $\varphi_{\overline{G}}^r(\{u_r\})$ is disjoint from W but contains v. Let $(r_k)_{k\in\mathbb{N}}$ be a growing sequence of radii such that $\varphi_{\overline{G}}^{r_k}(\{u_{r_k}\})$ converges to a limit H. Since v is contained in H and $H \cap W = \emptyset$, this limit H is neither empty nor the whole of $V(G)$. Hence we found a horoball H with $v \in H$ and $H \cap W = \emptyset$.

Now let v be in $S_G(W)$. By definition there is $r \in \mathbb{N}$ with $\varphi_G^r(\{v\}) \subseteq \varphi_G^r(W)$. Since $\varphi_G^r(\{v\})$ is finite, there is a finite subset $\widetilde{W}$ of W such that already $\varphi_G^r(\{v\}) \subseteq \varphi_G^r(\widetilde{W})$ and thus $v \in S_G^R(\widetilde{W})$ for all $R \geq r$. In particular, every ball in $\overline{G}$ with radius $R \geq r$ containing v has non-trivial intersection with $\widetilde{W}$ by Proposition 3.4. Since $\widetilde{W}$ is finite, this also means that every horoball in $\overline{G}$ containing v must intersect $\widetilde{W} \subseteq W$. Therefore v is not contained in $\bigcup\{H \in \mathrm{Hor}(\overline{G}) : H \cap W = \emptyset\}$. ∎

There is an analogy between sheltered sets in graphs and convex subsets of $\mathbb{R}^d$, as sheltered sets actually form an abstract convexity structure in the sense of [23]:

Proposition 3.6. *For any directed graph G the sheltered sets in G fulfill the following axioms:*

(a) *$\emptyset, V(G)$ are both sheltered.*

(b) *The family of sheltered sets is closed under arbitrary intersections.*

(c) *The union of an increasing chain of sheltered sets is sheltered.*

Furthermore, for any $W \subseteq V(G)$ the sheltered hull $S_G(W)$ is equal to the intersection of all sheltered sets containing W.

Proof. (a) This is clear by definition.

(b) Let $\mathcal{M}$ be a family of sheltered sets. By Proposition 3.5 we know that a set W is sheltered if and only if every element in its complement is contained in a horoball in $\overline{G}$ disjoint from W. For every $v \in V(G) \setminus \bigcap \mathcal{M}$ there is $W \in \mathcal{M}$ with $v \in V(G) \setminus W$ and hence there is a horoball H disjoint from W with $v \in H$. But then H is also disjoint from $\bigcap \mathcal{M}$, and thus $\bigcap \mathcal{M}$ is sheltered.

(c) Let $W_1 \subseteq W_2 \subseteq W_3 \subseteq \cdots$ be an increasing chain of sheltered sets with $W = \bigcup_{k=1}^{\infty} W_k$. Let $v \in V(G) \setminus W$ and hence $v \in V(G) \setminus W_k$ for all $k \in \mathbb{N}$. We want to find a horoball in $\overline{G}$ disjoint from W containing v. By Proposition 3.4 we know that for each $k \in \mathbb{N}$ we can find a ball $\varphi_{\overline{G}}^{r_k}(\{v_k\})$ which contains v and which is disjoint from W_k. By compactness we can take a subsequence of these balls converging to a horoball H in $\overline{G}$. Clearly $v \in H$ and if there would be $u \in H \cap W$, then for sufficiently large k we would have $u \in \varphi_{\overline{G}}^{r_k}(\{v_k\})$ and $u \in W_k$, since the later sets are monotonically increasing. But the set W_k and $\varphi_{\overline{G}}^{r_k}(\{v_k\})$ are disjoint, thus $H \cap W = \emptyset$. Therefore W is sheltered. ∎

Remark 3.7. By Proposition 3.5 and Proposition 3.6 given a directed graph G, the sheltered sets form a convexity structure generated by the complements of $\overline{G}$-horoballs in the sense that they are the smallest family of sets closed under intersections and unions of increasing chains which contains $\overline{G}$-horoball complements. Notice the analogy with the situation in $\mathbb{R}^n$ where the usual convexity structure is generated by the set of all half spaces (or equivalently the complements of half spaces). A similar concept of convex hull was considered in [7], but there the convexity structure is generated by the horoballs instead of their complements.

Remark 3.8. Since every $\overline{G}$-horoball is the union of $\overline{G}$-Busemann balls by Proposition 2.5, a set M is G-sheltered if and only if every point in its complements is contained in a $\overline{G}$-Busemann ball disjoint from M. In other words, the convexity structure of $\overline{G}$-sheltered sets is also generated by the complements of $\overline{G}$-Busemann balls.

Let us now turn our attention back to Cayley graphs. Let Γ be a finitely generated group and let A be a positively generating set containing the unit. For $g \in \Gamma$ we denote

$$|g|_A = d(1_\Gamma, g) = \min\{n \in \mathbb{N}_0 : g \in A^n\}.$$

To simplify notation we denote $S^r_{\mathrm{Cayley}(G,A)}$ and $S_{\mathrm{Cayley}(G,A)}$ by S^r_A and S_A.

A *dead end* for (Γ, A) is an element g from which no element of word length larger than $|g|_A$ can be reached via one of the generators. Equivalently, g is a dead end if and only if $\varphi_A(\{g\}) \subseteq A^{|g|_A}$. This notion is due to Bogopolskii, see [2].

Proposition 3.9. *For every $n \in \mathbb{N}$ with $S_A(A^n) \setminus A^n \neq \emptyset$, there is at least one dead end in $\Gamma \setminus A^n$.*

Proof. Let $w \in S_A(A^n) \setminus A^n$. Let $m \geq 1$ be the minimal positive integer such that $wA^m \subseteq A^{n+m}$. Then m is finite by the definition of $S_A(A^n)$. Since m was chosen minimal, there must be $v \in wA^{m-1} \setminus A^{n+m-1}$. Hence $|v|_A \geq n + m$ and therefore $vA \subseteq wA^m \subseteq A^{n+m} \subseteq A^{|v|_A}$. This shows that v is a dead end outside of A^n. ∎

Corollary 3.10. *If (Γ, A) has only finitely many dead ends, then $S_A(A^n) = A^n$ for sufficiently large n.*

Proof. If (Γ, A) has only finitely many dead ends, then there is $n \in \mathbb{N}$ such that all dead ends are contained in A^n and then $S_A(A^n) = A^n$ by Proposition 3.9. ∎

Remark 3.11. While some groups have "very few" dead ends, Šunić shows in [21, Theorem A.1 3] that every group has a generating set with respect to which it has at least one dead end.

Another bound on the size of the sheltered hull can be given in groups with more than one end. The *number of ends* of a finitely generated group (Γ, A) with a symmetric generating set is the minimal number n in $\mathbb{N} \cup \{\infty\}$ such that the removal of every finite vertex set from $\mathrm{Cayley}(\Gamma, A)$ leaves at most n infinite connected components. The number of ends of a group is independent of the generating set and is contained in $\{0, 1, 2, \infty\}$ by a result of Freudenthal.

Proposition 3.12. *If Γ is a group with more than one end and A is a symmetric generating set, then there is $C \in \mathbb{N}$ such that $S_A(A^n) \subseteq A^{n+C}$ for any sufficiently large n.*

Proof. The proof is an adaption of an argument of Lehnert [13, Theorem 2] showing that groups with more than one end have bounded dead end depth. The idea behind the proof is to find a uniform bound on the distance from group elements to their closest geodesic ray. Since Γ has more than one end, there must be $k \in \mathbb{N}$ such that removing A^k from Cayley(Γ, A) generates at least two infinite connected components. Set $C := 2k$, consider $n > k$ and let $u \in \Gamma \setminus A^{n+C}$. Let $m \in \mathbb{N}$ be arbitrary. Since Cayley(Γ, A) is invariant under translation by u, the removal of uA^k from Cayley(Γ, A) also generates at least two infinite connected components.

Take a geodesic ray γ starting at 1_Γ which eventually stays in a connected component not containing 1_Γ. This ray must cross uA^k. Let v be the last vertex on γ in uA^k. Let w be the $(m-k)$-th vertex on this ray after v. Then

$$\begin{aligned} d(1_\Gamma, w) &= d(1_\Gamma, v) + (m-k) \\ &\geq d(1_\Gamma, u) - k + m - k \\ &> n + C - 2k + m = n + m, \\ d(u, w) &\leq d(u, v) + d(v, w) \leq k + (m-k) = m, \end{aligned}$$

hence $w \in uA^m \setminus A^{n+m}$. Since m was arbitrary, this shows that uA^m is not contained in A^{n+m} for all m so $u \notin S_A(A^n)$. ∎

Question 3.13. Which finitely generated groups Γ have the property that for some finite positively generating set A the sheltered hull $S_A(W)$ is finite for any finite $W \subseteq \Gamma$? Does this depend on the positive generating set A?

Example 3.14. Consider the Heisenberg group H, given by the standard presentation

$$H = \langle a, b \mid [a, [a, b]], [b, [a, b]] \rangle.$$

This is the simplest example of a non-abelian finitely generated nilpotent group. With respect to the generating set $A = \{a, b, a^{-1}, b^{-1}, e\}$, there are finite sets whose sheltered hull is infinite. More precisely, any subsets of H which contains A^2 has an infinite sheltered hull. This is a consequence of the following result: If we abbreviate the commutator of a and b by $c := [a, b]$, then for any $m \in \mathbb{N}$ there exists $n \in \mathbb{N}$ such that $c^m A^n \subseteq A^{n+2}$. This last result can be extracted (with some additional arguments) for instance using [27, Proposition 5.3], which involves a certain "normal form" for elements of H.

On a first glance, there seems to be a strong resemblance between the infiniteness of $S_A(A^n)$ and unbounded dead end depth: If $w \in S_A(A^n)$, then there is $m \in \mathbb{N}$ such that $wA^m \subseteq A^{n+m}$. If w has dead end depth at least m, then $wA^m \subseteq A^{|w|}$. The next example shows at least that these conditions are not equivalent. We currently do not know, if bounded dead end depth implies finiteness of $S_A(A^n)$. In a preprint of this paper, we claimed to show that, but the alleged proof contained an error.

More precisely, we show that while the standard lamplighter group with standard generators has dead ends of unbounded depth, the sheltered hull of every finite set is nevertheless finite.

Example 3.15. The *lamplighter group* $L := \mathbb{Z}^2 \wr \mathbb{Z}$ is given by the presentation

$$L = \langle a, t \mid a^2, [a, t^{-n} a t^n] \rangle.$$

This is a finitely generated solvable group of exponential growth. The generator "a" corresponds to "switching a light", "t" corresponds to "moving the lamplighter". With respect to the generating set $A = \{a, t^{\pm 1}, e\}$, L has dead ends of unbounded depth. Indeed, the element

$$w = a^{(t^{-n})} \cdot \ldots \cdot a^{(t^{-1})} \cdot a \cdot a^{(t)} \cdot a^{(t^n)},$$

where $a^{(t^i)} = t^{-i} a t^i$, is a dead end of depth at least $n + 1$. The element w is depicted in Figure 2.

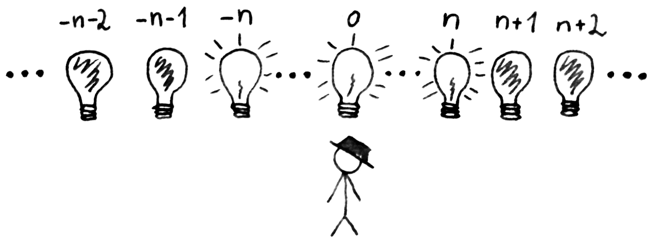


Figure 2. A dead end in the lamplighter group of depth n.

The element w corresponds to "the lamplighter standing at zero with lamps from $-n$ to n switched on". Note that w has length $6n + 1$ with respect to the generating set A because one needs n steps to move from the origin to position n, $2n$ steps to move to $-n$, and n steps to move back to the origin, and additionally there are $2n + 1$ lamps to switch on. Moving the lamplighter within the range from $-n$ to n only decreases the length of w, as we do not have to move the lamplighter back to the origin at the end of the movement. Switching lights off also decreases the length of w, hence $wA^n \subseteq A^{|w|}$ and w is a dead end of depth at least $n + 1$.

Nevertheless, the sheltered hull of every finite subset of the lamplighter group L with respect to the generating set A is finite, as shown by the following proposition.

Proposition 3.16. *The sheltered hull of every finite subset of the lamplighter group* $L = \mathbb{Z}^2 \wr \mathbb{Z} = \langle a, t \mid a^2, [a, t^{-n} a t^n] \rangle$ *with respect to the generators* $\{a, t^{\pm 1}, e\}$ *is finite.*

Proof. More precisely we will show that for every $m \in \mathbb{N}$ if $w \in S_A(A^m)$ then the lamplighter in w is positioned within $-m, \dots, m$ and there is no lamp switched on outside of $\{-m, \dots, m\}$.

Assume the lamplighter in w is positioned at $k \in \mathbb{Z}$ with $|k| > m$. Assume without loss of generality that $k > m$. The element wa^n has the lamplighter at position $k + n > m + n$, hence $wa^n \notin A^{m+n}$ for every n. Thus $w \notin S_A(A^m)$.

Now assume the lamplighter in w is positioned at $k \in \{-m, \dots, m\}$ and there is a light switched on in w at position ℓ outside of $\{-m, \dots, m\}$. If $\ell > 0$, consider the element wa^{-n}. It still has a light switched on at ℓ and the lamplighter is positioned at $k - n \leq m < \ell$. Hence wa^{-n} has length at least $\ell + \ell - (k - n)$, since the lamplighter has to get to ℓ and then back to $k - n$. But $2\ell - k + n > 2m - m + n = m + n$. Therefore $wa^{-n} \notin A^{m+n}$ and $w \notin S_A(A^m)$. If on the other hand $\ell < 0$, the same reasoning for wa^n leads to $w \notin S_A(A^m)$. Therefore $|S_A(A^m)| < (2m + 1)2^{2m+1} < \infty$. Now every finite set $M \subseteq L$ is contained in A^m for some m and $S_A(M) \subseteq S_A(A^m)$. ∎

4. Growth related conjugacy invariants for φ_G

In this section we will show that we can recover some "coarse geometric" properties of a graph G from the dynamics of the map φ_G. We begin by showing how to characterize the finite vertex sets "dynamically".

Definition 4.1. Suppose $\varphi : X \to X$ is a function. Define

$$\operatorname{Fin}(\varphi) := \left\{x \in X : |\varphi^{-r}(\{\varphi^{r+n}(x)\})| < \infty \text{ for all } r, n \in \mathbb{N}\right\}.$$

Lemma 4.2. *Let G be a countable, strongly connected locally finite directed graph. Then* $\operatorname{Fin}(\varphi_G) = \{M \subseteq V(G) : |M| < \infty\}$.

Proof. Let M be finite. Then $\varphi_G^{r+n}(M)$ is also finite for every $r, n \in \mathbb{N}$. Every subset $N \subseteq V(G)$ with $\varphi_G^r(N) = \varphi_G^{r+n}(M)$ must be a subset of $\varphi_G^{r+n}(M)$, hence there are only finitely many such sets. This shows $M \in \operatorname{Fin}(\varphi_G)$.

Now let $M \subseteq V(G)$ be infinite. We will show that $|\varphi_G^{-1}(\varphi_G^3(M))| = \infty$, and so $M \notin \operatorname{Fin}(\varphi_G)$. Since M is infinite and G is locally finite, we can choose an infinite 1-separated set $N \subseteq M$. By this we mean that N is an infinite subset of M such that $v_1 \notin \varphi_G(\{v_2\})$ for all $v_1, v_2 \in N$ with $v_1 \neq v_2$. Define

$$M' := \left(\varphi_G^2(M) \setminus \varphi_G(N)\right) \cup N,$$
$$M'' := \varphi_G^2(M) \setminus M' = \varphi_G(N) \setminus N.$$

The sets M' and M'' are clearly disjoint. From the fact that N is infinite and 1-separated, it follows that both M' and M'' are infinite. We will show that $U \cup M' \in \varphi_G^{-1}(\varphi_G^3(M))$ for every $U \subseteq M''$. Since $\{U \cup M' : U \subseteq M''\}$ is infinite, this will complete the proof.

Let $U \subseteq M''$. Clearly $\varphi_G(U \cup M') \subseteq \varphi_G^3(M)$. On the other hand

$$\begin{aligned}\varphi_G^3(M) &= \varphi_G\big((\varphi_G^2(M) \setminus \varphi_G(N)) \cup \varphi_G(N)\big)\\ &\subseteq \varphi_G(M') \cup \varphi_G^2(N)\\ &\subseteq \varphi_G(M') \cup M' \cup \varphi_G(N)\\ &\subseteq \varphi_G(M') \subseteq \varphi_G(M' \cup U).\end{aligned}$$

∎

Definition 4.3. Let (Γ, A) be a finitely generated group. Define

$$\begin{aligned}\mathcal{L}_A^k(M) &:= \{N \subseteq \Gamma : \varphi_A^k(N) = \varphi_A^k(M)\} = \varphi_A^{-k}(\{\varphi_A^k(M)\}),\\ L_A^k(M) &:= \log_2 |\mathcal{L}_A^k(M)|.\end{aligned}$$

We cannot access the size of subset $M \subseteq V(G)$ via the dynamics of φ_A and therefore will use $L_A^k(M)$ as a substitute. The next two lemmas show that this might be feasible. We start by showing that the map $M \mapsto L_A^k(M)$ is monotonous for fixed k.

