Groups Geom. Dyn. 17 (2023), 533–554
DOI 10.4171/GGD/708

Subgroups of $\mathrm{PL}_+ I$ which do not embed into Thompson's group F

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Abstract. We will give a general criterion—the existence of an F-*obstruction*—for showing that a subgroup of $\mathrm{PL}_+ I$ does not embed into Thompson's group F. An immediate consequence is that Cleary's "golden ratio" group F_τ does not embed into F, answering a question of Burillo, Nucinkis, and Reves. Our results also yield a new proof that Stein's groups $F_{p,q}$ do not embed into F, a result first established by Lodha using his theory of coherent actions. We develop the basic theory of F-obstructions and show that they exhibit certain rigidity phenomena of independent interest. In the course of establishing the main result of the paper, we prove a dichotomy theorem for subgroups of $\mathrm{PL}_+ I$. In addition to playing a central role in our proof, it is strong enough to imply both Rubin's reconstruction theorem restricted to the class of subgroups of $\mathrm{PL}_+ I$ and also Brin's ubiquity theorem.

1. Introduction

In this article, we aim to give a partial answer to the following question: *When does a group of piecewise linear homeomorphisms of the unit interval fail to embed into Richard Thompson's group F?* Thompson's group F is the subgroup of $\mathrm{PL}_+ I$ consisting of those functions whose breakpoints occur at dyadic rationals and whose slopes are powers of 2. We isolate the notion of an F-*obstruction* based on Poincaré's *rotation number* and show that subgroups of $\mathrm{PL}_+ I$ which contain F-obstructions do not embed into F.

In the course of proving the main result of the paper, we establish a dichotomy theorem for subgroups of $\mathrm{PL}_+ I$. This result seems likely to be of independent interest as it is already sufficiently powerful to prove both Brin's ubiquity theorem [3] and the restriction of Rubin's reconstruction theorem [18, Corollary 3.5 (c)] to the class of subgroups of $\mathrm{PL}_+ I$ (see also [17, Theorem 4] and [1, Theorem E16.3] which were precursors to [18]).

1.1. Rotation numbers and F-obstructions

Recall that if γ is a homeomorphism of the circle $\mathbb{R}/\mathbb{Z}$, then the *rotation number* of γ is defined to be

$$\lim_{n\to\infty} \frac{\tilde{\gamma}^n(x) - x}{n}$$

2020 Mathematics Subject Classification. Primary 20B07; Secondary 20B10, 20E07, 37C15.
Keywords. F-obstruction, piecewise linear, rotation number, Thompson's group, topological conjugacy.

modulo 1, where $\tilde{\gamma}\colon \mathbb{R} \to \mathbb{R}$ is a lift of γ (this limit always exists and, modulo 1, does not depend on x or the choice of $\tilde{\gamma}$). Observe that if γ is a rotation of $\mathbb{R}/\mathbb{Z}$ by $r \in (0,1)$, then we can take $\tilde{\gamma}(x) = x + r$ and the rotation number of γ is r. In fact, Poincaré showed that if γ is any homeomorphism such that no finite power has a fixed point, γ is semiconjugate to the irrational rotation specified by its rotation number. On the other hand, if γ^q has a fixed point for some q, then we can take $x \in \mathbb{R}$ and $\tilde{\gamma}$ such that $\tilde{\gamma}^q(x) = x + p$ for some $p \in \mathbb{Z}$ with $0 \le p < q$. It follows that the rotation number is p/q.

If $f, g \in \mathrm{Homeo}_+ I$ and $s \in I$ are such that

$$s < f(s) \le g(s) < f(g(s)) = g(f(s)),$$

then the *rotation number f modulo g at s* is the rotation number of the function on $[s, g(s))$ defined by $x \mapsto g^{-m}(f(x))$, where m is such that $s \le g^{-m}(f(x)) < g(s)$. This map is a homeomorphism of a circle when $[s, g(s))$ is given a suitable topology.

A pair (f, g) of elements of $\mathrm{PL}_+ I$ is an *F-obstruction* if there is s such that either:

- $s < f(s) \le g(s) < f(g(s)) = g(f(s))$ and the rotation number of f modulo g at s is irrational;
- $s > f(s) \ge g(s) > f(g(s)) = g(f(s))$ and the rotation number of f^{-1} modulo g^{-1} at $f(g(s))$ is irrational.

It follows from the work of Ghys and Sergiescu [12] that the standard way of representing F in $\mathrm{PL}_+ I$ does not contain any F-obstructions (see Section 3).

The main result of this paper is that the property of being an F-obstruction is preserved by monomorphisms into $\mathrm{PL}_+ I$.

Theorem 1.1. *If (f, g) is an F-obstruction and $\phi\colon \langle f, g\rangle \to \mathrm{PL}_+ I$ is a monomorphism, then $(\phi(f), \phi(g))$ is an F-obstruction. In particular, if $G \le \mathrm{PL}_+ I$ contains an F-obstruction, then G does not embed into F.*

1.2. A dichotomy for subgroups of $\mathrm{PL}_+ I$

Theorem 1.1 is first established for F-obstructions which generate a group with a single *orbital*—a component of support. The general case is then handled by way of a dichotomy theorem for subgroups of $\mathrm{PL}_+ I$. This dichotomy is strong enough to imply both Brin's ubiquity theorem [3] and a form of Rubin's reconstruction theorem [18, Corollary 3.5 (c)] for subgroups of $\mathrm{PL}_+ I$ (see Section 6).

If $G \le \mathrm{PL}_+ I$, then we say that J is a *resolvable* orbital of G if J is an orbital of G and $\{\mathrm{supt}(g) \cap J \mid g \in G\}$ forms a base for the topology on J. If $G \le \mathrm{Homeo}_+ I$, a partial function $\psi\colon I \to I$ is *G-equivariant* if its domain is G-invariant and for all $g \in G$ and $x \in \mathrm{dom}(\psi)$, $\psi(g(x)) = g(\psi(x))$. Our dichotomy theorem can now be stated as follows:

Theorem 1.2. *Suppose that $G \le \mathrm{PL}_+ I$ and J is a resolvable orbital of G. If K_i $(i < n)$ is a sequence of orbitals of G, then exactly one of the following is true:*

(1) *There is $g \in G$ whose support intersects J but is disjoint from K_i for all $i < n$.*

(2) *There is $i < n$ and a monotone surjection $\psi \colon K_i \to J$ which is G-equivariant.*

Remark 1.3. A more general result has been obtained independently by Brum, Matte Bon, Rivas, and Triestino [6, Corollary 5.17].

1.3. Corollaries of Theorem 1.1

Our original motivation for proving Theorem 1.1 is the following corollary, which answers Question 10.2 of [7] (see Section 7 for the definitions of F_τ and $F_{p,q}$).

Corollary 1.4. *Cleary's group F_τ does not embed into F.*

Theorem 1.1 also gives a new proof of the following result first established by Lodha using his theory of coherent actions.

Corollary 1.5 ([15]). *Stein's groups $F_{p,q}$ do not embed into F if p, q are relatively prime natural numbers.*

In the next corollary, we view $\mathrm{PL}_+ I$ as consisting of functions from $\mathbb{R}$ to $\mathbb{R}$ by defining its elements to be the identity outside of I. Here $F^{t \mapsto t-\xi}$ is the set of conjugates of elements of F by $t \mapsto t - \xi$.

Corollary 1.6. *If $0 < \xi < 1$ is irrational, then $\langle F \cup F^{t \mapsto t-\xi} \rangle$ does not embed into F.*

In the course of proving Theorem 1.1, we will also establish the following results. An F-obstruction is *basic* if the group it generates has connected support.

Theorem 1.7. *If two basic F-obstructions generate isomorphic groups, then the groups are topologically conjugate via a homeomorphism of their supports.*

Theorem 1.8. *If (f, g) is an F-obstruction, then F embeds into $\langle f, g \rangle$.*

Theorem 1.8 generalizes a result of Bleak [2, §3.3] which asserts that if $G \le \mathrm{PL}_+ I$ and the left or right group of germs at some point is nondiscrete—or equivalently noncyclic—then F embeds into G. Note that this implies the restriction [15, Theorem 1.6] to the class of subgroups of $\mathrm{PL}_+ I$.

We conjecture that the converse to Theorem 1.1 holds for finitely generated groups.

Conjecture 1.9. *If $G \le \mathrm{PL}_+ I$ is finitely generated and does not contain an F-obstruction, then G embeds into F.*

Notice that this conjecture implies every finitely generated subgroup of $\mathrm{PL}_+ I$ either contains a copy of F or else embeds into F; whether such a dichotomy holds was asked by Matthew Brin. A natural test case is the group $F_{2/3}$ consisting of those elements of $\mathrm{PL}_+ I$ having breakpoints in $\mathbb{Z}[1/6]$ and having slopes which are powers of $2/3$. It appears to be unknown both whether this group contains an F-obstruction and whether it embeds into F, see [8, Question 4.6].

This paper is organized as follows. After recalling some terminology and notation in Section 2 and establishing that F does not contain F-obstructions in Section 3, we prove Theorem 1.8 in Section 4. Section 5 contains the proof of Theorem 1.2. In Section 6, we use Theorem 1.2 to complete the proofs of Theorems 1.1 and 1.7 and give new derivations of both Brin's ubiquity theorem and Rubin's theorem for sub-groups of PL_+I. Finally, the computations needed for Corollaries 1.4–1.6 are presented in Section 7.

