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# Formal conjugacy growth in graph products I

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**Abstract.** In this paper we give a recursive formula for the conjugacy growth series of a graph product in terms of the conjugacy growth and standard growth series of subgraph products. We also show that the conjugacy and standard growth rates in a graph product are equal provided that this property holds for each vertex group. All results are obtained for the standard generating set consisting of the union of generating sets of the vertex groups.

## 1. Introduction

In this paper we obtain several results on conjugacy growth and languages in graph products with respect to their standard generating set: foremost, we find a formula for the conjugacy growth series for a graph product of groups as a function of the standard and conjugacy growth series of subgraph products, and in parallel we establish the equality of the standard and conjugacy growth rates if the same holds in each vertex group. En route to proving these results, we also study the shortlex conjugacy language for graph products.

The graph product construction generalizes both direct and free products. Given a finite simplicial graph with vertex set $V$ and for each vertex $v \in V$ an associated group $G_v$, the associated *graph product* $G_V$ is the group generated by the vertex groups with the added relations that elements of groups attached to adjacent vertices commute. Right-angled Artin groups (RAAGs) and Coxeter groups (RACGs) arise in this way as the graph products of infinite cyclic groups and cyclic groups of order two, respectively, and have been widely studied. Graph products were introduced by Green in her PhD thesis [12] and their (standard) growth series, based on the growth series of the vertex groups, were subsequently computed by Chiswell [4]. In particular, Chiswell showed that rationality of the standard growth series is preserved by the graph product construction.

The first conjugacy growth series computations appeared in the work of Rivin [18, 19] on free groups, and it is striking that, even for free groups with standard generating sets, the series are transcendental, and their formulas rather complicated. More generally and systematically, conjugacy growth series and languages featured in [2, 3, 5–7, 9, 17], where virtually abelian groups, Baumslag–Solitar groups, acylindrically hyperbolic groups, free and wreath products, and more, were explored. Except for virtually abelian groups, all

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the conjugacy growth series studied so far were shown to be transcendental. In contrast to Rivin's result for the standard conjugacy growth series for free groups, we note that the conjugacy geodesic growth series, which counts all the geodesics shortest with respect to conjugacy, is rational in free groups with respect to standard generating sets. Moreover, in the case of graph products, regularity of the pair consisting of the set of conjugacy geodesics and the set of geodesics is preserved by the graph product construction [6].

All groups in this paper are finitely generated, and all generating sets finite and inverse-closed. The *spherical*, or *standard, growth function* of a group $G$ with respect to a generating set $X$ records the size of the sphere of radius $n$ in the Cayley graph of $G$ with respect to $X$ for each $n \geq 0$, and the *spherical conjugacy growth function* counts the number of conjugacy classes intersecting the sphere of radius $n$ but not the ball of radius $n-1$. Taking the growth rate of the values given by the above functions produces the *spherical growth rate* and *spherical conjugacy growth rate* of $G$ with respect to $X$. Furthermore, the spherical standard growth and conjugacy growth series are those generating functions whose coefficients are the spherical growth function and spherical conjugacy growth function values, respectively. The exact meaning of the terminology used below and all necessary notation is given in Section 2.1.

The first main result of the paper gives a recursive formula for the spherical conjugacy growth series of a graph product based on the spherical growth and conjugacy growth series of the vertex groups.

**Theorem A.** *Let $G_V$ be a graph product group over a graph with vertex set $V$ and let $v \in V$ be a vertex. For each $v' \in V$ let $X_{v'}$ be an inverse-closed generating set for the vertex group $G_{v'}$. For each subset $S \subseteq V$ let $X_S = \bigcup_{v' \in S} X_{v'}$ be the generating set for the subgraph product $G_S$ on the subgraph induced by $S$. Let $\widetilde{\sigma}_S$ be the spherical conjugacy growth series and let $\sigma_S$ be the spherical growth series of $G_S$ with respect to $X_S$.*

*Then the conjugacy growth series of $G_V$ is given by*

$$\widetilde{\sigma}_V = \widetilde{\sigma}_{V \setminus \{v\}} + \widetilde{\sigma}_{\mathsf{Lk}(v)}(\widetilde{\sigma}_{\{v\}} - 1) + \sum_{S \subseteq \mathsf{Lk}(v)} \widetilde{\sigma}_S^{\mathcal{M}} \mathsf{N}\Big(\Big(\frac{\sigma_{\mathsf{Lk}(S) \setminus \{v\}}}{\sigma_{\mathsf{Lk}(v) \cap \mathsf{Lk}(S)}} - 1\Big)(\sigma_{\{v\}} - 1)\Big),$$

*where $\mathsf{Lk}(v)$ is the set of vertices adjacent to $v$, $\widetilde{\sigma}_S^{\mathcal{M}} = \sum_{S' \subseteq S} (-1)^{|S|-|S'|} \widetilde{\sigma}_{S'}$, and for any complex power series $f(z)$,*

$$\mathsf{N}(f)(z) := \sum_{k=1}^{\infty} \sum_{l=1}^{\infty} \frac{\phi(k)}{kl} (f(z^k))^l$$

*in which $\phi$ is the Euler totient function.*

*Moreover, if $\{v\} \cup \mathsf{Lk}(v) = V$, then $\widetilde{\sigma}_V = \widetilde{\sigma}_{\mathsf{Lk}(v)} \widetilde{\sigma}_{\{v\}}$.*

The proof of Theorem A employs the use of Möbius inversion formulas applied to languages of conjugacy representatives that arise from the amalgamated free product decomposition of a graph product. The second main result of the paper follows from many of the same techniques and shows that equality of the spherical growth and conjugacy growth rates is preserved by the graph product construction.

**Theorem B.** *Let $G_V$ be a graph product group over a graph with vertex set $V$ and assume that for each vertex $v \in V$ the spherical growth rate and spherical conjugacy growth rate of $G_v$, over a generating set $X_v$, are equal. Let $X_V = \bigcup_{v \in V} X_v$. Then the spherical growth rate and spherical conjugacy growth rate of $G_V$ with respect to $X_V$ are equal. Hence also the radii of convergence of the spherical and spherical conjugacy growth series of $G_V$ over $X_V$ are equal.*

It is interesting that many infinite discrete groups display the same behaviour as that in Theorem B, that is, the standard and conjugacy growth rates are equal. This is the case for hyperbolic [2] and relatively hyperbolic [11] groups, the wreath products (including lamplighter groups) in [17], and soluble Baumslag–Solitar groups $\mathrm{BS}(1, k)$ [5]. It is an intriguing question whether the equality of growth rates holds for larger classes of groups (such as acylindrically hyperbolic), or if there exists a common thread in the proofs of this equality for the different classes of groups mentioned above.

The proofs of the two main theorems revolve around methods that come from analytic combinatorics, such as the 'necklace' series associated to a language. Since these tools are not standard in group theory, we begin in Section 2 with a discussion of these tools. In Section 3 we provide background information as well as new results on languages associated to graph products of groups, including conjugacy and cyclic geodesics, and shortlex normal forms for conjugacy classes, that are used in the rest of the paper.

In Section 4.1 we establish in Proposition 4.3 a set of conjugacy geodesics (minimal length representatives, over the generators, for conjugacy classes) for a graph product group that contains at least one representative for each conjugacy class, and we determine when two elements of this set represent conjugate elements. The remainder of Section 4 contains the proofs of Theorems A and B, as well as Example 4.10, where the conjugacy growth series of a right-angled Coxeter group is computed using the formulas in the paper.

Further types of formulas for the spherical conjugacy growth series of graph products and an analysis of their algebraic complexity will be the subject of a subsequent paper.

## 2. Preliminaries and necklace languages

### 2.1. Notation and terminology

We use standard notation from formal language theory: where $X$ is a finite set, we denote by $X^*$ the set of all words over $X$, and call a subset of $X^*$ a language. We write $\lambda$ for the empty word, and denote by $X^+$ the set of all nonempty words over $X$ (so $X^* = X^+ \cup \{\lambda\}$). For each letter $a \in X$, we write $a^*$ and $a^+$ to denote the sets $\{a\}^*$ and $\{a\}^+$, respectively. For each word $w \in X^*$, let $l(w) = l_X(w) = |w|$ denote its length over $X$.

For a group $G$ with inverse-closed generating set $X$, let $\pi\colon X^* \to G$ be the natural projection onto $G$, and let $=$ denote equality between words and $=_G$ equality between group elements (so $w =_G v$ means $\pi(w) = \pi(v)$). For $g \in G$, the *length* of $g$, denoted $\|g\|$ $(= \|g\|_X)$, is the length of a shortest representative word for $g$ over $X$. A *geodesic* is a word $w \in X^*$ with $l(w) = \|\pi(w)\|$; we denote the set of all geodesics for $G$ with respect to $X$ by $\mathsf{Geo}(G, X)$.

Let $\sim$, or $\sim_G$, denote the equivalence relation on $G$ given by conjugacy, and $G/\!\sim$ its set of equivalence classes. Let $[g]_\sim$ denote the conjugacy class of $g \in G$ and $\|g\|_\sim$ denote its *length up to conjugacy*, that is,

$$\|g\|_\sim := \min\{\|h\| \mid h \in [g]_\sim\}.$$

We say that $g$ has *minimal length up to conjugacy* if $\|g\| = \|g\|_\sim$. A *conjugacy geodesic* is a word $w \in X^*$ with $l(w) = \|\pi(w)\|_\sim$; we denote the set of all conjugacy geodesics by $\mathsf{ConjGeo}(G, X)$.

Fix a total ordering of $X$, and let $\leq_{\mathrm{sl}}$ be the induced shortlex ordering of $X^*$ (for which $u <_{\mathrm{sl}} w$ if either $l(u) < l(w)$ or $l(u) = l(w)$ but $u$ precedes $w$ lexicographically). For each $g \in G$, the *shortlex normal form* of $g$ is the unique word $y_g \in X^*$ with $\pi(y_g) = g$ such that $y_g \leq_{\mathrm{sl}} w$ for all $w \in X^*$ with $\pi(w) = g$. For each conjugacy class $c \in G/\!\sim$, the *shortlex conjugacy normal form* of $c$ is the shortlex least word $z_c$ over $X$ representing an element of $c$; that is, $\pi(z_c) \in c$, and $z_c \leq_{\mathrm{sl}} w$ for all $w \in X^*$ with $\pi(w) \in c$. The *shortlex language* and *shortlex conjugacy language* for $G$ over $X$ are defined as

$$\mathsf{SL} = \mathsf{SL}(G, X) := \{y_g \mid g \in G\},$$
$$\mathsf{ConjSL} = \mathsf{ConjSL}(G, X) := \{z_c \mid c \in G/\!\sim\}.$$

Any language $L$ over $X$ gives rise to a *strict growth function* $\theta_L\colon \mathbb{N} \cup \{0\} \to \mathbb{N} \cup \{0\}$, defined by $\theta_L(n) := |\{w \in L \mid l(w) = n\}|$; an associated generating function, called the *strict growth series*, given by $F_L(z) := \sum_{n=0}^{\infty} \theta_L(n) z^n$; and an *exponential growth rate* $\mathsf{gr}_L = \lim_{n\to\infty} (\theta_L(n))^{1/n}$.

For the two languages above, the coefficient $\theta_{\mathsf{SL}}(n)$ is the number of elements of $G$ of length $n$, and $\theta_{\mathsf{ConjSL}}(n)$ is the number of conjugacy classes of $G$ whose shortest elements have length $n$. As in [6], we refer to the strict growth series of $\mathsf{SL}$ below as the *standard* or *spherical growth series*

$$\sigma(z) = \sigma_{(G,X)}(z) := F_{\mathsf{SL}(G,X)}(z) = \sum_{n=0}^{\infty} \theta_{\mathsf{SL}(G,X)}(n) z^n$$

and the strict growth series

$$\widetilde{\sigma}(z) = \widetilde{\sigma}_{(G,X)}(z) := F_{\mathsf{ConjSL}(G,X)}(z) = \sum_{n=0}^{\infty} \theta_{\mathsf{ConjSL}(G,X)}(n) z^n$$

of $\mathsf{ConjSL}$ as the *spherical conjugacy growth series*.

**Remark 2.1.** Note that the growth series in the paper will be often denoted as $\sigma$ and $\widetilde{\sigma}$ instead of $\sigma(z)$ or $\widetilde{\sigma}(z)$ due to the length of some of the formulas.

For the group $G$ and generating set $X$, the exponential growth rates of these two series give the *standard* or *spherical growth rate* of a group $G$ over $X$, namely

$$\rho = \rho(G, X) := \mathsf{gr}_{\mathsf{SL}(G,X)} = \lim_{n\to\infty} (\theta_{\mathsf{SL}(G,X)}(n))^{1/n},$$

and the *spherical conjugacy growth rate* given by

$$\widetilde{\rho} = \widetilde{\rho}(G, X) := \mathsf{gr}_{\mathsf{ConjSL}(G,X)} = \limsup_{n\to\infty} (\theta_{\mathsf{ConjSL}(G,X)}(n))^{1/n}.$$

## 2.2. Complex power series

In this section we recall some basic facts about power series in complex analysis (see for example [8, Chapter III, Sections 1 and 2]).

We denote the open disk of radius $r > 0$ centered at $c \in \mathbb{C}$ by $B(c, r) := \{z \in \mathbb{C} \colon |z - c| < r\}$. A *complex power series* is a function $f \colon B(0, r) \to \mathbb{C}$ of the form $f(z) = \sum_{n=0}^{\infty} a_n z^n$, where $a_n \in \mathbb{C}$ for all $n$. We express the fact that $a_n$ is the *coefficient* of $z^n$ in $f$ by writing

$$[z^n] f(z) := a_n.$$

The radius of convergence $\mathsf{RC}(f)$ of $f$ can be defined as

$$\mathsf{RC}(f) = \sup\{r \in \mathbb{R} \colon f(z) \text{ converges for all } z \in B(0, r)\},$$

or equivalently as

$$\mathsf{RC}(f) = \frac{1}{\limsup_{n\to\infty} \sqrt[n]{|a_n|}}. \tag{2.1}$$

If $\limsup_{n\to\infty} \sqrt[n]{|a_n|} = 0$ then $\mathsf{RC}(f) = +\infty$. If $\mathsf{RC}(f) > 0$, then $f$ is defined and converges absolutely at every point in the open disk $B(0, \mathsf{RC}(f))$.