Lemma 4.4. *Let $k \in \mathbb{N}$ and let $M, M' \subseteq V(G)$ with $M \subseteq M'$. Then $L_A^k(M) \leq L_A^k(M')$.*

Proof. Define a map $\Psi : \mathcal{L}_A^k(M) \to V(G)$ by $N \mapsto N \cup (M' \setminus S_A^k(M))$. For $N \in \mathcal{L}_A^k(M)$ we have $\varphi_A^k(N) = \varphi_A^k(M)$, hence $N \subseteq S_A^k(M)$. Therefore $\Psi(N) \cap S_A^k(M) = N$. This shows that the map is injective. To prove our claim, it is now enough to show that the image of Ψ is contained in $\mathcal{L}_A^k(M')$, or in other words, that $\varphi_A^k(\Psi(N)) = \varphi_A^k(M')$. Since $N \subseteq \Psi(N)$ and $\varphi_A^k(N) = \varphi_A^k(M)$, it is enough to show that $\varphi_A^k(M') \setminus \varphi_A^k(M) \subseteq \varphi_A^k(\Psi(N))$. Let $y \in \varphi_A^k(M') \setminus \varphi_A^k(M)$. There must be $x \in M'$ with $y \in \varphi_A^k(\{x\})$. Since $y \notin \varphi_A^k(M)$, this implies $\varphi_A^k(\{x\}) \not\subseteq \varphi_A^k(M)$, hence $x \notin S_A^k(M)$. Together this shows

$$y \in \varphi_A^k(\{x\}) \subseteq \varphi_A^k(M' \setminus S_A^k(M)) \subseteq \varphi_A^k(\Psi(N)).$$

∎

Recall that A is positively generating and contains the identity, hence there exists $q \in \mathbb{N}$ such $A^{-1} \subseteq A^q$.

Lemma 4.5. *Let (Γ, A) be a finitely generated group and let $q \in \mathbb{N}$ be as above. Let M be a finite non-empty subset of Γ and let k be a positive integer such that $M \subseteq A^k$. The following inequalities hold for every $r \in \mathbb{N}$:*

$$|\varphi_A^{r-1}(M)| \leq L_A^{(q+1)(r+k)}(\varphi_A^r(M)) \leq |S_A^{(q+1)(r+k)}(\varphi_A^r(M))|.$$

Proof. If $\varphi_A^{(q+1)(r+k)}(\{v\}) \subseteq \varphi_A^{(q+1)(r+k)}(\varphi_A^r(M))$, then by definition

$$v \in S_A^{(q+1)(r+k)}(\varphi_A^r(M)).$$

This establishes the upper bound on $L_A^{(q+1)(r+k)}(M)$. To prove the lower bound on $L_A^{(q+1)(r+k)}(M)$, we verify the identity

$$\varphi_A^{(q+1)(r+k)}(\varphi_A^r(M)) = \varphi_A^{(q+1)(r+k)}(\varphi_A^r(M) \setminus \varphi_A^{r-1}(M)).$$

Clearly the set on the right-hand side is contained in the set appearing on the left-hand side of the identity we wish to check. Since Γ is infinite, there is a point $y \in M' := \varphi_A^r(M) \setminus \varphi_A^{r-1}(M)$. Hence $1_\Gamma \in \varphi_{A^{-1}}^{r+k}(\{y\}) \subseteq \varphi_A^{q(r+k)}(M')$ and so

$$\varphi_A^{r-1}(M) \subseteq \varphi_A^{k+r-1+q(r+k)}(M') \subseteq \varphi_A^{(q+1)(r+k)}(M').$$

Thus

$$\begin{aligned}\varphi_A^{(q+1)(r+k)}(\varphi_A^r(M)) &= \left(\varphi_A^{(q+1)(r+k)+r}(M) \setminus \varphi_A^{r-1}(M)\right) \cup \varphi_A^{r-1}(M)\\ &\subseteq \varphi_A^{(q+1)(r+k)}(\varphi_A^r(M) \setminus \varphi_A^{r-1}(M)).\end{aligned}$$ ∎

We recall notions of strong domination and strong equivalence for functions on the integers, following [6, Chapter VI]:

Definition 4.6. A function $f_1 : \mathbb{N} \to \mathbb{N}$ is said to *strongly dominate* a function $f_2 : \mathbb{N} \to \mathbb{N}$ if there is a constant $C > 0$ such that $f_2(n) \leq f_1(Cn)$ for all $n \in \mathbb{N}$. Two functions f_1 and f_2 are said to be *strongly equivalent* if they strongly dominate each other.

It is easily verified that strong equivalence is indeed an equivalence relation.

Definition 4.7. The *growth function* of a finitely generated group (Γ, A) is given by

$$r \mapsto |A^r|.$$

Theorem 4.8. *Let (Γ, A) be a finitely generated group and let $q \in \mathbb{N}$ be such that $A^{-1} \subseteq A^q$. Let $M \subseteq \mathrm{Fin}(\varphi_A)$ be non-empty and let k and q be such that $M \subseteq A^k$ and $A^{-1} \subset A^q$. Then the growth function of (Γ, A) is strongly equivalent to the function*

$$r \mapsto L_A^{(q+1)rk}(\varphi_A^r(M)).$$

Proof. Using the left inequality in Lemma 4.5, we have

$$|\varphi_A^{2r-1}(M)| \leq L_A^{2(q+1)rk}(\varphi_A^{2r}(M)).$$

Since M is non-empty, we have $|A^{2r-1}| \leq |\varphi_A^{2r-1}(M)|$, and as $r \leq 2r-1$, it follows that

$$|A^r| \leq L_A^{2(q+1)rk}(\varphi_A^{2r}(M)).$$

On the other hand, using the right inequality in Lemma 4.5 and

$$S_A^{(q+1)rk}(M) \subseteq S_A^{(q+1)rk}(A^{r+k}) \subseteq A^{(q+1)rk+r+k},$$

we conclude that $L_A^{(q+1)rk}(\varphi_A^r(M)) \leq |A^{(q+3)kr}|$. ∎

Corollary 4.9. *Let (Γ_1, A_1) and (Γ_2, A_2) be finitely generated groups. If φ_{A_1} and φ_{A_2} are topologically conjugate, then Γ_1 and Γ_2 have strongly equivalent growth functions.*

Proof. Suppose $\Phi : \mathcal{P}(\Gamma_1) \to \mathcal{P}(\Gamma_2)$ is a topological conjugacy between φ_{A_1} and φ_{A_2}. Take a non-empty $M_1 \in \mathrm{Fin}(\varphi_{A_1})$, and let $M_2 = \Phi(M_1)$. Clearly $M_2 \in \mathrm{Fin}(\varphi_{A_2})$ is non-empty. Also, for any $r, q, k \in \mathbb{N}$,

$$L_{A_1}^{(q+1)rk}(\varphi_{A_1}^r(M_1)) = L_{A_2}^{(q+1)rk}(\Phi(\varphi_{A_1}^r(M_1))) = L_{A_2}^{(q+1)rk}(\varphi_{A_2}^r(M_2)).$$

Choosing k and q large enough to apply Theorem 4.8, it follows that the growth functions of (Γ_1, A_1) and (Γ_2, A_2) are equivalent. ∎

When $\Gamma = \mathbb{Z}^d$, we can say more:

Corollary 4.10. *Let A_1 and A_2 be positively generating sets of $\mathbb{Z}^{d_1}$ and $\mathbb{Z}^{d_2}$ respectively, both containing* 0. *If φ_{A_1} and φ_{A_2} are conjugate, then $d_1 = d_2$ and* $\mathrm{vol}_{d_1}(\mathrm{conv}(A_1)) = \mathrm{vol}_{d_2}(\mathrm{conv}(A_2))$.

Proof. The growth type of $\mathbb{Z}^d$ is $n \mapsto n^d$ and these growth types are different for pairwise different d. Recall that by [13] $(\mathbb{Z}^d, A)$ has only finitely many dead ends (the statement in [13] is only for symmetric generating sets, but the proof goes through for positively generating sets). By Corollary 3.10, we have $S_A(A^n) = A^n$ for sufficiently large n (we use multiplicative notation here for the group operation, even though the group is $\mathbb{Z}^d$).

Therefore, for $M \in \mathrm{Fin}(\varphi_A)$, q so that $A^{-1} \subseteq A^q$, k so that $M \subseteq A^k$ and large r,

$$\frac{1}{r^d}|A^{r-1}| \le \frac{1}{r^d} L_A^{(q+1)rk}(\varphi_A^r(M)) \le \frac{1}{r^d}|A^{r+k}|.$$

Sending r to infinity, the left and the right side of this inequality both converge to $\mathrm{vol}(\mathrm{conv}(A))$, a fact which follows directly from Proposition 8.1 below. Therefore

$$\lim_{r\to\infty} \frac{1}{r^d} \log_2 \left|\varphi_A^{-(q+1)rk}\left(\{\varphi_A^{(q+1)rk}(\varphi_A^r(M))\}\right)\right| = \mathrm{vol}(\mathrm{conv}(A)).$$ ∎

Definition 4.11. For $f : \mathbb{N} \to \mathbb{N}$ we call $\omega(f) = \lim_{n\to\infty} \sqrt[n]{f(n)}$ the exponential growth rate of f if this limit exists.

An argument very similar to that in the proof of Corollary 4.10 shows that for free groups of rank at least 2 the exponential growth is also "dynamically recognizable".

Corollary 4.12. *Let (Γ_1, A_1) and (Γ_2, A_2) be two finitely generated groups with infinitely many ends. If φ_{A_1} and φ_{A_2} are topologically conjugate, then* $\mathrm{Cayley}(\Gamma_1, A_1)$ *and* $\mathrm{Cayley}(\Gamma_2, A_2)$ *have the same exponential growth rate.*

5. Amenability

Throughout this section (Γ, A) will be a finitely generated group, and $q \in \mathbb{N}$ will be a constant such that $A^{-1} \subseteq A^q$. The aim of this section is to show that amenability of (Γ, A) can be characterized in terms of the dynamics of φ_A.

Recall that a sequence $(M_n)_{n\in\mathbb{N}}$ of finite subsets of Γ is called a *(right) Følner sequence* if for any $g \in \Gamma$, $|M_n g \setminus M_n|/|M_n| \to 0$ as $n \to \infty$. Existence of a Følner sequence is one of many equivalent conditions for amenability of a group [8]. The following lemma states two of many well-known equivalent conditions for a sequence of subsets to be a Følner sequence. See for instance [5, Chapter 4] for details.

Lemma 5.1. *For an increasing sequence $(M_n)_{n\in\mathbb{N}}$ of finite subsets of Γ the following are equivalent:*

(1) *M_n is a Følner sequence.*

(2) *There is $\ell \geq 1$ with $|\varphi_A^\ell(M_n)|/|M_n| \to 1$.*

(3) *There is $\ell \geq 1$ with $|\varphi_A^\ell(M_n) \setminus M_n|/|M_n| \to 0$.*

Lemma 5.2. *Let $k \in \mathbb{N}$ and let $M, N \subseteq \Gamma$ with $\varphi_A^k(M) \subseteq \varphi_A^k(N)$. Then*

$$\varphi_A^k\big((N \cap M) \cup (\varphi_A^{(q+1)k}(M) \setminus M)\big) = \varphi_A^k(\varphi_A^{(q+1)k}(M)).$$

Proof. Clearly

$$\varphi_A^k\big((N \cap M) \cup (\varphi_A^{(q+1)k}(M) \setminus M)\big) \subseteq \varphi_A^k(\varphi_A^{(q+1)k}(M)).$$

Let us show the other inclusion. Suppose $u \in \varphi_A^k(\varphi_A^{(q+1)k}(M))$. We need to show that

$$u \in \varphi_A^k\big((N \cap M) \cup (\varphi_A^{(q+1)k}(M) \setminus M)\big).$$

If $u \in \varphi_A^k(\varphi_A^{(q+1)k}(M) \setminus M)$, we are done. So assume that $u \notin \varphi_A^k(\varphi_A^{(q+1)k}(M) \setminus M)$. Thus,

$$u \in \varphi_A^k(M) \setminus \big(\varphi_A^k(\varphi_A^{(q+1)k}(M) \setminus M)\big).$$

In particular, because $\varphi_A^k(M) \subseteq \varphi_A^k(N)$, we have $u \in \varphi_A^k(N)$, so there exists $v \in N$ so that $u \in \varphi_A^k(\{v\})$. Hence

$$v \in \varphi_A^{qk}(\{u\}) \subseteq \varphi_A^{(q+1)k}(M).$$

But the assumption $u \notin \varphi_A^k(\varphi_A^{(q+1)k}(M) \setminus M)$ together with $u \in \varphi_A^k(\{v\})$ implies that $v \in M$. We conclude that $u \in \varphi_A^k(N \cap M)$. This completes the proof. ∎

Lemma 5.3. *Let $M \subseteq \mathrm{Fin}(\varphi_A)$ and $k \in \mathbb{N}$. If $\ell \geq (q+1)k$, then*

$$L_A^k(\varphi_A^\ell(M)) \leq L_A^k(\varphi_A^{(q+1)k}(M)) + |\varphi_A^{\ell+k}(M) \setminus M|.$$

Proof. Consider the map

$$\Psi : \mathcal{L}_A^k(\varphi_A^{(q+1)k}(M)) \times \mathcal{P}(\varphi_A^{\ell+k}(M) \setminus M) \to \varphi_A^{\ell+k}(M),$$
$$\Psi(W_1, W_2) := W_1 \triangle W_2.$$

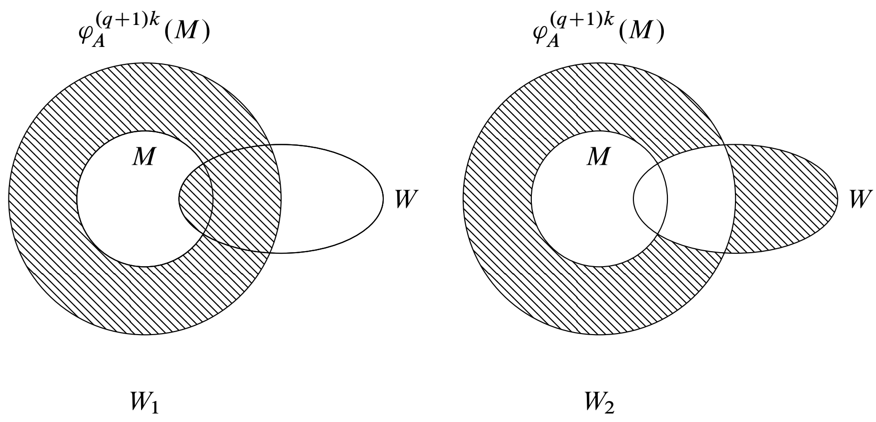


Figure 3. Illustration of the decomposition of W in the proof of Lemma 5.3.