2. Preliminaries

Throughout the paper, counting will start at 0 and i, j, k, l, m, n will only be used to denote integers. If A and B are subsets of an ordered set, we will sometimes write $A \le B$ to indicate that every element of A is less than every element of B.

As already mentioned, Homeo_+I is the collection of orientation preserving homeomorphisms of the unit interval $I := [0, 1]$. Homeo_+I is a group with the operation of composition. PL_+I is the subgroup of Homeo_+I consisting of those elements which are piecewise linear. If $f \in \mathrm{PL}_+I$, we say that s is a *breakpoint* of f if the derivative of f at s is undefined. If s is not a breakpoint of f, we will refer to $f'(s)$ as the *slope* of f at s. Thompson's group F consists of those elements of PL_+I whose slopes are integer powers of 2 and whose breakpoints are in $\mathbb{Z}[\frac{1}{2}]$. When there is a need to emphasize that we are working with this particular group and not an isomorphic copy, we will refer to it as the *standard model of* F. The reader is referred to [9] for the basic analysis of Thompson's group F and [4] for background on PL_+I.

Going forward, we will adopt the convention common in the literature that elements of Homeo_+I act on I from the right. Thus we will write xf for the application of $f \in \mathrm{Homeo}_+I$ to $x \in I$. If $f \in \mathrm{Homeo}_+I$, then the *support* of f is defined to be

$$\mathrm{supt}(f) := \{x \in I \mid xf \neq x\}.$$

If $A \subseteq \mathrm{Homeo}_+I$, then the support of A is defined to be

$$\mathrm{supt}\, A := \{x \in I \mid \exists g \in A\ (xg \neq x)\} = \cup\{\mathrm{supt}(g) \mid g \in A\}.$$

Notice that $\mathrm{supt}\, A = \mathrm{supt}\langle A\rangle$. We will write $\overline{\mathrm{supt}}\, A$ for the closure of $\mathrm{supt}\, A$. A connected component of the support of f is an *orbital* of f; similarly one defines the orbital of a subgroup of Homeo_+I. If f has a single orbital, we will say that f is a *bump*. If f is a bump and $sf > s$ for some (equivalently all) s in its support, then we say f is a *positive bump*; otherwise f is a *negative bump*.

If $f \in \mathrm{Homeo}_+I$ and $X \subseteq I$ is a union of orbitals and fixed points of f, then $f|_X \in \mathrm{Homeo}_+I$ is defined by

$$sf|_X := \begin{cases} sf & \text{if } s \in X, \\ s & \text{otherwise.} \end{cases}$$

This map will be referred to as the *projection to* X. If $G \le \mathrm{Homeo}_+ I$ and X is a union of orbitals and fixed points of G, then the *projection of* G *to* X is the image of G under the homomorphism $f \mapsto f|_X$; we will sometimes use "the projection of G to X" to refer to the homomorphism itself.

If f, g are elements of a group G, define $g^f := f^{-1} g f$ and $[f, g] := f^{-1} g^{-1} f g = f^{-1} f^g = (g^{-1})^f g$. It is easily checked that if $f, g \in \mathrm{PL}_+ I$, then $\mathrm{supt}(f^g) = \mathrm{supt}(f) g$. If A and B are sets of group elements, we will write $[A, B]$ for $\{[a, b] \mid a \in A \text{ and } b \in B\}$. The subgroup of G generated by $[G, G]$ is denoted by G'. If $G = G'$, then we say that G is *perfect*. If $G, H \le \mathrm{PL}_+ I$, we will say that G *commutes with* H if every element of G commutes with every element of H.

We finish this section with some well-known results which will be needed later in the paper.

Proposition 2.1 (see [9]). *If a and b are elements of a group and $[a^b, b^a] = [a^{ba^{-1}}, b^{ab^{-1}}]$ is the identity but $ab \ne ba$, then $\langle a, b\rangle$ is isomorphic to Thompson's group F.*[1] *In particular, if $s_0 < s_1 < t_0 < t_1$ and $a_0, a_1 \in \mathrm{Homeo}_+ I$ are such that $\mathrm{supt}(a_i) = (s_i, t_i)$ and $t_0 a_1 \le s_1 a_0$, then $\langle a_0, a_1\rangle$ is isomorphic to F.*

The next theorem is known as Brin's ubiquity theorem. If $G \le \mathrm{PL}_+ I$, J is an orbital of G and $g \in G$, we say g *approaches the left (right) end of* J if the closure of $\mathrm{supt}(g) \cap J$ contains the left (right) endpoint of J.

Theorem 2.2 ([3]). *Suppose that $G \le \mathrm{PL}_+ I$ and there is an orbital J of G such that some element of G approaches one end of J but not the other. Then there is a subgroup of G isomorphic to F.*

Lemma 2.3 ([4]). *If $G \le \mathrm{PL}_+ I$ and $a \in G'$, then $\overline{\mathrm{supt}}\,(a) \subseteq \mathrm{supt}\, G$.*

The next lemma is more or less established in [4] in the course of showing that nonabelian subgroups of $\mathrm{PL}_+ I$ contain infinite rank free abelian groups. We leave the details to the interested reader.

Lemma 2.4. *If G is a subgroup of $\mathrm{Homeo}_+ I$ and $X \subseteq \mathrm{supt}\, G$ is compact, then there exists $g \in G$ such that for all nonzero $k \in \mathbb{Z}$, $X g^k \cap X = \emptyset$.*

3. F does not contain F-obstructions

In this section, we will prove the following proposition.

Proposition 3.1. *No pair of elements of the standard model of F is an F-obstruction.*

[1] The standard presentation of F is $\langle x_0, x_1 \mid [x_0 x_1^{-1}, x_0^{-1} x_1 x_0], [x_0 x_1^{-1}, x_0^{-2} x_1 x_0^2]\rangle$. The presentation stated in Proposition 2.1 is obtained by the substitution $a := x_0 x_1^{-1}$ and $b := x_1^{-1}$. The proposition follows from this and the fact that the only proper quotients of F are abelian.

Proof. Recall that Thompson's group T consists of all piecewise linear homeomorphisms of $\mathbb{R}/\mathbb{Z}$ which map 0 to a dyadic rational, whose breakpoints are dyadic rationals, and whose slopes are powers of 2. Ghys and Sergiescu [12] have shown that every element of T has a rational rotation number. It therefore suffices to show that if $f, g \in F$ and $s \in I$ with $s < sf \leq sg < sfg = sgf$, then the associated homeomorphism γ defined in the introduction is topologically conjugate to an element of T.

Let s, f and g be given as above and let $s_0 < s$ be a dyadic rational such that $s < s_0 g$, noting that $sg < s_0 g^2$. By conjugating by an element of F and revising f, g, s, and s_0 if necessary, we can assume that for some k, $s_0 g = s_0 + 2^{-k}$ and $s_0 g < 1 - 2^{-k}$. By further conjugating by an element h of F which satisfies $th = t$ if $t \leq s_0 + 2^{-k}$ and $th = tg^{-1} + 2^{-k}$ if $s_0 + 2^{-k} \leq t \leq s_0 g^2$, we can additionally assume that if $s_0 \leq t \leq s_0 g$, then $tg = t + 2^{-k}$. (This conjugacy argument is essentially the *staircase algorithm* of [14].) Repeating this procedure on the interval $[s_0 g, s_0 g^2]$, we can assume without loss of generality that if $s_0 \leq t < s_0 g^2$, then $tg = t + 2^{-k}$.

The homeomorphism γ associated to this revised choice of f, g and s is topologically conjugate to the homeomorphism associated to the original choice of f, g, and s. Moreover, γ is a homeomorphism of $\mathbb{R}/2^{-k}\mathbb{Z}$ which maps dyadic rationals to dyadic rationals, whose breakpoints are dyadic rationals, and whose slopes are powers of 2. Clearly, γ is topologically conjugate to an element of T and hence by [12], γ has a rational rotation number. ■

4. F-obstructions yield copies of F

A key step in proving Theorem 1.1 is to demonstrate that if $f, g \in \mathrm{PL}_+ I$ is a basic F-obstruction and $J := \mathrm{supt}\langle f, g\rangle$, then J is a resolvable orbital of $\langle f, g\rangle$. When combined with Proposition 2.1, this readily yields many copies of F inside $\langle f, g\rangle$, establishing Theorem 1.8 as a byproduct. The first step is the following lemma.

Lemma 4.1. *If (f, g) is an F-obstruction, then f and g do not commute.*

Proof. Let s witness that (f, g) is an F-obstruction and let J be the orbital of $\langle f, g\rangle$ such that $s \in J$. If $f|_J$ and $g|_J$ commute, then by [5] there must be h such that

$$f|_J = h^p \quad \text{and} \quad g|_J = h^q$$

for integers p and q. This implies that the rotation number of f modulo g at s is $p/q \in \mathbb{Q}$, which is a contradiction. ■

For the duration of this section, fix a basic F-obstruction (f, g) and fix $s \in I$ which witnesses this. Specifically, set $C := [s, sg)$ and let $\gamma\colon C \to C$ be defined by

$$x\gamma := \begin{cases} xf & \text{if } xf < sg, \\ xfg^{-1} & \text{if } sg \leq xf. \end{cases}$$

Notice that if $s \le x < sg$ and $sg \le xf$, then $s \le xfg^{-1} < sg$; this last inequality holds since $xf < sgf = sfg \le sg^2$ by our hypothesis. Define a metric d on C by

$$d(x, y) := \min(y - x, sg - y + x - s)$$

whenever $x < y$.

