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Commensurations of the outer automorphism group of a universal Coxeter group

Yassine Guerch

Abstract. This paper studies the rigidity properties of the abstract commensurator of the outer automorphism group of a universal Coxeter group of rank n, which is the free product W_n of n copies of $\mathbb{Z}/2\mathbb{Z}$. We prove that for $n \geq 5$ the natural map $\mathrm{Out}(W_n) \to \mathrm{Comm}(\mathrm{Out}(W_n))$ is an isomorphism and that every isomorphism between finite index subgroups of $\mathrm{Out}(W_n)$ is given by a conjugation by an element of $\mathrm{Out}(W_n)$.

1. Introduction

Given a group G, the *abstract commensurator of* G, denoted by $\mathrm{Comm}(G)$, is the group of equivalence classes of isomorphisms between finite index subgroups of G. Two such isomorphisms are equivalent if they agree on some common finite index subgroup of their domain. Note that every automorphism of G induces an element of $\mathrm{Comm}(G)$, and in particular, the action of G on itself by global conjugation gives a homomorphism $G \to \mathrm{Comm}(G)$.

The abstract commensurator of G captures a notion of symmetry for the group that is weaker than its group of automorphisms. For instance, the abstract commensurator of $\mathbb{Z}^m$ is isomorphic to $\mathrm{GL}(m, \mathbb{Q})$ while the abstract commensurator of a nonabelian free group is not finitely generated (see [1]). However, some groups satisfy strong rigidity properties and the group $\mathrm{Comm}(G)$ is then not much larger than $\mathrm{Aut}(G)$ or G itself. For instance, the Mostow–Prasad–Margulis rigidity theorem and Margulis arithmeticity theorem (see, for instance, [37]) imply that if Γ is a lattice in a connected noncompact simple Lie group G with trivial center, and if $G \neq \mathrm{PSL}(2, \mathbb{R})$, then Γ is a finite index subgroup of $\mathrm{Comm}(\Gamma)$ if and only if Γ is not arithmetic, otherwise $\mathrm{Comm}(\Gamma)$ is dense in G. In the case of the extended mapping class group of a connected orientable closed surface S_g of genus g at least 3, we have an even stronger result due to Ivanov [26] since the natural homomorphism $\mathrm{Mod}^{\pm}(S_g) \to \mathrm{Comm}(\mathrm{Mod}^{\pm}(S_g))$ is an isomorphism. This result also extends to the case of the mapping class group of a connected orientable surface with genus equal to 2 and with at least two boundary components. In the context of the outer automorphism

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group of a free group F_N of rank N, Farb and Handel [10] proved that, for $N \geq 4$, the natural map from $\mathrm{Out}(F_N)$ to $\mathrm{Comm}(\mathrm{Out}(F_N))$ is an isomorphism and that every isomorphism between two finite index subgroups of $\mathrm{Out}(F_N)$ extends to an inner automorphism of $\mathrm{Out}(F_N)$. This result was later extended by Horbez and Wade [25] to the case $N = 3$ using a more geometric approach. Their techniques also enabled them to compute the abstract commensurator of many interesting subgroups of $\mathrm{Out}(F_N)$, like its Torelli subgroup. These rigidity results have been extended to other groups, such as handlebody groups [22] and big mapping class groups [2].

In this article, we are interested in the outer automorphism group of a universal Coxeter group. Let n be an integer greater than 1. Let $F = \mathbb{Z}/2\mathbb{Z}$ be a cyclic group of order 2 and $W_n = \ast_n F$ be a universal Coxeter group of rank n, that is, a free product of n copies of F. We prove the following theorem.

Theorem 1.1. *Let $n \geq 5$. The natural homomorphism*

$$\mathrm{Out}(W_n) \to \mathrm{Comm}(\mathrm{Out}(W_n))$$

is an isomorphism.

The group $\mathrm{Out}(W_2)$ is finite and the group $\mathrm{Out}(W_3)$ is isomorphic to $\mathrm{PGL}(2, \mathbb{Z})$. This gives an almost complete classification except for $n = 4$, where our proof for $n \geq 5$ cannot be immediately adapted to this case as $\mathrm{Out}(W_4)$ does not contain any direct product of two nonabelian free groups. Hence the case $n = 4$ remains open. One step towards the understanding of $\mathrm{Out}(W_4)$ is given in [12], where we proved that $\mathrm{Out}(W_4)$ has a nontrivial outer automorphism. The conclusion of Theorem 1.1 will therefore not be true if one can prove that this outer automorphism remains nontrivial for every finite index subgroup of $\mathrm{Out}(W_n)$. Theorem 1.1 is a major improvement of [12, Théorème 1.1] which states that, for $n \geq 5$, the only automorphisms of $\mathrm{Out}(W_n)$ are the global conjugations. In turn, Theorem 1.1 implies that every isomorphism between two finite index subgroups of $\mathrm{Out}(W_n)$ is given by a conjugation by an element of $\mathrm{Out}(W_n)$. The proof of the present Theorem 1.1 significantly differs from the one of [12, Théorème 1.1] since the proof of [12, Théorème 1.1] is based on the study of torsion subgroups of $\mathrm{Out}(W_n)$, whereas $\mathrm{Out}(W_n)$ is virtually torsion free (see [17, Corollary 5.5]).

Our proof of Theorem 1.1 is inspired by the proof of the similar result in the context of $\mathrm{Out}(F_N)$ given by Horbez and Wade [25]. However, new ideas are required in this situation. Indeed, to our knowledge, there is no way to compute the abstract commensurator of $\mathrm{Out}(W_n)$ by identifying it with a subgroup of $\mathrm{Out}(F_N)$. Moreover, the study of the restriction of automorphisms of W_n to some finite index nonabelian free subgroup of W_n is not sufficient to understand the abstract commensurator of $\mathrm{Out}(W_n)$, as it does not give information about finite index subgroups of $\mathrm{Out}(W_n)$. Finally, the proof of Horbez and Wade relies extensively on the possibility of writing a free group as an HNN extension, which is not possible in a universal Coxeter group. Instead, we use the fact that W_n can be written as a free product $W_n = A \ast B$, where B is a finite abelian subgroup of W_n.

We now sketch our proof of Theorem 1.1. Following a strategy that dates back to Ivanov's work [26], we study the action of $\mathrm{Out}(W_n)$ on various graphs which are *rigid*, that is, every graph automorphism is induced by an element of $\mathrm{Out}(W_n)$. These graphs include the spine K_n of the outer space of W_n as defined by Guirardel and Levitt in [17], generalizing Culler and Vogtmann's outer space of a free group [9], or the free splitting graph $\overline{K}_n$ of W_n (see [13, Theorems 1.1 and 1.2] and Section 2.2 for definitions). The proof of Theorem 1.1 relies on the action of $\mathrm{Out}(W_n)$ on a subset of the vertices of $\overline{K}_n$, called *the set of* W_k*-stars*. Let $k \in \{0, \dots, n-1\}$. A W_k-star is a free splitting S of W_n such that the underlying graph of the induced graph of groups $W_n \backslash S$ is a tree with $n-k$ edges, such that the degree of one of the vertices, called the *center*, is equal to $n-k$, and such that the group associated with the center is isomorphic to W_k and the groups associated with the leaves are all isomorphic to F. The W_k-stars are the analogue for W_n of the roses in the outer space of a free group. They play a significant role in the proof of other rigidity results for $\mathrm{Out}(W_n)$ (see [12, 13]).

This allows us to introduce a graph called the *graph of one-edge compatible* W_{n-2}*-stars*, and denoted by X_n. It is defined as follows: vertices are W_n-equivariant homeomorphism classes of W_{n-2}-stars, where two vertices $\mathcal{S}$ and $\mathcal{S}'$ are adjacent if there exist $S \in \mathcal{S}$ and $S' \in \mathcal{S}'$ such that S and S' have both a common refinement and a common collapse. We prove the following result.

Theorem 1.2. *Let $n \geq 5$. The natural homomorphism*

$$\mathrm{Out}(W_n) \to \mathrm{Aut}(X_n)$$

is an isomorphism.

Our proof of Theorem 1.2 requires the rigidity of another graph, called the *graph of* W_**-stars*, and denoted by X_n'. It is the graph whose vertices are the W_n-equivariant homeomorphism classes of W_k-stars with k varying in $\{0, \dots, n-2\}$, where two vertices $\mathcal{S}$ and $\mathcal{S}'$ are adjacent if there exist $S \in \mathcal{S}$ and $S' \in \mathcal{S}'$ such that S refines S' or conversely. We first show that every graph automorphism of X_n induces a graph automorphism of X_n' and that the induced map $\mathrm{Aut}(X_n) \to \mathrm{Aut}(X_n')$ is injective. Using the rigidity of X_n' (see Theorem 3.4), we show that any graph automorphism of X_n is induced by an element of $\mathrm{Out}(W_n)$.

We then show that every commensuration f of $\mathrm{Out}(W_n)$ induces a graph automorphism of X_n. Once we have that result, a general argument (see Proposition 2.1) gives the isomorphism between $\mathrm{Out}(W_n)$ and $\mathrm{Comm}(\mathrm{Out}(W_n))$. In order to construct such a homomorphism $\mathrm{Comm}(\mathrm{Out}(W_n)) \to \mathrm{Aut}(X_n)$, we first give an algebraic characterization of the stabilizers of equivalence classes of W_{n-2}-stars. The characterization relies on the examination of maximal abelian subgroups of $\mathrm{Out}(W_n)$ and of direct products of nonabelian free groups in $\mathrm{Out}(W_n)$. In particular, we prove (see Theorem 5.1), using the action of $\mathrm{Out}(W_n)$ on a simplicial complex called the *free factor complex of* W_n, the following result.