**Proposition 2.2.** *Let $f \neq 0$ be a complex power series such that $\mathsf{RC}(f) > 0$, $[z^n] f(z) \geq 0$ for all $n \in \mathbb{N}$, and $[z^0] f(z) = 0$. Then there exists a unique positive number $t > 0$ such that $f(t) = 1$; moreover,*

$$t = \inf\{|z| \colon z \in \mathbb{C},\ |f(z)| = 1\} = \sup\{r > 0 \colon |f(z)| \leq 1 \text{ for all } z \in B(0, r)\},$$

*and the infimum and supremum are attained.*

*Proof.* Write $f = \sum_{n=1}^{\infty} a_n z^n$, where $a_n \geq 0$ for all $n$ and $a_m \neq 0$ for at least one index $m$. For any complex number $z$ we have

$$|f(z)| = \left| \sum_{n=1}^{\infty} a_n z^n \right| \leq \sum_{n=1}^{\infty} a_n |z|^n = f(|z|),$$

hence if the series $f(z)$ diverges, then so does the series $f(|z|)$. Moreover, $f(|z|) \geq a_m |z|^m$ is unbounded as $|z|$ increases. Thus on the real interval $[0, \mathsf{RC}(f))$ the function $f$ is continuous, strictly increasing, and unbounded, and so there exists a unique $t \in [0, \mathsf{RC}(f))$ such that $f(t) = 1$. Now for any complex number $z$ satisfying $|z| \leq t$ the following holds:

$$|f(z)| \leq \sum_{n=1}^{\infty} a_n |z|^n \leq \sum_{n=1}^{\infty} a_n t^n = f(t) = 1. \qquad \blacksquare$$

## 2.3. Necklace set associated to a language

Let $X$ be a finite alphabet and $L$ be a language over $X$. Let $\mathbb{N}$ denote the positive integers, and $\mathbb{N}_0$ denote the nonnegative integers. For $n \in \mathbb{N}$, let $L^n$ denote the Cartesian product of $n$ copies of $L$. For $(l_1, \ldots, l_n) \in L^n$, the elements $l_j$ with $j \in \{1, \ldots, n\}$ are called the *components*, and the *length* of this $n$-tuple is defined to be $|(l_1, \ldots, l_n)| := \sum_{j=1}^{n} |l_j|$.

Let $C_n := \mathbb{Z}/n\mathbb{Z}$. The group $C_n$ acts on $L^n$ by cyclically permuting the entries of tuples in $L^n$, that is, $g \cdot (u_1, \ldots, u_n) := (u_{1+g}, \ldots, u_{n+g})$ for all $g \in C_n$ and $u_1, \ldots, u_n \in L$, where the index $i + g$ of $u_{i+g}$ is taken modulo $n$. Let $L^n/C_n$ denote the quotient by this action, and define the set of *necklaces over* $L$ as

$$\mathsf{Necklaces}(L) := \bigsqcup_{n=1}^{\infty} \left(L^n/C_n\right).$$

Since the length of an element in $L^n$ is preserved by cyclic permutation of its components, we extend the definition of length on $L^n$ to $\mathsf{Necklaces}(L)$.

In analogy with the growth of languages over an alphabet, any set $S$ together with a length function $|\cdot|: S \to \mathbb{N}_0$, satisfying the property that for each nonnegative integer the number of elements of that length is finite, has a strict growth function $\theta_S: \mathbb{N}_0 \to \mathbb{N}_0$, defined by $\theta_S(n) := |\{s \in S \mid |s| = n\}|$, and a strict growth series given by $F_S(z) := \sum_{n=0}^{\infty} \theta_S(n) z^n$.

Next we collect some identities among several strict growth series. Given $u \in L$, let $\mathsf{diag}(u)$ denote the diagonal element $\mathsf{diag}_n(u) := (u, u, \ldots, u)$ in $L^n$. Similarly, for $v = (v_1, \ldots, v_d) \in L^d$ and $m \in \mathbb{N}$, let $\mathsf{diag}_m(v)$ denote the element

$$\mathsf{diag}_m(v) := (v_1, \ldots, v_d, \ldots, v_1, \ldots, v_d)$$

of $L^{md}$. Note that whenever $n \neq n'$, the sets $L^n/C_n$ and $L^{n'}/C_{n'}$ are disjoint.

**Lemma 2.3.** *Let $L$ be a language and let $n \in \mathbb{N}$. Then the following hold:*

(1) $F_{\mathsf{Necklaces}(L)}(z) = \sum_{n=1}^{\infty} F_{L^n/C_n}(z)$.

(2) $F_{L^n}(z) = (F_L(z))^n$.

(3) $F_{\{\mathsf{diag}_m(u):u \in L^d\}}(z) = F_{L^d}(z^m)$.

(4) $[z^m](F_L(z))^d = [z^{mn}](F_L(z^n))^d$.

The following gives a computation of the strict growth series $F_{\mathsf{Necklaces}(L)}(z)$ from $F_L(z)$.

**Proposition 2.4.** *The growth series of the set of necklaces over a language $L$ is*

$$F_{\mathsf{Necklaces}(L)}(z) = \sum_{k=1}^{\infty}\sum_{l=1}^{\infty}\frac{\phi(k)}{kl}\big(F_L(z^k)\big)^l,$$

*where $\phi$ is the Euler totient function.*

*Proof.* For every $n, m \in \mathbb{N}$, the set $S^n(m) := \{w \in L^n : |w| = m\}$ is invariant under the cyclic permutation action of $C_n$ on $L^n$. Then the coefficient $[z^m]F_{L^n/C_n}(z)$ is the number of orbits in $S^n(m)$ under the action of $C_n$. For each $g \in C_n$, let $\mathsf{Fix}_{S^n(m)}(g)$ denote the set of elements of $S^n(m)$ that are fixed by the action of $g$. Using Burnside's lemma, we find

$$[z^m]F_{L^n/C_n}(z) = \frac{1}{n}\sum_{g\in C_n}|\mathsf{Fix}_{S^n(m)}(g)| = \frac{1}{n}\sum_{d|n}\sum_{\substack{1\le g\le n\\(g,n)=d}}|\mathsf{Fix}_{S^n(m)}(g)|.$$

In fact, whenever $d|n$, $1 \le g \le n$, $(g,n) = d$, and $w \in L^n$, then $w \in \mathsf{Fix}_{S^n(m)}(g)$ if and only if $w = \mathsf{diag}_{\frac{n}{d}}(v)$ for some $v \in L^d$ with $|v| = \frac{md}{n}$. In the case that $(g,n) = d$, we have

$$|\mathsf{Fix}_{S^n(m)}(g)| = [z^{\frac{md}{n}}]\big(F_{L^d}(z)\big) = [z^{\frac{md}{n}}]\big(F_L(z)\big)^d = [z^m]\big(F_L(z^{\frac{n}{d}})\big)^d,$$

where the second and third equalities apply Lemma 2.3 (2) and (4), respectively. Therefore we find

$$F_{L^n/C_n}(z) = \frac{1}{n}\sum_{d|n}|\{1 \le g \le n,\ (g,n) = d\}|\big(F_L(z^{\frac{n}{d}})\big)^d = \frac{1}{n}\sum_{d|n}\phi\Big(\frac{n}{d}\Big)\big(F_L(z^{\frac{n}{d}})\big)^d.$$

Finally, using Lemma 2.3 (1),

$$F_{\mathsf{Necklaces}(L)}(z) = \sum_{n=1}^{\infty}\frac{1}{n}\sum_{d|n}\phi\Big(\frac{n}{d}\Big)\big(F_L(z^{\frac{n}{d}})\big)^d = \sum_{k=1}^{\infty}\sum_{l=1}^{\infty}\frac{\phi(k)}{kl}\big(F_L(z^k)\big)^l. \qquad \blacksquare$$

Note that if the language $L$ contains the empty word, then the set $\mathsf{Necklaces}(L)$ contains infinitely many elements of length 0 and so the strict growth series $F_{\mathsf{Necklaces}(L)}(z)$ is nowhere defined. Thus for the remainder of the paper, every language $L$ for which we consider the series $F_{\mathsf{Necklaces}(L)}(z)$ is assumed not to contain the empty word, so that $F_L(0) = 0$.

**Remark 2.5.** For every $n \in \mathbb{N}$ and $m \in \mathbb{N}_0$, the strict growth functions for $L^n$ and $L^n/C_n$ satisfy $\frac{1}{n}\theta_{L^n}(m) \le \theta_{L^n/C_n}(m) \le \theta_{L^n}(m)$. Then Lemma 2.3 (1) and (2) yield

$$[z^m]\sum_{n=1}^{\infty}\frac{(F_L(z))^n}{n} \le [z^m]F_{\mathsf{Necklaces}(L)}(z) \le [z^m]\underbrace{\sum_{n=1}^{\infty}\big(F_L(z)\big)^n}_{\frac{F_L(z)}{1-F_L(z)}}.$$

**Corollary 2.6.** *Let $L$ be a nonempty language that does not contain the empty word. The radius of convergence of $F_{\mathsf{Necklaces}(L)}(z)$ is given by*

$$\mathsf{RC}(F_{\mathsf{Necklaces}(L)}(z)) = \inf\{|z| : z \in \mathbb{C},\ |F_L(z)| = 1\},$$

*which is the positive real number $t$ such that $F_L(t) = 1$.*

*Proof.* Remark 2.5 implies that

$$\mathsf{RC}\left(\sum_{n=1}^{\infty} \frac{(F_L(z))^n}{n}\right) \geq \mathsf{RC}(F_{\mathsf{Necklaces}(L)}(z)) \geq \mathsf{RC}\left(\sum_{n=1}^{\infty} (F_L(z))^n\right).$$

The convergence radius of the geometric series $\sum_{n>0} z^n$ is 1, and hence the series $\sum_{n=1}^{\infty} (F_L(z))^n$ converges for all $z$ satisfying $|F_L(z)| < 1$ and diverges for all $z$ such that $|F_L(z)| > 1$. Since the language $L$ is a subset of $X^*$ for a finite set $X$, we have $\theta_L(m) \leq |X|^m$ for all $m$, and so the radius of convergence of the strict growth series $F_L$ is at least $\frac{1}{|X|}$. By Proposition 2.2,

$$\begin{aligned}\mathsf{RC}\left(\sum_{n=1}^{\infty} (F_L(z))^n\right) &= \sup\{r > 0 : |F_L(z)| \leq 1 \text{ for all } z \in B(0,r)\} \\ &= \min\{|z| : z \in \mathbb{C},\ |F_L(z)| = 1\}, \end{aligned} \tag{2.2}$$

and moreover, $\mathsf{RC}(\sum_{n=1}^{\infty}(F_L(z))^n) = t$, where $t$ is equal to the unique positive real number such that $F_L(t) = 1$. But then $\sum_{n=1}^{\infty} \frac{F_L(t))^n}{n} = \sum_{n=1}^{\infty} \frac{1}{n}$ diverges, implying that $\mathsf{RC}(\sum_{n=1}^{\infty} \frac{(F_L(z))^n}{n}) \leq t$.

This implies that $\mathsf{RC}(F_{\mathsf{Necklaces}(L)}(z)) = t$, as required. ∎

**Example 2.7.** Let $L = \{c_1, \ldots, c_p\}$ be a finite subset of $X$; that is, $|c_i| = 1$ for all $i$. Then $F_L(z) = pz$ and the set $L$ can be viewed as a set of colors. In this case, Proposition 2.4 says that

$$F_{\mathsf{Necklaces}(L)}(z) = \sum_{k=1}^{\infty} \sum_{l=1}^{\infty} \frac{\phi(k)}{kl} p^l z^{kl}.$$

The coefficient of $z^m$ in this series is the number of necklaces that we can make with $m$ pearls, all with a color in $L$.

Proposition 2.4 leads us to the following definition.

**Definition 2.8.** For any complex power series $f$ with integer coefficients satisfying the condition $[z^0]f(z) = 0$, let

$$\mathsf{N}(f)(z) := \sum_{k=1}^{\infty} \sum_{l=1}^{\infty} \frac{\phi(k)}{kl} (f(z^k))^l = \sum_{k=1}^{\infty} \frac{-\phi(k)}{k} \log(1 - f(z^k)).$$

We note that if $f = F_L$ is the growth series of a nonempty language $L$ that does not contain the empty word, then by Proposition 2.4, $N(f) = N(F_L) = F_{\mathsf{Necklaces}(L)}$, and by Corollary 2.6, the radius of convergence $\mathsf{RC}(\mathsf{N}(f))$ is the unique positive number $t$ such that $f(t) = 1$.

**Example 2.9.** If $f(z) = z^K$ with $K > 0$, then $\mathsf{N}(f)(z) = \frac{z^K}{1-z^K}$ (see [20, Lemma 1 (1)]).

## 3. Graph products: background and languages

Let $\Gamma = (V, E)$ be a finite simple graph with vertex set $V$ and edge set $E$; that is, an undirected graph without loops or multiple edges.

For any nonempty subset $V' \subseteq V$, the *link* or *centralizing set* $\mathsf{Lk}(V')$ of $V'$ denotes the set of all vertices of $\Gamma$ that are adjacent to all of the vertices in $V'$. That is, for any vertex $v \in V$ the set

$$\mathsf{Lk}(v) := \{w \in V : \{v, w\} \in E\}$$

is the set of neighbours of $v$, and for any nonempty subset $V' \subseteq V$, we have

$$\mathsf{Lk}(V') := \bigcap_{v \in V'} \mathsf{Lk}(v).$$

We also set $\mathsf{Lk}(\emptyset) := V$.

For each vertex $v$ of $\Gamma$, let $G_v$ be a nontrivial group. The *graph product of the groups $G_v$ with respect to* $\Gamma$ is the quotient of their free product by the normal closure of the set of relators $[g_v, g_w]$ for all $g_v \in G_v$, $g_w \in G_w$ for which $\{v, w\}$ is an edge of $\Gamma$.

Given a graph product group $G$ over a graph $\Gamma = (V, E)$ and any subset $V' \subseteq V$, the *subgraph product* associated to $V'$ is the subgroup $G_{V'} := \langle G_v \mid v \in V' \rangle$ of $G$. By [12, Proposition 3.31], $G_{V'}$ is isomorphic to the graph product of the $G_v$ $(v \in V')$ on the induced subgraph of $\Gamma$ with vertex set $V'$. Note that $G_V = G$ and $G_\emptyset$ is the trivial group.

Suppose that each vertex group $G_v$ of the graph product has an inverse-closed generating set $X_v$ with $\varepsilon \notin X_v$. For each $V' \subseteq V$, let

$$X_{V'} := \bigsqcup_{v \in V'} X_v;$$

then $X_{V'}$ is an inverse-closed generating set for $G_{V'}$. A *syllable* of a word $w \in X_V^*$ is a subword $u$ of $w$ satisfying the properties that $u \in X_v^+$ for some $v \in V$ and $u$ is not contained in a strictly longer subword of $w$ that also lies in $X_v^*$.

For each $v \in V$ let $Y_v$ denote the particular generating set $Y_v := G_v \setminus \{\varepsilon\}$ for $G_v$. We denote the associated generating set for $G_{V'}$ by

$$Y_{V'} := \bigcup_{v \in V'} G_v \setminus \{\varepsilon\}.$$

Define a function $\zeta \colon X_V^* \to Y_V^*$ by setting $\zeta(w)$ to be the word obtained from $w \in X_V^*$ by replacing each syllable $u \in X_v^+$ of $w$ by the element of $Y_v$ represented by $u$.

**Definition 3.1.** For an element $g \in G_V$, the *support* of $g$ is the set

$$\mathsf{Supp}(g) := \bigcap_{V' \subseteq V \text{ and } g \in G_{V'}} V'.$$

For a word $w \in X_V^*$, the *support* $\mathsf{Supp}(w)$ of $w$ is the set of all vertices $v$ for which a letter of $X_v$ appears in $w$.

### 3.1. Geodesic languages and word operations

Over the generating set $Y_V$, one can obtain a geodesic representative of an element $g \in G_V$ from any other geodesic representative by iteratively swapping the order of consecutive letters from commuting vertex groups (see [12, Theorem 3.9] or [6, Proposition 3.3]). The support of $g$ can be realized as the set of all vertices $v$ for which a nontrivial element in $G_v$ appears in a geodesic word representative of $g$ over $Y_V$.

