Groups Geom. Dyn. (Online first)
DOI 10.4171/GGD/701

All finitely generated 3-manifold groups are Grothendieck rigid

Hongbin Sun

Abstract. In this paper, we prove that all finitely generated 3-manifold groups are Grothendieck rigid. More precisely, for any finitely generated 3-manifold group G and any finitely generated proper subgroup $H < G$, we show that the inclusion induced homomorphism $\hat{i}: \hat{H} \to \hat{G}$ on profinite completions is not an isomorphism.

1. Introduction

For a group G, its profinite completion is the inverse limit of the direct system of its finite quotients (see Section 2 for definition) and is denoted by $\hat{G}$. There is always a natural homomorphism $G \to \hat{G}$, and it is injective if and only if G is residually finite.

A group homomorphism $u: H \to G$ induces a homomorphism $\hat{u}: \hat{H} \to \hat{G}$ on their profinite completions (see Section 2 for definition). During his study of linear representations of groups, Grothendieck asked the following question in [5].

Problem 1.1. Let $u: H \to G$ be a homomorphism of finitely presented residually finite groups such that $\hat{u}: \hat{H} \to \hat{G}$ is an isomorphism. Is u an isomorphism?

For Problem 1.1, it suffices to consider the case where u is injective, since any nontrivial element in the kernel of u gives a nontrivial element in the kernel of $\hat{u}$ (since H is residually finite). So we can assume that H is a subgroup of G, and u is the inclusion homomorphism. According to convention, we rewrite the inclusion homomorphism as $i: H \to G$, and we may simply write it as $H \to G$ when no confusion is caused. Similarly, we use $\hat{H} \to \hat{G}$ to denote the inclusion induced homomorphism on profinite completions when it causes no confusion.

Now we review some terminologies introduced by Long and Reid in [8]. We assume all groups are finitely generated and residually finite unless otherwise stated. Let G be a group and let $H < G$ be a subgroup. We say that (G, H) is a *Grothendieck pair* if H is a proper subgroup of G, and the inclusion induced homomorphism $\hat{H} \to \hat{G}$ on profinite completions is an isomorphism. Thus, it provides a negative answer to Problem 1.1 if

2020 Mathematics Subject Classification. Primary 20E18; Secondary 57M05, 20E26.
Keywords. 3-manifold groups, profinite completions, Grothendieck rigidity, subgroup separability.

both G and H are finitely presented. Moreover, we say that G is *Grothendieck rigid* if for any finitely generated proper subgroup $H < G$, (G, H) is not a Grothendieck pair. In other words, G is Grothendieck rigid if for any finitely generated proper subgroup $H < G$, the inclusion induced homomorphism $\widehat{H} \to \widehat{G}$ is not an isomorphism.

In [10], Platonov and Tavgen' constructed Grothendieck pairs (G, H) consisting of finitely generated (but infinitely presented) residually finite groups, thus giving a partial negative answer to Problem 1.1. Then in [3], Bridson and Grunewald generalized the work in [10] to construct Grothendieck pairs (G, H) consisting of finitely presented residually finite groups, thus answering Problem 1.1 negatively.

Now we restrict to the category of finitely generated 3-manifold groups and their finitely generated subgroups. Note that all these groups are automatically finitely presented (by [13]) and residually finite (by [7] and the geometrization).

In [8], Long and Reid gave the first result on Grothendieck rigidity of 3-manifold groups. They proved that groups of all closed geometric 3-manifolds and all finite volume hyperbolic 3-manifold groups are Grothendieck rigid. Moreover, in [2], Boileau and Friedl proved that groups of all compact, connected, orientable, irreducible 3-manifolds with empty or tori boundary are Grothendieck rigid.

In this paper, we generalize the results in [8] and [2] to prove that all finitely generated 3-manifold groups are Grothendieck rigid.

Theorem 1.2. *Let M be a 3-manifold with finitely generated fundamental group $G = \pi_1(M)$. Then, for any finitely generated proper subgroup $H < G$, the inclusion induced homomorphism $\widehat{H} \to \widehat{G}$ on profinite completions is not an isomorphism. In other words, $\pi_1(M)$ is Grothendieck rigid.*

Note that our proof of Theorem 1.2 is independent of the proofs in [8] and [2]. The new ingredient is the author's work in [15], which characterizes separability of subgroups of 3-manifold groups (see Sections 2 and 3 for more details).

The starting point of our proof of Theorem 1.2 is a fundamental observation of Long and Reid in [8] (Lemma 2.2 in the present paper): if $H < G$ is a proper subgroup that is separable in G, then (G, H) is not a Grothendieck pair. So we only need to consider nonseparable subgroups $H < G = \pi_1(M)$. By the author's characterization of nonseparable subgroups of 3-manifold groups [15], there exists a subgroup $H_0 < H < G$, such that the normalizer of H_0 in H (denoted by $N_H(H_0)$) contains H_0 as a finite index subgroup, i.e., $[N_H(H_0) : H_0] < \infty$, while the normalizer of H_0 in G satisfies $[N_G(H_0) : H_0] = \infty$. Then we use tools in profinite groups and profinite graphs to prove that the above behavior of normalizers passes to the profinite completion, which implies that $\widehat{H} \to \widehat{G}$ is not an isomorphism.

The organization of this paper is summarized as follows. In Section 2, we review basic concepts on profinite completions of groups and subgroup separability, in particular, we recall Long and Reid's observation that separable subgroups do not give Grothendieck pairs. In Section 3, we consider a graph of group structure on H and the author's characterization of separability of $H < \pi_1(M)$. Then we construct the desired subgroup

$H_0 < H$, which is contained in a vertex group $H^v < H$ as a finite index subgroup. In Section 4, we review basic concepts on graphs of profinite groups and the associated actions on profinite Bass–Serre trees. Then we use the graph of profinite group structure on $\widehat{H}$ to prove that $[N_{\widehat{H}}(\widehat{H}_0) : \widehat{H}_0] < \infty$. Actually, in Sections 3 and 4, we need some mild assumptions on the 3-manifold M (Conditions 3.1), so that the characterization in [15] is applicable. In Section 5, we first prove the Grothendieck rigidity for groups of 3-manifolds satisfying Conditions 3.1, then we prove it for all finitely generated 3-manifold groups.

2. Preliminaries on profinite completions and subgroup separability

In this section, we first review some basic concepts on profinite completions of groups, then we review subgroup separability and prove the fundamental observation (Lemma 2.2) for the proof of Theorem 1.2.

2.1. Profinite completions of groups

All the following material on profinite groups can be found in Ribes and Zalesskii's book [12]. In the current paper, when we talk about groups, we mean abstract groups, and we will emphasize profinite groups when we mean it. The notation of any profinite group has a hat $\widehat{\ }$ on it, even if the profinite group is not the profinite completion of an abstract group.

Let G be a finitely generated group, and let $\mathcal{N}$ be the set of all finite index normal subgroups of G. For $N_1, N_2 \in \mathcal{N}$ with $N_1 < N_2$, there is a natural quotient homomorphism $G/N_1 \to G/N_2$. Then the *profinite completion of* G is defined to be the inverse limit of the family of finite quotients $\{G/N\}_{N\in\mathcal{N}}$:

$$\widehat{G} = \varprojlim_{N\in\mathcal{N}} G/N.$$

For each $N \in \mathcal{N}$, there is a natural surjective homomorphism $\pi_N : \widehat{G} \to G/N$.

We also have a compatible family of quotient homomorphisms $\{G \to G/N\}_{N\in\mathcal{N}}$, and it induces a homomorphism $G \to \widehat{G}$. This homomorphism is injective if and only if G is residually finite.

The profinite completion $\widehat{G} = \varprojlim_{N\in\mathcal{N}} G/N$ is a subset of the compact topological space $\Pi_{N\in\mathcal{N}} G/N$. Under the subspace topology, $\widehat{G}$ is a compact Hausdorff totally-disconnected topological group, and the image of the natural homomorphism $G \to \widehat{G}$ is dense in $\widehat{G}$.