We have to show that $\mathcal{L}_A^k(\varphi_A^\ell(M))$ is contained in the image of Ψ. Let $W \in \mathcal{L}_A^k(\varphi_A^\ell(M))$. Then (see Figure 3)

$$\begin{aligned} W &= W_1 \triangle W_2 \text{ with} \\ W_1 &:= \big((W \cap M) \cup (\varphi_A^{(q+1)k}(M) \setminus M)\big), \\ W_2 &:= \big(\varphi_A^{(q+1)k}(M) \triangle (M \cup W)\big). \end{aligned}$$

By Lemma 5.2 the set W_1 is contained in $\mathcal{L}_A^k(\varphi_A^{(q+1)k}(M))$. Since $W \in \mathcal{L}_A^k(\varphi_A^\ell(M)) \subseteq \mathcal{P}(\varphi_A^{\ell+k}(M))$ and $W_2 \cap M = \emptyset$, we also have $W_2 \subseteq \varphi_A^{\ell+k}(M) \setminus M$. ∎

Lemma 5.4. *Let H be a finite undirected graph such that every connected component contains at least two vertices. Then H has a vertex cover of size at most $|V(H)|/2$.*

Proof. Choose a spanning tree in every connected component. Since every one of these spanning trees is bipartite, we can pick the smaller of the two partition classes in each of them. Since all our spanning trees contain at least two vertices, the picked vertices form a vertex cover and we picked at most half of the vertices. ∎

Lemma 5.5. *Let $M \subseteq \mathrm{Fin}(\varphi_A)$. Then for all $k \geq q$, $\ell > 1$ we have $L_A^k(\varphi_A^\ell(M)) \geq \frac{1}{2}|M|$.*

Proof. Form the following graph H. The vertices of H are the elements of $\varphi_A(M)$ and we add an edge between u and v in $\varphi_A(M)$ if $u \in \varphi_A^q(\{v\})$ and $v \in \varphi_A^q(\{u\})$. Since for every $a \in A$ and $u \in M$ we have $u = uaa^{-1} \in \varphi_{A^{-1}}(\{ua\}) \subseteq \varphi_A^q(\{ua\})$ and $ua \in \varphi_A(\{u\})$, every connected component of H contains at least two vertices. Hence by Lemma 5.4 we can find a subset $W \subseteq \varphi_A(M)$ with $|W| \leq \frac{1}{2}|\varphi_A(M)|$ such that for all elements $v \in \varphi_A(M)$ there is a vertex $w \in W$ with $v \in \varphi_A^q(\{w\})$, hence $\varphi_A(M) \subseteq \varphi_A^k(W)$. Thus for every set $N \subseteq \varphi_A^\ell(M)$ with $(\varphi_A^\ell(M) \setminus \varphi_A(M)) \cup W \subseteq N$ we have $\varphi_A^k(N) = \varphi_A^k(\varphi_A^\ell(M))$. Since there are $2^{|\varphi_A(M)|-|W|}$ such sets, we have $L_A^k(\varphi_A^\ell(M)) \geq \frac{1}{2}|\varphi_A(M)| \geq \frac{1}{2}|M|$. ∎

Lemma 5.6. *Let $k \in \mathbb{N}$ and let $M \subseteq \mathrm{Fin}(\varphi_A)$. If $\ell \geq (q+2)k+3$, then*

$$L_A^k(\varphi_A^\ell(M)) \geq L_A^k(\varphi_A^{(q+1)k}(M)) + |\varphi_A^{\ell-1}(M) \setminus \varphi_A^{\ell-2}(M)|.$$

Proof. By definition of L_A^k, we need to prove that

$$|\mathcal{L}^k(\varphi_A^\ell(M))| \geq \left|\mathcal{L}_A^k(\varphi_A^{(q+1)k}(M)) \times \mathcal{P}(\varphi_A^{\ell-1}(M) \setminus \varphi_A^{\ell-2}(M))\right|.$$

We prove this by constructing an injective function

$$\Psi : \mathcal{L}_A^k(\varphi_A^{(q+1)k}(M)) \times \mathcal{P}(\varphi_A^{\ell-1}(M) \setminus \varphi_A^{\ell-2}(M)) \to \mathcal{L}_A^k(\varphi_A^\ell(M)).$$

This is given by

$$\Psi(Q,P) := (\varphi_A^\ell(M) \setminus \varphi_A^{\ell-1}(M)) \cup P \cup (\varphi_A^{\ell-2}(M) \setminus \varphi_A^{(q+2)k}(M)) \cup Q.$$

First we check that the image of this map lies indeed in $\mathcal{L}_A^k(\varphi_A^\ell(M))$. Let

$$(Q,P) \in \mathcal{L}_A^k(\varphi_A^{(q+1)k}(M)) \times \mathcal{P}(\varphi_A^{\ell-1}(M) \setminus \varphi_A^{\ell-2}(M)).$$

It is clear that $\varphi_A^k(\Psi(Q,P)) \subseteq \varphi_A^{k+\ell}(M)$. We also have

$$\begin{aligned}(\varphi_A^{\ell+k}(M) \setminus \varphi_A^{\ell-1}(M)) &\cup (\varphi_A^{\ell-2+k}(M) \setminus \varphi_A^{(q+2)k}(M)) \\ &= \varphi_A^{\ell+k}(M) \setminus \varphi_A^{(q+2)k}(M) \subseteq \varphi_A^k(\Psi(Q,P)).\end{aligned}$$

Finally, $\varphi_A^k(Q) = \varphi_A^{(q+2)k}(M)$, hence $\varphi_A^{k+\ell}(M) \subseteq \varphi_A^k(\Psi(Q,P))$.

It is now enough to check that Ψ is injective. This follows from

$$\begin{aligned}P &= \Psi(Q,P) \cap (\varphi_A^{\ell-1}(M) \setminus \varphi_A^{\ell-2}(M)),\\ Q &= \Psi(Q,P) \cap \varphi_A^{(q+2)k}(M).\end{aligned}$$

■

Theorem 5.7. *Let (Γ, A) be a finitely generated group and let $q \in \mathbb{N}$ be such that $A^{-1} \subseteq A^q$. Then Γ is amenable if and only if there is a sequence of finite sets $(M_n)_{n\in\mathbb{N}}$ in $\mathrm{Fin}(\varphi_A)$ such that*

$$\lim_{n\to\infty} \frac{L_A^q(\varphi_A^{(q+5)q}(M_n))}{L_A^q(\varphi_A^{(q+1)q}(M_n))} = 1. \tag{5.1}$$

Proof. Let Γ be amenable and let $(M_n)_{n\in\mathbb{N}}$ be a Følner sequence. By Lemmas 4.4, 5.3 and 5.5 we have

$$\begin{aligned}1 \leq \frac{L_A^q(\varphi_A^{(q+5)q}(M_n))}{L_A^q(\varphi_A^{(q+1)q}(M_n))} &\leq 1 + \frac{|\varphi_A^{(q+6)q}(M_n) \setminus M_n|}{L_A^q(\varphi_A^{(q+1)q}(M_n))}\\ &\leq 1 + \frac{2|\varphi_A^{(q+6)q}(M_n) \setminus M_n|}{|M_n|},\end{aligned}$$

where the last inequity follows from Lemma 5.5. By Lemma 5.1 the right-hand side converges to one and therefore (5.1) is satisfied.

On the other hand assume that $(M_n)_{n\in\mathbb{N}}$ is a sequence in $\mathrm{Fin}(\varphi_A)$ satisfying (5.1). By Lemma 5.6 we have

$$\begin{aligned}\frac{L_A^q(\varphi_A^{(q+5)q}(M_n))}{L_A^q(\varphi_A^{(q+1)q}(M_n))} &\geq 1+\frac{|\varphi_A^{q(q+5)-1}(M_n)\setminus\varphi_A^{q(q+5)-2}(M_n)|}{L_A^q(\varphi_A^{(q+1)q}(M_n))}\\ &\geq 1+\frac{|\varphi_A^{q(q+5)-1}(M_n)\setminus\varphi_A^{q(q+5)-2}(M_n)|}{|\varphi_A^{(q+2)q}(M_n)|}\\ &\geq 1+\frac{|\varphi_A^{q(q+5)-1}(M_n)\setminus\varphi_A^{q(q+5)-2}(M_n)|}{|\varphi_A^{(q+5)q-2}(M_n)|}\geq 1.\end{aligned}$$

Since we assumed that the left side of this inequality converges to one, this shows that

$$\frac{|\varphi_A(\varphi_A^{(q+5)q-2}(M_n))\setminus\varphi_A^{(q+5)q-2}(M_n)|}{|\varphi_A^{(q+5)q-2}(M_n)|}\to 0.$$

Hence by Lemma 5.1 the sequence $(\varphi_A^{(q+5)q-2}(M_n))_{n\in\mathbb{N}}$ is a Følner sequence and Γ is amenable. ∎

Corollary 5.8. *Let (Γ_1, A_1) and (Γ_2, A_2) be finitely generated groups such that φ_{A_1} and φ_{A_2} are topologically conjugate. If Γ_1 is amenable, then Γ_2 is amenable too.*

6. North-south dynamics

Definition 6.1. A homeomorphism $T : X \to X$ of a compact metric space X is said to have *north-south dynamics* if there are precisely two fixed points $x^+, x^- \in X$ for T such that $\lim_{n\to\infty} T^n(y) = x^+$ for every $y \in X\setminus\{x^-\}$ and $\lim_{n\to\infty} T^{-n}(y) = x^-$ for every $y \in X \setminus \{x^+\}$.

Two simple examples of homeomorphisms with north-south dynamics are the map $t \mapsto \sqrt{t}$ on the interval $[0, 1]$ and the map $n \mapsto n + 1$ on the two-point compactification $\mathbb{Z}_{\pm\infty} = \mathbb{Z} \cup \{+\infty, -\infty\}$.

For any homeomorphism $T : X \to X$ the map $S(T) : S(X) \to S(X)$ given by

$$S(T)(x,t) = (T(x), \sqrt{t})$$

has north-south dynamics. Here $S(X)$ is the suspension of the topological space X given by

$$\begin{gathered} S(X) = (X\times[0,1])/{\sim},\\ (x_1,t_1)\sim(x_2,t_2) \iff t_1=t_2=0 \text{ or } t_1=t_2=1 \text{ or } (x_1,t_1)=(x_2,t_2).\end{gathered}$$

So for instance the suspension of the d-dimensional sphere is the $(d+1)$-dimensional sphere.

In a similar way we can define $S_{\mathbb{Z}}(T) : S_{\mathbb{Z}}(X) \to S_{\mathbb{Z}}(X)$ by

$$S_{\mathbb{Z}}(T)(x,n) = (T(x), n+1),$$

where $S_{\mathbb{Z}}(X)$ is a disconnected analog of the suspension given by

$$S_{\mathbb{Z}}(X) = (X \times \mathbb{Z}_{\pm\infty})/\sim,$$
$$(x_1,n_1) \sim (x_2,n_2) \iff n_1 = n_2 = +\infty \text{ or } n_1 = n_2 = -\infty \text{ or } (x_1,n_1) = (x_2,n_2).$$

A complete characterization of north-south systems was obtained for many spaces, see [15] for a survey of known results. This includes the Cantor set for which the following uniqueness result was obtained in [14]. We include a short proof for self-containment.

Proposition 6.2. *Up to topological conjugacy there is a unique homeomorphism with north-south dynamics on the Cantor space* $\{0,1\}^{\mathbb{N}}$.

Proof. We will show that any north-south dynamics on a Cantor space is topologically conjugate to the "standard" north-south dynamics $\varphi(x,n) = (x, n+1)$, $\varphi : S_{\mathbb{Z}}(C) \to S_{\mathbb{Z}}(C)$ where C is a Cantor space and $S_{\mathbb{Z}}(C)$ is the "disconnected suspension" defined above. It is easy to check that $S_{\mathbb{Z}}(C)$ is compact, totally disconnected, second countable and has no isolated points, so it is a Cantor space.

Let X be a Cantor space and $T : X \to X$ be any homeomorphism with north-south dynamics. Let D be an clopen neighborhood of the unique positively attracting fixed point x^+ such that $X \setminus D$ is a neighborhood of the unique negatively attracting fixed point x^-. For every point $x \in X \setminus \{x^+, x^-\}$ the expression $n_D(x) = \sup\{n \in \mathbb{Z} : T^n(x) \in X \setminus D\}$ is a well defined integer. We want to show that

$$C := \left\{x \in X \setminus \{x^+, x^-\} : n_D(x) = 0\right\} = (X \setminus D) \cap \bigcap_{k=1}^{\infty} T^{-k}(D)$$

is a clopen set in X. For every point $x \in X \setminus \{x^+\}$ there is $k > 0$ such that $T^{-k}(x) \in X \setminus D$, hence $\bigcap_{k=1}^{\infty} T^k(D) \cap (X \setminus D)$ is empty. By compactness there must be $N \in \mathbb{N}$ such that already the finite intersection $\bigcap_{k=1}^{N} T^k(D) \cap (X \setminus D)$ is empty. In other words, $\bigcap_{k=1}^{N} T^k(D) \subseteq D$ and therefore

$$\bigcap_{k=1}^{N} T^{-k}(D) \subseteq T^{-(N+1)}(D).$$

Thus

$$\bigcap_{k=1}^{N} T^{-k}(D) = \bigcap_{k=1}^{N+1} T^{-k}(D) \subseteq T^{-(N+2)}(D).$$

By induction we obtain

$$\bigcap_{k=1}^{N} T^{-k}(D) = \bigcap_{k=1}^{\infty} T^{-k}(D).$$

This shows that $C = (X \setminus D) \cap \bigcap_{k=1}^{\infty} T^{-k}(D)$ is indeed a clopen subset of X hence a Cantor space itself. Notice that the orbit of every non-fixed point x of T intersects C in precisely one point, namely $T^{n_D(x)}(x)$, since $n_D(T(x)) = n_D(x) - 1$.

Let y^+ be the attracting and y^- the repelling fixed point of φ. We define

$$\Phi : X \to S_{\mathbb{Z}}(C), \quad \Phi(x) = \begin{cases} y^+ & \text{if } x = x^+, \\ y^- & \text{if } x = x^-, \\ (T^{n_D(x)}(x), -n_D(x)) & \text{otherwise.} \end{cases}$$

It is clear that Φ intertwines φ and T and that it is bijective. To complete the proof, we check that Φ is continuous.

Let $(x_n)_{n\in\mathbb{N}}$ be a sequence of points in X so that $\lim_{n\to\infty} x_n = x \in X$. If $x \notin \{x^-, x^+\}$, then $x \in T^{-n_D(x)}(C)$. Since $T^{-n_D(x)}(C)$ is open, it also contains x_n for sufficiently large n. For these n we have

$$\Phi(x_n) = (T^{-n_D(x_n)}(x_n), -n_D(x_n)) = (T^{-n_D(x)}(x_n), -n_D(x)) \to \Phi(x).$$

If $x = x^+$, then $n_D(x_n) \to -\infty$ and $\Phi(x_n) \to y^+$. Similarly, if $x = x^-$, then $n_D(x_n) \to \infty$ and $\Phi(x_n) \to y^-$. ■

7. The eventual image and the natural extension of φ_G

Let (X, φ) be a topological dynamical system, not necessarily invertible. Namely X is a compact topological space and $\varphi : X \to X$ is a continuous self-map. The *eventual image* (also called the *maximal attractor*) is given by $\mathrm{Evt}(\varphi) := \bigcap_{n=1}^{\infty} \varphi^n(X)$.

Proposition 7.1. *For a non-empty set $M \subseteq V(G)$ with $M \neq V(G)$ the following are equivalent:*

(1) *M is in the eventual image $\bigcap_{n=1}^{\infty} \varphi_G^n(\mathcal{P}(V(G)))$.*

(2) *M is the union of arbitrary large balls, i.e. for every r there is $M' \subseteq M$ such that $M = \varphi_G^r(M')$.*

(3) *M is a union of horoballs.*

(4) *M is the union of Busemann balls.*

Proof. Conditions (1) and (2) are obviously equivalent as for any $M' \subseteq V(G)$ and $n > 0$ we have $\varphi_G^n(M') = \bigcup_{w\in M'} \varphi_G^n(\{w\})$. Assume now that M is a union of arbitrary large balls. For every $v \in M$ there is a sequence of points w_k and increasing radii r_k such that

$v \in \varphi_G^{r_k}(\{w_k\}) \subseteq M$. Taking a limit along a subsequence, one obtains a horoball in M containing v. That every union of horoballs is a union of Busemann balls and vice versa follows directly from Proposition 2.5. Condition (4) implies (2) because every Busemann ball is an increasing union of arbitrary large balls. ■

In view of the above proposition, we call elements of $\mathrm{Evt}(\varphi_G)$ *horoballunions*.

The *natural extension* of a dynamical system (X, φ) (which is not necessarily injective nor surjective) is the dynamical system $(\hat{X}_\varphi, \hat{\varphi})$ where

$$\hat{X}_\varphi := \left\{\hat{x} \in X^{\mathbb{Z}} : \hat{x}_{n+1} = \varphi(\hat{x}_n) \text{ for all } n \in \mathbb{Z}\right\},$$

and $\hat{\varphi} : \hat{X}_\varphi \to \hat{X}_\varphi$ is the shift given by

$$\hat{\varphi}(\hat{x})_n = \varphi(\hat{x}_n).$$

Note that $\hat{\varphi}(\hat{x})_n = \hat{x}_{n+1}$ for every $x \in \hat{X}_\varphi$ and $n \in \mathbb{Z}$. It follows that $\hat{\varphi} : \hat{X}_\varphi \to \hat{X}_\varphi$ is a homeomorphism. The natural extension factors onto the eventual image via the projection $\hat{x} \mapsto \hat{x}_0$. Every morphism from an invertible system factors through the natural extension. In particular, any other invertible extension of the eventual image factors through the natural extension.

For a countably infinite graph G, the natural extension of $(\mathcal{P}(V(G)), \varphi_G)$ is topologically conjugate to the map $(x_v)_{v \in V(G)} \mapsto (x_v + 1)_{v \in V(G)}$ on the space $\widetilde{X}_G \subseteq \mathbb{Z}_{\pm\infty}^{V(G)}$, consisting of the two points $x^+ := (+\infty)^{V(G)}$ and $x^- := (-\infty)^{V(G)}$ and all the points $x \in \mathbb{Z}^{V(G)}$ that satisfy:

- For every $v \in V(G)$ there exists $w \in V(G)$ such that $(w, v) \in E(G)$ and $x_w = x_v + 1$.
- For every $(v, w) \in E(G)$ one has $x_v \leq x_w + 1$.

In the case of an undirected graph, we can say that the natural extension of $\varphi_G : \mathcal{P}(V(G)) \to \mathcal{P}(V(G))$ is "pointwise incrementing by 1 on the two-point compactification of the integer-valued 1-Lipschitz functions on $V(G)$ without local maxima". With this identification, the natural extension of the subsystem corresponding to the closure of the horoballs consists of $\mathbb{Z}$-valued functions which are vertical translations of horofunctions (with the two additional points x^- and x^+ "at infinity"), see Proposition 2.3.

