On continuous orbit equivalence rigidity for virtually cyclic group actions

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Abstract. We prove that any two continuous minimal (topologically free) actions of the infinite dihedral group on an infinite compact Hausdorff space are continuously orbit equivalent only if they are conjugate. We also show the above fails if we replace the infinite dihedral group by certain other virtually cyclic groups, e.g., the direct product of the integer group with any non-abelian finite simple group.

1. Introduction

Let G be a countable discrete group and X be a compact Hausdorff space. Denote by $\mathcal{C}$ a class of continuous actions of G on X. A natural and classical topic is to classify elements of $\mathcal{C}$ up to certain equivalence relation. In classical topological dynamical systems, the classification of $\mathbb{Z}$-actions, or amenable group actions in general, up to conjugacy, has been studied intensively. Historically, the notion of entropy was invented to distinguish Bernoulli shifts on finite bases with different sizes up to conjugacy (at first, in the measurable setting). Over the last decade, more and more attention has been put on a much wider class of acting groups, sofic groups, after the pioneering work of L. Bowen on sofic entropy in the measurable setting [3] and D. Kerr and H. Li shortly in the topological setting [14].

In contrast to the conjugacy relation, a systematic study of the notion of continuous orbit equivalence (see Definition 2.1) for topological free actions of a general group G [19] was initiated by X. Li during his study of crossed product C^*-algebras. Note that this notion and its weaker versions, e.g., topological orbit equivalence have been studied in special cases (see [4,10,27]) before Li's work. In particular, Giordano–Putnam–Skau [10] and Giordano–Matui–Putnam–Skau [8, 9] have proved a series of remarkable results on classification of minimal $\mathbb{Z}$-actions on the Cantor set up to topological orbit equivalence.

Although continuous orbit equivalence relation is, by definition, a priori weaker than the conjugacy relation, an interesting phenomenon is that these two notions may coincide in certain cases. To better describe this, say that a continuous action α of G on X is *continuous orbit equivalence rigid for $\mathcal{C}$* if for any continuous action $\beta \in \mathcal{C}$, α is continuously

orbit equivalent to β, implies that they are conjugate, denoted by $\alpha \overset{coe}{\sim} \beta \Rightarrow \alpha \overset{conj}{\sim} \beta$. Note that here, unlike in [19, Definition 2.5], we further assume the acting groups are the same.

Now, we take $\mathcal{C}$ to be the class of minimal topologically free actions and explain two motivations behind this work. First, continuous orbit equivalence rigidity has been established for various group actions, see, e.g., [4, 5, 7, 19, 25, 26]. However, as far as we know, the acting groups are assumed to be torsion-free in all written proofs (see also Remarks 2.5 and 4.7). Thus it is natural to ask whether continuous orbit equivalence rigidity can be established when the acting groups contain non-trivial torsion elements. Second, a well-known result due to Boyle–Tomiyama [4] says that two topologically transitive, topologically free $\mathbb{Z}$-actions (e.g., minimal free $\mathbb{Z}$-actions) are continuously orbit equivalent only if they are conjugate. This result has been extended to minimal equicontinuous actions of $\mathbb{Z}^d$ ($d \geq 2$) in [7]. More precisely, it is shown there that for minimal equicontinuous $\mathbb{Z}^d$-systems, continuous orbit equivalence implies that the systems are virtually piecewise conjugate. Moreover, actions of virtually cyclic groups have recently been considered in the study of dynamic asymptotic dimension [1] and Matui's HK conjecture [22, 23]. Thus, it is natural to study the analogue of Boyle–Tomiyama's above mentioned result for virtually cyclic group actions, i.e., to ask the following question.

Question 1.1. Can we establish continuous orbit equivalence rigidity for minimal topologically free actions of infinite virtually cyclic (but non-cyclic) groups?

In general, we could not hope continuous orbit equivalence rigidity holds true for all infinite virtually cyclic groups. Indeed, we have the following theorem.

Theorem 1.2. *Let F be any non-trivial finite group with trivial center, e.g., a non-abelian finite simple group. Then there exist two minimal free actions $\widetilde{\alpha}$ and $\widetilde{\alpha}'$ of $\mathbb{Z} \times F$ on an infinite compact Hausdorff space X such that $\widetilde{\alpha} \overset{coe}{\sim} \widetilde{\alpha}'$ but $\widetilde{\alpha} \overset{conj}{\nsim} \widetilde{\alpha}'$.*

On the positive side, when dealing with the infinite dihedral group, we do have such a rigidity theorem.

Theorem 1.3. *Let α and β be any two minimal (topologically free) actions of the infinite dihedral group $D_\infty = \mathbb{Z} \rtimes \mathbb{Z}/2\mathbb{Z}$ on an infinite compact Hausdorff space X. Then $\alpha \overset{coe}{\sim} \beta \Rightarrow \alpha \overset{conj}{\sim} \beta$.*

Note that minimal actions of D_∞ on an infinite compact Hausdorff space are automatically topologically free, see Proposition 2.6. Moreover, under the assumptions of the above theorem, we know that continuous orbit equivalence implies that the two actions have the same topological entropy. This can be compared with [17, Theorem 7.5] and [16, Theorem D], where it is shown that many non-virtually cyclic groups admit actions for which topological entropy is an invariant of continuous orbit equivalence. For possible generalizations of the theorems to other virtually cyclic groups, see the discussion in Section 5. Here, we are content not to discuss a general version of the above theorem. Below, we briefly describe strategy for the proofs.

To prove Theorem 1.2, we use the skew product construction to reduce it to a problem of constructing minimal actions admitting continuous cocycles into finite groups which are not coboundaries. On the positive side, one may compare the proof of Theorem 1.3 with the proof of Boyle–Tomiyama [4] for continuous orbit equivalence rigidity for minimal $\mathbb{Z}$-actions. Here, the basic strategy is as follows: in view of a criterion by X. Li [19, Proposition 4.4], it suffices to show that the orbit cocycle c is cohomologous to a group isomorphism of D_∞. The starting point for our proof is the characterization of bi-Lipschitz bijections of $\mathbb{Z}$ due to Benjamini–Shamov [2]. After restricting the orbit cocycle $c \colon \mathbb{Z} \times X \to D_\infty$ on the sub-$\mathbb{Z}$-action, we can extract from it a defect cocycle a which attains only finitely many values in D_∞. Here, a is defined to measure the defect between c and a group homomorphism from $\mathbb{Z}$ to D_∞. By twisting a with a suitable coboundary, we can assume the range of a is contained in $\mathbb{Z}$. If the sub-$\mathbb{Z}$-action is minimal, one can apply the well-known Gottschalk–Hedlund theorem [12] for minimal $\mathbb{Z}$-actions to deduce that a is actually a coboundary, which implies the original orbit cocycle c is trivial. For the general case where the sub-$\mathbb{Z}$-action is not minimal, we can apply Gottschalk–Hedlund theorem to each $\mathbb{Z}$-component, now the existence of reflection of $\mathbb{Z}$ in the group structure of D_∞ allows us to patch up untwisters for each component to get a global one.

The paper is organized into five sections. In Section 2, we explain terminologies used in this paper, including cocycles, continuous orbit equivalence, skew product actions and induced actions, and present examples of continuous actions. Then we give examples of minimal free actions admitting continuous cocycles which are not coboundaries in Section 3. The proof of Theorem 1.2 (resp. Theorem 1.3) is given in Section 4.1 (resp. Section 4.2). We conclude the paper with a several remarks on the possible generalization of the theorems in Section 5.

In this paper, G and H usually mean virtually cyclic groups or finitely generated groups, which is clear from the context. Moreover, we denote a target group by K. Unless otherwise specified, all cocycles are assumed to be continuous.

2. Preliminaries

2.1. Cocycles and continuous orbit equivalence

Let $G \curvearrowright X$ be a continuous action on a compact Hausdorff space and let K be a discrete group. Recall that a continuous map $c \colon G \times X \to K$ is called a *K-valued (continuous) cocycle* if $c(g_1 g_2, x) = c(g_1, g_2 x) c(g_2, x)$ for all $g_1, g_2 \in G$ and all $x \in X$. Two continuous cocycles $c_1, c_2 \colon G \times X \to K$ are *(continuously) cohomologous/equivalent* if there exists a continuous map $b \colon X \to K$ such that $c_1(g, x) = b(gx)^{-1} c_2(g, x) b(x)$ for all $g \in G$ and all $x \in X$. A continuous cocycle is called *trivial* (resp. *a coboundary*) if it is equivalent to a group homomorphism $\phi \colon G \to K$ (resp. the trivial group homomorphism $\phi \equiv e_K$ on G), which is treated as a constant cocycle on X.

Note that when $G = \mathbb{Z}$, a K-valued cocycle c for $\mathbb{Z} \curvearrowright X$ can be represented as a map $f : X \to K$. Indeed, for one direction, set $f(x) = c(1, x)$; for the other direction, we can define $c(1, x) = f(x)$ and use the cocycle identity to define $c(N, x)$ for all $N \in \mathbb{Z}$ and hence get a cocycle c. Sometimes, we may abuse our notation by calling f a cocycle.

Given two continuous actions $G \curvearrowright X$ and $K \curvearrowright Y$, we have the notion of continuous orbit equivalence [19, Definition 2.5] (abbreviated as coe for short), which was systematically studied by X. Li [19].

Definition 2.1 (Continuous orbit equivalence). We call two continuous actions $G \curvearrowright X$ and $K \curvearrowright Y$ continuously orbit equivalent, $G \curvearrowright X \overset{\text{coe}}{\sim} K \curvearrowright Y$, if there exists a homeomorphism $\phi : X \approx Y$ with inverse $\psi = \phi^{-1}$ and continuous maps $a : G \times X \to K$ and $b : K \times Y \to G$ such that

$$\phi(gx) = a(g, x)\phi(x), \quad \psi(hy) = b(h, y)\psi(y)$$

for all $g \in G$, $h \in H$, $x \in X$ and $y \in Y$.

If both a and b are further assumed to be group isomorphisms from G to K and from K to G, respectively, we call these two actions *conjugate*, written as $G \curvearrowright X \overset{\text{conj}}{\sim} K \curvearrowright Y$. Clearly, conjugacy implies coe.

Remark that if $K \curvearrowright Y$ is *topologically free* (i.e., for each $e \neq h \in K$, $\{y \in Y : hy \neq y\}$ is dense in Y), then a is a continuous cocycle, see [19, Lemma 2.8]. When both a and b are cocycles in the above definition, we simply call b the inverse cocycle of a.

The following fact is proved independently in [21] and [20], we sketch the proof for completeness.

