

Modular orbits on the representation spaces of compact abelian Lie groups

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Abstract. Let S be a closed surface of genus g greater than zero. In the present paper, we study the topological-dynamical action of the mapping class group on the \mathbb{T}^n -character variety giving necessary and sufficient conditions for $\text{Mod}(S)$ -orbits to be dense. As an application, such a characterisation provides a dynamical proof of the Kronecker's theorem concerning inhomogeneous Diophantine approximation.

1. Introduction

Let S be a closed, connected and oriented topological surface of genus g . Its fundamental group $\pi_1(S)$ admits the presentation

$$\langle \alpha_1, \beta_g, \dots, \alpha_g, \beta_g \mid \prod_{i=1}^g [\alpha_i, \beta_i] = 1 \rangle.$$

Let G be a connected Lie group and let $\text{Hom}(\pi_1(S), G)$ denote the set of representations of $\pi_1(S)$ in G . Such a set can be topologized with the compact-open topology, and the resulting space is commonly known as *representation space*. There is a natural action of G on $\text{Hom}(\pi_1(S), G)$ obtained by post-composing representations with inner automorphisms of G . The resulting quotient space $\text{Hom}(\pi_1(S), G)/G$ is a space canonically associated to S (or $\pi_1(S)$) and G . When G is an algebraic and reductive Lie group, the quotient space is commonly known as *G -character variety of $\pi_1(S)$* . It can be shown that $\text{Hom}(\pi_1(S), G)/G$ identifies with the moduli space of isomorphism classes of flat G -bundles over S .

We next consider the effect of changing the presentation of $\pi_1(S)$. This can be done by pre-composing any representation with an automorphism $\phi \in \text{Aut}(\pi_1(S))$ such that any representation ρ is sent to $\rho \circ \phi^{-1}$. We can, therefore, consider the action of $\text{Aut}(\pi_1(S))$ on $\text{Hom}(\pi_1(S), G)$. The actions of G and $\text{Aut}(\pi_1(S))$ can be combined together in the following way: For any pair $(\phi, g) \in \text{Aut}(\pi_1(S)) \times G$ and $\gamma \in \pi_1(S)$, we define

$$(\phi, g) \cdot \rho(\gamma) = g(\rho \circ \phi^{-1}(\gamma))g^{-1}.$$

It is clear that the action of the normal subgroup $\text{Inn}(\pi_1(S)) < \text{Aut}(\pi_1(S))$, consisting of inner automorphisms, is absorbed into the action of G . In other words, the action of $\text{Inn}(\pi_1(S))$ on $\text{Hom}(\pi_1(S), G)$ descends to the trivial action on $\text{Hom}(\pi_1(S), G)/G$. Thus, there is a well-defined action of the *outer automorphisms group* $\text{Out}(\pi_1(S)) = \text{Aut}(\pi_1(S))/\text{Inn}(\pi_1(S))$ on $\text{Hom}(\pi_1(S), G)/G$.

When S is a closed surface, the group $\text{Out}(\pi_1(S))$ has a very explicit geometric interpretation. Every homeomorphism of S determines an automorphism of $\pi_1(S)$. On the other hand, for closed surfaces, the Dehn–Nielsen–Baer theorem [16] states that every automorphism of $\pi_1(S)$ is induced by a homeomorphism of S – this is actually true also for the punctured torus, but it is no longer true for other surfaces with boundary. Now, homeomorphisms which are isotopic can be considered as equivalent and determine conjugate automorphisms of $\pi_1(S)$, i.e., a well-defined element of $\text{Out}(\pi_1(S))$. Conversely, automorphisms which are conjugate can be considered as equivalent, and determine isotopic homeomorphisms. Therefore, for any closed surface the following isomorphism holds:

$$\frac{\text{Homeo}(S)}{\text{Isotopy}} \cong \text{Out}(\pi_1(S)).$$

Definition 1.1. The *mapping class group* of S is defined as

$$\text{Mod}(S) = \frac{\text{Homeo}(S)}{\text{Isotopy}}.$$

The character variety can be endowed with a symplectic structure which is preserved by the $\text{Out}(\pi_1(S))$ -action, see Goldman [7]. By taking the volume form associated to the symplectic structure, we obtain a finite measure μ_S and the space $\text{Hom}(\pi_1(S), G)/G$ turns into a measured space on which the group $\text{Out}(\pi_1(S))$ acts preserving measure. The dynamic of this action is known for compact Lie groups.

Theorem 1.2 (Goldman, Pickrell–Xia). *Let G be a compact Lie group. Then the mapping class group acts ergodically on each connected components of $\text{Hom}(\pi_1(S), G)/G$ with respect to the finite measure μ_S .*

A direct consequence of ergodicity is that almost every $\text{Mod}(S)$ -orbit is dense. A more subtle problem concerns the topological dynamics of the mapping class group action on the space $\text{Hom}(\pi_1(S), G)/G$. The topological-dynamical problem is definitely more delicate since no longer we may ignore invariant subsets of measure zero. For instance, if H is a finite subgroup of G , the space $\text{Hom}(\pi_1(S), H)$ is finite and its image in $\text{Hom}(\pi_1(S), G)/G$ is an invariant closed subset under the action of the mapping class group. It follows that, even when the action of $\text{Mod}(S)$ is ergodic, not all the orbits are dense in $\text{Hom}(\pi_1(S), G)/G$. In [10, Problem 2.7], Goldman posed the following problem (we refer to it in the sequel as main problem).

Main Problem. Determine necessary and sufficient conditions on a general representation ρ for its orbit $\text{Mod}(S) \cdot \rho$ to be dense.

We introduce the following definition.

Definition 1.3. A representation $\rho: \pi_1(S) \rightarrow G$ is defined as *dense representation* if the image of ρ is dense in G .

For closed surfaces, the following claim is expected: If the image of a representation ρ is dense in G , then the $\text{Mod}(S)$ -orbit of ρ is dense in $\text{Hom}(\pi_1(S), G)/G$. This is currently true for representations in $\text{SU}(2)$. In fact, this case has been completely treated by Previ-te and Xia [19], and it is based on their earlier work [18], in which they considered the case of representations in $\text{SU}(2)$ for the punctured torus. For all other compact Lie groups, the problem is still open, and in the present work we provide a positive answer also in the case of the n -dimensional torus \mathbb{T}^n . The following theorem is in fact the main result of this work.

Theorem A. *Let S be a surface of genus $g \geq 1$ and $\pi_1(S)$ its fundamental group. Let $\rho: \pi_1(S) \rightarrow \mathbb{T}^n$ be a representation. Then the image of ρ is dense in \mathbb{T}^n if and only if the mapping class group orbit $\text{Mod}(S) \cdot \rho$ is dense in the representation space.*

The proof strongly relies on the explicit knowledge of the objects involved. In fact, the n -torus has a well-known description and, thanks to the abelian property, the character variety coincides with the representation space since the action of \mathbb{T}^n by conjugation is trivial. Even better, the representation space can be identified with a torus of suitable dimension, hence the description of the representation space – and then of the character variety – is very explicit. The main difficulties in the abelian case concern questions coming from number theory and ergodic theory, see Section 1.2.

1.1. Strategy of the proof and related results

A representation $\rho: \pi_1(S) \rightarrow \mathbb{T}^n$ yields a representation $\bar{\rho}: H_1(S, \mathbb{Z}) \rightarrow \mathbb{T}^n$, namely an element of the *homological representation space* $\text{Hom}(H_1(S, \mathbb{Z}), \mathbb{T}^n)$. The map associating to any representation ρ its homological representation defines a bijection between the representation space and the homological representation space. This essentially follows because the commutator group $[\pi_1(S), \pi_1(S)]$ is trivially a subgroup of $\ker(\rho)$ in the abelian case, and such a property is no longer true for a generic non-abelian Lie group. There is also a well-defined action of the symplectic group $\text{Sp}(2g, \mathbb{Z})$ on the space $\text{Hom}(H_1(S, \mathbb{Z}), \mathbb{T}^n)$ by precomposition. Given a representation $\rho: \pi_1(S) \rightarrow \mathbb{T}^n$ and its induced representation $\bar{\rho}: H_1(S, \mathbb{Z}) \rightarrow \mathbb{T}^n$, the $\text{Mod}(S)$ -orbit of ρ coincides with the $\text{Sp}(2g, \mathbb{Z})$ -orbit of $\bar{\rho}$. As an immediate consequence, we obtain an equivalent version of the main Theorem A, namely we have the following theorem.

Theorem B. *Let S be a surface of genus $g \geq 1$ and let $\bar{\rho}: H_1(S, \mathbb{Z}) \rightarrow \mathbb{T}^n$ be a representation. Then the image of $\bar{\rho}$ is dense in \mathbb{T}^n if and only if the symplectic group orbit $\text{Sp}(2g, \mathbb{Z}) \cdot \bar{\rho}$ is dense in the homological representation space.*

Along the way of our investigation, we shall remark the following result: The action of the Torelli group $\mathcal{I}(S)$ on the representation space $\text{Hom}(\pi_1(S), \mathbb{T}^n)$ is trivial.

For a proof of this fact, we refer to Proposition 2.8 below. This claim seems to be known, however, at the best knowledge of the authors, it has never been stated explicitly. In contrast to the abelian case, Bouilly has shown in [4] that for any connected, compact and semisimple Lie group, the action of the Torelli group on the character variety is ergodic.

Given a representation, we have reduced the problem to the study of $\text{Sp}(2g, \mathbb{Z})$ -orbit instead of modular orbits. This makes the study of orbits more understandable because the symplectic group is linear. We will make the action even more explicit by identifying a representation with a matrix in the space $M(n, 2g; \mathbb{T})$. Such a space will be introduced later on in Section 2.3. After these reductions, we shall see that we are in the position to apply Ratner's theorem for studying orbit closures. In particular, we shall derive our main Theorem A.

Remark 1.4. For the torus, the reader may notice that Theorems A and B are not only equivalent but actually the same statement in the strict sense. Indeed, in this very particular case the equalities $\pi_1(S) = H_1(S, \mathbb{Z})$ and $\text{Mod}(S) = \text{SL}(2, \mathbb{Z}) = \text{Sp}(2, \mathbb{Z})$ hold.

The strategy we propose for Theorem A is different to the one developed by Previte and Xia to show their main theorem [19, Theorem 1.4]. Let us briefly give some more details. Given a dense representation $\rho: \pi_1(S) \rightarrow \text{SU}(2)$ – Previte and Xia defined such a representation as *generic* (see [19, Definition 1.6]) – they firstly found a handle Σ , namely a one-holed torus, such that the restriction of ρ to $\pi_1 \Sigma$ is dense. After obtaining a dense handle, they proceed to demonstrate the base density theorem for the $(n + 2g - 2)$ -holed torus. A similar process in the abelian case is not possible because dense handles do not always exist, see the discussion in Appendix A.2. In the light of Proposition 2.8, we shall bypass this issue by looking at the $\text{Sp}(2g, \mathbb{Z})$ -action on the representation space as described above.

1.2. Connection with the Kronecker's approximation theorem

The dynamical result provided by Theorem A finds application in the theory of geometry of numbers. An important theorem in this topic is the Kronecker's theorem concerning inhomogeneous Diophantine approximation, see Section 5 below for the precise statement.

By fixing a presentation of $\pi_1(S)$, we can associate to any representation $\rho: \pi_1(S) \rightarrow \mathbb{T}^n$ a matrix $\Theta_\rho \in M(n, 2g; \mathbb{R})$; see Definition 2.14 and Section 2.3 below for the details. We shall prove that a representation ρ is dense if and only if the rows of the matrix Θ_ρ satisfy the hypothesis of Kronecker's theorem, this is our Theorem 3.1. On the other hand, our main result says that the modular orbit of a representation ρ is dense in the representation space if and only if ρ itself is a dense representation. As the representation space identifies with \mathbb{T}^{2ng} , see Section 2 below, Theorem B provides a dynamical proof of Kro-

necker's theorem in the cases where $l = m = 2g$ for some $g \geq 1$. More precisely, we have the following theorem.

Theorem C. *Let $m = 2g$ with $g \geq 1$. Let $b^{(i)} = (b_1^{(i)}, \dots, b_m^{(i)})$, where $i = 1, \dots, n$, be vectors of \mathbb{R}^m such that $b^{(1)}, \dots, b^{(n)}, \pi e_1, \dots, \pi e_m$ are linearly independent over \mathbb{Q} in the vector space \mathbb{R}^m . Let $A \in M(n, m; \mathbb{R})$ be a real matrix and let ε be a positive number. Then there is an element $K \in \text{Sp}(2g, \mathbb{Z})$ such that*

$$\|A - B\| < C\varepsilon \pmod{2\pi},$$

where C is a constant depending only on m and n , and the norm is any norm on the space $M(n, m; \mathbb{R})$.

This is a sharper simultaneous approximation result because, in principle, one can always find a matrix $K \in M(2g, \mathbb{Z})$ according to Kronecker's theorem. For the sake of comprehension, the proof of Theorem C is delayed until Section 5, when we have developed the theory and notation even further. As we shall see, the proof of this result reduces to proving the following characterisation.