Proposition 7.2. *Suppose G_1, G_2 are both countable, strongly connected locally finite directed graphs. Let $(\hat{X}_{G_i}, \hat{\varphi}_{G_i})$ denote the natural extensions of φ_{G_i} for $i = 1, 2$. If $\hat{X}_{G_1}$ and $\hat{X}_{G_2}$ have no isolated points, then $(\hat{X}_{G_1}, \hat{\varphi}_{G_1})$ is topologically conjugate to $(\hat{X}_{G_2}, \hat{\varphi}_{G_2})$.*

Proof. For any strongly connected, countable graph G the natural extension of φ_G has a unique attracting fixed point x^+ given by $x_n^+ = V(G)$ for every $n \in \mathbb{Z}$, and a unique repelling fixed point x^- given by $x_n^- = \emptyset$. The natural extension of φ_G acts on a closed subspace of $\mathcal{P}(V(G))^{\mathbb{Z}}$, so under the assumption of no isolated points it has north-south dynamics on the Cantor set. The result follows by Proposition 6.2. ■

8. The geometry of horoballs in $\mathbb{Z}^d$

In the following sections we will exclusively consider abelian groups ($(\mathbb{Z}^d, +)$ and $(\mathbb{R}^d, +)$), so we switch to additive notation. For $B, C \subseteq \mathbb{Z}^d$ or $B, C \subseteq \mathbb{R}^d$ we write $B + C$ for the Minkowski sum

$$B + C = \{b + c : b \in B, c \in C\}.$$

For $B \subseteq \mathbb{Z}^d$ or $B \subseteq \mathbb{R}^d$ and $n \in \mathbb{N}$ we abbreviate the n-fold Minkowski sum of B by nB:

$$nB = \underbrace{B + \cdots + B}_{n} = \{b_1 + \cdots + b_n : b_1, \ldots, b_n \in B\}.$$

Note that whenever $B \subseteq \mathbb{R}^d$ is convex, the n-fold Minkowski sum coincides with n-dilation, meaning $nB = \{nb : b \in B\}$ for $n \in \mathbb{N}$. We denote the closed ball of radius R in $\mathbb{R}^n$ around v by $B_R(v)$. We denote the convex hull of a subset $B \subseteq \mathbb{R}^d$ by $\mathrm{conv}(B)$, and by $\mathrm{Ext}(B)$ the extremal points (or vertices) of $\mathrm{conv}(B)$. Whenever we write $\mathrm{conv}(B)$ for a set $B \subset \mathbb{Z}^d$, we mean the convex hull of B in $\mathbb{R}^d$.

In the following, A will be a positively generating set of the additive group $\mathbb{Z}^d$. The set A will not necessarily be symmetric, but it will always contain 0. This last assumption is mostly for convenience, because for any positively generating set A, there exists a positive integer n_0 such that $0 \in nA$ for every n which is a positive integer multiple of n_0.

Iterated Minkowski sums of a finite positively generating set of $\mathbb{Z}^d$ are “roughly” equal to the integer points in the dilated convex hull. A proof of this fact appears in [12]. For completeness, we include a precise statement and a short proof based on the well-known Shapley–Folkman lemma (see for instance [4]). See [29] for a similar convexity-based proof of a closely related result, and also [9].

Proposition 8.1. *Let $A \subseteq \mathbb{Z}^d$ be a finite positively generating set with $0 \in A$. Then for any $n \in \mathbb{N}$*

$$nA \subseteq n\,\mathrm{conv}(A) \cap \mathbb{Z}^d,$$

and there exists $N \in \mathbb{N}$ so that for every $n > N$

$$(n - N)\,\mathrm{conv}(A) \cap \mathbb{Z}^d \subseteq nA.$$

Proof. By definition of the convex hull, $A \subseteq \mathrm{conv}(A)$, so $nA \subseteq n\,\mathrm{conv}(A) \cap \mathbb{Z}^d$. To prove the second part we apply the Shapley–Folkman lemma which implies that for any compact $B \subseteq \mathbb{R}^d$ and $m \geq d$

$$\mathrm{conv}(mB) = (m - d)B + d\,\mathrm{conv}(B).$$

Since $A \subset \mathbb{Z}^d$, it follows that

$$\mathrm{conv}(mA) \cap \mathbb{Z}^d = (m - d)A + (d\,\mathrm{conv}(A) \cap \mathbb{Z}^d).$$

Since A is positively generating and contains the identity, the sequence kA monotonically increases to $\mathbb{Z}^d$ as $k \to \infty$. Since $d\,\mathrm{conv}(A) \cap \mathbb{Z}^d$ is a finite subset of $\mathbb{Z}^d$, it follows that there exists $k \in \mathbb{N}$ so that $d\,\mathrm{conv}(A) \cap \mathbb{Z}^d \subseteq kA$. Together this implies

$$m\,\mathrm{conv}(A) \cap \mathbb{Z}^d = \mathrm{conv}(mA) \cap \mathbb{Z}^d \subseteq (m-d)A + kA. \qquad \blacksquare$$

Definition 8.2. For a face F of $\mathrm{conv}(A)$ let M_F denote the collection of $(d-1)$-dimensional faces F' of $\mathrm{conv}(A)$ that contain F.

Definition 8.3. For a $(d-1)$-dimensional face F of $\mathrm{conv}(A)$ let ℓ_F denote the inward facing unit normal vector of F, namely the unique $\ell_F \in \mathbb{R}^d$ satisfying

- $\|\ell_F\| = 1$,
- $\langle \ell_F, u - v\rangle = 0$ for all $u, v \in F$,
- $\langle \ell_F, u - v\rangle \geq 0$ for all $u \in \mathrm{conv}(A)$, $v \in F$.

Given a face F of $\mathrm{conv}(A)$, we define the *envelope of A with respect to F* to be

$$\mathrm{Env}_F = \mathrm{Env}_{F,A} := \bigcap_{F' \in M_F} \{v \in \mathbb{R}^d : \langle \ell_{F'}, v\rangle \geq 0\}.$$

It follows from the definition that

$$\mathrm{Env}_F = \Big\{ \sum_{v \in \mathrm{Ext}(F)} \sum_{u \in A} \alpha_{u,v}(u - v) : \alpha_{u,v} \in [0,\infty) \text{ for all } u \in A,\ v \in \mathrm{Ext}(F)\Big\}. \quad (8.1)$$

In particular, if $F = \{v\}$ is a zero-dimensional face of $\mathrm{conv}(A)$, then

$$\mathrm{Env}_F = \Big\{ \sum_{u \in A} t_u(u - v) : t \in [0,\infty)\Big\}.$$

Also, if F is a $(d-1)$-dimensional face of $\mathrm{conv}(A)$, then Env_F is a translation of the unique half-space containing A whose boundary is the affine hull of F.

In $\mathbb{Z}^d$ the structure of A-horoballs is essentially given by the convex hull of A. There is a one-to-one correspondence between faces of $\mathrm{conv}(A)$ and A-horoballs up to translation. The following theorem makes this precise.

Theorem 8.4. *Let $A \subseteq \mathbb{Z}^d$ be a finite generating set. For any face F of* $\mathrm{conv}(A)$ *there exists a unique A-horoball $H_F \subseteq \mathbb{Z}^d$ with the property that* $0 \in H_F$ *and*

$$\mathrm{Env}_F = \mathrm{conv}(H_F).$$

The horoball H_F is given by

$$H_F = \sum_{w \in \mathrm{Ext}(F)} \bigcup_{j=0}^{\infty} j(A - w). \quad (8.2)$$

Conversely, any A-horoball is of the form $v + H_F$ for some $v \in \mathbb{Z}^d$ and a face F of $\mathrm{conv}(A)$. *This face is uniquely determined.*

Note that H_F is precisely the semigroup generated by $\{(u-v) : u \in A,\ v \in \mathrm{Ext}(F)\}$. We split the proof into a number of simple lemmas.

Lemma 8.5. *For any face F of* $\mathrm{conv}(A)$ *we have*

$$\mathrm{conv}(H_F) = \mathrm{Env}_F\ .$$

Proof. For any finite set $B \subseteq \mathbb{R}^d$,

$$\mathrm{conv}\Big(\Big\{\sum_{v\in B} a_v v : a_v \in \mathbb{Z}_+\Big\}\Big) = \Big\{\sum_{v\in B} \alpha_v v : \alpha_v \in [0,\infty)\Big\}.$$

The result follows immediately from the expression (8.2) for H_F and the expression (8.1) for Env_F. ■

Lemma 8.6. *Let F be a face of* $\mathrm{conv}(A)$*. Then the set H_F given by* (8.2) *is an A-horoball.*

Proof. Let $m = |\mathrm{Ext}(F)|$. For any $j \in \mathbb{N}$ let

$$B_j := \sum_{w\in \mathrm{Ext}(F)} j(A-w).$$

Then

$$B_j = \Big(-j \sum_{w\in\mathrm{Ext}(F)} w\Big) + jmA$$

is a translate of jmA. Since $B_j \subseteq B_{j+1}$ for every j,

$$\lim_{j\to\infty} B_j = \bigcup_{j=1}^{\infty} B_j = \bigcup_{j=1}^{\infty} \sum_{w\in\mathrm{Ext}(F)} j(A-w) = H_F.$$

This shows H_F is an increasing union of translates of iterated sums of A. It is clearly non-empty and by Lemma 8.5 it is contained in a half-space, so $H_F \neq \mathbb{Z}^d$. This proves H_F is indeed a horoball. ■

We will need the following lemma only later but we prove it here since it is very close in spirit to the previous one.

Lemma 8.7. *For any face F of* $\mathrm{conv}(A)$ *there exists $v \in \mathbb{Z}^d$ so that*

$$\mathrm{Env}_F \cap \mathbb{Z}^d \subseteq v + H_F.$$

Proof. As in the proof of Lemma 8.6, for every $N \in \mathbb{N}$ we have

$$H_F = \bigcup_{j=N}^{\infty}\Big(\Big(-j\sum_{w\in\mathrm{Ext}(F)} w\Big) + jmA\Big), \tag{8.3}$$

where $m = |\mathrm{Ext}(F)|$. By Proposition 8.1 there exists $N \in \mathbb{N}$ so that for any $j \geq N$

$$(j-N)\,\mathrm{conv}(A) \cap \mathbb{Z}^d \subseteq jA.$$

From this it follows that for every $j \geq N$

$$\Big(-j \sum_{w \in \mathrm{Ext}(F)} w\Big) + (mj - N)\,\mathrm{conv}(A) \cap \mathbb{Z}^d \subseteq jmA + \Big(-j \sum_{w \in \mathrm{Ext}(F)} w\Big).$$

Taking the union over $j \geq N$, using (8.3), it follows that

$$\bigcup_{j=N}^{\infty} \underbrace{\Big(\Big(-j \sum_{w \in \mathrm{Ext}(F)} w\Big) + (mj - N)\,\mathrm{conv}(A) \cap \mathbb{Z}^d\Big)}_{=:W_j} \subseteq H_F.$$

For each j the set W_j consists precisely of vectors $u \in \mathbb{R}^d$ of the form $u = \sum_{v \in \mathrm{Ext}(A)} a_v v$ so that $\sum_{v \in \mathrm{Ext}(A)} a_v = -N$, $a_v \geq 0$ for $v \in \mathrm{Ext}(A) \setminus \mathrm{Ext}(F)$, and $a_w \geq -j$ for $w \in \mathrm{Ext}(F)$. Moreover, $u \in \mathrm{Env}_F$ if and only if it is of the form $u = \sum_{v \in \mathrm{Ext}(A)} a_v v$ with $\sum_{v \in \mathrm{Ext}(A)} a_v = 0$ and $a_v \geq 0$ for any $v \in \mathrm{Ext}(A) \setminus \mathrm{Ext}(F)$. It follows that for $v_0 \in \mathrm{Ext}(F)$,

$$\bigcup_j W_j = \mathrm{Env}_F - N v_0.$$

Thus

$$\mathrm{Env}_F \cap \mathbb{Z}^d \subseteq N v_0 + H_F. \qquad \blacksquare$$

Here is a simple criterion for a semigroup in $\mathbb{Z}^d$ to be a group:

Lemma 8.8. *Let $S \subseteq \mathbb{Z}^d$ be a semigroup. Then S is a group if and only if the convex hull of S is equal to the linear span of S. Equivalently, a semigroup $S \subseteq \mathbb{Z}^d$ is a group if and only if it is not contained in a proper half-space of its linear span.*

Proof. Any subgroup of $\mathbb{Z}^d$ is a lattice in its linear span, and thus its convex hull is equal to its linear span. Conversely, suppose $S \subseteq \mathbb{Z}^d$ is a semigroup and that the convex hull of S is equal to the linear span of S. Then the rational convex hull of S is equal to the rational span of S. Now fix $a \in S$. Then in particular, $-a$ is in the rational convex hull of S, so there exist $q_1, \dots, q_n \in \mathbb{Q} \cap [0, \infty)$ and $v_1, \dots, v_m \in S$ so that

$$-a = \sum_{i=1}^{m} q_i v_i.$$

Multiplying by the common denominator of the q_i's, we see that for some positive integer N and positive integers $n_1, \dots, n_m$ we have $-Na = \sum_{i=1}^m n_i v_i$. This shows that $-Na$ is in the semigroup. Thus $-a = -Na + (N-1)a$ is also in the semigroup S. So the semigroup S is closed under inverses, thus it is a group. $\blacksquare$

For a set $L \subseteq \mathbb{Z}^d$ we denote the stabilizer of L by

$$\mathrm{stab}(L) := \{v \in \mathbb{Z}^d : L + v = L\}.$$

The set $\mathrm{stab}(L)$ is always a subgroup of $\mathbb{Z}^d$.

Also, for a face F of $\mathrm{conv}(A)$ let L_F denote the linear span of $\{u - v : u, v \in F \cap A\}$. Equivalently,

$$L_F = \bigcap_{F' \in M_F} \{v \in \mathbb{R}^d : \langle \ell_{F'}, v \rangle = 0\}.$$

The following lemma identifies the stabilizers of horoballs in $\mathbb{Z}^d$.

Lemma 8.9. *For any face F of* $\mathrm{conv}(A)$,

$$\mathrm{stab}(H_F) = H_F \cap L_F.$$

Equivalently, $\mathrm{stab}(H_F)$ *is equal to the group generated by* $\{u - v : u, v \in F \cap A\}$. *In particular,* $\mathrm{stab}(H_F)$ *is a finite index subgroup of* $L_F \cap \mathbb{Z}^d$.

Proof. Set $G_F := H_F \cap L_F$. From (8.2) and the fact that F is a face of the convex hull of A,

$$G_F = \sum_{v \in \mathrm{Ext}(A) \cap F} \bigcup_{j=0}^{\infty} j((A \cap F) - v).$$

In particular, G_F is a semigroup. Also since any vector of the form $v - w$ with $v, w \in \mathrm{Ext}(A \cap F)$ is in G_F, the convex hull of G_F is equal to L_F, which contains the linear span of G_F. Thus by Lemma 8.8, G_F is a group and therefore contains any vector of the form $u - v$ with $u, v \in A \cap F$.

Now since $0 \in H_F$, for any $v \in \mathrm{stab}(H_F)$, $v \in v + H_F = H_F$. If $v \in H_F \setminus L_F$, then the convex hull of $H_F + v$ is contained in the interior of the convex hull of H_F, so $v \notin \mathrm{stab}(H_F)$. It follows that $\mathrm{stab}(H_F) \subseteq H_F \cap L_F$. Conversely, it follows directly from (8.2) that $u - v \in H_F$ for every $u, v \in A \cap F$. Since $\mathrm{stab}(H_F)$ is a group, this completes the proof. ∎

The following result, which will be used in Section 10, expresses the dynamics of φ_A on horoballs H_F.

Lemma 8.10. *For any face F of* $\mathrm{conv}(A)$ *and any $v \in$* $\mathrm{Ext}(F)$ *we have*

$$\varphi_A(H_F) = H_F + v.$$

Proof. Let F be a face of $\mathrm{conv}(A)$ and let $v \in \mathrm{Ext}(F)$. Since $v \in A$, we have

$$H_F + v \subseteq H_F + A = \varphi_A(H_F).$$

On the other hand, suppose $u \in \varphi_A(H_F) = H_F + A$. Then by definition of H_F, there exist $w, w_1, \ldots, w_j \in A$ and $v_1, \ldots, v_j \in \mathrm{Ext}(F)$ so that $u = \sum_{k=1}^{j}(w_k - v_k) + w$. It follows that

$$u = \sum_{k=1}^{j}(w_k - v_k) + (w - v) + v \in H_F + v.$$

This shows that $\varphi_A(H_F) \subseteq H_F + v$. ∎

Lemma 8.11. *Let $W \subseteq \mathrm{Ext}(A)$ be a non-empty subset. If F is the minimal face of* $\mathrm{conv}(A)$ *that contains W, then*

$$H_F = \sum_{v \in W} \bigcup_{j=0}^{\infty} j(A - v).$$

Proof. The point $\frac{1}{|W|} \sum_{w \in W} w$ is contained in the relative interior of F. Therefore we can represent it as the rational convex combination of the extremal points of F with all coefficients positive. Multiplying by the common denominator, we obtain $m \in \mathbb{N}$ and positive integer coefficients $(\beta_v)_{v \in \mathrm{Ext}(F)}$ such that

$$m \sum_{w \in W} w = \sum_{v \in \mathrm{Ext}(F)} \beta_v v \quad \text{and} \quad \sum_{v \in \mathrm{Ext}(F)} \beta_v = m|W|.$$

Note that for any $w \in W$, $\bigcup_{j=0}^{\infty} j(A - w) = \bigcup_{j=0}^{\infty} jm(A - w)$. Therefore

$$\begin{aligned}
\sum_{w \in W} \bigcup_{j=0}^{\infty} j(A - w) &= \bigcup_{j=0}^{\infty} jm|W|A - jm \sum_{w \in W} w \\
&= \bigcup_{j=0}^{\infty} jm|W|A - j \sum_{v \in \mathrm{Ext}(F)} \beta_v v \\
&= \sum_{v \in \mathrm{Ext}(F)} \bigcup_{j=0}^{\infty} j\beta_v (A - v) \\
&= \sum_{v \in \mathrm{Ext}(F)} \bigcup_{j=0}^{\infty} j(A - v) \\
&= H_F. \qquad \blacksquare
\end{aligned}$$

Proof of Theorem 8.4. We first show that any A-horoball is of the form $v + H_F$ for some $v \in \mathbb{Z}^d$ and a face F of $\mathrm{conv}(A)$. By Lemma 8.11 it suffices to show that any A-horoball is a translate of a set of the form $\sum_{w \in W} \bigcup_{j=1}^{\infty} j(A - w)$ for some $W \subseteq \mathrm{Ext}(V)$. Let L be a horoball containing 0. There is a sequence of sets of the form $-b_k + n_k A$ with increasing n_k and $b_k \in n_k A$ converging to L. By the Shapley–Folkman lemma as in the proof of Proposition 8.1, passing to a subsequence, we can ensure that there are $r \in \mathbb{N}$ and $s \in rA$ such that $b_k \in s + (n_k - r)\,\mathrm{Ext}(A)$. Hence we can find tuples $(\alpha_{v,k})_{v \in \mathrm{Ext}(A)}$ of non-negative integers such that

$$b_k = s + \sum_{v \in \mathrm{Ext}(A)} \alpha_{v,k} v \quad \text{and} \quad \sum_{v \in \mathrm{Ext}(A)} \alpha_{v,k} = n_k - r.$$

Again passing to a subsequence, we may assume that for every $v \in \mathrm{Ext}(A)$ the sequence $(\alpha_{v,k})_{k \in \mathbb{N}}$ is non-decreasing and that therefore the limit

$$\alpha_v := \lim_{k \to \infty} \alpha_{v,k} \in \mathbb{N}_0 \cup \{+\infty\}$$

exists. Let

$$W = \{v \in \mathrm{Ext}(A) : \alpha_v = +\infty\}$$

be the set of vertices v for which $(\alpha_{v,k})_{k\in\mathbb{N}}$ tends to $+\infty$. We may assume without loss of generality that $\alpha_{v,k} = 0$ for all $v \notin W$ and $k \in \mathbb{N}$ as this merely translates L. With this assumption, we claim that L is a translate of $\sum_{w\in W} \bigcup_{j=1}^{\infty} j(A - w)$.