In [6], Ciobanu and Hermiller give characterizations of the geodesics and conjugacy geodesics over the generating set $X_V$ in a graph product group $G_V$ using a collection of homomorphisms. For each $v \in V$, define a monoid homomorphism $\pi_v = \pi_v^X \colon X_V^* \to (X_v \cup \{\$\})^*$, where \$ denotes a letter not in $X_V$, by defining

$$\pi_v(a) := \begin{cases} a & \text{if } a \in X_v, \\ \$ & \text{if } a \in X_{V \setminus (\mathsf{Lk}(v) \cup \{v\})}, \\ 1 & \text{if } a \in X_{\mathsf{Lk}(v)}. \end{cases}$$

For the generating set $Y_V$ of $G_V$, we denote the associated map $\pi_v \colon Y_V^* \to (Y_v \cup \{\$\})^*$ by $\pi_v^Y$.

Given languages $L$, $L'$ over a finite set $X$, let $LL' := \{uv \colon u \in L,\ v \in L'\}$ (the concatenation of $L$ with $L'$), $L^+ := \bigcup_{n=1}^{\infty} L^n$ (where $L^n := L^{n-1}L$ for all $n$), and $L^* := L^+ \cup \{\lambda\}$. Also define

$$\mathsf{CycPerm}(L) := \{vu \colon uv \in L\}$$

to be the set of *cyclic permutations* of words in $L$.

**Lemma 3.2** ([6, Propositions 3.3 and 3.5]). *The set of geodesics in the graph product group $G_V$ with respect to the generating set $X_V$ is*

$$\mathsf{Geo}(G_V, X_V) = \bigcap_{v \in V} \pi_v^{-1}(\mathsf{Geo}(G_v, X_v)(\$\mathsf{Geo}(G_v, X_v))^*),$$

*and the set of conjugacy geodesics is*

$$\mathsf{ConjGeo}(G_V, X_V) = \bigcap_{v \in V} \pi_v^{-1}(\mathsf{ConjGeo}(G_v, X_v) \cup \mathsf{CycPerm}((\$\mathsf{Geo}(G_v, X_v))^+)).$$

For a group $G$ with inverse-closed generating set $X$, we say that a word $w \in X^*$ is *cyclically geodesic* over $X$ if every cyclic permutation of $w$ lies in $\mathsf{Geo}(G, X)$, and we denote

$$\mathsf{CycGeo}(G, X) := \{\text{cyclically geodesic words for } G \text{ over } X\}.$$

For the generating set $Y_V$, the fact that

$$\mathsf{Geo}(G_v, Y_v) = \mathsf{CycGeo}(G_v, Y_v) = \mathsf{ConjGeo}(G_v, Y_v) = Y_v \cup \{\lambda\}$$

for any vertex $v \in V$ together with Lemma 3.2 show that

$$\mathsf{ConjGeo}(G_V, Y_V) = \mathsf{CycGeo}(G_V, Y_V).$$

The following is also an immediate consequence of Lemma 3.2.

**Corollary 3.3.** *Let $G_V$ be a graph product group with generating set $X_V$ and let $V'$ be any subset of $V$. Then*

$$\begin{aligned} \mathsf{Geo}(G_V, X_V) \cap X_{V'}^* &= \mathsf{Geo}(G_{V'}, X_{V'}), \\ \mathsf{CycGeo}(G_V, X_V) \cap X_{V'}^* &= \mathsf{CycGeo}(G_{V'}, X_{V'}), \\ \mathsf{ConjGeo}(G_V, X_V) \cap X_{V'}^* &= \mathsf{ConjGeo}(G_{V'}, X_{V'}). \end{aligned}$$

We consider two sets of operations on words over $X_V$. The following (first) set of operations on words over $X_V$ preserve the group element being represented:

- *Local reduction*: $yuz \to ywz$ with $y, z \in X_V^*$, $u, w \in X_v^*$ for some $v \in V$, $u =_{G_v} w$, and $l(u) > l(w)$.
- *Local exchange*: $yuz \to ywz$ with $y, z \in X_V^*$, $u, w \in X_v^*$ for some $v \in V$, $u =_{G_v} w$, and $l(u) = l(w)$.
- *Shuffle*: $yuwz \to ywuz$ with $y, z \in X_V^*$, $u \in X_v^*$ for some $v \in V$ and $w \in X_{v'}^*$ for some $v' \in \mathsf{Lk}(v)$.

Whenever a word $x \in X_V^*$ can be obtained from another word $w \in X_V^*$ by a sequence of local exchanges and shuffles, we write $w \overset{\text{les}}{\to} x$, and whenever $x$ can be obtained from $w$ by a sequence of local reductions, local exchanges, and shuffles, we write $w \overset{\text{lrles}}{\to} x$.

**Lemma 3.4** ([6, Proposition 3.3]). *Let $x$ be a geodesic in the graph product group $G_V$ with respect to the generating set $X_V$ and let $w$ be a word over $X_V$ satisfying $w =_{G_V} x$. Then $w \overset{\text{lrles}}{\to} x$. Moreover, if $w$ is also in $\mathsf{Geo}(G_V, X_V)$, then $w \overset{\text{les}}{\to} x$.*

The following (second) set of operations on words over $X_V$ preserve the conjugacy class being represented:

- *Conjugate replacement*: $yuz \to ywz$ with $y, z \in X_V^*$, $u, w \in X_v^*$ for some $v \in V$, $\mathsf{Supp}(yz) \subseteq \mathsf{Lk}(v)$, and $u \sim_{G_v} w$.
- *Cyclic permutation*: $yu \to uy$ with $y \in X_V^*$ and $u \in X_v^*$ for some $v \in V$.

Whenever a word $x \in X_V^*$ can be obtained from another word $w \in X_V^*$ by a sequence of local reductions and exchanges, shuffles, conjugate replacements, and cyclic permutations, we write $w \stackrel{\text{lrlescrcp}}{\longrightarrow} x$.

In [10], Ferov shows the following.

**Lemma 3.5** ([10, Lemma 3.12]). *If $x$ and $y$ are cyclic geodesics in the graph product group $G_V$ with respect to the generating set $Y_V$, and $x \sim_{G_V} y$, then $x \stackrel{\text{lrlescrcp}}{\longrightarrow} y$. Moreover,* $\mathsf{Supp}(x) = \mathsf{Supp}(y)$ *and* $x \sim_{G_{\mathsf{Supp}(x)}} y$.

In fact, again using the fact that over the generating set $Y_v$ of a vertex group $G_v$ the geodesics and conjugacy geodesics are the words of length 0 or 1, Ferov's proof only uses shuffles, conjugate replacements consisting of replacing a single letter in a vertex generating set $Y_v$ by another letter in that set, and cyclic permutations. In the following, we extend Ferov's result to the generating set $X_V$.

**Corollary 3.6.** *Let $x$ be a cyclic geodesic in the graph product group $G_V$ with respect to the generating set $X_V$ and let $w$ be a word over $X_V$ satisfying $w \sim_{G_V} x$. Then $w \stackrel{\text{lrlescrcp}}{\longrightarrow} x$. Moreover, if $w$ is also in* $\mathsf{CycGeo}(G_V, X_V)$*, then* $\mathsf{Supp}(w) = \mathsf{Supp}(x)$ *and* $w \sim_{G_{\mathsf{Supp}(x)}} x$.

*Proof.* Starting from the word $w$, by repeatedly performing local reductions and exchanges, shuffles, and cyclic permutations, after a finite number of steps we must obtain a word $w_1$ for which no local reductions can occur in any further sequence. Then Lemma 3.4 shows that the word $w_1 \in \mathsf{CycGeo}(G_V, X_V)$.

Among all of the (finitely many) words that can be obtained from $w_1$ by shuffles, let $w'$ be a word with the minimum possible number of syllables (where $w'$ is chosen to be $w_1$ if $w_1$ already realizes the minimum). Cyclically permute $w'$ by a single letter, and repeat the syllable minimization process by shuffles. Repeat this process until a word $w_2$ is obtained for which no cyclic permutations of $w_2$ allow shuffles that decrease the number of syllables.

We claim that $\zeta(w_2) \in \mathsf{CycGeo}(G_V, Y_V)$. To show this, suppose instead that $\zeta(w_2) \notin \mathsf{CycGeo}(G_V, Y_V)$, and write $w_2 = u_1 \cdots u_n$ where the $u_i$ are the syllables of $w_2$. For each $1 \le i \le n$, let $v_i$ be the vertex for which $u_i \in X_{v_i}^+$ and let $g_i$ be the element of $G_{v_i} \setminus \{\varepsilon\}$ represented by $u_i$. Then $\zeta(w_2) = g_1 \cdots g_n$, and there is an index $j$ such that $g_{j+1} \cdots g_n g_1 \cdots g_j \notin \mathsf{Geo}(G_V, Y_V)$. Applying Lemma 3.4, the word $g_{j+1} \cdots g_n g_1 \cdots g_j$ admits a finite sequence of local shuffles leading to a local reduction. However, the corresponding sequence of shuffles of the cyclic permutation $u_{j+1} \cdots u_n u_1 \cdots u_j$ of $w_2$ leads to a word with fewer syllables, giving the required contradiction and proving the claim.

Similarly, there is a sequence of shuffles and cyclic permutations from $x$ to another word $x_2 \in \mathsf{CycGeo}(G_V, X_V)$ satisfying $\zeta(x_2) \in \mathsf{CycGeo}(G_V, Y_V)$. Now Lemma 3.5 says that $\zeta(w_2) \stackrel{\text{lrlescrcp}}{\longrightarrow} \zeta(x_2)$.

Construct a sequence of operations beginning from the word $w_2$ that follows the pattern of the sequence $\zeta(w_2) \stackrel{\text{lrlescrcp}}{\longrightarrow} \zeta(x_2)$, in which each shuffle of the form

$$\zeta(y)\zeta(p)\zeta(q)\zeta(z) \to \zeta(y)\zeta(q)\zeta(p)\zeta(z)$$

of letters in $Y_V$ is replaced by a shuffle of the corresponding syllables $ypqz \to yqpz$ in $X_V^+$, each cyclic permutation $\zeta(y)\zeta(a) \to \zeta(a)\zeta(y)$ by a letter $\zeta(a)$ in a vertex group generating set $Y_v$ is replaced by a cyclic permutation $ya \to ay$ by the corresponding syllable $a$ in $X_v^+$, and each conjugate replacement $\zeta(y)\zeta(p)\zeta(z) \to \zeta(y)\widetilde{q}\zeta(z)$ of a letter $\zeta(p)$ in a set $Y_v$ is replaced by conjugate replacement $ypz \to yqz$ of the corresponding syllable $p$ in $X_v^+$ by any geodesic $q \in \mathsf{Geo}(G_v, X_v)$ satisfying $q =_{G_V} \widetilde{q}$. Let $w_3$ be the word obtained from $w_2$ via this sequence of operations on words.

Now $\zeta(w_3) = \zeta(x_2)$, and each syllable of $w_3$ and of $x_2$ is geodesic. Hence there is a sequence of local exchanges from $w_3$ to $x_2$.

Combining all of the sequences of operations above shows that $w \xrightarrow{\text{lrlescrcp}} x$. Moreover, if $w \in \mathsf{CycGeo}(G_V, X_V)$, then we can take $w = w_1$. Since none of the operations in the sequence from $w = w_1$ to $x$ involve local reductions, and the conjugate replacements in the sequence must replace a word by another nonempty word over the same vertex group generating set, these operations do not alter the support, and moreover only involve conjugation by elements of $G_V$ whose support is in $\mathsf{Supp}(w)$. ∎

### 3.2. Shortlex and conjugacy representatives

We now have the tools to show that the results of Corollary 3.3 hold for the shortlex and conjugacy shortlex languages as well. A total ordering $<_V$ of the generating set $X_V$ of $G_V$ is called *compatible with* a total ordering $\ll$ of the vertex set $V$ of $\Gamma$ if for each vertex $v \in V$ there is a total ordering $<_v$ of the $X_v$ such that for all $a, b \in X_V$ we have $a < b$ if and only if either $\mathsf{Supp}(a) \ll \mathsf{Supp}(b)$ or $\mathsf{Supp}(a) = \mathsf{Supp}(b)$ and $a <_{\mathsf{Supp}(a)} b$.

**Proposition 3.7.** *Let $G_V$ be a graph product group with generating set $X_V$, let $V'$ be any subset of $V$. Let $<_{\mathrm{sl}}$ be a shortlex ordering on $X_V^*$ induced by an ordering compatible with a total ordering $\ll$ on $V$, and let the shortlex ordering on $X_{V'}^*$ be the restriction of the ordering $<_{\mathrm{sl}}$. Then*

$$\mathsf{SL}(G_V, X_V) \cap X_{V'}^* = \mathsf{SL}(G_{V'}, X_{V'})$$

*and*

$$\mathsf{ConjSL}(G_V, X_V) \cap X_{V'}^* = \mathsf{ConjSL}(G_{V'}, X_{V'}).$$

*Proof.* Suppose first that $w$ is a word in $\mathsf{SL}(G_V, X_V) \cap X_{V'}^*$. Then no shortlex smaller word over $X_V$ represents the same element of $G_V$, and so no shortlex smaller word over the subset $X_{V'}$ represents the same element of the subgroup $G_{V'}$; hence $w \in \mathsf{SL}(G_{V'}, X_{V'})$.

On the other hand, if $w \in \mathsf{SL}(G_{V'}, X_{V'})$, then it follows from Corollary 3.3 that $w \in \mathsf{Geo}(G_V, X_V) \cap X_{V'}^*$. Then Lemma 3.4 says that there is a sequence of operations $w \xrightarrow{\text{les}} x$ (in the group $G_V$ over the generating set $X_V$) from $w$ to the shortlex least word $x$ over $X_V$ representing the same element of $G_V$ as $w$. Since all of these operations also apply to the group $G_{V'}$ over the generating set $X_{V'}$, then $x \in \mathsf{SL}(G_V, X_V) \cap X_{V'}^*$. Moreover, since $w$ and $x$ are both shortlex least representatives in $X_{V'}^*$ of the same element of $G_{V'}$, then $w = x$, completing the proof of the first equality in Proposition 3.7.

Next note that if $w$ is a word in $\mathsf{ConjSL}(G_V, X_V) \cap X_{V'}^*$, then for all $g \in G_V$ we have $w \leq_{\rm sl} x_g$ for the shortlex least representative $x_g$ over $X_V$ of the element $gwg^{-1} \in G_V$. In the first part of this proof, we show that for all $g \in G_{V'}$, the word $x_g$ is also the shortlex least representative over $X_{V'}$ of the element $gwg^{-1} \in G_{V'}$. Hence $w \in \mathsf{ConjSL}(G_{V'}, X_{V'})$.

Finally, let $w \in \mathsf{ConjSL}(G_{V'}, X_{V'})$, and let $x$ be the element of $\mathsf{ConjSL}(G_V, X_V)$ satisfying $w \sim_{G_V} x$. Then Corollary 3.6 says that $w \overset{\text{lrlescrcp}}{\longrightarrow} x$. Again, all of these operations also apply to the group $G_{V'}$ over the generating set $X_{V'}$, and so $x \in \mathsf{ConjSL}(G_V, X_V) \cap X_{V'}^*$. Now $w$ and $x$ are both shortlex least representatives in $X_{V'}^*$ of the same conjugacy class of $G_{V'}$, and so $w = x$. ∎

The following result is useful for characterizing the shortlex least representatives of the elements of the graph product $G_V$, and in particular shows that shortlex normal forms have geodesic images under $\zeta$.