Let $u : H \to G$ be a group homomorphism. For any finite index normal subgroup $N \lhd G$, we have a homomorphism $H \xrightarrow{u} G \to G/N$, which factors through $H/u^{-1}(N)$. Since $u^{-1}(N)$ is a finite index normal subgroup of H, we have an induced homomorphism

$$u_N : \widehat{H} \xrightarrow{\pi_{u^{-1}(N)}} H/u^{-1}(N) \to G/N.$$

The family of homomorphisms $\{u_N : \widehat{H} \to G/N\}_{N \in \mathcal{N}}$ gives rise to a homomorphism

$$\widehat{u} \colon \widehat{H} \to \widehat{G} = \varprojlim_{N \in \mathcal{N}} G/N,$$

which is the *induced homomorphism* of $u \colon H \to G$ on their profinite completions.

If the homomorphism $H \to G$ is an inclusion, the image of the induced homomorphism $\widehat{H} \to \widehat{G}$ is the closure of $H \subset G \subset \widehat{G}$ in $\widehat{G}$. Moreover, $\widehat{H} \to \widehat{G}$ is injective if and only if for any finite index subgroup $H' < H$, there exists a finite index subgroup $G' < G$, such that $G' \cap H < H'$.

Here is a lemma that allows us to prove Grothendieck rigidity by passing to finite index normal subgroups.

Lemma 2.1. *Let G be a group and let $G' \lhd G$ be a finite index normal subgroup. If G' is Grothendieck rigid, then G is Grothendieck rigid.*

Proof. Let $H < G$ be a finitely generated subgroup with inclusion homomorphism $i \colon H \to G$. Let $\beta \colon G \to G/G'$ be the quotient homomorphism (to a finite group).

Suppose that $\widehat{i} \colon \widehat{H} \to \widehat{G}$ is an isomorphism, then [2, Lemma 2.8] implies that the inclusion induced homomorphism $\widehat{H \cap G'} \to \widehat{G}'$ is an isomorphism. Since G' is Grothendieck rigid, we must have $H \cap G' = G'$, i.e., $G' < H$ holds.

Since $[G : G'] < \infty$, H must be a finite index subgroup of G. Then the image of $\widehat{i} \colon \widehat{H} \to \widehat{G}$ is an open subgroup of $\widehat{G}$ with finite index $[G : H]$, by the fundamental correspondence between finite index subgroups of G and $\widehat{G}$, see [12, Proposition 3.2.2]. So we must have $H = G$. ∎

It is not hard to prove that Lemma 2.1 still holds if G' is only a finite index subgroup of G, but Lemma 2.1 is good enough for proving Theorem 1.2.

2.2. Subgroup separability

Now we turn to the concept of subgroup separability. For a group G and a subgroup $H < G$, we say that H *is separable in* G if for any $g \in G \setminus H$, there exists a homomorphism $\phi \colon G \to Q$ to a finite group, such that $\phi(g) \notin \phi(H)$. Moreover, we say a group G is LERF (locally extended residually finite) if all finitely generated subgroups of G are separable in G.

The starting point of our proof of Theorem 1.2 is the lemma below. It is essentially the result [8, Lemma 2.5], and we state it in a slightly weaker form. Although this lemma was already proved in [8], we still prove it here, since the proof is simple and this result plays a fundamental role in this paper.

Lemma 2.2. *Let G be a group, and let $H < G$ be a proper subgroup that is separable in G. Then (G, H) is not a Grothendieck pair.*

Proof. Since $H < G$ is a proper subgroup, there exists an element $g \in G \setminus H$. Since H is separable in G, there exists a finite index normal subgroup $N \lhd G$ such that for the quotient homomorphism $\phi: G \to G/N$, $\phi(g) \notin \phi(H)$ holds.

Let $i: H \to G$ be the inclusion, let $\bar{i}: H/N \cap H \to G/N$ be the induced homomorphism on quotient groups, and let $\hat{i}: \widehat{H} \to \widehat{G}$ be the induced homomorphism on profinite completions. Then we have the following commutative diagram:

$$\begin{tikzcd}
\widehat{H} \arrow[r, "\hat{i}"] \arrow[d, "\pi_{N\cap H}"] & \widehat{G} \arrow[d, "\pi_N"] \\
H/N \cap H \arrow[r, "\bar{i}"] & G/N.
\end{tikzcd}$$

Suppose that $\hat{i}: \widehat{H} \to \widehat{G}$ is an isomorphism. Since $\pi_N: \widehat{G} \to G/N$ is surjective, $\pi_N \circ \hat{i}: \widehat{H} \to G/N$ is surjective. However, by our construction of N, gN does not lie in the image of $\bar{i}: H/N \cap H \to G/N$, so $\bar{i} \circ \pi_{N\cap H}: \widehat{H} \to G/N$ is not surjective.

So we get a contradiction, thus (G, H) is not a Grothendieck pair. ∎

Lemma 2.2 immediately implies the following corollary, since both LERFness and Grothendieck rigidity only concern finitely generated subgroups.

Corollary 2.3. *Let G be a LERF group, then it is Grothendieck rigid.*

Obviously, Corollary 2.3 implies that all LERF 3-manifold groups are Grothendieck rigid. In particular, Agol's celebrated result [1] that all hyperbolic 3-manifold groups are LERF implies that all hyperbolic 3-manifold groups are Grothendieck rigid. This is actually the main result of [8], while Agol's result was not available when [8] was written.

Unfortunately, there are a number of 3-manifold groups that are not LERF. To prove the Grothendieck rigidity of these groups, it remains to prove that any nonseparable subgroup does not give a Grothendieck pair.

3. A graph of group structure on $H < \pi_1(M)$ and the construction of $H_0 < H$ for nonseparable $H < \pi_1(M)$

In this section, we first describe a graph of group structure on $H < \pi_1(M)$, then we review the author's characterization of separability of $H < \pi_1(M)$ and construct the desired subgroup $H_0 < H$ for a nonseparable $H < \pi_1(M)$.

In this and the next section, we restrict to 3-manifolds satisfying the following conditions, which form the essential case towards the proof of Theorem 1.2.

Conditions 3.1. (1) M is compact, orientable, irreducible and ∂-irreducible (may have some boundary component of genus at least 2).

(2) M has nontrivial torus decomposition and does not support the Sol geometry.

(3) Under the torus decomposition of M, no Seifert piece is the twisted I-bundle over Klein bottle.

For a 3-manifold M satisfying Condition 3.1 (1), it has a canonical torus decomposition (see [6, Theorem 1.9]): there exists a finite collection $\mathcal{T} \subset M$ of disjoint incompressible tori such that each component of $M \setminus \mathcal{T}$ is either atoroidal or a Seifert manifold, and a minimal such collection $\mathcal{T}$ is unique up to isotopy. Note that we do not take the torus-annulus decomposition (the theory of characteristic submanifolds) here. This is because of the fact that each decomposition annulus is contained in an atoroidal piece of $M \setminus \mathcal{T}$. Such a piece admits a complete hyperbolic structure (possibly of infinite volume) and is known to have a LERF group (by [1]).

By Condition 3.1 (2), $\mathcal{T}$ is not empty. Moreover, by Conditions 3.1 (2), (3) and the classification of Seifert fibering structures (see [6, Theorem 2.3]), each component of $M \setminus \mathcal{T}$ has a unique Seifert fibering structure, and its base orbifold has negative Euler characteristic.

Note that Conditions 3.1 (1) and (2) are assumed in [15, Theorem 1.3], which gives the characterization of separability of $H < \pi_1(M)$. Condition 3.1 (3) is a mild condition, and it is for convenience of our proof.

3.1. A graph of group structure on $H < \pi_1(M)$

Now we suppose that M is a 3-manifold satisfying Conditions 3.1 and describe a graph of group structure on a finitely generated subgroup $H < \pi_1(M)$. Given the (nonempty) torus decomposition $\mathcal{T}$ of M, M has a graph of space structure, and we denote the dual graph by Γ. Here each component M^v of $M \setminus \mathcal{T}$ (called a piece of M) corresponds to a vertex v of Γ, and each component T^e of $\mathcal{T}$ corresponds to an edge e of Γ. Then the fundamental group $\pi_1(M)$ has a graph of group structure with dual graph Γ. Here the vertex group corresponding to vertex v is $\pi_1(M^v)$, and the edge group corresponding to edge e is $\pi_1(T^e)$. We will review the profinite counterpart of graph of groups in Section 4.