First of all we have

$$\begin{aligned} -b_k + n_k A &= -s - \Big(\sum_{v\in W} \alpha_{v,k} v\Big) + n_k A \\ &= -s + rA + \sum_{v\in W} \alpha_{v,k}(A - v). \end{aligned}$$

Hence L is the increasing union of the sets $-b_k + n_k A$. Now

$$\begin{aligned} L = \bigcup_{k\in\mathbb{N}} -b_k + n_k A &= -s + rA + \bigcup_{k\in\mathbb{N}} \sum_{v\in W} \alpha_{v,k}(A - v) \\ &= -s + rA + \sum_{v\in W} \bigcup_{k\in\mathbb{N}} \alpha_{v,k}(A - v) \\ &= -s + rA + \sum_{v\in W} \bigcup_{k\in\mathbb{N}} k(A - v). \end{aligned}$$

Now choose $w \in W$. Then $rA = r(A - w) + rw$. Since $(A - w)$ is contained in the semigroup $\sum_{v\in W} \bigcup_{k\in\mathbb{N}} k(A - v)$ and contains 0, we have

$$r(A - w) + \sum_{v\in W} \bigcup_{k\in\mathbb{N}} k(A - v) = \sum_{v\in W} \bigcup_{k\in\mathbb{N}} k(A - v).$$

It follows that

$$L = -s + rw + \sum_{v\in W} \bigcup_{k\in\mathbb{N}} k(A - v).$$

This completes the proof that any A-horoball is a translate of some H_F. Let $v + H_F$ be an A-horoball for some face F of $\mathrm{conv}(A)$ and $v \in \mathbb{Z}^d$ such that $0 \in v + H_F$ and so that $\mathrm{conv}(v + H_F) = \mathrm{Env}_F$. By Lemma 8.5, $\mathrm{conv}(v + H_F) = v + \mathrm{Env}_F$. Thus, $v + \mathrm{Env}_F = \mathrm{Env}_F$, so $v \in L_F$. But $0 \in v + H_F$, so $-v \in H_F$. By Lemma 8.9 it follows that $v + H_F = H_F$. ∎

9. Horoballs in $\mathbb{Z}^d$ as a topological space and a topological dynamical system

In this section we make a brief digression from the study of φ_A in order to study the space $\mathrm{Hor}(\mathbb{Z}^d, A)$ as a compact totally disconnected topological space and as $\mathbb{Z}^d$-topological dynamical system.

For $u \in \mathbb{R}^d$ and $R > 0$ we let $B_R(u)$ denote the Euclidean ball of radius R centered at u.

Lemma 9.1. *There exists $R > 0$ such that for any face F of* $\operatorname{conv}(A)$*, if $u \in \operatorname{conv}(H_F)$, then $u \in \operatorname{conv}(H_F \cap B_R(u))$.*

Proof. Choose $u \in \operatorname{conv}(H_F)$. Then u is of the form

$$u = \sum_{v \in \operatorname{Ext}(F)} \sum_{w \in A} \alpha_{v,w}(w - v),$$

with $\alpha_{v,w} \in [0, \infty)$. For $\alpha \in \mathbb{R}$ let $[\alpha]_0 = \lfloor \alpha \rfloor$ and $[\alpha]_1 = \lceil \alpha \rceil$. Then u is in the convex hull of the set

$$A_u = \Big\{ \sum_{v \in \operatorname{Ext}(F)} \sum_{w \in A} [\alpha_{v,w}]_{f(v,w)}(w - v) : f \in \{0,1\}^{\operatorname{Ext}(F) \times A} \Big\}.$$

Let

$$A' = \Big\{ \sum_{v \in \operatorname{Ext}(F)} \sum_{w \in A} b_{v,w}(w - v) : b \in [-1,1]^{\operatorname{Ext}(F) \times A} \Big\}.$$

Then $A_u \subseteq H_F \cap (u + A')$. Choose $R > 0$ sufficiently large so that A' is contained in $B_R(0)$. Then $u \in \operatorname{conv}(H_F \cap B_R(u))$. ∎

Recall that for a face F of $\operatorname{conv}(A)$, M_F has been defined in Definition 8.2.

Lemma 9.2. *Suppose F_1, F_2 are faces of* $\operatorname{conv}(A)$*, and that $(v_i)_{i \in \mathbb{N}}$ is a sequence of points in $\mathbb{Z}^d$. Then $v_n + \operatorname{Env}_{F_1} \to \operatorname{Env}_{F_2}$ as $n \to \infty$ (with respect to the product topology on $\{0,1\}^{\mathbb{R}^d}$) if and only if $F_1 \subseteq F_2$ and for every $F' \in M_{F_1}$ we have*

$$\lim_{n \to \infty} \langle \ell_{F'}, v_n \rangle = \begin{cases} -\infty & \text{if } F' \in M_{F_1} \setminus M_{F_2}, \\ 0 & \text{if } F' \in M_{F_2}. \end{cases} \tag{9.1}$$

Proof. Note that $F_1 \subseteq F_2$ implies $M_{F_2} \subseteq M_{F_1}$. If $\langle \ell_{F'}, v_n \rangle < -R$ for every $F' \in M_{F_1} \setminus M_{F_2}$ and $\langle \ell_{F'}, v_n \rangle = 0$ for every $(d-1)$-dimensional face F' of $\operatorname{conv}(A)$ which contains F_2, then

$$(v_n + \operatorname{Env}_{F_1}) \cap B_R(0) = \operatorname{Env}_{F_2} \cap B_R(0).$$

This proves that the conditions $F_1 \subseteq F_2$ together with (9.1) imply that $v_n + \operatorname{Env}_{F_1} \to \operatorname{Env}_{F_2}$ as $n \to \infty$.

Conversely, suppose that $v_n + \operatorname{Env}_{F_1} \to \operatorname{Env}_{F_2}$ as $n \to \infty$. Since $0 \in \operatorname{Env}_{F_2}$, it follows that $0 \in v_n + \operatorname{Env}_{F_1}$ and thus $-v_n \in \operatorname{Env}_{F_1}$ for all sufficiently large n. This implies that $\operatorname{Env}_{F_1} \subseteq v_n + \operatorname{Env}_{F_1}$ for all sufficiently large n. Taking $n \to \infty$, we deduce that $\operatorname{Env}_{F_1} \subseteq \operatorname{Env}_{F_2}$, which implies $F_1 \subseteq F_2$. Let $F' \in M_{F_1}$. Define points

$$m_{F_1} := \frac{1}{|\operatorname{Ext}(F_1)|} \sum_{w \in \operatorname{Ext}(F_1)} w \in F_1 \subseteq F',$$

$$u := \sum_{v \in \operatorname{Ext}(F')} (v - m_{F_1}).$$

Suppose $F' \in M_{F_2}$. Let $\varepsilon > 0$. We have $\langle u, \ell_{F'} \rangle = 0$. For $F'' \in M_{F_1} \setminus \{F'\}$ we have $\langle u, \ell_{F''} \rangle > 0$. We thus can choose $\alpha > 0$ large enough such that for all these F'' we have $\langle \alpha u - \varepsilon \ell_{F'}, \ell_{F''} \rangle > 0$. Since $\langle \alpha u - \varepsilon \ell_{F'}, \ell_{F'} \rangle = -\varepsilon < 0$, our point $\alpha u - \varepsilon \ell_{F'}$ is not contained in Env_{F_2}, and hence also not in $v_n + \mathrm{Env}_{F_1}$ for sufficiently large $n \in \mathbb{N}$. As shown above, we may also assume that $-v_n \in \mathrm{Env}_{F_1}$. Therefore there must be $F'' \in M_{F_1}$ with $\langle \alpha u - \varepsilon \ell_{F'} - v_n, \ell_{F''} \rangle < 0$. Since $\langle -v_n, \ell_{F''} \rangle \geq 0$, this implies $\langle \alpha u - \varepsilon \ell_{F'}, \ell_{F''} \rangle < 0$. By our choice of α this can only happen for $F'' = F'$. But then $\langle \alpha u - \varepsilon \ell_{F'} - v_n, \ell_{F'} \rangle < 0$, thus $-\varepsilon = \langle \alpha u - \varepsilon \ell_{F'}, \ell_{F'} \rangle < \langle v_n, \ell_{F'} \rangle \leq 0$. Hence $\lim_{n \to \infty} \langle v_n, \ell_{F'} \rangle = 0$.

Suppose $F' \in M_{F_1} \setminus M_{F_2}$. For every $F'' \in M_{F_2}$ we have $\langle u, \ell_{F''} \rangle > 0$. It follows that for every $R > 0$, there exists $Q > 0$ such that $Qu - R\ell_{F'} \in v_n + \mathrm{Env}_{F_1}$ for all sufficiently large n. So there is $w_n \in \mathrm{Env}_{F_1}$ such that $v_n = Qu - R\ell_{F'} - w_n$. This implies that

$$\langle \ell_{F'}, v_n \rangle = \langle \ell_{F'}, Qu - R\ell_{F'} - w_n \rangle = -R - \langle \ell_{F'}, w_n \rangle \leq -R$$

and thus

$$\lim_{n \to \infty} \langle \ell_{F'}, v_n \rangle = -\infty,$$

completing the proof. ■

Lemma 9.3. *Let F be a face of* $\mathrm{conv}(A)$ *and let $(v_n)_{n \in \mathbb{N}}$ be a sequence of points in $\mathbb{Z}^d$. If*

$$\lim_{n \to \infty} \max_{F' \in M_F} \langle \ell_{F'}, v_n \rangle = +\infty, \tag{9.2}$$

then $(v_n + H_F) \to \emptyset$ as $n \to \infty$.

Proof. For any $R > 0$ there exists $Q > 0$ such that $\max_{F' \in M_F} \langle \ell_{F'}, v \rangle < Q$ for every $v \in B_R(0)$. It follows that whenever $\max_{F' \in M_F} \langle \ell_{F'}, v_n \rangle \geq Q$, we have $(v_n + H_F) \cap B_R(0) = \emptyset$. Thus condition (9.2) implies that $(v_n + H_F) \to \emptyset$ as $n \to \infty$. ■

Lemma 9.4. *Suppose F_1, F_2 are faces of* $\mathrm{conv}(A)$ *and that $(v_i)_{i \in \mathbb{N}}$ is a sequence of points in $\mathbb{Z}^d$. For ease of notation we allow $F_2 = \mathrm{conv}(A)$ (considered as a "d-dimensional face"), in which case we denote $H_{F_2} = \mathbb{Z}^d$. Then $(v_n + H_{F_1}) \to H_{F_2}$ as $n \to \infty$ if and only if $(v_n + \mathrm{conv}(H_{F_1})) \to \mathrm{conv}(H_{F_2})$ as $n \to \infty$ and there exists $N \in \mathbb{N}$ so that $v_n \in \mathrm{stab}(H_{F_2})$ for all $n > N$.*

Proof. Suppose $(v_n + H_{F_1}) \to H_{F_2}$. Let $R > 0$ be as in Lemma 9.1. Recall that $\mathrm{Env}_{F_1} = \mathrm{conv}(H_{F_1})$ and $\mathrm{Env}_{F_2} = \mathrm{conv}(H_{F_2})$. Let $u \in \mathbb{R}^n$. Since $(v_n + H_{F_1}) \to H_{F_2}$, it follows that there exists $N \in \mathbb{N}$ such that $H_{F_2} \cap B_R(u) = (v_n + H_{F_1}) \cap B_R(u)$ for all $n > N$. Thus, for all $n > N$, $\mathrm{conv}(H_{F_2} \cap B_R(u)) = \mathrm{conv}((v_n + H_{F_1}) \cap B_R(u))$. By Lemma 9.1, $u \in \mathrm{conv}(H_{F_2})$ if and only if

$$\begin{aligned} u \in \mathrm{conv}(H_{F_2} \cap B_R(u)) &= \mathrm{conv}((H_{F_1} + v_n) \cap B_R(u)) \\ &= v_n + \mathrm{conv}(H_{F_1} \cap B_R(u - v_n)) \end{aligned}$$

and this is the case if and only if $u \in v_n + \mathrm{conv}(H_{F_1})$. Hence

$$(v_n + \mathrm{conv}(H_{F_1})) \to \mathrm{conv}(H_{F_2}).$$

By Lemma 9.2 this implies that v_n is in the linear span of $\{(v-w) : v, w \in \mathrm{Ext}(F_2)\}$ for sufficiently large n and that $F_1 \subseteq F_2$. Since $v_n + H_{F_1} \to H_{F_2}$, it follows that $0 \in v_n + H_{F_1}$ for large n. So $-v_n \in H_{F_1}$. By Lemma 8.9, the intersection of the linear span of $\{(v-w) : v, w \in \mathrm{Ext}(F_2)\}$ with H_{F_1} is contained in $\mathrm{stab}(H_{F_2})$. We conclude that there exists $N \in \mathbb{N}$ so that $v_n \in \mathrm{stab}(H_{F_2})$ for all $n > N$.

Conversely, suppose that $v_n + \mathrm{conv}(H_{F_1}) \to \mathrm{conv}(H_{F_2})$ as $n \to \infty$ and there exists $N \in \mathbb{N}$ so that $v_n \in \mathrm{stab}(H_{F_2})$ for all $n > N$. By compactness, it suffices to show that for any converging subsequence $(v_{n_k} + H_{F_1})_{k\in\mathbb{N}}$, we have $(v_{n_k} + H_{F_1}) \to H_{F_2}$ as $k \to \infty$. Let $(v_{n_k} + H_{F_1})_{k\in\mathbb{N}}$ be a converging subsequence, so $(v_{n_k} + H_{F_1}) \to W$ for some $W \subseteq \mathbb{Z}^d$. By Lemma 9.2, $v_n + \mathrm{conv}(H_{F_1}) \to \mathrm{conv}(H_{F_2})$ implies $F_1 \subseteq F_2$. Since $v_n \in \mathrm{stab}(H_{F_2}) \subseteq H_{F_2}$, and H_{F_2} is a semigroup, it follows that $v_n + H_{F_1} \subseteq H_{F_2}$ for all $n > N$. This implies that $W \subseteq H_{F_2}$. By Theorem 8.4 the limit W must either be of the form $v + H_F$ for some face F of $\mathrm{conv}(A)$, or $\mathbb{Z}^d$ or $\emptyset$. But the first part of the proof shows that $\mathrm{conv}(W) = \mathrm{conv}(H_{F_2})$. It follows that $W = v + H_{F_2}$ for some $v \in \mathbb{Z}^d$ in the linear span of $\{(u-w) : u, w \in \mathrm{Ext}(F_2)\}$. We have already concluded that $v + H_{F_2} = W \subseteq H_{F_2}$. Using Lemma 8.9, the conditions $v + H_{F_2} \subseteq H_{F_2}$ and $v + \mathrm{conv}(H_{F_2}) = \mathrm{conv}(H_{F_2})$ together imply that $v \in \mathrm{stab}(H_{F_2})$ and $(v_{n_k} + H_{F_1}) \to H_{F_2}$. ■

Lemma 9.5. *Let F be a face of* $\mathrm{conv}(A)$ *and let $(v_i)_{i\in\mathbb{N}}$ be a sequence of points in $\mathbb{Z}^d$. Then:*

(1) *Suppose* $\lim_{n\to\infty} \max_{F'\in M_F} \langle \ell_{F'}, v_n\rangle = +\infty$. *Then $(v_n + H_F) \to \emptyset$ as $n \to \infty$.*

(2) *Suppose that* $\lim_{n\to\infty}\langle \ell_{F'}, v_n\rangle = -\infty$ *for every $(d-1)$-dimensional face F' of* $\mathrm{conv}(A)$ *containing F. Then $(v_n + H_F) \to \mathbb{Z}^d$ as $n \to \infty$.*

(3) *Otherwise, suppose that the limit* $\lim_{n\to\infty}\langle \ell_{F'}, v_n\rangle \in [-\infty, +\infty)$ *exists for every $F' \in M_F$ and let F'' denote the face of* $\mathrm{conv}(A)$ *which is the intersection of all $F' \in M_F$ such that* $\lim_{n\to\infty}\langle \ell_{F'}, v_n\rangle$ *is finite. Furthermore, suppose all but finitely many elements of $(v_n)_{n\in\mathbb{N}}$ belong to a fixed coset of* $\mathrm{stab}(H_{F''})$. *Then there exists $v \in \mathbb{Z}^d$ so that $v_n + H_F \to v + H_{F''}$ as $n \to \infty$.*

(4) *In all other cases, the sequence $v_n + H_F$ does not converge.*

Proof. (1) If $\lim_{n\to\infty} \max_{F'\in M_F} \langle \ell_{F'}, v_n\rangle = +\infty$, then $(v_n + H_F) \to \emptyset$ by Lemma 9.3.