With this metric, C is homeomorphic to a circle and γ is an orientation preserving homeomorphism of C. Our hypothesis is that the rotation number of γ is irrational. Notice that this implies that $sf \ne sg$ (otherwise this would give a rotation number of 0) and hence $sf < sg$.

Since γ is piecewise linear, Herman's variation of Denjoy's theorem [13] (see also [16]) implies the orbits of γ are dense and moreover that $\gamma = \alpha^{-1}\theta\alpha$ for some irrational rotation θ of C and some homeomorphism α of C. Since α is uniformly continuous, θ is an isometry, and $\gamma^n = \alpha^{-1}\theta^n\alpha$, we can witness the uniform continuity of γ^n independently of n: for every $\varepsilon > 0$ there is $\delta > 0$ such that for all $x, y \in C$ and $n \in \mathbb{Z}$ if $d(x, y) < \delta$, then $d(x\gamma^n, y\gamma^n) < \varepsilon$. Noting that this assertion remains unchanged if we swap the roles of (x, y) and $(x\gamma^n, y\gamma^n)$, we will sometimes employ the contrapositive of this implication: if $d(x, y) \ge \varepsilon$ and $n \in \mathbb{Z}$, then $d(x\gamma^n, y\gamma^n) \ge \delta$. Notice that $d(x, y) \le |x - y|$ and $d(x, y) = |x - y|$ if $|x - y| \le (sg - s)/2$. Since $fgf^{-1}g^{-1} \in \mathrm{PL}_+ I$ and $sfg = sgf$, there are $t > s$ and $c > 0$ such that $xfgf^{-1}g^{-1} = cx + (1 - c)s$ whenever $s \le x \le t$. If $c \le 1$, then $xfg \le xgf$ for all $x \in [s, t)$ and if $c \ge 1$, then $xfg \ge xgf$ for all $x \in [s, t)$.

Lemma 4.2. *There is $\delta > 0$ such that for all $n \ge 0$ and all $x < y$ in C with $|x - y| < \delta$:*

- *if $c \le 1$ and $x\gamma^n < y\gamma^n$, then there is $h \in \langle f, g\rangle$ such that $xh = x\gamma^n < y\gamma^n \le yh$;*
- *if $c \ge 1$ and $x\gamma^{-n} < y\gamma^{-n}$, then there is $h \in \langle f, g\rangle$ such that $xh = x\gamma^{-n} < y\gamma^{-n} \le yh$.*

Proof. First observe that $sf < sfgf^{-1} = sgff^{-1} = sg$ and hence if $s \le x^* < sf$, then $x^*gf^{-1} < sfgf^{-1} = sg$.

Claim 4.3. *There is $\delta > 0$ such that for all $s \le x < y < sg$ with $|x - y| < \delta$ and for all $n \in \mathbb{Z}$:*

- *if $x\gamma^n < y\gamma^n$, then $d(x\gamma^n, y\gamma^n) = x\gamma^n - y\gamma^n$;*
- *if $x\gamma^n > y\gamma^n$, then $d(x\gamma^n, y\gamma^n) = sg - x\gamma^n + y\gamma^n - s$.*

Proof. Let $\delta > 0$ be such that $\delta < (sg - s)/6$ and for all $s \le x, y < sg$ and $n \in \mathbb{Z}$ if $d(x, y) < \delta$, then $d(x\gamma^n, y\gamma^n) < (sg - s)/6$. This implies that whenever $s \le x, y < sg$ and $n \in \mathbb{Z}$ if $d(x, y) \ge (sg - s)/6$, then $d(x\gamma^n, y\gamma^n) \ge \delta$.

Now suppose that $s \le x < y < sg$ are given with $|x - y| < \delta$, and let z be the midpoint of the longest arc of C connecting $x\gamma^n$ and $y\gamma^n$. Since $d(x, y) < \delta$, $d(x\gamma^n, y\gamma^n) < (sg - s)/6$ and therefore $\min(d(x\gamma^n, z), d(z, y\gamma^n)) \ge (sg - s)/6$. It follows that

$$\min(d(x, z\gamma^{-n}), d(z\gamma^{-n}, y)) \ge \delta,$$

and therefore either $s \le z\gamma^{-n} < x$ or $y < z\gamma^{-n} < sg$—i.e., $z\gamma^{-n}$ is not between x and y in the cyclic order on C. Since γ preserves the cyclic order, z is not between $x\gamma^n$ and $y\gamma^n$ in the cyclic order.

Suppose that $n \in \mathbb{Z}$ and $x\gamma^n < y\gamma^n$. Then either $s \le z < x\gamma^n$ or $y\gamma^n < z < sg$. In the former case $x\gamma^n - z \ge d(x\gamma^n, z) \ge (sg - s)/6$ and in the latter case $z - y\gamma^n \ge d(y\gamma^n, z) \ge (sg - s)/6$. In either case

$$x\gamma^n - s + sg - y\gamma^n \ge (sg - s)/6 > d(x\gamma^n, y\gamma^n),$$

and hence $d(x\gamma^n, y\gamma^n) = y\gamma^n - x\gamma^n$.

On the other hand, if $n \in \mathbb{Z}$ is such that $x\gamma^n > y\gamma^n$, then $y\gamma^n < z < x\gamma^n$. This implies $(sg - s)/6 \le x\gamma^n - z \le x\gamma^n - y\gamma^n$, and therefore $d(x\gamma^n, y\gamma^n) = sg - x\gamma^n + y\gamma^n - s$. ■

Let $\varepsilon > 0$ be such that $\varepsilon < sg - sf$ and if $|x^* - y^*| < \varepsilon$ and $x^* f < sg \le y^* f$, then $y^* fg^{-1} < t$. Find $\delta > 0$ satisfying the conclusion of Claim 4.3 and such that additionally if $d(x^*, y^*) < \delta$, then $d(x^*\gamma^n, y^*\gamma^n) < \varepsilon$ for all $n \in \mathbb{Z}$.

We will now verify the conclusion of the lemma by induction on $n \ge 0$ under the assumption $c \le 1$; the case $c \ge 1$ is handled by an analogous computation. If $n = 0$, then we can take h to be the identity and there is nothing to show. Now suppose that $n > 0$, $x < y$ and $x\gamma^n < y\gamma^n$. If $sf \le x\gamma^n$, then $x\gamma^n = x\gamma^{n-1} f$ and $y\gamma^n = y\gamma^{n-1} f$. By our induction hypothesis, there is $h_0 \in \langle f, g\rangle$ such that $x\gamma^{n-1} = xh_0$ and $y\gamma^{n-1} \le yh_0$. Since f is order preserving, $y\gamma^{n-1} f \le yh_0 f$ and since $x\gamma^n = xh_0 f$, $h := h_0 f$ satisfies the conclusion of the lemma. Similarly, if $y\gamma^n < sf$, then $x\gamma^n = x\gamma^{n-1} fg^{-1}$ and $y\gamma^n = y\gamma^{n-1} fg^{-1}$ and we can apply our induction hypothesis to find $h_0 \in \langle f, g\rangle$ such that $xh_0 = x\gamma^{n-1}$ and $y\gamma^{n-1} \le yh_0$. It follows that $h := h_0 fg^{-1}$ satisfies

$$xh = xh_0 fg^{-1} = x\gamma^n < y\gamma^n = y\gamma^{n-1} fg^{-1} \le yh_0 fg^{-1} = yh.$$

Finally, suppose that $x\gamma^n < sf \le y\gamma^n$. By choice of δ and its property asserted in Claim 4.3, this implies that $d(x\gamma^n, y\gamma^n) = y\gamma^n - x\gamma^n$. It follows that $x\gamma^n = x\gamma^{n-1} fg^{-1}$ and $y\gamma^n = y\gamma^{n-1} f$. Observe that $x\gamma^{n-1} > y\gamma^{n-1}$, hence $n > 1$ and $d(x\gamma^{n-1}, y\gamma^{n-1}) = sg - x\gamma^{n-1} + y\gamma^{n-1} - s$. Since $d(x\gamma^{n-1}, y\gamma^{n-1}) < \varepsilon$, it follows that $sf < sg - \varepsilon < x\gamma^{n-1}$. Thus $x\gamma^n = x\gamma^{n-2} f^2 g^{-1}$ and $y\gamma^n = y\gamma^{n-2} fg^{-1} f$. Observe that $x\gamma^{n-2} < sgf^{-1} \le y\gamma^{n-2}$. By the induction hypothesis, there is $h_0 \in \langle f, g\rangle$ such that $xh_0 = x\gamma^{n-2}$ and $y\gamma^{n-2} \le yh_0$. Define $h = h_0 f^2 g^{-1}$. Since $d(x\gamma^{n-2}, y\gamma^{n-2}) < \varepsilon$ by our choice of δ and since $x\gamma^{n-2} f < sg \le y\gamma^{n-2} f$, it follows from our choice of ε that $s \le y\gamma^{n-2} fg^{-1} < t$. Therefore,

$$y\gamma^{n-2} fg^{-1} fg \le y\gamma^{n-2} fg^{-1} gf = y\gamma^{n-2} f^2.$$

Acting on the right by g^{-1} yields that

$$y\gamma^n = y\gamma^{n-2} fg^{-1} f \le y\gamma^{n-2} f^2 g^{-1} \le yh_0 f^2 g^{-1} = yh,$$

and hence h satisfies the conclusion of the lemma. ■

Proposition 4.4. *Suppose that (f, g) is a basic F-obstruction. There are dense sets $A, B \subseteq J := \operatorname{supt}\langle f, g\rangle$ such that if $a \in A$ and $b \in B$ with $a < b$, then there is $h \in \langle f, g\rangle$ such that $\operatorname{supt}(h) = (a, b)$. In particular, the support of $\langle f, g\rangle$ is a resolvable orbital of $\langle f, g\rangle$.*