Proposition D. *Theorem A holds if and only if Theorem C holds.*

1.3. Related dynamical problems

For a generic compact Lie groups G , the main issues one has to face are mainly two. First of all, the group G may not have a nice description. In fact, among all compact Lie groups we found the classical simple Lie groups belonging to the four families $SU(n + 1)$, $SO(2n + 1)$, $Sp(n)$, $SO(2n)$, but also the five exceptional Lie groups corresponding to the Dynkin diagrams G_2, F_4, E_6, E_7, E_8 which are harder to treat. The second issue comes from the fact that most of the representation spaces and their quotients $\text{Hom}(\pi_1(S), G)/G$ have no explicit description to work with. The abelian case is not the only one on which these issues vanish. Also in the case of $SU(2)$ they completely miss since both the group and the representation space were already well known in the literature. For open surfaces of positive genus and positive number of boundary components, there is a further problem to be addressed. Suppose S has genus g with boundary of k disjoint circles. A relative character variety is a slice of the space $\text{Hom}(\pi_1(S), G)/G$ subject to the condition imposed by some finite collection of k conjugacy classes. Like in the close case, the relative character variety carries a symplectic structure which is preserved by the mapping class group action. In [17], Pickrell and Xia established the ergodicity of the mapping class group action with respect to the symplectic measure for $k > 2$. Their result may be seen as the follow up of the Goldman's work [9], where he considered the cases of groups whose simple factors are locally isomorphic to $SU(2)$. Very recently, in [11], Goldman, Lawton and Xia announced a proof in the case of $SU(3)$ based on different techniques than in [17].

As for the modular orbits characterisation, the main problem posed by Goldman has been answered only in the case of $SU(2)$, as mentioned above. This work is therefore

a new and partial development of the wider program to understanding the dynamics of the mapping class group. In [3], Biswas, Koberda, Mj and Santharoubane considered the opposite problem of characterising representations having *finite* modular orbit. For any fixed Lie group G , they showed that any representation, with values in G , having finite modular orbit, has necessarily finite image in G . The case $G = \mathrm{SL}(2, \mathbb{C})$ has been handled also by Biswas, Gupta, Mj and Whang in [2].

The non-compact case is even more complicated and delicate; let us spend a few words. The current situation is different for non-compact Lie groups, and we do not expect an analogous theorem. For compact Lie groups G , the space $\mathrm{Hom}(\pi_1(S), G)/G$ has non-trivial homotopy type, and Theorem 1.2 says that the dynamics of the action of the mapping class is chaotic on each connected component. On the other hand, when G is a non-compact semisimple Lie group, the space $\mathrm{Hom}(\pi_1(S), G)/G$ contains open contractible components on which the action of the mapping class group is properly discontinuous. Often, these components correspond to locally homogeneous structures uniformizing S . A remarkable case is that of $\mathrm{PSL}(2, \mathbb{R})$. It is well known that the $\mathrm{PSL}(2, \mathbb{R})$ -character variety has $4g - 3$ connected components indexed by the Euler class, taking values in a finite set of \mathbb{Z} , where g denotes the genus of S , see [8]. Two of these components correspond to the Teichmüller spaces $\mathcal{T}(S)$ and $\mathcal{T}(\bar{S})$ of S (where \bar{S} is the surface S taken with the opposite orientation). The action of $\mathrm{Mod}(S)$ is known to be proper on these components and it is conjectured to be ergodic on the others. This conjecture is currently treated in the case of genus 2 surfaces thanks to recent results of Marché and Wolff. They proved that the conjecture about ergodicity is true for Euler number equal to ± 1 and decomposed the connected component of Euler number 0 into two subspace on which the mapping class group acts ergodically [14, 15]. As it may be easy to expect, even less is known about the topological dynamics of the mapping class group on the $\mathrm{PSL}(2, \mathbb{R})$ -character variety. In the case of genus two, one of the consequences of Marché and Wolff's result is the following claim: In each subspace of the character variety on which the action of the mapping class group is ergodic, there is a full measure subset of representations whose mapping class group orbit is dense in this subspace. Like in the compact case, we can pose the following question: Does a dense representation $\rho: \pi_1(S) \rightarrow \mathrm{PSL}(2, \mathbb{R})$ have dense $\mathrm{Mod}(S)$ -orbit? Answering to this question is even more tricky, and for surfaces with boundary we already know counterexamples – see [20] for an example in $\mathrm{SL}(2, \mathbb{C})$.

1.4. Structure of the paper

The paper is organised as follow. In Section 2, we begin with a description of the \mathbb{T}^n -character variety and then subsequently introduce the homological representation space and show the identification with the character variety. We finally describe the action of the symplectic group $\mathrm{Sp}(2g, \mathbb{Z})$ on the homological representation space. As a consequence, we shall derive Proposition 2.8 and the equivalence of Theorems A and B. In Section 3, we shall give a complete characterisation of dense representations in the n -dimensional torus by proving Theorem 3.1. In Section 4, we shall finally derive our main Theorem A. In the

last section, we prove Proposition D and indeed Theorem C establishing the connection of our dynamical result with the Kronecker's approximation theorem. We finally conclude with appendixes discussing some further aspects related to our project. In Appendix A, we discuss a direct approach to our problem which works for a fairly general class of representations. In Appendix B, we digress a little by providing a brief description of the relative \mathbb{T}^n -character variety for surfaces with one puncture and then we claim that our main results extend to one-punctured surfaces.

2. \mathbb{T}^n -character variety

In this work, we are interested in characterising the orbits of the $\text{Mod}(S)$ -action on $\text{Hom}(\pi_1(S), G)/G$, where G is a compact, connected and abelian Lie group. It is classical to see that any such a group is isomorphic to \mathbb{T}^n , the n -dimensional torus for some positive n , see, for instance, in [5, Corollary 3.7]. The specific interest for the abelian case comes from its connection with abstract harmonic analysis, the geometry of numbers and the theory of group actions on homogeneous spaces (connections with Ratner's theorem, see Section 4).

In the introduction, we have given a very brief view of the character variety for a generic compact Lie group G . In this section, we specialise the discussion for compact and connected abelian Lie groups. From the Lie theory, any such a group is known to be an n -dimensional torus, namely the product of n copies of the unit circle \mathbb{S}^1 . In the present work, \mathbb{S}^1 is seen as $\{e^{i\theta} \mid \theta \in [0, 2\pi)\}$, where $[0, 2\pi)$ carries the quotient topology obtained identifying the boundary points of the closed interval $[0, 2\pi]$. Consequently, the n -torus \mathbb{T}^n is defined as $\{(e^{i\theta_1}, \dots, e^{i\theta_n}) \mid \theta_i \in [0, 2\pi) \text{ for any } i = 1, \dots, n\}$ endowed with the product topology.

Let S be a closed surface, and let $\alpha_1, \beta_1, \dots, \alpha_g, \beta_g$ be any standard generating system of the fundamental group. The choice of a representation $\rho: \pi_1(S) \rightarrow \mathbb{T}^n$ amounts to choose for each generator an element of \mathbb{T}^n such that these elements satisfy the condition imposed by the presentation of the fundamental group of S . Since \mathbb{T}^n is an abelian group, the relation $[A_1, B_1] \cdots [A_g, B_g] = 1$ is automatically satisfied for any choice of $2g$ elements in $(A_1, B_1, \dots, A_g, B_g) \in \mathbb{T}^n$. Thus, the representation space can be identified with the full group $(\mathbb{T}^n)^{2g} \cong \mathbb{T}^{2ng}$. Even more, thanks again to the abelian property, the action of \mathbb{T}^n on $\text{Hom}(\pi_1(S), \mathbb{T}^n)$ by post-composition with inner automorphisms of \mathbb{T}^n is trivial. As a consequence, the \mathbb{T}^n -character variety coincides with the representation space.

2.1. Homological representations

Let $H_1(S, \mathbb{Z})$ be the first homology group. The close connection between the objects $\pi_1(S)$ and $H_1(S, \mathbb{Z})$ is well known, indeed the latter is known to be isomorphic to the abelianization of $\pi_1(S)$. The representation space $\text{Hom}(\pi_1(S), \mathbb{T}^n)$, as we have seen above, naturally identifies with the $2gn$ -dimensional torus assigning to any representa-

tion ρ the $2g$ -tuple $(\rho(\alpha_1), \rho(\beta_1), \dots, \rho(\alpha_g), \rho(\beta_g))$, where $\alpha_1, \beta_1, \dots, \alpha_g, \beta_g$ is a basis for $\pi_1(S)$. Every representation ρ fails to be injective, and its kernel $\ker(\rho)$ always contains the subgroup generated by the commutators since the target is abelian. So, ρ boils down to a representation

$$\bar{\rho}: H_1(S, \mathbb{Z}) \cong \frac{\pi_1(S)}{[\pi_1(S), \pi_1(S)]} \rightarrow \mathbb{T}^n, \quad \bar{\rho}([\gamma]) := \rho(\gamma).$$

In fact, let $\gamma \in \pi_1(S)$ and let $[\gamma]$ be its image via the canonical projection $p: \pi_1(S) \rightarrow H_1(S, \mathbb{Z})$. Let $\gamma + [\sigma_1, \sigma_2]$ be a representative of $[\gamma]$. Since the chain of equalities

$$\rho(\gamma + [\sigma_1, \sigma_2]) = \rho(\gamma)\rho([\sigma_1, \sigma_2]) = \rho(\gamma)$$

holds, the representation $\bar{\rho}$ is well-defined and the image does not depend on the choice of the representative. Furthermore, the image of ρ agrees with the image of $\bar{\rho}$ by construction.

Definition 2.1. The set $\text{Hom}(H_1(S, \mathbb{Z}), \mathbb{T}^n)$, endowed with the compact-open topology, is defined as the homological representation space.

Lemma 2.2. *The homological representation space $\text{Hom}(H_1(S, \mathbb{Z}), \mathbb{T}^n)$ identifies with the $2gn$ -dimensional torus \mathbb{T}^{2gn} .*

Proof. To any representation $\bar{\rho}$, we can assign the $2g$ -tuple defined as

$$(\bar{\rho}([\alpha_1]), \bar{\rho}([\beta_1]), \dots, \bar{\rho}([\alpha_g]), \bar{\rho}([\beta_g])),$$

where the collection $[\alpha_i], [\beta_i]$, $1 \leq i \leq g$, is a fixed basis of the homology group $H_1(S, \mathbb{Z})$. Conversely, since \mathbb{T}^{2gn} is an abelian group, for any $2g$ -tuple of $(\mathbb{T}^n)^{2g}$, for example, $(v_1, w_1, \dots, v_g, w_g)$, the universal property of free abelian groups implies the existence of a unique group homomorphism from $H_1(S, \mathbb{Z})$ into the n -torus \mathbb{T}^n which sends $[\alpha_i]$ to v_i and $[\beta_i]$ to w_i for every $i = 1, \dots, g$. ■

The implications of this lemma are quite simple, but of crucial importance. Upon choosing a basis for $\pi_1(S)$, the representation space $\text{Hom}(\pi_1(S), \mathbb{T}^n)$ identifies with the homological representation space $\text{Hom}(H_1(S, \mathbb{Z}), \mathbb{T}^n)$, and the identification is explicitly given by the association $\rho \mapsto \bar{\rho}$. According to this property, we derive the following lemma.

Lemma 2.3. *Let $\rho_1, \rho_2: \pi_1(S) \rightarrow \mathbb{T}^n$ be two representations. Then $\rho_1 \equiv \rho_2$ if and only if $\bar{\rho}_1 \equiv \bar{\rho}_2$.*

Proof. This is just a matter of definitions given so far. The necessary condition follows trivially. The sufficient condition follows from $\bar{\rho}([\gamma]) = \rho(\gamma)$ for any $\gamma \in \pi_1(S)$. ■

2.2. Actions of the symplectic group $\text{Sp}(2g, \mathbb{Z})$

In this section, we are going to describe the action of the symplectic group $\text{Sp}(2g, \mathbb{Z})$ both on the representation space and on the homological representation space.

2.2.1. The symplectic group $\mathrm{Sp}(2g, \mathbb{Z})$. We begin with recalling some standard notions. The algebraic intersection number

$$\cap: H_1(S, \mathbb{Z}) \times H_1(S, \mathbb{Z}) \rightarrow \mathbb{Z}$$

extends uniquely to a nondegenerate, alternating bilinear map

$$\cap: H_1(S, \mathbb{R}) \times H_1(S, \mathbb{R}) \rightarrow \mathbb{R}$$

which realises $H_1(S, \mathbb{R})$ as a symplectic vector space.

Definition 2.4. A collection of elements $[\alpha_i], [\beta_i], 1 \leq i \leq g$, of $H_1(S, \mathbb{Z}) < H_1(S, \mathbb{R})$ such that

$$[\alpha_i] \cap [\alpha_j] = [\beta_i] \cap [\beta_j] = 0, \quad [\alpha_i] \cap [\beta_j] = \delta_{ij}$$

for all i, j with $1 \leq i, j \leq g$ is called a *symplectic basis of the group* $H_1(S, \mathbb{Z})$ or a basis for the symplectic vector space $(H_1(S, \mathbb{Z}), \cap)$. We define a collection of curves α_i, β_i such that $\{[\alpha_i], [\beta_i]\}$ is a symplectic basis as *geometric symplectic basis* for $\pi_1(S)$.

The matrix associated to the antisymmetric bilinear form \cap on the basis $[\alpha_i], [\beta_i]$ is the $2g \times 2g$ blockwise diagonal matrix

$$J = \begin{pmatrix} J_o & & \\ & \ddots & \\ & & J_o \end{pmatrix} \quad \text{with } J_o = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

The symplectic linear group $\mathrm{Sp}(2g, \mathbb{R})$ is defined as the group of invertible matrices A satisfying the relation $AJA^T = J$, and we denote by $\mathrm{Sp}(2g, \mathbb{Z})$ the subgroup of those matrices with integer coefficients.