Proposition 2.2. *Let G and K be finitely generated groups. Let $G \curvearrowright X$ and $K \curvearrowright Y$ be two topological free actions on compact spaces. Assume they are continuously orbit equivalent, and $a : G \times X \to K$ is the associated continuous orbit cocycle, then $G \ni g \mapsto a(g, x) \in K$ is a bi-Lipschitz bijection for each $x \in X$ with respect to the right invariant word metrics on both G and K.*

Throughout the paper, a *bi-Lipschitz bijection* means a bijective map ϕ such that both ϕ and ϕ^{-1} are Lipschitz maps in the usual sense.

Proof. Let S and T be any symmetric generating sets for G and K, respectively. Let $|\cdot|_S$ and $|\cdot|_T$ be associated word lengths, and let $d_S(\cdot, \cdot)$ and $d_T(\cdot, \cdot)$ be the corresponding right invariant word length metrics on G and K, i.e., $d_S(g_1, g_2) = |g_2 g_1^{-1}|_S$ and $d_T(h_1, h_2) = |h_2 h_1^{-1}|_T$ for all $g_1, g_2 \in G$ and $h_1, h_2 \in K$. Then

$$\begin{aligned}
d_T(a(g_1, x), a(g_2, x)) &= |a(g_2, x)a(g_1, x)^{-1}|_T = |a(g_2 g_1^{-1}, g_1 x)|_T \\
&\leq |g_2 g_1^{-1}|_S \cdot \sup_{x \in X} \max_{s \in S} |a(s, x)|_T \\
&= d_S(g_1, g_2) \cdot \sup_{x \in X} \max_{s \in S} |a(s, x)|_T,
\end{aligned}$$

where the inequality follows from the cocycle identity. This shows that $g \mapsto a(g, x)$ is a Lipschitz map.

To see that $g \mapsto a(g, x)$ is a bijection and its inverse is a Lipschitz map, apply a similar argument to the inverse cocycle using [19, Lemma 2.10]. ∎

The following lemma is well known, but we present its proof for the reader's convenience.

Lemma 2.3. *Let $\alpha \colon \mathbb{Z} \curvearrowright X$ be a minimal action on a compact space X and $K_0 \subset K$ be a closed subgroup in a topological group. Let $c \colon \mathbb{Z} \times X \to K_0$ be a continuous cocycle such that $c(n, x) = f(\alpha_n(x)) f(x)^{-1}$ for some continuous map $f \colon X \to K$ and all $n \in \mathbb{Z}$, $x \in X$. Then there is a continuous map $f' \colon X \to K_0$ such that $c(n, x) = f'(\alpha_n(x)) f'(x)^{-1}$ holds for all $n \in \mathbb{Z}$ and all $x \in X$.*

Proof. Observe that the continuous map

$$x \in X \to K_0 f(x) \in K_0 \backslash K$$

is α-invariant. Hence minimality implies there exists some $k \in K$ such that $K_0 f(x) = K_0 k$ for all $x \in X$, i.e., $f(x) \in K_0 k$ for all $x \in X$. Define $f'(x) = f(x)k^{-1}$. We deduce that $c(n, x) = f'(\alpha_n(x)) f'(x)^{-1}$. ∎

2.2. Skew product actions

We recall the following well-known construction of skew product actions. Let $\sigma \colon H \to \mathrm{Aut}(K)$ be a group homomorphism. Let $c \colon H \curvearrowright X \to K$ be a *skew cocycle*, i.e., for all $h_1, h_2 \in H$ and all $x \in X$,

$$c(h_1 h_2, x) = c(h_1, h_2 x)\sigma_{h_1}(c(h_2, x)).$$

One can check that $c((h_1 h_2)h_3, x) = c(h_1(h_2 h_3), x)$ holds for all $h_i \in H$, $1 \le i \le 3$, and all $x \in X$. Besides, c is a skew cocycle if and only if the map $c' \colon H \curvearrowright X \to K \rtimes_\sigma H$ given by $c'(h, x) = (c(h, x), h)$ is a cocycle in the usual sense.

Given a group homomorphism $\sigma \colon H \to \mathrm{Aut}(K)$, an action $H \curvearrowright X$ and a skew cocycle c defined as above, we can define the *generalized skew product action*:

$$G := K \rtimes_\sigma H \curvearrowright X \times K, \quad (k, h)(x, k') := (hx, c(h, x)\sigma_h(k')k^{-1}).$$

Note that when σ is trivial, i.e., $\sigma \equiv \mathrm{id}_K$, skew cocycles (resp. skew product actions) are reduced to the usual cocycles (resp. usual skew product actions) for the group $K \times H$. In this paper, we usually take $H = \mathbb{Z}$, K be a finite group and σ be trivial. To emphasize the cocycle c, we usually denote $X \times K$ by $X \times_c K$.

2.3. Induced actions

Let H be a subgroup of G. Fix any lift map $L: G/H \to G$ such that $L(gH)H = gH$ for all $g \in G$. Consider the associated cocycle $c: G \times G/H \to H$ given by $c(g, g'H) = L(gg'H)^{-1}gL(g'H)$ for all $g, g' \in G$. Given any continuous action $H \curvearrowright Y$, then the *induced action* $G \curvearrowright G/H \times Y$ is defined as follows:

$$g(g'H, y) = (gg'H, c(g, g'H)y) \quad \text{for all } g, g' \in G \text{ and } y \in Y.$$

2.4. Examples of continuous actions

Odometer actions. Fix a sequence of strictly increasing positive integers $(n_i)_{i \geq 1}$ such that $n_i \mid n_{i+1}$ for all $i \geq 1$. Let

$$\varprojlim \mathbb{Z}/n_i\mathbb{Z} = \Big\{(x_i + n_i\mathbb{Z})_{i \geq 1} \in \prod_{i \geq 1} \mathbb{Z}/n_i\mathbb{Z} : n_i \mid (x_{i+1} - x_i) \; \forall i \geq 1\Big\}.$$

The *odometer action associated to* (n_i) is defined as the continuous action $\alpha: \mathbb{Z} \curvearrowright \varprojlim \mathbb{Z}/n_i\mathbb{Z}$ given by

$$\alpha_n((x_i + n_i\mathbb{Z})_i) = (n + x_i + n_i\mathbb{Z})_i, \quad x_i \in \mathbb{Z}.$$

Weakly mixing actions. A continuous action $\mathbb{Z} \curvearrowright X$ on a compact Hausdorff space is called *(topologically) weakly mixing* [11, p. 23] if the product action $\mathbb{Z} \curvearrowright X \times X$ is *transitive*, i.e., for every pair of non-empty open sets U, V in $X \times X$ there exists some $g \in \mathbb{Z}$ with $gU \cap V \neq \emptyset$. Examples of free minimal weakly mixing actions include the so-called Chacón system [11, p. 27].

It is well known and easy to check that for a continuous action $\mathbb{Z} \curvearrowright X$ on a compact Hausdorff space X, if X has no isolated points and there is a point in X with dense orbit, then $\mathbb{Z} \curvearrowright X$ is transitive. Conversely, for a compact metric space X, $\mathbb{Z} \curvearrowright X$ being transitive implies that it admits a point with dense orbit, see, e.g., [15, Proposition 7.9].

2.5. Subgroups of D_∞

Let $D_\infty = \mathbb{Z} \rtimes \frac{\mathbb{Z}}{2\mathbb{Z}} = \langle s \rangle \rtimes \langle t \rangle = \langle s, t \mid t^2, tsts \rangle$ be the infinite dihedral group. We record a simple lemma on the structure of its subgroups.

Lemma 2.4. *If H is a subgroup of D_∞, then either $H = \langle s^k, s^i t \rangle$ for some $k, i \in \mathbb{Z}$, $0 \leq |i| < k$, or $H = \langle s^k t \rangle$, or $H = \langle s^k \rangle$ for some $k \in \mathbb{Z}$. Moreover, all subgroups can be realized as $\phi_i(H)$ for some $i \in \mathbb{Z}$ and some $H \in \{k\mathbb{Z} \rtimes \frac{\mathbb{Z}}{2\mathbb{Z}}, k\mathbb{Z}, \frac{\mathbb{Z}}{2\mathbb{Z}} : k \in \mathbb{Z}\}$, where $\phi_i \in \mathrm{Aut}(G)$ is defined by $\phi_i(s) = s$, $\phi_i(t) = s^i t$.*

Proof. Write $H \cap \mathbb{Z} = \langle s^k \rangle$ for some $k \in \mathbb{Z}$. Note that the set of torsion elements of G is precisely $\{s^i t : i \in \mathbb{Z}\}$. If H contains no non-trivial torsion elements, then $H \subset \mathbb{Z}$, i.e., $H = H \cap \mathbb{Z} = \langle s^k \rangle$. If H contains some non-trivial torsion element, say $s^i t \in H$ for some $i \in \mathbb{Z}$. Without loss of generality, we can assume that $|i|$ is smallest among all non-trivial torsion elements in H.

Case 1. $i = 0$. Then clearly, $H = \langle s^k, t \rangle$.

Case 2. $|i| > 0$. If $k = 0$, then $H = \langle s^i t \rangle$. Indeed, assume not, then $s^j t \in H$ for some $j \neq i$, which implies that $(s^i t)(s^j t) = s^{i-j} \in H \cap \mathbb{Z}$, a contradiction. We can suppose that $k > 0$. Clearly, $0 < |i| < k$, then $H = \langle s^k, s^i t \rangle$. Indeed, assume $s^j t \in H$, then $(s^j t)(s^i t)^{-1} = s^{j-i} \in H \cap \mathbb{Z} = \langle s^k \rangle$, hence $s^j t \in \langle s^k, s^i t \rangle$.

The moreover part is now clear. ∎

Remark 2.5. Lemma 2.4 implies that the notion of structurally conjugacy between two odometer actions of D_∞ as introduced in [7, Definition 3.1] coincides with the usual conjugacy. Thus, from [7, Theorem 3.3] it follows that continuous orbit equivalence between two free odometer actions of D_∞ implies they are conjugate.

Next, let us observe the following holds, see also [22, Proposition 2.8].

Proposition 2.6. *If $\alpha: D_\infty \curvearrowright X$ is a minimal action on an infinite compact Hausdorff space X, then it is topologically free.*

Proof. Without loss of generality, we can assume α is not free. Write $D_\infty = \mathbb{Z} \rtimes \frac{\mathbb{Z}}{2\mathbb{Z}} = \langle s, t \mid t^2, tsts \rangle = \langle s \rangle \rtimes \langle t \rangle$. Then observe that the sub $\mathbb{Z} = \langle s \rangle$-action, i.e., $\alpha|_{\mathbb{Z}}: \mathbb{Z} \curvearrowright X$ is minimal.

