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Asymptotic property C of the wreath product $\mathbb{Z} \wr \mathbb{Z}$

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Abstract. Using the relationship between transfinite asymptotic dimension and asymptotic property C, we obtain that the wreath product $\mathbb{Z} \wr \mathbb{Z}$ has asymptotic property C. Specifically, we prove that the transfinite asymptotic dimension of the wreath product $\mathbb{Z} \wr \mathbb{Z}$ does not exceed $\omega + 1$.

1. Introduction

In coarse geometry, the asymptotic dimension (asdim) of a metric space is an important concept which was defined by Gromov for studying asymptotic invariants of discrete groups [4]. As a large scale analogue of Haver's property C in dimension theory, Dranishnikov introduced the notion of asymptotic property C and proved that every metric space of bounded geometry with asymptotic property C has property A [3]. Guentner, Tessera and Yu introduced the notion of finite decomposition complexity to study topological rigidity of manifolds [5] and proved that every metric space of bounded geometry with finite decomposition complexity has property A [6].

It follows by definition that every metric space with finite asymptotic dimension has asymptotic property C and finite decomposition complexity [3, 6]. But the inverse is not true, which means that there exists some infinite asymptotic dimension metric space X with both asymptotic property C and finite decomposition complexity. Therefore, how to classify the metric spaces with infinite asymptotic dimension into smaller categories becomes an interesting problem. Radul defined transfinite asymptotic dimension (trasdim), viewing it as a transfinite extension for asymptotic dimension, and proved that for every metric space X, it having asymptotic property C is equivalent to $\text{trasdim}(X) \leq \alpha$ for some countable ordinal number α [8].

The relation between asymptotic property C and finite decomposition complexity was studied by Dranishnikov and Zarichnyi [2]. As far as we know, there is no example of a group that makes a difference between asymptotic property C and finite decomposition complexity. The wreath product $\mathbb{Z} \wr \mathbb{Z}$ has finite decomposition complexity [6, 10], but $\mathbb{Z} \wr \mathbb{Z}$ does not have finite asymptotic dimension. So we are interested in the question of whether $\mathbb{Z} \wr \mathbb{Z}$ has asymptotic property C. In this paper, we prove that transfinite asymptotic dimension of $\mathbb{Z} \wr \mathbb{Z}$ does not exceed $\omega + 1$. Consequently, $\mathbb{Z} \wr \mathbb{Z}$ has asymptotic

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property C. The wreath product $\mathbb{Z} \wr \mathbb{Z}$ is the first finitely generated group we found with asymptotic property C and infinite asymptotic dimension.

The paper is organized as follows: In Section 2, we recall some definitions and properties of asymptotic property C, transfinite asymptotic dimension and wreath product. In Section 3, we prove that transfinite asymptotic dimension of the wreath product $\mathbb{Z} \wr \mathbb{Z}$ does not exceed $\omega + 1$.

2. Preliminaries

Our terminology concerning asymptotic dimension follows from [1].

2.1. Asymptotic property C

Let (X, d) be a metric space and $U, V \subseteq X$. Let

$$\operatorname{diam} U = \sup\{d(x, y) \mid x, y \in U\}$$

and

$$d(U, V) = \inf\{d(x, y) \mid x \in U,\ y \in V\}.$$

Let $R > 0$ and $\mathcal{U}$ be a family of subsets of X, $\mathcal{U}$ is said to be R-*bounded* if

$$\operatorname{diam} \mathcal{U} = \sup\{\operatorname{diam} U \mid U \in \mathcal{U}\} \leq R.$$

In this case, $\mathcal{U}$ is said to be *uniformly bounded*. For $r > 0$, $\mathcal{U}$ is said to be r-*disjoint* if

$$d(U, V) \geq r \quad \text{for every } U, V \in \mathcal{U} \text{ with } U \neq V.$$

In this paper, we denote the union $\cup\{U \mid U \in \mathcal{U}\}$ by $\cup\mathcal{U}$ and denote the family $\{U \mid U \in \mathcal{U}_1 \text{ or } U \in \mathcal{U}_2\}$ by $\mathcal{U}_1 \cup \mathcal{U}_2$. Let $\mathbb{N}$ be the set of all non-negative integers.

Definition 2.1 ([1]). A metric space X is said to have *finite asymptotic dimension* if there is such $n \in \mathbb{N}$ that for every $r > 0$, there exists a sequence of uniformly bounded families $\{\mathcal{U}_i\}_{i=0}^n$ of subsets of X such that the family $\bigcup_{i=0}^n \mathcal{U}_i$ covers X and each $\mathcal{U}_i$ is r-disjoint for $i \in \{0, 1, \ldots, n\}$. In this case, we say that $\operatorname{asdim}(X) \leq n$.

Definition 2.2 ([3]). A metric space X is said to have *asymptotic property C* if for every sequence $R_0 < R_1 < R_2 < \cdots < R_i < \cdots$ of positive real numbers, there exist such $n \in \mathbb{N}$ and uniformly bounded families $\mathcal{U}_0, \ldots, \mathcal{U}_n$ of subsets of X that each $\mathcal{U}_i$ is R_i-disjoint for $i \in \{0, 1, \ldots, n\}$ and the family $\bigcup_{i=0}^n \mathcal{U}_i$ covers X.

2.2. Transfinite asymptotic dimension

In [8], Radul generalized asymptotic dimension of a metric space X to transfinite asymptotic dimension, which is denoted by $\operatorname{trasdim}(X)$.