Remark 2.5. Here, the symplectic group $\mathrm{Sp}(2g, \mathbb{R})$ is the subgroup of $\mathrm{SL}(2g, \mathbb{R})$ of matrices preserving the alternating 2-form $\omega = e_1 \wedge e_2 + \cdots + e_{2g-1} \wedge e_{2g}$. The group $\mathrm{Sp}(2g, \mathbb{R})$ contains the g -times product $\mathrm{SL}(2, \mathbb{R}) \times \cdots \times \mathrm{SL}(2, \mathbb{R})$ as a proper subgroup. In turns, the group $\mathrm{Sp}(2g, \mathbb{Z})$ contains the g -times product $\mathrm{SL}(2, \mathbb{Z}) \times \cdots \times \mathrm{SL}(2, \mathbb{Z})$ as a proper subgroup. This property will be useful in the sequel, see Appendix A.

An orientation-preserving homeomorphism yields an isomorphism at the level of homology groups which preserves the intersection form \cap defined above. Since isotopic homeomorphisms induce the same map in homology, there is a representation

$$\mu: \mathrm{Mod}(S) \rightarrow \mathrm{Aut}^+(H_1(S, \mathbb{Z})) \cong \mathrm{SL}(2g, \mathbb{Z}).$$

As each homeomorphism preserves the intersection form \cap , the image of μ lies also inside $\mathrm{Sp}(2g, \mathbb{R})$. Therefore, the image of μ lies inside $\mathrm{SL}(2g, \mathbb{Z}) \cap \mathrm{Sp}(2g, \mathbb{R}) = \mathrm{Sp}(2g, \mathbb{Z})$. The representation $\mu: \mathrm{Mod}(S) \rightarrow \mathrm{Sp}(2g, \mathbb{Z})$ – usually called *symplectic representation* of $\mathrm{Mod}(S)$ – is surjective with kernel $\mathcal{I}(S)$. The subgroup $\mathcal{I}(S)$ is called *the Torelli subgroup* of $\mathrm{Mod}(S)$.

Remark 2.6. In the genus one case, the Torelli subgroup is trivial. Indeed, $\text{Mod}(T) \cong \text{SL}(2, \mathbb{Z}) \cong \text{Sp}(2, \mathbb{Z})$.

2.2.2. Comparison of the $\text{Mod}(S)$ -orbits with the $\text{Sp}(2g, \mathbb{Z})$ -orbits. We now consider the effect of changing the basis of $H_1(S, \mathbb{Z})$ pre-composing any homological representation with an automorphism $\phi \in \text{Aut}^+(H_1(S, \mathbb{Z}))$ such that any representation $\bar{\rho}$ is sent to $\bar{\rho} \circ \phi^{-1}$. We can, therefore, consider $\text{SL}(2g, \mathbb{Z})$ -action on $\text{Hom}(H_1(S, \mathbb{Z}), \mathbb{T}^n)$. Of course, this action restricts to an action of the symplectic group $\text{Sp}(2g, \mathbb{Z})$ on the same space. We are interested in studying the $\text{Sp}(2g, \mathbb{Z})$ -orbits in the homological representation space. The main goal of this section is to prove the following claim.

Proposition 2.7. *Let $\rho_1, \rho_2: \pi_1(S) \rightarrow \mathbb{T}^n$ be two representations and $\bar{\rho}_1, \bar{\rho}_2: H_1(S, \mathbb{Z}) \rightarrow \mathbb{T}^n$ be the induced representations. Suppose there is $\phi \in \text{Mod}(S)$ such that $\rho_2 = \rho_1 \circ \phi$. Then $\bar{\rho}_2 = \bar{\rho}_1 \circ \mu(\phi)$, where μ is the symplectic representation of $\text{Mod}(S)$.*

Proof. Let $\phi: \pi_1(S) \rightarrow \pi_1(S)$ be any element of $\text{Out}(\pi_1(S))$. As the image of any commutator is also a commutator, the mapping ϕ boils down to an isomorphism in homology $\mu(\phi): H_1(S, \mathbb{Z}) \rightarrow H_1(S, \mathbb{Z})$. Two mappings ϕ_1 and ϕ_2 boil down to the same isomorphism in homology if and only if $\phi_2 \circ \phi_1^{-1}$ descends to the identity map in homology, that is, $\phi_2 \circ \phi_1^{-1}$ is an element of the Torelli subgroup by a theorem of Johnson, [13]. Therefore, the association $\phi \mapsto \mu(\phi)$ defines the symplectic representation μ seen above. Look at the commutative diagram

$$\begin{array}{ccccc}
 H_1(S, \mathbb{Z}) & \xrightarrow{\bar{\rho}_2 = \overline{\rho_1 \circ \phi}} & & & \mathbb{T}^n \\
 p \uparrow & & & & \text{id} \uparrow \\
 \pi_1(S) & \xrightarrow{\phi} & \pi_1(S) & \xrightarrow{\rho_1} & \mathbb{T}^n \\
 \downarrow p & & \downarrow & & \downarrow \text{id} \\
 H_1(S, \mathbb{Z}) & \xrightarrow{\mu(\phi)} & H_1(S, \mathbb{Z}) & \xrightarrow{\bar{\rho}_1} & \mathbb{T}^n,
 \end{array}$$

where p is the canonical projection. As $\rho_2 = \rho_1 \circ \phi$ by assumption, it turns out $\bar{\rho}_2 = \overline{\rho_1 \circ \phi} = \bar{\rho}_1 \circ \mu(\phi)$ as desired. ■

2.2.3. Direct consequences. Proposition 2.7 leads to some interesting consequences that we are going to show. The first one concerns the action of the Torelli subgroup $\mathcal{I}(S)$ on the representation space $\text{Hom}(\pi_1(S), \mathbb{T}^n)$.

Proposition 2.8. *The action of $\mathcal{I}(S)$ on the representation space $\text{Hom}(\pi_1(S), \mathbb{T}^n)$ is trivial.*

Proof. Let $\rho_1 \in \text{Hom}(\pi_1(S), \mathbb{T}^n)$ be any representation and let $\phi \in \mathcal{I}(S)$. Set $\rho_2 = \phi \cdot \rho_1 = \rho_1 \circ \phi^{-1}$. Proposition 2.7 implies that $\bar{\rho}_1 = \bar{\rho}_2$ because $\mu(\phi) = 1$. We now invoke Lemma 2.3 to conclude $\rho_1 = \rho_2$, namely the action of ϕ is trivial. ■

Remark 2.9. An alternative argument is the following. Let γ and γ' be two isotopy classes of simple closed non-separating curves. In [13], Johnson noticed that γ and γ' are $\mathcal{I}(S)$ -equivalent if and only if they represent the same element in $H_1(S, \mathbb{Z})$. Fix a basis for $\pi_1(S)$ of simple closed non-separating curves, and let ρ_1, ρ_2 be two $\mathcal{I}(S)$ -equivalent representations, that is, $\rho_2 = \rho_1 \circ \phi$. For any generator γ , we have the following chain of equalities:

$$\rho_2(\gamma) = \bar{\rho}_2([\gamma]) = \overline{\rho_1 \circ \phi^{-1}([\gamma])} = \bar{\rho}_1(\phi^{-1}([\gamma])) = \bar{\rho}_1([\gamma]) = \rho_1(\gamma),$$

that imply $\rho_1 = \rho_2$ as desired.

The n -torus \mathbb{T}^n is a compact and connected Lie group, and hence mapping class group $\text{Mod}(S)$ acts ergodically on the representation space, see Theorem 1.2 in the introduction. As the action of the Torelli subgroup $\mathcal{I}(S)$ is trivial, the action of the quotient group is also well-defined and the following holds.

Proposition 2.10. *The action of*

$$\text{Sp}(2g, \mathbb{Z}) \cong \frac{\text{Mod}(S)}{\mathcal{I}(S)}$$

on $\text{Hom}(\pi_1(S), \mathbb{T}^n)$ is ergodic with respect to the finite measure μ_S .

As the homological representation spaces identifies with the representation space, it also carries a finite measure. Calling ι the identifying map, this finite measure can be seen as the pullback measure $\iota^* \mu_S$, where μ_S is the finite measure carried by the representation space.

Corollary 2.11. *The action of $\text{Sp}(2g, \mathbb{Z})$ on the space $\text{Hom}(H_1(S, \mathbb{Z}), \mathbb{T}^n)$ is ergodic with respect to the finite measure $\iota^* \mu_S$.*

As a final consequence, we have the following characterisation.

Proposition 2.12. *Let $\rho: \pi_1(S) \rightarrow \mathbb{T}^n$ be a representation and let $\bar{\rho}: H_1(S, \mathbb{Z}) \rightarrow \mathbb{T}^n$ be the homological representation induced by ρ . Then the mapping class group orbit $\text{Mod}(S) \cdot \rho$ is dense if and only if the symplectic group orbit $\text{Sp}(2g, \mathbb{Z}) \cdot \bar{\rho}$ is dense.*

Proof. Proposition 2.7 implies that the mapping class group orbit of ρ coincides with the symplectic group orbit of $\bar{\rho}$ via the identification $\rho \mapsto \bar{\rho}$. Therefore, one orbit is dense if and only if the other is dense. ■

Corollary 2.13. *Let S be a surface of genus $g \geq 1$. Then Theorem A holds if and only if Theorem B holds.*

2.3. The matrix presentation

The n -torus \mathbb{T}^n is also seen as the quotient of \mathbb{R}^n by the action of the lattice $2\pi\mathbb{Z}^n$, indeed the exponential map provides an identification between $\mathbb{R}^n/2\pi\mathbb{Z}^n$ and the n -torus

described above. We shall define the map

$$\exp: \mathbb{R}^n \rightarrow \mathbb{T}^n, \quad (\theta_1, \dots, \theta_n) \mapsto (e^{i\theta_1}, \dots, e^{i\theta_n}) \quad (2.1)$$

as the *canonical projection*. In the sequel, it turns out also useful to look at the n -torus as the quotient of \mathbb{R}^n with a suitable lattice $\Lambda = g \cdot (2\pi\mathbb{Z}^n)$, where $g \in \mathrm{SL}(n, \mathbb{Z})$. The reason of that will be discussed afterwards. We define $2\pi\mathbb{Z}^n$ as the *the standard lattice* – notice that this lattice is 2π times the usual standard lattice.

Fix a set of generators $\{\alpha_1, \beta_1, \dots, \alpha_g, \beta_g\}$ and consider \mathbb{T}^n as the quotient of \mathbb{R}^n by the action of the standard lattice. Let $\rho: \pi_1(S) \rightarrow \mathbb{T}^n$ be any representations and set

$$\rho(\alpha_i) = (e^{i\theta_{1,2i-1}}, \dots, e^{i\theta_{n,2i-1}}), \quad \rho(\beta_i) = (e^{i\theta_{1,2i}}, \dots, e^{i\theta_{n,2i}})$$

for any $i = 1, \dots, n$. The elements $\rho(\alpha_1), \rho(\beta_1), \dots, \rho(\alpha_g), \rho(\beta_g)$ generate the image of the representation ρ . Any generic element $\gamma \in \pi_1(S)$ may be seen as a word in the letters α_i, β_i for $i = 1, \dots, 2g$. Hence, since \mathbb{T}^{2g} is abelian,

$$\rho(\gamma) = \rho(w(\alpha_1, \beta_1, \dots, \alpha_g, \beta_g)) = \rho(\alpha_1)^{k_1} \dots \rho(\beta_g)^{k_{2g}}$$

for some $k_1, \dots, k_{2g} \in \mathbb{Z}$. In particular, the element $\rho(\gamma)$ can be computed with the following matrix multiplication:

$$\begin{pmatrix} \theta_{1,1} & \cdots & \theta_{1,i} & \cdots & \theta_{1,2g} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ \theta_{n,1} & \cdots & \theta_{n,i} & \cdots & \theta_{n,2g} \end{pmatrix} \begin{pmatrix} k_1 \\ \vdots \\ k_i \\ \vdots \\ k_{2g} \end{pmatrix}.$$

Definition 2.14. Let Θ_ρ be the matrix having as entries the values $\theta_{i,j} \in [0, 2\pi)$ with $i = 1, \dots, n$ and $j = 1, \dots, 2g$. We define Θ_ρ as *the matrix associated to ρ* with respect to the basis $\{\alpha_1, \beta_1, \dots, \alpha_g, \beta_g\}$ and the standard lattice $2\pi\mathbb{Z}^n$.

In what follows, we shall often identify a representation ρ with its associated matrix. Let us briefly see the reason why we are legitimated to do that. Consider the topological vector space $\mathrm{M}(n, 2g; \mathbb{R})$ and introduce an equivalence relation where $A \sim B$ if and only if $A - B = 2\pi H \in \mathrm{M}(n, 2g; 2\pi\mathbb{Z})$. The mapping ι associating to any ρ its associated matrix Θ_ρ provides an homeomorphism between the representation space $\mathrm{Hom}(\pi_1(S), \mathbb{T}^n)$ and the quotient space $\mathrm{M}(n, 2g, \mathbb{T})$. Moreover, the post-composition of the mapping $\rho \mapsto \bar{\rho}$ with ι^{-1} defines a homeomorphism between the spaces $\mathrm{Hom}(H_1(S, \mathbb{Z}), \mathbb{T}^n)$ and $\mathrm{M}(n, 2g, \mathbb{T})$.

Given a representation ρ , the matrix Θ_ρ depends on the choice of a set of generators for $\pi_1(S)$ and also on the choice of a lattice $\Lambda < \mathbb{R}^n$. Let us see how these choices affect Definition 2.14. We begin describing the effect of changing the set of generators of $\pi_1(S)$.

