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Self-similar abelian groups and their centralizers

Alex C. Dantas, Tulio M. G. Santos, and Said N. Sidki

Abstract. We extend results on transitive self-similar abelian subgroups of the group of automorphisms $\mathcal{A}_m$ of an m-ary tree $\mathcal{T}_m$ by Brunner and Sidki to the general case where the permutation group induced on the first level of the tree, has $s \geq 1$ orbits. We prove that such a group A embeds in a self-similar abelian group A^* which is also a maximal abelian subgroup of $\mathcal{A}_m$. The construction of A^* is based on the definition of a free monoid Δ of rank s of partial diagonal monomorphisms of $\mathcal{A}_m$. Precisely, $A^* = \overline{\Delta(B(A))}$, where $B(A)$ denotes the product of the projections of A in its action on the different s orbits of maximal subtrees of $\mathcal{T}_m$, and bar denotes the topological closure. Furthermore, we prove that if A is non-trivial, then $A^* = C_{\mathcal{A}_m}(\Delta(A))$, the centralizer of $\Delta(A)$ in $\mathcal{A}_m$. When A is a torsion self-similar abelian group, it is shown that it is necessarily of finite exponent. Moreover, we extend recent constructions of self-similar free abelian groups of infinite enumerable rank to examples of such groups which are also Δ-invariant for $s = 2$. In the final section, we introduce for $m = ns \geq 2$, a generalized adding machine a, an automorphism of $\mathcal{T}_m$, and show that its centralizer in $\mathcal{A}_m$ to be a split extension of $\langle a \rangle^*$ by $\mathcal{A}_s$. We also describe important $\mathbb{Z}_n[\mathcal{A}_s]$ submodules of $\langle a \rangle^*$.

1. Introduction

Our purpose in this paper is to study intransitive self-similar abelian groups and their centralizers, generalizing known results for the transitive case. The results confirm the importance of the centralizer as an analytic and discovery tool for groups and their representations.

Nekrashevych and Sidki characterized in [7] self-similar free abelian subgroups of *finite rank* of the group of automorphisms of the binary tree $\mathcal{T}_2$. Brunner and Sidki conducted in [3] a thorough study of *transitive* self-similar abelian subgroups of $\mathcal{A}_m$, the group of automorphisms of the m-ary tree $\mathcal{T}_m$, based on two facts. The first is that $\mathcal{A}_m$ is a topological group and for an abelian subgroup G of $\mathcal{A}_m$, its topological closure $\overline{G}$ is again abelian. The second, when G is a transitive self-similar abelian group, its closure under the monoid $\langle x \rangle$ generated by the *diagonal monomorphism*

$$x\colon \alpha \mapsto (\alpha, \alpha, \ldots, \alpha)$$

from $\mathcal{A}_m$ to its first level stabilizer, is again self-similar abelian.

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Actually, it was asserted erroneously in [3, Proposition 1, p. 459] that the topological and diagonal closure operations commute and then that the group A^* was a "combination" of these two closures applied to A. The precise statement is that in that paper A^* should have been, first the diagonal closure then the topological closure applied to A, which in reality was the order used.

Bartholdi and Sidki produced in [1] the first free abelian transitive self-similar group of infinite enumerable rank. Further transitive self-similar abelian groups appear in Kochloukova–Sidki [6, Theorem C] in the context of transitive self-similar metabelian groups as finitely generated $\mathbb{Z}Q$-modules of Krull dimension 1, where Q is a finitely generated abelian group.

Let G be a subgroup of $\mathcal{A}_m$. Consider $P = P(G) \leqslant \mathrm{Sym}(m)$ the permutation group, or group of activities, induced by G on $Y = \{1, \ldots, m\}$, and let

$$O_{(1)} = \{1, \ldots, m_1\}, \quad O_{(2)} = \{m_1 + 1, \ldots, m_1 + m_2\}, \quad \ldots,$$
$$O_{(s)} = \{m_1 + \cdots + m_{s-1} + 1, \ldots, m\}$$

of respective size $m_1, m_2, \ldots, m_s$ be the orbits of P in this action; we will view these orbits as ordered in the manner as they are written here. For this set of orbits, an element ξ of $\mathrm{Sym}(m)$ is called *rigid*, provided it permutes the set of orbits and is order preserving in the sense that for all $1 \leq i \leq s$ and all $k \leq l$ in $O_{(i)}$, $(k)\xi \leq (l)\xi$ in $O_{((i)\xi)}$.

The group P induces transitive permutation groups $P_{(i)}$ on $O_{(i)}$, $i = 1, \ldots, s$, and P is naturally identified with a sub-direct product of the $P_{(i)}$'s. We call $(m_1, \ldots, m_s)$ the *orbit-type* of G and $(P_{(1)}, \ldots, P_{(s)})$ the *permutation-type* of G. An element σ of P decomposes as $\sigma = \sigma_{(1)}\sigma_{(2)} \cdots \sigma_{(s)}$, where $\sigma_{(i)} \in P_{(i)}$.

Every $\alpha = (\alpha_1, \ldots, \alpha_m)\sigma \in G$ can be written as

$$\alpha = (\alpha_{(1)}, \ldots, \alpha_{(s)})\sigma,$$

where $\alpha_{(1)} = (\alpha_1, \ldots, \alpha_{m_1}), \ldots, \alpha_{(s)} = (\alpha_{m_1+\cdots+m_{s-1}+1}, \ldots, \alpha_m)$. Therefore, α can be factored in the form

$$\alpha = \alpha_{[1]} \cdots \alpha_{[s]},$$

where $\alpha_{[1]} = (\alpha_{(1)}, e, \ldots, e)\sigma_{(1)}, \ldots, \alpha_{[s]} = (e, \ldots, e, \alpha_{(s)})\sigma_{(s)}$; the factors $\alpha_{[i]}$ commute among themselves.

Define the subgroups of $\mathcal{A}_m$

$$G_{[1]} = \{(\alpha_{(1)}, e, \ldots, e)\sigma_{(1)} \mid \alpha \in G\},$$
$$\vdots$$
$$G_{[s]} = \{(e, \ldots, e, \alpha_{(s)})\sigma_{(s)} \mid \alpha \in G\}$$

and denote the group generated by the $G_{[i]}$'s by $B(G)$. Clearly, $B(G) = G_{[1]} \cdots G_{[s]}$ is a direct product of its factors and G is a sub-direct product of $B(G)$.

The symmetric group $\mathrm{Sym}(m)$ is naturally embedded in $\mathcal{A}_m$ as a group of rigid permutations of the set of maximal subtrees of $\mathcal{T}_m$; therefore, any subgroup of $\mathrm{Sym}(m)$ in this embedding is self-similar in $\mathcal{A}_m$.

In the intransitive setting, we need to substitute the monoid $\langle x \rangle$ by the monoid Δ freely generated by the *s-partial diagonal monomorphisms* from $\mathcal{A}_m$ to $\mathrm{Stab}_{\mathcal{A}_m}(1)$

$$x_i \colon \alpha \mapsto (e, \dots, e, \alpha, \dots, \alpha, e, \dots, e)$$

with α occurring in coordinates from the orbit $O_{(i)}$ and the trivial automorphism e occurring in the other positions.

Given $u \in \Delta$, we denote its length by $|u|$, set $\Delta_k = \{u \in \Delta \mid |u| = k\}$ for $k \geq 0$ and denote the closure of G under Δ by

$$\Delta(G) = \langle G^{\omega} \mid \omega \in \Delta \rangle.$$

Our results are as follows.

Theorem A. *Let $A \leqslant \mathcal{A}_m$ be a self-similar abelian group. Let $C_{\mathcal{A}_m}(\Delta(A))$ denote the centralizer of $\Delta(A)$ in $\mathcal{A}_m$ and define $A^* = \overline{\Delta(B(A))}$. Then*

(i) *$\Delta(A)$, $\overline{\Delta(A)}$, $\Delta(B(A))$ and A^* are again self-similar abelian groups and of the same permutation-type as A.*

(ii) *If A is nontrivial, then $A^* = C_{\mathcal{A}_m}(\Delta(A))$ and A^* is a maximal abelian subgroup of $\mathcal{A}_m$.*

Theorem B. *There exists a finitely generated subgroup H of $B(A)$ such that $A^* = \overline{\Delta(H)}$.*

Corollary 1.1. *An abelian torsion group of infinite exponent cannot have a faithful representation as a self-similar group.*

Problem 1.2. In view of the above corollary, we pose the question about the existence of torsion-free abelian subgroups of $\mathrm{Aut}(\mathcal{T}_m)$ which do not afford faithful representations as self-similar subgroups of $\mathrm{Aut}(\mathcal{T}_{m'})$ for any $m' \geq 2$.

Let j be a positive integer, $\alpha = (e, \dots, e, \alpha^{x^{j-1}})(1 \cdots m)$ and $D_m(j)$ be the group generated by $Q(\alpha) = \{\alpha, \alpha^x, \dots, \alpha^{x^{j-1}}\}$, the states of α. Then $D_m(j)$ is a free abelian group of rank j which is transitive self-similar and diagonally closed. In contrast, in the general case, we have the following result.