Theorem 1.3. *Let $n \geq 4$. The maximal number of factors in a direct product of nonabelian free groups contained in* $\mathrm{Out}(W_n)$ *is equal to* $n-3$.

One of the examples of such a maximal direct product of nonabelian free subgroups of $\mathrm{Out}(W_n)$ is the following. Let $W_n = \langle x_1, \ldots, x_n \rangle$ be a standard generating set for W_n and let $W = \langle x_1, x_2, x_3 \rangle$. For every $i \geq 4$ and every $w \in W$, let $F_{i,w}$ be the automorphism which fixes x_j for every $j \neq i$ and which sends x_i to wx_iw^{-1}. Let $[F_{i,w}]$ be the outer automorphism class of $F_{i,w}$ and let $H_i = \langle [F_{i,w}]_{w \in W} \rangle$. Then the group $\langle H_i \rangle_{i \geq 4}$ is a subgroup of $\mathrm{Out}(W_n)$ isomorphic to a direct product of $n-3$ groups isomorphic to W_3.

The complete characterization of stabilizers of equivalence classes of W_{n-2}-stars being quite technical, we do not give the complete statement in the introduction (see Propositions 6.11 and 7.10). However, we remark that this characterization relies on the following key points: the fact that stabilizers of equivalence classes of W_{n-2}-stars contain a maximal free abelian subgroup and the fact that it contains a direct product of $n-3$ nonabelian free groups. The characterization also features a study of the group of twists of a W_{n-2}-star, which is a direct product of two virtually nonabelian free groups by a result of Levitt [28] and such that each of which has finite index in the centralizer in $\mathrm{Out}(W_n)$ of the other.

This characterization being preserved by commensurations of $\mathrm{Out}(W_n)$, it induces a homomorphism from $\mathrm{Comm}(\mathrm{Out}(W_n))$ to the group $\mathrm{Bij}(VX_n)$ of bijections of the set of vertices of X_n. In order to show that this map extends to the edge set of X_n, we also present an algebraic characterization of compatibility of W_{n-2}-stars, which is essentially based on the fact that if the intersection of stabilizers of equivalence classes of W_{n-1}-stars contains a maximal abelian subgroup of $\mathrm{Out}(W_n)$, then the W_{n-1}-stars are pairwise compatible (see Propositions 6.13 and 8.1). We deduce that the map $\mathrm{Comm}(\mathrm{Out}(W_n)) \to \mathrm{Bij}(VX_n)$ extends to a map $\mathrm{Comm}(\mathrm{Out}(W_n)) \to \mathrm{Aut}(X_n)$, which completes our proof.

Finally, we prove in the appendix the rigidity of another natural graph endowed with an $\mathrm{Out}(W_n)$-action, called the *graph of W_{n-1}-stars*. It is the graph whose vertices are W_n-equivariant homeomorphism classes of W_{n-1}-stars, where two vertices $\mathcal{S}$ and $\mathcal{S}'$ are adjacent if there exist $S \in \mathcal{S}$ and $S' \in \mathcal{S}'$ such that S and S' have a common refinement. This graph arises naturally in the study of $\mathrm{Out}(W_n)$ and its action on the free splitting graph $\overline{K}_n$ as it is isomorphic to the full subgraph of $\overline{K}_n$ whose vertices are the equivalence classes of W_k-stars, with k varying in $\{0, \ldots, n-1\}$. This gives another geometric rigid model for $\mathrm{Out}(W_n)$ (see Theorem A.1).

2. Preliminaries

2.1. Commensurations

Let G be a group. The *abstract commensurator* of G, denoted by $\mathrm{Comm}(G)$, is the group whose elements are the equivalence classes of isomorphisms between finite index subgroups of G for the following equivalence relation. Two isomorphisms between finite index subgroups $f\colon H_1 \to H_2$ and $f'\colon H_1' \to H_2'$ are equivalent if they agree on some common finite index subgroup H of their domains. If f is an isomorphism between finite index subgroups, we denote by $[f]$ the equivalence class of f. The identity of $\mathrm{Comm}(G)$ is the equivalence class of the identity map on G. Let $[f], [f'] \in \mathrm{Comm}(G)$, and let $f\colon H_1 \to H_2$ and $f'\colon H_1' \to H_2'$ be representatives. The composition law $[f] \cdot [f']$ is given by $[f] \cdot [f'] = [f \circ f'|_{f'^{-1}(H_1) \cap H_1'}]$. Note that if H is a finite index subgroup of G, then the natural map $\mathrm{Comm}(G) \to \mathrm{Comm}(H)$ obtained by restriction is an isomorphism.

Two subgroups G_1 and G_2 in G are commensurable if $G_1 \cap G_2$ has finite index in both G_1 and G_2. Being commensurable is an equivalence relation. If H is a subgroup of G, we will denote by $[H]$ its commensurability class in G. The group $\mathrm{Comm}(G)$ acts on the set of all commensurability classes as follows. Let $[H]$ be the commensurability class of a subgroup H. Let $[f] \in \mathrm{Comm}(G)$ and let $f\colon H_1 \to H_2$ be a representative of $[f]$. Then we define $[f] \cdot [H]$ by setting $[f] \cdot [H] = [f(H \cap H_1)]$.

The next result, due to Horbez and Wade, gives a sufficient condition for $\mathrm{Comm}(G)$ to be rigid. It comes from ideas due to Ivanov when studying mapping class groups (see [26]). It requires the existence of a graph on which G acts by graph automorphisms.

Proposition 2.1 ([25, Proposition 1.1]). *Let G be a group. Let X be simplicial graph such that G acts on X by graph automorphisms. Let $\mathrm{Aut}(X)$ be the group of graph automorphisms of X. Assume that*

(1) *the natural homomorphism $G \to \mathrm{Aut}(X)$ is an isomorphism,*

(2) *given two distinct vertices v and w of X, the groups $\mathrm{Stab}_G(v)$ and $\mathrm{Stab}_G(w)$ are not commensurable in G,*

(3) *the sets $\mathcal{I} = \{[\mathrm{Stab}_G(v)] \mid v \in VX\}$, $\mathcal{J} = \{([\mathrm{Stab}_G(v)], [\mathrm{Stab}_G(w)]) \mid vw \in EX\}$ are $\mathrm{Comm}(G)$-invariant (in the latter case with respect to the diagonal action).*

Then any isomorphism $f\colon H_1 \to H_2$ between finite index subgroups of G is given by the conjugation by an element of G and the natural map $G \to \mathrm{Comm}(G)$ is an isomorphism.

2.2. Free splittings and free factor systems of W_n

Let n be an integer greater than 1. Let $F = \mathbb{Z}/2\mathbb{Z}$ be a cyclic group of order 2 and $W_n = \ast_n F$ be a universal Coxeter group of rank n. A *splitting* of W_n is a minimal, simplicial W_n-action on a simplicial tree S such that:

(1) The finite graph $W_n \backslash S$ is not empty and not reduced to a point.

(2) Vertices of S with trivial stabilizer have degree at least 3.

Here *minimal* means that W_n does not preserve any proper subtree of S. A splitting S of W_n is *free* if all edge stabilizers are trivial. A splitting S' is a *blow-up*, or equivalently a *refinement*, of a splitting S if S is obtained from S' by collapsing some edge orbits in S'. Two splittings are *compatible* if they have a common refinement. We define an equivalence class in the set of free splittings, where two splittings S and S' are equivalent if there exists a W_n-equivariant homeomorphism between them.

A *free factor system* of W_n is a set $\mathcal{F}$ of conjugacy classes of subgroups of W_n which arises as the set of all conjugacy classes of nontrivial point stabilizers in some (nontrivial) free splitting of W_n. Equivalently, there exist $k \in \mathbb{N}\backslash\{0,1\}$ and $[A_1], \dots, [A_k]$ conjugacy classes of nontrivial, proper subgroups of W_n such that $W_n = A_1 * \dots * A_k$ and $\mathcal{F} = \{[A_1], \dots, [A_k]\}$. The free factor system is *sporadic* if $k = 2$, and *nonsporadic* otherwise. The set of all free factor systems of W_n has a natural partial order, where $\mathcal{F} \leq \mathcal{F}'$ if every factor of $\mathcal{F}$ is conjugate into one of the factors of $\mathcal{F}'$. Remark that if $\{x_1, \dots, x_n\}$ is a standard generating set of W_n, then for every free factor system $\mathcal{F}$ of W_n and every $i \in \{1, \dots, n\}$, there exists $[A] \in \mathcal{F}$ such that x_i is conjugate into A. In other words, the free factor system $\{[x_1], \dots, [x_n]\}$ is a minimum for the partial order on the set of free factor systems of W_n.

Let $\mathcal{F}$ be a free factor system of W_n. We denote by $\mathrm{Out}(W_n, \mathcal{F})$ the subgroup of $\mathrm{Out}(W_n)$ consisting of all outer automorphisms that preserve all the conjugacy classes of subgroups in $\mathcal{F}$. If $\mathcal{F} = \{[A_1], \dots, [A_k]\}$, we denote by $\mathrm{Out}(W_n, \mathcal{F}^{(t)})$ the subgroup of $\mathrm{Out}(W_n, \mathcal{F})$ consisting of all outer automorphisms which have a representative whose restriction to each A_i with $i \in \{1, \dots, k\}$ is a global conjugation by some $g_i \in W_n$.