For any finitely generated subgroup $H < \pi_1(M)$, we take the corresponding covering space $\pi \colon M_H \to M$. Then $\pi^{-1}(\mathcal{T})$ induces a graph of space structure on M_H in the same manner as on M. Here each component of $\pi^{-1}(\mathcal{T})$ is either a torus, a cylinder, or a plane.

Since H is finitely generated, by [15, Lemma 3.1], there is a unique minimal codimension-0 connected submanifold $M_H^c \subset M_H$, such that it is a union of finitely many pieces of M_H, contains all pieces of M_H with nontrivial π_1, and the inclusion $M_H^c \to M_H$ induces an isomorphism of fundamental groups. Since H is isomorphic to $\pi_1(M_H^c)$, the graph of space structure on M_H^c (induced from M_H) gives a graph of group structure on H with finite dual graph Γ_H.

For the above graph of group structure on H, H acts naturally on the corresponding Bass–Serre tree T_H. Basically, the Bass–Serre tree T_H is the dual graph of the universal cover $\widetilde{M}_H^c$ of M_H^c, and the H-action on T_H is induced by its action on $\widetilde{M}_H^c$. The vertex (edge) stabilizers of this H-action on T_H are conjugations of vertex (edge) groups of H, and the quotient of T_H by this H-action is isomorphic to Γ_H. We will review the profinite counterpart of Bass–Serre theory in Section 4.

3.2. The construction of $H_0 < H$ for nonseparable $H < \pi_1(M)$

We continue to use notations from the previous subsection. Each piece M_H^v of M_H^c with nontrivial fundamental group covers a piece M^v of M. We are interested in the pieces M_H^v such that one of the following holds:

(1) M^v is a finite volume hyperbolic 3-manifold (i.e., an atoroidal manifold with tori boundary), and M_H^v corresponds to a virtual fiber surface subgroup of $\pi_1(M^v)$.

(2) M^v is a Seifert manifold, and the S^1-bundle (over 2-orbifold) structure on M^v lifts to an $\mathbb{R}$-bundle structure on M_H^v.

More precisely, in case (1), there is a compact surface Σ^v such that M_H^v is equal to $\Sigma^v \times \mathbb{R}$ (Σ^v is orientable) or $\Sigma^v \tilde{\times} \mathbb{R}$ (Σ^v is nonorientable). Moreover, the covering map $M_H^v \to M^v$ factors through a finite cover $N^v \to M^v$, such that N^v has a surface bundle over circle structure (with orientable fiber surface Σ^v) or a semi-bundle structure (a union of two twisted I-bundles over nonorientable surface Σ^v).

In case (2), M_H^v is homeomorphic to $S^v \times \mathbb{R}$ or $S^v \tilde{\times} \mathbb{R}$ for some surface S^v. If S^v is compact, a similar description as in case (1) holds, and we take $\Sigma^v = S^v$. If S^v is not compact, we take a compact connected subsurface $\Sigma^v \subset S^v$ such that the inclusion $\Sigma^v \to S^v$ induces an isomorphism of fundamental groups, each boundary component of Σ^v either lies in ∂S^v or lies in the interior of S^v, and no component of $S^v \setminus \Sigma^v$ is a disc or an annulus.

When $M_H^v = \Sigma^v \times \mathbb{R}$ or $\Sigma^v \tilde{\times} \mathbb{R}$ (case (1) and the first subcase of case (2)), Σ^v is called a *virtual fiber surface*. When $M_H^v = S^v \times \mathbb{R}$ or $S^v \tilde{\times} \mathbb{R}$ for a noncompact surface S_v (the second subcase of case (2)), the compact subsurface $\Sigma^v \subset S^v$ is called a *partial fiber surface*.

We consider each virtual fiber or partial fiber surface Σ^v constructed above as a (possibly nonproper) subsurface of M_H^v, then the inclusion $\Sigma^v \to M_H^v$ is a homotopy equivalence. Since we assumed that M_H^v has nontrivial fundamental group and Σ^v has boundary, Σ^v has nonpositive Euler characteristic.

If there is a component $C \subset \pi^{-1}(\mathcal{T})$ that intersects with two such subsurfaces Σ^v and Σ^w (on their boundaries), then C must be a cylinder, while $\Sigma^v \cap C$ and $\Sigma^w \cap C$ are isotopic curves on C. Then we isotopy Σ^v and Σ^w so that they intersect with C along the same curve, and we paste them together along this curve. By doing such pasting procedure along all cylinder components of $\pi^{-1}(\mathcal{T})$, whenever possible, we get the *almost fiber surface* $\Phi(H)$ defined in [15]. We can consider $\Phi(H)$ as a (possibly disconnected and nonproper) compact subsurface of $M_H^c \subset M_H$. By construction, $\Phi(H)$ has a natural graph of space structure. We denote the dual graph of $\Phi(H)$ by $\Gamma_{\Phi(H)}$, then it is naturally a subgraph of Γ_H (the dual graph of M_H^c).

In [15], the author proved the following characterization of separability of $H < \pi_1(M)$ (see [15, Theorem 1.3 and Remark 3.5]).

Theorem 3.2. *Let M be a 3-manifold satisfying Conditions 3.1, and let $H < \pi_1(M)$ be a finitely generated subgroup. Then there is a canonically defined group homomorphism $s\colon H_1(\Phi(H);\mathbb{Z}) \to \mathbb{Q}_+^\times$ that factors through $H_1(\Gamma_{\Phi(H)};\mathbb{Z})$, such that H is separable in $\pi_1(M)$ if and only if s is the trivial homomorphism. In particular, if H is not separable in $\pi_1(M)$, then $\Gamma_{\Phi(H)}$ contains a simple cycle.*

Here $\mathbb{Q}_+^\times$ denotes the group of positive rationals with the multiplicative operation. The homomorphism s is called the *generalized spirality character* of H, and its definition is not important for this paper. We will only use the "in particular" part of Theorem 3.2 in this paper.

At first, we prove the following lemma that provides many separable subgroups in $\pi_1(M)$.

Lemma 3.3. *Let M be a 3-manifold satisfying Conditions 3.1, and let $M^v \subset M$ be a piece of M under the torus decomposition. Then any finitely generated subgroup $H < \pi_1(M^v) < \pi_1(M)$ is separable in $\pi_1(M)$.*

Proof. We take the covering space $M_H \to M$ corresponding to the subgroup $H < \pi_1(M)$. For the graph of space structure on M_H, its dual graph is a tree.

Then the dual graph $\Gamma_{\Phi(H)}$ of the almost fiber surface $\Phi(H)$ is a subgraph of the dual graph of M_H. So $\Gamma_{\Phi(H)}$ is a union of trees (actually, a tree), and $H_1(\Gamma_{\Phi(H)};\mathbb{Z})$ is trivial. Then Theorem 3.2 implies that H is separable in $\pi_1(M)$. ∎

Now we construct the desired subgroup $H_0 < H$ for a nonseparable subgroup $H < \pi_1(M)$.

Proposition 3.4. *Let M be a 3-manifold satisfying Conditions 3.1, and let $H < \pi_1(M)$ be a finitely generated nonseparable subgroup. Under the graph of group structure on H, there is a vertex group $H^v < H$ and a subgroup $H_0 < H^v$ such that the following holds:*

(1) *The index of H_0 in H^v is either 1 or 2.*

(2) *H_0 is a non-abelian free group.*

(3) *Any finitely generated subgroup of H^v is separable in $\pi_1(M)$.*

(4) *The normalizer of H_0 in H is H^v, i.e., $N_H(H_0) = H^v$.*

(5) *The normalizer of H_0 in $\pi_1(M)$ satisfies $[N_{\pi_1(M)}(H_0) : H_0] = \infty$.*

Proof. By Theorem 3.2, the nonseparability of H in $\pi_1(M)$ implies that $\Gamma_{\Phi(H)}$ contains a simple cycle. Take any vertex v in this simple cycle, then it has degree at least 2 in $\Gamma_{\Phi(H)}$. This vertex v corresponds to a piece $\Sigma^v \subset \Phi(H)$, and Σ^v is contained in a piece M_H^v of M_H. We first claim that Σ^v is neither an annulus nor a Möbius band.