Theorem C. *Let A be a free abelian group.*

(i) *Suppose A has finite rank.*

(a) *If A is a non-transitive self-similar group, then $\Delta(A)$ is not finitely generated.*

(b) *Let A be a transitive self-similar subgroup of $\mathcal{A}_m$. Then the self-similar representation of A extends to a non-transitive self-similar representation into $\mathcal{A}_{m+1}$ having orbit-type $(m, 1)$ such that $\Delta(A)$, with respect to the second* *representation, contains a self-similar free abelian group of infinite enumerable rank.*

(ii) *Suppose A has infinite countable rank. Then A can be realized as a self-similar subgroup $\mathbf{A}$ of $\mathcal{A}_{m+1}$ of orbit-type $(m, 1)$, invariant with respect to $\Delta = \langle x_1, x_2 \rangle$, and is freely generated by the set U whose elements are*

$$\alpha_1 = (e, \ldots, e, \alpha_1, e)(1\,2 \cdots m)(m+1),$$
$$\alpha_{2i-1} = {\alpha_i}^{x_1} \ (i \geq 2), \quad \alpha_{2i} = {\alpha_i}^{x_2} \ (i \geq 1).$$

Moreover, ${\alpha_1}^m = {\alpha_1}^{x_1}$, $U = \{\alpha_1\} \cup \{\alpha_1 \cdot x_2 \Delta\}$. Also $\mathbf{A} = \Delta(\langle \alpha_1 \rangle)$, and $C_{\mathcal{A}_m}(\mathbf{A})$ is a free abelian pro-m group,

$$C_{\mathcal{A}_m}(\mathbf{A}) = \overline{\mathbf{A}} \simeq \prod_{i \geq 1} \alpha_i \mathbb{Z}_m.$$

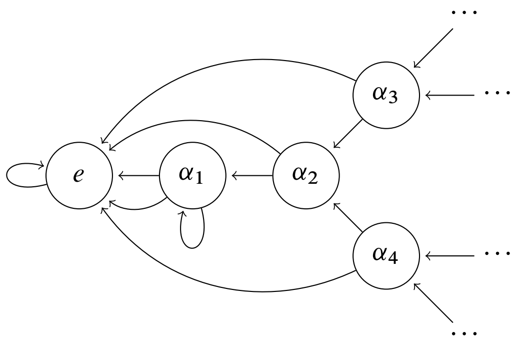

Figure 1. Diagram of the automaton **A**.

Problem 1.3. Is there a self-similar free abelian group of uncountable rank?

The final section is an application of the general theory developed in this paper to a self-similar cyclic group generated by a generalized adding machine which we define below. Other examples of self-similar cyclic groups can be found in the previous version of the present paper [5, Section 5].

Let $m = ns \geq 2$. The generalized adding machine a is an element of $\mathrm{Aut}(\mathcal{T}_m)$ defined as:

$$a = (a_{(1)}, a_{(2)}, \ldots, a_{(s)})p_1 p_2 \cdots p_s, \quad a_{(i)} = (e, e, \ldots, a) \quad \text{for } 1 \leq i \leq s,$$
$$p_1 = (1, 2, \ldots, n), \ p_2 = (n+1, n+2, \ldots, 2n), \ \ldots, \ p_s = ((s-1)n+1, \ldots, sn).$$

We will show that $C(A)$ is a semidirect product of a $\mathbb{Z}_n[\mathrm{Aut}(\mathcal{T}_s)]$-module by $\mathrm{Aut}(\mathcal{T}_s)$ and exhibit some of its important submodules. Define the following subgroups of A^*:

$$V = \langle a_{[i]} \cdot u \mid 1 \leq i \leq s,\, u \in \Delta \rangle,$$
$$V_k = \langle a_{[i]} \cdot u \mid 1 \leq i \leq s-1,\, u \in \Delta_k \rangle \quad \text{for } k \geq 0$$

and ${V_k}^{\#}$ is the normal closure of V_k under the action of $\Delta(S)$. We gather the results in the following theorem.

Theorem D. (i) $C(A) = A^* \cdot \overline{\Delta(S)}$, *where* S *is the group of permutations of the set of the* s *orbits* $O_{(j)}$ *which is isomorphic to* $\mathrm{Sym}(s)$, $\overline{\Delta(S)}$ *corresponds to* $\mathrm{Aut}(\mathcal{T}_s)$, *and* A^* *is a* $\mathbb{Z}_n[\mathrm{Aut}(\mathcal{T}_s)]$*-module.*

(ii) V *is freely generated by* $U = \{a\} \cup \{a_{[i]} \cdot u \mid 1 \leq i \leq s-1, u \in \Delta\}$.

(iii) $A^* = \overline{V} = A \cdot \mathbb{Z}_n \oplus \bigoplus_{1 \leq i \leq s-1} \{a_{[i]} \cdot (u\mathbb{Z}_n) \mid u \in \Delta\}$.

(iv) ${V_0}^{\#} = A \oplus V_0$, *and for* $k \geq 1$, ${V_k}^{\#} = {V_0}^{\#} \cdot n^k \oplus \bigoplus_{1 \leq j \leq k} V_j \cdot n^{k-j}$.

(v) ${V_k}^{\#}$ *is* $\overline{\Delta(S)}$*-invariant for all* k.

2. Preliminaries

2.1. Self-similar groups

Let $Y = \{1, \ldots, m\}$ be a finite alphabet with $m \geq 2$ letters. The monoid of finite words $\widehat{Y}$ over Y has a structure of a rooted m-ary tree denoted by $\mathcal{T}(Y)$ or $\mathcal{T}_m$. The incidence relation on $\mathcal{T}_m$ is given by: (u, v) is an edge if and only if there exists a letter y such that $v = uy$. The empty word $\emptyset$ is the root of the tree, and the level i is the set of all words of length i. Two vertices u, v of $\mathcal{T}_m$ are *comparable* provided u is a suffix of v or vice versa, v is a suffix of u.

The automorphism group $\mathcal{A}_m$ (or $\mathrm{Aut}(\mathcal{T}_m)$) of $\mathcal{T}_m$ is isomorphic to the restricted wreath product recursively defined by $\mathcal{A}_m = \mathcal{A}_m \wr \mathrm{Sym}(Y)$. An automorphism α of $\mathcal{T}_m$ has the form $\alpha = (\alpha_1, \ldots, \alpha_m)\sigma(\alpha)$, where the state α_i belongs to $\mathcal{A}_m$ and $\sigma\colon \mathcal{A}_m \to \mathrm{Sym}(Y)$ is the permutational representation of $\mathcal{A}_m$ on Y, seen as first level of the tree $\mathcal{T}_m$; $\sigma(\alpha)$ is referred to as the *activity* of α. The action of $\alpha = (\alpha_1, \ldots, \alpha_m)\sigma(\alpha) \in \mathcal{A}_m$ on a word $y_1 y_2 \cdots y_n$ of length n is given recursively by $(y_1)^{\sigma(\alpha)}(y_2 \cdots y_n)^{\alpha_{y_1}}$.

The recursively defined subset of $\mathcal{A}_m$

$$Q(\alpha) = \{\alpha\} \bigcup_{i=1}^{m} Q(\alpha_i)$$

is called the set of *states* of α. A subgroup G of $\mathcal{A}_m$ is *self-similar* (or *state-closed*, or *functionally recursive*) if $Q(\alpha)$ is a subset of G for all α in G and is *transitive* if its action on Y is transitive.

2.2. Self-similarity data

A *virtual endomorphism* of an abstract group G is a homomorphism $f: H \to G$ from a subgroup H of finite index in G. Let G be a group and consider

$$\begin{aligned}
\mathbf{H} &= (H_i \leqslant G \mid [G : H_i] = m_i \ (1 \leq i \leq s)),\\
\mathbf{m} &= (m_1, \ldots, m_s), \quad m = m_1 + \cdots + m_s,\\
\mathbf{F} &= (f_i: H_i \to G \text{ virtual endomorphisms} \mid 1 \leq i \leq s).
\end{aligned}$$

We will call $(\mathbf{m}, \mathbf{H}, \mathbf{F})$ a G*-data* or a *data for* G. The $\mathbf{F}$*-core* is the largest subgroup of $\cap_{i=1}^{s} H_i$ which is normal in G and f_i-invariant for all $i = 1, \ldots, s$. In the case where $\mathbf{F}$-core is trivial, we say that $\mathbf{F}$ is *free*.

The following approach to produce intransitive self-similar groups was given in [4].

Proposition 2.1. *Given a group G, $m \geq 2$ and a G-data $(\mathbf{m}, \mathbf{H}, \mathbf{F})$. Then the data provides a self-similar representation of G on the m-tree with kernel the $\mathbf{F}$-core,*

$$\left\langle K \leqslant \bigcap_{i=1}^{s} H_i \mid K \lhd G, K^{f_i} \leqslant K, \forall i = 1, \ldots, s \right\rangle.$$

There exists a variety of G-data. Among these we distinguish: G-data $(\mathbf{m}, \mathbf{H}, \mathbf{F})$ is *simple* if the $\mathbf{F}$-core is trivial; *strongly simple* if every f_i-invariant subgroup of H_i is trivial; *recurrent*, if in addition each virtual endomorphism f_i is an epimorphism. Furthermore, we say that a recurrent G-data is *strongly recurrent* if

$$f_i\colon\ H_i \cap \bigcap_{j \neq i} \ker(f_j) \to G$$

is an epimorphism for any f_i in $\mathbf{F} = (f_1, \ldots, f_s)$.

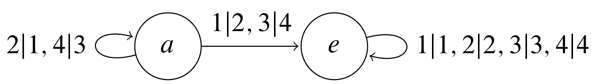

Figure 2. Diagram of the double adding machine a.