Definition 2.3 ([8]). Let $\operatorname{Fin}\mathbb{N}$ denote the collection of all finite, nonempty subsets of $\mathbb{N}$ and let $M \subseteq \operatorname{Fin}\mathbb{N}$. For $\sigma \in \{\varnothing\} \cup \operatorname{Fin}\mathbb{N}$, let

$$M^\sigma = \{\tau \in \operatorname{Fin}\mathbb{N} \mid \tau \cup \sigma \in M \text{ and } \tau \cap \sigma = \varnothing\}.$$

Let M^a abbreviate $M^{\{a\}}$ for $a \in \mathbb{N}$. Define the ordinal number $\operatorname{Ord} M$ inductively as follows:

$$\begin{aligned}
\operatorname{Ord} M &= 0 && \Leftrightarrow M = \varnothing,\\
\operatorname{Ord} M &\le \alpha && \Leftrightarrow \forall a \in \mathbb{N},\ \operatorname{Ord} M^a \le \beta \text{ for some } \beta < \alpha,\\
\operatorname{Ord} M &= \alpha && \Leftrightarrow \operatorname{Ord} M \le \alpha \text{ and for any } \gamma < \alpha,\ \operatorname{Ord} M \le \gamma \text{ is not true},\\
\operatorname{Ord} M &= \infty && \Leftrightarrow \text{for any ordinal number } \alpha,\ \operatorname{Ord} M \le \alpha \text{ is not true}.
\end{aligned}$$

Definition 2.4 ([8]). Given a metric space (X, d), define the following collection:

$$A(X,d) = \big\{\sigma \in \operatorname{Fin}\mathbb{N} \mid \text{there are no uniformly bounded families } \mathcal{U}_i \text{ for } i \in \sigma \\ \text{such that each } \mathcal{U}_i \text{ is } i\text{-disjoint and } \textstyle\bigcup_{i\in\sigma} \mathcal{U}_i \text{ covers } X\big\}.$$

The *transfinite asymptotic dimension* of X is defined by $\operatorname{trasdim}(X) = \operatorname{Ord} A(X, d)$.

Remark 2.5. It is easy to see that transfinite asymptotic dimension is a generalization of finite asymptotic dimension. That is, $\operatorname{trasdim}(X) \le n$ if and only if $\operatorname{asdim}(X) \le n$ for each $n \in \mathbb{N}$.

Lemma 2.6 ([8]). *Let X be a metric space. X has asymptotic property C if and only if* $\operatorname{trasdim}(X) \le \alpha$ *for some countable ordinal number α.*

Lemma 2.7 ([11]). *Given a metric space X with* $\operatorname{asdim}(X) = \infty$ *and $k \in \mathbb{N}$, the following are equivalent:*

- $\operatorname{trasdim}(X) \le \omega + k$*;*
- *for every $n \in \mathbb{N}$, there exists $m(n) \in \mathbb{N}$ such that for every $d > 0$, there are uniformly bounded families $\mathcal{U}_{-k}, \mathcal{U}_{-k+1}, \dots, \mathcal{U}_{m(n)}$ satisfying the following conditions:*
  - *$\mathcal{U}_i$ is n-disjoint for $i \in \{-k, -k+1, \dots, 0\}$,*
  - *$\mathcal{U}_j$ is d-disjoint for $j \in \{1, 2, \dots, m(n)\}$,*
  - *the family $\bigcup_{i=-k}^{m(n)} \mathcal{U}_i$ covers X.*

2.3. Wreath product

Let G be a finitely generated group with finite generating set S. For any $g \in G$, let $|g|_S$ be the length of the shortest word representing g in elements of $S \cup S^{-1}$. We say that $|\cdot|_S$ is *word-length function* for G with respect to S. The *left-invariant word-metric* d_S on G is induced by word-length function, i.e., for every $g, h \in G$,

$$d_S(g,h) = |g^{-1}h|_S.$$

The *Cayley graph* is the graph whose vertex set is G, one vertex for each element in G, and any two vertices $g, h \in G$ are incident with an edge if and only if $g^{-1}h \in S \cup S^{-1}$.

Let G and N be finitely generated groups and let $1_G \in G$ and $1_N \in N$ be their identity elements. The *support* of a function $f: N \to G$ is the set

$$\mathrm{supp}(f) = \{x \in N \mid f(x) \neq 1_G\}.$$

The direct sum $\bigoplus_N G$ of groups G (or restricted direct product) is the group of functions

$$C_0(N, G) = \{f: N \to G \text{ with finite support}\}.$$

There is a natural action of N on $C_0(N, G)$: for all $a \in N$, $x \in N$, $f \in C_0(N, G)$,

$$a(f)(x) = f(xa^{-1}).$$

The semidirect product $C_0(N, G) \rtimes N$ is called the *restricted wreath product* and is denoted by $G \wr N$. We recall that the product in $G \wr N$ is defined by the formula

$$(f, a)(g, b) = (fa(g), ab) \quad \text{for every } f, g \in C_0(N, G) \text{ and } a, b \in N.$$

Note that $(f, a)^{-1} = (a^{-1}(f^{-1}), a^{-1})$.