**Lemma 3.8.** *Let $<_{\rm sl}$ be a shortlex ordering on words over the generating set $X_V$ of the graph product group $G_V$ induced by an ordering compatible with a total ordering $\ll$ on $V$, let $\ll_{sl}$ be a shortlex ordering on $Y_V^*$ compatible with $\ll$, and let $u \in X_V^*$. Then $u \in \mathsf{SL}(G_V, X_V)$ if and only if ($\zeta(u) \in \mathsf{SL}(G_V, Y_V)$ and each syllable of $u$ is in $\mathsf{SL}(G_v, X_v)$ for some $v \in V$).*

*Proof.* Suppose first that $u \in \mathsf{SL}(G_V, X_V)$. Lemma 3.2 shows that

$$u \in \bigcap_{v \in V} (\pi_v^X)^{-1}(\mathsf{Geo}(G_v, X_v)(\$\mathsf{Geo}(G_v, X_v))^*),$$

and since any two $X_v$ letters of $u$ whose images under $\pi_v^X$ are consecutive must also be consecutive in the shortlex normal form $u$, then

$$u \in \bigcap_{v \in V} (\pi_v^X)^{-1}(\mathsf{SL}(G_v, X_v)(\$\mathsf{SL}(G_v, X_v))^*).$$

Moreover, if $\zeta(u)$ is not geodesic, then there exist two nonadjacent letters of $\zeta(u)$ in the same subset $X_v$ (for some $v$) that can be shuffled together so that a local reduction can be applied; hence the corresponding two syllables of $u$ can be shuffled together, and so $u$ is not a shortlex least representative of an element of $G_V$. Similarly, if $\zeta(u)$ is geodesic but not in $\mathsf{SL}(G_V, Y_V)$, then Lemma 3.4 says that there is a sequence of shuffles (since local exchanges cannot alter an element of $\mathsf{Geo}(G_V, Y_V)$) from $\zeta(u)$ to its shortlex normal form in $\mathsf{SL}(G_V, Y_V)$. Applying the same shuffles to the corresponding syllables of $u$ results in a word over $X_V$ that is smaller in the order $<_{\rm sl}$, contradicting that $u \in \mathsf{SL}(G_V, X_V)$. Hence $\zeta(u) \in \mathsf{SL}(G_V, Y_V)$.

Next suppose instead that $\zeta(u) \in \mathsf{SL}(G_V, Y_V)$ and each syllable of $u$ is in $\mathsf{SL}(G_v, X_v)$ for some $v \in V$. Lemma 3.2 says that for each vertex $v \in V$,

$$\pi_v^Y(\zeta(u)) \in (Y_v \cup \{\lambda\})(\$(Y_v \cup \{\lambda\}))^*,$$

and hence no two distinct syllables of $u$ with support $v$ can be shuffled to be adjacent. Thus each $\pi_v^X(u)$ has the form $u_1\$u_2\cdots\$u_n$ for some $n \geq 1$, where each $u_i$ is a syllable of $u$, and so $\pi_v^X(u) \in \mathtt{Geo}(G_v, X_v)(\$\mathtt{Geo}(G_v, X_v))^*$. Now Lemma 3.2 says that $u \in \mathtt{Geo}(G_V, X_V)$.

Let $u' \in \mathtt{SL}(G_V, X_V)$ satisfy $u' =_{G_V} u$; that is, let $u'$ be the shortlex normal form for the group element represented by $u$. By the first part of this proof, we have $\zeta(u') \in \mathtt{SL}(G_V, Y_V)$, and so $\zeta(u) = \zeta(u')$. Lemma 3.4 says that $u \stackrel{\text{les}}{\to} u'$. For each $v \in V$, shuffles applied to $u$ cannot change the image of the homomorphism $\pi_v^X$, and so $\pi_v^X(u) = \pi_v^X(u')$. Moreover, since $\zeta(u) = \zeta(u')$ is a geodesic over $Y_V$, no sequence of shuffles applied to $u$ or $u'$ can result in fewer syllables. Hence the syllables of both $u$ and $u'$ are the same, the syllables lie in $\mathtt{SL}(G_V, X_V)$, and they occur in the same order. Therefore $u = u'$, and so $u \in \mathtt{SL}(G_V, X_V)$. ∎

### 3.3. Decomposition of graph products into amalgamated products, admissible transversals, and growth formulas

The computation of the standard growth series of a graph product by Chiswell in [4] involves decomposing the graph product into an amalgamated product, and applying the concept of "admissible subgroups". In this section we give a brief summary of these results, and describe a language representing an admissible transversal for a subgraph product in a graph product.

Each graph product over a graph with more than one vertex can be decomposed as an amalgamated product of graph products of groups over the graph product of an appropriate centralizing set.

**Lemma 3.9** ([4, 12]). *Let $G_V$ be a graph product of groups, and let $v \in V$. Using the inclusion maps from $G_{\mathsf{Lk}(v)}$ into both $G_{V\setminus\{v\}}$ and $G_{\mathsf{Lk}(v)\cup\{v\}} = G_{\mathsf{Lk}(v)} \times G_v$, the group $G_V$ can be decomposed as the amalgamated product*

$$G_V = G_{V\setminus\{v\}} *_{G_{\mathsf{Lk}(v)}} (G_{\mathsf{Lk}(v)} \times G_v).$$

**Definition 3.10** ([1, 14]). Let $G$ be a group, $H$ a subgroup of $G$, $X$ an inverse-closed generating set of $G$ and $Y$ an inverse-closed generating set of $H$. The group $H$ is *admissible in $G$ with respect to the pair $(X, Y)$* if $Y \subset X$ and there exists a right transversal $U_{H\setminus G} \subseteq G$ for $H$ in $G$ such that whenever $g = hu$ with $g \in G$, $h \in H$ and $u \in U_{H\setminus G}$, then $\|g\|_X = \|h\|_Y + \|u\|_X$. Note that the transversal contains the identity as representative of $H$ since $g = \varepsilon$ implies $\varepsilon \in U_{H\setminus G}$. We say that $U_{H\setminus G}$ is an *admissible* right transversal of $H$ in $G$ with respect to $(X, Y)$.

**Remark 3.11.** For an admissible subgroup $H = \langle Y\rangle$ of $G = \langle X\rangle$ with admissible transversal $U_{H\setminus G}$, the spherical growth series satisfy the relation $\sigma_{(G,X)} = \sigma_{(H,Y)}\sigma_{(U_{H\setminus G},X)}$, where $\sigma_{(U_{H\setminus G},X)}$ denotes the growth series of the elements of the transversal $U_{H\setminus G}$ with respect to $X$.

The next lemma shows the relationship between the spherical growth series of a free product of groups amalgamated along a common admissible subgroup, and the spherical growth series of the factor and amalgamating subgroups.

**Lemma 3.12** ([1, 14]). *Let $G$, $K$ be groups and let $H$ be a subgroup of both $G$ and $K$. Let $X$, $Y$ and $Z$ be inverse-closed generating sets of $G$, $H$ and $K$, respectively. Suppose that $H$ is admissible in both $G$ and $K$ with respect to the pairs $(X, Y)$ and $(Z, Y)$, respectively. Let $A$ be the amalgamated product $A := G *_H K$ and let $W := X \cup Z$. Then*

$$\frac{1}{\sigma_{(A,W)}} = \frac{1}{\sigma_{(G,X)}} + \frac{1}{\sigma_{(K,Z)}} - \frac{1}{\sigma_{(H,Y)}}.$$

**Remark 3.13.** Given groups $G_i = \langle X_i \rangle$ for $i = 1, 2$, it follows directly from Definition 3.10 that $G_1$ is admissible in the direct product group $G_1 \times G_2$ with respect to the pair of generating sets $(X_1 \cup X_2, X_1)$, with admissible transversal $\{\varepsilon\} \times G_2$.

Recall that if each vertex group $G_v$ of a graph product on a graph with vertex set $V$ has an inverse-closed generating set $X_v$, then for each $V' \subseteq V$, the subgraph product $G_{V'}$ has generating set $X_{V'} := \bigcup_{v \in W} X_v$. Using these generating sets, any subgraph product $G_{V'}$ is an admissible subgroup in $G_V$ with respect to the pair $(X_V, X_{V'})$ (see [4], [16, Proposition 14.4]).

The following formula for computing the spherical growth series of a graph product from spherical growth series of subgraph products is an immediate consequence of Lemmas 3.12 and 3.9 and Remarks 3.11 and 3.13; this formula was obtained by Chiswell in [4, proof of Proposition 1]. This recursive formula is the analog for spherical growth series of our formula in Theorem A for spherical conjugacy growth series.

**Corollary 3.14.** *Let $G_V$ be a graph product group over a graph with vertex set $V$ and let $v \in V$. For each $v' \in V$ let $X_{v'}$ be an inverse-closed generating set for the vertex group $G_{v'}$, and for each $S \subseteq V$ let $\sigma_S$ be the spherical growth series for the subgraph product $G_S$ on the subgraph induced by $S$, over the generating set $X_S = \bigcup_{v' \in S} X_{v'}$. Then*

$$\sigma_V = \frac{\sigma_{\mathsf{Lk}(v)}\sigma_{V\setminus\{v\}}\sigma_{\{v\}}}{\sigma_{\mathsf{Lk}(v)}\sigma_{\{v\}} + \sigma_{V\setminus\{v\}} - \sigma_{V\setminus\{v\}}\sigma_{\{v\}}}.$$

In the following, we provide a set of representatives for a specific admissible transversal for a subgraph product in a graph product, which we will use in our proofs in Section 4.

**Lemma 3.15.** *Let $G_V$ be a graph product with vertex set $V$, for each $v \in V$ let $X_v$ be an inverse-closed generating set for $G_v$, and let $V' \subseteq V$. Let $<_{\mathrm{sl}}$ be a shortlex ordering on $X_V^*$ compatible with a total ordering $\ll$ on $V$ satisfying $a \ll b$ for all $a \in V'$ and $b \in V \setminus V'$. Then the set of words*

$$\widehat{U}_{G_{V'}\setminus G_V} := \{\lambda\} \cup (\mathsf{SL}(G_V, X_V) \cap (X_{V\setminus V'} X_V^*))$$

*is a set of unique representatives of an admissible right transversal $U_{G_{V'}\setminus G_V}$ for the subgraph product group $G_{V'}$ in $G_V$ with respect to the pair $(X_V, X_{V'})$.*

*Proof.* Let $g$ be any element of $G_V$, and let $y$ be the shortlex normal form of $g$. Then there is a factorization $y = y_1 y_2$ where $y_1$ is the longest prefix of $y$ lying in $X^*_{V'}$. Now either $y_2 = \lambda$, or else the first letter of $y_2$ lies in $X_{V\setminus V'}$. Since every subword of a shortlex normal form is again a shortlex least representative of a group element, then $y_2 \in \widehat{U}_{G_{V'}\setminus G_V}$. Hence $\widehat{U}_{G_{V'}\setminus G_V}$ contains representatives of elements in every coset.

Next suppose that $w \in \mathsf{SL}(G_{V'}, X_{V'})$ and $u \in \widehat{U}_{G_{V'}\setminus G_V}$; in this paragraph, we show that $wu$ is a geodesic in $G_V$ over $X_V$ using Lemma 3.2. Given $v \in V \setminus V'$, the image of $wu$ under the homomorphism $\pi_v$ associated to $v$ satisfies $\pi_v(wu) = \pi_v(w)\pi_v(u)$, where $\pi_v(w) \in \$^*$ and $\pi_v(u) \in \mathsf{Geo}(G_v, X_v)(\$\mathsf{Geo}(G_v, X_v))^*$ by Lemma 3.2 since $u$ is a shortlex normal form and hence a geodesic. On the other hand, given $v \in V'$, we have $\pi_v(wu) = \pi_v(w)\pi_v(u)$ where $\pi_v(w) \in \mathsf{Geo}(G_v, X_v)(\$\mathsf{Geo}(G_v, X_v))^*$ since $w$ is a geodesic. Either $\mathsf{Supp}(u) \subseteq \mathsf{Lk}(v)$, in which case $\pi_v(u) = \lambda$, or else we can write the shortlex normal form $u = u_1 c u_2$ for some $u_1 \in X^*_{\mathsf{Lk}(v)}$ and $c \in X_{V\setminus \mathsf{Lk}(v)}$. In the latter case, we show that $c \notin X_v$: if $c \in X_v$, then $u_1 c u_2 =_{G_V} c u_1 u_2$ because $u_1$ commutes with every element in $G_v$. However, the first letter $b$ of $u$ (and therefore $u_1$) lies in $X_{V\setminus V'}$ by the definition of $\widehat{U}_{G_{V'}\setminus G_V}$, and so $c < b$. This contradicts the fact that $u = u_1 c u_2$ is a shortlex least representative, and consequently $c \notin X_v$. Therefore the first letter of $\pi_v(u)$ is \$, and in this case the image of the geodesic $u$ satisfies $\pi_v(u) \in (\$\mathsf{Geo}(G_v, X_v))^*$. Hence in all cases we have $\pi_v(wu) \in \mathsf{Geo}(G_v, X_v)(\$\mathsf{Geo}(G_v, X_v))^*$. Then Lemma 3.2 shows that $wu$ is geodesic.

Now suppose that $wu =_{G_V} w'u'$ for some $w, w' \in \mathsf{SL}(G_{V'}, X_{V'})$ and $u, u' \in \widehat{U}_{G_{V'}\setminus G_V}$. Then $u =_{G_V} w''u'$, where $w''$ is the element of $\mathsf{SL}(G_{V'}, X_{V'})$ representing $w^{-1}w'$. By the preceding paragraph, $u$ and $w''u'$ are geodesics representing the same element of $G_V$. Lemma 3.4 shows that $u \xrightarrow{\text{les}} w''u'$; that is, $w''u'$ can be obtained from $u$ by a sequence of local exchanges and shuffles. Suppose that $w'' \neq \lambda$, and let $v \in V'$ be the support of the first letter $a$ of $w''$. Then the first letter of $\pi_v(w''u')$ is $a$, and the argument in the previous paragraph shows that either $\pi_v(u) = \lambda$ or the first letter of $\pi_v(u)$ is \$. Note that the shuffle operation does not change the image of any word under the $\pi_v$ homomorphism, and the only change possible under a local exchange is the replacement of one subword of $X^*_v$ by another of the same length. Hence the word $w''u'$ cannot be obtained from $u$; this contradiction shows that $w'' = \lambda$. Therefore $w =_{G_V} w'$ (and so $w = w'$). Consequently, we also have $u =_{G_V} u'$, and since $u, u'$ are shortlex normal forms, $u = u'$ as well. Thus each coset has only one representative in $\widehat{U}_{G_{V'}\setminus G_V}$, completing the proof that this is a set of unique representatives of an admissible transversal. ∎

## 4. The conjugacy growth series of a graph product

In this section we will first determine a set of conjugacy geodesic representatives of the conjugacy classes of a graph product, in Section 4.1. Then in Section 4.2 we establish preservation of equality of standard and conjugacy growth rates by a graph product, and in Section 4.3 we derive the recursive formula for the spherical conjugacy growth series.

### 4.1. Conjugacy geodesic representatives of conjugacy classes

In Proposition 4.1 we apply the characterisation of geodesics and conjugacy geodesics in graph products from Lemma 3.2 to the amalgamated product decomposition of Lemma 3.9.

Throughout Section 4.1 we will assume the following.