Indeed, note that the number of minimal $\mathbb{Z}$-components in X is either one or two (see the discussion before case I in the proof of Theorem 1.3 for details). If we have two minimal $\mathbb{Z}$-components, say X_0 and tX_0, then $\alpha|_{\mathbb{Z}}$ on each component is free and so is α, contradicting our assumption.

Now, as the action is minimal and X is infinite, every stabilizer subgroup must be of infinite index in D_∞. By Lemma 2.4, we deduce that for each $x \in X$, if the stabilizer group $\mathrm{Stab}(x)$ is not trivial, then $\mathrm{Stab}(x) = \{e, s^n t\}$ for some $n \in \mathbb{Z}$. To check topological freeness of α, we only need to show that for each $n \in \mathbb{Z}$, the closed set $X_n := \{x \in X : s^n tx = x\}$ is nowhere dense.

Note that $x \in X_n$ if and only if $\mathrm{Stab}(x) = \{e, s^n t\}$. Now, assume that $\exists \emptyset \neq U \subsetneq X_n$, where U is an open set. Then we can pick any $y \in X \setminus U$ and use $\mathbb{Z} \curvearrowright X$ is minimal to deduce that for each N, $\exists |m| > N$ such that $s^m y \in U$. Hence, $\{e, s^n t\} = \mathrm{Stab}(s^m y) = s^m \mathrm{Stab}(y) s^{-m}$. Since $\mathrm{Stab}(y) = \{e\}$ or $\{e, s^k t\}$ for some $k \in \mathbb{Z}$, we deduce that $2m = n - k$, which gives us a contradiction since m can be arbitrarily large. ∎

2.6. Bi-Lipschitz bijections of $\mathbb{Z}$

We need the following result from [2].

Theorem 2.7 (Benjamini–Shamov). *Let $f: \mathbb{Z} \to \mathbb{Z}$ be a bi-Lipschitz bijection ($\mathbb{Z}$ is equipped with its usual metric, namely $\rho(x, y) := |x - y|$). Then either $\sup_{x \in \mathbb{Z}} |f(x) - x| < +\infty$ or $\sup_{x \in \mathbb{Z}} |f(x) + x| < +\infty$. More precisely, $f(x) = \pm x + \mathrm{const} + r(x)$, where $\sup_{x \in \mathbb{Z}} |r(x)| < \infty$.*

2.7. Gottschalk–Hedlund theorem

Another ingredient that we need is the classical Gottschalk–Hedlund theorem [12, Theorem 14.11]. We record the following version as presented in [13, Theorem 2.9.4, p. 102] and explain how to use it in our setting. Note that the proof of [13, Theorem 2.9.4, p. 102] still works without the metrizability assumption on the space.

Theorem 2.8 (Gottschalk–Hedlund). *Let X be a compact Hausdorff space, $f: X \to X$ a minimal continuous map and $g: X \to \mathbb{R}$ continuous such that*

$$\sup_{n \in \mathbb{N}} \left| \sum_{i=0}^{n} g \circ f^i(x_0) \right| < \infty$$

for some $x_0 \in X$. Then there is a continuous $\phi: X \to \mathbb{R}$ such that $\phi \circ f - \phi = g$.

Now, let $f: X \to X$ be a minimal continuous action and $c: \mathbb{Z} \times X \to \mathbb{R}$ be a continuous cocycle for this action. Set $g(x) = c(1, x)$, then a calculation shows that

$$\sum_{i=0}^{n} g \circ f^i(x_0) = c(n+1, x_0).$$

Thus, Gottschalk–Hedlund theorem actually shows that if c is a bounded cocycle, i.e., $\sup_{(n,x) \in \mathbb{Z} \times X} |c(n, x)| < \infty$, then it is a continuous coboundary map.

3. Existence of cocycles that are not coboundaries

To proof Theorem 1.2, we need to construct free minimal actions that admit continuous cocycles which are not coboundaries with values into finite groups. We collect some results along this direction.

Proposition 3.1. *Let $\mathbb{Z} \curvearrowright X$ be a minimal weakly mixing and continuous action on a compact metric space X. Then for each non-trivial group homomorphism ϕ from $\mathbb{Z}$ to F, where F is a finite abelian group, ϕ is not a coboundary.*

Proof. Suppose that $\phi(s) = L(sx) - L(x)$ for some continuous map $L: X \to F$ and all $s \in \mathbb{Z}$, $x \in X$. Then $L(sx) - L(x) = L(sy) - L(y)$ for all $x, y \in X$. Therefore, $L'(x, y) := L(x) - L(y)$ is a continuous map which is constant on each orbit of any point in $X \times X$. By assumption, we know that L' is constant since the diagonal action $\mathbb{Z} \curvearrowright X \times X$ admits a point with dense orbit. Thus $L'(x, y) = L'(x, x) = 0$ for all $x, y \in X$, i.e., L is a constant function on X. Hence $\phi(s) = 0$ for all $s \in \mathbb{Z}$, i.e., ϕ is a trivial homomorphism, a contradiction. ∎

For the next result, we need the following known characterization of coboundaries using essential values for continuous cocycles in [18], which was adapted from Schmidt's notion for measurable cocycles in [24].

Let K be a locally compact group with the unit element e. By K_∞ we denote the one point compactification of K, i.e., $K_\infty = K \cup \{\infty\}$. We can extend the group operation from the group K onto the set K_∞ by $g \cdot \infty = \infty$ for all $g \in K_\infty$. One can check that the operation $K_\infty \times K_\infty \ni (g_1, g_2) \to g_1 g_2 \in K_\infty$ is continuous.

Definition 3.2 ([18, Definition 3.1]). Let $T : \mathbb{Z} \curvearrowright X$ be a continuous action on a compact space X. Let $c : \mathbb{Z} \times X \to K$ be a continuous cocycle. We say that $r \in K_\infty$ is an *essential value of ϕ* if for each non-empty open $U \subset X$ and each neighborhood V of r there exists $N \in \mathbb{Z}$ such that
$$U \cap T^{-N} U \cap \{x \in X : c(N, x) \in V\} \neq \emptyset.$$

The set of all essential values of c we denote by $E_\infty(c)$. Put also $E(c) = E_\infty(c) \cap K$.

Proposition 3.3. *Let $\mathbb{Z} \curvearrowright X$ be a minimal action and K be a compact abelian group, e.g., a finite abelian group. Let $c : \mathbb{Z} \times X \to K$ be a continuous cocycle. Then $E(c) = \{e\}$ if and only if c is a coboundary, where e denotes the neutral element in K.*

Proof. $\Rightarrow$ See [18, Lemma 3.1 (ii)]. $\Leftarrow$ One can check this directly using the definition of $E(c)$. ∎

The following result might be known to experts; since we do not find a reference, we include the proof.

Proposition 3.4. *Let $(\mathbb{Z}, X)$ be a minimal odometer action, say $X = \varprojlim \mathbb{Z}/n_i \mathbb{Z}$, where $n_i \mid n_j$ for all $i \leq j$ and $\lim_{j \to \infty} n_j = \infty$. Let p be a prime number. Then the following statements are equivalent:*

(1) *Every continuous map $f : X \to \mathbb{Z}/p\mathbb{Z}$ gives rise to a continuous coboundary.*

(2) $\sup_i \operatorname{ord}(p, n_i) = \infty$, *where $\operatorname{ord}(p, n_i) := \max_{p^k \mid n_i} k$.*

Proof. (1) $\Rightarrow$ (2): By Proposition 3.3, we know that for all continuous $f : X \to \mathbb{Z}/p\mathbb{Z}$, $E(f) = \{0\}$, i.e., for all $0 \neq c \in \mathbb{Z}/p\mathbb{Z}$, there exists some non-empty open set $U \subset X$ such that for all $n \in \mathbb{Z}$, we have $x \in U \cap T^{-n} U \Rightarrow f(x) + f(Tx) + \cdots + f(T^{n-1}x) \neq c$.

Now, denote by $\pi_i : X \to \mathbb{Z}/n_i \mathbb{Z}$ the natural projection map, i.e.,
$$\pi_i((x_i + n_i \mathbb{Z})_i) = x_i + n_i \mathbb{Z}.$$

Suppose that $\sup_i \operatorname{ord}(p, n_i)$ is bounded, let us take some j such that
$$\operatorname{ord}(p, n_j) = \sup_i \operatorname{ord}(p, n_i)$$

and assume $n_j > 1$, then we define a map $f : X \to \mathbb{Z}/p\mathbb{Z}$ as the composition
$$X \xrightarrow{\pi_j} \mathbb{Z}/n_j \mathbb{Z} \xrightarrow{f'} \mathbb{Z}/p\mathbb{Z},$$

where $f'(i + n_j \mathbb{Z}) = \delta_{i,1} \in \mathbb{Z}/p\mathbb{Z}$ for all $0 \leq i \leq n_1 - 1$. Take $c = 1 \in \mathbb{Z}/p\mathbb{Z}$, then we find some $U \subset X$ as above. We can shrink U if necessary to assume $U = \pi_k^{-1}(i_0 + n_k \mathbb{Z})$

for some $k > j$ and i_0. Now, take any $x = (x_i + n_i\mathbb{Z})_i \in U$ and $n = n_k\ell$ for some integer ℓ such that $\frac{n_k}{n_j}\ell \equiv 1 \bmod p$. (This is possible since $p \nmid \frac{n_k}{n_j}$ by our choice of j and p is a prime number.) Note that $n_k \mid n$ implies that $T^n x \in U$. Now, a calculation shows that

$$f(x) + f(Tx) + \cdots + f(T^{n-1}x) = \sum_{s=0}^{n-1} f'(x_j + s + n_j\mathbb{Z})$$

$$= 1 \cdot \frac{n}{n_j} = \frac{n_k\ell}{n_j} = 1 = c \in \mathbb{Z}/p\mathbb{Z}.$$

This is a contradiction.

(2) $\Rightarrow$ (1): The proof is similar as above. By Proposition 3.3, we aim to show that for all continuous $f: X \to \mathbb{Z}/p\mathbb{Z}$ and all $0 \neq c \in \mathbb{Z}/p\mathbb{Z}$, there exists some non-empty set $U \subset X$ such that for all $n \in \mathbb{Z}$, $x \in U \cap T^{-n}U \Rightarrow f(x) + f(Tx) + \cdots + f(T^{n-1}x) \neq c$.