Let S and T be finite generating sets for G and N, respectively. Let $e \in C_0(N, G)$ denote the constant function with the constant value 1_G. For every $v \in N$ and $b \in G$, let $\delta_v^b: N \to G$ be the δ-function, i.e.,

$$\delta_v^b(v) = b \quad \text{and} \quad \delta_v^b(x) = 1_G \quad \text{for } x \neq v.$$

Note that $a(\delta_v^b) = \delta_{va}^b$ and hence

$$(\delta_v^b, 1_N) = (e, v)(\delta_{1_N}^b, 1_N)(e, v^{-1}).$$

Since any function $f \in C_0(N, G)$ can be presented as $\delta_{v_1}^{b_1} \cdots \delta_{v_k}^{b_k}$,

$$(f, 1_N) = (\delta_{v_1}^{b_1}, 1_N) \cdots (\delta_{v_k}^{b_k}, 1_N) \quad \text{and} \quad (f, u) = (f, 1_N)(e, u).$$

Thus the set

$$\widetilde{S} = \{(\delta_{1_N}^s, 1_N),\ (e, t) \mid s \in S, t \in T\}$$

is a generating set for $G \wr N$.

An explicit formula for the word-length of wreath products with $\mathbb{Z}$ was found by Parry.

Lemma 2.8 ([7, 9]). *Let H be a finitely generated group and $x = (f, n) \in H \wr \mathbb{Z}$, $m = \min\{k \in \mathbb{Z} \mid f(k) \neq 1_H\}$, $M = \max\{k \in \mathbb{Z} \mid f(k) \neq 1_H\}$. The word-length of x satisfies*

$$|x| = \begin{cases} |n| & \text{if } f = e, \\ \sum_{i \in \mathbb{Z}} |f(i)| + L_{\mathbb{Z}}(x) & \text{otherwise}, \end{cases}$$

where e is the identity of $\bigoplus_{l \in \mathbb{Z}} H$, $L_{\mathbb{Z}}(x)$ denotes the length of the shortest path starting from 0, ending at n and passing through m and M in the (canonical) Cayley graph of $\mathbb{Z}$.

Remark 2.9. By the formula of the word-length, we obtain that for every $x = (f, n_1)$, $y = (g, n_2) \in \mathbb{Z} \wr \mathbb{Z}$,

$$d(x,y) = |x^{-1}y| = |(n_1^{-1}(f^{-1}g), n_1^{-1}n_2)| = \begin{cases} |n_1 - n_2| & \text{if } f = g, \\ \sum_{i \in \mathbb{Z}} |f(i) - g(i)| + L_{\mathbb{Z}}(x^{-1}y) & \text{otherwise,} \end{cases}$$

where $L_{\mathbb{Z}}(x^{-1}y)$ denotes the length of the shortest path starting from 0, ending at $n_2 - n_1$ and passing through all vertices in the support of $n_1^{-1}(f^{-1}g)$ in the (canonical) Cayley graph of $\mathbb{Z}$.

Note that $d(x, y) \geq |n_1 - n_2|$.

2.4. Saturated union

Definition 2.10 ([1]). Let $\mathcal{U}$ and $\mathcal{V}$ be families of subsets of X. For $r > 0$, the r-*saturated union* of $\mathcal{V}$ with $\mathcal{U}$ is defined as

$$\mathcal{V} \bigcup_r \mathcal{U} = \{N_r(V; \mathcal{U}) \mid V \in \mathcal{V}\} \cup \{U \in \mathcal{U} \mid d(U, \mathcal{V}) > r\},$$

where $N_r(V; \mathcal{U}) = V \cup (\cup\{U \in \mathcal{U} \mid d(U, V) \leq r\})$ and $d(U, \mathcal{V}) > r$ means that for every $V \in \mathcal{V}$, $d(U, V) > r$.

Lemma 2.11 ([1]). *Let $\mathcal{U}$ be an r-disjoint and R-bounded family of subsets of X with $R \geq r$. Let $\mathcal{V}$ be a $5R$-disjoint, D-bounded family of subsets of X. Then the family $\mathcal{V} \bigcup_r \mathcal{U}$ is r-disjoint, $(D + 2R + 2r)$-bounded and $\mathcal{V} \bigcup_r \mathcal{U}$ covers $\cup(\mathcal{V} \cup \mathcal{U})$.*

3. Main results

Inspired by the technique from [12], we obtain the following lemma.

Let $\mathbb{Z}^+$ be the set of all positive integers. For $a \in \mathbb{Z}$ and $k \in \mathbb{Z}^+$, let

$$X_{a,k} = \Big(\bigoplus_{\mathbb{Z}} \mathbb{Z}\Big) \times ([a, a+k] \cap \mathbb{Z}) = \Big\{(f, n) \;\Big|\; f \in \bigoplus_{\mathbb{Z}} \mathbb{Z}, n \in [a, a+k] \cap \mathbb{Z}\Big\},$$

and assume that $X_{a,k}$ is a subspace of the metric space $\mathbb{Z} \wr \mathbb{Z}$ with the left-invariant word-metric.

Lemma 3.1. *For every $k, m \in \mathbb{Z}^+$, there exists $B = B(k, m) > 0$ satisfying the condition that for any $a \in \mathbb{Z}$, there exist B-bounded families*

$$\mathcal{U}_0, \mathcal{U}_1, \ldots, \mathcal{U}_{(3k+1)2^{3k}}$$

of subsets of $X_{a,k}$ such that each $\mathcal{U}_i$ is m-disjoint for $i \in \{1, 2, \ldots, (3k+1)2^{3k}\}$, $\mathcal{U}_0$ is k-disjoint and the family $\bigcup_{i=0}^{(3k+1)2^{3k}} \mathcal{U}_i$ covers $X_{a,k}$.