**Hypothesis A.** *Let $G_V$ be a graph product group, with generating set $X_V$, and let $v \in V$ be a vertex for which $\{v\} \cup \mathsf{Lk}(v) \subsetneq V$. Let $<_{\mathrm{sl}}$ be a shortlex ordering on $X_V^*$ that is compatible with a total ordering $\ll$ on $V$ satisfying $x \ll y$ for all $x \in \mathsf{Lk}(v)$ and $y \in V \setminus (\mathsf{Lk}(v) \cup \{v\})$, and let $\widehat{U} := \widehat{U}_{G_{\mathsf{Lk}(v)} \backslash G_{V\setminus\{v\}}}$ be the admissible transversal set of representatives for $G_{\mathsf{Lk}(v)}$ in $G_{V\setminus\{v\}}$ with respect to $(X_{V\setminus\{v\}}, X_{\mathsf{Lk}(v)})$ from Lemma 3.15.*

**Proposition 4.1.** *Let $G_V$ and $v \in V$ satisfy Hypothesis A. Suppose that $u_i \in \widehat{U} \setminus \{\lambda\}$ and $c_i \in \mathsf{Geo}(G_v, X_v) \setminus \{\lambda\}$ for all $i$, $b \in \mathsf{Geo}(G_{\mathsf{Lk}(v)}, X_{\mathsf{Lk}(v)})$, $\widetilde{b} \in \mathsf{ConjGeo}(G_{\mathsf{Lk}(v)}, X_{\mathsf{Lk}(v)})$, and $\mathsf{Supp}(\widetilde{b}) \subseteq \mathsf{Lk}(\bigcup_{i=1}^n \mathsf{Supp}(u_i))$. Then*

(1) *The words $bu_1c_1 \cdots u_nc_n$ and $bc_0u_1c_1 \cdots u_nc_n$ are geodesics in $G_V$ over $X_V$.*

(2) *The word $\widetilde{b}u_1c_1 \cdots u_nc_n$ is a conjugacy geodesic in $G_V$ over $X_V$.*

*Proof.* Let $w = u_1c_1 \cdots u_nc_n$. We consider the images of the words $bw$, $bc_0w$, and $\widetilde{b}w$ under the $\pi_{v'} = \pi_{v'}^{X_V}$ maps, for $v' \in V$, in turn.

In the case that $v' = v$, note that

$$\pi_v(b) = \pi_v(\widetilde{b}) = \lambda, \quad \pi_v(c_i) = c_i \in \mathsf{Geo}(G_v, X_v), \quad \pi_v(u_i) \in \$^+,$$

where the latter containment follows from the fact that the first letter of $u_i$ lies in the generating set $X_{V\setminus(\mathsf{Lk}(v)\cup\{v\})}$, and hence the word $\pi_v(u_i)$ is nonempty. Then

$$\pi_v(bw) = \pi_v(\widetilde{b}w) = \pi_v(w) = \$^{i_1}c_1 \cdots \$^{i_n}c_n \in (\$\mathsf{Geo}(G_v, X_v))^*$$

for some natural numbers $i_1, \ldots, i_n$, and $\pi_v(bc_0w) \in \mathsf{Geo}(G_v, X_v)(\$\mathsf{Geo}(G_v, X_v))^*$.

Next consider the case that $v' \in V \setminus (\mathsf{Lk}(v) \cup \{v\})$. Applying Lemma 3.2 to $u_i$ (since $u_i$ is a geodesic in $G_V$ over $X_V$ by Corollary 3.3), we have

$$\pi_{v'}(b), \pi_{v'}(\widetilde{b}) \in \$^*, \quad \pi_{v'}(c_i) \in \$^+, \quad \pi_{v'}(u_i) \in \mathsf{Geo}(G_{v'}, X_{v'})(\$\mathsf{Geo}(G_{v'}, X_{v'}))^*.$$

Hence in this case, $\pi_{v'}(bw), \pi_{v'}(bc_0w), \pi_{v'}(\widetilde{b}w) \in (\mathsf{Geo}(G_{v'}, X_{v'})\$)^*$.

Finally, suppose that $v' \in \mathsf{Lk}(v)$. Let $a_i$ be the first letter of the word $u_i$; then $\mathsf{Supp}(a_i) \subseteq V \setminus (\mathsf{Lk}(v) \cup \{v\})$. If the word $\pi_{v'}(u_i)$ were to start with a letter $a$ in $X_{v'}$, then $u_i$ can be shuffled to a word beginning with $a$, contradicting the fact that $u_i \in \widehat{U}$ is a shortlex normal form and $a <_{\mathrm{sl}} a_i$ in the shortlex ordering (compatible with $\ll$). Hence $\pi_{v'}(u_i)$ is either $\lambda$ or starts with \$. In this case (applying Lemma 3.2 and Corollary 3.3 again), we have $\pi_{v'}(c_i) = \lambda$ and

$$\pi_{v'}(b) \in \mathsf{Geo}(G_{v'}, X_{v'})(\$\mathsf{Geo}(G_{v'}, X_{v'}))^*, \quad \pi_{v'}(u_i) \in (\$\mathsf{Geo}(G_{v'}, X_{v'}))^*.$$

Moreover, either $v' \notin \mathsf{Supp}(\tilde{b})$ and $\pi_{v'}(\tilde{b}) \in \$^*$, or else $v' \in \mathsf{Supp}(\tilde{b}) \subseteq \mathsf{Lk}(\bigcup_{i=1}^n \mathsf{Supp}(u_i))$ and hence (by Lemma 3.2 and Corollary 3.3)

$$\pi_{v'}(\tilde{b}) \in \mathsf{ConjGeo}(G_{v'}, X_{v'}) \cup \mathsf{CycPerm}((\$\mathsf{Geo}(G_{v'}, X_{v'}))^+)$$

and $\pi_{v'}(u_i) = \lambda$ for all $i$.

Thus for all $v' \in V$ we have $\pi_{v'}(w) \in (\$\mathsf{Geo}(G_{v'}, X_{v'}))^*$,

$$\pi_{v'}(bw), \pi_{v'}(bc_0w) \in \mathsf{Geo}(G_{v'}, X_{v'})(\$\mathsf{Geo}(G_{v'}, X_{v'}))^*,$$

and

$$\pi_{v'}(\tilde{b}w) \in \mathsf{ConjGeo}(G_{v'}, X_{v'}) \cup \mathsf{CycPerm}((\$\mathsf{Geo}(G_{v'}, X_{v'}))^+).$$

Lemma 3.2 then completes the proof of (1) and (2). ■

A *piecewise subword* of a word $w \in X_V^*$ is a word over $X_V$ of the form $b_1 \cdots b_k$ such that $w = d_0b_1d_1 \cdots b_kd_k$ for some words $d_0, \dots, d_k \in X_V^*$. A piecewise subword $b'$ of $w$ is *proper* if $b' \neq w$. In the following lemma, we show that multiplying the geodesics in Proposition 4.1 on the right by a word $w$ over $X_{\mathsf{Lk}(v)}$ yields an element represented by another such geodesic in which a piecewise subword of $w$ occurs on the left.

**Lemma 4.2.** *Let $G_V$ and $v \in V$ satisfy Hypothesis A. Suppose that $u_1 \in \hat{U}$, $u_i \in \hat{U} \setminus \{\lambda\}$ for all $i > 1$, $b \in \mathsf{SL}(G_{\mathsf{Lk}(v)}, X_{\mathsf{Lk}(v)})$, $c_n \in \mathsf{SL}(G_v, X_v)$, and $c_i \in \mathsf{SL}(G_v, X_v) \setminus \{\lambda\}$ for all $i < n$. Then*

(1) *$u_1c_1 \cdots u_nc_nb$ is equal in $G_V$ to a word of the form $b'u_1'c_1 \cdots u_n'c_n$ satisfying $u_1' \in \hat{U}$, with $u_1' = \lambda$ if and only if $u_1 = \lambda$, $u_i' \in \hat{U} \setminus \{\lambda\}$ for all $i > 1$, and $b'$ is a piecewise subword of $b$. Moreover, if $\mathsf{Supp}(b) \nsubseteq \mathsf{Lk}(\bigcup_{i=1}^n \mathsf{Supp}(u_i))$, then $b'$ is a proper piecewise subword of $b$.*

(2) *$bu_1c_1 \cdots u_nc_n$ can be conjugated by an element of $G_{\mathsf{Supp}(b)}$ to an element of $G_V$ represented by a word of the form $\tilde{b}u_1'c_1 \cdots u_n'c_n$ satisfying $u_1' \in \hat{U}$ and $u_1' = \lambda$ if and only if $u_1 = \lambda$, $u_i' \in \hat{U} \setminus \{\lambda\}$ for all $i > 1$, $\tilde{b} \in \mathsf{ConjSL}(G_{\mathsf{Lk}(v)}, X_{\mathsf{Lk}(v)})$, and $\mathsf{Supp}(\tilde{b}) \subseteq \mathsf{Lk}(\bigcup_{i=1}^n \mathsf{Supp}(u_i'))$.*

*Proof.* We begin by proving statement (1) in the special case that $n = 1$, $u_1 \in \hat{U} \setminus \{\lambda\}$, $c_1 = \lambda$, and $b \in X_{v'}$ is a single letter, with $v' \in \mathsf{Lk}(v)$.

*Case* 1. Suppose that $\mathsf{Supp}(b) \subseteq \mathsf{Lk}(\mathsf{Supp}(u_1))$. Then $u_1b =_{G_V} bu_1$, which has the required form.

*Case* 2. Suppose that $\mathsf{Supp}(b) \nsubseteq \mathsf{Lk}(\mathsf{Supp}(u_1))$. Then we can write $u_1 = xyz$, where $z$ is the maximal suffix of $u_1$ satisfying $\mathsf{Supp}(b) \subseteq \mathsf{Lk}(\mathsf{Supp}(z))$, and $y$ is a syllable of $u_1$.

*Case* 2a. Suppose that $\mathsf{Supp}(y) \neq \mathsf{Supp}(b)$. Then $b$ is a syllable of the word $u_1b$ and $\zeta(u_1b) = \zeta(u_1)\zeta(b)$. Since $u_1 \in \mathsf{SL}(G_{V\setminus\{v\}}, X_{V\setminus\{v\}})$, Lemma 3.8 says that $\zeta(u_1) \in \mathsf{SL}(G_{V\setminus\{v\}}, Y_{V\setminus\{v\}})$ and each syllable of $u_1$ is in $\mathsf{SL}(G_{\hat{v}}, X_{\hat{v}})$ for some $\hat{v}$. For all $\hat{v} \in V$

we have $\pi^Y_{\hat{v}}(\zeta(u_1 b)) = \pi^Y_{\hat{v}}(\zeta(u_1))\pi^Y_{\hat{v}}(\zeta(b))$, and Lemma 3.2 says that $\pi^Y_{\hat{v}}(\zeta(u_1)) \in \mathsf{Geo}(G_{\hat{v}}, Y_{\hat{v}})(\$\mathsf{Geo}(G_{\hat{v}}, Y_{\hat{v}}))^*$.

For each $\hat{v} \neq v'$, either $\pi^Y_{\hat{v}}(\zeta(u_1 b)) = \pi^Y_{\hat{v}}(\zeta(u_1))$, or $\pi^Y_{\hat{v}}(\zeta(u_1 b)) = \pi^Y_{\hat{v}}(\zeta(u_1))$. Also $\pi^Y_{v'}(\zeta(u_1 b)) = \pi^Y_{v'}(\zeta(x))\$\zeta(b)$. Hence

$$\zeta(u_1 b) \in \bigcap_{\hat{v} \in V} (\pi^Y_{\hat{v}})^{-1}(\mathsf{Geo}(G_{\hat{v}}, Y_{\hat{v}})(\$\mathsf{Geo}(G_{\hat{v}}, Y_{\hat{v}}))^*),$$

and by Lemma 3.2 the word $\zeta(u_1 b)$ is a geodesic in $G_V$ over $Y_V$. Now Lemma 3.4 says that there is a sequence of shuffles from $\zeta(u_1 b)$ to its shortlex normal form. Let $u' \in X^*_{V\setminus\{v\}}$ be the word obtained from $u_1 b$ by performing the same shuffles to the associated syllables of $u_1 b$. Then $\zeta(u') \in \mathsf{SL}(G_{V\setminus\{v\}}, Y_{V\setminus\{v\}})$ and each syllable of $u'$ is (either $b$ or a syllable of $u_1$ and hence) in the shortlex language of its vertex group. Now Lemma 3.8 says that $u' \in \mathsf{SL}(G_{V\setminus\{v\}}, X_{V\setminus\{v\}})$. Moreover, since shuffles cannot alter the image of a word under a $\pi^X_{\hat{v}}$ map, and since $\pi^X_{\hat{v}}(u_1 b)$ is either the empty word or starts with a \$ for every $\hat{v} \in \mathsf{Lk}(v)$ (by definition of $\widehat{U}$ and the choice of the ordering $<_{\mathrm{sl}}$ compatible with $\ll$), the same is true for the shuffled word $u'$. Hence $u' \in X_{V\setminus(\{v\}\cup\mathsf{Lk}(v))} X^*_{V\setminus\{v\}}$, and so $u' \in \widehat{U}$. Therefore $u_1 b =_{G_V} u'$ for a word $u' \in \widehat{U}$ in case 2a.

*Case* 2b. Suppose that $\mathsf{Supp}(y) = \mathsf{Supp}(b)$. Let $y'$ be the shortlex normal form for $yb$. Since $y$ and $z$ commute and $xyz$ is in shortlex form, the rightmost syllable of $x$ and the leftmost syllable of $z$ cannot have the same support, and so (irrespective of whether or not $y'$ is the empty word) the syllables of $xy'z$ are either $y'$ or syllables of $x$ or of $z$ and hence are syllables of $u_1$. Thus each syllable of $xy'z$ is in $\mathsf{SL}(G_{\hat{v}}, X_{\hat{v}})$ for some $\hat{v}$. Following an argument similar to that in case 2a, the word $\zeta(u_1) \in \mathsf{SL}(G_{V\setminus\{v\}}, Y_{V\setminus\{v\}})$, and the word $\zeta(xy'z)$ is obtained from $\zeta(u_1) = \zeta(xyz)$ either by an exchange of a letter $\zeta(y)$ for a letter $\zeta(y')$ if $y' \neq \lambda$, in which case the word $\zeta(xy'z)$ is again in $\mathsf{SL}(G_{V\setminus\{v\}}, Y_{V\setminus\{v\}})$, or else by removal of the letter $\zeta(y)$, if $y' = \lambda$. In the latter situation, an argument similar to that in case 2a, using the maps $\pi^Y_{\hat{v}}$, can be used to show that the word $\zeta(xy'z) = \zeta(xz)$ is geodesic, and moreover is in $\mathsf{SL}(G_{V\setminus\{v\}}, Y_{V\setminus\{v\}})$. Hence Lemma 3.8 shows that $xy'z \in \mathsf{SL}(G_{V\setminus\{v\}}, X_{V\setminus\{v\}})$. Since the first letter $a$ of the word $u_1 = xyz$ lies in $X_{V\setminus(\{v\}\cup\mathsf{Lk}(v))}$, the subword $x$ is nonempty and the first letter of the word $u' := xyz$ is also $a$. Therefore $u_1 b =_{G_V} u'$ for a word $u' \in \widehat{U}$ in case 2b also.