A $(W_n, \mathcal{F})$*-tree* is an $\mathbb{R}$-tree equipped with a W_n-action by isometries and such that every subgroup of W_n whose conjugacy class belongs to $\mathcal{F}$ is elliptic. A *free splitting of* W_n *relative to* $\mathcal{F}$ is a free splitting of W_n such that every free factor in $\mathcal{F}$ is elliptic. A *free factor of* $(W_n, \mathcal{F})$ is a subgroup of W_n which arises as a point stabilizer in a free splitting of W_n relative to $\mathcal{F}$. A free factor of $(W_n, \mathcal{F})$ is *proper* if it is nontrivial, not equal to W_n and not conjugate to an element of $\mathcal{F}$. An element $g \in W_n$ is $\mathcal{F}$*-peripheral* (or simply *peripheral* if there is no ambiguity) if it is conjugate into one of the subgroups of $\mathcal{F}$, and $\mathcal{F}$*-nonperipheral* otherwise. In particular, for every free factor system $\mathcal{F}$ of W_n, and every element $x \in W_n$ appearing in a standard generating set of W_n, we see that x is $\mathcal{F}$-peripheral.

2.3. The outer space of $(W_n, \mathcal{F})$

Recall the definition of the *unprojectivized outer space of* $(W_n, \mathcal{F})$, denoted by $\mathcal{O}(W_n, \mathcal{F})$ and introduced by Guirardel and Levitt in [17]. It is the set of all $(W_n, \mathcal{F})$-equivariant isometry classes $\mathcal{S}$ of metric simplicial trees with a nontrivial action of W_n, with trivial arc stabilizers and such that a subgroup is elliptic if and only if it is peripheral. The set $\mathcal{O}(W_n, \mathcal{F})$ is equipped with the Gromov–Hausdorff equivariant topology introduced in [31]. The *projectivized outer space of* $(W_n, \mathcal{F})$, denoted by $\mathbb{P}\mathcal{O}(W_n, \mathcal{F})$, is defined as the space of homothety classes of trees in $\mathcal{O}(W_n, \mathcal{F})$. The spaces $\mathcal{O}(W_n, \mathcal{F})$ and $\mathbb{P}\mathcal{O}(W_n, \mathcal{F})$ come equipped with a right action of $\mathrm{Out}(W_n, \mathcal{F})$ given by precomposition of the actions.

The space $\mathbb{P}\mathcal{O}(W_n, \mathcal{F})$ has a natural structure of a simplicial complex with missing faces. Indeed, every element $\mathcal{S} \in \mathbb{P}\mathcal{O}(W_n, \mathcal{F})$ defines an open simplex as follows. Let S be a representative of $\mathcal{S}$ such that the sum of the edge lengths of $W_n \backslash S$ is equal to 1. We associate an open simplex by varying the lengths of the edges, so that the sum of the edge lengths is still equal to 1. A homothety class $\mathcal{S}' \in \mathbb{P}\mathcal{O}(W_n, \mathcal{F})$ of a splitting S' defines a codimension 1 face of the simplex associated with $\mathcal{S}$ if we can obtain S' from some representative S of $\mathcal{S}$ by contracting one orbit of edges in S.

The closure $\overline{\mathcal{O}(W_n, \mathcal{F})}$ of outer space in the space of all isometry classes of minimal nontrivial W_n-actions on $\mathbb{R}$-trees, equipped with the Gromov–Hausdorff equivariant topology, was identified in [24] with the space of all *very small* $(W_n, \mathcal{F})$*-trees*, which are the $(W_n, \mathcal{F})$-trees whose arc stabilizers are either trivial, or cyclic, root-closed and nonperipheral, and whose tripod stabilizers are trivial. The space $\mathbb{P}\overline{\mathcal{O}(W_n, \mathcal{F})}$ equipped with the quotient topology is compact (see [24, Theorem 1]).

We recall the definition of a simplicial complex on which the space $\mathbb{P}\mathcal{O}(W_n, \mathcal{F})$ retracts $\mathrm{Out}(W_n, \mathcal{F})$-equivariantly, called the *spine of outer space of* $(W_n, \mathcal{F})$ and denoted by $K(W_n, \mathcal{F})$. It is the flag complex whose vertices are the W_n-equivariant homeomorphism classes $\mathcal{S}$ of free splittings relative to $\mathcal{F}$ with the property that, if $S \in \mathcal{S}$, then all elliptic subgroups in S are peripheral. Two vertices $\mathcal{S}$ and $\mathcal{S}'$ in $K(W_n, \mathcal{F})$ are linked by an edge if there exist $S \in \mathcal{S}$ and $S' \in \mathcal{S}'$ such that S refines S' or conversely. There is an embedding $F \colon K(W_n, \mathcal{F}) \hookrightarrow \mathbb{P}\mathcal{O}(W_n, \mathcal{F})$ whose image is the barycentric spine of $\mathbb{P}\mathcal{O}(W_n, \mathcal{F})$. We will from now on identify $K(W_n, \mathcal{F})$ with $F(K(W_n, \mathcal{F}))$. If $\mathcal{F}$ consists of exactly n copies of F, we simply write K_n for $K(W_n, \mathcal{F})$. In this case, the dimension of the simplicial complex K_n is $n-2$. Indeed, if $\mathcal{S}$ is an equivalence class of a free splitting S in K_n such that the number of edges of $W_n \backslash S$ is minimal, then, the number of edges in $W_n \backslash S$ is equal to $n-1$. If $\mathcal{S}$ is an equivalence class of a free splitting S in K_n such that the number of edges of $W_n \backslash S$ is maximal, then $W_n \backslash S$ has n leaves and every vertex of $W_n \backslash S$ that is not a leaf has degree equal to 3. As S is a tree, this shows that the number of edges in $W_n \backslash S$ is equal to $2n-3$. Since, every splitting S of K_n collapses onto a splitting S' such that $W_n \backslash S'$ has $n-1$ edges, we see that the dimension of K_n is equal to $2n-3-(n-1) = n-2$.

The *free splitting graph of* W_n, denoted by $\bar{K}_n$, is the following graph. The vertices of $\bar{K}_n$ are the W_n-equivariant homeomorphism classes of free splittings. Two distinct equivalence classes $\mathcal{S}$ and $\mathcal{S}'$ are joined by an edge in $\bar{K}_n$ if there exist $S \in \mathcal{S}$ and $S' \in \mathcal{S}'$ such that S refines S' or conversely. The free splitting graph of W_n is the 1-skeleton of the closure of K_n in the space of free splittings of W_n. The group $\mathrm{Aut}(W_n)$ acts on $\bar{K}_n$ on the right by precomposition of the action. As $\mathrm{Inn}(W_n)$ acts trivially on $\bar{K}_n$, the action of $\mathrm{Aut}(W_n)$ induces an action of $\mathrm{Out}(W_n)$ on $\bar{K}_n$.

2.4. The free factor graph of $(W_n, \mathcal{F})$

Let $\mathcal{F}$ be a free factor system of W_n. We now define a Gromov hyperbolic graph on which $\mathrm{Out}(W_n, \mathcal{F})$ acts by isometries. The *free factor graph relative to* $\mathcal{F}$, denoted by $\mathrm{FF}(W_n, \mathcal{F})$, is the following graph. Its vertices are the W_n-equivariant homeomorphism classes of free splittings of W_n relative to $\mathcal{F}$. Two equivalence classes $\mathcal{S}$ and $\mathcal{S}'$ are joined by an edge if there exist $S \in \mathcal{S}$ and $S' \in \mathcal{S}'$ such that S and S' are compatible or share a common nonperipheral elliptic element. The free factor graph is always hyperbolic (see [3, 16, 21]). The next proposition is due to Guirardel and Horbez. Here, if H is a subgroup of $\mathrm{Out}(W_n)$ and if $\mathcal{F}$ is a free factor system of W_n, we say that $\mathcal{F}$ is *H-periodic* if there exists a finite index subgroup H' of H such that $H'(\mathcal{F}) = \mathcal{F}$.

Proposition 2.2 ([16, Theorem 5.1]). *Let $n \geq 3$ and let $\mathcal{F}$ be a nonsporadic free factor system of W_n. Let H be a subgroup of $\mathrm{Out}(W_n, \mathcal{F})$ which acts on $\mathrm{FF}(W_n, \mathcal{F})$ with bounded orbits. Then there exists an H-periodic free factor system $\mathcal{F}'$ such that $\mathcal{F} \leq \mathcal{F}'$ and $\mathcal{F} \neq \mathcal{F}'$.*

The Gromov boundary of $\mathrm{FF}(W_n, \mathcal{F})$ has been described in terms of *relatively arational trees* (see the work of Reynolds [33] for the definition of an arational tree in the context of a free group, the work of Bestvina and Reynolds and the work of Hamenstädt [4, 20] for the description of the boundary in the case of a free group, and the work of Guirardel and Horbez [16] in the case of a free product). A $(W_n, \mathcal{F})$-tree T is *arational* if no proper $(W_n, \mathcal{F})$-free factor acts elliptically on T and, for every proper $(W_n, \mathcal{F})$-free factor A, the A-minimal invariant subtree of T (that is the union of the axes of the loxodromic elements of A for the action of W_n on T, see [8, Proposition 3.1]) is a simplicial A-tree in which every nontrivial point stabilizer can be conjugated into one of the subgroups of $\mathcal{F}$. We equip each arational $(W_n, \mathcal{F})$-tree with the *observers' topology*: this is the topology on a tree T such that a basis of open sets is given by the connected components of the complements of points in T. We equip the set of arational $(W_n, \mathcal{F})$-trees with an equivalence relation, where two arational $(W_n, \mathcal{F})$-trees are equivalent if they are W_n-equivariantly homeomorphic with the observers' topology.