There are recurrent G-data $(\mathbf{m}, \mathbf{H}, \mathbf{F})$ which are not strongly recurrent. Indeed, consider A the group generated by the double-adding machine $a = (e, a, e, a)(1\,2)(3\,4)$, then $\mathrm{Fix}_A(1) = \mathrm{Fix}_A(3) = \langle a^2 \rangle$ and the A-data $((2,2), (\mathrm{Fix}_A(1), \mathrm{Fix}_A(3)), (\pi_1, \pi_3))$ is recurrent but not strongly recurrent. The automorphism a was first considered in [2, Section 2.1]. Its generalization for the m-tree ($m \geq 2$) is given in Section 5.

2.3. Layer closure

Given a subgroup P of $\mathrm{Sym}(m)$, the set of elements α of $\mathcal{A}_m$ whose states have their activities are from P forms a self-similar subgroup $L_m(P)$ of $\mathcal{A}_m$ called the *layer closure* of P in $\mathcal{A}_m$. We simplify $L_m(P)$ to $L(P)$ when the context is clear. $L(P)$ inherits from the wreath product structure of $\mathcal{A}_m$ its structure $L(P) = (\times_m L(P)) \cdot P$. Let G be a self-similar subgroup of $\mathcal{A}_m$ of orbit-type $(m_1, m_2, \dots, m_s)$ and corresponding permutation-type $(P_1, P_2, \dots, P_s)$. Then the following factorization holds.

Lemma 2.2 ([4, Lemma 2.1]).

$$L(P) = L(P_1)L(P_2)\cdots L(P_s),$$

where the factors are two by two permutable.

3. Operators acting on automorphisms of m-trees

Since partial diagonal monomorphisms of $\mathcal{A}_m$ such as $\alpha \mapsto (\alpha, e, \dots, e)$ play an important role in our paper, we approach the subject more generally. Define

$$\begin{aligned} \mathrm{Func}(m) &= \{f : \mathcal{A}_m \to \mathcal{A}_m \text{ function}\},\\ \mathrm{End}(m) &= \{f : \mathcal{A}_m \to \mathcal{A}_m \text{ endomorphism}\},\\ \mathrm{Mon}(m) &= \{f : \mathcal{A}_m \to \mathcal{A}_m \text{ monomorphism}\}. \end{aligned}$$

As is usually observed, Func(m) is closed under two binary operations: the composition of functions "·" defined by

$$(a)(f \cdot g) = ((a)f)g$$

and the non-commutative "+" defined by

$$(a)(f + g) = ((a)f)((a)g)$$

for all $f, g \in \mathrm{Func}(m)$ and $a \in \mathcal{A}_m$.

End(m) is closed under "·" and Mon(m) is a submonoid of End(m). We note that for $f, g, h \in \mathrm{Func}(m)$,

$$h \cdot (f + g) = h \cdot f + h \cdot g \quad \text{and} \quad (f + g) \cdot h = f \cdot h + g \cdot h$$

provided $h \in \mathrm{End}(m)$.

Lemma 3.1. *Let $f, g \in \mathrm{End}(m)$ and let $h = f + g$.*

(i) *Then $h \in \mathrm{End}(m)$ if and only if $(\mathcal{A}_m)^f$ commutes element-wise with $(\mathcal{A}_m)^g$.*

(ii) *Suppose $h \in \mathrm{End}(m)$. Then $h \in \mathrm{Mon}(m)$ if and only if ${\mathcal{A}_m}^f \cap {\mathcal{A}_m}^g = \{e\}$.*

The proofs are straightforward.

3.1. Monomorphisms of $\mathcal{A}_m$

Let $(\mathcal{A}_m)\kappa$ denote the group induced by conjugation of $\mathcal{A}_m$ on itself. Since $m \geq 2$, the center of $\mathcal{A}_m$ is trivial, so $(\mathcal{A}_m)\kappa$ is isomorphic to $\mathcal{A}_m$ and is contained in $\mathrm{Mon}(m)$. The tree structure induces certain additional monomorphisms of $\mathcal{A}_m$ defined as follows. First, a *connecting set* M of $\mathcal{T}(Y)$ is a subset of vertices of $\mathcal{T}(Y)$ such that every element of $\mathcal{T}(Y)$ is comparable to some element of M and different elements of M are incomparable. Some examples for $Y = \{1, 2\}$ are: $M = \{\emptyset\}$; $M = Y$; $M = \{1, 21, 221, \ldots, 2^i \cdot 1, \ldots\}$.

Given a connecting set M and a non-empty subset N of M (called *partial connecting set*), define the monomorphism

$$\delta_N \colon \mathcal{A}_m \to \mathcal{A}_m, \quad a \mapsto (a)\delta_N = b,$$

where b is such that b_u is equal to a for all $u \in N$ and to the identity e for all u in $M \setminus N$. Then δ_N is in $\mathrm{Mon}(\mathcal{A}_m)$. When $w \in \widehat{Y}$, we simplify the notation $\delta_{\{w\}}$ to δ_w. Define $\delta(Y)$ to be the monoid $\langle \delta_i \mid i \in Y \rangle$. When Y is fixed, we simplify $\delta(Y)$ to δ and denote the image of a subgroup H of $\mathcal{A}_m$ under the action of δ by $\delta(H) = H^\delta = \langle (a) \cdot w \mid a \in H \text{ and } w \in \delta \rangle$.

3.2. The monoid $\delta(Y)$

Lemma 3.2. *For all $u, v \in \widehat{Y}$, $\delta_u \delta_v = \delta_{vu}$, and therefore*

$$\delta = \langle \delta_i \mid i \in \widehat{Y} \rangle$$

is a free monoid of rank m.

Proof. We check $(a)\delta_u\delta_v = ((a)\delta_u)\delta_v = (a)\delta_{vu}$ for all $a \in \mathcal{A}_m$ and freeness follows. ∎

Lemma 3.3. *Let $U = \{u_i \mid 1 \leq i \leq k\}$ be a set of incomparable elements of $\mathcal{T}_m$. Then, $\sum_{u \in U} \delta_u \in \mathrm{Mon}(m)$, $\{\delta_u \mid u \in U\}$ is a commutative set and*

$$(\mathcal{A}_m)^{\delta_v} \cap \langle (\mathcal{A}_m)^{\delta_u} \mid u \in U, u \neq v \rangle = \{e\}.$$

Proof. The argument is as that of Lemma 3.1. ∎

Lemma 3.4 (Permutability relations). *Let $r \in \mathcal{A}_m$ and $w \in \mathcal{T}_m$. Then*

$$\delta_w \cdot (r)\kappa = (r_w)\kappa \cdot \delta_{(w)r}$$

and therefore the following monoid factorization holds:

$$\langle (\mathcal{A}_m)\kappa, \delta(Y) \rangle = (\mathcal{A}_m)\kappa \cdot \delta(Y).$$

Proof. Let $|w| = k$. The development of r at the k-th level of the tree is $r = (r_u \mid |u| = k)\sigma$ for some permutation σ of the k-th level. Then as

$$(a)(\delta_w) = (e, \ldots, e, a, e, \ldots, e)$$

an element of $\mathrm{Stab}_{\mathcal{A}_m}(k)$ and has a in the w-th coordinate, we compute

$$\begin{aligned}(a)(\delta_w)\cdot(r)\kappa &= ((a)\delta_w)^r = (e,\ldots,e,a,e,\ldots,e)^r\\ &= (e,\ldots,a^{r_w},e,\ldots,e)^{\sigma} = (e,\ldots,e,a^{r_w},e,\ldots,e),\end{aligned}$$

where in the last equation a^{r_w} is in the $(w)^{\sigma} = (w)r$ coordinate. Thus,

$$\begin{aligned}(a)(\delta_w)\cdot(r)\kappa &= (a^{r_w})(\delta_{(w)r}) = (a)((r_w)\kappa\cdot(\delta_{(w)r})),\\ \delta_w\cdot(r)\kappa &= (r_w)\kappa\cdot(\delta_{(w)r}).\end{aligned}$$

■

3.3. Partial diagonal monomorphisms in $\delta(Y)$

Let $\pi = \{Y_j \mid j = 1,\ldots,s\}$ be a partition of Y. Define $\delta(\pi) = \langle \delta_N \mid N \in \pi\rangle$.

In the particular case where π is the set of orbits $\{O_{(1)},\ldots,O_{(s)}\}$ of $G \leqslant \mathcal{A}_m$ on Y, denote the partial diagonal monomorphisms associated to G and its set of orbits by

$$x_1 := \lambda_{O_{(1)}},\ldots,x_s := \lambda_{O_{(s)}}.$$

Order the elements of the orbit $O_{(i)}$ as $i1 < i2 < \cdots < im_i$.

Lemma 3.5. *Let $r \in \mathcal{A}_m$ and x_i in Δ.*

(i) *Then*

$$x_i\cdot(r)\kappa = [(r_{i1})\kappa\cdot(\delta_{(i1)r})]\cdot[(r_{i2})\kappa\cdot(\delta_{(i2)r})]\cdots[(r_{im_i})\kappa\cdot(\delta_{(im_i)r})].$$

(ii) *If the states $r_{i1},\ldots,r_{im_i}$ of r centralize an element a of $\mathcal{A}_m$, then*

$$(a)x_i\cdot(r)\kappa = (a)x_l,$$

where $O_{(l)} = O_{((i)r)}$.