2.3.1. The effect of changing basis. Given two basis $\mathcal{B} = \{\alpha_1, \beta_1, \dots, \alpha_g, \beta_g\}$ and $\mathcal{B}' = \{\alpha'_1, \beta'_1, \dots, \alpha'_g, \beta'_g\}$ of $\pi_1(S)$, we define Θ_ρ and Θ'_ρ the matrices associated to $\rho: \pi_1(S) \rightarrow \mathbb{T}^n$ with respect to \mathcal{B} and \mathcal{B}' , respectively. Every generator α'_l and β'_l is a finite word in the letters $\alpha_1, \beta_1, \dots, \alpha_g, \beta_g$, so there are integers a_{ij} with $i, j \in \{1, \dots, 2g\}$ such that

$$\rho(\alpha'_l) = \rho(\alpha_1)^{a_{2l-11}} \dots \rho(\beta_g)^{a_{2l-12g}} \quad \text{and} \quad \rho(\beta'_l) = \rho(\alpha_1)^{a_{2l1}} \dots \rho(\beta_g)^{a_{2l2g}}.$$

Setting A as the integral matrix (a_{ij}) with $i, j \in \{1, \dots, 2g\}$, a direct computation shows that Θ'_ρ equals $\Theta_\rho \cdot A$. Likewise, α_l, β_l are also finite words in the letters $\alpha'_1, \beta'_1, \dots, \alpha'_g, \beta'_g$. Hence, there exist integers b_{ij} such that

$$\rho(\alpha_l) = \rho(\alpha'_1)^{b_{2l-11}} \dots \rho(\beta'_g)^{b_{2l-12g}} \quad \text{and} \quad \rho(\beta_l) = \rho(\alpha'_1)^{b_{2l1}} \dots \rho(\beta'_g)^{b_{2l2g}}.$$

Setting B as the integral matrix (b_{ij}) with $i, j \in \{1, \dots, 2g\}$, the same computation implies Θ_ρ equals $\Theta'_\rho \cdot B$.

It worth noticing $\Theta_\rho = \Theta'_\rho \cdot AB$ and the matrices A, B satisfy the equation $AB = I_{2g}$ implying that A, B are unimodular. As the matrix Θ_ρ can be singular, we cannot directly deduce that $AB = I_{2g}$, hence let us give a glimpse of why this is true.

Instead of working in $\pi_1(S)$, we look at the situation in the first homology group $H_1(S, \mathbb{Z}) \cong \mathbb{Z}^{2g}$. Let us consider α_1 as a word $w(\alpha'_1, \beta'_1, \dots, \alpha'_g, \beta'_g)$; then

$$[\alpha_1] = [\alpha'_1]^{b_{11}} [\beta'_1]^{b_{12}} \dots [\alpha'_g]^{b_{12g-1}} [\beta'_g]^{b_{12g}},$$

where b_{lj} with $j = 1, \dots, 2g$ are as above. On the other hand, any $[\alpha'_l]$ and $[\beta'_l]$ are of the form

$$\begin{aligned} [\alpha'_l] &= [\alpha_1]^{a_{2l-11}} [\beta_1]^{a_{2l-12}} \dots [\alpha_g]^{a_{2l-12g-1}} [\beta_g]^{a_{2l-12g}}, \\ [\beta'_l] &= [\alpha_1]^{a_{2l1}} [\beta_1]^{a_{2l2}} \dots [\alpha_g]^{a_{2l2g-1}} [\beta_g]^{a_{2l2g}}, \end{aligned}$$

where a_{ij} are as above. Replacing each $[\alpha'_l]$ and $[\beta'_l]$ inside $[w] = [\alpha_1]$, for any $l = 1, \dots, g$, we obtain

$$[\alpha_1] = [\alpha_1]^{k_1} [\beta_1]^{k_2} \dots [\alpha_g]^{k_{2g-1}} [\beta_g]^{k_{2g}}.$$

As \mathbb{Z}^{2g} is torsion-free, we can deduce that $k_1 = 1$ and $k_2 = \dots = k_{2g} = 0$. On the other hand, it is straightforward to see that $k_m = \sum_{r=1}^{2g} b_{1r} a_{rm}$. Applying the same reasoning to any other generator, we get the desire conclusion.

Remark 2.15. The matrices A and B found above may not have any geometrical meaning. Indeed, for closed surfaces the action of $\text{Aut}(\pi_1(S))$ is not transitive on the set of basis of $\pi_1(S)$, and then two different basis may not be related by any automorphisms of $\pi_1(S)$. This means that not all matrices in $\text{SL}(2g, \mathbb{Z})$ have a geometrical interpretation. As we shall see, a matrix has a geometrical meaning, that is, induced by a homeomorphism of S , if and only if it is symplectic; see Proposition 2.10 above.

2.3.2. The effect of changing the basis of the lattice. We begin noticing that the j -th column of the matrix Θ_ρ corresponds to the vector of coordinates of a lift of the j -th generator of $\rho(\pi_1(S))$ with respect to the standard lattice. Given any lattice Λ with basis $\{v_1, \dots, v_n\}$, there is a matrix $g \in \mathrm{SL}(n, \mathbb{Z})$ such that $\Lambda = g \cdot (2\pi\mathbb{Z}^n)$. In particular, $g(e_i) = v_i$. To change the basis means to change the coordinates of the vectors forming the columns of the matrix Θ_ρ . Therefore, with respect to the lattice Λ , the matrix associated to ρ has the form $g\Theta_\rho$. In the sequel, we shall need to consider the matrix Θ_ρ with respect to a lattice Λ different to the standard one. We therefore extend the notation in the following way: We denote by $\Theta_\rho(\Lambda)$ the associated matrix with respect to the lattice Λ . We shall use again the notation Θ_ρ when the lattice is the standard one.

2.3.3. The \mathbb{Z} -row rank of the associated matrix. We now introduce the following numerical invariant concerning the associated matrix Θ_ρ . As we shall see, such an invariant give us a way to characterise dense representations in \mathbb{T}^n completely.

Definition 2.16. Let $M \in \mathrm{M}(n, m; \mathbb{R})$. We define the \mathbb{Z} -row rank of M as the dimension of the \mathbb{Z} -module generated by the rows of M . We denote it by $\mathrm{rk}_{\mathbb{Z}}(M)$.

We observe that the \mathbb{Z} -row rank is not invariant by transposition.

Lemma 2.17. Let $M \in \mathrm{M}(n, m; \mathbb{R})$. The \mathbb{Z} -row rank $\mathrm{rk}_{\mathbb{Z}}(M)$ of M is invariant under the left action of $\mathrm{SL}(n, \mathbb{Z})$. Similarly, $\mathrm{rk}_{\mathbb{Z}}(M)$ is invariant under the right action of $\mathrm{SL}(m, \mathbb{Z})$.

Proof. We prove the first claim. Let $k = \mathrm{rk}_{\mathbb{Z}}(M) \leq n$. Define Z as the subset of \mathbb{Z}^n of those vectors v such that $vM = 0$. Notice that Z is a \mathbb{Z} -module of dimension $n - k$. Let A be any matrix in $\mathrm{SL}(n, \mathbb{Z})$, and we compute AM . It is easy to check that the j -th row is given by the linear combination $\sum_{i=1}^n a_{ji}(m_{i1}, \dots, m_{im})$. Suppose there is a vector $\mu = (\mu_1, \dots, \mu_n)$ such that $\mu AM = 0$, then a straightforward computation shows that $\mu A \in Z$, that is, $\mu = vA^{-1}$ for some $v \in Z$. Therefore, the subset $Z \cdot A^{-1} \subset \mathbb{Z}^n$ is the set of vectors μ such that $\mu AM = 0$ and it has dimension $n - k$ over \mathbb{Z} . Therefore, $\mathrm{rk}_{\mathbb{Z}}(AM) = k$. Similarly, the second claim follows applying an analogous reasoning. ■

Given a representation $\rho: \pi_1(S) \rightarrow \mathbb{T}^n$, the following claims are direct consequences of the lemma above applied to the matrix Θ_ρ .

Corollary 2.18. Let $\rho: \pi_1(S) \rightarrow \mathbb{T}^n$ be a representation and let Θ_ρ be the matrix associated to ρ with respect to some basis of $\pi_1(S)$. The \mathbb{Z} -row rank of Θ_ρ is well-defined and it does not depend on any choice of a basis for $\pi_1(S)$ nor on the choice of any lattice.

Let v_1, \dots, v_k be vectors in \mathbb{R}^n . In the sequel, we will use the following definition.

Definition 2.19. We say that a \mathbb{Z} -module generated by v_1, \dots, v_k is $\pi\mathbb{Q}$ -free if and only if

$$\langle v_1, \dots, v_k \rangle_{\mathbb{Z}} \cap \pi\mathbb{Q}^n = \{(0, \dots, 0)\}.$$

Keeping this definition in mind, we finally state the following proposition.

Proposition 2.20. *Let $\rho: \pi_1(S) \rightarrow \mathbb{T}^n$ be a representation and let Θ_ρ be the matrix associated to ρ with respect to some basis of $\pi_1(S)$. The \mathbb{Z} -module $\langle \Theta_j: j = 1, \dots, n \rangle_{\mathbb{Z}}$ generated by the rows of Θ_ρ is $\pi\mathbb{Q}$ -free if and only if the \mathbb{Z} -module $\langle (A\Theta_\rho)_j: j = 1, \dots, n \rangle_{\mathbb{Z}}$ generated by the rows of $A\Theta_\rho$ is $\pi\mathbb{Q}$ -free, where $A \in \mathrm{SL}(n, \mathbb{Z})$. Similarly, $\langle \Theta_j: j = 1, \dots, n \rangle_{\mathbb{Z}}$ is $\pi\mathbb{Q}$ -free if and only if the \mathbb{Z} -module $\langle (\Theta_\rho B)_j: j = 1, \dots, n \rangle_{\mathbb{Z}}$ is $\pi\mathbb{Q}$ -free, where $B \in \mathrm{SL}(2g, \mathbb{Z})$.*

Proof. Look at the matrix $A\Theta_\rho$ and suppose there are $\lambda_1, \dots, \lambda_n \in \mathbb{Z}$ such that

$$\sum_{j=1}^n \lambda_j \left(\sum_{i=1}^n a_{ji}(\theta_{i,1}, \dots, \theta_{i,2g}) \right) \in \pi\mathbb{Q}^{2g}.$$

A simple manipulation of the formula above shows that

$$\sum_{j=1}^n \lambda_j \left(\sum_{i=1}^n a_{ji}(\theta_{i,1}, \dots, \theta_{i,2g}) \right) = \sum_{i=1}^n \left(\sum_{j=1}^n \lambda_j a_{ji} \right) (\theta_{i,1}, \dots, \theta_{i,2g}),$$

implying the existence of some $\mu_1, \dots, \mu_n \in \mathbb{Z}$ such that

$$\sum_{j=1}^n \mu_j (\theta_{i,1}, \dots, \theta_{i,2g}) \in \pi\mathbb{Q}^{2g}.$$

The proof of the second claim works similarly: Suppose there are $\lambda_1, \dots, \lambda_n \in \mathbb{Z}$ such that

$$\sum_{i=1}^n \lambda_i \left(\sum_{j=1}^{2g} \theta_{i,j}(b_{j1}, \dots, b_{j2g}) \right) \in \pi\mathbb{Q}^{2g}.$$

The same manipulation shows that

$$\sum_{i=1}^n \lambda_i \left(\sum_{j=1}^{2g} \theta_{i,j}(b_{j1}, \dots, b_{j2g}) \right) = \sum_{j=1}^{2g} \left(\sum_{i=1}^n \lambda_i \theta_{i,j} \right) (b_{j1}, \dots, b_{j2g}),$$

implying $\sum_{i=1}^n \lambda_i \theta_{i,j} \in \pi\mathbb{Q}$ for any $j = 1, \dots, 2g$. That is, $(\lambda_1, \dots, \lambda_n)\Theta_\rho \in \pi\mathbb{Q}^{2g}$. ■

2.4. Remarks and comments on the modular action

In this section, we collect a couple of final remarks about the $\mathrm{Sp}(2g, \mathbb{Z})$ -action.

2.4.1. Explicit description of the modular action. The action of the mapping class group on the representation space is defined by pre-composition of any representation with an automorphism $\phi \in \mathrm{Out}(\pi_1(S))$, namely any representation ρ is sent to $\rho \circ \phi^{-1}$ under this action. Since the Torelli group acts trivially on the representation space, the action of mapping class group boils down to an action of the group $\mathrm{Sp}(2g, \mathbb{Z})$ which agrees with the $\mathrm{Sp}(2g, \mathbb{Z})$ -action on $\mathrm{Hom}(H_1(S, \mathbb{Z}), \mathbb{T}^n)$; this is a consequence of Proposition 2.8.

In Section 2.3, we have identified the representation space with $M(n, 2g; \mathbb{T})$ by using the mapping ι associating to any representation ρ its matrix Θ_ρ . We use such a mapping to transfer the action of $\text{Mod}(S)$ on $\text{Hom}(\pi_1(S), \mathbb{T}^n)$ to an action of $\text{Sp}(2g, \mathbb{Z})$ on $M(n, 2g; \mathbb{T})$. Since any $\phi \in \mathcal{I}(S)$ leaves ρ fixed, the matrix associated to $\rho' = \phi \cdot \rho = \rho \circ \phi^{-1}$ agrees with Θ_ρ ; this is a consequence of Proposition 2.8. Any coset $\phi \mathcal{I}(S)$ defines a unique matrix A in $\text{Sp}(2g, \mathbb{Z})$. In the light of the discussion given at Section 2.3.1, the matrix associated to $\rho' = \phi \cdot \rho$ is $\Theta_{\rho'} = \Theta_\rho A^{-1}$. Therefore, the action of $\text{Sp}(2g, \mathbb{Z})$ on $M(n, 2g; \mathbb{T})$ is defined as $A \cdot \Theta_\rho = \Theta_\rho A^{-1}$. As the mapping ι is a homeomorphism, it is clear that $\text{Sp}(2g, \mathbb{Z})$ -orbit of ρ is dense in the representation space if and only if the $\text{Mod}(S)$ -orbit of Θ_ρ is dense in $M(n, 2g; \mathbb{T})$.