Theorem 2.3 ([16, Theorem 3.4]). *Let $n \geq 3$. Let $\mathcal{F}$ be a nonsporadic free factor system of W_n. The Gromov boundary of $\mathrm{FF}(W_n, \mathcal{F})$ is $\mathrm{Out}(W_n, \mathcal{F})$-equivariantly homeomorphic to the space of all equivalence classes of arational $(W_n, \mathcal{F})$-trees.*

Lemma 2.4 ([14, Proposition 13.5]). *Let $n \geq 3$. Let $\mathcal{F}$ be a nonsporadic free factor system of W_n, and let H be a subgroup of $\mathrm{Out}(W_n, \mathcal{F})$. If H fixes a point in $\partial_\infty \mathrm{FF}(W_n, \mathcal{F})$, H has a finite-index subgroup that fixes the homothety class of an arational $(W_n, \mathcal{F})$-tree.*

2.5. Groups of twists

Let S be a splitting of W_n, let $v \in VS$, let e be an edge with origin v, and let z be an element of the centralizer $C_{G_v}(G_e)$ of G_v in G_e. We define the *twist by z around e* to be the automorphism $D_{e,z}$ of W_n defined as follows (see [28]). Let $\overline{S}$ be the splitting obtained from S by collapsing all the edges of S outside of the orbit of e. Then $\overline{S}$ is a tree. Let $\overline{e}$ be the image of e in $\overline{S}$ and let $\overline{v}$ be the image of v in $\overline{S}$. Let $\overline{w}$ be the endpoint of $\overline{e}$ distinct from $\overline{v}$. The automorphism $D_{e,z}$ is defined to be the unique automorphism that acts as the identity on $G_{\overline{v}}$ and as conjugation by z on $G_{\overline{w}}$. The element z is called the *twistor of $D_{e,z}$*. It is well defined up to composing on the right by an element of $C_{W_n}(G_{\overline{w}}) \cap C_{G_v}(G_e)$. The *group of twists of S* is the subgroup of $\mathrm{Out}(W_n)$ generated by all twists around oriented edges of S.

We now give a description of the stabilizer of a point in $\overline{K}_n$ due to Levitt. If $\mathcal{S} \in V\overline{K}_n$, we denote by $\mathrm{Stab}(\mathcal{S})$ the stabilizer of $\mathcal{S}$ under the action of $\mathrm{Out}(W_n)$. Let S be a representative of $\mathcal{S}$. We denote by $\mathrm{Stab}^0(\mathcal{S})$ the subgroup of $\mathrm{Stab}(\mathcal{S})$ consisting of all elements $F \in \mathrm{Out}(W_n)$ such that the graph automorphism induced by F on $W_n \backslash S$ is the identity.

Proposition 2.5 ([28, Propositions 2.2, 3.1 and 4.2]). *Let $n \geq 4$ and $\mathcal{S} \in V\overline{K}_n$. Let S be a representative of $\mathcal{S}$ and let $v_1, \dots, v_k$ be the vertices of $W_n \backslash S$ with nontrivial associated groups. For $i \in \{1, \dots, k\}$, let G_i be the group associated with v_i.*

(1) *The group* $\mathrm{Stab}^0(\mathcal{S})$ *fits in an exact sequence*

$$1 \to \mathcal{T} \to \mathrm{Stab}^0(\mathcal{S}) \to \prod_{i=1}^{k} \mathrm{Out}(G_i) \to 1,$$

where $\mathcal{T}$ is the group of twists of S.

(2) *The group* $\mathrm{Stab}^0(\mathcal{S})$ *is isomorphic to*

$$\prod_{i=1}^{k} G_i^{\deg(v_i)-1} \rtimes \mathrm{Aut}(G_i),$$

where $\mathrm{Aut}(G_i)$ *acts on $G_i^{\deg(v_i)-1}$ diagonally.*

(3) *The group of twists $\mathcal{T}$ of S is isomorphic to*

$$\mathcal{T} \simeq \bigoplus_{i=1}^{k} G_i^{\deg(v_i)}/Z(G_i),$$

where the center $Z(G_i)$ of G_i is embedded diagonally in $G_i^{\deg(v_i)}$.

Remark 2.6. In [28, Proposition 2.2], Levitt shows that the kernel of the natural homomorphism $\mathrm{Stab}^0(\mathcal{S}) \to \prod_{i=1}^{k} \mathrm{Out}(G_i)$ given by the action on the vertex groups is generated by *bitwists*. Since edge stabilizers are trivial, the group of bitwists is equal to the group of twists. More generally (see [28, Proposition 2.3]), if the outer automorphism group of every edge stabilizer is finite (in particular, if edge stabilizers are isomorphic to $\mathbb{Z}$ or to F), then the group of twists is a finite index subgroup of the group of bitwists.

Finally, if the centralizer in W_n of an edge stabilizer is trivial, then the group of bitwists about this edge is trivial. Therefore, if the edge stabilizer is not cyclic, then the group of bitwists about this edge is trivial. In all cases, we see that, for every equivalence class $\mathcal{S}$ of a splitting S of W_n, the group of twists of $\mathcal{S}$ is a finite index subgroup of the group of bitwists of W_n.

We establish one last fact about twists around edges whose centralizer is cyclic (see [7, Lemma 5.3] for a similar statement in the context of the outer automorphism group of a nonabelian free group).

Lemma 2.7. *Let $n \geq 3$ and let $\mathcal{S}$ be the equivalence class of a splitting S. Suppose that there exists an edge e of S with cyclic stabilizer and let D be the outer automorphism class of a twist about e. Let $H_{\mathcal{S}}$ be the subgroup of* $\mathrm{Stab}^0(\mathcal{S})$ *which induces the identity on the edge stabilizer G_e of e. Then D is central in $H_{\mathcal{S}}$. In particular,* $\mathrm{Stab}^0(\mathcal{S})$ *has a finite index subgroup $H_{\mathcal{S}}$ such that D is central in $H_{\mathcal{S}}$.*

Proof. Let U be a splitting onto which S collapses (or S itself if S does not have a nontrivial collapse), and let $\mathcal{U}$ be its equivalence class. Then $\mathrm{Stab}^0(\mathcal{S}) \subseteq \mathrm{Stab}^0(\mathcal{U})$. Thus, we may suppose, up to collapsing all orbits of edges of S except the one containing e, that S has exactly one orbit of edges. Let v and w be the two endpoints of e and let G_v and G_w be their edge stabilizers. Let $f \in H_{\mathcal{S}}$ and let F be a representative of f such that $F(G_v) = G_v$, $F(G_w) = G_w$ and $F|_{G_e} = \mathrm{id}_{G_e}$ (this representative exists since $f \in H_{\mathcal{S}}$). Let $z \in C_{G_v}(G_e)$ be such that $D_{e,z}$ is a representative of D. Then, since $F(z) = z$, for every $x \in W_n$, we have $D_{e,z} \circ F \circ D_{e,z}^{-1}(x) = F(x)$. Hence f and D commute, and D is central in $H_{\mathcal{S}}$. Since the outer automorphism group of a cyclic group is finite, we see that $H_{\mathcal{S}}$ is a finite index subgroup of $\mathrm{Stab}^0(\mathcal{S})$. This concludes the proof. ∎

3. Geometric rigidity in the graph of W_k-stars

We start by defining W_k*-stars*, which are the main splittings of interest in this article.

Definition 3.1. Let $n \geq 3$, and let $k \geq 1$ be an integer.

(1) A free splitting S is a *k-edge free splitting* if $W_n \backslash S$ has exactly k edges.

(2) Suppose that $0 \leq k \leq n-2$. A *W_k-star* is an $(n-k)$-edge free splitting such that:

- the underlying graph of $W_n \backslash S$ has $n-k+1$ vertices and one of them, called the *center of* $W_n \backslash S$, has degree exactly $n-k$,
- the group associated with the center of $W_n \backslash S$ is isomorphic to W_k (we use the convention that $W_0 = \{1\}$ and that $W_1 = F$),
- the group associated with any leaf of $W_n \backslash S$ is isomorphic to F.

(3) A *W_{n-1}-star* is a one-edge free splitting S such that one of the vertex groups of $W_n \backslash S$ is isomorphic to W_{n-1} while the other vertex group is isomorphic to F.

Note that, in [13], a W_{n-1}-star is called an *F-one-edge free splitting*. Using Proposition 2.5 (2), we see that, if $k \in \{0, \dots, n-2\}$, and if $\mathcal{S}$ is the equivalence class of a W_k-star, then the group $\mathrm{Stab}^0(\mathcal{S})$ is isomorphic to $W_k^{n-k-1} \rtimes \mathrm{Aut}(W_k)$.