Proof. Let $a \in \mathcal{A}_m$. Then

$$\begin{aligned}(a)x_i\cdot(r)\kappa &= [(a)\delta_{i1}(a)\delta_{i2}\cdots(a)\delta_{im_i}]^r = [(a)\delta_{i1}]^r[(a)\delta_{i2}]^r\cdots[(a)\delta_{im_i}]^r\\ &= [(a^{r_{i1}})\delta_{(i1)r}][(a^{r_{i2}})\delta_{(i2)r}]\cdots[(a^{r_{im_i}})\delta_{(im_i)r}].\end{aligned}$$

In the last equality, we used Lemma 3.4.

Item (ii) is a consequence of (i). ■

3.4. Some relations between the operations, Δ, Stab, $^-$ and B

Let A be an abelian self-similar subgroup of $\mathcal{A}_m$. As we often need to compose the above operations in our manipulations of A, for the sake of reference, in the following, we separate facts about some of them.

Lemma 3.6. *Let $x_i \in \Delta$ and $k \geq 1$. Then,* $\mathrm{Stab}_{A^{x_i}}(k) = (\mathrm{Stab}_A(k-1))^{x_i}$.

Proof. Straightforward. ■

Lemma 3.7. (i) $\Delta(\overline{A}) \leqslant \overline{\Delta(A)}$*;*

(ii) *for* $m = 2$ *and* $A = \langle a = (1\,2)\rangle$*, the above inclusion is proper.*

Proof. (i) The topological closure of A is the infinite product

$$\overline{A} = A \cdot \mathrm{Stab}_A(1) \cdots \mathrm{Stab}_A(i) \cdots .$$

Therefore, for any $w \in \Delta$, $(\overline{A})^w = A^w \cdot (\mathrm{Stab}_A(1))^w \cdots (\mathrm{Stab}_A(i))^w \cdots$ which, by the previous lemma, is a subset of $\overline{A^w}$.

(ii) Since for $A = \langle (1\,2)\rangle \leqslant \mathcal{A}_2$, $\overline{A} = A$, we have

$$\Delta(\overline{A}) = \{a^{p(x)} \mid p(x) \in \mathbb{Z}/2\mathbb{Z}\,[x]\},$$

whereas

$$\overline{\Delta(A)} = \{a^{f(x)} \mid f(x) \in \mathbb{Z}/2\mathbb{Z}\,[[x]]\}. \qquad \blacksquare$$

Lemma 3.8. *Let* $w \in \Delta$*.*

(i) $B(A^w) = A^w$ *if* $|w| \geq 1$*;*

(ii) $B(\Delta(A)) = B(A) \cdot \Delta(A)$*,* $\Delta(B(\Delta(A))) = B(\Delta(A))$*.*

Proof. (i) $(A^{w'x_i})_{[k]} = \{e\}$ if k is different from i, and

$$(A^{w'x_i})_{[k]} = A^{w'x_i} \quad \text{if } k = i.$$

(ii) By direct computation, we have

$$\begin{aligned} B(\Delta(A)) &= B(A)\langle B(A^w) \mid |w| \geq 1\rangle = B(A)\langle A^w \mid |w| \geq 1\rangle \\ &= B(A)A\langle A^w \mid |w| \geq 1\rangle = B(A)\Delta(A). \end{aligned} \qquad \blacksquare$$

4. The centralizer of self-similar abelian groups

We start with a description of the structure of centralizers of self-similar abelian groups and then proceed to prove Theorem A.

The very first step is to describe the structure of the centralizer of subgroups of finite symmetric groups.

Lemma 4.1. *Let* Q *be a subgroup of* $\mathrm{Sym}(m)$ *with the set of orbits* $O_{(i)}$*,* $1 \leq i \leq s$*. Then the centralizer of* Q *in* $\mathrm{Sym}(m)$ *has the structure*

$$C_{\mathrm{Sym}(m)}(Q) = C_{\mathrm{Sym}(m_1)}(Q_{(1)}) \cdots C_{\mathrm{Sym}(m_s)}(Q_{(s)}) \cdot S(Q),$$

where $S(Q)$ *is the subgroup of rigid permutations in* $\mathrm{Sym}(m)$ *with respect to the given set of orbits. Let* $\{J_1, \ldots, J_t\}$ *be the partition of the orbits of* Q *with equal length. Then,* $S(Q)$ *is isomorphic to the direct product* $\mathrm{Sym}(J_1) \times \cdots \times \mathrm{Sym}(J_t)$*. In the case where* Q *is abelian,* $C_{\mathrm{Sym}(m)}(Q) = B(Q)S(Q)$*.*

Given a subgroup G of $\mathcal{A}_m$, we write $C(G)$ for $C_{\mathcal{A}_m}(G)$ and define

$$S(G) = \{\xi \in S(P(G)) \mid C(G) \cap \mathrm{Stab}_{\mathcal{A}_m}(1)\xi \text{ non-empty}\}.$$

Since $S(P(G)) = S(P(B(G))$, any lifting of ξ in $S(G)$ to $C_{\mathcal{A}_m}(G)$ can be modified by an element of $B(G)$ so as to induce a rigid permutation of the orbits, in this sense that we choose each ξ to be rigid and define $R(G)$ to be the group generated by these liftings. We note that any two liftings of ξ differ by an element of $\mathrm{Stab}_G(1)$. If ξ itself is in $C_{\mathcal{A}_m}(G)$, then ${G_{[i]}}^{\xi} = G_{[j]}$, where ${O_{(i)}}^{\xi} = O_{(j)}$, and therefore ξ normalizes $B(G)$.

Proposition 4.2. *Let A be an abelian subgroup of $\mathcal{A}_m$ and recall $B(A)$, $C(A)$, $R(A)$. Then,*

(i) $C(A) = \mathrm{Stab}_{C(A)}(1)B(A)R(A)$*;*

(ii) *the subgroup* $\mathrm{Stab}_{C(A)}(1)B(A)$ *is normal in* $C(A)$*;*

(iii) $B(A)$ *centralizes* $\mathrm{Stab}_{C(A)}(1)$.

Suppose furthermore that A is self-similar. Then,

(iv) $\mathrm{Stab}_A(1)$ *is a subgroup of* $A^{x_1} \cdot A^{x_2} \cdots A^{x_s}$*;*

(v) $C(A)$ *and* $\mathrm{Stab}_{C(A)}(1)$ *are* Δ*-invariant;*

(vi) $\Delta(A)$, $\Delta(B(A))$ *and* $\overline{\Delta(B(A))}$ *are self-similar abelian groups of the same permutation-type as* A.

Proof. (i) This item follows from the fact that $C(A)$ modulo $\mathrm{Stab}_{C(A)}(1)$ is isomorphic to a subgroup of $C_{\mathrm{Sym}(m)}(P(A))$ as in Lemma 4.1.

(ii) Given $r = (r_1, \ldots, r_m)\xi \in R(A)$, where $\xi \in S(P(A))$, and $h \in A_{[i]}$ written as $h = h' \cdot \sigma$, where $h' \in \mathrm{Stab}_{\mathcal{A}_m}(1)$ and $\sigma \in P_{(i)}$, we have $\sigma^{\xi} \in P_{(i^{\xi})}$. As there exists $g \in A_{[i^{\xi}]}$ such that $g = g' \cdot \sigma^{\xi}$, we have $(h^r) \cdot g^{-1} = (h')^r \cdot (g')^{-1} \in \mathrm{Stab}_{C(A)}(1)$, and therefore the subgroup $\mathrm{Stab}_{C(A)}(1)B(A)$ is normal in $C(A)$.

(iii) Let $f = (f_1, f_2, \ldots, f_m) = f_{[1]} f_{[2]} \cdots f_{[s]} \in \mathrm{Stab}_{\mathcal{A}_m}(1)$ and $a = a_{[1]} \cdots a_{[s]} \in A$. Then,

$$f^a = {f_{[1]}}^{a_{[1]}} \cdots {f_{[s]}}^{a_{[s]}}.$$

Thus, $f \in C(A)$ if and only if $f_{[i]}$ centralizes $A_{[i]}$ for all i. As $f_{[i]}$ centralizes $A_{[j]}$ for all $j \neq i$, it follows that $f \in C(A)$ if and only if $f_{[i]}$ centralizes $B(A)$ for all i.

(iv) An element of $\mathrm{Stab}_A(1)$ has the form $a = (a_{(1)}, a_{(2)}, \ldots, a_{(s)})$. As $P_{(i)}$ is transitive on $O_{(i)}$, conjugation of a by elements $A_{[i]}$ shows that the entries of $a_{(i)}$ are all equal.

(v) Given $c \in C(A)$, easily $c^{x_i} \in C(A)$. Therefore, $C(A)^{\Delta} = C(A)$. Since

$$(\mathrm{Stab}_{C(A)}(1))^{\Delta} \leqslant C(A)^{\Delta} = C(A) \quad \text{and} \quad (\mathrm{Stab}_{C(A)}(1))^{w} \leqslant \mathrm{Stab}_{C(A)}(1)$$

for $w \in \Delta$ of length at least 1, we have

$$(\mathrm{Stab}_{C(A)}(1))^{\Delta} = \mathrm{Stab}_{C(A)}(1)).$$

(vi) Since A is a self-similar abelian group, $\Delta(A)$ is also self-similar abelian. By induction on the length of elements of Δ, the fact that $\Delta(B(A))$ is abelian is reducible to $A_{[i]}$ commutes with $(A_{[j]})^w$ for all i, j. Clearly, $A_{[i]}$ commutes with $(A_{[j]})^w$ if $w = e$ or $w = w' \cdot x_k$ for $k \neq i$. In the case $k = i$, $A_{[i]}$ commutes with $(A_{[j]})^w$ if and only if $A_{[i]}$ commutes with $(A_{[j]})^{w'}$. Further, since $P(\Delta(B(A))) = P(\Delta(A)) = P(A)$, it follows that $\Delta(B(A))$ and $\Delta(A)$ are of the same permutation-type as A. Since the topological closure of abelian groups are also abelian, $\overline{\Delta(B(A))}$ is abelian and is also self-similar. ∎

4.1. The case where $R(A)$ consists of rigid permutations

Naturally, the complexity of $C(A)$ depends upon that of $R(A)$. In the case $R(A) = S(A)$, we obtain finer structural results.