Proof. Without loss of generality, we assume that $m \geq k$. Let $B = 13m + 5m^2 + 2^{6m+4}$ and let

$$\mathcal{V}_0 = \{[(2n-1)m, 2nm) \cap \mathbb{Z} \mid n \in \mathbb{Z}\} \quad \text{and} \quad \mathcal{V}_1 = \{[2nm, (2n+1)m) \cap \mathbb{Z} \mid n \in \mathbb{Z}\}.$$

Then $\mathcal{V}_0, \mathcal{V}_1$ are m-disjoint, m-bounded and the family $\mathcal{V}_0 \cup \mathcal{V}_1$ covers $\mathbb{Z}$. Let $S = k + m$. For $l = 1, 2, 3, \ldots, 2^{2m}$, let

$$\begin{aligned}\mathcal{C}_l &= \{[(2^{2m}(n-1)+l)2S - m, (2^{2m}n + l)2S - m - k) \cap \mathbb{Z} \mid n \in \mathbb{Z}\},\\ \mathcal{D}_l &= \{[(2^{2m}n + l)2S - m - k, (2^{2m}n + l)2S - m) \cap \mathbb{Z} \mid n \in \mathbb{Z}\}.\end{aligned}$$

Then each $\mathcal{C}_l$ is k-disjoint, 2^{3m+3}-bounded and $\mathcal{D}_l$ is $(2^{2m+1}k + 2^{2m+1}m - k)$-disjoint, k-bounded. The family $\mathcal{D}_l \cup \mathcal{C}_l$ covers $\mathbb{Z}$ and the family $\bigcup_{l=1}^{2^{2m}} \mathcal{D}_l$ is m-disjoint. Let

$$\mathcal{W}_l = \left\{\prod_{i=1}^{2m} V_i \mid V_i \in \mathcal{V}_{\varphi(l)_i}, i \in \{1, 2, 3, \ldots, 2m\}\right\},$$

where φ is a bijection from $\{1, 2, 3, \ldots, 2^{2m}\}$ to $\{0, 1\}^{2m}$. Then each $\mathcal{W}_l$ is m-disjoint for $l \in \{1, 2, 3, \ldots, 2^{2m}\}$ and

$$\bigcup_{l=1}^{2^{2m}} \mathcal{W}_l \text{ is disjoint, i.e., for every } W, \widetilde{W} \in \bigcup_{l=1}^{2^{2m}} \mathcal{W}_l \text{ and } W \neq \widetilde{W},\ W \cap \widetilde{W} = \varnothing.$$

Define $\pi_1, \pi_2 \colon \mathbb{Z}^{2m} \to \mathbb{Z}^m$ by

$$\pi_1((x_i)_{i=1}^{2m}) = (x_i)_{i=1}^{m} \quad \text{and} \quad \pi_2((x_i)_{i=1}^{2m}) = (x_{m+i})_{i=1}^{m}.$$

For $\widetilde{x} \in \bigoplus_{\mathbb{Z}} \mathbb{Z}$ and $((C_i)_{i=1}^{3k+1}, W) \in \bigcup_{l=1}^{2^{2m}} (\mathcal{C}_l^{3k+1} \times \mathcal{W}_l)$, let $U(\widetilde{x}, (C_i)_{i=1}^{3k+1}, W)$ denote the set of all $x \in \bigoplus_{\mathbb{Z}} \mathbb{Z}$ satisfying the following conditions:

- $x_{a-k-m+i} = \widetilde{x}_i$ for $i \in \mathbb{Z}$ with $i < 0$,
- $(x_{a-k-m-1+i})_{i=1}^{m} \in \pi_1(W)$,
- $(x_{a-k-1+i})_{i=1}^{3k+1} \in \prod_{i=1}^{3k+1} C_i$,
- $(x_{a+2k+i})_{i=1}^{m} \in \pi_2(W)$,
- $x_{a+2k+m+1+i} = \widetilde{x}_i$ for $i \in \mathbb{Z}$ with $i \geq 0$.

Let

$$\mathcal{U}_0 = \left\{U(\widetilde{x}, (C_i)_{i=1}^{3k+1}, W) \times ([a, a+k] \cap \mathbb{Z}) \;\middle|\; \widetilde{x} \in \bigoplus_{\mathbb{Z}} \mathbb{Z},\right.$$
$$\left.((C_i)_{i=1}^{3k+1}, W) \in \bigcup_{l=1}^{2^{2m}} (\mathcal{C}_l^{3k+1} \times \mathcal{W}_l)\right\}.$$

To define $\mathcal{U}_1, \mathcal{U}_2, \mathcal{U}_3, \ldots, \mathcal{U}_{(3k+1)2^{3k}}$, we let $\rho\colon \{1, 2, 3, \ldots, 2^{3k}\} \to \{0, 1\}^{3k}$ be a bijection, $t \in \{1, 2, 3, \ldots, 2^{3k}\}$ and $s \in \{a-k, a-k+1, \ldots, a+2k\}$. For $\widetilde{x} \in \bigoplus_{\mathbb{Z}} \mathbb{Z}$, $(V_i)_{i=1}^{3k} \in \prod_{i=1}^{3k} \mathcal{V}_{\rho(t)_i}$ and $(D_s, W) \in \bigcup_{l=1}^{2^{2m}} (\mathcal{D}_l \times \mathcal{W}_l)$, let $U_s(\widetilde{x}, (V_i)_{i=1}^{3k}, D_s, W)$ denote the set of all $x \in \bigoplus_{\mathbb{Z}} \mathbb{Z}$ satisfying the following conditions:

- $x_{a-k-m+i} = \widetilde{x}_i$ for $i \in \mathbb{Z}$ with $i < 0$,
- $(x_{a-k-m-1+i})_{i=1}^{m} \in \pi_1(W)$,
- $x_{a-k-1+i} \in V_i$ for $i \in \{1, 2, 3, \ldots, s-a+k\}$,
- $x_s \in D_s$,
- $x_{a-k+i} \in V_i$ for $i \in \{s-a+k+1, s-a+k+2, \ldots, 3k\}$,
- $(x_{a+2k+i})_{i=1}^{m} \in \pi_2(W)$,
- $x_{a+2k+m+1+i} = \widetilde{x}_i$ for $i \in \mathbb{Z}$ with $i \geq 0$.

Let

$$\mathcal{U}_{2^{3k}(s-a+k)+t} = \Big\{ U_s(\widetilde{x}, (V_i)_{i=1}^{3k}, D_s, W) \times ([a, a+k] \cap \mathbb{Z}) \;\Big|\; \widetilde{x} \in \bigoplus_{\mathbb{Z}} \mathbb{Z}, \; (V_i)_{i=1}^{3k} \in \prod_{i=1}^{3k} \mathcal{V}_{\rho(t)_i}, (D_s, W) \in \bigcup_{l=1}^{2^{2m}} (\mathcal{D}_l \times \mathcal{W}_l) \Big\}.$$

Step 1. Firstly, we will prove that $\mathcal{U}_0$ is k-disjoint and $B(m)$-bounded. To prove that $\mathcal{U}_0$ is k-disjoint, let

$$U = U(\widetilde{x}, (C_i)_{i=1}^{3k+1}, W) \in \mathcal{U}_0, \quad U' = U(\widetilde{x}', (C_i')_{i=1}^{3k+1}, W') \in \mathcal{U}_0 \quad \text{and} \quad U \neq U'.$$

Let $x = (f, n_1) \in U$, $y = (g, n_2) \in U'$.

Case 1: $(\widetilde{x}, W) \neq (\widetilde{x}', W')$. Then there exists $j < a-k$ or $j > a+2k$ such that $f(j) \neq g(j)$, i.e., $(f^{-1}g)(j) \neq 0$, which implies that there exists $j < a-k-n_1 \leq -k$ or $j > a+2k-n_1 \geq k$ such that $n_1^{-1}(f^{-1}g)(j) \neq 0$. It follows that $L_{\mathbb{Z}}(x^{-1}y) \geq k$. And hence $d(x, y) \geq k$.

Case 2: $(\widetilde{x}, W) = (\widetilde{x}', W')$. Since $\bigcup_{l=1}^{2^{2m}} \mathcal{W}_l$ is disjoint, $W = W' \in \mathcal{W}_l$ for some unique $l \in \{1, 2, 3, \ldots, 2^{2m}\}$. Then

$$(C_i)_{i=1}^{3k+1}, \quad (C_i')_{i=1}^{3k+1} \in \mathcal{C}_l^{3k+1} \quad \text{and} \quad (C_i)_{i=1}^{3k+1} \neq (C_i')_{i=1}^{3k+1}.$$

It follows that there exists $i \in \{1, 2, 3, \ldots, 3k+1\}$ such that $C_i \neq C_i'$ and $C_i, C_i' \in \mathcal{C}_l$, which implies that $|f(a-k-1+i) - g(a-k-1+i)| \geq k$ since $\mathcal{C}_l$ is k-disjoint. Hence $d(x, y) \geq k$. So $d(U, U') \geq k$, i.e., $\mathcal{U}_0$ is k-disjoint.

To prove that $\mathcal{U}_0$ is $B(m)$-bounded, let $x = (f, n_1)$, $z = (h, n_3) \in U$.

If $f = h$, then $d(x, z) = |n_3 - n_1| \leq k \leq m$. Otherwise, since

$$\begin{aligned}\sum_{i \in \mathbb{Z}} |f(i) - h(i)| &\leq 2m^2 + (3k+1)2^{3m+3} \leq 2m^2 + (3m+1)2^{3m+3} \\ &\leq 2m^2 + 2^{6m+4}\end{aligned}$$

and

$$\operatorname{supp} n_1^{-1}(f^{-1}h) \subseteq [a-k-m-n_1, a+2k+m-n_1] \subseteq [-2k-m, 2k+m] \subseteq [-3m, 3m],$$

then $L_{\mathbb{Z}}(x^{-1}z) \leq 12m$, $d(x,z) \leq 12m + 2m^2 + 2^{6m+4} < B(m)$, i.e., $\operatorname{diam} U \leq B(m)$. So $\mathcal{U}_0$ is $B(m)$-bounded.

Step 2. Now we will prove that for every $t \in \{1,2,3,\dots,2^{3k}\}$ and $s \in \{a-k, a-k+1, \dots, a+2k\}$, $\mathcal{U}_{2^{3k}(s-a+k)+t}$ is m-disjoint and $B(m)$-bounded.