Note that, if S is a W_k-star with $k \in \{0, \dots, n-2\}$ and S' is a splitting on which S collapses, then there exists $\ell \in \{k, \dots, n-1\}$ such that S' is a W_ℓ-star. In particular, for every $k \in \{0, \dots, n-2\}$, if S is a W_k-star, then every one-edge free splitting on which S collapses is a W_{n-1}-star. A similar statement is also true for refinements of W_k-stars (see Lemma 3.8).

3.1. Rigidity of the graph of W_*-stars

We introduce in this section a graph, *the graph of one-edge compatible* W_{n-2}*-stars*, on which $\mathrm{Out}(W_n)$ acts by simplicial automorphisms. We prove that this graph is a rigid geometric model for $\mathrm{Out}(W_n)$. The proof relies on the study of the rigidity of an additional graph on which $\mathrm{Out}(W_n)$ acts, the *graph of* W_**-stars*, to be defined after Theorem 3.3.

Definition 3.2. (1) The *graph of* W_{n-2}*-stars*, denoted by $\widetilde{X}_n$, is the graph whose vertices are the W_n-equivariant homeomorphism classes of W_{n-2}-stars, where two equivalence classes $\mathcal{S}$ and $\mathcal{S}'$ are joined by an edge if there exist $S \in \mathcal{S}$ and $S' \in \mathcal{S}'$ such that S and S' are compatible.

(2) The *graph of one-edge compatible* W_{n-2}*-stars*, denoted by X_n, is the graph whose vertices are the W_n-equivariant homeomorphism classes of W_{n-2}-stars where two equivalence classes $\mathcal{S}$ and $\mathcal{S}'$ are joined by an edge if there exist $S \in \mathcal{S}$ and $S' \in \mathcal{S}'$ such that S and S' have a common refinement which is a W_{n-3}-star.

Note that the adjacency in the graph X_n is equivalent to having both a common collapse (which is a W_{n-1}-star) and a common refinement. The graph X_n is a subgraph of $\widetilde{X}_n$. The group $\mathrm{Aut}(W_n)$ acts on $\widetilde{X}_n$ and X_n by precomposition of the action. As $\mathrm{Inn}(W_n)$ acts trivially on X_n, the action of $\mathrm{Aut}(W_n)$ induces an action of $\mathrm{Out}(W_n)$. We denote by $\mathrm{Aut}(X_n)$ the group of graph automorphisms of X_n. In Section 3.2, we prove the following theorem.

Theorem 3.3. *Let $n \geq 5$. The natural homomorphism*

$$\mathrm{Out}(W_n) \to \mathrm{Aut}(X_n)$$

is an isomorphism.

In order to prove this theorem, we take advantage of the action of $\mathrm{Out}(W_n)$ on another graph, namely the *graph of* W_**-stars*, denoted by X_n'. The vertices of this graph are the W_n-equivariant homeomorphism classes of W_k-stars, with k varying in $\{0, \dots, n-2\}$. Two equivalence classes $\mathcal{S}$ and $\mathcal{S}'$ are joined by an edge if there exist $S \in \mathcal{S}$ and $S' \in \mathcal{S}'$ such that S refines S' or conversely. Note that we have a natural embedding $X_n' \hookrightarrow \bar{K}_n$. We identify from now on X_n' with its image in $\bar{K}_n$. In this section, we prove the following theorem.

Theorem 3.4. *Let $n \geq 5$. The natural homomorphism*

$$\mathrm{Out}(W_n) \to \mathrm{Aut}(X_n')$$

is an isomorphism.

Theorem 3.4 relies on the fact that X_n' contains a rigid subgraph, namely the graph of $\{0\}$-stars and F-stars, and denoted by L_n. The vertices of this graph are the W_n-equivariant homeomorphism classes of $\{0\}$-stars and F-stars. Two equivalence classes $\mathcal{S}$ and $\mathcal{S}'$ are joined by an edge if there exist $S \in \mathcal{S}$ and $S' \in \mathcal{S}'$ such that S refines S' or conversely.

We recall the following theorem.

Theorem 3.5 ([13, Theorem 3.1, Corollary 3.2]). *Let $n \geq 4$. Let f be an automorphism of L_n preserving the set of $\{0\}$-stars and the set of F-stars. Then f is induced by the action of a unique element γ of $\mathrm{Out}(W_n)$. In particular, for every $n \geq 5$, the natural homomorphism*

$$\mathrm{Out}(W_n) \to \mathrm{Aut}(L_n)$$

is an isomorphism.

The strategy in order to prove Theorem 3.4 is to show that every automorphism of X_n' preserves L_n and that the natural map $\mathrm{Aut}(X_n') \to \mathrm{Aut}(L_n)$ is injective.

Remark 3.6. Using the same techniques, we may prove that the *graph of W_{n-1}-stars* is rigid. This is done in the appendix (see Theorem A.1).

First we recall a theorem due to Scott and Swarup.

Theorem 3.7 ([34, Theorem 2.5]). *Let $n \geq 4$. Any set $\{S_1, \ldots, S_k\}$ of pairwise nonequivalent, pairwise compatible, one-edge free splittings of W_n has a unique refinement S such that $W_n \backslash S$ has exactly k edges. Moreover, the equivalence class of S only depends on the equivalence classes of $S_1, \ldots, S_k$. If S is a free splitting such that $W_n \backslash S$ has exactly k edges, then S refines exactly k pairwise nonequivalent one-edge free splittings.*

We also need the following lemma concerning refinements of W_k-stars.

Lemma 3.8. *Let $k, \ell \in \{0, \ldots, n-1\}$ and let S and S' be respectively a W_k-star and a W_ℓ-star. If S and S' have a common refinement, then there exists $j \in \{0, \ldots, n-2\}$ and a W_j-star S'' which refines both S and S'. Moreover, S'' can be chosen such that S'' is a refinement of S and S' with the minimal number of orbits of edges.*

Proof. Let $S_1, \ldots, S_{n-k}$ be $n-k$ W_{n-1}-stars onto which S collapses and let $S_1', \ldots, S_{n-\ell}'$ be $n-\ell$ W_{n-1}-stars onto which S' collapses. Then the set $\{S_1, \ldots, S_{n-k}, S_1', \ldots, S_{n-\ell}'\}$ is a set of pairwise compatible W_{n-1}-stars. For every $s \in \{1, \ldots, n-k\}$ and every $t \in \{1, \ldots, n-\ell\}$, let $\mathcal{S}_s$ be the equivalence class of S_s and $\mathcal{S}_t'$ be the equivalence class of S_t'. Let $n-j = |\{\mathcal{S}_1, \ldots, \mathcal{S}_{n-k}, \mathcal{S}_1', \ldots, \mathcal{S}_{n-\ell}'\}|$. By Theorem 3.7, there exists a free splitting S'' with $n-j$ edges which refines every W_{n-1}-star of the set $\{S_1, \ldots, S_{n-k}, S_1', \ldots, S_{n-\ell}'\}$. But, as F is freely indecomposable, a common refinement of two W_{n-1}-stars U and U' is obtained from U by blowing-up an edge at the vertex of $W_n \backslash U$ whose associated group is isomorphic to W_{n-1}. Since U' is also a W_{n-1}-star, this common refinement has two orbits of edges and the two corresponding leaves have a stabilizer isomorphic to F, hence it is a W_{n-2}-star. The same argument shows that, if U_0 is a W_{n-1}-star and if U_1 is a W_k-star with $k \in \{1, \ldots, n-1\}$ compatible with U_0, then a common refinement of U_0 and U_1 with a minimal number of orbits of edges is either a W_k-star (if the equivalence classes of U_0 and U_1 are adjacent in $\bar{K}_n$) or a W_{k-1}-star. Therefore, by induction on $i \in \{1, \ldots, n-\ell\}$, we see that a common refinement of $\{S_1, \ldots, S_{n-k}, S_1', \ldots, S_{n-\ell}'\}$

with the minimal number of orbits of edges is a W_j-star. This shows that S'' is a W_j-star. This concludes the proof. ∎

Lemma 3.8 implies that the set of W_k-stars with k varying in $\{0,\dots,n-1\}$ is closed under taking collapse and taking refinement with a minimal number of orbits of edges.

Lemma 3.9. *Let $n \geq 5$. For every $f \in \mathrm{Aut}(X_n')$, we have $f(L_n) = L_n$. Moreover, if $f|_{L_n} = \mathrm{id}_{L_n}$, then $f = \mathrm{id}_{X_n'}$.*

Proof. Let $f \in \mathrm{Aut}(X_n')$. The fact that $f(L_n) = L_n$ follows from the fact that vertices of $K_n \cap X_n'$ in X_n' are characterized by the fact that they are the vertices with finite valence. The proof is identical to the proof of [13, Proposition 5.1].

Now suppose that $f|_{L_n} = \mathrm{id}_{L_n}$ and let $\mathcal{S}$ be the equivalence class of a W_{n-2}-star S. Let us prove that $f(\mathcal{S}) = \mathcal{S}$. Let $\{x_1,\dots,x_n\}$ be a standard generating set of W_n such that the free factor decomposition of W_n induced by S is

$$W_n = \langle x_1\rangle * \langle x_2,\dots,x_{n-1}\rangle * \langle x_n\rangle.$$

Let $\mathcal{X}$ be the equivalence class of the F-star X depicted in Figure 1.