Proof. We will first show that there is $A_0 \subseteq [s, sg)$ which is dense in $[s, sg)$ such that if $a \in A_0$, then $(a, sg]$ is an initial part of the support of some element of $\langle f, g\rangle$. By Lemma 4.1, the commutator $[f, g]$ is not the identity and by Lemma 2.3, the infimum of its support is in J. Let $h_0 \in \langle f, g\rangle$ be such that $p := \inf \operatorname{supt}([f, g]^{h_0})$ is in (s, sg). Such h_0 exists since every $\langle f, g\rangle$-orbit of a point in J intersects $[s, sg)$. Let $q > p$ be such that $(p, q]$ is an initial part of the support of $[f, g]^{h_0}$. Let $\delta > 0$ satisfy the conclusions of Lemma 4.2 and Claim 4.3 and moreover satisfy for all $x < y$ in C:

- if $d(x, y) \geq q - p$, then for all n, $d(x\gamma^n, y\gamma^n) \geq \delta$;
- if $d(x, y) < \delta$ and $x\gamma^n > y\gamma^n$, then $x\gamma^{n-1} < y\gamma^{n-1}$.

There are now two cases depending on whether $c \leq 1$.

If $c \leq 1$, then define $A_0 := \{p\gamma^n \mid n \geq 0\}$. By Herman's variation of Denjoy's theorem [13] (see also [16]), A_0 is dense in $[s, sg)$.

Claim 4.5. *If $a \in A_0$, then either $(a, a + \delta]$ or $(a, sg]$ is an initial part of the support of some element of $\langle f, g\rangle$.*

Proof. If $a \in A_0$, let $n \geq 0$ be such that $a = p\gamma^n$. If $a < q\gamma^n$, then by choice of δ there is $h \in \langle f, g\rangle$ such that

$$a = ph = p\gamma^n < a + \delta \leq q\gamma^n \leq qh.$$

Thus, $(a, a + \delta]$ is an initial segment of the support of $[f, g]^{h_0 h}$. If $p\gamma^n > q\gamma^n$, then by choice of δ we have that $af^{-1} = p\gamma^{n-1} < q\gamma^{n-1}$ and there is h such that $p\gamma^{n-1} = ph$ and $q\gamma^{n-1} \leq qh$. It follows that $(af^{-1}, q\gamma^{n-1}] \subseteq (ph, qh]$ is an initial part of the support of $[f, g]^{h_0 h}$ and hence $(a, sg]$ is an initial part of the support of $[f, g]^{h_0 h f}$. ■

Now suppose that $a \in A_0$ and use the density of A_0 to select a sequence $a_0 = a < a_1 < \cdots < a_k = sg$ such that if $i < k$ then $a_i \in A_0$ and $a_{i+2} - a_i < \delta$ if $i < k - 1$. For each $i < k$, let $h_i \in \langle f, g\rangle$ be such that $(a_i, a_{i+1}]$ is an initial part of $\operatorname{supt}(h_i)$ and $a_{i+2} \leq a_{i+1}h_i$ if $0 \leq i \leq k - 2$. If $h := \prod_{i<k-1} h_i$, then $(a, a_1] \subseteq \operatorname{supt}(h)$ and $sg \leq a_1 h$. It follows h has $(a, sg]$ as an initial part of its support.

If $tfg \geq tgf$, then we define $A_0 := \{p\gamma^n \mid n \leq 0\}$ and an analogous argument gives the desired conclusion. Next, using a similar argument construct an analogous dense $B_0 \subseteq [s, sg)$ such that if $b \in B_0$, then there is an element of $\langle f, g\rangle$ whose support has $[s, b)$ as a final segment. If $a \in A_0$ and $b \in B_0$ with $a < b$, then let h_0 and h_1 be such that $(a, sg]$ is an initial segment of the support of h_0 and $[s, b)$ is a final segment of the support of h_1. Furthermore, select h_0 and h_1 such that $a < bh_0 < ah_1 < b$ and observe that

$$[h_0, h_1] = h_0^{-1}h_1^{-1}h_0h_1 = h_0^{-1} \cdot (h_0)^{h_1} = (h_1^{-1})^{h_0} \cdot h_1.$$

Since to the left of ah_1, the product $h_0^{-1} \cdot (h_0)^{h_1}$ acts as h_0^{-1} followed by a function which is the identity to the left of ah_1, $[h_0, h_1]$ is increasing on $(a, ah_1]$ and the identity to the left of a. Similarly, to the right of $bh_0 < ah_1$, $[h_0, h_1] = (h_1^{-1})^{h_0} \cdot h_1$ acts as h_1 and so is increasing on $[bh_0, b)$ and is the identity to the right of b. Hence $\operatorname{supt}([h_0, h_1]) = (a, b)$ and $[h_0, h_1]$ is increasing on (a, b).

Finally, define $A := A_0\langle f, g\rangle$ and $B := B_0\langle f, g\rangle$ and observe that A and B are both dense in J. Let $(x, y) \subseteq J$ be a maximal open interval containing (s, sg) such that if $a \in A \cap (x, y)$ and $b \in B \cap (x, y)$ with $a < b$, then there is an element of $\langle f, g\rangle$ with support (a, b). It suffices to show that $(x, y) = J$. Suppose for contradiction that this is not true—then either x or y are in J.

If $x \in J$, let $h \in \{f^{\pm 1}, g^{\pm 1}\}$ be such that $xh \in (x, y)$; such h exists since $(s, sg) \subseteq (x, y)$. Notice that $x' := xh^{-1} < x$. Let $x' < a < b < y$ with $a \in A$ and $b \in B$. It suffices to show that (a, b) is the support of an element of $\langle f, g\rangle$ as this will contradict the maximality of (x, y). If $x < a$, then (a, b) is the support of an element of $\langle f, g\rangle$ by our choice of (x, y). Similarly, if $x' < a < b \leq x$, then $x < ah < bh < y$ and there is $h_0 \in \langle f, g\rangle$ with support (ah, bh). It follows that $h_0^{h^{-1}}$ has support (a, b). If $x' < a \leq x < b$, then $x < ah \leq xh$. Let $b' \in B$ be such that $xh < b' < \min(bh, y)$. Let $h_0 \in \langle f, g\rangle$ be a positive bump with support (ah, b'), noting that $h_0^{h^{-1}}$ has support $(a, b'h^{-1})$. Let $a' \in A$ be such that $a' < b'h^{-1} < b$ and let h_1 be a positive bump with support (a', b). Now $h_0^{h^{-1}} h_1$ is a positive bump with support (a, b). This gives the desired contradiction. The case $y \in J$ is handled by an analogous argument. ∎

Theorem 1.8 is an immediate consequence of Proposition 4.4 and Brin's ubiquity theorem [3].

5. A dichotomy for subgroups of $\mathrm{PL}_+ I$

In this section, we will prove Theorem 1.2.

Theorem 1.2. *Suppose that $G \leq \mathrm{PL}_+ I$ and J is a resolvable orbital of G. If K_i $(i < n)$ is a sequence of orbitals of G, then exactly one of the following is true:*

(1) *There is $g \in G$ whose support intersects J but is disjoint from K_i for all $i < n$.*

(2) *There is $i < n$ and a monotone surjection $\psi\colon K_i \to J$ which is G-equivariant.*

This theorem will be proved through a series of lemmas. The first gives a criterion for the existence of an equivariant surjection between orbitals of a subgroup of $\mathrm{Homeo}_+ I$.

Lemma 5.1. *Suppose that J is a resolvable orbital of $G \leq \mathrm{Homeo}_+ I$ and K is an orbital of G. If there are nonempty open intervals U and V such that*

(1) *$\bar{U} \subseteq J$ and $\bar{V} \subseteq K$, and*

(2) *for all $g \in G$, $Ug \cap U \neq \emptyset$ if and only if $Vg \cap V \neq \emptyset$,*

then there is a G-equivariant surjection $\psi\colon K \to J$ which is monotone.

Proof. Define

$$V^* := \cup\{Vh \mid h \in G \text{ and } Uh \subseteq U\}$$

and observe that V^* is an open interval containing V.