2.4.2. The modular action commutes with the change of lattice. Given a representation $\rho: \pi_1(S) \rightarrow \mathbb{T}^n$, the main goal of the present paper is to study its orbit under the action of the mapping class group. This reduces to studying the orbit of the matrix Θ_ρ naturally attached to ρ in the space $M(n, 2g; \mathbb{T})$. However, the matrix Θ_ρ depends on the lattice chosen, and hence the orbit could depend on the chosen lattice. The aim of this paragraph is to point out that this is not the case; indeed, the change of lattice commutes with the modular action. Each element Θ in the space $M(n, 2g; \mathbb{T})$ can be thought as the data of n vectors $\Theta_i \in \mathbb{T}^{2g}$ corresponding to the rows of Θ . By adopting this point of view, the space $M(n, 2g; \mathbb{R})$ identifies with $\mathbb{T}^{2g} \times \dots \times \mathbb{T}^{2g}$. There is a left action of the group G defined as

$$G = \left\{ \begin{pmatrix} A & & \\ & \ddots & \\ & & A \end{pmatrix} : A \in \text{Sp}(2g, \mathbb{Z}) \right\} \cong \text{Sp}(2g, \mathbb{Z}) < \text{SL}(2gn, \mathbb{Z})$$

on the $2gn$ -dimensional torus induced by the natural right action of the symplectic group in the matrix space $M(n, 2g; \mathbb{T})$. Using this new perspective, one can easily verify that any change of the action of $\text{Sp}(2g, \mathbb{Z})$. Indeed, any change of lattice $h \in \text{SL}(n, \mathbb{Z})$ can be seen as an element of the group H defined as

$$H = \left\{ \begin{pmatrix} h_{11} \mathbf{I}_{2g} & \cdots & h_{1n} \mathbf{I}_{2g} \\ \vdots & \ddots & \vdots \\ h_{n1} \mathbf{I}_{2g} & \cdots & h_{nn} \mathbf{I}_{2g} \end{pmatrix}, \text{ where } \begin{pmatrix} h_{11} & \cdots & h_{1n} \\ \vdots & \ddots & \vdots \\ h_{n1} & \cdots & h_{nn} \end{pmatrix} \in \text{SL}(n, \mathbb{Z}) \right\}$$

$$\cong \text{SL}(n, \mathbb{Z}) < \text{SL}(2gn, \mathbb{Z}).$$

Since H commutes with the group G defined above, the action of $\text{Sp}(2g, \mathbb{Z})$ commutes with the change of lattice.

3. Characterising dense representations

In this section, we provide a complete characterisation of *dense representations* by providing necessary and sufficient conditions. From Section 1, we recall that a representation

$\rho: \pi_1(S) \rightarrow \mathbb{T}^n$ is *dense* if the subgroup $\rho(\pi_1(S))$ is dense in \mathbb{T}^n . We have seen in the previous section that, upon choosing a basis of the fundamental group and a lattice, each representation is represented by a well-define matrix Θ_ρ . Along this section, we fix an arbitrary basis for the fundamental group and consider \mathbb{T}^n as the quotient of \mathbb{R}^n with the standard lattice.

Theorem 3.1. *Let $\rho: \pi_1(S) \rightarrow \mathbb{T}^n$ be a representation and let Θ_ρ be its associated matrix. Then ρ is dense in \mathbb{T}^n if and only if $\text{rk}_{\mathbb{Z}}(\Theta_\rho) = n$ and the rows of Θ_ρ generate a $\pi\mathbb{Q}$ -free \mathbb{Z} -module.*

We can notice that the necessary condition means that the \mathbb{Z} -module generated by the rows of the matrix Θ_ρ does not intersect $\pi\mathbb{Q}^{2g}$ and is equivalent to say that row rank over \mathbb{Z} of the matrix

$$\begin{pmatrix} \Theta_\rho \\ \pi \cdot \mathbf{I}_{2g} \end{pmatrix} = \begin{pmatrix} \theta_{1,1} & \cdots & \theta_{1,2g} \\ \vdots & \ddots & \vdots \\ \theta_{n,1} & \cdots & \theta_{n,2g} \\ \pi & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \pi \end{pmatrix} \quad (3.1)$$

is maximal, namely $2g + n$. Before proving the theorem, we need a preliminary lemma.

Lemma 3.2. *Suppose that $\rho: \pi_1(S) \rightarrow \mathbb{T}^n$ is dense. Then each representation*

$$\rho_k = \pi_k \circ \rho,$$

where π_k is the projection to k -th factor, is dense.

Proof. Suppose there is k for which the representation ρ_k is not dense. Then there is an open subset $A \subset \mathbb{S}^1$ such that $A \cap \rho_k(\pi_1(S)) = \emptyset$. Suppose without loss of generality that $k = 1$. Then

$$(A \times \mathbb{T}^{n-1}) \cap \rho(\pi_1(S)) = \emptyset.$$

In particular, ρ is not dense, hence a contradiction. ■

Proof of Theorem 3.1. Assume ρ has a dense image and suppose the \mathbb{Z} -module generated by the rows intersect $\pi\mathbb{Q}^{2g}$, that is,

$$\text{rk}_{\mathbb{Z}} \begin{pmatrix} \Theta_\rho \\ \pi \cdot \mathbf{I}_{2g} \end{pmatrix} < 2g + n.$$

Thus, there is a row Θ_i of Θ such that

$$\sum_{\substack{j=1 \\ j \neq i}}^n \lambda_j \Theta_j + (\lambda_{n+1}\pi, \dots, \lambda_{n+2g}\pi) = \lambda_i \Theta_i$$

with λ_i different to zero. Such a summation can be rewritten as

$$\sum_{j=1}^n \lambda_j \Theta_j = (\lambda_{n+1}\pi, \dots, \lambda_{n+2g}\pi)$$

for some $\lambda_j \in \mathbb{Z}$ and not all zero. Consider the matrix $M \in M(n, \mathbb{Z}) \cap GL(n, \mathbb{Q})$ defined as

$$M = I_n + (\lambda_i - 1)E_{ii} - \sum_{\substack{j=1 \\ j \neq i}}^n \lambda_j E_{ij} = \begin{pmatrix} 1 & 0 & \cdots & \cdots & \cdots & 0 \\ 0 & 1 & \cdots & \cdots & \cdots & \cdots \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ -\lambda_1 & -\lambda_2 & \cdots & \lambda_i & \cdots & -\lambda_n \\ \cdots & \cdots & \cdots & \cdots & 1 & 0 \\ 0 & 0 & \cdots & \cdots & 0 & 1 \end{pmatrix},$$

where the $E_{ij} = (e_{kl})$ are the matrices with coefficients $e_{kl} = \delta_{ki}\delta_{lj}$. The matrix M defines a linear homeomorphism, say f_M , of \mathbb{R}^n with respect to the canonical basis because $\det M = -\lambda_i$ which is different from zero. The mapping f_M sends \mathbb{Z}^n to itself and descends to a finite-degree covering $\bar{f}_M: \mathbb{T}^n \rightarrow \mathbb{T}^n$ – in fact the degree coincides with the determinant of M . In particular, the following equation holds: $\pi \circ f_M = \bar{f}_M \circ \exp$, where $\exp: \mathbb{R}^n \rightarrow \mathbb{T}^n$ denotes as usual the canonical projection. Consider now the \mathbb{Z} -module $\langle \Theta_j: j = 1, \dots, n \rangle_{\mathbb{Z}}$ generated by the rows of Θ_ρ . A straightforward computation shows that its image via the mapping f_M is the \mathbb{Z} -module generated by the vectors

$$\left\langle \begin{pmatrix} \theta_{1,1} \\ \vdots \\ \lambda_{n+1}\pi \\ \vdots \\ \theta_{n,1} \end{pmatrix}, \dots, \begin{pmatrix} \theta_{1,2g} \\ \vdots \\ \lambda_{n+2g}\pi \\ \vdots \\ \theta_{n,2g} \end{pmatrix} \right\rangle_{\mathbb{Z}}.$$

Let us point out the following fact: As ρ is assumed to be dense in the torus, the image via the canonical projection in \mathbb{T}^n of \mathbb{Z} -module generated by the rows of Θ_ρ fills a dense subset of \mathbb{T}^n , namely the image of ρ . As M commutes with the action of $2\pi\mathbb{Z}^n$ and pass through to the quotient as a finite-degree covering map of the \mathbb{T}^n , the \mathbb{Z} -module $M \cdot \langle \Theta_j: j = 1, \dots, n \rangle_{\mathbb{Z}}$ is mapped on a dense subset of the torus. On the other hand, the projection of the i -th factor is discrete. Lemma 3.2 implies the desire contradiction.

We now prove the opposite implication and again argue by contradiction. Suppose ρ does not have a dense image in the n -torus, then its closure is a k -dimensional submanifold, say S_0 , of dimension $k < n$. We note that S_0 may not be connected in general. Indeed, any closed subgroup of \mathbb{T}^n is homeomorphic to $\mathbb{T}^d \times \frac{\mathbb{Z}}{m_1\mathbb{Z}} \times \cdots \times \frac{\mathbb{Z}}{m_{n-d}\mathbb{Z}}$, that is, a finite collection of inhomogeneous tori. Assume first S_0 be connected; we shall deduce the general case later on. The subspace S_0 lifts to a linear subspace \tilde{S}_0 of \mathbb{R}^n which of course contains the \mathbb{Z} -module $\langle \Theta^j: j = 1, \dots, 2g \rangle_{\mathbb{Z}}$ generated by the columns of Θ_ρ . We now invoke the following lemma.

Lemma 3.3. *There is $g \in \mathrm{SL}(n, \mathbb{Z})$ such that*

$$g \cdot \langle \Theta^j : j = 1, \dots, 2g \rangle_{\mathbb{Z}} < \langle e_1, \dots, e_k \rangle_{\mathbb{R}},$$

where the e_i 's are the vectors of the canonical basis of \mathbb{R}^n .

Assume the lemma holds. The \mathbb{Z} -module $g \cdot \langle \Theta^j : j = 1, \dots, 2g \rangle_{\mathbb{Z}}$ is contained in the first factor of $\mathbb{T}^n = \mathbb{T}^k \times \mathbb{T}^{n-k}$, and then Θ_ρ cannot have maximal row rank over \mathbb{Z} . As a consequence, the matrix given in equation (3.1) cannot have maximal row rank over \mathbb{Z} . The general case follows by applying the same reasoning to the component S_0° of the subspace S_0 containing the identity, which contains a finite-index \mathbb{Z} -module of $\langle \Theta^j : j = 1, \dots, 2g \rangle_{\mathbb{Z}}$. In the general case, $g \cdot \langle \Theta^j : j = 1, \dots, 2g \rangle_{\mathbb{Z}}$ is contained in $\mathbb{T}^n = \mathbb{T}^k \times F$, where F is isomorphic to the finite group $\frac{\mathbb{Z}}{m_1\mathbb{Z}} \times \dots \times \frac{\mathbb{Z}}{m_{n-d}\mathbb{Z}}$. Let us proceed with the proof of Lemma 3.3.

Proof of Lemma 3.3. If \tilde{S}_0° is contained in $\langle e_{\sigma(1)}, \dots, e_{\sigma(k)} \rangle_{\mathbb{R}}$ for some $\sigma \in \mathfrak{S}_n$, then it is sufficient to rename the coordinates. This corresponds to a matrix g obtained by product of elementary matrices. Assume \tilde{S}_0° is not contained in any such a space. Let x_i be the intersection of \tilde{S}_0° with the affine space $e_i + \mathbb{R}^{n-k}$, and let d_i be its Euclidean distance to \mathbb{R}^d . Then x_i has the following form:

$$x_i = (0, \dots, 1, \dots, 0, t_1, \dots, t_{n-k}),$$

where $t_i \in \mathbb{Q}$. In fact, if this had been not true, then S_0° would have been a dense subspace of dimension $k + 1$ in the torus. As a consequence, $d_i \in \mathbb{Q}$ for any $i = 1, \dots, k$ and \tilde{S}_0° is described by $n - k$ equations with integer coefficients. Look at the set $\tilde{S}_0^{\circ} \cap \mathbb{Z}^n$. This is a lattice in \tilde{S}_0° , and there is a basis v_1, \dots, v_k made of integer vectors. We invoke [6, p. 14, Corollary 3] to claim the existence of $n - k$ vectors v_{k+1}, \dots, v_n such that the vectors v_1, \dots, v_n gathered together form a basis for \mathbb{Z}^n . Since $\mathrm{SL}(n, \mathbb{Z})$ acts transitively on the space of lattices, there is g such that

$$g \cdot \langle \Theta^j : j = 1, \dots, 2g \rangle_{\mathbb{Z}} < g \cdot \tilde{S}_0^{\circ} = \langle e_1, \dots, e_k \rangle_{\mathbb{R}}.$$

This concludes the proof of Lemma 3.3 and indeed the proof of Theorem 3.1. ■

From the proof, we deduce that the row rank of the matrix Θ_ρ has a very explicit geometric interpretation, in fact it coincides with the dimension of the subspace containing the image of ρ . Of course, the proof does not depend on the presentation of $\pi_1(S)$ nor on the lattice chosen. Let us prove these facts.