Let

$$U = U_s(\tilde{x}, (V_i)_{i=1}^{3k}, D_s, W) \times ([a, a+k] \cap \mathbb{Z}) \in \mathcal{U}_{2^{3k}(s-a+k)+t},$$
$$U' = U_s(\tilde{x'}, (V_i')_{i=1}^{3k}, D_s', W') \times ([a, a+k] \cap \mathbb{Z}) \in \mathcal{U}_{2^{3k}(s-a+k)+t}$$

and $U \neq U'$. Let $x = (f, n_1) \in U$, $y = (g, n_2) \in U'$.

Case 1: $D_s \neq D_s'$. Since $\bigcup_{l=1}^{2^{2m}} \mathcal{D}_l$ is m-disjoint, $|f(s) - g(s)| \geq m$. So $d(x,y) \geq m$.

Case 2: $D_s = D_s'$. Then $D_s = D_s' \in \mathcal{D}_l$ for some unique $l \in \{1,2,3,\dots,2^{2m}\}$. By definition of $\mathcal{U}_{2^{3k}(s-a+k)+t}$, $W, W' \in \mathcal{W}_l$. If $W \neq W'$, then $\sum_{i\in\mathbb{Z}} |f(i) - g(i)| \geq m$ since $\mathcal{W}_l$ is m-disjoint. If $W = W'$, then

$$(\tilde{x}, (V_i)_{i=1}^{3k}) \neq (\tilde{x'}, (V_i')_{i=1}^{3k}).$$

It follows that at least one of the following two situations holds:

(a) $(V_i)_{i=1}^{3k} \neq (V_i')_{i=1}^{3k}$. In this case, there exists $j \in \{1,2,3,\dots,3k\}$ such that $V_j \neq V_j' \in \mathcal{V}_{\rho(t)_j}$. Then $|f(j) - g(j)| \geq m$ since $\mathcal{V}_{\rho(t)_j}$ is m-disjoint. So $d(x,y) \geq m$.

(b) $\tilde{x} \neq \tilde{x}'$. Then there exists $j < a-k-m$ or $j > a+2k+m$ such that $f(j) \neq g(j)$, i.e., $(f^{-1}g)(j) \neq 0$, which implies that there exists $j < a-k-m-n_1 \leq -m$ or $j > a+2k+m-n_1 \geq m$ such that $n_1^{-1}(f^{-1}g)(j) \neq 0$. It follows that $L_{\mathbb{Z}}(x^{-1}y) \geq m$. So $d(x,y) \geq m$.

Then $d(U,U') \geq m$, and hence $\mathcal{U}_i$ is m-disjoint.

Let $x = (f, n_1), z = (h, n_3) \in U$. If $f = h$, then $d(x,z) = |n_3 - n_1| \leq k \leq m$. Otherwise, since

$$\sum_{i\in\mathbb{Z}} |f(i) - h(i)| \leq 2m^2 + (3k+1)m \leq 2m^2 + (3m+1)m,$$

and

$$\operatorname{supp} n_1^{-1}(f^{-1}h) \subseteq [a-k-m-n_1, a+2k+m-n_1] \subseteq [-2k-m, 2k+m] \subseteq [-3m, 3m]$$

implies that $L_{\mathbb{Z}}(x^{-1}z) \leq 12m$, $d(x,z) \leq 2m^2 + (3m+1)m + 12m < B(m)$.

So $\operatorname{diam} U \leq B(m)$, i.e., $\mathcal{U}_i$ is $B(m)$-bounded.

Step 3. Finally, we will prove that $\bigcup_{i=0}^{(3k+1)2^{3k}} \mathcal{U}_i$ covers X.

Indeed, for every $x = (f, n) \in X \setminus \cup\mathcal{U}_0$, there exist a unique $l \in \{1, \dots, 2^{2m}\}$ and $W \in \mathcal{W}_l$ such that

$$(f(a-k-m), \dots, f(a-k-1)) \in \pi_1(W)$$

and

$$(f(a+2k+1), \dots, f(a+2k+m)) \in \pi_2(W).$$

Since $x \notin \cup\mathcal{U}_0$, we have

$$(f(a-k-1+i))_{i=1}^{3k+1} \notin \cup\left\{ \prod_{i=1}^{3k+1} C_i \mid (C_i)_{i=1}^{3k+1} \in \mathcal{C}_l^{3k+1} \right\}.$$

Then there exists $s \in \{a-k, a-k+1, \dots, a+2k\}$ such that $f(s) \notin \cup\mathcal{C}_l$. Since $\mathcal{C}_l \cup \mathcal{D}_l$ covers $\mathbb{Z}$, there exists $D_s \in \mathcal{D}_l$ such that $f(s) \in D_s$. Since $\mathcal{V}_0 \cup \mathcal{V}_1$ covers $\mathbb{Z}$, we can take $t \in \{1, 2, 3, \dots, 2^{3k}\}$ such that $(f(a-k), f(a-k+1), f(a-k+2), \dots, f(s-1), f(s+1), \dots, f(a+2k)) \in \prod_{i=1}^{3k} V_i$, where $V_i \in \mathcal{V}_{\rho(t)_i}$, $i \in \{1, 2, 3, \dots, 3k\}$. So $x \in \cup\mathcal{U}_{2^{3k}(s-a+k)+t}$. ■

Theorem 3.2. *Let X be the metric space $\mathbb{Z} \wr \mathbb{Z}$ with the left-invariant word-metric. Then* trasdim $X \leq \omega + 1$. *Consequently, $\mathbb{Z} \wr \mathbb{Z}$ has asymptotic property C.*

Proof. By Lemma 2.7, it suffices to show that for every $k, m \in \mathbb{N}$, there are uniformly bounded families $\mathcal{V}_{-1}, \mathcal{V}_0, \dots, \mathcal{V}_{(6k+2)2^{3k}}$ such that $\mathcal{V}_i$ is k-disjoint for $i = -1, 0$, $\mathcal{V}_j$ is m-disjoint for $j = 1, 2, \dots, (6k+2)2^{3k}$ and the family $\bigcup_{i=-1}^{(6k+2)2^{3k}} \mathcal{V}_i$ covers X.