Proposition 4.3. *Suppose $R(A) = S(A)$. Then,*

(i) *$B(A)$ is normal in $C(A)$ and $\Delta(B(A))$ is normalized by $\Delta(R(A))$.*

(ii) *Define $H = \Delta(B(A))\Delta(R(A))$. Then $\overline{\Delta(R(A))}$ normalizes $\overline{\Delta(B(A))}$, and $\overline{H} = \overline{\Delta(B(A))\Delta(R(A))}$.*

Proof. (i) As $R(A) = S(A)$, conjugation by elements of $R(A)$ induces permutations of the set of $A_{[i]}$'s, and therefore $R(A)$ normalizes $B(A)$. Hence $B(A)$ is a normal subgroup of $C(A)$. We calculate for $b \in B(A)$ the conjugates of b^v by ξ^w for $b = a_{[i]} \in A_{[i]}$ and $v, w \in \Delta$. As $B(A)$ is normal in $C(A)$, we may assume $|v| > 0$ and write $v = v'x_i$ for some i. Also, since for any $g \in \mathcal{A}_m$, $(g^{x_k})^\xi = g^{x_l}$, where $l = k^\xi$, we can assume $|w| > 0$ and write $w = w'x_j$. Now check that $(b^{w'x_i})^{\xi^{v'x_j}} = (b^{w'})^{x_i}$ if $j = i$ and $(b^{w'x_i})^{\xi^{v'x_j}} = b^{w'x_i}$ otherwise. The result follows by induction on the lengths of v and w.

(ii) Since $P(A)$ and $P(S(A))$ intersect trivially, it follows that their layered closures $L(P(A))$, $L(P(S(A)))$ also intersect trivially. Therefore, for all $k \geq 0$, the k-th stabilizers of $\Delta(B(A))$ and $\Delta(R(A))$ intersect trivially, and thus

$$\mathrm{Stab}_H(k) = \mathrm{Stab}_{\Delta(B(A))}(k) \cdot \mathrm{Stab}_{\Delta(R(A))}(k).$$

Hence, $\overline{\Delta(R(A))}$ normalizes $\overline{\Delta(B(A))}$. ∎

We prove Theorem A in a more detailed form.

Theorem 4.4. *Let $A \leqslant \mathcal{A}_m$ be a nontrivial abelian self-similar group and let $A^* = \overline{\Delta(B(A))}$ and $C = C_{\mathcal{A}_m}(\Delta(A))$. Then*

(i) *C leaves each orbit $O_{(i)}$ invariant (that is, $R(\Delta(A)) = \{e\}$) and*

$$C = \mathrm{Stab}_C(1)B(A);$$

(ii) $\mathrm{Stab}_C(1) = (C)^{x_1}(C)^{x_2}\cdots(C)^{x_s}$*;*

(iii) *$A^* = C_{\mathcal{A}_m}(\Delta(A))$ and A^* is a maximal abelian subgroup of $\mathcal{A}_m$ and is the unique one containing $\Delta(A)$.*

Proof. Denote $\Delta(A)$ by K.

(i) We need to prove $R(K) = \{e\}$. Suppose, by contradiction, that

$$r = (r_{(1)}, \dots, r_{(s)})\xi \in R(K)$$

with $\xi \neq e$, then r centralizes $K = \Delta(A)$. Let $\alpha \in A$ be nontrivial. Since $\xi \neq e$, there are $O_{(i)} \neq O_{(j)}$ such that $\xi \colon O_{(i)} \mapsto O_{(j)}$. Recall $O_{(i)} = (i_1, i_2, \dots, i_{m_i})$. Then $(\alpha^{x_i})^r = \alpha^{x_i}$, and on the other hand, $(\alpha^{x_i})^r = (e, \dots, e, \alpha^{r_{i1}}, \dots, \alpha^{r_{im_i}}), e, \dots, e)^{\xi}$; the identity e is in positions outside $O_{(i)}$, after conjugation by ξ, the identities are shifted to positions outside $O_{(j)}$. Therefore, $\alpha = e$ which is a contradiction. Hence, $C = \mathrm{Stab}_C(1)B(K)$. By Lemma 3.8, $B(K) = B(A)\Delta(A)$, and therefore

$$C = \mathrm{Stab}_C(1)B(A). \tag{4.1}$$

By Proposition 4.2, $B(A)$ is central in C.

(ii) Let $\gamma = (\gamma_{(1)}, \dots, \gamma_{(s)}) \in \mathrm{Stab}_C(1)$, $w \in \Delta$ and $a \in A$. Then,

$$e = [\gamma, a^{wx_i}] = (e, \dots, e, [\gamma_{(i)}, a^w], e, \dots, e), \quad \gamma_{(i)} \in C(A^{\Delta}) = C.$$

Therefore, $\gamma \in (C)^{x_1} \cdots (C)^{x_s}$. Thus,

$$\mathrm{Stab}_C(1) = C^{x_1} C^{x_2} \cdots C^{x_s}. \tag{4.2}$$

(iii) We substitute (4.1) in (4.2) and collect $B(A)^{x_i}$ in the right-hand side to obtain

$$\mathrm{Stab}_C(1) = \prod_{w \in \Delta, |w|=1} (\mathrm{Stab}_C(1))^w \prod_{w \in \Delta, |w|=1} (B(A))^w.$$

Thus,

$$\mathrm{Stab}_C(1) = \mathrm{Stab}_C(2) \prod_{w \in \Delta, |w|=1} (B(A))^w.$$

More generally,

$$\mathrm{Stab}_C(1) = \prod_{w \in \Delta, |w|=k} (\mathrm{Stab}_C(1))^w \prod_{w \in \Delta, |w|=k} (B(A))^w, \tag{4.3}$$

and

$$\mathrm{Stab}_C(1) = \mathrm{Stab}_C(k+1) \prod_{w \in \Delta, |w|=k} (B(A))^w. \tag{4.4}$$

Let $g \in C$. Then, as $C_{\mathcal{A}_m}(\Delta(A)) = \mathrm{Stab}_C(1)B(A)$, there exists $b^{(0)} \in B(A)$ such that

$$g(b^{(0)})^{-1} \in \mathrm{Stab}_C(1)$$

and using (4.3), there exists $b^{(1)} \in \prod_{w \in \Delta, |w|=1} (B(A))^w$ such that

$$g(b^{(0)})^{-1}(b^{(1)})^{-1} \in \prod_{w \in \Delta, |w|=1} (\mathrm{Stab}_C(1))^w \leqslant \mathrm{Stab}_C(2).$$

Iterating, we produce $b^{(k)}$ in $\prod_{w\in\Delta,\,|w|=k}(B(A))^{w}$ such that

$$g(b^{(0)})^{-1}(b^{(1)})^{-1}\cdots(b^{(k)})^{-1}\in \mathrm{Stab}_C(k+1).$$

Thus we produce the infinite product $b=\cdots b^{(k)}\cdots b^{(1)}b^{(0)}\in\overline{\Delta(B(A))}$, and

$$gb^{-1}\in\bigcap_{k\geq 0}\mathrm{Stab}_C(k)=\{e\}.$$

Hence, $g=b$ and $C_{\mathcal{A}_m}(\Delta(A))=\overline{\Delta(B(A))}$.

Now suppose that M is a maximal abelian subgroup of $\mathcal{A}_m$ and $\Delta(A)\leqslant M$. Then, $C_{\mathcal{A}_m}(M)\leqslant C_{\mathcal{A}_m}(\Delta(A))=A^*$. Thus, $C_{\mathcal{A}_m}(M)$ is abelian and so, $C_{\mathcal{A}_m}(M)=M\leqslant A^*$ and $M=A^*$. ■

Next we prove Theorem B.

Theorem B. *There exists a finitely generated subgroup H of $B(A)$ such that $A^*=\overline{\Delta(H)}$.*

Proof. Choose a generating set $\{\sigma_{ij}\mid 1\leq j\leq l_i\}$ for each $P_{(i)}$ and choose $\beta_{ij}\in B(A)$ having activity σ_{ij}. Define $H=\langle\beta_{ij}\mid 1\leq i\leq s,\ 1\leq j\leq l_i\rangle$. Then $B(A)\leqslant \mathrm{Stab}_{B(A)}(1)H$ and by Proposition 4.2, $\mathrm{Stab}_{B(A)}(1)\leqslant B(A)^{x_1}\cdots B(A)^{x_s}$.