If Σ^v is a virtual fiber surface and M_H^v covers a hyperbolic piece $M^v \subset M$ of a finite volume, then such a virtual fiber surface cannot be an annulus or a Möbius band. If Σ^v is a virtual fiber surface and M_H^v covers a Seifert piece $M^v \subset M$, then Σ^v is a finite cover of the base orbifold of M^v. By Conditions 3.1 (2) and (3), the base orbifold of M^v has negative Euler characteristic, so Σ^v is neither an annulus nor a Möbius band. If Σ^v is a partial fiber surface, then it is a nonproper subsurface of M_H^v, thus at least one boundary component of Σ^v is contained in the interior of M_H^v. Since v has degree at least 2, $\Sigma^v \cap \partial M_H^v$ has at least two components. Thus, Σ^v has at least 3 boundary components, which makes it neither an annulus nor a Möbius band.

Since Σ^v is neither an annulus nor a Möbius band, $\pi_1(\Sigma^v)$ is a non-abelian free group. By our construction, the inclusion $\Sigma^v \to M_H^v$ is a homotopy equivalence. We take the vertex subgroup $H^v < H$ to be $\pi_1(M_H^v) \cong \pi_1(\Sigma^v)$, which is a non-abelian free group. If Σ^v is an orientable surface, we simply take $H_0 = H^v$. If Σ^v is nonorientable, we take H_0 to be the group of the orientable double cover $\widetilde{\Sigma}^v \to \Sigma^v$, then H_0 is an index-2 (normal) subgroup of H^v. So H_0 and H^v satisfy conditions (1) and (2).

Since H^v is contained in a vertex subgroup $\pi_1(M^v)$ of $\pi_1(M)$, any finitely generated subgroup of H^v is also contained in $\pi_1(M^v)$. By Lemma 3.3, such a subgroup is separable in $\pi_1(M)$, thus condition (3) holds.

At first, it is clear that H^v is contained in the normalizer of H_0. For the H-action on its Bass–Serre tree T_H, $H^v < H$ is the stabilizer of a vertex $\hat{v} \in T_H$. For any $h \in N_H(H_0)$, we have $h^{-1}H_0h = H_0$. Since $H_0 < H^v$ stabilizes $\hat{v}$, H_0 also lies in the stabilizer of $h(\hat{v}) \in T_H$. Suppose that $h(\hat{v}) \neq \hat{v}$, then H_0 stabilizes the nontrivial subtree of T_H spanned by $\hat{v}$ and $h(\hat{v})$, and in particular H_0 stabilizes an edge of T_H. So H_0 is contained in a conjugation of $H^e < H$ for some edge $e \in \Gamma_H$. Since each edge group H^e is a subgroup of $\mathbb{Z}^2 \cong \pi_1(T^2)$, H_0 is isomorphic to a subgroup of $\mathbb{Z}^2$. It contradicts condition (2) that H_0 is a non-abelian free group, so we must have $h(\hat{v}) = \hat{v}$. Then h lies in the stabilizer of $\hat{v}$, thus $h \in H^v$. So we have $N_H(H_0) = H^v$, thus condition (4) holds.

If M_H^v covers a finite volume hyperbolic piece $M^v \subset M$, then the covering map factors through a finite cover N^v of M^v, such that M_H^v corresponds to a fiber subgroup of $\pi_1(N^v)$. More precisely, if Σ^v is orientable, then $M_H^v = \Sigma^v \times \mathbb{R}$ and $N^v = \Sigma^v \times I/(x,0) \sim (\phi^v(x), 1)$ for some pseudo-Anosov map $\phi^v\colon \Sigma^v \to \Sigma^v$. So $\pi_1(N^v)$ is contained in the normalizer of $H_0 = \pi_1(\Sigma^v)$ in $\pi_1(M)$. If Σ^v is nonorientable, then N^v is a union of two twisted I-bundles over Σ^v and the normalizer of $\pi_1(\Sigma^v)$ in $\pi_1(N^v)$ is actually $\pi_1(\Sigma^v)$. Recall that $\widetilde{\Sigma}^v$ is the orientable double cover of Σ^v, and $H_0 = \pi_1(\widetilde{\Sigma}^v) < \pi_1(\Sigma^v) = H^v$. We take the double cover $\widetilde{N}^v$ of N^v such that it is a $\widetilde{\Sigma}^v$-bundle over the circle. Then $\pi_1(\widetilde{N}^v) < \pi_1(M)$ is contained in the normalizer of $H_0 = \pi_1(\widetilde{\Sigma}^v)$ in $\pi_1(M)$.

If M_H^v covers a Seifert piece $M^v \subset M$, then the fiber subgroup of $\pi_1(M^v)$ (isomorphic to $\mathbb{Z}$) intersects with $H^v = \pi_1(\Sigma^v)$ trivially. If Σ^v is orientable, then the fiber subgroup commutes with $H_0 = \pi_1(\Sigma^v)$; if Σ^v is nonorientable, then the fiber subgroup does not commute with $H^v = \pi_1(\Sigma^v)$, but it commutes with $H_0 = \pi_1(\widetilde{\Sigma}^v)$.

In all these cases, we have checked that the normalizer of H_0 in $\pi_1(M)$ always contains H_0 as an infinite index subgroup, thus condition (5) holds. ■

4. The graph of profinite group structure on $\widehat{H}$ and the normalizer of $\widehat{H}_0$ in $\widehat{H}$

In this section, we still assume that the 3-manifold M satisfies Conditions 3.1. We will first review basic concepts on graphs of profinite groups and the profinite Bass–Serre theory, then we will apply the theory to $H < \pi_1(M)$ and prove that the normalizer of $\widehat{H}_0$ in $\widehat{H}$ contains $\widehat{H}_0$ as a finite index subgroup (for $H_0 < H < \pi_1(M)$ constructed in Section 3).

4.1. Graph of profinite groups

In this section, we review basic concepts on graphs of profinite groups and the profinite Bass–Serre theory. The readers can find more details on this topic in Ribes' book [11].

Definition 4.1. A *profinite graph* is a quadruple $(\Gamma, V(\Gamma), d_0, d_1)$ such that the following holds:

(1) Γ is a nonempty profinite space (an inverse limit of finite discrete spaces).

(2) $V(\Gamma) \subset \Gamma$ is a nonempty closed subset.

(3) $d_0, d_1 \colon \Gamma \to V(\Gamma)$ are two continuous functions such that $d_0|_{V(\Gamma)}$ and $d_1|_{V(\Gamma)}$ are both identity on $V(\Gamma)$.

The edge set of this profinite graph is $E(\Gamma) = \Gamma \setminus V(\Gamma)$, which may not be a closed subset of Γ. In the case where Γ is a finite set, this notion of profinite graphs coincides with the notion of directed graphs in the usual sense, and we will also use the notion in Definition 4.1 for (directed) finite graphs.

Now we give the definition of graph of profinite groups over finite graphs. The theory of graph of profinite groups over infinite profinite graphs is more complicated and will not be used in this paper.

Definition 4.2. A graph $(\widehat{\mathcal{G}}, \Gamma)$ of profinite groups $\widehat{\mathcal{G}}$ over a finite connected graph Γ consists of the following data:

(1) For any vertex $v \in V(\Gamma)$ and edge $e \in E(\Gamma)$, there are associated profinite groups $\widehat{\mathcal{G}}^v$ (vertex group) and $\widehat{\mathcal{G}}^e$ (edge group), respectively.

(2) For each edge $e \in E(\Gamma)$, there are injective homomorphisms $\partial_0^e \colon \widehat{\mathcal{G}}^e \to \widehat{\mathcal{G}}^{d_0(e)}$ and $\partial_1^e \colon \widehat{\mathcal{G}}^e \to \widehat{\mathcal{G}}^{d_1(e)}$. Here $d_0(e)$ and $d_1(e)$ are the two vertices of Γ adjacent to e.