Without loss of generality, we assume that $m = kp$ for some $p \in \mathbb{N}$. For $i \in \mathbb{Z}$, let $A_i = [ik, ik+k] \cap \mathbb{Z}$. By Lemma 3.1, there exist $B_1(m) > 0$ and $B_1(m)$-bounded families

$$\mathcal{U}_0^{ip+1}, \mathcal{U}_1^{ip+1}, \dots, \mathcal{U}_{(3k+1)2^{3k}}^{ip+1}$$

satisfying each $\mathcal{U}_j^{ip+1}$ is m-disjoint for $j \in \{1, 2, \dots, (3k+1)2^{3k}\}$, $\mathcal{U}_0^{ip+1}$ is k-disjoint and $\bigcup_{j=0}^{(3k+1)2^{3k}} \mathcal{U}_j^{ip+1}$ covers $(\bigoplus_{\mathbb{Z}} \mathbb{Z}) \times A_{ip+1}$. By Lemma 3.1, there exist $D_1(m) > 0$ and $D_1(m)$-bounded families

$$\mathcal{U}_0^{ip+2}, \mathcal{U}_1^{ip+2}, \dots, \mathcal{U}_{(3k+1)2^{3k}}^{ip+2}$$

satisfying each $\mathcal{U}_j^{ip+2}$ is $5B_1(m)$-disjoint for $j \in \{1, 2, \dots, (3k+1)2^{3k}\}$, $\mathcal{U}_0^{ip+2}$ is k-disjoint and $\bigcup_{j=0}^{(3k+1)2^{3k}} \mathcal{U}_j^{ip+2}$ covers $(\bigoplus_{\mathbb{Z}} \mathbb{Z}) \times A_{ip+2}$. For $j \in \{1, 2, \dots, (3k+1)2^{3k}\}$, let

$$\mathcal{V}_j^{ip+2} = \mathcal{U}_j^{ip+2} \bigcup_m \mathcal{U}_j^{ip+1}.$$

Then $\mathcal{V}_j^{ip+2}$ is m-disjoint and $B_2(m)$-bounded by Lemma 2.11, where $B_2(m) = D_1(m) + 2B_1(m) + 2m$. By Lemma 3.1, there exist $D_2(m) > 0$ and $D_2(m)$-bounded families

$$\mathcal{U}_0^{ip+3}, \mathcal{U}_1^{ip+3}, \dots, \mathcal{U}_{(3k+1)2^{3k}}^{ip+3}$$

satisfying each $\mathcal{U}_j^{ip+3}$ is $5B_2(m)$-disjoint for $j \in \{1, 2, \ldots, (3k+1)2^{3k}\}$, $\mathcal{U}_0^{ip+3}$ is k-disjoint and $\bigcup_{j=0}^{(3k+1)2^{3k}} \mathcal{U}_j^{ip+3}$ covers $(\bigoplus_{\mathbb{Z}} \mathbb{Z}) \times A_{ip+3}$. For $j \in \{1, 2, \ldots, (3k+1)2^{3k}\}$, let

$$\mathcal{V}_j^{ip+3} = \mathcal{U}_j^{ip+3} \bigcup_m \mathcal{V}_j^{ip+2}.$$

Then $\mathcal{V}_j^{ip+3}$ is m-disjoint and $B_3(m)$-bounded by Lemma 2.11, where $B_3(m) = D_2(m) + 2B_2(m) + 2m$. After finitely many steps, we obtain m-disjoint and $B_p(m)$-bounded families

$$\mathcal{V}_1^{ip+p}, \mathcal{V}_2^{ip+p}, \ldots, \mathcal{V}_{(3k+1)2^{3k}}^{ip+p}.$$

Note that $\mathcal{V}_j^{ip+p}$ is a family of subsets of $(\bigoplus_{\mathbb{Z}} \mathbb{Z}) \times (\bigcup_{j=1}^{p} A_{ip+j})$ and

$$\left(\bigcup_{j=1}^{p} \mathcal{U}_0^{ip+j}\right) \cup \left(\bigcup_{j=1}^{(3k+1)2^{3k}} \mathcal{V}_j^{ip+p}\right) \quad \text{covers} \left(\bigoplus_{\mathbb{Z}} \mathbb{Z}\right) \times \left(\bigcup_{j=1}^{p} A_{ip+j}\right).$$

For $j \in \{1, 2, \ldots, (3k+1)2^{3k}\}$, let

$$\mathcal{V}_j = \bigcup_{n \in \mathbb{Z}} \mathcal{V}_j^{2np+p} \quad \text{and} \quad \mathcal{V}_{j+(3k+1)2^{3k}} = \bigcup_{n \in \mathbb{Z}} \mathcal{V}_j^{(2n+1)p+p}.$$