Choose large enough j such that f factors through $\pi_j: X \to \mathbb{Z}/n_j\mathbb{Z}$, i.e., $f = \pi_j \circ f'$ for some map $f': \mathbb{Z}/n_j\mathbb{Z} \to \mathbb{Z}/p\mathbb{Z}$. Since $\sup_i \operatorname{ord}(p, n_i) = \infty$, we can find $k > j$ such that $\operatorname{ord}(p, n_k) > \operatorname{ord}(p, n_j)$. Then let $U = \pi_k^{-1}(0 + n_k\mathbb{Z})$. For each $n \in \mathbb{Z}$, if $x \in U \cap T^{-n}U$, then $x_k = 0 = x_k + n$ in $\mathbb{Z}/n_k\mathbb{Z}$, so $n_k \mid n$. Now, a calculation shows that

$$f(x) + f(Tx) + \cdots + f(T^{n-1}x)$$

$$= \sum_{s=0}^{n-1} f'(x_j + s + n_j\mathbb{Z}) = \left(\sum_{s=0}^{n_j-1} f'(s + n_j\mathbb{Z}) \right) \frac{n}{n_j}$$

$$= \left[\left(\sum_{s=0}^{n_j-1} f'(s + n_j\mathbb{Z}) \right) \frac{n}{n_k} \right] \frac{n_k}{n_j} \in p\mathbb{Z} \quad (\text{since } \operatorname{ord}(p, n_k) > \operatorname{ord}(p, n_j)).$$

Thus, $0 = f(x) + f(Tx) + \cdots + f(T^{n-1}x) \neq c$ in $\mathbb{Z}/p\mathbb{Z}$. ∎

From the above proposition, we can deduce that the free minimal odometer action $\mathbb{Z} \curvearrowright \varprojlim \frac{\mathbb{Z}}{2^i\mathbb{Z}} := X$ admits a continuous $\mathbb{Z}/3\mathbb{Z}$-valued cocycle that is not a coboundary.

4. Proofs

4.1. Non-rigidity part

Theorem 1.2 is a direct corollary of the following proposition.

Proposition 4.1. *Let $\alpha: \mathbb{Z} \curvearrowright X$ be a minimal free action on a compact Hausdorff space. Let $c, c': \mathbb{Z} \times X \to F$ be two continuous cocycles (with respect to the action α) into a finite non-trivial group F. Consider the associated skew product actions $\tilde{\alpha}: F \times \mathbb{Z} \curvearrowright X \times_c F$ and $\tilde{\alpha}': F \times \mathbb{Z} \curvearrowright X \times_{c'} F$ as defined in Section 2.2. Then the following holds:*

    (i)   *Both $\tilde{\alpha}$ and $\tilde{\alpha}'$ are minimal free actions.*

(ii) $\widetilde{\alpha} \overset{\text{coe}}{\sim} \widetilde{\alpha}'$.

(iii)   *If $c' \equiv e \in F$, the neutral element in F, then $\widetilde{\alpha} \overset{\text{conj}}{\sim} \widetilde{\alpha}'$ implies that c is cohomologous to a group homomorphism from $\mathbb{Z}$ into $C(F)$.*

(iv)   *For each F with trivial center, there exist α, c and c' such that $\widetilde{\alpha} \overset{\text{conj}}{\not\sim} \widetilde{\alpha}'$.*

Proof. (i) We leave it as an exercise.

(ii) Recall that $\widetilde{\alpha} \colon F \times \mathbb{Z} \curvearrowright X \times_c F$ is defined as follows:

$$\widetilde{\alpha}_{(f,n)}(x, f') = (\alpha_n(x), c(n, x) f' f^{-1}) \quad \text{for all } f, f' \in F, n \in \mathbb{Z} \text{ and } x \in X.$$

Define $\theta \colon (F \times \mathbb{Z}) \times (X \times_c F) \to F \times \mathbb{Z}$ by setting

$$\theta((t, n), (x, f)) = (t f^{-1} c(n, x)^{-1} c'(n, x) f, n) \quad \text{for all } n \in \mathbb{Z}, t \in F \text{ and } x \in X.$$

One can check that it is a continuous cocycle with respect to the skew product action $\widetilde{\alpha}$. Indeed,

$$\begin{aligned}
\theta&((t_1, n_1)(t_2, n_2), (x, f)) \\
&= \theta((t_1 t_2, n_1 + n_2), (x, f)) \\
&= (t_1 t_2 f^{-1} c(n_1 + n_2, x)^{-1} c'(n_1 + n_2, x) f, n_1 + n_2), \\
\theta&((t_1, n_1), \widetilde{\alpha}_{(t_2, n_2)}(x, f)) \theta((t_2, n_2), (x, f)) \\
&= \theta((t_1, n_1), (\alpha_{n_2}(x), c(n_2, x) f t_2^{-1})) \theta((t_2, n_2), (x, f)) \\
&= (t_1 t_2 f^{-1} c(n_2, x)^{-1} c(n_1, \alpha_{n_2}(x))^{-1} c'(n_1, \alpha_{n_2}(x)) c(n_2, x) f t_2^{-1}, n_1) \\
&\quad \times (t_2 f^{-1} c(n_2, x)^{-1} c'(n_2, x) f, n_2) \\
&= (t_1 t_2 f^{-1} c(n_2, x)^{-1} c(n_1, \alpha_{n_2}(x))^{-1} c'(n_1, \alpha_{n_2}(x)) c'(n_2, x) f, n_1 + n_2) \\
&= (t_1 t_2 f^{-1} c(n_1 + n_2, x)^{-1} c'(n_1 + n_2, x) f, n_1 + n_2).
\end{aligned}$$

Thus,

$$\theta((t_1, n_1)(t_2, n_2), (x, f)) = \theta((t_1, n_1), \widetilde{\alpha}_{(t_2, n_2)}(x, f)) \theta((t_2, n_2), (x, f)).$$

Moreover, one can verify that

$$\widetilde{\alpha}_{(t,n)}(x, f) = \widetilde{\alpha}'_{\theta((t,n),(x,f))}(x, f).$$

By symmetry, one can also define the inverse cocycle for θ, hence $\widetilde{\alpha} \overset{\text{coe}}{\sim} \widetilde{\alpha}'$.

(iii) First, we observe that for each $\Phi \in \mathrm{Aut}(F \times \mathbb{Z})$, we have $\Phi(t, 1) = (\varepsilon(t) g, \pm 1)$ for some $\varepsilon \in \mathrm{Aut}(F)$ and $g \in C(F)$, the center of F. Indeed, since F is finite, $\Phi(F \times \{0\}) = F \times \{0\}$, thus we define $\varepsilon = \Phi|_{F \times \{0\}}$. Next, we can write $\Phi(e, 1) = (g, \pm 1)$ for some $g \in F$. From $\Phi((e, 1)(t, 0)) = \Phi((t, 0)(e, 1))$ for all $t \in F$, we deduce that $g \in C(F)$. Note that $\Phi(t, n) = (\varepsilon(t) g^n, \pm n)$ for all $n \in \mathbb{Z}$ and $t \in F$. For such Φ, we simply write $\Phi = (\varepsilon(\cdot) g, \pm)$.

Next, assume $\widetilde{\alpha}$ is conjugate to $\widetilde{\alpha}'$, say via a homeomorphism $\tau\colon X \times_c F \to X \times_{c'} F$ and a group isomorphism $(\varepsilon(\cdot)g, \pm)\colon F \times \mathbb{Z} \cong F \times \mathbb{Z}$ defined in the last paragraph. Then we can write

$$\tau((x, f')) = (\phi(x, f'), \psi(x, f'))$$

for some continuous maps $\phi\colon X \times_c F \to X$ and $\psi\colon X \times_c F \to F$.

Fix any $(x, f') \in X \times_c F$ and $(f, n) \in F \times \mathbb{Z}$, the $F \times \mathbb{Z}$-equivariance of τ shows that for all $x \in X$, $f, f' \in F$ and all $n \in \mathbb{Z}$, we have

$$\begin{aligned}
\alpha_{\pm n}(\phi(x, f')) &= \phi(\alpha_n(x), c(n, x)f'f^{-1}), \\
c'(\pm n, \phi(x, f'))\psi(x, f')g^{-n}\varepsilon(f)^{-1} &= \psi(\alpha_n(x), c(n, x)f'f^{-1}).
\end{aligned} \tag{4.1}$$

Since f is arbitrary in the first expression in (4.1), we deduce that ϕ only depends on its first coordinate, so we can directly write $\phi(x, -) = \phi(x)$ and hence $\alpha_{\pm n}(\phi(x)) = \phi(\alpha_n(x))$. Using τ is a homeomorphism, we can check that $\phi|_X$ is also a homeomorphism.

Now, set $f = c(n, x)$ and $f' = e$ into the second line in (4.1). We get that

$$c'(\pm n, \phi(x))\psi(x, e)g^{-n}\varepsilon(c(n, x))^{-1} = \psi(\alpha_n(x), e).$$

Thus,

$$c'(\pm n, \phi(x)) = \psi(\alpha_n(x), e)\varepsilon(c(n, x))g^n\psi(x, e)^{-1}. \tag{4.2}$$

Since $\alpha_{\pm n}(\phi(x)) = \phi(\alpha_n(x))$, (4.2) can be rewritten as

$$c'(\pm n, x) = \psi(\phi^{-1}(\alpha_{\pm n}(x)), e)\varepsilon(c(n, \phi^{-1}(x)))g^n\psi(\phi^{-1}(x), e)^{-1},$$

or simply

$$c'(n, x) = \psi'(\alpha_n(x))\varepsilon(c(\pm n, \phi^{-1}(x)))g^{\pm n}\psi'(x)^{-1} \quad \text{for all } n \in \mathbb{Z} \text{ and } x \in X, \tag{4.3}$$

where $\psi'(x) := \psi(\phi^{-1}(x), e)$.

Now, let $c \equiv e \in F$, the neutral element in F, then (4.3) is the same as saying c' is cohomologous to a group homomorphism $\mathbb{Z} \ni n \mapsto g^{\pm n} \in C(F)$.

(iv) Set $c \equiv e$. We are left to construct α and c such that $\widetilde{\alpha} \overset{\text{conj}}{\nsim} \widetilde{\alpha}'$. It suffices to make sure the above identity (4.3) fails. Since $C(F)$ is trivial, we deduce $g = e$. It suffices to find some α and a cocycle $c'\colon \mathbb{Z} \times X \to F$ (with respect to α) which is not a coboundary. By Cauchy theorem, we can find some non-trivial $t \in F$ such that $\mathbb{Z}/p\mathbb{Z} \cong \langle t \rangle \subset F$ for some prime number p. By Lemma 2.3, it suffices to find some α and a continuous cocycle $c'\colon \mathbb{Z} \times X \to \mathbb{Z}/p\mathbb{Z}$ which is not a coboundary.