For convenience, we assume that all vertex and edge groups $\widehat{\mathcal{G}}^v$ and $\widehat{\mathcal{G}}^e$ are finitely generated, so all their finite index subgroups are open due to [9]. The profinite fundamental group $\Pi_1(\widehat{\mathcal{G}}, \Gamma)$ of $(\widehat{\mathcal{G}}, \Gamma)$ is defined to be the profinite completion of the abstract fundamental group $\pi_1(\widehat{\mathcal{G}}, \Gamma)$ of $(\widehat{\mathcal{G}}, \Gamma)$ (with $\widehat{\mathcal{G}}^v$ and $\widehat{\mathcal{G}}^e$ considered as abstract groups).

Recall that the abstract fundamental group $\pi_1(\widehat{\mathcal{G}}, \Gamma)$ of $(\widehat{\mathcal{G}}, \Gamma)$ is defined by the following process. One first fix a maximal spanning tree $T \subset \Gamma$, then take $t^e = 1$ for each edge $e \in E(T)$ and take a stable letter t^e for each edge $e \in E(\Gamma) \setminus E(T)$. Then the abstract fundamental group $\pi_1(\widehat{\mathcal{G}}, \Gamma)$ is defined to be the quotient of the free product of all vertex groups and a free group by the following relations:

$$(*_{v\in V(\Gamma)}\widehat{\mathcal{G}}^v) * (*_{e\in E(\Gamma)\setminus E(T)}\mathbb{Z}\langle t^e\rangle)/\langle\langle (t^e)^{-1}\partial_0^e(g)t^e(\partial_1^e(g))^{-1} : e \in E(\Gamma), g \in \widehat{\mathcal{G}}^e\rangle\rangle.$$

Here $\mathbb{Z}\langle t^e\rangle$ denotes an infinite cyclic group generated by t^e.

Given the maximal spanning tree $T \subset \Gamma$, we have natural homomorphisms

$$\widehat{\mathcal{G}}^v \to \Pi_1(\widehat{\mathcal{G}}, \Gamma) \quad \text{and} \quad \widehat{\mathcal{G}}^e \xrightarrow{\partial_0^e} \widehat{\mathcal{G}}^{d_0(e)} \to \Pi_1(\widehat{\mathcal{G}}, \Gamma)$$

to the profinite fundamental group $\Pi_1(\widehat{\mathcal{G}}, \Gamma)$. In contrast with the case of abstract groups, the homomorphisms $\widehat{\mathcal{G}}^v \to \Pi_1(\widehat{\mathcal{G}}, \Gamma)$ and $\widehat{\mathcal{G}}^e \to \Pi_1(\widehat{\mathcal{G}}, \Gamma)$ may not be injective in general.

Now we work on a graph of abstract groups $(\mathcal{G}, \Gamma)$ over finite graph Γ. Given $(\mathcal{G}, \Gamma)$, we have the following two sequences of constructions. One can first take profinite completions of vertex and edge groups of $(\mathcal{G}, \Gamma)$ to get a graph of profinite groups $(\widehat{\mathcal{G}}, \Gamma)$ (assuming each homomorphism $\widehat{\mathcal{G}}^e \to \widehat{\mathcal{G}}^v$ is injective), then take the profinite fundamental group $\Pi_1(\widehat{\mathcal{G}}, \Gamma)$. Alternatively, one can first take the abstract fundamental group $\pi_1(\mathcal{G}, \Gamma)$, then take its profinite completion $\pi_1(\widehat{\mathcal{G}}, \Gamma)$.

In general, these two sequences of constructions do not give isomorphic profinite groups, and it requires the following condition of efficiency on $(\mathcal{G}, \Gamma)$.

Definition 4.3. A finite graph of abstract groups $(\mathcal{G}, \Gamma)$ is *efficient* if the following holds:

(1) The abstract fundamental group $\pi_1(\mathcal{G}, \Gamma)$ is residually finite.

(2) For any $m \in \Gamma$ (a vertex or an edge), $\mathcal{G}^m$ is separable in $\pi_1(\mathcal{G}, \Gamma)$.

(3) For any $m \in \Gamma$ and any finite index subgroup $K < \mathcal{G}^m$, there is a finite index subgroup $N < \pi_1(\mathcal{G}, \Gamma)$ such that $N \cap \mathcal{G}^m < K$.

A generalization of [12, Exercise 9.2.7] gives the following result, see also [16, Theorem 5.6].

Theorem 4.4. *Let $(\mathcal{G}, \Gamma)$ be a finite graph of abstract groups that is efficient. Then there is a natural isomorphism*

$$\pi_1(\widehat{\mathcal{G}}, \Gamma) \xrightarrow{\cong} \Pi_1(\widehat{\mathcal{G}}, \Gamma).$$

Moreover, for any $m \in \Gamma$, the natural homomorphism $\widehat{\mathcal{G}^m} \to \Pi_1(\widehat{\mathcal{G}}, \Gamma)$ is injective.

Next, for a graph of profinite groups $(\widehat{\mathcal{G}}, \Gamma)$, we assume that $\widehat{\mathcal{G}}^m \to \Pi_1(\widehat{\mathcal{G}}, \Gamma)$ is injective for any $m \in \Gamma$, and we still use $\widehat{\mathcal{G}}^m$ to denote the image.

For a graph of profinite groups $(\widehat{\mathcal{G}}, \Gamma)$, its profinite Bass–Serre tree $\mathcal{T}_{(\widehat{\mathcal{G}},\Gamma)}$ is defined as follows.

Definition 4.5. (1) The profinite set $\mathcal{T}_{(\widehat{\mathcal{G}},\Gamma)}$ is the disjoint union of left cosets of vertex and edge groups

$$\mathcal{T}_{(\widehat{\mathcal{G}},\Gamma)} = \bigcup_{m\in\Gamma} \Pi_1(\widehat{\mathcal{G}}, \Gamma)/\widehat{\mathcal{G}}^m,$$

and the vertex set $V(\mathcal{T}_{(\widehat{\mathcal{G}},\Gamma)})$ is the disjoint union of left cosets of vertex groups

$$V(\mathcal{T}_{(\widehat{\mathcal{G}},\Gamma)}) = \bigcup_{v\in V(\Gamma)} \Pi_1(\widehat{\mathcal{G}},\Gamma)/\widehat{\mathcal{G}}^v.$$

(2) We only need to define maps $d_0, d_1 \colon \mathcal{T}_{(\widehat{\mathcal{G}},\Gamma)} \to V(\mathcal{T}_{(\widehat{\mathcal{G}},\Gamma)})$ on the edge set $E(\mathcal{T}_{(\widehat{\mathcal{G}},\Gamma)})$ (which is closed in $\mathcal{T}_{(\widehat{\mathcal{G}},\Gamma)}$). For any edge $e \in \Gamma$, we define

$$d_0(g\widehat{\mathcal{G}}^e) = g\widehat{\mathcal{G}}^{d_0(e)}, \quad d_1(g\widehat{\mathcal{G}}^e) = gt^e\widehat{\mathcal{G}}^{d_1(e)}.$$

It is routine to check that d_0 and d_1 are well-defined.

The profinite graph $\mathcal{T}_{(\widehat{\mathcal{G}},\Gamma)}$ is actually a *profinite tree*. The reader can find the definition of profinite trees in [11, Section 2.4] and the proof that $\mathcal{T}_{(\widehat{\mathcal{G}},\Gamma)}$ is a profinite tree in [11, Section 6.3].

We also have a natural $\Pi_1(\widehat{\mathcal{G}},\Gamma)$-action on the profinite Bass–Serre tree $\mathcal{T}_{(\widehat{\mathcal{G}},\Gamma)}$, which is defined by $g(h\widehat{\mathcal{G}}^m) = (gh)\widehat{\mathcal{G}}^m$ for any $g, h \in \Pi_1(\widehat{\mathcal{G}},\Gamma)$ and $m \in \Gamma$. It is obvious that the stabilizer of any element $g\widehat{\mathcal{G}}^m \in \mathcal{T}_{(\widehat{\mathcal{G}},\Gamma)}$ is $g\widehat{\mathcal{G}}^m g^{-1} < \Pi_1(\widehat{\mathcal{G}},\Gamma)$.

Now we give an application of the profinite Bass–Serre theory. This lemma is useful for proving Theorem 1.2 for reducible 3-manifolds.