This completes the proof of the special case. For the general case of part (1), let $w = u_1 c_1 \cdots u_n c_n$ and write $b = b_1 \cdots b_m$ with each $b_i \in X_{\mathsf{Lk}(v)}$. Starting with the word $wb$, shuffle $b_1$ to the left until either $b_1$ reaches the left side of the word, or $b_1$ reaches a subword $u_j$ such that $\mathsf{Supp}(b_1) \not\subseteq \mathsf{Supp}(u_j)$, in which case the special case above is applied to replace $u_j$ by another element of $\widehat{U}$. Iterating this for the letters $b_2$ through $b_m$ completes the proof of (1).

Note that although cyclic conjugation of $bw$ to $wb$ and then applying the process from part (1) above results in a word $b'\hat{u}_1 c_1 \cdots \hat{u}_n c_n =_{G_V} wb$ with each $\hat{u}_i \in \widehat{U}$ and $b'$ a piecewise subword of $b$ that is potentially shorter than $b$, it is possible that $\mathsf{Supp}(b') \not\subseteq \mathsf{Lk}(\bigcup_{i=1}^n \mathsf{Supp}(\hat{u}_i))$.

Iterate this process of cyclically conjugating the maximal prefix in $X^*_{\mathsf{Lk}(v)}$ to the right side of the word and applying the algorithm above. Since the word length of the prefix in $X^*_{\mathsf{Lk}(v)}$ can only strictly decrease finitely many times, after finitely many steps, the procedure must reach a word of the form $b''w' = b''u'_1c_1 \cdots u'_nc_n$ such that the algorithm above applied to $w'b''$ results in $b''w'$; that is, $\mathsf{Supp}(b'') \subseteq \mathsf{Lk}(\bigcup_{i=1}^n \mathsf{Supp}(u'_i)) \cap \mathsf{Supp}(b)$. Finally, let $\widetilde{b} \in \mathsf{ConjSL}(G_{\mathsf{Supp}(b)}, X_{\mathsf{Supp}(b)})$ be the shortlex least word representing an element of the conjugacy class of $G_{\mathsf{Supp}(b)}$ containing $b''$; Corollary 3.3 shows that $\widetilde{b} \in \mathsf{ConjSL}(G_{\mathsf{Lk}(v)}, X_{\mathsf{Lk}(v)})$ as well. Now there is an element $g \in G_{\mathsf{Supp}(b)}$ such that $gb''g^{-1} =_{G_{\mathsf{Supp}(b)}} \widetilde{b}$, and so $\widetilde{b}w' =_{G_V} gb''w'g^{-1}$ is a conjugate of $wb$ by an element of $G_{\mathsf{Supp}(b)}$ as well. This completes the proof of (2). ∎

Following the notation in [15, Section IV.2], a sequence $a_1, \ldots, a_n$ (with $n \geq 0$) of elements of the amalgamated product $G = A *_C B$ is *reduced* if each $a_i$ is in one of two subgroups $A$ or $B$, successive $a_i$ are not in the same subgroup, if $n = 1$ then $a_1 \neq \varepsilon$, and if $n > 1$ then no $a_i$ is in $C$. This sequence is *cyclically reduced* if every cyclic permutation of the sequence is reduced.

In the following, we apply the normal form and conjugacy normal form theorems [15, Theorems IV.2.6 and IV.2.8] for sequences in free products with amalgamation to establish conjugacy representatives for every conjugacy class of a graph product, and to determine when two of these conjugacy geodesics represent the same conjugacy class.

**Proposition 4.3.** *Let $G_V$ and $v \in V$ satisfy Hypothesis A. Then*

(1) *For each element $g \in G_V$ there exists a conjugacy geodesic $w \in X_V^*$ representing the conjugacy class $[g]_{\sim,G_V}$, with $w$ either of the form*

$$w = \widetilde{b}u_1c_1 \cdots u_nc_n, \quad \text{where } n > 0,\ u_i \in \widehat{U} \setminus \{\lambda\},\ c_i \in \mathsf{SL}(G_v, X_v) \setminus \{\lambda\},$$
$$\widetilde{b} \in \mathsf{ConjSL}(G_{\mathsf{Lk}(v)}, X_{\mathsf{Lk}(v)}), \text{ and } \mathsf{Supp}(\widetilde{b}) \subseteq \mathsf{Lk}\left(\bigcup_{i=1}^n \mathsf{Supp}(u_i)\right), \tag{†}$$

*or else of the form*

$$w \in \mathsf{ConjSL}(G_{V\setminus\{v\}}, X_{V\setminus\{v\}}) \cup \mathsf{ConjSL}(G_{\{v\}\cup\mathsf{Lk}(v)}, X_{\{v\}\cup\mathsf{Lk}(v)}). \tag{‡}$$

(2) *Two words $w_1, w_2 \in X_V^*$ that are each of the form (†) or (‡) represent conjugate elements of $G_V$ if and only if either $w_1 = w_2$, or the words can be written as $w_1 = \widetilde{b}u_1c_1 \cdots u_nc_n$ and $w_2 = \widetilde{b}'u'_1c'_1 \cdots u'_{n'}c'_{n'}$ in the form (†) such that*

(i) *$\widetilde{b} = \widetilde{b}'$ and $n = n'$, and*

(ii) *there is an index $j$ such that $u_i = u'_{i+j}$ and $c_i = c'_{i+j}$ for all $i$, where the indices are considered modulo $n$.*

*Proof.* Let $g$ be any element of $G_V$. Using Lemma 3.9 and the normal form theorem for amalgamated products (see for example [15, Theorem IV.2.6]), the element $g$ is represented by a word of the form $x = \widehat{b}\widehat{u}_1\widehat{c}_1 \cdots \widehat{u}_n\widehat{c}_n$ for some $n \geq 0$, $\widehat{u}_i \in \widehat{U}$ for all $i$

with $\widehat{u}_i \neq \lambda$ for all $i > 1$, $\widehat{c}_i \in \mathsf{SL}(G_v, X_v)$ for all $i$ with $\widehat{c}_i \neq \lambda$ for all $i < n$, and $\widehat{b} \in \mathsf{SL}(G_{\mathsf{Lk}(v)}, X_{\mathsf{Lk}(v)})$. By Lemma 4.2 (2), $g$ is conjugate in $G_V$ to another element $g'$ represented by a word of the form $x' := \widehat{b}'\widehat{u}'_1\widehat{c}_1 \cdots \widehat{u}'_n\widehat{c}_n$ satisfying $\widehat{u}'_1 \in \widehat{U}$, $\widehat{u}'_i \in \widehat{U} \setminus \{\lambda\}$ for $i > 1$, $\widehat{b}' \in \mathsf{ConjSL}(G_{\mathsf{Lk}(v)}, X_{\mathsf{Lk}(v)})$, and $\mathsf{Supp}(\widetilde{b}) \subseteq \mathsf{Lk}(\bigcup_{i=1}^n \mathsf{Supp}(\widehat{u}'_i))$.

If $n = 0$, then $x'$ is of the form (‡). Suppose instead that $n > 0$.

If both $\widehat{u}'_1$ and $\widehat{c}_n$ are not the empty word, then $x'$ is in the form (†). If both $\widehat{u}'_1$ and $\widehat{c}_n$ are the empty word, then $g$ is conjugate to the element of $G_V$ represented by $\widehat{b}'\widehat{u}'_n\widehat{c}_1 \cdots \widehat{u}'_{n-1}\widehat{c}_{n-1}$, which is of the form (†) (or (‡) if $n = 1$).

On the other hand, if exactly one of $\widehat{u}'_1, \widehat{c}_n$ is equal to $\lambda$, then using the fact that $g$ is also conjugate to $g'' =_{G_V} \widehat{b}'\widehat{u}'_n\widehat{c}_n\widehat{u}'_1\widehat{c}_1 \cdots \widehat{u}'_{n-1}\widehat{c}_{n-1}$, we can replace any consecutive $\widehat{c}_n\widehat{c}_1$ by the shortlex least representative of this element in $G_v$ over $X_v$, and we can replace any consecutive $\widehat{b}'\widehat{u}'_n\widehat{u}'_1$ (or $\widehat{b}'\widehat{u}'_n\widehat{u}'_2$ if $\widehat{u}'_1 = \lambda$ and $\widehat{c}_n\widehat{c}_1 =_{G_v} \lambda$) by $du''$ for some $d \in \mathsf{Geo}(G_{\mathsf{Lk}(v)}, X_{\mathsf{Lk}(v)})$ and $u'' \in \widehat{U}$, since $\widehat{U}$ is a set of representatives of a transversal.

We repeat this process iteratively; that is, at each step we conjugate by a word over $X_{\mathsf{Lk}(v)}$ in order to apply Lemma 4.2 (2), and then (cyclically) conjugate by the maximal suffix in $\widehat{U} \cdot \mathsf{SL}(G_v, X_v)$, shuffling this word past the maximal prefix in $\mathsf{SL}(G_{\mathsf{Lk}(v)}, X_{\mathsf{Lk}(v)})$, and combining terms in $\widehat{U}$ and/or $\mathsf{SL}(G_v, X_v)$. At the end apply a final conjugation by a word over $X_{\mathsf{Lk}(v)}$ in order to apply Lemma 4.2 (2) a last time.

After a finite number of iterations this process must stop, resulting either in a word of the form (†), or else in a word over one of the alphabets $X_{V\setminus\{v\}}$ or $X_{\mathsf{Lk}(v)\cup\{v\}}$. In the latter case, further conjugation shows that $g$ is conjugate to a word of the form (‡).

Finally, Proposition 4.1 shows that all words of the form (†) are conjugacy geodesics, and Proposition 3.7 shows that all words of the form (‡) are conjugacy geodesics, for the group $G_V$ over the generating set $X_V$, completing the proof of item (1).

For the proof of statement (2), we start by noting that it is straightforward to check that if (i) and (ii) hold, then $w_1 = \widetilde{b}u_1c_1 \cdots u_nc_n \sim_{G_V} w_2 = \widetilde{b}'u'_1c'_1 \cdots u'_{n'}c'_{n'}$.

Now suppose that $w_1, w_2$ each have the form (†) or (‡) and represent conjugate elements of $G_V$. Corollary 3.6 shows that any two conjugacy geodesics for $G_V$ over $X_V$ that represent the same conjugacy class must have the same support. Hence either $w_1, w_2$ are both of the form (‡), in which case $w_1 = w_2$ is the shortlex least representative of their conjugacy class in the subgroup, or both have the form (†).

In the latter case, we write $w_1 = \widetilde{b}u_1c_1 \cdots u_nc_n$ and $w_2 = \widetilde{b}'u'_1c'_1 \cdots u'_{n'}c'_{n'}$ in (†) form, where the sequences $(\widetilde{b}u_1), c_1, \ldots, u_n, c_n$ and $(\widetilde{b}'u'_1), c'_1, \ldots, u_{n'}, c_{n'}$ are cyclically reduced sequences of length at least 2. The conjugacy theorem for free products with amalgamation (see for example [15, Theorem IV.2.8]) implies that any two cyclically reduced sequences of length at least 2 representing conjugate elements of the amalgamated product $G_V = G_{V\setminus\{v\}} *_{G_{\mathsf{Lk}(v)}} G_{\mathsf{Lk}(v)\cup\{v\}}$ must have the same length $n = n'$, and moreover there exist a $d \in \mathsf{SL}(G_{\mathsf{Lk}(v)}, X_{\mathsf{Lk}(v)})$ and an index $0 \leq j \leq n-1$ such that either

$$w_2 =_{G_V} d(u_{j+1}c_{j+1} \cdots u_nc_n(\widetilde{b}u_1)c_1 \cdots u_jc_j)d^{-1} \tag{4.1}$$

or

$$w_2 =_{G_V} d(c_ju_{j+1}c_{j+1} \cdots u_nc_n(\widetilde{b}u_1)c_1 \cdots u_j)d^{-1}. \tag{4.2}$$

We assume that $d$ has been chosen to be of minimal length; that is, no word of shorter length over $X_{\mathsf{Lk}(v)}$ satisfies equation (4.1) or equation (4.2).

If equation (4.2) holds, then since the support of $\widetilde{b}$ is in the centralizing sets of the supports of all of the $u_i$, we have

$$w_2 =_{G_V} d\widetilde{b}(c_j u_{j+1} c_{j+1} \cdots u_n c_n u_1 c_1 \cdots u_j) d^{-1},$$

and then Lemma 4.2 (1) says that

$$w_2 =_{G_V} (d\widetilde{b}\widehat{d}) c_j \widehat{u}_{j+1} c_{j+1} \cdots \widehat{u}_n c_n \widehat{u}_1 c_1 \cdots \widehat{u}_j$$

for a piecewise subword $\widehat{d}$ of $d^{-1}$ and elements $\widehat{u}_1, \ldots, \widehat{u}_n \in \widehat{U}$. Let $\widehat{b}$ be the shortlex least representative of $d\widetilde{b}\widehat{d}$. Then the normal form theorem for amalgamated products says that $\widetilde{b}' = \widehat{b}$ and $u'_1 = c_j$ is the first coset representative in the two representations of $w_2$. However, this contradicts the fact that $u'_1 \in \widehat{U} \setminus \{\lambda\}$ and $c_j \in \mathsf{SL}(G_v, X_v) \setminus \{\lambda\}$, since these sets are disjoint. Hence equation (4.1) must hold.

We now claim that $\mathsf{Supp}(d) \subseteq \mathsf{Lk}(\bigcup_{i=1}^n \mathsf{Supp}(u_i))$. To prove this claim, we suppose to the contrary that this containment does not hold. Again using the fact that $\mathsf{Supp}(\widetilde{b}') \subseteq \mathsf{Lk}(\bigcup_{i=1}^n \mathsf{Supp}(u_i))$ and Lemma 4.2 (1), we have

$$w_2 =_{G_V} (d\widehat{b}\widehat{d}) \widehat{u}_{j+1} c_{j+1} \cdots \widehat{u}_n c_n \widehat{u}_1 c_1 \cdots \widehat{u}_j c_j$$

for a proper piecewise subword $\widehat{d}$ of $d^{-1}$ and elements $\widehat{u}_1, \ldots, \widehat{u}_n \in \widehat{U}$. Note that $|\widehat{d}| < |d|$. Let $\widehat{b}$ be the element of $\mathsf{SL}(G_{\mathsf{Lk}(v)}, X_{\mathsf{Lk}(v)})$ representing $d\widetilde{b}\widehat{d}$. Now the normal form theorem for amalgamated products says that $n = n'$, $u'_i = \widehat{u}_{i+j}$ and $c'_i = c_{i+j}$ for all $i$ (where the indices are considered modulo $n$), and $\widetilde{b}' = \widehat{b}$. Moreover, since $w_2$ is in the form (†), we have $\mathsf{Supp}(\widehat{b}) \subseteq \mathsf{Lk}(\bigcup_{i=1}^n \mathsf{Supp}(\widehat{u}_i))$. Hence

$$\begin{aligned} w_2 &=_{G_V} \widehat{u}_{j+1} c_{j+1} \cdots \widehat{u}_n c_n \widehat{u}_1 c_1 \cdots \widehat{u}_j c_j \widehat{b} \\ &=_{G_V} \widehat{u}_{j+1} c_{j+1} \cdots \widehat{u}_n c_n \widehat{u}_1 c_1 \cdots \widehat{u}_j c_j (d\widetilde{b}\widehat{d}) \\ &=_{G_V} \widehat{d}^{-1} (u_{j+1} c_{j+1} \cdots u_n c_n \widetilde{b} u_1 c_1 \cdots u_j c_j) \widehat{d}, \end{aligned}$$

and so $\widehat{d}^{-1}$ is a shorter word satisfying equation (4.1), giving the required contradiction.