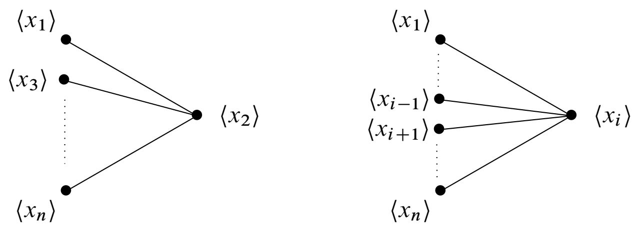

Figure 1. The F-stars X (on the left) and X' (on the right) of the proof of Lemma 3.9.

We see that $\mathcal{S}$ and $\mathcal{X}$ are adjacent in X_n'. Therefore, as $f(\mathcal{X}) = \mathcal{X}$, we see that $f(\mathcal{S})$ and $\mathcal{X}$ are adjacent in X_n'.

Let $\mathcal{S}'$ be the equivalence class of a W_{n-2}-star adjacent to $\mathcal{X}$ and distinct from $\mathcal{S}$. Then, as $\mathcal{X}$ and $\mathcal{S}'$ are adjacent, there exist distinct $i,j \in \{1,\dots,n\}$ with $i,j \neq 2$ and a representative S' of $\mathcal{S}'$ such that the free factor decomposition of W_n induced by S' is

$$W_n = \langle x_i\rangle * \langle x_1,\dots,\widehat{x_i},\dots,\widehat{x_j},\dots,x_n\rangle * \langle x_j\rangle.$$

Since $\mathcal{S} \neq \mathcal{S}'$, we may suppose that $i \notin \{1,n\}$. But then $\mathcal{S}$ is adjacent to the equivalence class $\mathcal{X}'$ of the F-star X' depicted in Figure 1 whereas $\mathcal{S}'$ is not adjacent to $\mathcal{X}'$. Since $f(\mathcal{X}') = \mathcal{X}'$, this shows that $f(\mathcal{S}) \neq \mathcal{S}'$.

Finally, let $k \in \{2,\dots,n-3\}$ and let $\mathcal{S}^{(2)}$ be the equivalence class of a W_k-star $S^{(2)}$ which is adjacent to $\mathcal{X}$. We prove that $f(\mathcal{S}) \neq \mathcal{S}^{(2)}$. Since $k \leq n-3$, the underlying graph of $W_n \backslash S^{(2)}$ has at least 3 edges. Therefore, there exists $i \notin \{1,n\}$ and a leaf v of the underlying graph of $W_n \backslash S^{(2)}$ such that the preimage by the marking of $W_n \backslash S^{(2)}$ of the generator of the group associated with v is x_i. But then the equivalence class $\mathcal{S}^{(2)}$ is not adjacent to the equivalence class $\mathcal{X}'$ of the F-star X' depicted in Figure 1. As $\mathcal{S}$ is adjacent to $\mathcal{X}'$ and as $f(\mathcal{X}') = \mathcal{X}'$, we see that $f(\mathcal{S}) \neq \mathcal{S}^{(2)}$. Therefore, $f(\mathcal{S}) = \mathcal{S}$.

The above paragraphs show that f fixes pointwise the set of equivalence classes of W_{n-2}-stars. Let $k \in \{2, \dots, n-3\}$ and let $\mathcal{T}$ be the equivalence class of a W_k-star T. By Theorem 3.7, the equivalence class $\mathcal{T}$ is uniquely determined by the set of W_{n-1}-stars on which T collapses. Since two distinct equivalence classes of W_{n-2}-stars are adjacent in $\overline{K}_n$ to distinct pairs of equivalence classes of W_{n-1}-stars, the equivalence class $\mathcal{T}$ is uniquely determined by the set of W_{n-2}-stars on which it collapses. Since f fixes pointwise the set of equivalence classes of W_{n-2}-stars, we see that $f(\mathcal{T}) = \mathcal{T}$. Hence $f = \mathrm{id}_{X'_n}$. This concludes the proof. ■

Proof of Theorem 3.4. Let $n \geq 5$. We first prove the injectivity. By Theorem 3.5, the homomorphism $\mathrm{Out}(W_n) \to \mathrm{Aut}(L_n)$ is injective. Moreover, it factors through $\mathrm{Out}(W_n) \to \mathrm{Aut}(X'_n) \to \mathrm{Aut}(L_n)$. We therefore deduced the injectivity of $\mathrm{Out}(W_n) \to \mathrm{Aut}(X'_n)$. We now prove the surjectivity. Let $f \in \mathrm{Aut}(X'_n)$. By Lemma 3.9, we have a homomorphism $\Phi \colon \mathrm{Aut}(X'_n) \to \mathrm{Aut}(L_n)$ defined by restriction. By Theorem 3.5, the automorphism $\Phi(f)$ is induced by an element $\gamma \in \mathrm{Out}(W_n)$. Since the homomorphism $\mathrm{Aut}(X'_n) \to \mathrm{Aut}(L_n)$ is injective by Lemma 3.9, f is induced by γ. This concludes the proof. ■

3.2. Rigidity of the graph of one-edge compatible W_{n-2}-stars

In this section, we prove Theorem 3.3. In order to do so, we construct an injective homomorphism

$$\mathrm{Aut}(X_n) \to \mathrm{Aut}(X'_n).$$

First, we need to show some technical results concerning the graph X_n. Indeed, let Δ be a triangle (that is, a cycle of length 3) in X_n, and let $\mathcal{S}_1$, $\mathcal{S}_2$ and $\mathcal{S}_3$ be the vertices of this triangle. By Theorem 3.7, for every $i \in \{1, 2, 3\}$, there exists $S_i \in \mathcal{S}_i$ such that S_1, S_2 and S_3 have a common refinement S, and we suppose that S has the minimal number of orbits of edges among the common refinements of S_1, S_2 and S_3. Since S_1, S_2 and S_3 are W_{n-2}-stars, there exists $k \in \{0, \dots, n-3\}$ such that S is a W_k-star. By definition of the adjacency in X_n, the splitting S is either a W_{n-3}-star or a W_{n-4}-star (see Figure 2). Our first result shows that we can distinguish these two types of triangles.

Lemma 3.10. *Let $n \geq 5$. Let $\mathcal{S}_1, \mathcal{S}_2$ and $\mathcal{S}_3$ be three equivalence classes of W_{n-2}-stars which are pairwise adjacent in X_n. Let S_1, S_2 and S_3 be representatives of $\mathcal{S}_1$, $\mathcal{S}_2$ and $\mathcal{S}_3$ which have a common refinement S. Suppose that S is the refinement of S_1, S_2 and S_3 which has the minimal number of orbit of edges. Then S is a W_{n-4}-star if and only if there exists an equivalence class $\mathcal{S}_4$ of a W_{n-2}-star S_4 distinct from $\mathcal{S}_1$, $\mathcal{S}_2$ and $\mathcal{S}_3$ such that, for every $i \in \{1, 2, 3\}$, the equivalence classes $\mathcal{S}_i$ and $\mathcal{S}_4$ are adjacent in X_n.*

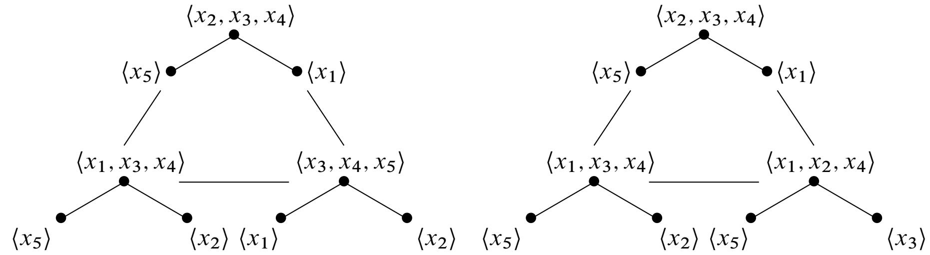

Figure 2. Two triangles in X_n, one corresponding to a W_{n-3}-star (on the left) and one corresponding to a W_{n-4}-star (on the right).

Proof. Suppose first that S is a W_{n-4}-star. Let $\{x_1, \dots, x_n\}$ be a standard generating set of W_n such that the free factor decomposition of W_n induced by S is

$$W_n = \langle x_1 \rangle * \langle x_2 \rangle * \langle x_3 \rangle * \langle x_4 \rangle * \langle x_5, \dots, x_n \rangle.$$

Since being adjacent in X_n is equivalent to having a common refinement which is a W_{n-3}-star and having a common collapse which is a W_{n-1}-star, the W_{n-2}-stars S_1 and S_2 share a common collapse S' which is a W_{n-1}-star. Let $\mathcal{S}'$ be the equivalence class of S'. We claim that there exists an orbit of edges E in S_3 such that the splitting obtained from S_3 by collapsing every orbit of edges of S_3 except E is in $\mathcal{S}'$. Indeed, suppose towards a contradiction that this is not the case. Then, as for every $i \in \{1,2\}$, the equivalence classes $\mathcal{S}_i$ and $\mathcal{S}_3$ are adjacent in X_n, we see that, for every $i \in \{1,2\}$, the splittings S_i and S_3 share a common collapse onto a W_{n-1}-star S_i'. Recall that we supposed that there does not exist an orbit of edges E in S_3 such that the splitting obtained from S_3 by collapsing every orbit of edges of S_3 except E is in $\mathcal{S}'$. This implies that for every $i \in \{1,2\}$, the equivalence class $\mathcal{S}_i'$ of S_i' is distinct from $\mathcal{S}'$. Since S_1 and S_2 are W_{n-2}-stars, they collapse onto exactly 2 distinct W_{n-1}-stars. Therefore, for every $i \in \{1,2\}$, the equivalence classes $\mathcal{S}'$ and $\mathcal{S}_i'$ are the two equivalence classes of W_{n-1}-stars onto which S_i collapses. It follows that a common refinement of S_1', S_2' and S' is also a common refinement of S_1, S_2 and S_3. But a common refinement of S_1', S_2' and S' is a W_{n-3}-star. This contradicts the fact that S has the minimal number of edges among common refinements of S_1, S_2 and S_3. Thus S_3 collapses onto a W_{n-1}-star in the equivalence class $\mathcal{S}'$. Let $j \in \{1, \dots, 4\}$ be such that the free factor decomposition of W_n induced by S' is

$$W_n = \langle x_j \rangle * \langle x_1, \dots, \widehat{x_j}, \dots, x_n \rangle.$$