Therefore,

$$\begin{aligned} A^* &= \mathrm{Stab}_{A^*}(1)B(A)=\mathrm{Stab}_{A^*}(1)H,\\ \mathrm{Stab}_{A^*}(1) &= (A^*)^{x_1}\cdots(A^*)^{x_s}=(\mathrm{Stab}_{A^*}(1)H)^{x_1}\cdots(\mathrm{Stab}_{A^*}(1)H)^{x_s}\\ &= \prod_{w\in\Delta,\,|w|=1}(\mathrm{Stab}_{A^*}(1))^{w}\prod_{w\in\Delta,\,|w|=1}H^{w}. \end{aligned}$$

Thus, we can substitute H for $B(A)$ in our arguments in the previous theorem to obtain $A^*=\overline{\Delta(H)}$. ■

An equivalent form of Corollary 1.1 is next.

Corollary 4.5. *Let A be a torsion self-similar abelian group. Then A has finite exponent.*

Proof. As $A^*=\overline{\Delta(H)}$, and H has finite order, it follows that A^* has the same exponent as H, and therefore so does A. ■

4.2. Free abelian groups

The purpose of this section is constructing a self-similar free abelian group of infinite countable rank which is $\Delta=\langle x_1,x_2\rangle$-invariant and showing that its topological closure is a free abelian pro-m group.

Proposition 4.6. *Let A be a self-similar abelian group of degree $m=m_1+\cdots+m_s$ and orbit-type $\mathbf{m}=(m_1,\ldots,m_s)$ with $s\geq 2$. Then $A=\Delta(A)$ if and only if there exists a strongly recurrent A-data $(\mathbf{m},\mathbf{H},\mathbf{F})$.*

Proof. If $A = \Delta(A)$, it is enough to consider the A-data $(\mathbf{m}, \mathbf{H}, \mathbf{F})$ defined by

$$H_1 = \mathrm{Fix}_A(1),\ H_2 = \mathrm{Fix}_A(m_1 + 1),\ \dots,\ H_s = \mathrm{Fix}_A(m_1 + \dots + m_{s-1} + 1),$$

and

$$f_1 = \pi_1,\ f_2 = \pi_{m_1+1},\ \dots,\ f_s = \pi_{m_1+\dots+m_{s-1}+1},$$

where π_i is the projection on the i-th coordinate.

Suppose that we are given a strongly recurrent data $(\mathbf{m}, \mathbf{H}, \mathbf{F})$ for A. Let $T_1, \dots, T_s$ transversal of $H_1, \dots, H_s$ in A, respectively. We will show that given $a \in A$ there exists $b \in A$ such that $b^\varphi = a^{\varphi x_i}$ for all $i = 1, \dots, s$, where $\varphi \colon A \to \mathcal{A}_m$ is the representation induced by the A-data $(\mathbf{m}, \mathbf{H}, \mathbf{F})$ and the transversal $T_1, \dots, T_s$. Indeed, since $f_i \colon H_i \cap \bigcap_{j \neq i} \ker(f_j) \to A$ is onto there exists $b \in H_i \cap \bigcap_{j \neq i} \ker(f_j)$ such that $b^{f_i} = a$. Note that $H_i t b = H_i t$ for any $t \in T_i$, then

$$b^\varphi = (e, \dots, e, a^\varphi, \dots, a^\varphi, e, \dots, e) = a^{\varphi x_i}.$$

The result follows. ∎

Now we prove Theorem C.

Theorem C. *Let A be a free abelian group.*

(i) *Suppose A has finite rank.*

   (a) *If A is a non-transitive self-similar group, then $\Delta(A)$ is not finitely generated.*

   (b) *Let A be a transitive self-similar subgroup of $\mathcal{A}_m$. Then the self-similar representation of A extends to a non-transitive self-similar representation into $\mathcal{A}_{m+1}$ having orbit-type $(m, 1)$ such that $\Delta(A)$, with respect to the second representation, contains a self-similar free abelian group of infinite enumerable rank.*

(ii) *Suppose A has infinite countable rank. Then A can be realized as a self-similar subgroup $\mathbf{A}$ of $\mathcal{A}_{m+1}$ of orbit-type $(m, 1)$, invariant with respect to $\Delta = \langle x_1, x_2 \rangle$, and is freely generated by the set U whose elements are*

$$\alpha_1 = (e, \dots, e, \alpha_1, e)(1\,2 \cdots m)(m + 1),$$
$$\alpha_{2i-1} = {\alpha_i}^{x_1}\ (i \geq 2), \quad \alpha_{2i} = {\alpha_i}^{x_2}\ (i \geq 1).$$

Moreover, ${\alpha_1}^m = {\alpha_1}^{x_1}$, $U = \{\alpha_1\} \cup \{\alpha_1 \cdot x_2 \Delta\}$. Also $\mathbf{A} = \Delta(\langle \alpha_1 \rangle)$, and $C_{\mathcal{A}_m}(\mathbf{A})$ is a free abelian pro-m group,

$$C_{\mathcal{A}_m}(\mathbf{A}) = \bar{\mathbf{A}} \simeq \prod_{i \geq 1} \alpha_i \mathbb{Z}_m.$$

Proof. (i) Let us prove (a). Suppose that $K = \Delta(A)$ is finitely generated. Then $K = \Delta(K)$ and by Proposition 4.6 there exists a strongly recurrent K-data $(\mathbf{m}, \mathbf{H}, \mathbf{F})$. Since K is an abelian group, we have that $f_i|_{H_i \cap \bigcap_{j \neq i} \ker(f_j)}$ is injective, $H_i \cap \bigcap_{j \neq i} \ker(f_j) \simeq K$ and the index

$$\Big[K\colon H_i \cap \bigcap_{j \neq i} \ker(f_j)\Big]$$

is finite for all $i = 1, \dots, s$. Therefore, $\ker(f_1) \cap \cdots \cap \ker(f_s) \neq \{e\}$; a contradiction.

Let us prove (b). Let H be a subgroup of index m in A, and let $f\colon H \to A$ be a virtual endomorphism whose core is trivial. Consider the free abelian group of infinite rank $A^\omega = \oplus_{i=1}^{\infty} A_i$, where $A_i = A$ for all i. Write $H_1 = H \oplus (\oplus_{i=2}^{\infty} A_i)$, $H_2 = A^\omega$, and for $i = 1, 2$, define the homomorphisms $f_i\colon H_i \to A^\omega$:

$$f_1\colon (h, a_2, a_3, \dots) \mapsto (h^f, a_2, a_3, \dots),$$
$$f_2\colon (a_1, a_2, a_3, \dots) \mapsto (a_2, a_3, a_4, \dots).$$

It follows that the A^ω-data

$$(\mathbf{m+1} = (m, 1),\ \mathbf{H} = (H_1, H_2),\ \mathbf{F} = \{f_1, f_2\})$$

has a trivial **F**-core. Let $\varphi\colon A^\omega \to \mathcal{A}_{m+1}$ be the self-similar representation of A^ω induced by the above A^ω-data. Then,

$$A_1^\varphi \leq (A^\omega)^\varphi \leq \Delta(A_1^\varphi),$$

and A_1^φ is self-similar.

(ii) Let $m \geq 2$ be an integer and let $\{a_1, a_2, a_3, \dots\}$ be a free basis of A. Consider $H_1 = \langle a_1^m, a_2, a_3, \dots \rangle$ and $H_2 = A$. Define the homomorphisms $f_i\colon H_i \to A$ $(i = 1, 2)$ which extend the maps

$$f_1\colon a_1^m \mapsto a_1, \quad a_{2i} \mapsto e\ (i \geq 1), \quad a_{2i-1} \mapsto a_i\ (i \geq 2),$$
$$f_2\colon a_{2i-1} \mapsto e\ (i \geq 1), \quad a_{2i} \mapsto a_i\ (i \geq 1).$$

Note that the A-data $((m, 1), (H_1, H_2), (f_1, f_2))$ is strongly recurrent. By Proposition 4.6, the induced representation

$$\varphi\colon A \to \mathcal{A}_{m+1}$$

is faithful and $\mathbf{A} = A^\varphi$ is closed under $\Delta = \langle x_1, x_2 \rangle$; recall the action of x_1, x_2 on $\mathcal{A}_{m+1}$,

$$c^{x_1} = (c, \dots, c, e), \quad c^{x_2} = (e, \dots, e, c).$$

From the representation φ, we have $\alpha_i := a_i^\varphi$ $(i \in \mathbb{N})$, where

$$\alpha_1 = (e, \dots, e, \alpha_1, e)(1\,2 \cdots m)(m+1),$$
$$\alpha_{2i-1} = {\alpha_i}^{x_1} \quad (i \geq 2),$$
$$\alpha_{2i} = {\alpha_i}^{x_2} \quad (i \geq 1).$$

The free abelian group $\mathbf{A}$ is freely generated by the ordered set $U = \{\alpha_i \mid i \in \mathbb{N}\}$. The subgroup $\langle \alpha_1 \rangle$ is self-similar, and $\alpha_1^{x_1} = \alpha_1^m$. It is clear that, given $n \in \mathbb{N}$, there exists w in Δ such that $\alpha_n = \alpha_1^w$ and $\mathbf{A} = \Delta(\langle \alpha_1 \rangle) = \langle \alpha_1 \rangle \oplus \langle {\alpha_1}^w \mid w \in x_2 \cdot \Delta \rangle$. Furthermore, $B(\mathbf{A}) = \mathbf{A}$ and $R(\mathbf{A}) = \{e\}$. Therefore, by Theorem 4.4, $C(\mathbf{A}) = \mathbf{A}^* = \overline{\Delta(\mathbf{A})}$.