Proof of independence on the chosen basis. Let $\langle \Theta^j : j = 1, \dots, 2g \rangle_{\mathbb{Z}}$ be the \mathbb{Z} -module generated by the columns of Θ_ρ . In Section 2.3.1, we have seen that the effect of changing a basis of generators corresponds to multiply on the right the matrix Θ_ρ with a matrix $A \in \mathrm{SL}(2g, \mathbb{Z})$. Since the row rank of Θ_ρ is invariant under the action by right multiplication of $\mathrm{SL}(2g, \mathbb{Z})$, the matrices Θ_ρ and $\Theta_\rho A$ have the same row rank. Furthermore, in the light of

Corollary 2.20, $\pi\mathbb{Q}$ -freedom is also invariant under the right action of $\mathrm{SL}(2g, \mathbb{Z})$. On the other hand, let S be the closure of the subspace of \mathbb{T}^n generated by the columns of Θ_ρ . Its lift \tilde{S} is a linear (possibly improper) subspace of \mathbb{R}^n described by $n - k$ equations. As the columns of $\Theta_\rho A$ satisfy the same equations, the image is unaffected by the change of basis. This proves the independence on the basis chosen. ■

Proof of independence on the chosen lattice. Given two lattices Λ_1 and Λ_2 , there always exists an element of $A \in \mathrm{SL}(n, \mathbb{Z})$ mapping the first lattice on the second one because the action of $\mathrm{SL}(n, \mathbb{Z})$ is transitive on the set of lattices. Such a map descends to a homeomorphism of the n -torus and hence the \mathbb{Z} -module $\langle \Theta^j : j = 1, \dots, 2g \rangle_{\mathbb{Z}}$ projects to a dense subset of the torus if and only if its image via A projects to dense subset as well. On the other hand, it is immediate to verify that the row rank of the matrix $\Theta_\rho(\Lambda_1)$ equals the one of $\Theta_\rho(\Lambda_2)$ because the row rank is invariant under the action by left multiplication of $\mathrm{SL}(n, \mathbb{Z})$. Again, Corollary 2.20 implies $\pi\mathbb{Q}$ -freedom is invariant under the left action of $\mathrm{SL}(n, \mathbb{Z})$. Hence the conclusion. ■

We finally provide a couple of explicit examples.

Example 3.4. Let S be a surface of genus 2, and let $\rho: \pi_1(S) \rightarrow \mathbb{T}^2 \cong \mathbb{S}^1 \times \mathbb{S}^1$ be the representation such that $\rho(a_1) = \rho(a_2) = (e^{i\varphi}, e^{i\varphi})$, where $\varphi \in \mathbb{R} \setminus \pi\mathbb{Q}$, and $\rho(b_1) = \rho(b_2) = (1, 1)$.

The matrix Θ has the following form:

$$\begin{pmatrix} \varphi & 0 & \varphi & 0 \\ \varphi & 0 & \varphi & 0 \end{pmatrix}.$$

If $\gamma \in \pi_1(S)$, then $\rho(\gamma) = \rho(a_1)^{k_1} \rho(b_1)^{k_2} \rho(a_2)^{k_3} \rho(b_2)^{k_4}$ with $k_i \in \mathbb{Z}$. Consider the vector $v = (k_1, k_2, k_3, k_4)$, then

$$\Theta \cdot v = ((k_1 + k_3)\varphi, (k_1 + k_3)\varphi).$$

Viewing the 2-torus as a complex with one 0-cell, four 1-cells and one 2-cell, the image of ρ is densely contained the main diagonal. Both projections are dense in \mathbb{S}^1 , but the image does not fill \mathbb{T}^2 . Notice that the row rank of Θ over \mathbb{Z} is one as the dimension of the smallest subspace containing $\rho(\pi_1(S))$.

Example 3.5. Let S be a surface of genus 2, and let $\rho: \pi_1\Sigma \rightarrow \mathbb{T}^2 \cong \mathbb{S}^1 \times \mathbb{S}^1$ be the representation such that $\rho(a_1) = \rho(a_2) = (e^{i\varphi}, 1)$ and $\rho(b_1) = \rho(b_2) = (1, e^{i\varphi})$ with $\varphi \in \mathbb{R} \setminus \pi\mathbb{Q}$.

The matrix Θ has the following form:

$$\begin{pmatrix} \varphi & 0 & \varphi & 0 \\ 0 & \varphi & 0 & \varphi \end{pmatrix}.$$

If $\gamma \in \pi_1(S)$, then $\rho(\gamma) = \rho(a_1)^{k_1} \rho(b_1)^{k_2} \rho(a_2)^{k_3} \rho(b_2)^{k_4}$ with $k_i \in \mathbb{Z}$. Consider the vector $v = (k_1, k_2, k_3, k_4)$, then

$$\Theta \cdot v = ((k_1 + k_3)\varphi, (k_2 + k_4)\varphi).$$

Viewing the 2-torus as a complex with one 0-cell, four 1-cells and one 2-cell, the image of ρ densely fills the torus. Notice that the rank of Θ is two in this case and both projections are dense.

4. $\mathrm{Sp}(2g, \mathbb{Z})$ -action and orbit closures

The group $\mathrm{Sp}(2g, \mathbb{Z})$ acts on the space $\mathrm{Hom}(H_1(S, \mathbb{Z}), \mathbb{T}^n)$ by precomposition. We have seen in Section 2.3 that, up to a choice of a symplectic basis, this latter space identifies with the space $\mathrm{M}(n, 2g; \mathbb{T})$. In this section, we would like to study the orbit closures of an element of $\mathrm{M}(n, 2g; \mathbb{T})$ under the action of $\mathrm{Sp}(2g, \mathbb{Z})$. The first thing we notice is that a subset $\Omega \subset \mathrm{M}(n, 2g; \mathbb{R})$ is invariant under the action $\mathrm{Sp}(2g, \mathbb{Z}) \times \mathrm{M}(n, 2g; 2\pi\mathbb{Z})$ if and only if its projection onto $\mathrm{M}(n, 2g; \mathbb{T})$ is $\mathrm{Sp}(2g, \mathbb{Z})$ -invariant. This simple remark legitimates us to study the orbit closures on the universal cover, that is, $\mathrm{M}(n, 2g; \mathbb{R})$.

Let us consider the group $G = \mathrm{Sp}(2g, \mathbb{R}) \times \mathrm{M}(n, 2g; \mathbb{R})$. Given two elements (A, a) and (B, b) , their product is defined as $(A, a) \cdot (B, b) = (AB, bA^{-1} + a)$. The group G acts transitively on the space $\mathrm{M}(n, 2g; \mathbb{R})$ with the action being defined as $(A, a) \cdot p = pA^{-1} + a$; indeed, a point $p \in \mathrm{M}(n, 2g; \mathbb{R})$ may be regarded as the couple (I, p) . There is a natural identification between the space $\mathrm{M}(n, 2g; \mathbb{R})$ with the G/U , where U is the stabiliser of any point. It is straightforward to check that the zero matrix is nothing but $\mathrm{Sp}(2g, \mathbb{R})$. The subgroup $\Gamma = \mathrm{Sp}(2g, \mathbb{Z}) \times \mathrm{M}(n, 2g; 2\pi\mathbb{Z})$ is a lattice in G and acts in the obvious way on G/U . Under these conditions, we are in the right position to apply Ratner's theorem, see [21], which we state as follows according to our setting.

Ratner's theorem. *Let G, U, Γ be as above and let $p \in \mathrm{M}(n, 2g; \mathbb{R}) = G/U$ be such that $p = \gamma U$. Then there is a closed subgroup H_γ such that the following holds:*

- $U_\gamma = \gamma U \gamma^{-1} \leq H_\gamma$,
- $\Gamma \cap H_\gamma$ is a lattice in H_γ ,
- $\overline{\Gamma \cdot p} = \Gamma H_\gamma p$.

Notice that γ can be taken as (I, p) . Since our goal here is to classify the closures of Γ -orbits of any point in the space of matrices $\mathrm{M}(n, 2g; \mathbb{R})$, we just need to figure out which subgroups of G may be provided by Ratner's theorem. To this purpose, let us consider the projection $\Phi: \mathrm{Sp}(2g, \mathbb{R}) \times \mathrm{M}(n, 2g; \mathbb{R}) \rightarrow \mathrm{Sp}(2g, \mathbb{R})$. Given a point p in $\mathrm{M}(n, 2g; \mathbb{R})$, the group H_γ is isomorphic to the semidirect product $H_\gamma = U_\gamma \ltimes K_\gamma$, where K_γ is defined as $\ker \Phi \cap H_\gamma$, that is, the kernel of the mapping Φ restricted to H_γ . Notice that the image

of H_γ under the mapping Φ is the whole group $\mathrm{Sp}(2g, \mathbb{R})$ because $H_\gamma \geq U_\gamma \cong \mathrm{Sp}(2g, \mathbb{R})$. In particular, $H_\gamma \cong \mathrm{Sp}(2g, \mathbb{R}) \times K_\gamma$.

Let us proceed on understanding K_γ . The first thing we notice is that any change of lattice $h \in \mathrm{SL}(n, \mathbb{Z})$ extends to a homeomorphism ϕ_h of G defined as

$$\phi_h: G \rightarrow G, \quad \phi_h(A, a) = (A, ha).$$

This is an automorphism of G , and its restriction to $\ker \Phi$, where Φ is the projection just defined above, is linear and corresponds to a change of lattice in $\mathrm{M}(n, 2g; \mathbb{R})$. In particular, the relation

$$\Gamma \cdot (hp) = \phi_h(\Gamma \cdot p)$$

holds for any $p \in \mathrm{M}(n, 2g; \mathbb{R})$. As a consequence of Lemma 3.3, there is an element $h \in \mathrm{SL}(n, \mathbb{Z})$ such that $h \cdot p$ is of the form

$$\begin{pmatrix} \theta_{1,1} & \cdots & \theta_{1,2g} \\ \vdots & \ddots & \vdots \\ \theta_{k,1} & \cdots & \theta_{k,2g} \\ \pi q_{k+1,1} & \cdots & \pi q_{k+1,2g} \\ \vdots & \ddots & \vdots \\ \pi q_{n,1} & \cdots & \pi q_{n,2g} \end{pmatrix} = \begin{pmatrix} \Theta_o \\ \pi Q \end{pmatrix}, \quad (4.1)$$

where

- $\Theta_o \in \mathrm{M}(k, 2g; \mathbb{R})$ for some $0 \leq k \leq n$,
- $\pi Q \in \mathrm{M}(n-k, 2g; \pi\mathbb{Q})$,
- the rows $\{(\theta_{i,1}, \dots, \theta_{i,2g})\}_{i=1, \dots, k}$ are linearly independent over \mathbb{Z} ,
- $\langle (\theta_{i,1}, \dots, \theta_{i,2g}) : i = 1, \dots, k \rangle$ is $\pi\mathbb{Q}$ -free,

and hence it is sufficient to study K_γ for $\gamma = (I, p)$ and p is a matrix in form (4.1). Furthermore, it will be sufficient to study the closures of Γ -orbits for matrices in these form. We also notice that K_γ is a linear subspace of $\mathrm{M}(n, 2g; \mathbb{R})$ invariant under the action of U_γ by conjugation. In fact, suppose $\gamma = (I, p)$, let $q = (I, q) \in K_\gamma$ be any point and let $(A, p - pA^{-1})$ be a generic element of U_γ . Then

$$(A, p - pA^{-1}) \cdot (I, q) \cdot (A^{-1}, p - pA) = (I, qA^{-1}) \in K_\gamma$$

as claimed. The following lemma implies our main Theorem A for representations of closed surface groups into the unit circle S^1 .

Lemma 4.1. *Let $\bar{\theta} = (\theta_1, \dots, \theta_{2g}) \in \mathbb{R}^{2g}$. If $\bar{\theta} \in \pi\mathbb{Q}^{2g}$, then $(\mathrm{Sp}(2g, \mathbb{Z}) \times 2\pi\mathbb{Z}^{2g}) \cdot \bar{\theta}$ is discrete in \mathbb{R}^{2g} . Otherwise, if $\bar{\theta} \in \mathbb{R}^{2g} \setminus \pi\mathbb{Q}^{2g}$, then $\mathrm{Sp}(2g, \mathbb{Z}) \cdot \bar{\theta}$ is dense in \mathbb{T}^{2g} and hence $(\mathrm{Sp}(2g, \mathbb{Z}) \times 2\pi\mathbb{Z}^{2g}) \cdot \bar{\theta}$ is dense in \mathbb{R}^{2g} .*

Proof. Let Λ be the subgroup of \mathbb{R} generated by the entries of $\bar{\theta}$ and consider Λ^{2g} . The first claim is easy to establish. In this case, Λ^{2g} is a lattice in \mathbb{R}^{2g} containing $2\pi\mathbb{Z}^{2g}$ and preserved by $\mathrm{Sp}(2g, \mathbb{Z})$. Now observe that the $\mathrm{Sp}(2g, \mathbb{Z})$ -orbit of $\bar{\theta}$ is contained in Λ^{2g} . Suppose $\bar{\theta} \in \mathbb{R}^{2g} \setminus \pi\mathbb{Q}^{2g}$, hence there exists $\theta_i \in \mathbb{R} \setminus \pi\mathbb{Q}$. There is an element in $\mathrm{Sp}(2g, \mathbb{Z})$ such that all the entries are $\mathbb{R} \setminus \pi\mathbb{Q}$. We may assume $\theta_i \in [0, 2\pi)$ for all $i = 1, \dots, 2g$. Let $\theta^* \in \mathbb{T}^{2g}$ be any point. For each couple $(\theta_{2i-1}, \theta_{2i})$, where $i = 1, \dots, g$, there are two integers k_i, h_i such that the couple $(k_i\theta_{2i-1} + \theta_{2i}, (k_i h_i - 1)\theta_{2i-1} + h_i\theta_{2i})$ is closed to $(\theta_{2i-1}^*, \theta_{2i}^*)$. Therefore, the $\mathrm{Sp}(2g, \mathbb{Z})$ -orbit of θ is dense in \mathbb{T}^{2g} and hence $\mathrm{Sp}(2g, \mathbb{Z}) \times 2\pi\mathbb{Z}^{2g} \cdot \theta$ is dense in \mathbb{R}^{2g} as desired. ■

Before proving the general case, we need the following proposition in which we describe the group K_γ .