We can apply Proposition 3.1 or Proposition 3.4 to construct a suitable free minimal and weakly mixing or free minimal odometer action $\mathbb{Z} \curvearrowright X$ and a continuous cocycle c which is not a coboundary. ∎

4.2. Rigidity part

We are ready to prove Theorem 1.3.

Proof of Theorem 1.3. Let $D_\infty = \mathbb{Z} \rtimes \frac{\mathbb{Z}}{2\mathbb{Z}} = \langle s \rangle \rtimes \langle t \rangle$. Denote by S the symmetric generating set $\{s^\pm, t^\pm\}$ for D_∞. Denote by $|\cdot|_S$ the word length on G with respect to S. Note that $|s^n t|_S = n + 1$ and $|s^n|_S = n$. Let $d(\cdot, \cdot)$ be the right-invariant word metric on D_∞ with respect to S, i.e., $d(g_1, g_2) := |g_2 g_1^{-1}|_S$. Let $c: D_\infty \times X \to D_\infty$ be the continuous orbit cocycle associated to a given continuous orbit equivalence.

Claim 4.2. *For all $x \in X$, the following holds: either* $\sup_{n \in \mathbb{Z}} d(c(s^n, x), s^n) < \infty$ *or* $\sup_{n \in \mathbb{Z}} d(c(s^n, x), s^{-n}) < \infty$.

Proof. Define $\pi: \mathbb{Z} \to D_\infty$ by setting $\pi(s^{2n}) = s^n$ and $\pi(s^{2n+1}) = ts^n$ for all $n \in \mathbb{Z}$. One can verify that π is a bi-Lipschitz bijection. In fact,

$$\frac{|n - m|}{2} \leq d(\pi(s^n), \pi(s^m)) \leq 2|n - m| \quad \text{for all } n, m \in \mathbb{Z}.$$

Fix any $x \in X$ and write $\phi_x(g) = c(g, x)$ for all $g \in D_\infty$. Define $\phi: \mathbb{Z} \to \mathbb{Z}$ by setting $\phi = \pi^{-1} \circ \phi_x \circ \pi$. Clearly, ϕ is a bi-Lipschitz bijection as well by Proposition 2.2. From Theorem 2.7, we deduce that either $\sup_{n \in \mathbb{Z}} |\phi(n) - n| < \infty$ or $\sup_{n \in \mathbb{Z}} |\phi(n) + n| < \infty$. This implies Claim 4.2 holds. ∎

Define X_+ and X_- in the following way:

$$X_+ = \{x \in X : \sup_{n \in \mathbb{Z}} d(c(s^n, x), s^n) < \infty\},$$

$$X_- = \{x \in X : \sup_{n \in \mathbb{Z}} d(c(s^n, x), s^{-n}) < \infty\}.$$

Clearly, $X_+ \cap X_- = \emptyset$ since s has infinite order. From Claim 4.2, we know that $X = X_+ \sqcup X_-$.

Claim 4.3. *Both X_+ and X_- are clopen subsets of X.*

Proof. We follow the idea while dealing with $\mathbb{Z}$-actions in [4]. For each $r \geq 1$, define $B(r) := \{ts^i, s^i : |i| \leq r\} \subset D_\infty$. Note that $B(r) \nearrow D_\infty$ as $r \to \infty$.

First, for all $N > 0$, there exists $K > 0$ such that $B(N) \subset c(B(K), x)$ for all $x \in X$. Indeed, since $D_\infty \ni g \mapsto c(g, x) \in D_\infty$ is a bijection, we can find $K_x > 0$ such that $B(N) \subset c(B(K_x), x)$ for all $x \in X$. Moreover, by continuity of c, we can find a small open neighborhood $V_x \ni x$ such that $B(N) \subset c(B(K_x), y)$ for all $y \in V_x$. Since $X = \cup_{x \in X} V_x$, compactness of X implies we can find a finite subcover $X = \cup_{i=1}^n V_{x_i}$, set $K = \max_{1 \leq i \leq n} K_{x_i}$.

Let $N := \max_{x \in X, g \in S} |c(g, x)|$, and K as above for this N. Then for all $x \in X$,

$$\text{either} \quad c(s^{>K}, x) \subseteq \{s^{>0}, ts^{>0}\} \text{ and } c(s^{<-K}, x) \subseteq \{s^{<0}, ts^{<0}\}$$

$$\text{or} \quad c(s^{>K}, x) \subseteq \{s^{<0}, ts^{<0}\} \text{ and } c(s^{<-K}, x) \subseteq \{s^{>0}, ts^{>0}\}.$$

Here, $c(s^{>K}, x) := \{c(s^i, x): i > K\}$, and $c(s^{-K}, x)$ is similarly defined; $\{s^{>0}, ts^{>0}\} :=$ $\{s^i, ts^j: i > 0, \ j > 0\}$ and $\{s^{<0}, ts^{<0}\}$ is similarly defined.

To see this, for each $|k| > K$, we have $c(s^k, x) \notin B(N)$ as $c(-, x)$ is bijective. Since $|c(s^{\pm 1}, s^k x)| \leq N$, from $c(s^{\pm 1 + k}, x) = c(s^{\pm 1}, s^k x)c(s^k, x)$, we deduce that $c(s^{\pm 1 + k}, x)$ and $c(s^k, x)$ must have the same sign for the s-exponent when writing them as elements in $t\mathbb{Z} = t\langle s \rangle$ or $\mathbb{Z} = \langle s \rangle$. Then apply Claim 4.2 to see that if $c(s^{>K}, x) \subseteq \{s^{>0}, ts^{>0}\}$ holds, then $x \in X_+$ and thus $c(s^{-K}, x) \subseteq \{s^{<0}, ts^{<0}\}$ is automatic.

Now, denote by $X'_+ := \{x \in X: c(s^{>K}, x) \subseteq \{s^{>0}, ts^{>0}\}, c(s^{<-K}, x) \subseteq \{s^{<0}, ts^{<0}\}\}$ and $X'_- := \{x \in X: c(s^{>K}, x) \subseteq \{s^{<0}, ts^{<0}\}, c(s^{<-K}, x) \subseteq \{s^{>0}, ts^{>0}\}\}$. From above, we have shown $X'_+ \sqcup X'_- = X = X_+ \sqcup X_-$. It is easy to see that $X'_+ = X_+$ and $X'_- = X_-$. Hence, to finish the proof, we just need to observe that both X'_+ and X'_- are closed as c is continuous and D_∞ is countable and discrete. ∎

Define $a: \mathbb{Z} \times X \to D_\infty$ by setting

$$a(s^n, x) = \begin{cases} c(s^n, x)s^{-n} & \text{if } x \in X_+, \\ c(s^n, x)s^n & \text{if } x \in X_-. \end{cases}$$

Claim 4.4. *The map a is a continuous cocycle taking finitely many values as n changes.*

Proof. From Claim 4.2, we know that a takes only finitely many values. Fix any $x \in X_+$ and $n \in \mathbb{Z}$. We have

$$a(s^{m+n}, x) = c(s^{m+n}, x)s^{-m-n} = c(s^m, s^n x)c(s^n, x)s^{-n}s^{-m}$$
$$= \begin{cases} a(s^m, s^n x)s^m a(s^n, x)s^{-m} & \text{if } s^n x \in X_+, \\ a(s^m, s^n x)s^{-m} a(s^n, x)s^{-m} & \text{if } s^n x \in X_-. \end{cases}$$

Case 1. $s^n x \in X_+$. Since both $a(s^{m+n}, x)$ and $a(s^m, s^n x)$ lie in a finite set as m changes, we deduce that $\{s^m a(s^n, x)s^{-m}: m \in \mathbb{Z}\}$ is a finite set. Therefore, we have $a(s^n, x) \in \mathbb{Z}$ and thus $a(s^{m+n}, x) = a(s^m, s^n x)a(s^n, x)$.

Case 2. $s^n x \in X_-$. Since both $a(s^{m+n}, x)$ and $a(s^m, s^n x)$ lie in a finite set as m changes, we deduce that $\{s^{-m} a(s^n, x)s^{-m}: m \in \mathbb{Z}\}$ is a finite set. Thus, $a(s^n, x) \in \mathbb{Z}t$. Once again, this implies that $a(s^{m+n}, x) = a(s^m, s^n x)a(s^n, x)$. To sum up, we have shown that for all $x \in X_+$, $a(s^{m+n}, x) = a(s^m, s^n x)a(s^n, x)$ holds for all $m, n \in \mathbb{Z}$.

Similarly, one can show this also holds for all $x \in X_-$. We include the details for completeness.

Fix any $x \in X_-$ and $n \in \mathbb{Z}$. We have

$$a(s^{m+n}, x) = c(s^{m+n}, x)s^{m+n} = c(s^m, s^n x)c(s^n, x)s^{m+n}$$
$$= \begin{cases} a(s^m, s^n x)s^m a(s^n, x)s^m & \text{if } s^n x \in X_+, \\ a(s^m, s^n x)s^{-m} a(s^n, x)s^m & \text{if } s^n x \in X_-. \end{cases}$$

(i) $s^n x \in X_+$. Since $a(s^{m+n}, x)$ and $a(s^m, s^n x)$ take finitely many values as m changes, we get that $\{s^m a(s^n, x) s^m : m \in \mathbb{Z}\}$ is a finite set. Therefore, $a(s^n, x) \in \mathbb{Z}t$, which implies that $a(s^{m+n}, x) = a(s^m, s^n x) a(s^n, x)$.

(ii) $s^n x \in X_-$. This case can be checked similarly. ■

Define $D : X \to D_\infty$ by

$$D(x) = \begin{cases} t & \text{if } x \in X_-, \\ e & \text{if } x \in X_+. \end{cases}$$

From Claim 4.3, we know D is a continuous map. Moreover, one can verify that

$$D(s^n x) a(s^n, x) D(x)^{-1} \in \mathbb{Z} \quad \text{for all } x \in X \text{ and } n \in \mathbb{Z}.$$

Indeed, one can check this by considering four cases depending on $x \in X_\pm$ and $s^n x \in X_\pm$. We explain the proof for the case $x \in X_+$ and $s^n x \in X_-$ below, the other three cases can be checked similarly.

By assumption, there exists a finite subset $F \subset D_\infty$ such that $c(s^k, x) s^{-k} \in F \ni c(s^k, s^n x) s^k$ for all $k \in \mathbb{Z}$. From the cocycle identity $c(s^n, x) = c(s^k, s^n x)^{-1} c(s^{k+n}, x)$, we deduce that $c(s^n, x) = s^k g_k s^{k+n}$ for some $g_k \in F^{-1} F$ for all $k \in \mathbb{Z}$. Taking a sufficiently large k, we deduce that $g_k \in t\mathbb{Z} = t\langle s \rangle$ and thus $c(s^n, x) \in t\langle s \rangle$. Therefore, $D(s^n x) a(s^n, x) D(x)^{-1} = tc(s^n, x) s^{-n} \in \mathbb{Z} = \langle s \rangle$.