(2) & (3) Note that (2) is essentially a particular case of (3) if we agree that $F'' = \mathrm{conv}(A)$ when $\lim_{n\to\infty}\langle \ell_{F'}, v_n\rangle = -\infty$ for all $F' \in M_F$. By Lemma 9.2,

$$v_n + \mathrm{conv}(H_F) \to v + \mathrm{conv}(H_{F''}).$$

Using the fact that $v_n - v \in \mathrm{stab}(H_{F''})$ for all $n \geq N$, by Lemma 9.4, it follows that $v_n + H_F \to v + H_{F''}$.

(4) Let $(v_i)_{i\in\mathbb{N}}$ be a sequence of points in $\mathbb{Z}^d$, and let us suppose that the sequence $(v_n + H_F)_{n\in\mathbb{N}}$ converges. By compactness this sequence converges if and only if all of its converging subsequences have the same limit. Then by Theorem 8.4 the possible limit points are $\emptyset$, $\mathbb{Z}^d$ and $v + H_{F''}$ for faces F'' of $\mathrm{conv}(A)$ and $v \in \mathbb{Z}^d$.

If $\lim_{n\to\infty} \max_{F'\in M_F} \langle \ell_{F'}, v_n\rangle = +\infty$, we are in case (1). Otherwise we know from Lemma 9.4 that $(v_n + \mathrm{conv}(H_F)) \to (v + \mathrm{conv}(H_{F''}))$ and that $v_n \in (v + \mathrm{stab}(H_{F''}))$ for all sufficiently large $n \in \mathbb{N}$. By Lemma 9.2 we can conclude that if $(v_n + H_F)_{n\in\mathbb{N}}$ converges, we are either in case (1), case (2) or case (3). ∎

For a topological space X let X' denote the set of accumulation points of X. The *k-th Cantor–Bendixson derivative* is inductively defined as

$$X^{(0)} = X \quad \text{and} \quad X^{(k+1)} = (X^{(k)})'.$$

Later in Section 11 we will also need the α-th Cantor–Bendixson derivative for an ordinal number α. For successor ordinals the same definition as above holds and for limit ordinals α it is defined as

$$X^{(\alpha)} = \bigcap_{\beta<\alpha} X^{(\beta)}.$$

A topological space X has *Cantor–Bendixson rank* α if and only if α is the smallest ordinal with $X^{(\alpha)} = X^{(\alpha+1)}$.

Using Theorem 8.4 and Lemma 9.5, we can summarize the topological structure of $\mathrm{Hor}(\mathbb{Z}^d, A)$ as follows:

Theorem 9.6. *Let $A \subseteq \mathbb{Z}^d$ be a finite positively generating set. Then the closure of* $\mathrm{Hor}(\mathbb{Z}^d, A)$ *in $\mathcal{P}(\mathbb{Z}^d)$ is a countable compact subset of $\mathcal{P}(\mathbb{Z}^d)$ having Cantor–Bendixson rank $d+1$. Furthermore,*

$$\overline{\mathrm{Hor}(\mathbb{Z}^d, A)}^{(d)} = \{\emptyset, \mathbb{Z}^d\},$$

and for $0 \le k < d$ the isolated points of $\overline{\mathrm{Hor}(\mathbb{Z}^d, A)}^{(k)}$ *are precisely*

$$\{v + H_F : F \text{ is a } k\text{-dimensional face of } \mathrm{conv}(A) \text{ and } v \in \mathbb{Z}^d\}.$$

Proof. Let F, F' be faces of $\mathrm{conv}(A)$. By Lemma 9.5, the horoball $H_{F'}$ is a limit point of the orbit of H_F under translations if and only if $F \subsetneq F'$. This show that $v + H_F$ is an isolated point of $\overline{\mathrm{Hor}(\mathbb{Z}^d, A)}^{(k)}$ if and only if F is a k-dimensional face of $\mathrm{conv}(A)$. ∎

Corollary 9.7. *Let $d_1, d_2 \in \mathbb{N}$. Suppose $A_i \subseteq \mathbb{Z}^{d_i}$ are finite positively generating sets for $i = 1, 2$. Then* $\overline{\mathrm{Hor}(\mathbb{Z}^{d_1}, A_1)}$ *is homeomorphic to* $\overline{\mathrm{Hor}(\mathbb{Z}^{d_2}, A_2)}$ *if and only if $d_1 = d_2$.*

Proof. Lemma 9.5 shows that for any finite positively generating set A of $\mathbb{Z}^d$, if $X_A = \overline{\mathrm{Hor}(\mathbb{Z}^d, A)}$, then $X_A^{(d)} = \{\emptyset, \mathbb{Z}^d\}$ consists of 2 points. It follows from a theorem of Mazurkiewicz and Sierpiński [16] that any such countable compact metrizable topological space of Cantor–Bendixson rank $d+1$ is homeomorphic to the ordinal $2\omega^d + 1$, with the order topology. ∎

Theorem 9.8. *Suppose $A_1, A_2 \subseteq \mathbb{Z}^d$ are finite positively generating sets. Then the $\mathbb{Z}^d$ actions by translations on* $\overline{\mathrm{Hor}(\mathbb{Z}^d, A_1)}$ *and* $\overline{\mathrm{Hor}(\mathbb{Z}^d, A_2)}$ *are topologically conjugate if and only if there is a bijection Φ between the faces of* $\mathrm{conv}(A_1)$ *and the faces of* $\mathrm{conv}(A_2)$

such that for every face F of $\mathrm{conv}(A_1)$*, the following two conditions hold:*

$$\sum_{u,v \in F \cap A_1} \mathbb{Z}(u-v) = \sum_{u',v' \in \Phi(F) \cap A_2} \mathbb{Z}(u'-v')$$

(in other words, the group generated by differences of elements in $F \cap A_1$ is equal to the group generated by differences of elements in $\Phi(F) \cap A_2$) and

$$\mathrm{Env}_{F,A_1} = \mathrm{Env}_{\Phi(F),A_2}\,.$$

Proof. The case $d = 1$ is somewhat degenerate as the convex hull of any finite set is an interval. We leave it for the reader to check that for any finite positively generating set $A \subset \mathbb{Z}$, the shift action on $\overline{\mathrm{Hor}(\mathbb{Z}, A)}$ is topologically conjugate to the action of the shift on $\overline{\mathrm{Hor}(\mathbb{Z}, \{-1, 0, 1\})}$. So from now on we assume $d \geq 2$.

By Theorem 9.6, the orbits of the $\mathbb{Z}^d$ action on $\overline{\mathrm{Hor}(\mathbb{Z}^d, A_i)}$ are the two fixed points $\emptyset, \mathbb{Z}^d$ together with the orbits of H_F where F runs over the faces of $\mathrm{conv}(A_i)$. Suppose $\phi : \overline{\mathrm{Hor}(\mathbb{Z}^d, A_1)} \to \overline{\mathrm{Hor}(\mathbb{Z}^d, A_2)}$ is a topological conjugacy of the $\mathbb{Z}^d$ actions. By Theorem 9.6, the $\mathbb{Z}^d$-orbits of $\mathrm{Hor}(\mathbb{Z}^d, A_i)$ which are isolated points in $\overline{\mathrm{Hor}(\mathbb{Z}^d, A)}^{(k)}$ (with $0 \leq k < d$) are in bijection with the k-dimensional faces of $\mathrm{conv}(A_i)$. Thus, for each k-dimensional face F of $\mathrm{conv}(A_1)$ there exist $v_F \in \mathbb{Z}^d$ and a k-dimensional face $\Phi(F)$ of $\mathrm{conv}(F_2)$ such that $\phi(H_F) = v_F + H_{\Phi(F)}$. Clearly Φ is a bijection between the faces of $\mathrm{conv}(A_1)$ and the faces of $\mathrm{conv}(A_2)$. Furthermore, by Lemma 9.5 for faces F, F' of $\mathrm{conv}(A_1)$, $H_{F'}$ is an accumulation point of the orbit of H_F if and only if $F \subseteq F'$. Thus the bijection Φ respects the incidence relations between the faces of the polytopes $\mathrm{conv}(A_1)$ and $\mathrm{conv}(A_2)$. In other words, the polytopes $\mathrm{conv}(A_1)$ and $\mathrm{conv}(A_2)$ are *combinatorially isomorphic*. Since ϕ is a topological conjugacy, we must have $\mathrm{stab}(H_F) = \mathrm{stab}(\phi(H_F))$. By Lemma 8.9, the stabilizer of H_F is equal to the group generated by differences of elements in $F \cap A_1$. This shows that the group generated by differences of elements in $F \cap A_1$ is equal to the group generated by differences of elements in $\Phi(F) \cap A_2$. The set of two fixed points $\{\emptyset, \mathbb{Z}^d\}$ is obviously mapped bijectively into itself via Φ. Using Lemma 9.5, we conclude that $\mathbb{Z}^d$ has the property that for any $H \in \overline{\mathrm{Hor}(\mathbb{Z}^d, A_i)}$ the set of directions v in which H tends to $\mathbb{Z}^d$ is convex, whereas (for $d \geq 2$) the set of directions v in which a vertex horoball tends to $\emptyset$ is not convex. This implies (for $d \geq 2$) that $\Phi(\mathbb{Z}^d) = \mathbb{Z}^d$ and so $\Phi(\emptyset) = \emptyset$. By Lemma 9.3, $v_n + H_F \to \emptyset$ if and only if the distance between $-v_n$ and $\mathrm{conv}(H_F)$ tends to ∞. This shows that $\mathrm{conv}(H_F) = \mathrm{conv}(H_{\Phi(F)})$. By Lemma 8.5 this implies that $\mathrm{Env}_{F,A_1} = \mathrm{Env}_{\Phi(F),A_2}$.

Conversely, suppose that there is a bijection Φ between the faces F of $\mathrm{conv}(A_1)$ and the faces of $\mathrm{conv}(A_2)$ as above. Define $\phi : \overline{\mathrm{Hor}(\mathbb{Z}^d, A_1)} \to \overline{\mathrm{Hor}(\mathbb{Z}^d, A_2)}$ by $\phi(\emptyset) = \emptyset$, $\phi(\mathbb{Z}^d) = \mathbb{Z}^d$ and $\phi(v + H_F) = v + H_{\Phi(F)}$ for $v \in \mathbb{Z}^d$ and F a face of $\mathrm{conv}(A_1)$. By the assumption that the group generated by differences of elements in $F \cap A_1$ is equal to the group generated by differences of elements in $\Phi(F) \cap A_2$ and using Lemma 8.9, we get that $v_1 + H_F = v_2 + H_F$ if and only if $v_1 + H_{\Phi(F)} = v_2 + H_{\Phi(F)}$, so the map ϕ is well defined and it is a bijection (here we also use Theorem 9.6). It is clear that the map

ϕ is equivariant. Combining Lemma 9.5 with the assumption that $\mathrm{Env}_{F,A_1} = \mathrm{Env}_{\Phi(F),A_2}$ for every face F of $\mathrm{conv}(A_1)$, it follows that $(v_n + H_F) \to (v + H_{F'})$ if and only if $(v_n + H_{\Phi(F)}) \to (v + H_{\Phi(F')})$. The analogous statements for $\mathbb{Z}^d$ and $\emptyset$ hold as well. This shows that ϕ is a homeomorphism. ∎

Example 9.9. Consider the generating set depicted on the left side of Figure 4. Rotation by 90° gives a generating set whose space of horoballs together with the action by translation is conjugate to the original one, but none of the horoballs agree.

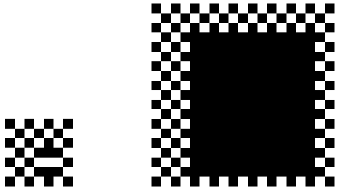


Figure 4. A generating set A of $\mathbb{Z}^2$, whose convex hull is a centered square, and the ball A^3.

From Theorem 9.8 it follows that there is an algorithm to decide given two finite positively generating sets $A_1, A_2 \subseteq \mathbb{Z}^d$ if the $\mathbb{Z}^d$ actions by translation on $\mathrm{Hor}(\mathbb{Z}^d, A_1)$ and $\mathrm{Hor}(\mathbb{Z}^d, A_2)$ are topologically conjugate, as one can reduce this to several applications of the following problems:

- Given $v \in \mathbb{Z}^d$ and a finite set $C \subseteq \mathbb{Z}^d$, decide if v is in the convex cone generated by C (this is a linear programming problem).
- Given $v \in \mathbb{Z}^d$ and a finite set $C \subseteq \mathbb{Z}^d$, decide if v is in the group generated by C (this amounts to solving a system of linear Diophantine equations).

10. The natural extension of φ_A, $A \subseteq \mathbb{Z}^d$

Our next goal is to show that for $G = \mathrm{Cayley}(\mathbb{Z}^d, A)$ with $d \geq 2$ the natural extension of φ_A is perfect. This will take the remainder of this section.

Lemma 10.1. *For every finite positively generating set $A \subseteq \mathbb{Z}^d$ the set*

$$\widetilde{X}_A = \left\{ \bigcup_{i=1}^{m} (w_i + H_{\{v_i\}}) : m \geq |\mathrm{Ext}(A)|,\ w_1, \dots, w_m \in \mathbb{Z}^d,\ \{v_1, \dots, v_m\} = \mathrm{Ext}(A) \right\}$$

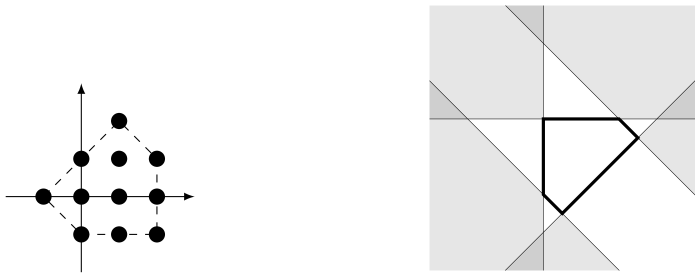
is dense in the eventual image of φ_A.

Proof. It is clear that $\widetilde{X}_A$ consists of horoballunions and hence is a subset of the eventual image of φ_A. Let $R > 0$. There is $n \in \mathbb{N}$ such that $(-nv + H_{\{v\}}) \cap B_R(0) = \emptyset$ for every $v \in \mathrm{Ext}(A)$. Let M be a set in the eventual image of φ_A, hence M is the union of translates of horoballs of the form $H_{\{v\}}$ for $v \in \mathrm{Ext}(A)$. The unions of finitely many of these horoballs cover $M \cap B_R(0)$. Denote this union by M_1. Then $M_2 := M_1 \cup \bigcup_{v \in V}(-nv + H_{\{v\}}) \in \widetilde{X}_A$. Furthermore, $M_2 \cap B_R(0) = M \cap B_R(0)$. Therefore $\widetilde{X}_A$ is dense in the eventual image of φ_A. ■

Definition 10.2. For a finite positively generating set $A \subseteq \mathbb{Z}^d$ define $\check{A}$ by

$$\check{A} := \bigcup_{v \in \mathrm{Ext}(A)} (-v + \mathrm{Env}_{\{v\},A}),$$

see Figure 5 for an illustration.



(a) A generating set A of $\mathbb{Z}^2$.   **(b)** $-\operatorname{conv}(A)$ together with $\check{A}$.

Figure 5. Illustration of the proof of Lemma 10.3.