To calculate $\overline{\mathbf{A}}$, we note that for $k \geq 0$,

$$\mathrm{Stab}_{\mathbf{A}}(k) = \langle \alpha_1^{m^k}, \alpha_2^{m^{k-1}}, \ldots, \alpha_{k-1}^m, \alpha_{k+2}, \alpha_{k+3}, \ldots \rangle,$$

and $\mathrm{Stab}_{\mathbf{A}}(k) \setminus \mathrm{Stab}_{\mathbf{A}}(k+1)$ is the set of coset representatives

$$\big(\langle \alpha_1^{m^k} \rangle \setminus \langle \alpha_1^{m^{k+1}} \rangle\big) \cdot \big(\langle \alpha_2^{m^{k-1}} \rangle \setminus \langle \alpha_2^{m^k} \rangle\big) \cdots \big(\langle \alpha_{k+1} \rangle \setminus \langle \alpha_{k+1}^m \rangle\big).$$

Therefore, $\overline{\mathbf{A}} = \prod_{i \in \mathbb{N}} \overline{\langle \alpha_i \rangle}$ and each $\overline{\langle \alpha_i \rangle}$ is isomorphic to $\mathbb{Z}_m$. Since U converges to the identity automorphism, we conclude that $\overline{\mathbf{A}}$ is the free abelian pro-m group generated by U [8, Example 3.3.8 (b), (d)]. ∎

5. The n-adding machine of multiplicity s

Let m be an integer ≥ 2 and let $m = sn$ be a factorization in positive integers s, n. We follow the model of the adding machine in defining for the pair (s, n), the following automorphism in $\mathcal{A}_m$:

$$a = (a_{(1)}, \ldots, a_{(s)})p,$$

where the n-blocks are defined by

$$\begin{aligned} a_{(1)} &= a_{(2)} = \cdots = a_{(s)} = (e, e, \ldots, e, a), \\ p &= p_1 \cdots p_s, \\ p_1 &= (1, 2, \ldots, n), \quad p_2 = (n+1, n+2, \ldots, 2n), \quad \ldots, \\ p_s &= ((s-1)n+1, (s-1)n+2, \ldots, sn). \end{aligned}$$

Then $a = a_{[1]} \cdot a_{[2]} \cdots a_{[s]}$, where $a_{[i]} = (e, \ldots, e, a_{(i)}, e, \ldots, e)p_i$.

Remark 5.1. (1) The A-data in the given representation of A is as follows:

$$\begin{aligned} m_1 &= m_2 = \cdots = m_s = n, \\ H_1 &= H_2 = \cdots = H_s = \langle a^n \rangle, \\ f_1 &= f_2 = \cdots = f_s, \quad f_i \colon a^n \mapsto a. \end{aligned}$$

Then, the A-data is strongly simple and recurrent.

(2) It was proven in [2, Theorem 3] that transitive self-similar finitely generated torsion-free nilpotent subgroups of $\mathcal{A}_m$ have solvability degree bounded by the number of prime divisors (counting multiplicity) of m, and the same holds for the

nilpotency class in the case of strongly simple representations. In the same paper, it was shown that the transitivity condition is necessary. Indeed, the centralizer in $\mathcal{A}_4$ of the double adding machine,

$$a = (e, a, e, a)(1\,2)(3\,4),$$

was shown to contain an infinite family of self-similar 2-generated metabelian torsion-free nilpotent groups such their set of nilpotency class is unbounded.

(3) The highly regular definition of a enables us to find b, an s-th root of it in $\mathrm{Aut}(\mathcal{T}_m)$, such that $\langle b \rangle$ is *transitive* self-similar.
One such root is

$$b = (e, \dots, e, a)q,$$

where

$$q = (1, s+1, \dots, (n-1)s+1, 2, s+2, \dots, (n-1)s+2, \dots, s, 2s, \dots, ns).$$

Clearly,

$$C = C_{\mathcal{A}_m}(b) = \mathrm{Stab}_C(1) + \langle b \rangle.$$

Given $c = (c_1, c_2, \dots, c_{m-1}, c_m)$, we have

$$c^b = (c_1, c_2, \dots, c_{m-1}, {c_m}^a)^q = ({c_m}^a, c_{s+1}, \dots, c_{(n-1)s}).$$

The equation $c^b = c$ implies $c_i = c_1$ for all i and $c_1 \in C_{\mathcal{A}_m}(a)$. Therefore,

$$C_{\mathcal{A}_m}(b) = (C_{\mathcal{A}_m}(a))^x + \langle b \rangle,$$

where $x = x_1 + x_2 + \cdots + x_s$ is the full diagonal monomorphism.

We recall,

$$R(A) = S(A) = S \simeq \mathrm{Sym}(s), \quad C(A) = \overline{\Delta(B(A))} \cdot \overline{\Delta(S)}, \quad \overline{\Delta(S)} \simeq \mathrm{Aut}(\mathcal{T}_s).$$

The group $\Delta(S)$ corresponds to the finitary subgroup of $\mathrm{Aut}(\mathcal{T}_s)$, and $\overline{\langle \Delta_l(S) \mid l \geq k \rangle}$ corresponds to $\mathrm{Stab}_{\mathrm{Aut}(\mathcal{T}_s)}(k)$.

Let $R = \mathbb{Z}[\Delta]$ *be the integral monoid ring of* Δ, or equivalently, *the polynomial ring in non-commuting variables* x_i $(1 \leq i \leq s)$. The ring $R = \mathbb{Z}[\Delta]$ decomposes as

$$R = \mathbb{Z} + R \cdot x_1 + R \cdot x_2 + \cdots + R \cdot x_s.$$

If μ is a non-zero (reduced) polynomial, its degree is the maximum length of a Δ-term in μ. We will view $\Delta(B(A))$ additively and denote it by V, which then becomes an R-module generated by $\{a_{[i]} \mid 1 \leq i \leq s\}$. Since a is central, we choose it to replace $a_{[s]}$ in the free generating $\{a_{[1]}, a_{[2]}, \dots, a_{[s]}\}$ set of $B(A)$.

Proposition 5.2. *The group V is $\mathbb{Z}$-freely generated by*

$$U = \{a\} \cup \{a_{[i]} \cdot u \,(1 \leq i \leq s-1),\, u \in \Delta\}.$$

Proof. Since $a = a_{[1]} + a_{[2]} + \cdots + a_{[s]}$, we write $a' = a_{[1]} + a_{[2]} + \cdots + a_{[s-1]}$ and we write our first additive relation as $a_{[s]} = a - a'$. Further additive relations are

$$\begin{aligned} a \cdot x_i &= a_{[i]} \cdot n \quad (1 \leq i \leq s-1), \\ a \cdot x_s &= a_{[s]} \cdot n = (a - a') \cdot n = a \cdot n - a' \cdot n. \end{aligned}$$

It follows that

$$V = A + \sum_{i=1}^{s-1} a_{[i]} \cdot \mathbb{Z}[\Delta].$$

In order to formalize reductions in V, we look at its presentation with respect to its generating set $\{a, a_{[i]}, \ldots, a_{[s]}\}$.

Let M be the free-R module freely generated by $\{y, y_1, \ldots, y_s\}$; denote $y' = y_1 + y_2 + \cdots + y_{s-1}$. Then, the map

$$\phi\colon\, y \to a, \quad y_i \to a_{[i]} \quad (1 \leq i \leq s-1)$$

extends to an R-epimorphism $\phi\colon M \to V$.

Let $K = \ker(\phi)$. We note that K has the following recursive property inherited from automorphisms of the tree:

$$\mu = \mu_1 \cdot x_1 + \mu_2 \cdot x_2 + \cdots + \mu_s \cdot x_s \in K \quad \text{implies } \mu_1, \mu_2, \ldots, \mu_s \in K.$$

Let I be the R-module contained in K generated by

$$\{y_s - y + y',\, y \cdot x_i - y_i \cdot n \,(1 \leq i \leq s-1),\, y \cdot x_s - (y - y')n\}.$$

Given $\mu \in M/I$, we can assume it to be written as $\mu = \eta_1 + \eta_2$, where

$$\eta_1 = y \cdot k_0 + \sum_{i=1}^{s-1} y_i \cdot k_i, \quad \eta_2 = \sum_{i=1}^{s-1} y_i \cdot \left(\sum_{j=1}^{s} \rho(ij) \cdot x_j \right)$$

for some integers k_i and polynomials $\rho(ij) \in \mathbb{Z}[\Delta]$.

We note that as $\phi(\mu) = 0$ and as $\phi(\eta_2)$ belongs to the first level stabilizer of the tree, then $\phi(\eta_1)$ also belongs to the first level stabilizer of the tree.