Thus, the map $a' : \mathbb{Z} \times X \to \mathbb{Z} \subset D_\infty$ defined by

$$a'(s^n, x) := D(s^n x) a(s^n, x) D(x)^{-1}$$

is a continuous cocycle taking finitely many values in $\mathbb{Z}$.

Take a $\mathbb{Z}$-minimal component, say X_0, i.e., $X_0 = \overline{\mathrm{Orb}(\mathbb{Z}, x_0)}$ for some $x_0 \in X$ is minimal with respect to the sub-$\mathbb{Z}$-action. Then since $\mathbb{Z} \lhd D_\infty$, we know $tX_0 = \overline{\mathrm{Orb}(\mathbb{Z}, tx_0)}$ is also a minimal $\mathbb{Z}$-component. Moreover, $X_0 \cup tX_0$ is a G-invariant closed subset, hence $X = X_0 \cup tX_0$ by minimality assumption. Clearly, either $X_0 \cap tX_0 = \emptyset$ or $X_0 = tX_0$.

At this stage, we need to split the proof into two cases.

Case I. $X_0 = tX_0$, i.e., the sub-$\mathbb{Z}$-action $\mathbb{Z} \curvearrowright X$ is still minimal. We can think of a' as a cocycle taking values in $\mathbb{R}$ via the natural inclusion $\mathbb{Z} \hookrightarrow \mathbb{R}$. As we assume $\mathbb{Z} \curvearrowright X$ is still minimal, we can apply Gottschalk–Hedlund theorem, i.e., Theorem 2.8 and Lemma 2.3 to deduce there exists some continuous map $L : X \to \mathbb{Z}$ such that $a'(s^n, x) = L(s^n x)^{-1} L(x)$. Thus, letting $L'(x) = L(x) D(x)$, we deduce that

$$a(s^n, x) = L'(s^n x)^{-1} L'(x).$$

Therefore,

$$c(s^n, x) = \begin{cases} L'(s^n x)^{-1} L'(x) s^n & \text{if } x \in X_+, \\ L'(s^n x)^{-1} L'(x) s^{-n} & \text{if } x \in X_- \end{cases}$$
$$= L'(s^n x)^{-1} s^n L'(x) \quad \text{for all } x \in X \text{ and } n \in \mathbb{Z}.$$

Here, the last equality can be checked by observing that $L(x) \in \mathbb{Z}$ which is abelian and t acts on $\mathbb{Z}$ as a reflection.

Claim 4.5. *There exists some $k \in \mathbb{Z}$ such that $c(t, x) = L'(tx)^{-1}(s^k t)L'(x)$ for all $x \in X$.*

Proof. Notice that on the one hand,

$$(s^{-n}, x) = L'(s^{-n}x)^{-1}s^{-n}L'(x),$$

on the other hand,

$$
\begin{aligned}
c(s^{-n}, x) = c(ts^n t^{-1}, x) &= c(t, s^n t^{-1}x)c(s^n, t^{-1}x)c(t^{-1}, x) \\
&= c(t, s^n t^{-1}x)L'(s^n t^{-1}x)^{-1}s^n L'(t^{-1}x)c(t^{-1}, x).
\end{aligned}
$$

Thus,

$$s^{-n}(L'(x)c(t^{-1}, x)^{-1}L'(t^{-1}x)^{-1})s^{-n} = L'(s^{-n}x)c(t, s^n t^{-1}x)L'(s^n t^{-1}x)^{-1}. \quad (4.4)$$

Since the right-hand side of the above takes only finitely many values as n changes, we deduce that $L'(x)c(t^{-1}, x)^{-1}L'(t^{-1}x)^{-1} \in \mathbb{Z}t$. Write $L'(x)c(t^{-1}, x)^{-1}L'(t^{-1}x)^{-1} = s^{k(x)}t$ for some $k(x) \in \mathbb{Z}$. It is clear from definition that $X \ni x \mapsto k(x) \in \mathbb{Z}$ is continuous. Next, from (4.4), we can deduce that $c(t^{-1}, x)^{-1} = L'(x)^{-1}s^{k(x)}tL'(t^{-1}x)$ and $c(t, s^n t^{-1}x) = L'(s^{-n}x)^{-1}s^{k(x)}tL'(s^n t^{-1}x)$. Since $t^2 = e$, the above implies that $k(x) = k(s^n x)$ for all $n \in \mathbb{Z}$ and $x \in X$. Since the sub-$\mathbb{Z}$-action is minimal and $k(\cdot)$ is continuous, we get that $k(x) \equiv k$ for all $x \in X$. ∎

To sum up, we have shown that

$$c(s^n, x) = L'(s^n x)^{-1}s^n L'(x), \ c(t, x) = L'(tx)^{-1}s^k t L'(x) \quad \text{for all } n \in \mathbb{Z}, \ x \in X.$$

Clearly, this is equivalent to say c is cohomologous to $\phi \in \text{Aut}(D_\infty)$, where ϕ is given by $\phi(s) = s$ and $\phi(t) = s^k t$ as mentioned in Lemma 2.4. Finally, we can apply [19, Proposition 4.4] to conclude that the two actions are conjugate.

Case II. $X_0 \cap tX_0 = \emptyset$, i.e., $X = X_0 \sqcup tX_0$ and hence X_0 is a clopen subset. We apply Gottschalk–Hedlund theorem to the minimal subactions $\mathbb{Z} \curvearrowright X_0$ and $\mathbb{Z} \curvearrowright tX_0$. Argue similarly to case I, we can find two continuous maps $L'': X_0 \to D_\infty$ and $L': tX_0 \to D_\infty$ such that

$$c(s^n, x) = \begin{cases} L''(s^n x)^{-1}s^n L''(x) & \text{for all } x \in X_0 \text{ and } n \in \mathbb{Z}, \\ L'(s^n x)^{-1}s^n L'(x) & \text{for all } x \in tX_0 \text{ and } n \in \mathbb{Z}. \end{cases} \quad (4.5)$$

Fix any $x \in X_0$, we compute as follows:

$$
\begin{aligned}
c(s^{-n}, x) &= L''(s^{-n}x)^{-1}s^{-n}L''(x), \\
c(s^{-n}, x) = c(ts^n t^{-1}, x) &= c(t, s^n t^{-1}x)c(s^n, t^{-1}x)c(t^{-1}, x) \\
&= c(t, s^n t^{-1}x)[L'(s^n t^{-1}x)^{-1}s^n L'(t^{-1}x)]c(t^{-1}, x).
\end{aligned}
$$

Thus, we deduce

$$L''(s^{-n}x)c(t, s^n t^{-1}x)L'(s^n t^{-1}x)^{-1} = s^{-n}[L''(x)c(t^{-1}, x)^{-1}L'(t^{-1}x)^{-1}]s^{-n}.$$

As $\{L''(s^{-n}x)c(t, s^n t^{-1}x)L'(s^n t^{-1}x)^{-1}: n \in \mathbb{Z}\}$ is a finite set, we deduce that

$$L''(x)c(t^{-1}, x)^{-1}L'(t^{-1}x)^{-1}$$
$$= L''(s^{-n}x)c(t, s^n t^{-1}x)L'(s^n t^{-1}x)^{-1} \in \mathbb{Z}t \quad \text{for all } n \in \mathbb{Z}.$$

Let us write

$$L''(x)c(t^{-1}, x)^{-1}L'(t^{-1}x)^{-1} = s^{k(x)}t \tag{4.6}$$

for some map $k: X_0 \to \mathbb{Z}$. Clearly, $k(\cdot)$ is a continuous map. Moreover, since X_0 is $\mathbb{Z} = \langle s \rangle$-invariant, from the above identity, we deduce that

$$L''(s^{-n}x)c(t^{-1}, s^{-n}x)^{-1}L'(t^{-1}s^{-n}x)^{-1} = s^{k(s^{-n}x)}t,$$
$$[L''(s^{-n}x)c(t, s^n t^{-1}x)L'(s^n t^{-1}x)^{-1}]^{-1} = [s^{k(x)}t]^{-1}.$$

Multiply the above two expressions and use $t^2 = e$, we deduce that $e = s^{k(s^{-n}x)-k(x)}$, i.e., $k(s^{-n}x) = k(x)$ for all $n \in \mathbb{Z}$. Since X_0 is $\mathbb{Z} = \langle s \rangle$-minimal, we deduce that k is constant, say $k(x) = k$ for all $x \in X_0$. Then, (4.6) can be simplified to

$$L'(tx) = L'(t^{-1}x) = t^{-1}s^{-k}L''(x)c(t^{-1}, x)^{-1}.$$

From (4.5), a calculation shows that for each $x \in X_0$,

$$c(s^n, tx) = L'(s^n tx)^{-1}s^n L'(tx) = L'(ts^{-n}x)^{-1}s^n L'(tx)$$
$$= [t^{-1}s^{-k}L''(s^{-n}x)c(t^{-1}, s^{-n}x)^{-1}]^{-1}s^n[t^{-1}s^{-k}L''(x)c(t^{-1}, x)^{-1}]$$
$$= c(t^{-1}, s^{-n}x)L''(s^{-n}x)^{-1}s^{-n}L''(x)c(t^{-1}, x)^{-1}.$$

In fact, using cocycle identity, we can compute the full expression for c. More precisely, for all $x \in X_0$, we have

$$\begin{aligned}
c(s^n, x) &= L''(s^n x)^{-1}s^n L''(x), \\
c(s^n t, x) &= c(t^{-1}, s^{-n}x)L''(s^{-n}x)^{-1}s^{-n}L''(x), \\
c(s^n, tx) &= c(t^{-1}, s^{-n}x)L''(s^{-n}x)^{-1}s^{-n}L''(x)c(t^{-1}, x)^{-1}, \\
c(s^n t, tx) &= L''(s^n x)^{-1}s^n L''(x)c(t, tx).
\end{aligned} \tag{4.7}$$

Now, we remind that the two actions considered in Theorem 1.3 are α and β. But for simplicity, we have omitted writing α for the first action. So, we have $gx = \beta_{c(g,x)}(x)$ for all $g \in D_\infty$ and $x \in X$. Define a map

$$X = X_0 \sqcup tX_0 \xrightarrow{\phi} X, \quad \phi(x) = \beta_{L''(x)}(x), \quad \phi(tx) = \beta_{t^{-1}L''(x)}(x) \quad \text{for all } x \in X_0.$$

Clearly, ϕ is continuous as X_0 is clopen and both c and $L'': X_0 \to D_\infty$ are continuous.