Claim 5.2. *For all $g \in G$:*

(1) *$Ug \cap U \neq \emptyset$ if and only if $V^*g \cap V^* \neq \emptyset$;*

(2) *$\overline{V^*} \subseteq K$;*

(3) *$Ug \subseteq U$ if and only if $V^*g \subseteq V^*$;*

(4) *if $\overline{Ug} \subseteq U$, then $\overline{V^*g} \subseteq V^*$.*

Proof. Let $g \in G$. If $U \cap Ug \neq \emptyset$, then $\emptyset \neq V \cap Vg \subseteq V^* \cap V^*g$. Next, suppose that $x \in V^* \cap V^*g$ for some g and let h_0 and h_1 be such that $Uh_0 \cup Uh_1 \subseteq U$ and $x \in Vh_0 \cap Vh_1 g$. It follows that $V \cap Vh_1 g h_0^{-1} \neq \emptyset$, which implies $U \cap Uh_1 g h_0^{-1} \neq \emptyset$ which in turn implies $Uh_0 \cap Uh_1 g \subseteq U \cap Ug \neq \emptyset$. This establishes (1).

Observe that if $\overline{V^*}$ contains an endpoint of K, then for any $g \in G$, $V^*g \cap V^* \neq \emptyset$. On the other hand, since J is a resolvable orbital of G and $\overline{U} \subseteq J$, there is $g \in G$ such that $Ug \cap U = \emptyset$. Thus (2) follows from (1).

We will now prove (3). First suppose that $Ug \subseteq U$ for some $g \in G$. If $y \in V^*g$, let h be such that $Uh \subseteq U$ and $yg^{-1} \in Vh$. Then $Uhg \subseteq Ug \subseteq U$ and so $y \in Vhg \subseteq V^*$. Suppose now Ug is not contained in U. Since J is a resolvable orbital of G, there is $h \in G$ such that Uh intersects Ug but not U. It follows from (1) that V^*h intersects V^*g but not V^* and hence that V^*g is not contained in V^*.

Finally, suppose that $\overline{Ug} \subseteq U$ for some $g \in G$. Since J is a resolvable orbital of G, there are $h_0, h_1 \in G$ such that:

- $Uh_0 \cap Uh_1 = \emptyset$,
- $Uh_0 \cup Uh_1 \subseteq U$,
- Uh_0 and Uh_1 intersect Ug but neither are contained in Ug.

It follows from items (1) and (3) that these same conditions hold of V^* in place of U. This implies that the endpoints of V^*g are contained in $V^*h_0 \cup V^*h_1$ and hence that $\overline{V^*g} \subseteq V^*$. ■

By replacing V with V^* if necessary, we can assume that V has the additional properties of V^* in Claim 5.2—these will be referred to as the *revised hypotheses on U and V*.

Define ψ to consist of all pairs $(x, y) \in K \times J$ such that for all $g \in G$, whenever $y \in Ug$, $x \in \overline{Vg}$. To see that ψ is a (partial) function, suppose $y_0 \neq y_1$ are in J. Since J is a resolvable orbital of G there are $g_i, h_i \in G$ such that $y_i \in Ug_i \subseteq \overline{Ug_i} \subseteq Uh_i$ and $Uh_0 \cap Uh_1 = \emptyset$. By our revised hypotheses, $\overline{Vg_0} \cap \overline{Vg_1} = \emptyset$. Hence there is no x such that (x, y_0) and (x, y_1) are in ψ. It also follows immediately from the definition that $(x, y) \in \psi$ if and only if $(xg, yg) \in \psi$ and hence ψ is G-equivariant.

Next, let us say that two intervals are *linked* if they intersect and neither is a subset of the other. Observe that for any $g \in G$, U and Ug are linked if and only if V and Vg are linked. Also, a pair of intervals is linked if and only if each contains an endpoint of the other. If A and B are intervals, then we will write $A <_l B$ to mean that the pair A, B is linked and the left endpoint of A is less than the left endpoint of B. Clearly, if A and B is a linked pair of intervals, then exactly one of $A <_l B$ or $B <_l A$.

Claim 5.3. *Either for all $g \in G$, $U <_l Ug$ implies $V <_l Vg$ or for all $g \in G$, $U <_l Ug$ implies $Vg <_l V$.*

Proof. Observe that if $U <_l Ug$, then $Ug^{-1} <_l U$. Hence if the claim is false, there are $g_0, g_1 \in G$ such that $Ug_0 <_l U <_l Ug_1$ and yet $V <_l Vg_0, Vg_1$. Since J is a resolvable orbital of G, there is $h \in G$ such that $\mathrm{supt}(h) \cap J \subset U$ and $Ug_0 \cap Ug_1 h = \emptyset$. Since $Uh = U$, $Vh = V$ and because the right endpoint of V is in Vg_1, it is also in $Vg_1 h$. In particular, this right endpoint is in both Vg_0 and $Vg_1 h$ while Ug_0 and $Ug_1 h$ are disjoint, contrary to our hypothesis. ∎

Claim 5.4. *The function ψ is monotone.*

Proof. Suppose that for all $g, h \in G$, $Ug <_l Uh$ implies $Vg <_l Vh$. Let $\psi(x_0) = y_0 < y_1 = \psi(x_1)$ and let $g_i \in G$ be such that $y_i \in Ug_i$ and $Ug_0 \cap Ug_1 = \emptyset$. By resolvability of G on J, there is h such that Uh links both Ug_0 and Ug_1. In particular, $Ug_0 <_l Uh <_l Ug_1$, which implies $Vg_0 <_l Vh <_l Vg_1$. Since $x_i \in \overline{Vg_i}$ and $Vg_0 \cap Vg_1 = \emptyset$, it follows that $x_0 \leq x_1$; since ψ is a (partial) function we must have $x_0 < x_1$. Similarly, if $Ug <_l Uh$ always implies $Vh <_l Vg$, then ψ is monotone decreasing. ∎

Claim 5.5. *The function ψ is a surjection from K onto J.*

Proof. In order to see that ψ is a surjection, let $y \in J$ be given. By assumption,

$$\mathscr{F} := \{\overline{Vg} \mid g \in G \text{ and } y \in Ug\}$$

is a pairwise intersecting collection of intervals. By our revised hypotheses on U and V and by the resolvability of G on J, $\mathscr{F}$ contains elements whose closure is contained in K. Thus $\psi^{-1}(y) = \bigcap \mathscr{F}$ is a nonempty interval.

Now suppose that $x \in K$. Since ψ is a surjection, its domain is nonempty; since ψ is G-equivariant, its domain X contains elements both to the left and right of x. Set

$$x_0 := \sup\{s \in X \mid s \leq x\}, \quad x_1 := \inf\{s \in X \mid x \leq s\}.$$

Notice that since ψ is a monotone surjection, $\psi(x_0) = \psi(x_1)$. Since we have shown ψ-preimages of points are intervals, $x \in \mathrm{dom}(\psi)$. ∎

Our strategy for proving Theorem 1.2 will now be to carefully select a subgroup $H \leq G$ whose support has nice properties which will allow us to define intervals U and V as in Lemma 5.1. The first step toward this goal is the following lemma. Recall that a group G is *perfect* if $G' = G$.

Lemma 5.6. *Suppose that J is a resolvable orbital of $G \leq \mathrm{PL}_+I$ and K is an orbital of G. There is $H \leq G$ such that:*

(1) *H is perfect;*

(2) *H has finitely many orbitals;*

(3) $\operatorname{supt} H \cap J$ *is a resolvable orbital of H with closure contained in J;*

(4) $\operatorname{supt} H \cap K$ *has closure contained in K.*

This lemma will itself be proved though a series of lemmas.

Lemma 5.7. *If $A, B \leq \mathrm{Homeo}_+I$ and for each $a \in A$ we have $\overline{\operatorname{supt}}\,(a) \subseteq \operatorname{supt} B$, then $A' \leq \langle [[A, B], [A, B]] \rangle$.*

Proof. Suppose that $a_0, a_1 \in A$ are arbitrary; it suffices to show that

$$[a_0, a_1] \in [[A, B], [A, B]].$$

Set

$$X := \overline{\operatorname{supt}}\,\{a_0, a_1\}.$$

By Lemma 2.4, there is $b \in B$ such that both Xb and Xb^{-1} are disjoint from X. Since $(a_1^{-1})^b$ is supported on Xb and $(a_0^{-1})^{b^{-1}}$ is supported on Xb^{-1}, these terms commute with each other and with a_0 and a_1, which are supported on X. Thus

$$[[b^{-1}, a_0], [b, a_1]] = [(a_0^{-1})^{b^{-1}} a_0, (a_1^{-1})^b a_1] = [a_0, a_1]$$

is in $[[A, B], [A, B]]$ as desired. ∎

Lemma 5.8. *Let G be a subgroup of PL_+I. If $\operatorname{supt} G' = \operatorname{supt} G$, then $\operatorname{supt} G'' = \operatorname{supt} G$ and G'' is perfect.*

Proof. If $a \in G'$, then by Lemma 2.3

$$\overline{\operatorname{supt}}\,(a) \subseteq \operatorname{supt} G = \operatorname{supt} G'.$$

Thus we can apply Lemma 5.7 to $A = B = G'$ and obtain that

$$G'' \leq \langle [[G', G'], [G', G']] \rangle \leq G'''.$$

Thus $G'' = G'''$ and G'' is perfect. To see that $\operatorname{supt} G'' = \operatorname{supt} G$, suppose that $x \in \operatorname{supt} G$. By assumption, there is $g \in G'$ such that $x \in \operatorname{supt}(g)$. By Lemmas 2.3 and 2.4, there is $f \in G'$ such that the supports of g and g^f are disjoint. It follows that $xg = x[f, g]$ and therefore that x is in the support of $[f, g] \in G''$. ∎