If μ is non-zero, we define its degree as

$$\partial(\mu) = |k_0| + \max\{\deg(\rho(ij)) \mid 1 \leq i \leq s-1, 1 \leq j \leq s\}.$$

Suppose $K \neq I$ and choose $\mu \in K \setminus I$ minimizing $\partial(\mu)$. As $\phi(\eta_1)$ belongs to the first level stabilizer of the tree, we see from its multiplicative form

$$\begin{aligned} a^{k_0} \cdot {a_{[1]}}^{k_1} \cdots {a_{[s-1]}}^{k_{s-1}} &= (a_{[1]} \cdots a_{[s-1]} a_{[s]})^{k_0} \cdot {a_{[1]}}^{k_1} \cdots {a_{[s-1]}}^{k_{s-1}} \\ &= {a_{[1]}}^{k_0 + k_1} \cdots {a_{[s-1]}}^{k_0 + k_{s-1}} \cdot {a_{[s]}}^{k_0} \end{aligned}$$

that ${p_1}^{k_0+k_1} \cdots {p_{s-1}}^{k_0+k_{s-1}} \cdot {p_s}^{k_0} = e$ is the trivial permutation. We conclude that the exponents $k_0 + k_1, \ldots, k_0 + k_{s-1}, k_0$ are multiples of n, and so we write

$$k_i = nk_i' \quad (0 \le i \le s-1).$$

Now, since $a_{[i]} \cdot n = a \cdot x_i$ $(1 \le i \le s)$, we have

$$\phi(\eta_1) = a \cdot (k_0' + k_1') \cdot x_1 + \cdots + a \cdot (k_0' + k_{s-1}') \cdot x_{s-1} + a \cdot k_0' \cdot x_s.$$

Thus we can assume

$$\eta_1 = y(k_0' + k_1')x_1 + \cdots + y(k_0' + k_{s-1}')x_{s-1} + yk_0'x_s.$$

When this new form of η_1 is substituted into μ, we find

$$\mu = \mu_{(1)} \cdot x_1 + \mu_{(2)} \cdot x_2 + \cdots + \mu_{(s)} \cdot x_s,$$

where

$$\begin{aligned}
\mu_{(1)} &= y \cdot (k_0' + k_1') + y_1 \cdot \rho(11) + \cdots + y_{s-1} \cdot \rho((s-1)1),\\
&\vdots\\
\mu_{(s-1)} &= y \cdot (k_0' + k_{s-1}') + y_1 \cdot \rho(1(s-1)) + \cdots + y_{s-1}\rho((s-1)(s-1)),\\
\mu_{(s)} &= y \cdot k_0' + y_1 \cdot \rho(1s) + \cdots + y_{s-1}\rho((s-1),s)
\end{aligned}$$

all elements of K.

Next, we note that $\partial(\mu_s) < \partial(\mu)$, or $\mu_s = 0$. By the minimality of μ, we have $\mu_s = 0$; that is,

$$\{k_0', \rho(is), \ldots, \rho((s-1),s)\} = \{0\}.$$

Hence, substituting $k_0' = 0$ in $\mu_{(i)}$, $i \ne s$, we conclude similarly that $\mu(j) = 0$ for $1 \le j \le s-1$, and thus, $\mu = 0$; a contradiction is reached. ∎

5.1. Topological closure

Define $V_{-1} = A$, recall $V_k = \langle a_{[i]} \cdot u \mid 1 \le i \le s-1,\ u \in \Delta_k \rangle$ and define

$$\widetilde{V}_0 = V_{-1} \oplus V_0.$$

In order to determine the topological closure of $V = \Delta(B(A))$, first we find

$$\mathrm{Stab}_V(k) = (\widetilde{V}_0 \cdot n^k) \oplus \bigoplus_{i=1}^{k-1} (V_i \cdot n^{k-i}) \oplus \bigoplus_{i \ge k} V_i$$

for $k \ge 1$. Next we choose the coset representatives $C_0 = V \setminus \mathrm{Stab}_V(1) = \widetilde{V}_0 \setminus (\widetilde{V}_0 \cdot n)$, and for $k \ge 1$,

$$\begin{aligned}
C_k &= \mathrm{Stab}_V(k) \setminus \mathrm{Stab}_V(k+1)\\
&= (\widetilde{V}_0 \cdot n^k) \setminus (\widetilde{V}_0 \cdot n^{k+1}) + \sum_{i=1}^{k} (V_i \cdot n^{k-i}) \setminus (V_i \cdot n^{k+1-i}).
\end{aligned}$$

By writing infinite products $c_0 \cdot c_1 \cdots c_k \cdots$ with $c_i \in C_i$ and collecting the V_i terms, we obtain

$$\bar{V} = \overline{\widetilde{V}_0} \oplus \bigoplus_{i \geq 1} \overline{V_i}.$$

Indeed,

$$\bar{V} = a\mathbb{Z}_n \oplus \bigoplus_{i=1}^{s-1} \{a_{[i]}\mathbb{Z}_n \mid u \in \Delta,\ |u| \geq 0\}.$$

5.2. Submodules of A^*

The group $\mathrm{Aut}(\mathcal{T}_s)$ acts permutationally on the $\mathbb{Z}$-modules W_k freely generated by the vertices of the k-th level of the tree $\mathcal{T}_s$. The kernel of the action on W_k is $\mathrm{Stab}_{\mathrm{Aut}(\mathcal{T}_s)}(k)$. The natural "permutational" space W for $\mathrm{Aut}(\mathcal{T}_s)$ is defined as the inverse limit of the ordered set of spaces

$$\{W_0 + W_1 + \cdots + W_k \mid k \geq 0\}.$$

In contrast with W_k, we consider ${V_k}^{\#}$ ($k \geq 0$), the normal closure of V_k under the action of $\Delta(S)$, and show it to have a *comet-like* appearance.

Theorem 5.3. *For $k = 0$,*

$${V_0}^{\#} = \widetilde{V}_0 = A \oplus V_0.$$

For $k \geq 1$,

$${V_k}^{\#} = {V_0}^{\#} \cdot n^k \oplus \bigoplus_{i=1}^{k} V_i \cdot n^{k-i}.$$

Moreover, ${V_k}^{\#}$ is centralized by $\Delta_l(S)$ for $l > k$, and both ${V_k}^{\#}$, $\overline{{V_k}^{\#}} (= {V_k}^{\#}\mathbb{Z}_n)$ are invariant with respect to $\overline{\Delta(S)} \simeq \mathrm{Aut}(\mathcal{T}_s)$.

Proof. Let $\xi \in S$ and $u, w \in \Delta$. We will determine $(a_{[i]} \cdot u)^{\xi^w}$ for $1 \leq i \leq s-1$.

We recall Lemma 3.5 (ii): Let $1 \leq i \leq s$. Then, given $g \in \mathrm{Aut}(\mathcal{T}_m)$ and $\xi \in S$, $\xi \colon g^{x_i} \to g^{x_{(i)\xi}}$. Let v be the largest common suffix of w and u, and write $w = w'v$, $u = u'v$. Then

$$\xi^w \colon a_{[i]} \cdot (u'v) \mapsto (a_{[i]} \cdot u')^{\xi^{w'}} \cdot v.$$

It can be verified directly that

$$(a_{[i]} \cdot u')^{\xi^{w'}} = \begin{cases} \text{(i)} & a_{[(i)\xi)]} \cdot u' & \text{if } u' = w' = 1; \\ \text{(ii)} & a_{[i]} \cdot u' & \text{if } u' = 1,\ w' \neq 1; \\ \text{(iii)} & a_{[i]} \cdot u' & \text{if } u' \neq 1 \neq w'; \\ \text{(iv)} & a_{[i]} \cdot (u'' \cdot x_{(l)\xi}) & \text{if } u' = u''x_l,\ w' = 1. \end{cases}$$

Thus, for $|u| = k$, we have $(a_{[i]} \cdot u')^{\xi^w} \cdot v \in V_k$, unless $u = w$ and $(i)\xi = s$. In the latter case, the image is

$$(a_{[i]} \cdot u)^{\xi^w} = a_{[i]}^{\xi^w} \cdot u = a_{[(i)\xi]} \cdot u = a_{[s]} \cdot u.$$

On varying ξ, we obtain

$$V_k{}^{\#} = \langle a_{[s]} \cdot \Delta_k \rangle \oplus V_k,$$

in particular,

$$V_0{}^{\#} = \langle a_{[s]} \rangle \oplus V_0 = A \oplus V_0.$$

More generally, since $a_{[s]} \cdot u = (a - a') \cdot u$, we obtain

$$V_k{}^{\#} = \langle a \cdot \Delta_k \rangle \oplus V_k.$$

Moreover, for $1 \leq l \leq s$, $a \cdot (x_l \cdot u') = (a \cdot x_l) \cdot u' = (a_{[l]} \cdot n)u'$. Hence, for $k \geq 1$,

$$\begin{aligned} V_k{}^{\#} &= (\langle a_{[s]} \cdot \Delta_{k-1} \rangle \cdot n) \oplus (V_{k-1} \cdot n) \oplus V_k \\ &= (\langle a \cdot \Delta_{k-1} \rangle \cdot n) \oplus (V_{k-1} \cdot n) \oplus V_k = (V_{k-1}{}^{\#} \cdot n) \oplus V_k. \end{aligned}$$

We note from the actions of ξ^w on $a_{[i]} \cdot u$ spelled out above, that the action is trivial if $|w| > |u|$. Since

$$\overline{\Delta(S)} = \prod_{j \geq 0} \Delta_j(S),$$

it follows that $V_k{}^{\#}$ is $\overline{\Delta(S)}$-invariant, and so is $\overline{V_k{}^{\#}}$. ∎

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Alex C. Dantas
Departamento de Matemática, Universidade de Brasília, 70910-900 Brasilia, DF, Brazil;
alexcdan@gmail.com

Tulio M. G. Santos
Instituto Federal Goiano, 73840-000 Campos Belos, GO, Brazil; tulio.gentil@ifgoiano.edu.br

Said N. Sidki
Departamento de Matemática, Universidade de Brasília, 70910-900 Brasilia, DF, Brazil;
ssidki@gmail.com