Claim 4.6. *The map ϕ is the desired conjugacy between the two actions, i.e., ϕ is a homeomorphism and $\phi(gx) = \beta_g(\phi(x))$ for all $g \in D_\infty$ and $x \in X$.*

Proof. The conjugacy identity can be checked by computation using $\beta_{c(g,x)}(x) = gx$ and the above expression for $c(g,x)$ in (4.7). We are left to show that ϕ is a bijection.

Injectivity. First, we check that $\phi|_{X_0}$ is injective. Indeed, take any $x, y \in X_0$, suppose $\phi(x) = \phi(y)$, that is, $\beta_{L''(x)}(x) = \beta_{L''(y)}(y)$, i.e., $\beta_{L''(y)^{-1}L''(x)}(x) = y$. Since $D_\infty \ni g \mapsto c(g,x) \in D_\infty$ is a bijection by Proposition 2.2, we can find some $g \in D_\infty$ such that $c(g,x) = L''(y)^{-1}L''(x)$.

Thus, $y = \beta_{c(g,x)}(x) = gx$. Since $x, y \in X_0$ and $X_0 \cap tX_0 = \emptyset$, we deduce that $g \in \mathbb{Z}$, say $g = s^n$. Hence, $c(s^n, x) = L''(s^n x)^{-1}L''(x)$. In view of (4.7), we deduce $s^n = e$, i.e., $x = y$. Clearly, this also shows that $\phi|_{tX_0}$ is injective.

It remains for us to show that $\phi(x) \neq \phi(ty)$ for any $x, y \in X_0$. Assume not, then $\beta_{L''(x)}(x) = \beta_{t^{-1}L''(y)}(y)$, i.e., $\beta_{L''(y)^{-1}tL''(x)}(x) = y$. We can find some $g \in D_\infty$ such that $c(g,x) = L''(y)^{-1}tL''(x)$. Hence, $y = \beta_{c(g,x)}(x) = gx$. So $g \in \mathbb{Z}$, say $g = s^n$. Thus, $c(s^n, x) = L''(s^n x)^{-1}tL''(x)$. In view of (4.7), this implies $s^n = t$, a contradiction.

Surjectivity. First, we observe that the second action also has two distinct minimal $\mathbb{Z}$-components. Suppose not, we can apply the proof of case I to the second action to see the first action is conjugate to the second one.

Thus, these two actions must have the same number of minimal $\mathbb{Z}$-components (as each automorphism of D_∞ must fix the subgroup $\mathbb{Z} = \langle s \rangle$ globally), but this contradicts our assumption in case II.

Let us write the two minimal $\mathbb{Z}$-components of the second action as Y_0 and $\beta_t(Y_0)$, i.e., $X = Y_0 \sqcup \beta_t(Y_0)$. Now, since $\phi(gx) = \beta_g(\phi(x))$ for all $g \in D_\infty$ and all $x \in X$, we know $\phi(X_0), \phi(tX_0) \in \{Y_0, \beta_t(Y_0)\}$. As ϕ is injective and $X_0 \cap tX_0 = \emptyset$, we deduce that $\{\phi(X_0), \phi(tX_0)\} = \{Y_0, \beta_t(Y_0)\}$, so ϕ is surjective.

This concludes the proof of Theorem 1.3. ∎

Remark 4.7. From [5, 6], we know that for every finitely generated one-ended group G, its full shifts $G \curvearrowright A^G$ for finite A are continuous cocycle superrigid actions with respect to any countable target groups. It was mentioned in [5, Corollary 5] that this can be combined with [19, Theorem 1.6] to deduce that the full shifts as above are continuous orbit equivalence rigid actions if G is further assumed to be torsion-free and amenable. In fact, this further assumption is unnecessary. Indeed, we just observe that from continuous cocycle superrigidity, we deduce $c(g,x) = L(gx)^{-1}\phi(g)L(x)$. Then take x to be any fixed point for this full shift action and use $g \mapsto c(g,x)$ is a bijection for all $x \in X$, we deduce that $g \mapsto \phi(g)$ is automatically a group isomorphism of G. Hence, we can just apply [19, Proposition 4.4] instead to deduce that for every finitely generated one-ended group G, its full shifts as above are continuous orbit equivalence rigid actions.

5. Concluding remarks

Here are several remarks on the theorems and possible generalizations.

(1) Let F be any non-trivial finite abelian group. From the proof of Proposition 4.1, we know that if $\mathbb{Z} \curvearrowright X$ is a minimal action admitting a continuous cocycle $c: \mathbb{Z} \times X \to F$ which is not cohomologous to a group homomorphism from $\mathbb{Z}$ to $C(F)$, the center of F, then we can construct two continuously orbit equivalent but not conjugate actions.

In fact, for each non-trivial finite abelian group F such a minimal action $\mathbb{Z} \curvearrowright X$ and a cocycle as above do exist (by taking $\mathbb{Z} \curvearrowright X$ to be a well-chosen Toeplitz system [11, Example 1.20], as shown to me by Prof. Lemańczyk). By combining this with Lemma 2.3, we can deduce that for each finite non-trivial group F, $F \times \mathbb{Z}$ admits two continuously orbit equivalent but not conjugate minimal actions.

(2) The construction in the proof of Proposition 4.1 (ii) can be modified to deal with a general semi-direct product group $G = F \rtimes_\sigma \mathbb{Z}$ for a finite group F. Since $\mathrm{Aut}(F \rtimes_\sigma \mathbb{Z})$ becomes more involved and the existence of minimal actions that admit nontrivial skew cocycles into finite groups is not clear to us, we do not study the full generality here.

(3) Let $G = D_\infty$. Then we take a $\mathbb{Z}$-minimal component, for example X_0, i.e., $X_0 = \overline{\mathrm{Orb}(\mathbb{Z}, x_0)}$ for some $x_0 \in X$ is minimal with respect to the sub-$\mathbb{Z}$-action. Then since $\mathbb{Z} \lhd G$, we know $tX_0 = \overline{\mathrm{Orb}(\mathbb{Z}, tx_0)}$ is also a minimal $\mathbb{Z}$-component. Moreover, $X_0 \cup tX_0$ is a G-invariant closed subset, hence $X = X_0 \cup tX_0$ if we assume $G \curvearrowright X$ is minimal.

We observe below that once we have two minimal $\mathbb{Z}$-components, the G-action is an induced action.

Proposition 5.1. *Under the above notations and assumptions, if $X_0 \cap tX_0 = \emptyset$, then the action $G \curvearrowright X = X_0 \sqcup tX_0$ is conjugate to the induced action $G \curvearrowright G/\mathbb{Z} \times X_0$ associated to a cocycle $\delta: G \times G/\mathbb{Z} \to \mathbb{Z}$.*

Proof. Indeed, consider the natural lift map $L: G/\mathbb{Z} \to G$ which given by $L(\mathbb{Z}) = e$ and $L(t\mathbb{Z}) = t$. Then for the associated cocycle $\delta: G \times G/\mathbb{Z} \to \mathbb{Z}$ given by $\delta(g, g'\mathbb{Z}) = L(gg'\mathbb{Z})^{-1} g L(g'\mathbb{Z})$ we have

$$\delta(s^n, \mathbb{Z}) = s^n, \quad \delta(s^n, t\mathbb{Z}) = s^{-n}, \quad \delta(s^n t, \mathbb{Z}) = s^{-n}, \quad \delta(s^n t, t\mathbb{Z}) = s^n$$

for all $n \in \mathbb{Z}$. Now, define $\psi: X \to G/\mathbb{Z} \times X_0$ by setting

$$\psi(x) = \begin{cases} (\mathbb{Z}, x) & \text{if } x \in X_0, \\ (t\mathbb{Z}, tx) & \text{if } x \in tX_0. \end{cases}$$

Clearly, ψ is a bijection and continuous (as X_0 is clopen). One can check that $\psi(gx) = g\psi(x)$ for all $g \in G$ and $x \in X$.

Hence, ψ is a conjugacy between $G \curvearrowright X = X_0 \sqcup t X_0$ and the induced action $G \curvearrowright G/\mathbb{Z} \times X_0$ for a free minimal action $\mathbb{Z} \curvearrowright X_0$ (i.e., the sub-$\mathbb{Z}$-action) with respect to the above chosen cocycle c. ∎

Next, consider the two actions $G \curvearrowright G/\mathbb{Z} \times X_0$ induced from any two minimal topologically free $\mathbb{Z}$-actions on X_0 defined above, and let ϕ be a homeomorphism which induces a continuous orbit equivalence between these two actions. Suppose we know that $\phi = \mathrm{id}$, then one can show that the two actions are conjugate directly. This shows that the main difficulty for dealing with case II in the proof of Theorem 1.3 is that we do not know, a priori, how the homeomorphism behaves on the two minimal $\mathbb{Z}$-components.

Before proving the above assertion, let us fix some notation for the induced actions. Fix any lift $L: G/\mathbb{Z} \to G$, i.e., $L(g\mathbb{Z})\mathbb{Z} = g\mathbb{Z}$ for all $g \in G$. We can further assume $L(\mathbb{Z}) = e$ and $L(t\mathbb{Z}) = t$. Then let $\delta: G \times G/\mathbb{Z} \to \mathbb{Z}$ be the associated cocycle $\delta(g, s\mathbb{Z}) = L(gs\mathbb{Z})^{-1} g L(\mathbb{Z})$. Observe that $\delta(s, \mathbb{Z}) = L(s\mathbb{Z})^{-1} s L(\mathbb{Z}) = s$ for all $s \in \mathbb{Z}$.

Fix a minimal topologically free action $\alpha: \mathbb{Z} \curvearrowright X_0$, then the induced action $\tilde{\alpha}: G \curvearrowright X = G/\mathbb{Z} \times X_0$ is defined as follows: $\tilde{\alpha}_s(g\mathbb{Z}, x) := (sg\mathbb{Z}, \alpha_{\delta(s, g\mathbb{Z})}(x))$ for all $s, g \in G$ and $x \in X_0$.

Now it suffices to show the following holds.

Proposition 5.2. *Let $\alpha, \beta: \mathbb{Z} \curvearrowright X_0$ be two minimal topologically free actions. Also let $\tilde{\alpha}, \tilde{\beta}: G \curvearrowright X := G/\mathbb{Z} \times X_0$ be the associated induced actions as recalled above. Then the following holds:*

(1) $\tilde{\alpha} \overset{\mathrm{coe}}{\sim} \tilde{\beta}$ *via the identity homeomorphism* $\Rightarrow \alpha \overset{\mathrm{coe}}{\sim} \beta$;

(2) $\alpha \overset{\mathrm{coe}}{\sim} \beta \Rightarrow \tilde{\alpha} \overset{\mathrm{conj}}{\sim} \tilde{\beta}$.