Since for every $i_1, i_2 \in \mathbb{Z}$ with $i_1 \neq i_2$,

$$d\left(\left(\bigoplus_{\mathbb{Z}} \mathbb{Z}\right) \times \bigcup_{j=1}^{p} A_{2i_1 p+j}, \left(\bigoplus_{\mathbb{Z}} \mathbb{Z}\right) \times \bigcup_{j=1}^{p} A_{2i_2 p+j}\right) \geq pk = m$$

and

$$d\left(\left(\bigoplus_{\mathbb{Z}} \mathbb{Z}\right) \times \bigcup_{j=1}^{p} A_{(2i_1+1)p+j}, \left(\bigoplus_{\mathbb{Z}} \mathbb{Z}\right) \times \bigcup_{j=1}^{p} A_{(2i_2+1)p+j}\right) \geq pk = m,$$

each $\mathcal{V}_j$ is m-disjoint and uniformly bounded for $j \in \{1, 2, \ldots, (6k+2)2^{3k}\}$. Let

$$\mathcal{V}_0 = \bigcup_{n \in \mathbb{Z}} \mathcal{U}_0^{2n} \quad \text{and} \quad \mathcal{V}_{-1} = \bigcup_{n \in \mathbb{Z}} \mathcal{U}_0^{2n+1}.$$

Since

$$\left\{\left(\bigoplus_{\mathbb{Z}} \mathbb{Z}\right) \times A_{2n} \mid n \in \mathbb{Z}\right\}$$

is k-disjoint and $\mathcal{U}_0^{2n}$ is a family of subsets of $(\bigoplus_{\mathbb{Z}} \mathbb{Z}) \times A_{2n}$ and

$$\left\{\left(\bigoplus_{\mathbb{Z}} \mathbb{Z}\right) \times A_{2n+1} \mid n \in \mathbb{Z}\right\}$$

is k-disjoint and $\mathcal{U}_0^{2n+1}$ is a family of subsets of $(\bigoplus_{\mathbb{Z}} \mathbb{Z}) \times A_{2n+1}$, $\mathcal{V}_0$ and $\mathcal{V}_{-1}$ is k-disjoint and uniformly bounded.

Finally, we will prove that $\bigcup_{i=-1}^{(6k+2)2^{3k}} \mathcal{V}_i$ covers $\mathbb{Z} \wr \mathbb{Z}$.
Since

$$\begin{aligned}\mathbb{Z} \wr \mathbb{Z} &= \Big(\bigoplus_{\mathbb{Z}} \mathbb{Z}\Big) \rtimes \mathbb{Z} = \bigcup_{n\in\mathbb{Z}} \Big(\bigoplus_{\mathbb{Z}} \mathbb{Z}\Big) \times A_n = \bigcup_{i\in\mathbb{Z}} \bigcup_{j=1}^{p} \Big(\bigoplus_{\mathbb{Z}} \mathbb{Z}\Big) \times A_{ip+j} \\ &= \bigcup_{i\in\mathbb{Z}} \Big(\bigoplus_{\mathbb{Z}} \mathbb{Z}\Big) \times \bigcup_{j=1}^{p} A_{ip+j}\end{aligned}$$

and

$$\Big(\bigcup_{j=1}^{p} \mathcal{U}_0^{ip+j}\Big) \cup \Big(\bigcup_{j=1}^{(3k+1)2^{3k}} \mathcal{V}_j^{ip+p}\Big) \quad \text{covers } \Big(\bigoplus_{\mathbb{Z}} \mathbb{Z}\Big) \times \bigcup_{j=1}^{p} A_{ip+j},$$

we obtain that

$$\Big(\bigcup_{i\in\mathbb{Z}} \bigcup_{j=1}^{p} \mathcal{U}_0^{ip+j}\Big) \cup \Big(\bigcup_{i\in\mathbb{Z}} \bigcup_{j=1}^{(3k+1)2^{3k}} \mathcal{V}_j^{ip+p}\Big) \quad \text{covers } \mathbb{Z} \wr \mathbb{Z}.$$

Note that

$$\bigcup_{i\in\mathbb{Z}} \bigcup_{j=1}^{p} \mathcal{U}_0^{ip+j} = \bigcup_{n\in\mathbb{Z}} \mathcal{U}_0^{n} = \Big(\bigcup_{n\in\mathbb{Z}} \mathcal{U}_0^{2n}\Big) \cup \Big(\bigcup_{n\in\mathbb{Z}} \mathcal{U}_0^{2n+1}\Big) = \mathcal{V}_0 \cup \mathcal{V}_{-1}$$

and

$$\begin{aligned}\bigcup_{i\in\mathbb{Z}} \bigcup_{j=1}^{(3k+1)2^{3k}} \mathcal{V}_j^{ip+p} &= \Big(\bigcup_{j=1}^{(3k+1)2^{3k}} \bigcup_{n\in\mathbb{Z}} \mathcal{V}_j^{2np+p}\Big) \cup \Big(\bigcup_{j=1}^{(3k+1)2^{3k}} \bigcup_{n\in\mathbb{Z}} \mathcal{V}_j^{(2n+1)p+p}\Big) \\ &= \Big(\bigcup_{j=1}^{(3k+1)2^{3k}} \mathcal{V}_j\Big) \cup \Big(\bigcup_{j=1}^{(3k+1)2^{3k}} \mathcal{V}_{j+(3k+1)2^{3k}}\Big) \\ &= \bigcup_{j=1}^{(6k+2)2^{3k}} \mathcal{V}_j.\end{aligned}$$

Therefore, $\bigcup_{i=-1}^{(6k+2)2^{3k}} \mathcal{V}_i$ covers $\mathbb{Z} \wr \mathbb{Z}$. ∎

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