Proposition 4.2. *Let $p \in X$ be any point in the form given in equation (4.1) and let k be the number of lines not in $\pi\mathbb{Q}^{2g}$. Let H_γ be the group provided by Ratner's theorem, where $\gamma = (I, p)$. Then K_γ is trivial or $K_\gamma = \mathrm{M}(k, 2g; \mathbb{R})$.*

Proof. Let p be any point in $\mathrm{M}(n, 2g; \mathbb{R})$. Assume p be different from the zero matrix for which the claim trivially holds. Let us begin with the case $p = \pi Q \in \mathrm{M}(n, 2g; \pi\mathbb{Q})$, that means $k = 0$. We claim K_γ to be trivial. Let $\gamma = (I, p)$ and let H_γ be the group provided by Ratner's theorem. The orbit $\Gamma \cdot p$ lies in the subgroup of $\mathrm{M}(n, 2g; \mathbb{R})$ generated by the matrices $\pi q_{ij} E_{ij}$, where πq_{ij} are the entries of p , which is discrete and closed. This means that $\bar{\Gamma} \cdot p = \Gamma \cdot p$ and implies H_γ is the stabiliser of p . Therefore, $H_\gamma = U_\gamma$ and hence K_γ is trivial. Notice that this argument generalises the first case of the previous lemma. Let us now assume $k > 0$. The linear space K_γ is completely determined by Θ_o ; indeed, the block πQ does not give any contribution. In this case, the orbit $\Gamma \cdot p$ is no longer closed, and the $\mathrm{Sp}(2g, \mathbb{Z})$ -orbit of p is contained in some linear subspace of $\mathrm{M}(k, 2g; \mathbb{R})$ of dimension $2gl$, where l is the dimension of the linear space generated by the rows of Θ_o . Hence K_γ contains V as a proper subspace. We can notice that V is $\mathrm{Sp}(2g, \mathbb{R})$ -invariant but $V \cap \mathrm{M}(k, 2g; 2\pi\mathbb{Z})$ is not a lattice because the \mathbb{Z} -module, generated by the rows of Θ_o , is $\pi\mathbb{Q}$ -free. For the same reason, the minimal linear space containing V and a lattice is $\mathrm{M}(k, 2g; \mathbb{R})$, hence $K_\gamma = \mathrm{M}(k, 2g; \mathbb{R})$. ■

From the proof of Proposition 4.2, we can deduce the following corollary.

Corollary 4.3. *Let $p \in (n, 2g; \mathbb{R})$ be any point in the form given in equation (4.1) and let k be the number of lines not in $\pi\mathbb{Q}^{2g}$. Then there exists a closed connected subgroup $H \leq \mathbb{T}^n$ of dimension k such that $\bar{\Gamma} \cdot p$ projects into a finite union of inhomogeneous tori of dimension k corresponding to cosets of H . In particular, the modular orbit of a dense representation $\rho: \mathrm{H}_1(S, \mathbb{Z}) \rightarrow \mathbb{T}^n$ is dense in the representation space.*

This corollary implies Theorem B and indeed Theorem A. In Appendix A, we will study the modular orbits by applying a direct approach without relying on Ratner's theory.

5. An application: approximation result

The aim of this final section is to present Proposition D and indeed Theorem C. Let us begin by recalling the statement of Kronecker's theorem as formulated in [12, Section 26.19 (e)]. The reader may also consult [1, Section 1.12 (iii)] for another one-dimensional version of Kronecker's theorem.

Kronecker's approximation theorem. *Let $b^{(i)} = (b_1^{(i)}, \dots, b_m^{(i)})$ with $i = 1, \dots, n$ be vectors of \mathbb{R}^m such that $b^{(1)}, \dots, b^{(n)}, \pi e_1, \dots, \pi e_m$ are linearly independent over \mathbb{Q} in the vector space \mathbb{R}^m (where the e_j 's form the canonical basis of \mathbb{R}^m). Let a_1, \dots, a_n be any real numbers and let ε be a positive number. Then there is an element $(k_1, \dots, k_m) \in \mathbb{Z}^m$ such that*

$$\left| a_i - \sum_{l=1}^m k_l b_l^{(i)} \right| < \varepsilon \pmod{2\pi} \quad (5.1)$$

for every $i = 1, \dots, n$.

For a real a , the expression $|a| < \varepsilon \pmod{2\pi}$ means that $|a - 2k\pi| < \varepsilon$ for some integer k . From equation (5.1) above, one can easily infer the equivalent estimate

$$\|(a_1, \dots, a_n)^\top - 2\pi(h_1, \dots, h_n)^\top - B(k_1, \dots, k_m)^\top\| < C\varepsilon,$$

where $(h_1, \dots, h_n) \in \mathbb{Z}^n$, B is the matrix having $b^{(i)}$'s as rows, C is a real constant depending only on n , and $\|\cdot\|$ is any norm on \mathbb{R}^n . Kronecker's theorem generalises to simultaneous approximation of l given real vectors $a^{(j)} = (a_{1j}, \dots, a_{nj})^\top$, where $j = 1, \dots, l$. Indeed, for any $\varepsilon > 0$ there is a matrix $K \in M(m, l; \mathbb{Z})$ such that

$$\|A - 2\pi H - BK\| < C\varepsilon,$$

where A is the matrix having $a^{(j)}$'s as columns, $H \in M(n, l; \mathbb{Z})$ and C is a constant depending only on l, n . That is,

$$\|A - BK\| < \varepsilon \pmod{2\pi}.$$

Let S be a closed surface of genus greater than zero, let $\rho: \pi_1(S) \rightarrow \mathbb{T}^n$ be a representation and let Θ_ρ be the associated matrix in the sense of Definition 2.14.

Proposition D. *The following are equivalent:*

- (1) $\text{Mod}(S) \cdot \rho$ is dense in the representation space.
- (2) For any matrix $A \in M(n, 2g; \mathbb{R})$ and any $\varepsilon > 0$, there is a matrix $g \in \text{Sp}(2g, \mathbb{Z})$ such that

$$\|A - \Theta_\rho g\| < \varepsilon \pmod{2\pi}.$$

Proof. Each representation is identified with its associated matrix and the representation space with $M(n, 2g; \mathbb{T})$. Suppose $\text{Mod}(S) \cdot \Theta_\rho$ is not dense in the representation space.

Then there is an open set U such that $\text{Mod}(S) \cdot \Theta_\rho \cap U = \emptyset$. Let A be any matrix in U , and ε be a strictly positive real number such that the open ball $B_\varepsilon(A) \subset U$. Then, for any $g \in \text{Mod}(S)$ the estimate $|A - \Theta_\rho g| > \varepsilon \pmod{2\pi}$ holds. As the action of the Torelli subgroup is trivial by Proposition 2.8, the action of the mapping class group coincides with the action of $\text{Sp}(2g, \mathbb{Z})$. Therefore, Theorem C implies Theorem A.

Suppose $\text{Mod}(S) \cdot \Theta_\rho$ is dense in the representation space. In that case, for any $A \in \mathbb{M}(n, 2g; \mathbb{T})$ and for any $\varepsilon > 0$, the mapping class group orbit intersects the open set $B_\varepsilon(A) \subset \mathbb{M}(n, 2g; \mathbb{T})$, i.e., there is an element $g \in \text{Mod}(S)$ such that $g^{-1} \cdot \Theta_\rho = \Theta_\rho g \in B_\varepsilon(A)$. In particular, $\|A - \Theta_\rho g\| < \varepsilon \pmod{2\pi}$. Once again, by Proposition 2.8, the matrix g can be taken in $\text{Sp}(2g, \mathbb{Z})$, and so Theorem A implies Theorem C as desired. ■

A. Dense orbits and further discussion

In this appendix, we are going to prove Theorem A for almost every representation without relying on Ratner's theorem. We begin consider the genus one case and we shall use it to extend the discussion to surfaces of arbitrary genus.

A.1. Direct proof of Theorem B for almost every representations

The set of matrices $\mathbb{M}(n, 2g; \mathbb{T})$ contains, as a proper subset, the space \mathcal{D} of all of those matrices of the form

$$\begin{pmatrix} \theta_1 & \theta_2 & \cdots & \theta_{2i-1} & \theta_{2i} & \cdots & \theta_{2g-1} & \theta_{2g} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \lambda_j \theta_1 & \lambda_j \theta_2 & \cdots & \lambda_j \theta_{2i-1} & \lambda_j \theta_{2i} & \cdots & \lambda_j \theta_{2g-1} & \lambda_j \theta_{2g} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \lambda_n \theta_1 & \lambda_n \theta_2 & \cdots & \lambda_n \theta_{2i-1} & \lambda_n \theta_{2i} & \cdots & \lambda_n \theta_{2g-1} & \lambda_n \theta_{2g} \end{pmatrix}, \quad (\text{A.1})$$

where $(\theta_1, \theta_2, \dots, \theta_{2g-1}, \theta_{2g}) \in \mathbb{R}^{2g} \setminus \pi\mathbb{Q}^{2g}$ is the lift of $(e^{i\theta_1}, e^{i\theta_2}, \dots, e^{i\theta_{2g-1}}, e^{i\theta_{2g}}) \in \mathbb{T}^{2g}$ contained in $[0, 2\pi)^{2g}$ and the reals $\{1, \lambda_2, \dots, \lambda_n\} \subset \mathbb{R}$ are linearly independent over \mathbb{Q} .

Lemma A.1. *The space \mathcal{D} is dense in $\mathbb{M}(n, 2g; \mathbb{T})$.*

Proof. Let $\lambda_2, \dots, \lambda_n$ be real numbers such that $1, \lambda_2, \dots, \lambda_n$ are linearly independent over \mathbb{Q} . Let us consider the mapping $\varphi: \mathbb{R}^{2g} \rightarrow \mathbb{T}^{2gn}$ defined by

$$\begin{aligned} & (\theta_1, \theta_2, \dots, \theta_{2g-1}, \theta_{2g}) \\ & \mapsto ((e^{i\theta_1}, \dots, e^{i\theta_{2g}}), (e^{i\lambda_2\theta_1}, \dots, e^{i\lambda_2\theta_{2g}}), \dots, (e^{i\lambda_n\theta_1}, \dots, e^{i\lambda_n\theta_{2g}})). \end{aligned}$$

This mapping factors through a mapping $\bar{\varphi}: \mathbb{R}^{2g} \rightarrow \mathbb{R}^{2gn}$ such that $\varphi = \exp \circ \bar{\varphi}$ and \exp is the exponential mapping as in equation (2.1) introduced in Section 2.3. The image of $\bar{\varphi}$ is a $2g$ -dimensional linear subspace. Since the reals $\{1, \lambda_2, \dots, \lambda_n\}$ are linearly independent

over \mathbb{Q} , then the projection via the exponential mapping is dense in \mathbb{T}^{2g_n} . The space \mathcal{D} is defined as the union of the images for each possible subset $\{\lambda_2, \dots, \lambda_n\} \subset \mathbb{R}$ such that $1, \lambda_2, \dots, \lambda_n$ are linearly independent over \mathbb{Q} . Therefore, \mathcal{D} is dense. ■

The following lemma is easy to establish and the proof is left to the reader.

Lemma A.2. *The space \mathcal{D} is $\mathrm{Sp}(2g, \mathbb{Z})$ -invariant.*

Let us consider first surfaces of genus one. Let T be the genus one surface, and let $\rho: \pi_1 T \rightarrow \mathbb{T}^n$ be a dense representation, and Θ_ρ be its associated matrix with respect to some basis $\{\alpha, \beta\}$ and the standard lattice of \mathbb{R}^n . Let Ω_ρ be the $\mathrm{SL}(2, \mathbb{Z})$ -orbit of Θ_ρ in $M(n, 2; \mathbb{T})$. The associated matrix Θ_ρ has the form

$$\Theta_\rho = \begin{pmatrix} \theta_1 & \theta_2 \\ \vdots & \vdots \\ \lambda_i \theta_1 & \lambda_i \theta_2 \\ \vdots & \vdots \\ \lambda_n \theta_1 & \lambda_n \theta_2 \end{pmatrix}, \tag{A.2}$$

where $(\theta_1, \theta_2) \in \mathbb{R}^2$ is the lift of $(e^{i\theta_1}, e^{i\theta_2}) \in \mathbb{T}^2$ contained in $[0, 2\pi)^2$ and $\lambda_i \in \mathbb{R}$, for any $i = 2, \dots, n$, are linearly independent over \mathbb{Q} . Set

$$\bar{\Theta}_\rho = \begin{pmatrix} \Theta_\rho \\ \pi \cdot I_2 \end{pmatrix}.$$

Since the representation ρ is assumed to have a dense image, the matrix $\bar{\Theta}_\rho$ has maximal row rank, that is, $\mathrm{rk}_{\mathbb{Z}} = n + 2$. This implies the following properties of the matrix Θ_ρ above:

- (i) The real numbers θ_1 and θ_2 cannot be both elements of $\pi\mathbb{Q}$. If this were the case, the row rank of the matrix $\bar{\Theta}_\rho$ would fail to be maximal, contradicting our assumptions. In the case one of them is an element of $\pi\mathbb{Q}$, we can change the basis in such a way they are both elements of $\mathbb{R} \setminus \pi\mathbb{Q}$. Indeed, assume without loss of generality that $\theta_2 \in \pi\mathbb{Q}$. The Dehn twist \mathcal{T}_α along α maps the curve β to $\alpha\beta$ and hence $\rho(\beta)$ is mapped to $\rho(\alpha\beta)$. The second column of Θ_ρ changes accordingly, and the element in position $(1, 2)$ of $\Theta_{\mathcal{T}_\alpha, \rho}$ is nothing else than $\theta_1 + \theta_2$. As $\theta_1 \notin \pi\mathbb{Q}$, the same necessarily holds for $\theta_1 + \theta_2$. In what follows, we shall assume both $\theta_1, \theta_2 \notin \pi\mathbb{Q}$.
- (ii) The real numbers $\pi, \theta_1, \dots, \lambda_i \theta_1, \dots, \lambda_n \theta_1$ are linearly independent over \mathbb{Q} . Indeed, if this were not the case, then one can easily check that $\bar{\Theta}_\rho$ does not have maximal rank. This implies that the subgroup of \mathbb{T}^n generated by the vector $(\theta_1, \dots, \lambda_i \theta_1, \dots, \lambda_n \theta_1)$ is dense in \mathbb{T}^n (see [1, Exercise 1.13]), meaning that \mathbb{T}^n is monothetic (that is, \mathbb{T}^n contains a dense cyclic subgroup). The same holds also for the real numbers $\pi, \theta_2, \dots, \lambda_i \theta_2, \dots, \lambda_n \theta_2$.