Proof. (1) Let $c: G \times X \to G$ be the continuous orbit cocycle with respect to $\tilde{\alpha}$, i.e., $\tilde{\alpha}_g(\tilde{x}) = \tilde{\beta}_{c(g, \tilde{x})}(\tilde{x})$, where $\tilde{x} = (s\mathbb{Z}, x) \in X$ is any point. Then a calculation using the definition of the induced actions shows that

$$gs\mathbb{Z} = c(g, \tilde{x})s\mathbb{Z}, \quad \alpha_{\delta(g, s\mathbb{Z})}(x) = \beta_{\delta(c(g, \tilde{x}), s\mathbb{Z})}(x) \quad \forall g, s \in G, \ \forall x \in X_0.$$

This implies that $c(g, \tilde{x})\mathbb{Z} = g\mathbb{Z}$ as $\mathbb{Z} \lhd G$. In particular, $c(s, \tilde{x}) \in \mathbb{Z}$ for all $s \in \mathbb{Z}$.

Now, we define the map $\theta: \mathbb{Z} \times X_0 \to \mathbb{Z}$ given by $\theta(s, x) = c(s, \tilde{x})$, where $\tilde{x} := (\mathbb{Z}, x)$. Clearly, θ is well-defined and continuous. Then, we verify that θ is a cocycle with respect to α. Take any $s_1, s_2 \in \mathbb{Z}$ and $x \in X_0$, we have

$$\theta(s_1 s_2, x) = c(s_1 s_2, \tilde{x}) = c(s_1, \tilde{\alpha}_{s_2}(\tilde{x}))c(s_2, \tilde{x}),$$

$$\theta(s_1, \alpha_{s_2}(x))\theta(s_2, x) = c(s_1, \widetilde{\alpha_{s_2}(x)})c(s_2, \tilde{x}).$$

It suffices to check that $\tilde{\alpha}_{s_2}(\tilde{x}) = \widetilde{\alpha_{s_2}(x)}$. Recall that

$$\tilde{x} = (\mathbb{Z}, x) \quad \text{and} \quad \tilde{\alpha}_{s_2}(\tilde{x}) = (s_2\mathbb{Z}, \alpha_{\delta(s_2, \mathbb{Z})}x) = (\mathbb{Z}, \alpha_{s_2}(x)).$$

Meanwhile, $\widetilde{\alpha_{s_2}(x)} = (\mathbb{Z}, \alpha_{s_2}(x))$.

Now, let us check that θ gives us a coe between α and β. Indeed, we have

$$\alpha_s(x) = \alpha_{\delta(s,\mathbb{Z})}(x) = \beta_{\delta(c(s,\tilde{x}),\mathbb{Z})}(x) = \beta_{L(c(s,\tilde{x})\mathbb{Z})^{-1}c(s,\tilde{x})L(\mathbb{Z})}(x)$$
$$= \beta_{L(s\mathbb{Z})^{-1}\theta(s,x)}(x) = \beta_{\theta(s,x)}(x) \quad \forall s \in \mathbb{Z}, \ \forall x \in X_0.$$

(2) Let $\theta: \mathbb{Z} \times X_0 \to \mathbb{Z}$ be the cocycle from the coe. By [4], we know that $\theta(s,x) = f(\alpha_s(x))^{-1}s^{\pm}f(x)$ for some continuous map $f: X \to \mathbb{Z}$. One can check $X_0 \ni x \mapsto \beta_{f(x)}(x) \in X_0$ intertwines α_s with $\beta_{s^{\pm}}$, which implies $Y \ni (g\mathbb{Z}, x) \mapsto (g\mathbb{Z}, \beta_{f(x)}(x)) \in X$ intertwines $\tilde{\alpha_g}$ with $\tilde{\beta}_{\tau(g)}$. Here, $\tau \in \mathrm{Aut}(G)$ is defined as follows:

$$\tau = \begin{cases} \mathrm{id} & \text{if } \theta(s,x) = f(\alpha_s(x))^{-1}sf(x) \text{ for all } s \in \mathbb{Z}, \\ \phi & \text{if } \theta(s,x) = f(\alpha_s(x))^{-1}s^{-1}f(x) \text{ for all } s \in \mathbb{Z}. \end{cases}$$

Here, $\tau \in \mathrm{Aut}(G)$ is determined by $\tau(s) = s^{-1}$ for all $s \in \mathbb{Z}$ and $\tau(t) = t$ for the reflection $t \in G$. ∎

References

[1] M. Amini, K. Li, D. Sawicki, and A. Shakibazadeh, Dynamic asymptotic dimension for actions of virtually cyclic groups. *Proc. Edinb. Math. Soc. (2)* **64** (2021), no. 2, 364–372 Zbl 1478.37037 MR 4277767

[2] I. Benjamini and A. Shamov, Bi-Lipschitz bijections of $\mathbb{Z}$. *Anal. Geom. Metr. Spaces* **3** (2015), no. 1, 313–316 Zbl 1325.26011 MR 3412164

[3] L. Bowen, Measure conjugacy invariants for actions of countable sofic groups. *J. Amer. Math. Soc.* **23** (2010), no. 1, 217–245 Zbl 1201.37005 MR 2552252

[4] M. Boyle and J. Tomiyama, Bounded topological orbit equivalence and C^*-algebras. *J. Math. Soc. Japan* **50** (1998), no. 2, 317–329 Zbl 0940.37004 MR 1613140

[5] N.-P. Chung and Y. Jiang, Continuous cocycle superrigidity for shifts and groups with one end. *Math. Ann.* **368** (2017), no. 3–4, 1109–1132 Zbl 1384.37007 MR 3673649

[6] D. B. Cohen, Continuous cocycle superrigidity for the full shift over a finitely generated torsion group. *Int. Math. Res. Not. IMRN* **2020** (2020), no. 6, 1610–1620 Zbl 1477.22014 MR 4089429

[7] M. I. Cortez and K. Medynets, Orbit equivalence rigidity of equicontinuous systems. *J. Lond. Math. Soc. (2)* **94** (2016), no. 2, 545–556 Zbl 1367.37007 MR 3556453

[8] T. Giordano, H. Matui, I. F. Putnam, and C. F. Skau, Orbit equivalence for Cantor minimal $\mathbb{Z}^2$-systems. *J. Amer. Math. Soc.* **21** (2008), no. 3, 863–892 Zbl 1254.37012 MR 2393431

[9] T. Giordano, H. Matui, I. F. Putnam, and C. F. Skau, Orbit equivalence for Cantor minimal $\mathbb{Z}^d$-systems. *Invent. Math.* **179** (2010), no. 1, 119–158   Zbl 1293.37004   MR 2563761

[10] T. Giordano, I. F. Putnam, and C. F. Skau, Topological orbit equivalence and C^*-crossed products. *J. Reine Angew. Math.* **469** (1995), 51–111   Zbl 0834.46053   MR 1363826

[11] E. Glasner, *Ergodic theory via joinings*. Math. Surveys Monogr. 101, Amer. Math. Soc., Providence, RI, 2003   Zbl 1038.37002   MR 1958753

[12] W. H. Gottschalk and G. A. Hedlund, *Topological dynamics*. Amer. Math. Soc. Colloq. Publ. 36, Amer. Math. Soc., Providence, R.I., 1955   Zbl 0067.15204   MR 0074810

[13] A. Katok and B. Hasselblatt, *Introduction to the modern theory of dynamical systems*. Encyclopedia Math. Appl. 54, Cambridge University Press, Cambridge, 1995   Zbl 0878.58020   MR 1326374

[14] D. Kerr and H. Li, Entropy and the variational principle for actions of sofic groups. *Invent. Math.* **186** (2011), no. 3, 501–558   Zbl 1417.37041   MR 2854085

[15] D. Kerr and H. Li, *Ergodic theory. Independence and dichotomies*. Springer Monogr. Math., Springer, Cham, 2016   Zbl 1396.37001   MR 3616077

[16] D. Kerr and H. Li, Entropy, products, and bounded orbit equivalence. *Ergodic Theory Dynam. Systems* (2021),   DOI 10.1017/etds.2021.154

[17] D. Kerr and R. Tucker-Drob, Dynamical alternating groups, stability, property Gamma, and inner amenability. To appear in *Ann. Sci. Éc. Norm. Supér. (4)*

[18] M. Lemańczyk and M. K. Mentzen, Topological ergodicity of real cocycles over minimal rotations. *Monatsh. Math.* **134** (2002), no. 3, 227–246   Zbl 1002.54023   MR 1883503

[19] X. Li, Continuous orbit equivalence rigidity. *Ergodic Theory Dynam. Systems* **38** (2018), no. 4, 1543–1563   Zbl 1390.37007   MR 3789176

[20] X. Li, Dynamic characterizations of quasi-isometry and applications to cohomology. *Algebr. Geom. Topol.* **18** (2018), no. 6, 3477–3535   Zbl 06990070   MR 3868227

[21] K. Medynets, R. Sauer, and A. Thom, Cantor systems and quasi-isometry of groups. *Bull. Lond. Math. Soc.* **49** (2017), no. 4, 709–724   Zbl 1377.37020   MR 3725491

[22] E. Ortega and E. Scarparo, Almost finiteness and homology of certain non-free actions. *Groups Geom. Dyn.* (2022),   DOI 10.4171/GGD/656

[23] E. Scarparo, Homology of odometers. *Ergodic Theory Dynam. Systems* **40** (2020), no. 9, 2541–2551   Zbl 1460.46065   MR 4130816

[24] K. Schmidt, *Cocycles on ergodic transformation groups*. Macmillan Lectures in Math. 1, Macmillan Co. of India, Ltd., Delhi, 1977   Zbl 0421.28017   MR 0578731

[25] K. Schmidt, The cohomology of higher-dimensional shifts of finite type. *Pacific J. Math.* **170** (1995), no. 1, 237–269   Zbl 0866.28016   MR 1359979

[26] K. Schmidt, Tilings, fundamental cocycles and fundamental groups of symbolic $\mathbf{Z}^d$-actions. *Ergodic Theory Dynam. Systems* **18** (1998), no. 6, 1473–1525   Zbl 0915.58030   MR 1658619

[27] Y. Suzuki, Amenable minimal Cantor systems of free groups arising from diagonal actions. *J. Reine Angew. Math.* **722** (2017), 183–214   Zbl 1356.43001   MR 3589352

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