Lemma 5.9. *If $G \leq \mathrm{PL}_+I$ is perfect and $H \leq G$ is a normal subgroup with* $\operatorname{supt} H = \operatorname{supt} G$, *then $G = H$.*

Proof. Since H is a normal subgroup of G, $\langle[[G,H],[G,H]]\rangle \le H$. By Lemma 2.3, $\overline{\operatorname{supt}}(g) \subseteq \operatorname{supt} G = \operatorname{supt} H$ for every $g \in G$. Applying Lemma 5.7 to $A = G$ and $B = H$, we obtain that $G = G' \le \langle[[G,H],[G,H]]\rangle \le H$. ∎

Lemma 5.10. *Let $H_0 \le \mathrm{PL}_+ I$ and J be an orbital of H_0 such that $H_0|_J$ is both perfect and the normal closure of a single element. Then there exists a perfect subgroup H of H_0 with finitely many orbitals such that $H|_J = H_0|_J$.*

Proof. Since $H_0|_J$ is perfect $H_0''|_J = H_0|_J$. Fix $h \in H_0''$ with the normal closure of $h|_J$ in $H_0|_J$ equal to $H_0|_J$. Let H_1 be the normal closure of h in H_0'' and let H_2 be the normal closure of h in H_1. Define $\mathscr{U}$ to be the orbitals of H_0 that are also orbitals of H_2 and set $U := \cup\mathscr{U}$. Define $\mathscr{V}$ to be the orbitals of H_0 which intersect $\operatorname{supt} H_2$ but are not orbitals of H_2 and set $V := \cup\mathscr{V}$.

Claim 5.11. $\overline{\operatorname{supt}}\, H_2|_V$ *is contained in* $\operatorname{supt} H_0$.

Proof. We will first argue that if L is an orbital of $H_1|_V$, then the closure of L is contained in the support of H_0. To see this, suppose L is an orbital of $H_1|_V$. If L were an orbital of H_0, then it would also be an orbital of H_0' since $H_1 \le H_0'$. Hence Lemma 5.8 would imply that $H_0''|_L$ is perfect and Lemma 5.9 would imply $H_0''|_L = H_1|_L$. Since this in turn would imply $H_2|_L = H_1|_L = H_0''|_L$ and that L is in $\mathscr{U}$, this is impossible. Thus L is not an orbital of H_0.

Let $g \in H_0$ be such that $Lg \not\subseteq L$. Since Lg is contained in the support of H_0 and L is an orbital of H_1, Lg must be disjoint from L, and in particular the closure of L is contained in $\operatorname{supt} H_0$. Now let X be the closure of the union of the orbitals of $H_1|_V$ which intersect $\operatorname{supt}(h)$. We have shown that $X \subseteq \operatorname{supt} H_0$. Since X is H_1-invariant, $\operatorname{supt} H_2|_V \subseteq X$. ∎

By Claim 5.11, we can find $g \in H_0$ such that $V \cap \operatorname{supt} H_2 \cap \operatorname{supt} H_2^g = \emptyset$. Define $H := \langle[H_2^g, H_2]\rangle$, noting that $\operatorname{supt} H \subseteq U$. By Lemma 5.8, $H_0''|_U$ is perfect. By Lemma 5.9, $H_1|_U = H_0''|_U$ and therefore $H_2|_U = H_0''|_U$. Since H_0'' is a normal subgroup of H_0, we have $H_0''^g = H_0''$. Putting this all together, we obtain

$$\begin{aligned} H = H|_U &= \langle[H_2^g|_U, H_2|_U]\rangle = \langle[H_0''^g|_U, H_0''|_U]\rangle \\ &= \langle[H_0''|_U, H_0''|_U]\rangle = H_0'''|_U = H_0''|_U. \end{aligned}$$

Thus $H = H_0''|_U$ is perfect and has finitely many components of support. ∎

Lemma 5.12. *Suppose that $A, B \le \mathrm{PL}_+ I$ are perfect groups and N is the normal closure of B in $\langle A \cup B\rangle$. If S denotes the union of the orbitals of $\langle A \cup B\rangle$ which are not orbitals of N, then $A|_S$ is contained in $\langle A \cup B\rangle$.*

Proof. We will first show that for all $a \in A$ there exists $b \in N$ such that $ab|_S$ is in $\langle A \cup B\rangle$. Define A_0 to be the set of all $a \in A$ such that there exists $b \in N$ with $ab|_S$ is in $\langle A \cup B\rangle$. Since A is perfect, it suffices to show that $[A, A] \subseteq A_0$ and that A_0 is a group.

Toward showing $[A, A] \subseteq A_0$, let $f, g \in A$, and let us set X equal to the closure of $\operatorname{supt}\{f, g\} \setminus S$. Since A is perfect, Lemma 2.3 implies X is contained in $\operatorname{supt} N$. By Lemma 2.4, there is $h \in N$ such that $Xh \cap X = \emptyset$. Since f and g^h have disjoint supports, $[f, g^h]$ agrees with the identity on $I \setminus S$. In particular, $[f, g^h]|_S = [f, g^h]$ is in $\langle A \cup B\rangle$. We can rewrite $[f, g^h]$ as $[f, g](h^{-1})^{(f^g)} h^{fg} (h^{-1})^g h$. Since $(h^{-1})^{(f^g)} h^{fg} (h^{-1})^g h$ is in N we have shown that $[f, g]$ is in A_0.

It remains to show that A_0 is closed under composition and taking inverses. Let $a_0, a_1 \in A_0$ and fix $b_0, b_1 \in N$ with $a_0 b_0|_S$ and $a_1 b_1|_S$ in $\langle A \cup B\rangle$. Since $a_0 a_1 b_0^{a_1} b_1|_S = (a_0 b_0)|_S (a_1 b_1)|_S$ is in $\langle A \cup B\rangle$ and $b_0^{a_1} b_1$ is in N, it follows that $a_0 a_1 \in A_0$. Since $(a_0 b_0)^{-1}|_S = a_0^{-1}(b_0^{-1})^{a_0^{-1}}|_S$ is in $\langle A \cup B\rangle$ and $(b_0^{-1})^{a_0^{-1}}$ is in N, it follows that $a_0^{-1} \in A_0$. Since $a_0, a_1 \in A_0$ were arbitrary, A_0 is closed under multiplication and taking inverses and hence is a group. Thus $A_0 = A$.

Next we will show that $N|_S$ is contained in $\langle A \cup B\rangle$. Notice that this is sufficient to complete the proof since if $a \in A$, then for some $b \in N$, $(ab)|_S$ is in $\langle A \cup B\rangle$ and hence $a|_S = (ab)|_S (b^{-1})|_S$ is in $\langle A \cup B\rangle$. Since B is perfect, it is generated by $[B, B]$ and hence N is the normal closure of $[B, B]$ in $\langle A \cup B\rangle$. Thus it suffices to show that if $b_0, b_1 \in B$, then $[b_0, b_1]|_S$ is in $\langle A \cup B\rangle$. Toward this end, let $b_0, b_1 \in B$ be arbitrary. Observe that any endpoint of a connected component U of S is a limit point of $U \setminus \operatorname{supt} N$. Thus there is a set $X \subseteq S$ which is a finite union of intervals with endpoints in $S \setminus \operatorname{supt} N$ such that $\operatorname{supt}\{b_0, b_1\} \cap S \subseteq X$.

We now claim that X is contained in the support of A. If there were $x \in X$ fixed by every element of A, then x is in some component L of the support of B. In this case, however, the union of the translates of L by elements of $\langle A \cup B\rangle$ is an orbital of both N and $\langle A \cup B\rangle$, contradicting that L is contained in S. Thus it must be that X is contained in the support of A.

By Lemma 2.4, there is $a \in A$ such that $Xa \cap X = \emptyset$. Thus there is $g \in N$ such that $h := (ag)|_S$ is in $\langle A \cup B\rangle$. We will be finished once we show that $[[h, b_0], b_1] = [b_0, b_1]|_S$. Define $Y := S \setminus X$ and $Z := I \setminus S$ and set $c := (b_0^{-1})^h$. Observe that X, Y, and Z are all invariant under b_0, b_1 and c. Furthermore, $[h, b_0] = c b_0$ agrees with b_0 on X and the identity on Z. Since b_1 agrees with the identity on Y, we have that $[[h, b_0], b_1] = [c b_0, b_1]$ coincides with $[b_0, b_1]$ on X and is the identity elsewhere. Since $[b_0, b_1]$ is the identity on Y, it follows that $[[h, b_0], b_1] = [b_0, b_1]|_S$. ∎

Proof of Lemma 5.6. Let $s < t$ be in J and set $U := (s, t)$. Define $H_0 := \{g \in G \mid \operatorname{supt}(g) \cap J \subseteq U\}''$. Since J is a resolvable orbital of G, $H_0|_J$ is perfect by Lemma 5.8. It is easily checked that U is a resolvable orbital of H_0 and hence Lemma 5.9 implies that $H_0|_J$ is the normal closure of a single element. By Lemma 5.10, there is $H \leq H_0$ such that H is perfect, has finitely many orbitals, and $H|_J = H_0|_J$.