Let $\mathcal{S}_4$ be the equivalence class of the W_{n-2}-star S_4 whose induced free factor decomposition is

$$W_n = \langle x_j \rangle * \langle x_1, \dots, \widehat{x_5}, \dots, \widehat{x_j}, \dots, x_n \rangle * \langle x_5 \rangle.$$

Then, for every $i \in \{1,2,3\}$, the equivalence classes $\mathcal{S}_4$ and $\mathcal{S}_i$ are adjacent in X_n.

Conversely, suppose that S is a W_{n-3}-star. Let $\{x_1, \dots, x_n\}$ be a standard generating set of W_n such that the free factor decomposition of W_n induced by S is

$$W_n = \langle x_1 \rangle * \langle x_2 \rangle * \langle x_3 \rangle * \langle x_4, \dots, x_n \rangle.$$

Then, up to reordering, we may suppose that, for every $i \in \{1, 2, 3\}$ the free factor decomposition of W_n induced by S_i is

$$W_n = \langle x_i \rangle * \langle x_{i+1} \rangle * \langle x_1, \dots, \widehat{x_i}, \widehat{x_{i+1}}, \dots, x_n \rangle,$$

where, for $i = 3$, the index $i + 1$ is taken modulo 3. Let $\mathcal{S}'$ be the equivalence class of a W_{n-2}-star S' adjacent to $\mathcal{S}_1$ in X_n and distinct from $\mathcal{S}_2$ and $\mathcal{S}_3$. Then, up to changing the representative S', there exists $j \in \{1, 2\}$ such that S' collapses onto the W_{n-1}-star whose associated free factor decomposition is

$$W_n = \langle x_j \rangle * \langle x_1, \dots, \widehat{x_j}, \dots, x_n \rangle.$$

If $j = 1$, then, as $\mathcal{S}'$ is distinct from $\mathcal{S}_1$ and $\mathcal{S}_3$, we see that $\mathcal{S}'$ is not adjacent to $\mathcal{S}_2$ in X_n. If $j = 2$, then, as $\mathcal{S}'$ is distinct from $\mathcal{S}_1$ and $\mathcal{S}_2$, we see that $\mathcal{S}'$ is not adjacent to $\mathcal{S}_3$ in X_n. In both cases, we see that there exists $i \in \{2, 3\}$ such that $\mathcal{S}'$ is not adjacent to $\mathcal{S}_i$. This concludes the proof. ∎

Corollary 3.11. *Let $n \geq 5$. Let $k \geq 4$ and let $\mathcal{S}_1, \dots, \mathcal{S}_k$ be k equivalences classes of W_{n-2}-stars which are pairwise adjacent in X_n. For $i \in \{1, \dots, k\}$, let S_i be a representative of $\mathcal{S}_i$. Let S be a refinement of $S_1, \dots, S_k$ whose number of orbits of edges is minimal. Then S is a W_{n-k-1}-star.*

Proof. For all distinct $i, j \in \{1, \dots, k\}$, the equivalence classes $\mathcal{S}_i$ and $\mathcal{S}_j$ are adjacent in X_n. Hence, for every distinct $i, j \in \{1, \dots, k\}$, there exists a common refinement of S_i and S_j which is a W_{n-3}-star. This implies that, for every $p \in \{1, \dots, k\}$ and for all $i_1, \dots, i_p \in \{1, \dots, k\}$, a common refinement of $S_{i_1}, \dots, S_{i_p}$ is obtained from a common refinement of $S_{i_1}, \dots, S_{i_{p-1}}$ whose number of orbits of edges is minimal by adding at most one orbit of edges. We claim that a common refinement of $S_{i_1}, \dots, S_{i_p}$ whose number of orbits of edges is minimal has exactly $p + 1$ orbits of edges. Indeed, otherwise there would exist $i, j, \ell \in \{1, \dots, k\}$ pairwise distinct such that a W_{n-3}-star which refines both S_i and S_j also refines S_ℓ. This is not possible by Lemma 3.10 since $k \geq 4$. This proves the claim. Taking $p = k$ concludes the proof. ∎

Proposition 3.12. *Let $n \geq 5$. There exists a* $\mathrm{Out}(W_n)$*-equivariant injective homomorphism* $\widetilde{\Phi}\colon \mathrm{Aut}(X_n) \to \mathrm{Aut}(X_n')$.

Proof. We first exhibit a map $\Phi\colon \mathrm{Aut}(X_n) \to \mathrm{Bij}(VX_n')$. Let $f \in \mathrm{Aut}(X_n)$. Let $\mathcal{S}$ be the equivalence class of a W_k-star S with $k \in \{0, \dots, n-2\}$. If $k = n-2$, then we set $\Phi(f)(\mathcal{S}) = f(\mathcal{S})$. If $k \leq n-3$, let S_0 be a W_{n-1}-star refined by S. Let $S_1, \dots, S_{n-k-1}$ be the W_{n-2}-stars such that, for every $i \in \{1, \dots, n-k-1\}$, S refines S_i and S_i refines S_0

(see Figure 3). For every $i \in \{1, \dots, n-k-1\}$, let $\mathcal{S}_i$ be the equivalence class of S_i, and let T_i be a representative of $f(\mathcal{S}_i)$. By Corollary 3.11, if $n-k-1 \geq 4$, the W_{n-2}-stars $T_1, \dots, T_{n-k-1}$ are refined by a W_k-star T'. This W_k-star is unique up to W_n-equivariant homeomorphism by Theorem 3.7. In the case where $k = n-3$, we have $n-k-1 = 2$ and, since $f(\mathcal{S}_1)$ and $f(\mathcal{S}_2)$ are adjacent in X_n, the splittings T_1 and T_2 are refined by a W_{n-3}-star T' and it is unique up to W_n-equivariant homeomorphism by Theorem 3.7. Finally, when $k = n-4$, Lemma 3.10 implies that a common refinement of T_1, T_2 and T_3 with the minimal number of orbits of edges is a W_{n-4}-star T', and it is unique up to W_n-equivariant homeomorphism by Theorem 3.7. In all cases, let $\mathcal{T}'$ be the equivalence class of T'. We set $\Phi(f)(\mathcal{S}) = \mathcal{T}'$.

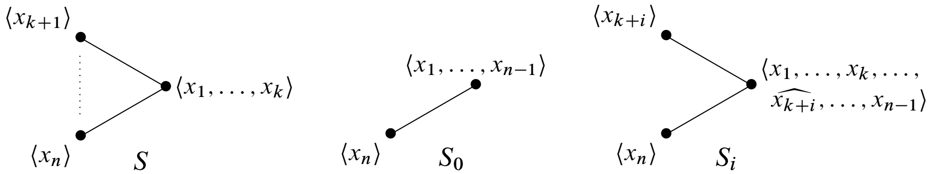

Figure 3. The construction of the map $\mathrm{Aut}(X_n) \to \mathrm{Aut}(X_n')$.

We now prove that Φ is well defined. Let $k \in \{0, \dots, n-2\}$ and let $\mathcal{S}$ be the equivalence class of a W_k-star S. Let S_0 and S_0' be two distinct W_{n-1}-stars onto which S collapses and let $\mathcal{S}_0$ and $\mathcal{S}_0'$ be their equivalence classes. Let $S_1, \dots, S_{n-k-1}$ be the W_{n-2}-stars such that, for every $i \in \{1, \dots, n-k-1\}$, S refines S_i and S_i refines S_0 and let $S_1', \dots, S_{n-k-1}'$ be the W_{n-2}-stars such that, for every $i \in \{1, \dots, n-k-1\}$, S refines S_i' and S_i' refines S_0'. For $i \in \{1, \dots, n-k-1\}$, let $\mathcal{S}_i$ be the equivalence class of S_i and let $\mathcal{S}_i'$ be the equivalence class of S_i'. For every $i \in \{1, \dots, n-k-1\}$, let T_i be a representative of $f(\mathcal{S}_i)$ and let T_i' be a representative of $f(\mathcal{S}_i')$. Let T be a W_k-star which refines $T_1, \dots, T_{n-k-1}$ and let T' be a W_k-star which refines $T_1', \dots, T_{n-k-1}'$. Finally, let $\mathcal{T}$ be the equivalence class of T and let $\mathcal{T}'$ be the equivalence class of T'. We claim that $\mathcal{T} = \mathcal{T}'$. Indeed, we first remark that there exist $i, j \in \{1, \dots, n-k-1\}$ such that $\mathcal{S}_i = \mathcal{S}_j'$: it is the equivalence class of the W_{n-2}-star which refines both S_0 and S_0'. Up to reordering, we may suppose that $i = j = 1$, that $S_1 = S_1'$ and that $T_1 = T_1'$. Therefore, both T and T' collapse onto T_1.