Lemma 10.3. *Let $A \subseteq \mathbb{Z}^d$ be a finite positively generating set. Let $m \geq |\mathrm{Ext}(A)|$, $w_1, \dots, w_m \in \mathbb{Z}^d$ and $\{v_1, \dots, v_m\} = \mathrm{Ext}(A)$. Set $M_n := \bigcup_{i=1}^m (w_i - nv_i + H_{\{v_i\}}) \in \widetilde{X}_A$. Then $M_n \in \varphi_A^{-n}(\{M_0\})$ and $\frac{1}{n}M_n$ converges to $\check{A}$ with respect to the Hausdorff metric, i.e. for all ε and n large enough we have $\frac{1}{n}M_n \subseteq \check{A} + B_\varepsilon(0)$ and $\check{A} \subseteq \frac{1}{n}M_n + B_\varepsilon(0)$.*

Proof. From Lemma 8.10 it follows that $M_n \in \varphi_A^{-n}(\{M_0\})$. Also there is $N > 0$ such that $\mathrm{Env}_{\{v\}} \subseteq B_{\varepsilon/2}(0) + (\mathrm{Env}_{\{v\}} \cap \frac{1}{n}\mathbb{Z}^d)$ for every $n > N$ and every $v \in \mathrm{Ext}(A)$. By Lemma 8.7 this implies that for every $v \in \mathrm{Ext}(A)$ there exists $u_v \in \mathbb{Z}^d$ such that

$$\mathrm{Env}_{\{v\}} \cap \mathbb{Z}^d \subseteq u_v + H_{\{v\}},$$

hence

$$\mathrm{Env}_{\{v\}} \subseteq B_{\varepsilon/2}(0) + \Big(\frac{1}{n}u_v + \frac{1}{n}H_{\{v\}}\Big).$$

Finally, for large enough n we have

$$\bigcup_{i=1}^m \Big(\frac{1}{n}w_i - v_i + \frac{1}{n}H_{v_i}\Big) + B_{\varepsilon/2}(0) \supseteq \bigcup_{i=1}^m \Big(\frac{1}{n}u_v - v_i + \frac{1}{n}H_{v_i}\Big)$$

and

$$\begin{aligned}\frac{1}{n}M_n + B_\varepsilon(0) &= \bigcup_{i=1}^{m}\Big(\frac{1}{n}w_i - v_i + \frac{1}{n}H_{v_i}\Big) + B_\varepsilon(0)\\ &\supseteq \bigcup_{i=1}^{m}\Big(\frac{1}{n}u_v - v_i + \frac{1}{n}H_{v_i}\Big) + B_{\varepsilon/2}(0)\\ &\supseteq \check{A}.\end{aligned}$$

On the other hand $\mathrm{Env}_{\{v\}} \supseteq H_{\{v\}}$ for every $v \in \mathrm{Ext}(A)$ and for sufficiently large n

$$\frac{1}{n}M_n = \bigcup_{i=1}^{m}\Big(\frac{1}{n}w_i - v_i + \frac{1}{n}H_{v_i}\Big) \subseteq \check{A} + B_\varepsilon(0). \qquad \blacksquare$$

Lemma 10.4. *Let $d \geq 2$ and let $A \subseteq \mathbb{Z}^d$ be a finite positively generating set and let $v \in \mathrm{Ext}(A)$. Then there is a point $p \in \mathbb{R}^d$ and an $\varepsilon > 0$ such that $B_\varepsilon(p) \cap \check{A} = \emptyset$ and $(B_\varepsilon(p) + H_{\{v\}}) \cap \mathrm{conv}(-A) = \emptyset$.*

Proof. Let F be a $(d-1)$-dimensional face of $\mathrm{conv}(A)$ containing v. Set

$$p_\delta := \frac{1}{|\mathrm{Ext}(F)|}\sum_{w \in \mathrm{Ext}(F)} -w + \delta \ell_F.$$

For all $w \in B_\varepsilon(p_\delta) + H_{\{v\}}$ we have $\langle \ell_F, w\rangle \geq \langle \ell_F, -v\rangle - \varepsilon + \delta$ but for $w \in \mathrm{conv}(-A)$ we have $\langle \ell_F, w\rangle \leq \langle \ell_F, -v\rangle$. Hence for $\varepsilon < \delta$ we have

$$(B_\varepsilon(p_\delta) + H_{\{v\}}) \cap \mathrm{conv}(-A) = \emptyset.$$

To show that $B_\varepsilon(p_\delta) \cap \check{A} = \emptyset$ for sufficiently small δ, we have to show that p_δ is not contained in $-\tilde{v} + \mathrm{Env}_{\{\tilde{v}\}}$ for every point $\tilde{v} \in \mathrm{Ext}(A)$. In other words, we have to show that $p_\delta + \tilde{v} \notin \mathrm{Env}_{\{\tilde{v}\}}$. Let $\widetilde{F} \neq F$ be a $(d-1)$-dimensional face of A containing $\tilde{v}$. Then all points $u \in \mathrm{Env}_{\{\tilde{v}\}}$ have $\langle \ell_{\widetilde{F}}, u\rangle \geq 0$ but

$$\langle \ell_{\widetilde{F}}, p_\delta + \tilde{v}\rangle = \frac{1}{|\mathrm{Ext}(F)|}\sum_{w \in \mathrm{Ext}(F)} \langle \ell_{\widetilde{F}}, \tilde{v} - w\rangle + \delta\langle \ell_{\widetilde{F}}, \ell_F\rangle$$

which is negative for small δ because $\langle \ell_{\widetilde{F}}, \tilde{v} - w\rangle$ is non-positive for all $w \in \mathrm{Ext}(F)$ and negative for at least one $w \in \mathrm{Ext}(F)$. $\blacksquare$

Proposition 10.5. *For any finite positively generating set $A \subseteq \mathbb{Z}^d$, $d \geq 2$, the natural extension of φ_A is perfect.*

Proof. By Lemma 10.1 it suffices to show that for any $M \in \widetilde{X}_A$ and $R > 0$ there exist $n \in \mathbb{N}$, $W_1, W_2 \in \varphi_A^{-n}(\{M\})$ and $W_2 \neq W_1$, both in the eventual image of φ_A, such that

$$M \cap B_R(0) = \varphi_A^n(W_1) \cap B_R(0) = \varphi_A^n(W_2) \cap B_R(0).$$

Let $M = \bigcup_{i=1}^m (w_i + H_{v_i})$ with $w_1, \dots, w_m \in \mathbb{Z}^d$ and $\{v_1, \dots, v_m\} = \mathrm{Ext}(A)$. If we define, as in Lemma 10.3,

$$M_n := \bigcup_{i=1}^m (w_i - nv_i + H_{\{v_i\}}),$$

for any $n \in \mathbb{N}$, then $M_n \in \varphi_A^{-n}(\{M\})$. By Lemma 10.4 there are $\varepsilon > 0$, a point $p \in \mathbb{R}^d$ and a point $v \in \mathrm{Ext}(A)$ such that

$$B_\varepsilon(p) \cap \check{A} = \emptyset \quad \text{and} \quad (B_\varepsilon(p) + \mathrm{Env}_{\{v\}}) \cap \mathrm{conv}(-A) = \emptyset.$$

Since $\frac{1}{n} M_n$ converges to $\check{A}$ by Lemma 10.3, we can choose n large enough such that

$$\frac{1}{n} M_n \cap B_{\varepsilon/2}(p) = \emptyset, \quad \frac{1}{n}\mathbb{Z}^d \cap B_{\varepsilon/2}(p) \neq \emptyset, \quad \frac{R}{n} < \frac{\varepsilon}{2}.$$

Let $w \in B_{n\varepsilon/2}(p) \cap \mathbb{Z}^d$ and set $W_1 := M_n$, $W_2 := M_n \cup (w + H_{\{v\}})$. Since $W_1 \cap B_{n\varepsilon/2}(p) = \emptyset$ and $w \in W_2 \cap B_{n\varepsilon/2}(p)$, we have $W_2 \neq W_1$. We also have

$$\varphi_A^n(W_2) = M \cup (w + nv + H_{\{v\}}).$$

It remains to show that $(w + nv + H_{\{v\}}) \cap B_R(0) = \emptyset$. We know that

$$(B_{\varepsilon/2}(p) + \mathrm{Env}_{\{v\}}) \cap (B_{\varepsilon/2}(0) + \mathrm{conv}(-A)) = \emptyset.$$

Hence

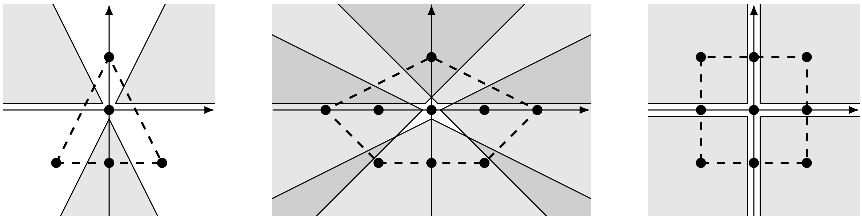
$$\begin{aligned}(B_{n\varepsilon/2}(np) + \mathrm{Env}_{\{v\}}) \cap (n\,\mathrm{conv}(-A) + B_{n\varepsilon/2}(0)) &= \emptyset,\\ (w + H_{\{v\}}) \cap (-nv + B_R(0)) &= \emptyset,\\ (w + nv + H_{\{v\}}) \cap B_R(0) &= \emptyset.\end{aligned}$$ ∎

Combining Proposition 7.2 and Proposition 10.5, we conclude:

Corollary 10.6. *For any $d > 1$ and any finite positively generating set $A \subset \mathbb{Z}^d$ that contains 0, the natural extension of φ_A is topologically conjugate to the (unique) north-south system on the Cantor set.*

11. The eventual image of φ_A, $A \subseteq \mathbb{Z}^d$

In this section we still consider the dynamics of $\varphi_A : \mathcal{P}(\mathbb{Z}^d) \to \mathcal{P}(\mathbb{Z}^d)$ where A is a positive generating set of $\mathbb{Z}^d$ containing 0. Our goal is to show that the Cantor–Bendixson rank of the eventual image is a non-trivial invariant. More precisely, we show by examples that for $A \subseteq \mathbb{Z}^2$, the Cantor–Bendixson rank of $\mathrm{Evt}(\varphi_A)$ can be 0, 1 or ω. We suspect that these are the only possibilities, at least for $d = 2$. Recall that we call elements of $\mathrm{Evt}(\varphi_A)$ *horoballunions*.



(a) A_1 **(b)** A_2 **(c)** A_3

Figure 6. Three generating sets whose eventual images have different Cantor–Bendixson rank, together with the envelopes corresponding to the extremal points of the convex hull, moved slightly outward for greater clarity.

Example 11.1. Consider the positive generating sets A_1, A_2 and A_3 of $\mathbb{Z}^2$ depicted in Figure 6. The eventual image of φ_{A_i} has the following structure.

(1) $\mathrm{Evt}(\varphi_{A_1})$ is perfect by Proposition 11.5 below.

(2) $\mathrm{Evt}(\varphi_{A_2})$ has Cantor–Bendixson rank 1 by Proposition 11.6 below.

(3) $\mathrm{Evt}(\varphi_{A_3})$ has Cantor–Bendixson rank ω by Proposition 11.7 below.

We now turn to prove the statement claimed regarding the example above.

Here is some ad-hoc terminology:

Definition 11.2. Let $W \subseteq \mathbb{Z}^d$ be finite and $M \in \mathrm{Evt}(\varphi_A)$. We call $\tilde{M} \in \mathrm{Evt}(\varphi_A)$ a W-*approximation* of M if $\tilde{M} \cap W = M \cap W$.

Definition 11.3. A horoballunion $M \in \mathrm{Evt}(\varphi_A)$ is called *deficient*, if there is a finite set of horoballs $u_i + H_{\{v_i\}}, i \in I$ such that $M = \bigcup_{i \in I}(u_i + H_{\{v_i\}})$ and $\bigcup_{i \in I} \mathrm{Env}_{\{v_i\}} \neq \mathbb{R}^d$. Denote the set of deficient horoballunions by D_A.

Lemma 11.4. *The set of all deficient horoballunions has no isolated points.*

Proof. Let $M = \bigcup_{i \in I}(u_i + H_{\{v_i\}})$ be a finite deficient union of horoballs. Set

$$U := \Big\{ w \in \bigcup_{i \in I} \mathrm{Env}_{\{v_i\}} : \|w\| = 1 \Big\}.$$

Let u be an element in the boundary of U considered as a subset of $S^{d-1} = \{w \in \mathbb{R}^d : \|w\| = 1\}$. We can find $v \in \{v_i : i \in I\}$ such that $u \in \mathrm{Env}_{\{v\}}$. Then $-u \not\in \mathrm{Env}_{\{v\}}$ and we can find $\tilde{u} \in \mathbb{Z}^d \setminus \bigcup_{i \in I} \mathrm{Env}_{\{v_i\}}$ such that still $-\tilde{u} \not\in \mathrm{Env}_{\{v\}}$. Hence, for every finite $W \subseteq \mathbb{Z}^d$ we can find $t_0 > 0$ such that for all $t > t_0$ we have $(t\tilde{u} + \mathrm{Env}_{\{v\}}) \cap W = \emptyset$. Since $M = \bigcup_{i \in I}(u_i + H_{\{v_i\}})$, by our choice of $\tilde{u}$ we can find $t_1 \in \mathbb{N}$, $t_1 > t_0$ such that $t_1\tilde{u} \not\in M$, hence $M \cup (t_1\tilde{u} + H_{\{v\}})$ is a W-approximation of M which is different from M but still deficient. ∎

The following results will show that there are cases where the Cantor–Bendixson rank is 0 or 1.

Proposition 11.5. *If $A \subseteq \mathbb{Z}^2$ is a positively generating set with $0 \in A$ and $\operatorname{conv}(A)$ is a triangle, then $\operatorname{Evt}(\varphi_A)$ is perfect.*

Proof. Let A be as above. Because the sum of the angles of a triangle is π, which is strictly less than 2π, every finite union of A-horoballs is deficient. Hence by Lemma 11.4, D_A is a dense subset of $\operatorname{Evt}(\varphi_A)$ without isolated points. ∎

Proposition 11.6. *Suppose $A \subseteq \mathbb{Z}^2$ is a positively generating set with $0 \in A$ that satisfies the following properties:*

(1) *For every $w \in \mathbb{R}^2 \setminus \{0\}$ there exists $v \in \operatorname{Ext}(A)$ so that w is contained in the interior of $\operatorname{Env}_{\{v\}}$.*
(2) *For every $v \in \operatorname{Ext}(A)$ there exists $w \in \mathbb{R}^2 \setminus \{0\}$ which is not contained in*

$$\bigcup_{v' \in \operatorname{Ext}(A) \setminus \{v\}} \operatorname{Env}_{\{v'\}} .$$

Then the Cantor–Bendixson rank of $\operatorname{Evt}(\varphi_A)$ is equal to 1.

Proof. Let A be as above. We will prove the result by showing that any horoballunion M which is not in the closure of D_A is isolated in $\operatorname{Evt}(\varphi_A)$. Let M be such a horoballunion. There exists a finite set $W \subseteq \mathbb{Z}^2$ such that all W-approximations of M are non-deficient. Consider the horoballunions of the form $\bigcup_{w \in (M \cap W)} (w + H_{\{v_w\}})$ with $\{v_w : w \in M \cap W\} = \operatorname{Ext}(A)$. There are only finitely many of them, they are all cofinite by (1), and by (2) every W-approximation of M contains one of them. Therefore we can find a finite set $\widetilde{W}$ such that all $\widetilde{W}$-approximations of M contain the complement of $\widetilde{W}$. Thus M is isolated. This shows that the isolated points of $\operatorname{Evt}(\varphi_A)$ are precisely all points not in $\overline{D_A}$. ∎

The following result demonstrates the case of Cantor–Bendixson rank ω.

Proposition 11.7. *Suppose $n \in \mathbb{N}$ and let $A = \{-1, 0, 1\}^2 \subseteq \mathbb{Z}^2$. Then $\operatorname{Evt}(\varphi_A)$ has Cantor–Bendixson rank ω.*

Proof. Let $A = \{-1, 0, 1\}^2 \subseteq \mathbb{Z}^2$. Denote

$$B = \big\{(-1, 0), (1, 0), (0, -1), (0, 1)\big\} \subseteq \mathbb{Z}^2.$$

Let a *ray* be a set of the form $\{(x, y) + tv : t \in \mathbb{N}\}$ with $(x, y) \in \mathbb{Z}^2$ and $v \in B$. For $M \in \operatorname{Evt}(\varphi_A)$ define the *rank* of M as follows. If M is in the closure of D_A, the rank of M is ∞. If M is not in the closure of D_A, the rank of M is the maximal number of pairwise disjoint rays contained in the complement of M.

Let us first establish that for $M \notin \overline{D_A}$, the rank is indeed finite. Let $M \notin \overline{D_A}$, hence there is a finite set W such that all W-approximations of M are non-deficient. Hence there must be $w_v \in \mathbb{Z}^2$ for $v \in \operatorname{Ext}(A)$ such that all W-approximations of M contain $\bigcup_{v\in\operatorname{Ext}(A)}(w_v + H_{\{v\}})$. But no ray contained in $\{(0, y) + t(1, 0) : t \geq 0\}$ with y between the y-coordinates of $w_{(-1,-1)}$ and $w_{(-1,1)}$ can be contained in the complement of M. Hence there can be only finitely many pairwise disjoint rays with direction $(1, 0)$ in the complement of M. Similarly for the other directions.

Now we want to show that the set of all horoballunions having rank k or larger is closed. Let $M \notin \overline{D_A}$. We have to show that every sufficiently good approximation of M has rank at most that of M. As above there is a finite set W and there are $w_v \in \mathbb{Z}^2$ for $v \in \operatorname{Ext}(A)$ such that all W-approximations of M contain

$$\widetilde{M} := \bigcup_{v\in\operatorname{Ext}(A)} (w_v + H_{\{v\}}).$$

Therefore $\widetilde{M}$ has finite rank larger than or equal to the rank of M. Let

$$R = \{(x, y) + t(1, 0) : t \in \mathbb{N}\}$$

be a ray contained in M but disjoint from $\widetilde{M}$. It is now enough to show that every sufficiently good approximation of M also contains R. Set

$$L := \{(x, y) + t(1, 0) : t \in \mathbb{Z}\}.$$

If $L \cap (X \setminus M) \neq \emptyset$, then every sufficiently good approximation of M by a horoballunion will also contain the ray R.