Lemma 4.6. *Let $\widehat{G}_1,\dots,\widehat{G}_k$ be profinite groups, let $\Pi_{i=1}^k \widehat{G}_i$ be their profinite free product, and let $\widehat{H} < \widehat{G}_1$ be a nontrivial closed subgroup. Then the normalizer of $\widehat{H}$ in $\Pi_{i=1}^k \widehat{G}_i$ equals the normalizer of $\widehat{H}$ in $\widehat{G}_1$.*

Proof. We take a finite connected graph Γ that is a chain of k vertices $v_1,\dots,v_k$ and $k-1$ edges $e_1,\dots,e_{k-1}$. For any vertex v_i, we take $\widehat{\mathcal{G}}^{v_i} = \widehat{G}_i$; for any edge e_j, we take $\widehat{\mathcal{G}}^{e_j}$ to be the trivial group. Then each homomorphism $\widehat{G}^e \to \widehat{G}^v$ is injective since all edge groups are trivial.

By [11, Proposition 5.1.6], the profinite free product $\Pi_{i=1}^k \widehat{G}_i$ is isomorphic to the profinite fundamental group of $(\widehat{\mathcal{G}},\Gamma)$, and each $\widehat{G}_i$ injects into $\Pi_{i=1}^k \widehat{G}_i$. Then $\Pi_{i=1}^k \widehat{G}_i$ acts on the profinite Bass–Serre tree $\mathcal{T}_{(\widehat{\mathcal{G}},\Gamma)}$.

Since $\widehat{H} < \widehat{G}_1$, $\widehat{H}$ lies in the stabilizer of the vertex $\hat{v} = \widehat{G}_1 \in \mathcal{T}_{(\widehat{\mathcal{G}},\Gamma)}$. For any $g \in \Pi_{i=1}^k \widehat{G}_i$ that normalizes $\widehat{H}$, we have $g^{-1}\widehat{H}g = \widehat{H}$. By considering the action of $\Pi_{i=1}^k \widehat{G}_i$ on $\mathcal{T}_{(\widehat{\mathcal{G}},\Gamma)}$, $\widehat{H}$ also lies in the stabilizer of $g(\hat{v}) \in \mathcal{T}_{(\widehat{\mathcal{G}},\Gamma)}$.

We suppose that $g(\hat{v}) \neq \hat{v}$. Since $\widehat{H}$ stabilizes both $\hat{v}$ and $g(\hat{v})$, it stabilizes the minimal subtree $[\hat{v}, g(\hat{v})]$ of $\mathcal{T}_{(\widehat{\mathcal{G}},\Gamma)}$ spanned by $\hat{v}$ and $g(\hat{v})$ of [11, Theorem 4.1.5]. Since $\hat{v}$ and $g(\hat{v})$ are two different vertices of $\mathcal{T}_{(\widehat{\mathcal{G}},\Gamma)}$, the subtree $[\hat{v}, g(\hat{v})]$ contains an edge $\hat{e}$ (by the definition of profinite trees), and $\widehat{H}$ stabilizes $\hat{e}$. However, this is impossible, since the stabilizer of any edge in $\mathcal{T}_{(\widehat{\mathcal{G}},\Gamma)}$ is the trivial group, while $\widehat{H}$ is nontrivial.

Then we must have $\widehat{G}_1 = \hat{v} = g(\hat{v}) = g\widehat{G}_1$, thus $g \in \widehat{G}_1$. So the normalizer of $\widehat{H}$ in $\Pi_{i=1}^k \widehat{G}_i$ is contained in the normalizer of $\widehat{H}$ in $\widehat{G}_1$, and they must be equal to each other. ∎

4.2. Normalizer of $\widehat{H}_0$ in $\widehat{H}$

For a finitely generated subgroup of a 3-manifold group $H < \pi_1(M)$, we first prove that the graph of group structure on $H < \pi_1(M)$ is efficient. Then we prove that, for any non-abelian subgroup $H_0 < H^v$ of a vertex group $H^v < H$, the normalizer of $\widehat{H}_0$ in $\widehat{H}$ is contained in $\widehat{H}^v$.

We first prove that the graph of group structure on H is efficient, thus this graph of group structure on H behaves nicely when passing to profinite completion.

Proposition 4.7. *Let M be a 3-manifold satisfying Conditions 3.1, and let $H < \pi_1(M)$ be a finitely generated subgroup. Then the graph of group structure on H is efficient.*

Proof. We need to check the three conditions in Definition 4.3.

Condition (1) follows from the fact that all finitely generated 3-manifold groups are residually finite, see [7].

Since each vertex or edge group H^m is contained in a vertex group of $\pi_1(M)$, by Lemma 3.3, it is separable in $\pi_1(M)$. Then by a simple algebraic argument, each H^m is separable in H, thus condition (2) holds.

Let $K < H^m$ be a finite index subgroup, then K is contained in a vertex subgroup of $\pi_1(M)$. By the argument for condition (2), K is separable in H. We take a finite left transversal $h_0 = e, h_1, \dots, h_k$ of K in H^m. Then since K is separable in H, there is a finite index subgroup $N < H$ such that $K < N$, and $h_1, \dots, h_k \notin N$. Then we must have $N \cap H^m = K$, thus condition (3) holds. ∎

For a 3-manifold M, a finitely generated subgroup $H < \pi_1(M)$ and a non-abelian subgroup $H_0 < H^v$ of a vertex group $H^v < H$, we prove that the normalizer of $\widehat{H}_0$ in $\widehat{H}$ is contained in $\widehat{H}^v$. (We are aiming to apply this result to the subgroups $H_0 < H^v < H$ constructed in Proposition 3.4.) The proof is parallel to the proof of Proposition 3.4 (3).

Proposition 4.8. *Let M be a 3-manifold satisfying Conditions 3.1, and let $H < \pi_1(M)$ be a finitely generated subgroup. Let $H_0 < H^v$ be a finitely generated non-abelian subgroup of a vertex group $H^v < H$. Then the inclusion induced homomorphisms $\widehat{H}_0 \to \widehat{H}$ and $\widehat{H}^v \to \widehat{H}$ are both injective. By denoting the images of these two embeddings by $\widehat{H}_0$ and $\widehat{H}^v$ respectively, the normalizer of $\widehat{H}_0$ in $\widehat{H}$ satisfies $N_{\widehat{H}}(\widehat{H}_0) < \widehat{H}^v$.*

Proof. By Lemma 3.3 and the proof of Proposition 4.7, the inclusions $H_0 \to H$ and $H^v \to H$ satisfy the injectivity criterion in Section 2.1, so both $\widehat{H}_0 \to \widehat{H}$ and $\widehat{H}^v \to \widehat{H}$ are injective.

At first, the graph of group structure $(\mathcal{H}, \Gamma)$ on H gives rise to a graph of profinite groups $(\widehat{\mathcal{H}}, \Gamma)$ over the same finite graph Γ. Here each vertex (edge) group of $(\widehat{\mathcal{H}}, \Gamma)$ is the profinite completion of the corresponding vertex (edge) group of $(\mathcal{H}, \Gamma)$.

By Theorem 4.4 and Proposition 4.7, $\widehat{H}$ is isomorphic to the profinite fundamental group $\Pi_1(\widehat{\mathcal{H}}, \Gamma)$ of the graph of profinite group $(\widehat{\mathcal{H}}, \Gamma)$. So $\widehat{H}$ acts on the profinite Bass–

Serre tree $\mathcal{T}_{(\widehat{\mathcal{H}},\Gamma)}$, such that each vertex (edge) stabilizer is conjugate to a vertex (edge) subgroup of $(\mathcal{H},\Gamma)$. In particular, $\widehat{H}_0 < \widehat{H}^v$ stabilizes the vertex $\widehat{v} = \widehat{H}^v \in \mathcal{T}_{(\widehat{\mathcal{H}},\Gamma)}$.

For any $h \in N_{\widehat{H}}(\widehat{H}_0)$, we have $h^{-1}\widehat{H}_0 h = \widehat{H}_0$. By considering the action of $\widehat{H}$ on $\mathcal{T}_{(\widehat{\mathcal{H}},\Gamma)}$, $\widehat{H}_0$ also stabilizes the vertex $h(\widehat{v}) \in \mathcal{T}_{(\widehat{\mathcal{H}},\Gamma)}$.