If the closure of $\operatorname{supt} H \cap K$ is contained in K, then H satisfies the conclusion of the lemma. If the closure of $\operatorname{supt} H \cap K$ contains an endpoint of K, then let $g \in G$ be such that $Ug \cap U = \emptyset$. Let N be the normal closure of H^g in $\langle H \cup H^g\rangle$ and let S be the union of the orbitals of $\langle H \cup H^g\rangle$ which are not also orbitals of N. Observe that $U \subseteq S$

and that the closure of $S \cap K$ does not contain the endpoints of K. Applying Lemma 5.12 to $A = H$ and $B = H^g$, the projection $H|_S$ is contained in $\langle H \cup H^g \rangle \leq G$ and satisfies the conclusion of the lemma. ■

Proof of Theorem 1.2. The theorem is proved by induction on n with the bulk of the proof dedicated to the base case $n = 1$. Suppose that $n = 1$ and write K for K_0. Applying Lemma 5.6, fix a perfect subgroup $H \leq G$ with finitely many orbitals such that:

- $U :=$ supt $H \cap J$ is a nonempty interval with closure contained in J;
- the closure of supt $H \cap K$ is contained in K;
- U is a resolvable orbital of H;
- the number of orbitals of H is minimized.

If supt $H \cap K = \emptyset$, then the first alternative of the theorem holds. Suppose now that this is not the case. Define V to be the leftmost component of supt $H \cap K$. By Lemma 5.1, it suffices to show that for all $g \in G$, $Ug \cap U \neq \emptyset$ if and only if $Vg \cap V \neq \emptyset$.

Claim 5.13. *For all $g \in G$, $Ug \cap U \neq \emptyset$ implies $Vg \cap V \neq \emptyset$.*

Proof. Suppose first for contradiction that for some $g \in G$, $Ug \cap U \neq \emptyset$ but $Vg \cap V = \emptyset$. By replacing g with g^{-1} if necessary, we can assume that V is disjoint from the support of H^g. Let N denote the normal closure of H^g in $\langle H \cup H^g \rangle$ and let S be the union of all orbitals of $\langle H \cup H^g \rangle$ which are not orbitals of N. Observe that since U is a resolvable orbital of H and $U \cap Ug \neq \emptyset$, it follows that U is disjoint from S. On the other hand, V is contained in S. Thus applying Lemma 5.12 to $A = H$ and $B = H^g$ yields that $H|_S$ is a perfect subgroup of G. Consequently, if $R =$ supt $H \setminus S$, then $H_0 := H|_R$ is also contained in G. Since H_0 is also perfect, satisfies $H_0|_U = H|_U$ and has fewer orbitals than H, we have contradicted our choice of H to minimize the number of its orbitals. ■

Claim 5.14. *For all $g \in G$, $Ug \cap U = \emptyset$ implies $Vg \cap V = \emptyset$.*

Proof. Suppose first for contradiction that for some $g \in G$, $Ug \cap U = \emptyset$ but $Vg \cap V \neq \emptyset$. As in the previous claim, let N denote the normal closure of H^g in $\langle H \cup H^g \rangle$ and let S be the union of all orbitals of $\langle H \cup H^g \rangle$ which are not orbitals of N. This time U is contained in S and V is disjoint from S. Lemma 5.12 implies $H_0 := H|_S$ is a perfect subgroup of G. Since H_0 has fewer orbitals than H, is perfect, and satisfies $H_0|_J = H|_J$, we again contradict our choice of H. ■

Now suppose that n is given and that the statement of the theorem is true for n. Let K_i ($i < n + 1$) be orbitals of G. We need to show that if (2) fails, then there is $g \in G$ whose support intersects J but not K_i for any $i < n + 1$. By our inductive assumption, there is $g_0 \in G$ whose support intersects J but is disjoint from K_i for $i < n$. Additionally there is $g_1 \in G$ whose support intersects J but is disjoint from K_n. Since J is a resolvable orbital of G, there is h such that g_0^h does not commute with g_1. It follows that $g = [g_0^h, g_1]$ is as desired. ■

6. Consequences of the dichotomy theorem

In this section, we will give proofs of Theorems 1.1 and 1.7 using Theorem 1.2. We will also illustrate the utility of Theorem 1.2 by deriving some known results as corollaries.

Proof of Theorem 1.1. Suppose that (f, g) is an F-obstruction and $\phi\colon \langle f, g\rangle \to \mathrm{PL}_+ I$ is an embedding. By rescaling and translating if necessary, we can assume that the supports of $\langle f, g\rangle$ and $\langle \phi(f), \phi(g)\rangle$ are disjoint. Let J be an orbital of $\langle f, g\rangle$ such that for some $s \in J$, the rotation number of f modulo g at s is irrational. Let K_i $(i < n)$ list the orbitals of $\langle \phi(f), \phi(g)\rangle$. Observe that since ϕ is an injection, the first alternative of Theorem 1.2 applied to $G := \langle f\phi(f), g\phi(g)\rangle$ cannot hold. Therefore, there is $i < n$ and a G-equivariant monotone surjection $\psi\colon K_i \to J$; let $t \in K_i$ be minimal such that $\psi(t) = s$. It is easily checked that $t\phi(f)\phi(g) = t\phi(g)\phi(f)$. Since rotation numbers are preserved by semiconjugacy, it follows that the rotation number of $\phi(f)$ modulo $\phi(g)$ at t is irrational and hence that $(\phi(f), \phi(g))$ is an F-obstruction. ∎

We will now recall the statement and context of Rubin's reconstruction theorem. Suppose that X is a locally compact Hausdorff space and G is a group of homeomorphisms of X. The group G's action on X is *locally dense* if X has no isolated points and whenever $x \in U \subseteq X$ with U open,

$$\{xg \mid (g \in G) \text{ and } (\mathrm{supt}(g) \subseteq U)\}$$

is somewhere dense. It is easily checked that if $X \subseteq I$ is an interval, then this is equivalent to X being a resolvable orbital of G. Rubin's theorem asserts that if X and Y are locally compact and if $G \leq \mathrm{Homeo} X$ and $H \leq \mathrm{Homeo} Y$ are such that the actions of G and H on their underlying spaces are locally dense, then any isomorphism between G and H is induced by a unique homeomorphism of X and Y (this is Corollary 3.5 (c) of [18]).

We will now show how to derive Rubin's theorem when G and H are subgroups of $\mathrm{PL}_+ I$. Notice that Theorem 1.7 is an immediate consequence of this result and Proposition 4.4.

Corollary 6.1 ([18]). *Suppose that $G, H \leq \mathrm{PL}_+ I$ are nontrivial and each acts on its support in a locally dense manner. If $\phi\colon G \to H$ is an isomorphism, then there is a unique homeomorphism $\psi\colon \mathrm{supt}\, G \to \mathrm{supt}\, H$ such that for all $x \in \mathrm{supt}\, G$, $\psi(xg) = \psi(x)\phi(g)$.*

Remark 6.2. Both McCleary [17] and Bieri–Strebel [1] had previously proved similar reconstruction theorems for subgroups of $\mathrm{PL}_+ I$, although under different dynamical hypotheses.

Proof. First observe that since the action of $G \leq \mathrm{PL}_+ I$ on its support is locally dense, then the only G-equivariant maps between orbitals of G are the identity functions. This in particular implies that ψ is unique if it exists. To prove existence, suppose $G, H \leq \mathrm{PL}_+ I$ and $\phi\colon G \to H$ are as in the statement of the corollary. By replacing G and H by rescaled translates if necessary, we can assume that the supports of G and H are disjoint.

Define $\Gamma := \{g\phi(g) \mid g \in G\}$ and let J be an orbital of G. Observe that we are finished once we have shown that there is a Γ-equivariant homeomorphism between $\operatorname{supt} G$ and $\operatorname{supt} H$. Furthermore, it suffices to show that for each orbital J of G, there is a unique orbital K of H for which there is a Γ-equivariant homeomorphism between J and K. The statement with the roles of G and H reversed must also hold and ψ is then obtained by pasting together these local homeomorphisms.

Fix $g_0 \in G$ such that $\operatorname{supt}(g_0)$ is nonempty with closure contained in J. Let K_i $(i < n)$ list the orbitals of H which intersect $\operatorname{supt}(\phi(g_0))$—there are only finitely many such orbitals since $\phi(g_0) \in \mathrm{PL}_+ I$. Since J is a resolvable orbital of G, the only element of $G|_J$ which commutes with every conjugate of g_0 is the identity. Apply Theorem 1.2 to the group Γ and observe that the first alternative cannot hold since if g_J is not the identity, g fails to commute with g_0^h for some $h \in G$. Since the support of $\phi(g_0^h)$ is contained in $\bigcup_{i<n} K_i$, it must be that $\phi(g)|_{K_i}$ is nontrivial for some $i < n$. Thus there is $i < n$ and a Γ-equivariant surjection $\theta\colon K_i \to J$. Similarly, there is a Γ-equivariant monotone surjection ϑ from some orbital J' of G to K_i. By the observation made at the start of the proof, $J' = J$ and $\vartheta = \theta^{-1}$. In particular, $\vartheta\colon J \to K$ is the desired Γ-equivariant homeomorphism. ∎

Remark 6.3. It should be noted that Theorem 1.2 is false if we replace $\mathrm{PL}_+ I$ with $\mathrm{Homeo}_+ I$. For example, since F is orderable [10], there is $G \leq \mathrm{Homeo}_+ I$ which is isomorphic to F such that every nonidentity element of G has only isolated fixed points; such G cannot be semiconjugate to the standard copy of F. It would be interesting to know if there are broader contexts in which Theorem 1.2 holds.