Now since $\mathsf{Supp}(d) \subseteq \mathsf{Lk}(\bigcup_{i=1}^n \mathsf{Supp}(u_i))$, then

$$w_2 = (d\widetilde{b}d^{-1}) u_{j+1} c_{j+1} \cdots u_n c_n u_1 c_1 \cdots u_j c_j,$$

and the normal form theorem for amalgamated products says that $n = n'$, $u'_i = u_{i+j}$ and $c'_i = c_{i+j}$ for all $i$ (where the indices are considered modulo $n$), and $\widetilde{b}' =_{G_V} d\widetilde{b}d^{-1}$. Since both $\widetilde{b}$ and $\widetilde{b}'$ are in $\mathsf{ConjSL}(G_{\mathsf{Lk}(v)}, X_{\mathsf{Lk}(v)})$ and represent conjugate elements of $G_{\mathsf{Lk}(v)}$, then $\widetilde{b} = \widetilde{b}'$ as well. ∎

### 4.2. Equality of the standard and conjugacy growth rates

In this section we show in Theorem B that the class of groups for which the standard and conjugacy growth rates are equal is closed with respect to the graph product construction.

Recall that $\sigma_{(G,X)}(z)$ and $\widetilde{\sigma}_{(G,X)}(z)$ denote the spherical growth series and spherical conjugacy growth series, respectively, for a group $G$ with respect to a generating set $X$.

**Notation 4.4.** Let $G_V$ be a graph product and assume that every vertex group $G_v$ has an inverse-closed generating set $X_v$. For each $V' \subseteq V$, let $X_{V'} := \bigcup_{v \in V'} X_v$ and write

$$\sigma_{V'}(z) := \sigma_{(G_{V'}, X_{V'})}(z) \quad \text{and} \quad \widetilde{\sigma}_{V'}(z) := \widetilde{\sigma}_{(G_{V'}, X_{V'})}(z).$$

We begin with a corollary of Corollary 3.14.

**Corollary 4.5.** *Let $G_V$ be a graph product group over a graph with vertex set $V$, and let $v \in V$ be a vertex. For each $v' \in V$ let $X_{v'}$ be an inverse-closed generating set for the vertex group $G_{v'}$, and let $X_V = \bigcup_{v' \in V} X_{v'}$.*

*Let $U = U_{G_{\mathsf{Lk}(v)} \backslash G_{V \setminus \{v\}}}$ be the admissible right transversal for $G_{\mathsf{Lk}(v)}$ in $G_{V\setminus\{v\}}$ with respect to the pair of generating sets $(X_{V\setminus\{v\}}, X_{\mathsf{Lk}(v)})$ from Lemma 3.15, and let $\sigma_U$ be the strict growth series of the elements of $U$ with respect to $X_V$. Using Notation 4.4, then*

$$\sigma_V = \sigma_{\mathsf{Lk}(v)} \frac{\sigma_U \sigma_{\{v\}}}{\sigma_{\{v\}} + \sigma_U - \sigma_U \sigma_{\{v\}}}.$$

*Moreover, the radius of convergence of $\sigma_V$ satisfies*

$$\mathsf{RC}(\sigma_V) = \min \big\{ \mathsf{RC}(\sigma_{\mathsf{Lk}(v)}), \mathsf{RC}(\sigma_U), \mathsf{RC}(\sigma_{\{v\}}), \\ \inf\{|z| : \sigma_{\{v\}}(z) + \sigma_U(z) - \sigma_U(z)\sigma_{\{v\}}(z) = 0\} \big\}.$$

*Proof.* If $V = \mathsf{Lk}(v) \cup \{v\}$, then $G_V = G_v \times G_{\mathsf{Lk}(v)}$ and $U = \{\varepsilon\}$. From Remark 3.11, the spherical growth series of a direct product of groups is the product of the spherical growth series of the factors, and so in this case we have $\sigma_U = 1$ and $\sigma_V = \sigma_{\{v\}} \sigma_{\mathsf{Lk}(v)}$, as required.

Next assume that $V \neq \mathsf{Lk}(v) \cup \{v\}$ and so $U_{\mathsf{Lk}(v) \backslash (V \setminus \{v\})} \neq \{\varepsilon\}$. Note from Remark 3.11 that $\sigma_{V \setminus \{v\}} = \sigma_{\mathsf{Lk}(v)} \sigma_U$. Corollary 3.14 and Lemma 3.15 give the required equality between the series. Since the radius of convergence of a product is the minimum of the radii of convergence of the factors, we obtain the claim about $\mathsf{RC}(\sigma_V)$. ∎

**Remark 4.6.** Recall (equation (2.1) in Section 2.2) that the exponential growth rate of the growth series of a language $L$ over a finite set $X$ is the reciprocal of the radius of convergence of the series; that is,

$$\mathsf{gr}_L = 1/\mathsf{RC}(F_L).$$

Thus for a group $G$ with generating set $X$ the spherical and spherical conjugacy growth rates can be computed from the radii of convergence of the corresponding growth series by $\rho = 1/\mathsf{RC}(\sigma)$ and $\widetilde{\rho} = 1/\mathsf{RC}(\widetilde{\sigma})$.

**Proposition 4.7.** *Let $G_V$ be a graph product. For any set of vertices $V' \subseteq V$, the spherical conjugacy growth rates satisfy the inequality $\widetilde{\rho}(G_{V'}, X_{V'}) \leq \widetilde{\rho}(G_V, X_V)$, and the radii of convergence satisfy $\mathsf{RC}(\widetilde{\sigma}_V) \leq \mathsf{RC}(\widetilde{\sigma}_{V'})$.*

*Proof.* Let $<_{\mathrm{sl}}$ be a shortlex ordering on $X_V^*$ that is compatible with a total ordering $\ll$ on $V$ satisfying $v' < v$ for all $v' \in V'$ and $v \in V \setminus V'$, and let the shortlex ordering on $X_{V'}^*$ be the restriction of the shortlex ordering on $X_V^*$. From Proposition 3.7, we have $\mathsf{ConjSL}(G_V, X_V) \cap X_{V'}^* = \mathsf{ConjSL}(G_{V'}, X_{V'})$, and in particular $\mathsf{ConjSL}(G_{V'}, X_{V'}) \subseteq \mathsf{ConjSL}(G_V, X_V)$. This implies the inequality on exponential growth rates. Then Remark 4.6 gives the inequality for the radii of convergence. ■

We are now ready to complete the proof of Theorem B, restated here with the notation from this section.

**Theorem B.** *Let $G_V$ be a graph product group over a graph with vertex set $V$ and assume that for each vertex $v \in V$ the spherical and spherical conjugacy growth rates of $G_v$ are equal; that is, $\rho(G_v, X_v) = \widetilde{\rho}(G_v, X_v)$ for all $v \in V$. Then*

$$\rho(G_V, X_V) = \widetilde{\rho}(G_V, X_V)$$

*and hence also $\mathsf{RC}(\sigma_V) = \mathsf{RC}(\widetilde{\sigma}_V)$.*

*Proof.* Note that Remark 4.6 shows that the equality for the two growth rates follows from equality of the two radii of convergence, and vice versa. The proof is by induction on the number of vertices $|V|$. If $|V| = 1$, the result is part of the hypothesis. So assume $|V| \geq 2$.

Suppose that the graph $\Gamma$ underlying the graph product is complete. Then $G_V$ is the direct product of the vertex groups and the spherical and spherical conjugacy growth series satisfy

$$\sigma_{(G_V,X_V)}(z) = \prod_{v \in V} \sigma_{(G_v,X_v)}(z) \quad \text{and} \quad \widetilde{\sigma}_{(G_V,X_V)}(z) = \prod_{v \in V} \widetilde{\sigma}_{(G_v,X_v)}(z),$$

so the radius of convergence of this product is the minimum of the radii of convergence of the factors; thus $\rho(G_V, X_V) = \max\{\rho(G_v, X_v) \mid v \in V\}$, and $\widetilde{\rho}(G_V, X_V) = \max\{\widetilde{\rho}(G_v, X_v) \mid v \in V\}$. Hence $\mathsf{RC}(\sigma_V) = \mathsf{RC}(\widetilde{\sigma}_V)$ and $\rho(G_V, X_V) = \widetilde{\rho}(G_V, X_V)$ in this direct product case.

For the remainder of this proof we assume that there are vertices $v, v' \in V$ such that $v$ and $v'$ are not connected by an edge. By the induction hypothesis and Proposition 4.7, we have

$$\mathsf{RC}(\widetilde{\sigma}_V) \leq \mathsf{RC}(\widetilde{\sigma}_{\mathsf{Lk}(v)}) = \mathsf{RC}(\sigma_{\mathsf{Lk}(v)}). \tag{4.3}$$

Also by induction $\mathsf{RC}(\sigma_{\{v\}}) = \mathsf{RC}(\widetilde{\sigma}_{\{v\}})$, and so by Proposition 4.7 we have

$$\mathsf{RC}(\widetilde{\sigma}_V) \leq \mathsf{RC}(\sigma_{\{v\}}). \tag{4.4}$$

Let $<_{\mathrm{sl}}$ be a shortlex ordering on $X_V^*$ that is compatible with an ordering $\ll$ on $V$ satisfying $x \ll y$ for all $x \in \mathsf{Lk}(v)$ and $y \in V \setminus \mathsf{Lk}(v)$. Let $\widehat{U} := \widehat{U}_{G_{\mathsf{Lk}(v)} \setminus G_{V \setminus \{v\}}}$ be the set of representatives for an admissible transversal $U$ of $G_{\mathsf{Lk}(v)}$ in $G_{V\setminus\{v\}}$ with respect to $(X_{V\setminus\{v\}}, X_{\mathsf{Lk}(v)})$ defined in Lemma 3.15. Since $\widehat{U} \subset \mathsf{SL}(G_V, X_V)$, the growth series satisfy $\sigma_U = F_{\widehat{U}}$.

Fix an element $d \in \mathsf{SL}(G_v, X_v)$ of length 1, and consider the language $L = \{ud \mid u \in \widehat{U} \setminus \{\lambda\}\}$. Proposition 4.3 shows that distinct elements of $L$ represent distinct conjugacy classes. Hence the elements of $L$ of length $m$ are in bijection with the set of conjugacy classes in $G_V$ represented by words in $L$ of length $m$; since Proposition 4.3 also shows that the words in $L$ are conjugacy geodesics, then the representatives in $\mathsf{ConjSL}(G_V, X_V)$ of these conjugacy classes also have length $m$. Hence the strict growth functions satisfy $\theta_{\mathsf{ConjSL}(G_V,X_V)}(m) \geq \theta_L(m) = \theta_{\widehat{U}}(m-1)$ for all $m > 1$, and so the radii of convergence satisfy

$$\mathsf{RC}(\widetilde{\sigma}_V) \leq \mathsf{RC}(\sigma_U). \tag{4.5}$$

Similarly, consider the language $L = \{uc \mid u \in \widehat{U} \setminus \{\lambda\}, c \in \mathsf{SL}(G_v, X_v) \setminus \{\lambda\}\}$. Proposition 4.3 shows that the elements of $\mathsf{Necklaces}(L)$ of length $m$ are in bijection with the conjugacy classes in $G_V$ represented by words of the form $u_1c_1 \cdots u_nc_n$ of length $m$, where each $u_i \in \widehat{U} \setminus \{\lambda\}$ and $c_i \in \mathsf{SL}(G_v, X_v) \setminus \{\lambda\}$, and Proposition 4.3 shows that these words are also conjugacy geodesics. Hence the strict growth functions satisfy

$$\theta_{\mathsf{ConjSL}(G_V,X_V)}(m) \geq \theta_{\mathsf{Necklaces}(L)}(m)$$

for all $m \geq 1$, and therefore

$$\mathsf{RC}(\widetilde{\sigma}_V) \leq \mathsf{RC}(F_{\mathsf{Necklaces}(L)}).$$

By Corollary 2.6, $\mathsf{RC}(F_{\mathsf{Necklaces}(L)})$ is $\inf\{|z| : z \in \mathbb{C}, |F_L(z)| = 1\}$, and the growth series of $L$ in this case is $F_L(z) = (\sigma_U(z) - 1)(\sigma_{\{v\}}(z) - 1)$. Since $F_L(z) = 1$ if and only if $\sigma_{\{v\}}(z) + \sigma_U(z) - \sigma_U(z)\sigma_{\{v\}}(z) = 0$, this yields

$$\mathsf{RC}(\widetilde{\sigma}_V) \leq \inf\{|z| : z \in \mathbb{C}, \sigma_{\{v\}}(z) + \sigma_U(z) - \sigma_U(z)\sigma_{\{v\}}(z) = 0\}. \tag{4.6}$$

In combination with inequalities (4.3), (4.4), (4.5), and (4.6) above, Corollary 4.5 shows that $\mathsf{RC}(\widetilde{\sigma}_V) \leq \mathsf{RC}(\sigma_V)$.

On the other hand, since in any group the number of conjugacy classes represented by a conjugacy geodesic of a given length is at most the number of group elements of that length, $\mathsf{RC}(\widetilde{\sigma}_V) \geq \mathsf{RC}(\sigma_V)$, yielding the equality of the two radii of convergence. ∎

The following result of Gekhtman and Yang [11, Corollary 1.2] is also an immediate consequence of Theorem B.

**Corollary 4.8.** *Let $G$ be a right-angled Artin or Coxeter group; that is, a graph product in which the vertex groups are cyclic of infinite order or of order 2, respectively. Then for the Artin or Coxeter generating set, respectively, the spherical conjugacy growth rate of $G$ is the same as the spherical growth rate of $G$.*

### 4.3. The conjugacy growth series formula

In this section we prove Theorem A, giving a recursive formula for the spherical conjugacy growth series $\widetilde{\sigma}_V$ of a graph product group $G_V$ in terms of the spherical conjugacy and spherical growth series $\widetilde{\sigma}_{V'}$ and $\sigma_{V'}$ for the subgraph products $G_{V'}$ where $V' \subsetneq V$.

We begin with an application of the inclusion-exclusion principle. Given a graph product group $G_V$ on a graph with vertex set $V$, we view $\widetilde{\sigma}$ as a function $\widetilde{\sigma}\colon \mathcal{P}(V) \to \mathbb{Z}[[z]]$ to the ring of formal power series, where $\widetilde{\sigma}_S = \widetilde{\sigma}_{(G_S, X_S)}$ is the evaluation of $\widetilde{\sigma}$ at the subset $S \subset V$. Recall from Proposition 3.7 that for each $S \subseteq V$ the spherical conjugacy growth series $\widetilde{\sigma}_S$ is the growth series of the language $\mathsf{ConjSL}(G_S, X_S) = \mathsf{ConjSL}(G_V, X_V) \cap X_S^*$; hence the series $\widetilde{\sigma}_S$ is also the contribution in $\widetilde{\sigma}_V$ of the conjugacy classes having shortlex conjugacy representative with support contained in $S$.