We suppose that $\widehat{v}$ and $h(\widehat{v})$ are two different vertices of $\mathcal{T}_{(\widehat{\mathcal{H}},\Gamma)}$. Then they span a minimal subtree $[\widehat{v}, h(\widehat{v})]$ in $\mathcal{T}_{(\widehat{\mathcal{H}},\Gamma)}$, and $\widehat{H}_0$ stabilizes this subtree. Since the subtree $[\widehat{v}, h(\widehat{v})]$ is not trivial, it contains an edge $\hat{e} \in \mathcal{T}_{(\widehat{\mathcal{H}},\Gamma)}$. Then $\widehat{H}_0$ is contained in the stabilizer of $\hat{e}$. However, it is impossible, since any edge group $\widehat{H}^e$ is abelian (the profinite completion of a subgroup of $\mathbb{Z}^2$), but $\widehat{H}_0$ is not abelian.

We must have $\widehat{H}^v = \widehat{v} = h(\widehat{v}) = h\widehat{H}^v$, thus $h \in \widehat{H}^v$. So $N_{\widehat{H}}(\widehat{H}_0) < \widehat{H}^v$ holds. ∎

5. Proof of the Grothendieck rigidity

We will prove Theorem 1.2 in this section. To prove Theorem 1.2, we first prove the following proposition, which covers the crucial case for Theorem 1.2.

Proposition 5.1. *Let M be a 3-manifold satisfying Conditions 3.1 with $G = \pi_1(M)$. Then G is Grothendieck rigid.*

Proof. Let $H < G$ be a finitely generated proper subgroup of G. We need to prove that the inclusion induced homomorphism $\hat{i}\colon \widehat{H} \to \widehat{G}$ is not an isomorphism.

If H is separable in G, then by Lemma 2.2, the inclusion induced homomorphism $\widehat{H} \to \widehat{G}$ is not an isomorphism.

If H is not separable in G, then by Proposition 3.4, there exists a finite index (index 1 or 2) non-abelian subgroup $H_0 < H^v$ of a vertex group $H^v < H$, such that $N_H(H_0) = H^v$ and $[N_G(H_0) : H_0] = \infty$. Then by Proposition 4.8, we have $N_{\widehat{H}}(\widehat{H}_0) < \widehat{H}^v$, and

$$[N_{\widehat{H}}(\widehat{H}_0) : \widehat{H}_0] \le [\widehat{H}^v : \widehat{H}_0] = [H^v : H_0] < \infty.$$

Here the equality follows from the correspondence between finite index subgroups of a group and its profinite completion.

For the inclusion homomorphisms

$$H_0 \to H^v \to H \xrightarrow{i} G,$$

we have induced homomorphisms on profinite completions

$$\widehat{H}_0 \to \widehat{H}^v \to \widehat{H} \xrightarrow{\hat{i}} \widehat{G}.$$

By Proposition 4.8, the first two homomorphisms on profinite completions are injective, so we can consider $\widehat{H}_0$ and $\widehat{H}^v$ as subgroups of $\widehat{H}$.

By Proposition 3.4 (3), $H_0 < G$ is separable, so $\widehat{i}(\widehat{H}_0) \cap G = H_0$ holds. By Proposition 3.4 (5), we have $[N_G(H_0) : H_0] = \infty$. Since H_0 is dense in $\widehat{i}(\widehat{H}_0) < \widehat{G}$, $N_G(H_0) < N_{\widehat{G}}(\widehat{i}(\widehat{H}_0))$ holds. Therefore,

$$[N_{\widehat{G}}(\widehat{i}(\widehat{H}_0)) : \widehat{i}(\widehat{H}_0)] \geq [N_{\widehat{G}}(\widehat{i}(\widehat{H}_0)) \cap G : \widehat{i}(\widehat{H}_0) \cap G] \geq [N_G(H_0) : H_0] = \infty.$$

Suppose that $\widehat{i}\colon \widehat{H} \to \widehat{G}$ is an isomorphism. Then we have

$$\infty = [N_{\widehat{G}}(\widehat{i}(\widehat{H}_0)) : \widehat{i}(\widehat{H}_0)] = [N_{\widehat{i}(\widehat{H})}(\widehat{i}(\widehat{H}_0)) : \widehat{i}(\widehat{H}_0)] = [N_{\widehat{H}}(\widehat{H}_0) : \widehat{H}_0] < \infty.$$

It is impossible, so $\widehat{i}\colon \widehat{H} \to \widehat{G}$ is not an isomorphism. ∎

Proposition 5.1 covers the crucial case of Theorem 1.2, and we are ready to prove Theorem 1.2 now.

Proof of Theorem 1.2. Step I. We suppose that M is compact, orientable, irreducible and ∂-irreducible.

If M has trivial torus decomposition, then M is either a Seifert manifold or a (possibly infinite volume) hyperbolic 3-manifold. By [14] and [1], respectively, $\pi_1(M)$ is LERF, so $\pi_1(M)$ is Grothendieck rigid.

If M supports the Sol geometry, since $\pi_1(M)$ is LERF, $\pi_1(M)$ is Grothendieck rigid.

If M has nontrivial torus decomposition and does not support the Sol geometry, then M has a double cover $M' \to M$ that satisfies Conditions 3.1. To get such a double cover, for each piece of M homeomorphic to the twisted I-bundle over Klein bottle, we take its double cover homeomorphic to $T^2 \times I$; for any other piece, we take two copies of the same piece. Then we can paste all these pieces together to get a desired double cover $M' \to M$. By Proposition 5.1, $\pi_1(M')$ is Grothendieck rigid. Then Lemma 2.1 implies that $\pi_1(M)$ is Grothendieck rigid.

Step II. We suppose that M is compact and orientable.

We take the sphere-disc decomposition of M, then $\pi_1(M)$ is a free product of groups of compact, orientable, irreducible, ∂-irreducible 3-manifolds $G_1 = \pi_1(M_1), \ldots, G_n = \pi_1(M_n)$ and a free group F_r.

By the argument as in Step I, we can take a double cover $M' \to M$, if necessary, such that no piece of M_i (under the torus decomposition) is homeomorphic to the twisted I-bundle over Klein bottle. Then by Lemma 2.1, it suffices to prove that $\pi_1(M')$ is Grothendieck rigid. By abusing notation, we still use M to denote this double cover.

By the Kurosh subgroup theorem, for any finitely generated subgroup $H < \pi_1(M) \cong (*_{i=1}^n G_i) * F_r$, it has an induced free product structure $H = (*_{j=1}^m H_j) * F_s$. Here each H_j is a finitely generated nontrivial group, and it equals $H \cap g_j G_{i_j} g_j^{-1}$ for some $g_j \in G$ and $i_j \in \{1, \ldots, n\}$.

If H is separable in G, by Lemma 2.2, (G, H) is not a Grothendieck pair. So we suppose that H is not separable in G. By [4], we know that some H_j is not separable in $g_j G_{i_j} g_j^{-1}$. Up to conjugation and permuting indices, we can assume that $H_1 = H \cap G_1$ is not separable in $G_1 = \pi_1(M_1)$.

Since geometric 3-manifolds and infinite volume hyperbolic 3-manifolds (with finitely generated groups) have LERF groups, as in Step I, M_1 has nontrivial torus decomposition and does not support the Sol geometry, so M_1 satisfies Conditions 3.1. By applying Propositions 3.4 and 4.8 to $H_1 < G_1 = \pi_1(M_1)$, there is a finitely generated non-abelian subgroup $H_{1,0} < H_1$ such that the following hold:

(1) Any finitely generated subgroup of $H_{1,0}$ is separable in G_1.

(2) $[N_{\widehat{H}_1}(\widehat{H}_{1,0}) : \widehat{H}_{1,0}] < \infty$.

(3) $[N_{G_1}(H_{1,0}) : H_{1,0}] = \infty$.