We begin with considering the $\mathrm{SL}(2, \mathbb{Z})$ action on the space $\mathrm{M}(n, 2; \mathbb{R})$ seen as the universal cover of $\mathrm{M}(n, 2; \mathbb{T})$, see also Remark 2.14. Given the matrix Θ_ρ as in (A.2), there is a unique lift, say $\Theta(\rho)$, in $\mathrm{M}(n, 2; \mathbb{R})$ which is still of form (A.2). Notice that such a matrix is the unique one who has all the entries in the interval $[0, 2\pi)$. Let us finally denote by $\Omega(\rho)$ the $\mathrm{SL}(2, \mathbb{Z})$ -orbit of $\Theta(\rho)$ in $\mathrm{M}(n, 2; \mathbb{R})$.

Since $\Theta(\rho)$ is of form (A.2), an easy computation shows that the matrix $A \cdot \Theta(\rho) \in \Omega(\rho)$ is still of form (A.2) for any $A \in \mathrm{SL}(2, \mathbb{Z})$, that is, the i -th row of $A \cdot \Theta(\rho)$ is λ_i -times $(\theta_1, \theta_2)A^{-1}$. Therefore, we can deduce that $\Omega(\rho)$ is contained in some proper linear subspace S of \mathbb{R}^{2n} . In fact, the coefficients of any matrix $A \cdot \Theta(\rho) \in \Omega(\rho)$ satisfy the homogeneous linear system

$$\mathcal{S}: \begin{cases} \theta_{2,1} - \lambda_2 \theta_{1,1} = 0, \\ \vdots \\ \theta_{n,1} - \lambda_n \theta_{1,1} = 0, \\ \theta_{22} - \lambda_2 \theta_{12} = 0, \\ \vdots \\ \theta_{n2} - \lambda_n \theta_{12} = 0 \end{cases}$$

in $2n - 2$ equations and $2n$ variables. Hence, S is defined as the full space of solutions of the linear system \mathcal{S} . Let us consider the subspace S . Since each of the λ_i is taken as an element of $\mathbb{R} \setminus \mathbb{Q}$, the subspace S necessarily meets the lattice $\mathrm{M}(n, 2; 2\pi\mathbb{Z})$ only at the origin. Hence, the projection of the subspace S into the space $\mathrm{M}(n, 2; \mathbb{T})$ densely fills a closed subspace K of $\mathrm{M}(n, 2; \mathbb{T})$. We finally claim that K cannot be a proper subspace. We notice that, due to the nature of the linear system \mathcal{S} , the subspace S splits as the direct product $V_1 \times V_2$ inside the vector space $\mathbb{R}^n \times \mathbb{R}^n \cong \mathrm{M}(n, 2; \mathbb{R})$. Then the image of S into the space $\mathrm{M}(n, 2; \mathbb{T})$ lies inside a closed subgroup of the form $H_1 \times H_2$, where $H_i < \mathrm{M}(n, 1; \mathbb{T}) \cong \mathbb{T}^n$, for $i = 1, 2$. Notice that K is a proper subgroup of $\mathrm{M}(n, 2; \mathbb{T})$ if and only if H_i is a proper subgroup of $\mathrm{M}(n, 1; \mathbb{T})$. Therefore, the proof of the final claim boils down to showing that H_i cannot be a proper subgroup for both $i = 1, 2$. As the group H_1 contains the vector $\exp(\theta_1, \lambda_2 \theta_1, \dots, \lambda_n \theta_1)$, then it contains also the subgroup $\{\exp(t(\theta_1, \lambda_2 \theta_1, \dots, \lambda_n \theta_1)) \mid t \in \mathbb{Z}\}$ and thus its closure which we know to be equal to the full space \mathbb{T}^n . In the same fashion, we can prove $H_2 = \mathbb{T}^n$. As a result, $K = \mathrm{M}(n, 2; \mathbb{T})$ and the $\mathrm{SL}(2, \mathbb{Z})$ -orbit of Θ_ρ is dense in $\mathrm{M}(n, 2; \mathbb{T})$ as desired.

The general case for surfaces of genus greater than one works similarly, and we rely on the following remark.

Remark A.3. Let S be any surface of genus $g \geq 2$. The space $\mathrm{Hom}(\pi_1(S), \mathbb{T}^n)$ naturally splits as the direct sum of g copies of $\mathrm{Hom}(\pi_1 T, \mathbb{T}^n)$, where T denotes the 2-torus (we do not use here the blackboard notation since T is considered only as a topological surface regardless of its group structure). The basis $\{\alpha_1, \beta_1, \dots, \alpha_g, \beta_g\}$ of $\pi_1(S)$ we fixed satisfies moreover the equalities $i(\alpha_i, \alpha_j) = i(\beta_i, \beta_j) = 0$, and $i(\alpha_i, \beta_j) = \delta_{ij}$ for all i, j with $1 \leq i, j \leq g$. We may associate to any representation $\rho: \pi_1(S) \rightarrow \mathbb{T}^n$ the g -tuple

of representations (ρ_1, \dots, ρ_g) where ρ_i is the restriction of ρ to the handle generated by α_i, β_i . Such a mapping defines then an isomorphism

$$\text{Hom}(\pi_1(S), \mathbb{T}^n) \cong \bigoplus_{i=1}^g \text{Hom}(\langle \alpha_i, \beta_i \rangle, \mathbb{T}^n),$$

which depends on the basis chosen. This decomposition is a consequence of the fact that a surface of genus g is the connected sum of a surface of genus $g - 1$ and a torus T along with the property that each representation ρ sends all simple closed separating curves to the identity. A recursive argument leads to the desired conclusion. We finally notice that the group $\text{SL}(2, \mathbb{Z}) \times \dots \times \text{SL}(2, \mathbb{Z}) < \text{Sp}(2g, \mathbb{Z})$ introduced in Remark 2.5 acts diagonally on this space; the i -th copy of $\text{SL}(2, \mathbb{Z})$ acts on $\text{Hom}(\langle \alpha_i, \beta_i \rangle, \mathbb{T}^n)$.

Given any matrix, say A , of the form as in equation (A.1), up to replace it with $A g^{-1}$ with $g \in \text{Sp}(2g, \mathbb{Z})$, we may assume, without loss of generality, that at least one of $\theta_{2i-1}, \theta_{2i} \notin \pi\mathbb{Q}$. Under this condition, observations (i) and (ii) hold for each pair of columns

$$\begin{pmatrix} \theta_{2i-1} & \theta_{2i} \\ \vdots & \vdots \\ \lambda_j \theta_{2i-1} & \lambda_j \theta_{2i} \\ \vdots & \vdots \\ \lambda_n \theta_{2i-1} & \lambda_n \theta_{2i} \end{pmatrix}.$$

Therefore, the action of the g -times product $\text{SL}(2, \mathbb{Z}) \times \dots \times \text{SL}(2, \mathbb{Z}) < \text{Sp}(2g, \mathbb{Z})$ provides a dense orbit inside the space $M(n, 2g; \mathbb{T})$ as desired.

A.2. Finding curve generating dense subgroups

All the representations ρ considered in Section A.1 above are characterised by the following property: Each column of the associated matrix Θ_ρ generates a dense subgroup of \mathbb{T}^n . It turns out that for any such a representation one can find infinitely many curves whose image generates a dense subgroup in \mathbb{T}^n . This led the authors to ask themselves: Given a dense representation $\rho: \pi_1(S) \rightarrow \mathbb{T}^n$, can we find a simple closed curve γ such that $\langle \rho(\gamma) \rangle$ is dense in \mathbb{T}^n ? Remind that a vector $(e^{i\theta_1}, \dots, e^{i\theta_n}) \in \mathbb{T}^n$ generates a dense subgroup if and only if $\pi, \theta_1, \dots, \theta_n$ are linearly independent over \mathbb{Q} . As a corollary of Lemma 4.1, we deduce the following.

Lemma A.4. *Let $\rho: \pi_1(S) \rightarrow \mathbb{S}^1$ be a dense representation. Then there always exists a simple closed curve γ such that $\overline{\langle \rho(\gamma) \rangle} = \mathbb{S}^1$.*

However, for $n \geq 2$, the scenario changes completely. Indeed, for any n we can find examples of dense representations which do not have any curve generating a dense subgroup in \mathbb{T}^n .

Example A.5. Let S be a surface of genus g and $\rho: \pi_1(S) \rightarrow \mathbb{T}^2$ be the representation having as the associated matrix $\Theta_\rho \in \mathbf{M}(2, 2g; \mathbb{T})$ the following:

$$\begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & \cdots & 0 & 0 \end{pmatrix} \in \mathbf{M}(2, 2g; \mathbb{T}).$$

One can show that $\bar{\Theta}_\rho$ has maximal rank and hence ρ is a dense representation. However, no curve is applied by ρ to a vector generating a dense subgroup.

Example A.6. Let S be a surface of genus g and $\rho: \pi_1(S) \rightarrow \mathbb{T}^n$ be the representation having as the associated matrix $\Theta_\rho \in \mathbf{M}(n, 2g; \mathbb{T})$ the following:

$$\begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & \cdots & 0 & 0 \\ \theta_3 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \theta_n & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \end{pmatrix} \in \mathbf{M}(n, 2g; \mathbb{T}),$$

where $1, \theta_3, \dots, \theta_n$ are linearly independent over \mathbb{Q} . One can show that $\bar{\Theta}_\rho$ has maximal rank and hence ρ is a dense representation. However, no curve is applied by ρ to a vector generating a dense subgroup.

B. Surfaces with one puncture

Let us now discuss the case of the one-holed torus Σ . We denote by $\pi_1 \Sigma \cong \langle \alpha, \beta \rangle$ the fundamental group of Σ . Also in this case, the choice of a representation consists in choosing for each generator an element of \mathbb{T}^n . The representation space $\text{Hom}(\pi_1 \Sigma, \mathbb{T}^n)$ trivially identifies with the space $\mathbb{T}^n \times \mathbb{T}^n$. For each choice of an element \mathbf{c} , the relative representation variety $\text{Hom}_{\mathbf{c}}(\pi_1 \Sigma, \mathbb{T}^n)$ is defined as the preimage of \mathbf{c} via the commutator map $k: \mathbb{T}^n \times \mathbb{T}^n \rightarrow \mathbb{T}^n$. Thus, as a consequence of the abelian property, the relative representation space is empty for any $\mathbf{c} \neq (1, \dots, 1)$ and coincides with the full representation variety when $\mathbf{c} = (1, \dots, 1)$. Once again, the action of \mathbb{T}^n by inner automorphisms is trivial and hence the character variety trivially coincides with the representation space. As a consequence, the space $\text{Hom}(\pi_1 \Sigma, \mathbb{T}^n)$ naturally identifies with the space $\text{Hom}(\pi_1 T, \mathbb{T}^n)$. The equalities $\text{Mod}(T) = \text{Mod}(\Sigma) = \text{SL}(2, \mathbb{Z})$ are well known, and the actions of $\text{Mod}(T)$ and $\text{Mod}(\Sigma)$ on the representation spaces associated to T and Σ , respectively, coincide. Therefore, we have the following proposition.

Proposition B.1. *Theorems A and B hold for the torus T if and only if they hold for the one-holed torus Σ .*

More generally, the main results of the present work extend to surfaces of higher genus and with one boundary component. Indeed, let $S_{g,1}$ be a surface of genus g and

one boundary component. We have already seen above that this is true for the one-holed torus Σ , see Proposition B.1. The general claim follows because, since \mathbb{T}^n is abelian, one can establish an identification between the representations spaces $\text{Hom}(\pi_1(S), \mathbb{T}^n)$ and $\text{Hom}(\pi_1(S_{g,1}), \mathbb{T}^n)$. Since the mapping class group coincides with the pure mapping class group for one-puncture surfaces the following proposition also holds.

Proposition B.2. *Theorems A and B hold for a closed surface of genus g if and only if they hold for the one-holed surface of genus g .*

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