Next we will derive Brin's ubiquity theorem from Theorem 1.2.

Corollary 6.4 ([3]). *Suppose $G \leq \mathrm{PL}_+ I$ and K is an orbital of G such that some element of G approaches one end of K but not the other. Then F embeds into G.*

Proof. By replacing G with a rescaled translate if necessary, we can assume that the support of G is contained in $(1/2, 1)$. Let a and b be the generators for the rescaled standard model of F with support $(0, 1/2)$. Let K_i $(i < n)$ list the orbitals of G so that $K_0 = K$. The hypothesis combined with Lemma 2.4 readily yields a pair $f, g \in G$ such that $f|_K$ and $g|_K$ satisfy the same relations as a and b.

Define $\Gamma := \langle af, bg \rangle$ and apply Theorem 1.2 to the group Γ, the distinguished orbital $J := (0, 1/2)$, and the orbitals K_i $(i < n)$. There is a subset $X \subseteq \{0, \ldots, n-1\}$ and Γ-equivariant monotone surjections $\psi_i\colon K_i \to J$ for $i \in X$ and $h \in \Gamma$ such that if $i < n$ is not in X, then $h|_{K_i}$ is the identity; let $\psi\colon \bigcup_{i \in X} K_i \to J$ be the common extension of the ϕ_i's. Using that J is a resolvable orbital of Γ and arguing as in the proof of Proposition 4.4, there is h_0 in the normal closure of h in Γ such that $h_0|_J$ is a positive bump. Observe that the image of the support of h_0 under ψ is the union of $(s_0, t_0) := \operatorname{supt}(h_0) \cap J$ and a finite set E. Let g be such that $Eg \cap E = \emptyset$ and $s_0 < s_0 g < t_0 < t_0 g$. It is now easily checked that for some $m > 0$, $a := h_0^m$ and $b := (h_0^g)^{-m}$ are as in Proposition 2.1. ∎

Remark 6.5. While we used Brin's ubiquity theorem to prove Theorem 1.8, it is not required for the proof of Theorem 1.2. Even so, the purpose of deriving Corollaries 6.1 and 6.4 from Theorem 1.2 is not to give new proofs of these facts but rather to demonstrate the ways in which Theorem 1.2 can be used and the utility that resides in it.

7. Some examples

In this section, we will prove Corollaries 1.4–1.6. Recall that Cleary's group F_τ is the subgroup of PL_+I consisting of those elements whose singularities are in $\mathbb{Z}[\tau]$ and whose slopes are powers of τ, where τ is the solution to $\tau^2 = \tau + 1$ with $\tau > 1$ [11]. If $1 < p < q$ are relatively prime integers, then Stein's group $F_{p,q}$ is the subgroup of PL_+I consisting of those elements whose singularities are in $\mathbb{Z}[\frac{1}{p}, \frac{1}{q}]$ and whose slopes are the product of a power of p and a power of q [19].

The following observations will allow us to show that Cleary's and Stein's groups contain F-obstructions.

Observation 7.1. *Suppose that $f, g \in \mathrm{Homeo}_+I$ and for some s and $0 < \xi < \eta$, $xf = x + \xi$ and $xg = x + \eta$ whenever $s \le x \le s + \eta$. Then the rotation number of f modulo g at s is defined and equals ξ/η.*

Observation 7.2. *Suppose that $f, g \in \mathrm{Homeo}_+I$ and for some $s_0 < s_1$ and $1 < a < b$, $xf = a(x - s_0) + s_0$ and $xg = b(x - s_0) + s_0$ whenever $s_0 \le x \le s_1$. If $s \in (s_0, s_1)$ is such that $sg \le s_1$, then the rotation number of f modulo g at s is defined and equals* $\log_b(a)$

The second observation is a consequence of the first by conjugating f and g by $\log_b \frac{x - s_0}{s}$. If $1 < p < q$ are relatively prime integers, then $\log_q(p)$ is irrational. Since $F_{p,q}$ contains elements which have slope p and q near 0, Corollary 1.5 follows from Observation 7.2 and Theorem 1.1.

We now turn to Cleary's group F_τ. Define $f, g \in F_\tau$ by

$$xf := \begin{cases} x\tau & \text{if } 0 \le x \le \tau^{-3}, \\ x + \tau^{-2} - \tau^{-3} & \text{if } \tau^{-3} \le x \le \tau^{-1}, \\ x\tau^{-1} + \tau^{-2} & \text{if } \tau^{-1} \le x \le 1, \end{cases}$$

$$xg := \begin{cases} x\tau^{2} & \text{if } 0 \le x \le \tau^{-4}, \\ x + \tau^{-2} - \tau^{-4} & \text{if } \tau^{-4} \le x \le \tau^{-1}, \\ x\tau^{-2} + \tau^{-1} & \text{if } \tau^{-1} \le x \le 1. \end{cases}$$

If we set $s := \tau^{-3}$, then

$$sf = \tau^{-2} < \tau^{-2} + \tau^{-3} - \tau^{-4} = \tau^{-1} - \tau^{-4} = sg < \tau^{-1}.$$

It follows from Observation 7.1 that the rotation number of f modulo g at s is defined and equals

$$\frac{\tau^{-2}-\tau^{-3}}{\tau^{-2}-\tau^{-4}} = \frac{\tau^{2}-\tau}{\tau^{2}-1} = \tau^{-1}.$$

Since τ^{-1} is irrational, (f, g) is an F-obstruction.

Finally, we wish to show that the group generated by $F \cup F^{t\mapsto t-\xi}$ contains an F-obstruction whenever $0 < \xi < 1$. Recall that F is the subgroup of $\mathrm{PL}_+ I$ consisting of those elements whose singularities occur at dyadic rationals and whose slopes are powers of 2. Let ξ be given and let n be such that $2^{-n} < \xi < 1 - 2^{-n+2}$. Observe that the following functions f, g_0, and g_1 are in either F or $F^{t\mapsto t-\xi}$:

$$xf := \begin{cases} 2x+\xi & \text{if } -\xi \le x \le 2^{-n}-\xi, \\ x+2^{-n} & \text{if } 2^{-n}-\xi \le x \le 1-2^{-n+1}-\xi, \\ 2^{-1}(x+1-\xi) & \text{if } 1-2^{-n+1}-\xi \le x \le 1-\xi, \\ x & \text{otherwise,} \end{cases}$$

$$xg_0 := \begin{cases} 2x & \text{if } 0 \le x \le 2^{-1}-2^{-n-1}, \\ x+2^{-1}-2^{-n-1} & \text{if } 2^{-1}-2^{-n-1} \le x \le 2^{-1}, \\ 2^{-n}x+1-2^{-n} & \text{if } 2^{-1} \le x \le 1, \\ x & \text{otherwise,} \end{cases}$$

$$xg_1 := \begin{cases} 2^{n}(x+\xi)-\xi & \text{if } -\xi \le x \le 2^{-n-1}-\xi, \\ x+2^{-1}-2^{-n-1} & \text{if } 2^{-n-1}-\xi \le x \le 2^{-n}-\xi, \\ 2^{-1}(x+\xi)+2^{-1}-\xi & \text{if } 2^{-n}-\xi \le x \le 1-\xi, \\ x & \text{otherwise.} \end{cases}$$

Set $g := g_0 g_1$. Observe that by our choice of n, if $0 \le x \le (1-\xi)/2$, then

$$xf = x + 2^{-n}, \quad xg = x + \frac{1-\xi}{2}.$$

Since $0f = 2^{-n} < (1-\xi)/2 = 0g$, it follows from Observation 7.2 that the rotation number of f modulo g at 0 is defined and equals $2^{-n+1}/(1-\xi)$, which is irrational. Hence (f, g) is an F-obstruction.

Remark 7.3. We do not know if $\langle \bigcup_{q\in\mathbb{Q}} F^{t\mapsto t-q} \rangle$ embeds into F. We conjecture it does not. Note that if

$$G \le \Big\langle \bigcup_{q\in\mathbb{Q}} F^{t\mapsto t-q} \Big\rangle$$

is finitely generated, then G is conjugate to a subgroup of F. Specifically, if $X \subseteq \frac{1}{n}\mathbb{Z}$, then $\langle \bigcup_{q\in X} F^{t\mapsto t-q} \rangle$ is conjugate to a subgroup of the real line model of F via the map $t \mapsto nt$.

Acknowledgments. The authors would like to thank Collin Bleak, Matthew Brin, José Burillo, Martin Kassabov, Lewis Molyneux, Brita Nucinkis, Paula Barja Romero, and Pavel Turek for suggesting comments and corrections to earlier versions of this paper. They would also like to thank the referee for carefully reading of the paper and offering suggestions.

Funding. The research of the second author is support by NSF grant DMS-1854367.

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Received 27 March 2021.

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