Define $f\colon \mathcal{P}(V) \to \mathbb{Z}[[z]]$ by setting $f(T)$ to be the contribution in $\widetilde{\sigma}_V$ of the conjugacy classes having shortlex conjugacy representative with support exactly $T$. Then for any subset $S \subseteq V$, we have $\widetilde{\sigma}_S = \sum_{S' \subseteq S} f(S')$. Now the Möbius inversion principle (an extension of the principle of inclusion-exclusion; see, for example, [21, Example 3.8.3], [13, (3.1.2)]) says that $f(S) = \widetilde{\sigma}_S^{\mathcal{M}}$, where $\widetilde{\sigma}_S^{\mathcal{M}} := \sum_{S' \subseteq S} (-1)^{|S|-|S'|} \widetilde{\sigma}_{S'}$ is the function that is the *Möbius inverse* of $f$, yielding the following.

**Lemma 4.9.** *Let $G_V$ be a graph product with generating set $X_V$ and let $S \subseteq V$. Let $<_{\mathrm{sl}}$ be a shortlex ordering on $X_V^*$ compatible with a total ordering on $V$. The contribution in $\widetilde{\sigma}_V$ of the conjugacy classes having shortlex conjugacy representative with support exactly $S$ is given by*

$$\widetilde{\sigma}_S^{\mathcal{M}} = \sum_{S' \subseteq S} (-1)^{|S|-|S'|} \widetilde{\sigma}_{S'}.$$

Recall from Definition 2.8 that

$$\mathsf{N}(f)(z) := \sum_{k=1}^{\infty} \sum_{l=1}^{\infty} \frac{\phi(k)}{kl} (f(z^k))^l = \sum_{k=1}^{\infty} \frac{-\phi(k)}{k} \log(1 - f(z^k))$$

for any complex power series $f$ with integer coefficients satisfying $[z^0] f(z) = 0$, and recall from Proposition 2.4 that the function $\mathsf{N}$ maps the growth series of a language $L$ to the growth series of the necklace language $\mathsf{Necklaces}(L)$.

The following paraphrased statement of Theorem A (using the notation above) provides a recursive formula for computing the conjugacy growth series of a graph product.

**Theorem A.** *Let $G_V$ be a graph product group over a graph with vertex set $V$ and let $v \in V$ be a vertex. Then the conjugacy growth series of $G_V$ is given by*

$$\widetilde{\sigma}_V = \widetilde{\sigma}_{V \setminus \{v\}} + \widetilde{\sigma}_{\mathsf{Lk}(v)}(\widetilde{\sigma}_{\{v\}} - 1) + \sum_{S \subseteq \mathsf{Lk}(v)} \widetilde{\sigma}_S^{\mathcal{M}} \mathsf{N}\Big(\Big(\frac{\sigma_{\mathsf{Lk}(S) \setminus \{v\}}}{\sigma_{\mathsf{Lk}(v) \cap \mathsf{Lk}(S)}} - 1\Big)(\sigma_{\{v\}} - 1)\Big).$$

*Moreover, if $\{v\} \cup \mathsf{Lk}(v) = V$, then $\widetilde{\sigma}_V = \widetilde{\sigma}_{\mathsf{Lk}(v)} \widetilde{\sigma}_{\{v\}}$.*

*Proof.* In the case that $\{v\} \cup \mathsf{Lk}(v) = V$, the graph product group is a direct product $G_V = G_{\mathsf{Lk}(v)} \times G_v$, and so the conjugacy growth series for $G_V$ is the product of the corresponding series for the factors [6, Proposition 2.1]. Since $\mathsf{Lk}(v) = V \setminus \{v\}$, then the sets $\mathsf{Lk}(v) \cap \mathsf{Lk}(S)$ and $\mathsf{Lk}(S) \setminus \{v\}$ are equal, and so $\mathsf{N}((\frac{\sigma_{\mathsf{Lk}(S)\setminus\{v\}}}{\sigma_{\mathsf{Lk}(v)\cap\mathsf{Lk}(S)}} - 1)(\sigma_{\{v\}} - 1)) = \mathsf{N}(0) = 0$. Hence the theorem holds in this case.

For the remainder of this proof $\{v\} \cup \mathsf{Lk}(v) \neq V$. Let $<_{\mathrm{sl}}$ be a shortlex ordering on $X_V$ compatible with an ordering $\ll$ on $V$ satisfying $x \ll y \ll v$ for all $x \in \mathsf{Lk}(v)$ and $y \in V \setminus (\{v\} \cup \mathsf{Lk}(v))$. Let $\widehat{U} := \widehat{U}_{G_{\mathsf{Lk}(v)} \setminus G_{V\setminus\{v\}}}$ be the set of representatives for the admissible transversal $U$ for $G_{\mathsf{Lk}(v)}$ in $G_{V\setminus\{v\}}$ with respect to $(X_{V\setminus\{v\}}, X_{\mathsf{Lk}(v)})$ from Lemma 3.15. Propositions 4.3 and 3.7, together with the fact that the shortlex conjugacy normal form set for the direct product $G_{\mathsf{Lk}(v)} \times G_v$ is the concatenation of the shortlex conjugacy normal form sets for the two factor groups, show that $\widetilde{\sigma}_V$ is equal to the growth series of the language

$$\mathsf{ConjSL}(G_{V\setminus\{v\}}, X_{V\setminus\{v\}}) \sqcup \mathsf{ConjSL}(G_{\mathsf{Lk}(v)}, X_{\mathsf{Lk}(v)})[\mathsf{ConjSL}(G_v, X_v) \setminus \{\lambda\}] \sqcup L_\dagger$$

over $X_V$, where the language $L_\dagger$ is a set of conjugacy class representatives containing exactly one word of the form (†), as defined in Proposition 4.3, for each equivalence class with respect to the equivalence in Proposition 4.3 (2). (Note that although we have not shown that the words in $L_\dagger$ are in $\mathsf{ConjSL}(G_V, X_V)$, Proposition 4.3 shows that they are conjugacy geodesic representatives for their conjugacy classes.) Hence $\widetilde{\sigma}_V = \widetilde{\sigma}_{V\setminus\{v\}} + \widetilde{\sigma}_{\mathsf{Lk}(v)}(\widetilde{\sigma}_{\{v\}} - 1) + F_{L_\dagger}$, where $F_{L_\dagger}$ is the growth series of the language $L_\dagger$.

Using Proposition 4.3 (2), and the concept of necklaces from Section 2.3, the growth series of $L_\dagger$ equals the growth series of the disjoint union

$$\bigsqcup_{S\subseteq \mathsf{Lk}(v)} \{b \in \mathsf{ConjSL}(G_{\mathsf{Lk}(v)}, X_{\mathsf{Lk}(v)}) \colon \mathsf{Supp}(b) = S\} \times \mathsf{Necklaces}(\widehat{U}_S\ C),$$

where $\widehat{U}_S := \widehat{U} \cap X^*_{\mathsf{Lk}(S)} \setminus \{\lambda\}$ is the set of nonempty words in $\widehat{U}$ whose support is contained in $\mathsf{Lk}(S)$, and $C = \mathsf{SL}(G_v, X_v) \setminus \{\lambda\}$.

The growth series of $\{b \in \mathsf{ConjSL}(G_{\mathsf{Lk}(v)}, X_{\mathsf{Lk}(v)}) \colon \mathsf{Supp}(b) = S\}$ is given by $\widetilde{\sigma}_S^{\mathcal{M}}$, from Lemma 4.9 and Proposition 3.7. The growth series of the set $C = \mathsf{SL}(G_v, X_v) \setminus \{\lambda\}$ is $\sigma_v - 1$.

By the definition of $\widehat{U}$ from Lemma 3.15, we obtain

$$\begin{aligned}\widehat{U}_S &= (\mathsf{SL}(G_{V\setminus\{v\}}, X_{V\setminus\{v\}}) \cap X_{(V\setminus\{v\})\setminus \mathsf{Lk}(v))} X^*_{V\setminus\{v\}}) \cap X^*_{\mathsf{Lk}(S)} \\ &= \mathsf{SL}(G_{\mathsf{Lk}(S)\setminus\{v\}}, X_{\mathsf{Lk}(S)\setminus\{v\}}) \cap X_{(\mathsf{Lk}(S)\setminus\{v\})\setminus(\mathsf{Lk}(v)\cap \mathsf{Lk}(S))} X^*_{\mathsf{Lk}(S)\setminus\{v\}},\end{aligned}$$

where the second equality follows from Proposition 3.7. Now Lemma 3.15 shows that $\widehat{U}_S \cup \{\lambda\}$ is a set of shortlex representatives of the admissible transversal for the subgroup $G_{\mathsf{Lk}(v)\cap \mathsf{Lk}(S)}$ in $G_{\mathsf{Lk}(S)\setminus\{v\}}$ with respect to $(X_{\mathsf{Lk}(S)\setminus\{v\}}, X_{\mathsf{Lk}(v)\cap\mathsf{Lk}(S)})$. Following the same counting argument as in Remark 3.11, admissibility of this transversal implies that the concatenation $\mathsf{SL}(G_{\mathsf{Lk}(v)\cap\mathsf{Lk}(S)}, X_{\mathsf{Lk}(v)\cap\mathsf{Lk}(S)})(\widehat{U}_S \cup \{\lambda\})$ is a set of (unique) geodesic representatives for the elements of $G_{\mathsf{Lk}(S)\setminus\{v\}}$ over $X_{\mathsf{Lk}(S)\setminus\{v\}}$, and so

$$F_{\widehat{U}_S} = \frac{\sigma_{\mathsf{Lk}(S)\setminus\{v\}}}{\sigma_{\mathsf{Lk}(v)\cap\mathsf{Lk}(S)}} - 1$$

(where as usual $F_{\widehat{U}_S}$ is the growth series of the language $\widehat{U}_S$).

In view of the growth series formula for necklaces in Proposition 2.4 and Definition 2.8, the contribution of $F_{L_\dagger}$ to $\widetilde{\sigma}_V$ is

$$\sum_{S\subseteq\mathsf{Lk}(v)} \widetilde{\sigma}_S^{\mathcal{M}}\mathsf{N}\Big(\Big(\frac{\sigma_{\mathsf{Lk}(S)\setminus\{v\}}}{\sigma_{\mathsf{Lk}(v)\cap\mathsf{Lk}(S)}} - 1\Big)(\sigma_{\{v\}} - 1)\Big). \qquad \blacksquare$$

We end this section with an example application of Theorems A and B to a right-angled Coxeter group.

**Example 4.10.** Suppose that $\Gamma = (V, E)$ is the finite simple graph with vertex set $V = \{v_1, v_2, v_3, v_4\}$ and edge set $E = \{\{v_1, v_2\}, \{v_2, v_3\}, \{v_3, v_4\}\}$; that is, $\Gamma$ is a path graph made up of 3 edges. For each $1 \le i \le 4$ let $G_{v_i} = \langle a_i \mid a_i^2 = 1\rangle$ be a cyclic group of order 2 with inverse-closed generating set $X_{v_i} = \{a_i\}$.

We compute spherical and spherical conjugacy growth series for several (virtually cyclic) subgraph products directly. For each vertex group $G_{v_i}$ the growth series satisfy $\sigma_{\{v_i\}} = \widetilde{\sigma}_{\{v_i\}} = 1 + z$. The subgraph product $G_\emptyset$ is the trivial group with $\sigma_\emptyset = 1$. The group $G_{V\setminus\{v_1\}}$ is the direct product of $G_{v_3}$ with the infinite dihedral group $G_{\{v_2,v_4\}}$, and so the growth series satisfy $\sigma_{V\setminus\{v_1\}} = \sigma_{\{v_2,v_4\}}\sigma_{\{v_3\}}$ and $\widetilde{\sigma}_{V\setminus\{v_1\}} = \widetilde{\sigma}_{\{v_2,v_4\}}\widetilde{\sigma}_{\{v_3\}}$. The series for the dihedral group are

$$\sigma_{\{v_2,v_4\}}(z) = 1 + \frac{2z}{1-z} = \frac{1+z}{1-z} \quad\text{and}\quad \widetilde{\sigma}_{\{v_2,v_4\}}(z) = \frac{1+2z-2z^3}{1-z^2}.$$

We apply Corollary 3.14 with the choice of vertex $v = v_1$. Using the fact that $\mathsf{Lk}(v) = \mathsf{Lk}(v_1) = \{v_2\}$, we have

$$\begin{aligned}\sigma_V &= \frac{\sigma_{\{v_2\}}\sigma_{V\setminus\{v_1\}}\sigma_{\{v_1\}}}{\sigma_{\{v_2\}}\sigma_{\{v_1\}} + \sigma_{V\setminus\{v_1\}} - \sigma_{V\setminus\{v_1\}}\sigma_{\{v_1\}}} \\ &= \frac{(1+z)\big(\big(\frac{1+z}{1-z}\big)(1+z)\big)(1+z)}{(1+z)^2 + \frac{(1+z)^2}{1-z} - \big(\frac{(1+z)^2}{1-z}\big)(1+z)} = \frac{(1+z)^2}{1-2z}.\end{aligned}$$

Now Theorem B (or Corollary 4.8) says that the radius of convergence of the spherical conjugacy growth series is $\mathsf{RC}(\widetilde{\sigma}_V) = \mathsf{RC}(\sigma_V) = \frac{1}{2}$, and so the spherical conjugacy growth rate is $\widetilde{\rho}(G_V, X_V) = \rho(G_V, X_V) = 2$.

To obtain an exact formula for $\widetilde{\sigma}_V$, we apply Theorem A with $v = v_1$. Since $\mathsf{Lk}(v) = \{v_2\}$, $\mathsf{Lk}(\emptyset) = V$, and $\mathsf{Lk}(v_2) = \{v_1, v_3\}$, we have

$$\begin{aligned}\widetilde{\sigma}_V &= \widetilde{\sigma}_{V\setminus\{v_1\}} + \widetilde{\sigma}_{\{v_2\}}(\widetilde{\sigma}_{\{v_1\}} - 1) + \widetilde{\sigma}_\emptyset^{\mathcal{M}}\mathsf{N}\Big(\Big(\frac{\sigma_{V\setminus\{v_1\}}}{\sigma_{\{v_2\}\cap V}} - 1\Big)(\sigma_{\{v_1\}} - 1)\Big) \\ &\quad + \widetilde{\sigma}_{\{v_2\}}^{\mathcal{M}}\mathsf{N}\Big(\Big(\frac{\sigma_{\{v_1,v_3\}\setminus\{v_1\}}}{\sigma_{\{v_2\}\cap\{v_1,v_3\}}} - 1\Big)(\sigma_{\{v_1\}} - 1)\Big).\end{aligned}$$

Computing the Möbius inverses gives

$$\widetilde{\sigma}_{\emptyset}^{\mathcal{M}} = (-1)^{0-0}\widetilde{\sigma}_{\emptyset} = 1, \quad \widetilde{\sigma}_{\{v_2\}}^{\mathcal{M}} = (-1)^{1-1}\widetilde{\sigma}_{\{v_2\}} + (-1)^{1-0}\widetilde{\sigma}_{\emptyset} = 1 + z - 1 = z.$$

Plugging these and the series for the subgraph products into the expression for $\widetilde{\sigma}_V$ above yields

$$\widetilde{\sigma}_V = \Big(\frac{1+2z-2z^3}{1-z^2}\Big)(1+z) + (1+z)z + \mathsf{N}\Big(\Big(\frac{2z}{1-z}\Big)z\Big) + z\mathsf{N}(z^2).$$

Now Example 2.9 says that $\mathsf{N}(z^2) = \frac{z^2}{1-z^2}$, and so this simplifies to

$$\widetilde{\sigma}_V = \Big(\frac{1+4z+3z^2-2z^3-3z^4}{1-z^2}\Big) + \mathsf{N}\Big(\frac{2z^2}{1-z}\Big).$$

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