Then we have the following commutative diagram:

$$\begin{tikzcd}
\widehat{H}_{1,0} \arrow[r, "\widehat{j}_1"] & \widehat{H}_1 \arrow[r, "\widehat{i}_1"] \arrow[d, "\widehat{k}_H"'] & \widehat{G}_1 \arrow[d, "\widehat{k}_G"'] \\
 & \widehat{H} \arrow[r, "\widehat{i}"] & \widehat{G}.
\end{tikzcd}$$

Here the horizontal homomorphisms are induced by inclusions $j_1 \colon H_{1,0} \to H_1$, $i_1 \colon H_1 \to G_1$ and $i \colon H \to G$, as subgroups. The vertical homomorphisms are induced by inclusions $k_H \colon H_1 \to H$ and $k_G \colon G_1 \to G$ as free factors.

Since H_1 and G_1 are free factors of H and G, respectively, $\widehat{k}_H$ and $\widehat{k}_G$ are both injective. By condition (1), all finitely generated subgroups of $H_{1,0}$ are separable in G_1, so they are all separable in H_1, and $\widehat{j}_1 \colon \widehat{H}_{1,0} \to \widehat{H}_1$ is injective. Then we have the following sequences of subgroups: $\widehat{H}_{1,0} < \widehat{H}_1 < \widehat{H}$ and $\widehat{G}_1 < \widehat{G}$. We will drop off the homomorphisms $\widehat{j}_1, \widehat{k}_H, \widehat{k}_G$, and $\widehat{i}_1$ is simply the restriction of $\widehat{i}$ on $\widehat{H}_1 < \widehat{H}$.

Since $H_{1,0}$ is separable in G_1, then $\widehat{i}(\widehat{H}_{1,0}) \cap G_1 = H_{1,0}$. Since $\widehat{G}_1$ is a profinite free factor of $\widehat{G}$, by Lemma 4.6, we have $N_{\widehat{G}}(\widehat{i}(\widehat{H}_{1,0})) = N_{\widehat{G}_1}(\widehat{i}(\widehat{H}_{1,0}))$. Hence,

$$\begin{aligned}
[N_{\widehat{G}}(\widehat{i}(\widehat{H}_{1,0})) : \widehat{i}(\widehat{H}_{1,0})] &= [N_{\widehat{G}_1}(\widehat{i}(\widehat{H}_{1,0})) : \widehat{i}(\widehat{H}_{1,0})] \\
&\geq [N_{\widehat{G}_1}(\widehat{i}(\widehat{H}_{1,0})) \cap G_1 : \widehat{i}(\widehat{H}_{1,0}) \cap G_1] \\
&\geq [N_{G_1}(H_{1,0}) : H_{1,0}] = \infty.
\end{aligned}$$

Here the last equality follows from condition (3) above.

On the other hand, since H_1 is a free factor of H, by applying Lemma 4.6 again, we have $N_{\widehat{H}}(\widehat{H}_{1,0}) = N_{\widehat{H}_1}(\widehat{H}_{1,0})$. Therefore,

$$[N_{\widehat{H}}(\widehat{H}_{1,0}) : \widehat{H}_{1,0}] = [N_{\widehat{H}_1}(\widehat{H}_{1,0}) : \widehat{H}_{1,0}] < \infty.$$

Here the last inequality follows from condition (2) above.

Suppose that $\widehat{i} \colon \widehat{H} \to \widehat{G}$ is an isomorphism. Then we have

$$\begin{aligned}
\infty &= [N_{\widehat{G}}(\widehat{i}(\widehat{H}_{1,0})) : \widehat{i}(\widehat{H}_{1,0})] = [N_{\widehat{i}(\widehat{H})}(\widehat{i}(\widehat{H}_{1,0})) : \widehat{i}(\widehat{H}_{1,0})] \\
&= [N_{\widehat{H}}(\widehat{H}_{1,0}) : \widehat{H}_{1,0}] < \infty.
\end{aligned}$$

We obtained a contradiction, so $\widehat{\imath}\colon \widehat{H} \to \widehat{G}$ is not an isomorphism.

Step III. General case. If M is compact and orientable, then the Grothendieck rigidity follows from Step II.

If M is orientable but not compact, then we take a Scott core $C \subset M$ [13]. Here C is a compact connected codimension-0 submanifold of M such that the inclusion map induces an isomorphism on fundamental groups. Then Step II implies that $\pi_1(C)$ is Grothendieck rigid, and so is $\pi_1(M)$.

If M is nonorientable (either compact or noncompact), then we take the orientable double cover $M' \to M$. By the previous case, $\pi_1(M')$ is Grothendieck rigid. Then Lemma 2.1 implies that $\pi_1(M)$ is Grothendieck rigid.

This completes the proof of Theorem 1.2. ■

Funding. The author is partially supported by Simons Foundation Collaboration Grant 615229.

References

[1] I. Agol, The virtual Haken conjecture (with an appendix by Agol, Daniel Groves, and Jason Manning). *Doc. Math.* **18** (2013), 1045–1087 Zbl 1286.57019 MR 3104553

[2] M. Boileau and S. Friedl, Grothendieck rigidity of 3-manifold groups. *Groups Geom. Dyn.* **13** (2019), no. 4, 1133–1150 Zbl 1445.57014 MR 4033499

[3] M. R. Bridson and F. J. Grunewald, Grothendieck's problems concerning profinite completions and representations of groups. *Ann. of Math. (2)* **160** (2004), no. 1, 359–373 Zbl 1083.20023 MR 2119723

[4] R. G. Burns, On finitely generated subgroups of free products. *J. Aust. Math. Soc.* **12** (1971), 358–364 Zbl 0218.20026 MR 0311778

[5] A. Grothendieck, Représentations linéaires et compactification profinie des groupes discrets. *Manuscripta Math.* **2** (1970), 375–396 Zbl 0239.20065 MR 262386

[6] A. Hatcher, Notes on basic 3-manifold topology. Available at https://pi.math.cornell.edu/~hatcher/3M/3M.pdf

[7] J. Hempel, Residual finiteness for 3-manifolds. In *Combinatorial group theory and topology (Alta, Utah, 1984)*, pp. 379–396, Ann. of Math. Stud. 111, Princeton University Press, Princeton, NJ, 1987 Zbl 0772.57002 MR 895623

[8] D. D. Long and A. W. Reid, Grothendieck's problem for 3-manifold groups. *Groups Geom. Dyn.* **5** (2011), no. 2, 479–499 Zbl 1250.57027 MR 2782181

[9] N. Nikolov and D. Segal, On finitely generated profinite groups. I. Strong completeness and uniform bounds. *Ann. of Math. (2)* **165** (2007), no. 1, 171–238 Zbl 1126.20018 MR 2276769

[10] V. P. Platonov and O. I. Tavgen', Grothendieck's problem on profinite completions and representations of groups. *K-Theory* **4** (1990), no. 1, 89–101 Zbl 0722.20020 MR 1076526

[11] L. Ribes, *Profinite graphs and groups*. Ergeb. Math. Grenzgeb. (3) 66, Springer, Cham, 2017 Zbl 1457.20001 MR 3677322

[12] L. Ribes and P. Zalesskii, *Profinite groups*. 2nd edn., Ergeb. Math. Grenzgeb. (3) 40, Springer, Berlin, 2010 Zbl 1197.20022 MR 2599132

[13] G. P. Scott, Compact submanifolds of 3-manifolds. *J. Lond. Math. Soc. (2)* **7** (1973), 246–250 Zbl 0266.57001 MR 326737

[14] P. Scott, Subgroups of surface groups are almost geometric. *J. Lond. Math. Soc. (2)* **17** (1978), no. 3, 555–565 Zbl 0412.57006 MR 494062

[15] H. Sun, A characterization on separable subgroups of 3-manifold groups. *J. Topol.* **13** (2020), no. 1, 187–236 Zbl 1448.57024 MR 4138737

[16] G. Wilkes, Profinite rigidity of graph manifolds and JSJ decompositions of 3-manifolds. *J. Algebra* **502** (2018), 538–587 Zbl 1404.57039 MR 3774902

Received 5 March 2021.

Hongbin Sun

Department of Mathematics, Rutgers University, New Brunswick, Hill Center, Busch Campus, Piscataway, NJ 08854, USA; hongbin.sun@rutgers.edu