If, on the other hand, $L \subseteq M$, we will show that every sufficiently good approximation of M by horoballunions must contain L. Assume there are arbitrary good approximations of M by horoballunions not containing L. It is easy to see that in this case either the upper or lower half space defined by L is contained in M. Without loss of generality assume it is the lower half space, call it S, and that S is the maximal one contained in M. Every sufficiently good approximation of M must contain $((x_1, y_1) + H_{(1,1)}) \cup ((x_{-1}, y_{-1}) + H_{(-1,1)})$ for points $(x_1, y_1) \in \mathbb{Z}^2$ and $(x_{-1}, y_{-1}) \in \mathbb{Z}^2$ with $y_1 > y$, $y_{-1} > y$ and $x_1 < x_{-1}$. This is due to the fact that M is not contained in $\overline{D_A}$ and that S is the maximal lower half space contained in M. Consider a vertical line

$$V = \{(a, 0) + t(0, -1) : t \in \mathbb{Z}\}$$

with $x_1 < a < x_{-1}$. It is now enough to show that $S \cap V$ is contained in every sufficiently good approximation of M. We have to treat two cases.

(1) The left or the right half space defined by V is contained in M. Assume it is the left one and call it $\widetilde{S}$. As discussed above, since M is not in the closure of D_A, there must be a point $\tilde{w} \in \mathbb{Z}^2 \setminus (S \cup \widetilde{S})$ such that $\tilde{w} + H_{\{(1,1)\}}$ is contained in every sufficiently good approximation of M. But then $V \cap S \subseteq \tilde{w} + H_{\{(1,1)\}}$ is also contained in every sufficiently good approximation of M. A similar argument works if the right half space defined by V is contained in M.

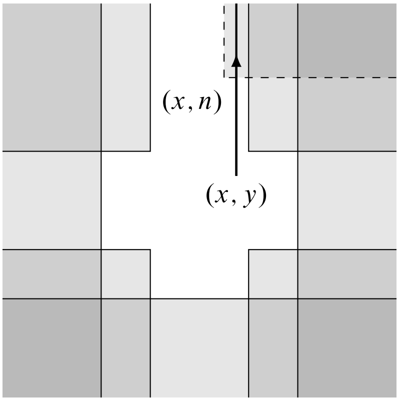


Figure 7. Approximating a horoballunion of rank $k + 1$ by one of rank k.

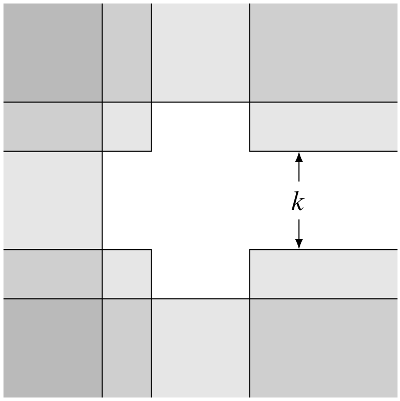


Figure 8. An element of $\mathrm{Evt}(\varphi_A)$ with rank k.

(2) Neither the left nor the right half space defined by V is contained in M. Hence $V \cap (\mathbb{Z}^2 \setminus M) \neq \emptyset$, as discussed above, so $V \cap S$ is contained in every sufficiently good approximation of M.

This shows that the whole of S is contained in every sufficiently good approximation of M. All in all we thus showed that the horoballunions having rank k or larger form a closed subset of $\mathrm{Evt}(\varphi_A)$.

Now we have to show that every horoballunion of rank $k + 1$ can be arbitrarily well approximated by a horoballunion of rank k. Assume that $\mathbb{Z}^2 \setminus M$ contains a ray with direction $(0, 1)$ and let $\{(x, y) + t(0, 1) : t \in \mathbb{N}\}$ be the rightmost of them. Then $(M \cup ((x, n) + H_{(-1,-1)}))_{n\in\mathbb{N}}$ is sequence of horoballunions converging to M whose rank is k for sufficiently large n. See Figure 7 for an illustration.

By induction this shows that $\mathrm{Evt}(\varphi_A)^{(k)}$ consists of all horoballunions of rank at least k and the isolated points in this set are those of rank precisely k. We also saw that

$$\bigcap_{k\in\mathbb{N}_0} (\mathrm{Evt}(\varphi_A))^{(k)} = \overline{D_A}.$$

Finally, see Figure 8 for an element in $\mathrm{Evt}(\varphi_A)^{(k)}$, so all these sets are non-empty.

All in all we therefore showed that $\mathrm{Evt}(\varphi_A)$ has rank ω. ■

12. Factoring and non-factoring results for φ_A, $A \subseteq \mathbb{Z}^d$

So far we presented several invariants for topological conjugacy for systems of the form $(\mathcal{P}(\Gamma), \varphi_A)$, but we do not have a complete solution for this isomorphism problem, even for the case $\Gamma = \mathbb{Z}^d$. The following is another seemingly innocent question, which we have been unable to resolve.

Question 12.1. Given two finite positively generating sets $A_1 \subseteq \mathbb{Z}^{d_1}$ and $A_2 \subseteq \mathbb{Z}^{d_2}$, when does $(\mathcal{P}(\mathbb{Z}^{d_1}), \varphi_{A_1})$ factor onto $(\mathcal{P}(\mathbb{Z}^{d_2}), \varphi_{A_2})$?

In fact, apart from the case $d_1 = 1$ and $d_2 > 1$, we do not know of any example where the answer is false. On the other hand, we know very few examples of non-trivial factor maps between such systems.

The rest of the section presents feeble partial results on the above question.

It is a folklore result that any factor map between topological dynamical systems induces a factor map between the eventual images and between the natural extensions. The simple proof (below) is a compactness argument.

Lemma 12.2. *Let X, Y be compact Hausdorff topological spaces and $f : X \to X$ and $g : Y \to Y$ be continuous maps, and let $\pi : X \to Y$ be a continuous surjective map such that $\pi \circ f = g \circ \pi$. Then:*

(1) *The restriction of π to* $\mathrm{Evt}(f)$ *is onto* $\mathrm{Evt}(g)$.
(2) *Let $\hat{\pi} : \hat{X}_f \to \hat{X}_g$ be given by $\hat{\pi}((x_n)_{n\in\mathbb{Z}})_k = \pi(x_k)$. Then $\hat{\pi}$ is a factor map from $(\hat{X}_f, \hat{f})$ onto $(\hat{X}_g, \hat{g})$.*

Proof. For every $y \in \mathrm{Evt}(g)$ we can find $(y_n)_{n\in\mathbb{Z}} \in \hat{X}_g$ with $y_0 = y$ by compactness. It is therefore enough to show that $\hat{\pi}$ is surjective. Let $(y_n)_{n\in\mathbb{Z}} \in \hat{X}_g$. For every $k \in \mathbb{N}$ there is $x^k \in X$ with $y_{-k} = \pi(x^k)$. Set

$$x_n^k := \begin{cases} x^k & \text{if } n < -k, \\ f^{k+n}(x^k) & \text{if } n \geq -k. \end{cases}$$

Let x be a limit of some subsequence of $((x_n^k)_{n\in\mathbb{Z}})_{k\in\mathbb{N}}$. Then $x \in \hat{X}_f$ and $(y_n)_{n\in\mathbb{Z}} = \hat{\pi}((x_n)_{n\in\mathbb{Z}})$. ∎

Proposition 12.3. *There is no factor map from $(\mathcal{P}(\mathbb{Z}), \varphi_{\{-1,0,1\}})$ onto $(\mathcal{P}(\mathbb{Z}^2), \varphi_{\{-1,0,1\}^2})$.*

Proof. By Lemma 12.2 it suffices to show that the eventual image of $(\mathcal{P}(\mathbb{Z}), \varphi_{\{-1,0,1\}})$ does not factor onto the eventual image of $(\mathcal{P}(\mathbb{Z}^2), \varphi_{\{-1,0,1\}^2})$. This follows from the fact that the eventual image of $(\mathcal{P}(\mathbb{Z}), \varphi_{\{-1,0,1\}})$ is countable, whereas the eventual image of $(\mathcal{P}(\mathbb{Z}^2), \varphi_{\{-1,0,1\}^2})$ is not:

The eventual image of $(\mathcal{P}(\mathbb{Z}), \varphi_{\{-1,0,1\}})$ consists of elements of the form

$$\{\mathbb{Z} \setminus (s,t) : s \leq t\}$$

with $s, t \in \mathbb{Z} \cup \{-\infty, \infty\}$. In particular, it is countable.

On the other hand the eventual image of $(\mathcal{P}(\mathbb{Z}), \varphi_{\{-1,0,1\}})$ is uncountable, as it contains the pairwise different elements $\bigcup_{i\in I}((i,-i) + H_{\{(1,1)\}})$ for $I \subseteq \mathbb{Z}$. ∎

We finish with some easy cases where there is a factor map.

Proposition 12.4. *There is a factor map from*

$$(\mathrm{Evt}(\varphi_{\{-1,0,1\}^2}), \varphi_{\{-1,0,1\}^2}) \quad \textit{onto} \quad (\mathrm{Evt}(\varphi_{\{-1,0,1\}}), \varphi_{\{-1,0,1\}}).$$

Proof. Define a map π from $(\mathrm{Evt}(\varphi_{\{-1,0,1\}^2}), \varphi_{\{-1,0,1\}^2})$ to $\mathcal{P}(\mathbb{Z})$ by

$$\pi(M) := \big\{k \in \mathbb{Z} : (k,k) \in M \text{ or } M \notin X_{|k|}\big\}$$

where

$$\begin{aligned} X_k := \big\{M \in \mathcal{P}(\mathbb{Z}) : \exists s,t \in \mathbb{Z} \text{ s.t. } & M \cap \{-k,\dots,k\}^2 \\ &= \big(((s,s) + H_{\{(-1,-1)\}}) \cup ((t,t) + H_{\{(1,1)\}})\big) \cap \{-k,\dots,k\}^2\big\}. \end{aligned}$$

Let us check that π indeed defines a factor map from $(\mathrm{Evt}(\varphi_{\{-1,0,1\}}), \varphi_{\{-1,0,1\}})$: The map π is continuous, because whether $k \in \pi(M)$ depends only on $M \cap \{-k,\dots,k\}^2$. Recall that

$$\begin{aligned} \mathrm{Evt}(\varphi_{\{-1,0,1\}}) = \big\{\mathbb{Z} \setminus (s,t) : s < t\big\} &\cup \big\{\mathbb{Z} \setminus (s,+\infty) : s \in \mathbb{Z}\big\} \\ \cup \big\{Z \setminus (-\infty,t) : t \in \mathbb{Z}\big\} &\cup \big\{\mathbb{Z}, \emptyset\big\}. \end{aligned}$$

Now for any $s < t$,

$$\begin{aligned} \pi\big(((s,s) + H_{\{(-1,-1)\}}) \cup ((t,t) + H_{\{(1,1)\}})\big) &= \mathbb{Z} \setminus (s,t), \\ \pi((s,s) + H_{\{(-1,-1)\}}) &= \mathbb{Z} \setminus (s,+\infty), \\ \pi((t,t) + H_{\{(1,1)\}}) &= \mathbb{Z} \setminus (-\infty,t), \\ \pi(\mathbb{Z}^2) = \mathbb{Z}, \quad \pi(\emptyset) &= \emptyset. \end{aligned}$$

Thus the image of π contains $\mathrm{Evt}(\varphi_{\{-1,0,1\}})$. It remains to show that π intertwines the dynamics. For this we record the following easy facts about the sets $X_k, k \in \mathbb{N}$.

(1) $X_{k+1} \subseteq X_k$.

(2) If $M \in X_{k+1}$, then $\varphi_{\{-1,0,1\}^2}(M) \in X_k$.

(3) If $\varphi_{\{-1,0,1\}^2}(M) \in X_k$ and there is $\ell \in \{-k,\dots,k\}$ with $(\ell,\ell) \notin \varphi_{\{-1,0,1\}^2}(M)$, then $M \in X_{k+1}$.

We first show that $\varphi_{\{-1,0,1\}}(\pi(M)) \subseteq \pi(\varphi_{\{-1,0,1\}^2}(M))$. Let $k \in \varphi_{\{-1,0,1\}}(\pi(M))$, so we must have $\{k-1,k,k+1\} \cap \pi(M) \neq \emptyset$. Therefore either

$$\big\{(k-1,k-1),(k,k),(k+1,k+1)\big\} \cap M \neq \emptyset$$

or

$$M \notin X_{|k-1|} \cap X_{|k|} \cap X_{|k+1|}.$$

In the first case, $(k,k) \in \varphi_{\{-1,0,1\}^2}(M)$ and $k \in \pi(\varphi_{\{-1,0,1\}^2}(M))$. In the second case, by (1), $M \notin X_{|k|+1}$ and thus by (3) either $\varphi_{\{-1,0,1\}^2}(M) \notin X_{|k|}$ or $(k,k) \in \varphi_{\{-1,0,1\}^2}(M)$. In both cases we have $k \in \pi(\varphi_{\{-1,0,1\}^2}(M))$.

Now we show $\varphi_{\{-1,0,1\}}(\pi(M)) \supseteq \pi(\varphi_{\{-1,0,1\}^2}(M))$. Let $k \in \pi(\varphi_{\{-1,0,1\}^2}(M))$, so $(k,k) \in \varphi_{\{-1,0,1\}^2}(M)$ or $\varphi_{\{-1,0,1\}^2}(M) \notin X_{|k|}$. In the first case

$$\big\{(k-1,k-1),(k,k),(k+1,k+1)\big\} \cap M \neq \emptyset,$$

hence $\{k-1,k,k+1\} \cap \pi(M) \neq \emptyset$ and thus $k \in \varphi_{\{-1,0,1\}}(\pi(M))$. In the second case $\varphi_{\{-1,0,1\}^2}(M) \notin X_{|k|}$ hence $M \notin X_{|k|+1}$. Therefore either $k+1 \in \pi(M)$ or $k-1 \in \pi(M)$, depending on the sign of k, and thus finally $k \in \varphi_{\{-1,0,1\}}(\pi(M))$. ∎

Let $G_{\mathbb{N}}$ denote the subgraph of Cayley$(\mathbb{Z}, \{-1,0,1\})$ induced by $\mathbb{N}$, i.e.,

$$V(G_{\mathbb{N}}) = \mathbb{N},$$
$$E(G_{\mathbb{N}}) = \big\{(n,n+1) : n \in \mathbb{N}\big\} \cup \big\{(n+1,n) : n \in \mathbb{N}\big\} \cup \big\{(n,n) : n \in \mathbb{N}\big\}.$$

Proposition 12.5. *Let* (Γ, A) *be a finitely generated group without dead ends and with* $A = A^{-1}$. *Then* $(\mathcal{P}(\Gamma), \varphi_A)$ *factors onto* $(\mathcal{P}(\mathbb{N}), \varphi_{G_{\mathbb{N}}})$.

Proof. The factor map is given by $\pi : \mathcal{P}(\Gamma) \to \mathcal{P}(\mathbb{N})$ with

$$0 \in \pi(M) \iff 1_\Gamma \in M \text{ and}$$
$$k \in \pi(M) \iff (A^k \setminus A^{k-1}) \cap M \neq \emptyset \text{ for } k > 0.$$ ∎

Proposition 12.6. *Let* $A = \{-1,0,1\} \subseteq \mathbb{Z}$. *There is factor map from* $(\mathcal{P}(\mathbb{Z}), \varphi_{A^n})$ *to* $(\mathcal{P}(\mathbb{Z}), \varphi_A)$ *for all* $n \in \mathbb{N}$.

Proof. The factor map is given by $\pi : \mathcal{P}(\mathbb{Z}) \to \mathcal{P}(\mathbb{Z})$ with

$$\pi(M) = \big\{k \in \mathbb{Z} : \{kn, \dots, kn+(n-1)\} \cap M \neq \emptyset\big\}.$$ ∎

More generally, with $A = \{-1,0,1\} \subseteq \mathbb{Z}$, as pointed out by the anonymous referee in response to a question posed in a preliminary version, for every $i < j$, $i, j \in \mathbb{N}$ there is a factor map π from $(\mathcal{P}(\mathbb{Z}), \varphi_{A^j})$ to $(\mathcal{P}(\mathbb{Z}), \varphi_{A^i})$ which is given by

$$(ki+r) \in \pi(M) \iff (kj+(r+j-i)) \in M \text{ whenever } k \in \mathbb{Z}, 0 < r < i,$$

and

$$ki \in \pi(M) \iff \{kj, \dots, kj+(j-i)\} \cap M \neq \emptyset \text{ for every } k \in \mathbb{Z}.$$

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Jeremias Epperlein
Fakultät für Informatik und Mathematik, Universität Passau, Innstr. 33, 94032 Passau, Germany;
jeremias.epperlein@uni-passau.de

Tom Meyerovitch
Department of Mathematics, Ben Gurion University of the Negev, P.O.B. 653,
Be’er Sheva 8410501, Israel; mtom@math.bgu.ac.il