Let $U_2, \dots, U_{n-k-1}$ be the W_{n-3}-stars such that, for every $j \in \{2, \dots, n-k-1\}$, the W_{n-3}-star U_j refines S_1 and U_j is refined by S. For every $j \in \{2, \dots, n-k-1\}$, there exist $\ell, \ell' \in \{2, \dots, n-k-1\}$ such that S_ℓ and $S_{\ell'}'$ are refined by U_j. Therefore, the map $g\colon \{2, \dots, n-k-1\} \to \{2, \dots, n-k-1\}$ sending ℓ to ℓ' is a bijection. Thus, we may suppose that g is the identity, that is, we may suppose that $j = \ell = \ell'$. It follows that for every $j \in \{2, \dots, n-k-1\}$, the equivalence class of the W_{n-3}-star which refines S_1 and S_j is the same one as the equivalence class of the W_{n-3}-star which refines S_1 and S_j'. Therefore, for every $i \in \{2, \dots, n-k-1\}$, the set $\{\mathcal{S}_1, \mathcal{S}_i, \mathcal{S}_i'\}$ defines a triangle in X_n which corresponds to the equivalence class of a W_{n-3}-star. By Lemma 3.10, for every

$i \in \{2, \ldots, n-k-1\}$, the set $\{f(\mathcal{S}_1), f(\mathcal{S}_i), f(\mathcal{S}_i')\}$ defines a triangle in X_n which corresponds to the equivalence class of a W_{n-3}-star. Thus, up to changing the representative T_i', for every $i \in \{1, \ldots, n-k-1\}$, the W_{n-3}-star which refines T_1 and T_i is the same one as the W_{n-3}-star which refines T_1 and T_i'. As $\mathcal{T}$ and $\mathcal{T}'$ are characterized by the set of equivalence classes of W_{n-3}-stars which collapses onto T_1 and on which T and T' collapse, we see that $\mathcal{T} = \mathcal{T}'$. Therefore, the map $\Phi(f)\colon VX_n' \to VX_n'$ is well defined. As $\Phi(f) \circ \Phi(f^{-1}) = \Phi(f \circ f^{-1}) = \mathrm{id}$, we see that $\Phi(f)$ is a bijection.

We now prove that the map $\Phi\colon \mathrm{Aut}(X_n) \to \mathrm{Bij}(VX_n')$ induces a monomorphism $\widetilde{\Phi}\colon \mathrm{Aut}(X_n) \to \mathrm{Aut}(X_n')$. Let $f \in \mathrm{Aut}(X_n)$ and let us prove that $\Phi(f)$ preserves EX_n'. Let $\mathcal{S}$, $\mathcal{S}'$ be adjacent vertices in X_n'. Up to exchanging the roles of $\mathcal{S}$ and $\mathcal{S}'$, we may suppose that there exist $S \in \mathcal{S}$ and $S' \in \mathcal{S}'$ such that S' collapses onto S. Let $k, \ell \in \{1, \ldots, n-2\}$ be such that S is a W_k-star and S' is a $W_{k-\ell}$-star. Let S_0 be a W_{n-1}-star such that S refines S_0. Let $S_1, \ldots, S_{n-k-1}$ be the W_{n-2}-stars such that, for every $i \in \{1, \ldots, n-k-1\}$, S refines S_i and S_i refines S_0. As S' refines S, there exist ℓ W_{n-2}-stars $S_{n-k}, \ldots, S_{n-k+\ell-1}$ such that the W_{n-2}-stars $S_1, \ldots, S_{n-k+\ell-1}$ are the $n-k+\ell-1$ W_{n-2}-stars which collapse onto S_0 and which are refined by S'. For every $i \in \{1, \ldots, n-k+\ell-1\}$, let $\mathcal{S}_i$ be the equivalence class of S_i. By definition of $\Phi(f)$, there exist a representative T of $\Phi(f)(\mathcal{S})$ and representatives $T_1, \ldots, T_{n-k-1}$ of $f(\mathcal{S}_1), \ldots, f(\mathcal{S}_{n-k-1})$ such that T is a common refinement of $T_1, \ldots, T_{n-k-1}$. Moreover, there exist a representative T' of $\Phi(f)(\mathcal{S}')$ and representatives $T_{n-k}, \ldots, T_{n-k+\ell-1}$ of $f(\mathcal{S}_{n-k}), \ldots, f(\mathcal{S}_{n-k+\ell-1})$ such that this T' is a common refinement of $f(\mathcal{S}_1), \ldots, f(\mathcal{S}_{n-k+\ell-1})$. As $\{f(\mathcal{S}_1), \ldots, f(\mathcal{S}_{n-k-1})\}$ is a subset of $\{f(\mathcal{S}_1), \ldots, f(\mathcal{S}_{n-k+\ell-1})\}$, we see that $f(\mathcal{S})$ and $f(\mathcal{S}')$ are adjacent. This shows that the application $\Phi(f)\colon VX_n \to VX_n'$ induces a homomorphism $\widetilde{\Phi}\colon \mathrm{Aut}(X_n) \to \mathrm{Aut}(X_n')$. Finally, the facts that $\widetilde{\Phi}$ is injective and is $\mathrm{Out}(W_n)$-equivariant follow from the fact that, for every equivalence class $\mathcal{S}$ of W_{n-2}-stars, we have $f(\mathcal{S}) = \Phi(f)(\mathcal{S})$. This concludes the proof. ∎

Proof of Theorem 3.3. Let $n \geq 5$. We first prove the injectivity. By Theorem 3.4, the homomorphism $\mathrm{Out}(W_n) \to \mathrm{Aut}(X_n')$ is injective. Moreover, it factors through $\mathrm{Out}(W_n) \to \mathrm{Aut}(X_n) \to \mathrm{Aut}(X_n')$. We therefore deduce the injectivity of $\mathrm{Out}(W_n) \to \mathrm{Aut}(X_n)$. We now prove the surjectivity. Let $f \in \mathrm{Aut}(X_n)$. By Proposition 3.12, we have a homomorphism $\widetilde{\Phi}\colon \mathrm{Aut}(X_n) \to \mathrm{Aut}(X_n')$. By Theorem 3.4, the automorphism $\widetilde{\Phi}(f)$ is induced by an element $\gamma \in \mathrm{Out}(W_n)$. Since the homomorphism $\mathrm{Aut}(X_n) \to \mathrm{Aut}(X_n')$ is injective by Proposition 3.12, f is induced by γ. This concludes the proof. ∎

4. The group of twists of a W_{n-1}-star

In this section, we study the centralizers in $\mathrm{Out}(W_n)$ of twists about a W_{n-1}-star. We first show that to a free factor of W_n isomorphic to W_{n-1}, one can associate a canonical equivalence class of W_{n-1}-star (see Lemma 4.4). We then show that, for an outer automorphism f in the stabilizer of the equivalence class $\mathcal{S}$ of a W_{n-1}-star, there exists a canonical representative F of f such that f commutes with a twist T of $\mathcal{S}$ if and only if F fixes the twistor of T (see Lemma 4.12). We first need some preliminary results about stabilizers of free factors of W_n isomorphic to W_{n-1}.

Let $\{x_1, \dots, x_n\}$ be a standard generating set of W_n. For distinct $i, j \in \{1, \dots, n\}$, let $\sigma_{j,i}\colon W_n \to W_n$ be the automorphism sending x_j to $x_i x_j x_i$ and, for $k \neq j$, fixing x_k. For distinct $i, j \in \{1, \dots, n\}$, let $(i\ j)$ be the automorphism of W_n switching x_i and x_j and, for $k \neq i, j$, fixing x_k. The following theorem is due to Mühlherr.

Theorem 4.1 ([30, Theorem B]). *Let $n \geq 2$. The set $\{\sigma_{i,j} \mid i \neq j\} \cup \{(i\ j) \mid i \neq j\}$ is a generating set of* $\mathrm{Aut}(W_n)$.

We now introduce a finite index subgroup of $\mathrm{Out}(W_n)$ which will be used throughout the remainder of this paper. For all $i, j \in \{1, \dots, n\}$ distinct, both $\sigma_{i,j}$ and $(i\ j)$ preserve the set of conjugacy classes $\{[x_1], \dots, [x_n]\}$. Since $\{\sigma_{i,j} \mid i \neq j\} \cup \{(i\ j) \mid i \neq j\}$ generates $\mathrm{Aut}(W_n)$ by Theorem 4.1, we see that we have a well-defined homomorphism $\mathrm{Out}(W_n) \to \mathrm{Bij}(\{[x_1], \dots, [x_n]\})$. Let $\mathrm{Out}^0(W_n)$ be the kernel of this homomorphism. The group $\mathrm{Out}^0(W_n)$ has finite index in $\mathrm{Out}(W_n)$. We will mostly work in $\mathrm{Out}^0(W_n)$ from now on because of the following lemma.

Lemma 4.2. *Let $n \geq 3$ and let $f \in \mathrm{Out}^0(W_n)$. Suppose that f fixes the equivalence class $\mathcal{S}$ of a free splitting S. Then the graph automorphism of the underlying graph of $W_n \backslash S$ induced by f is the identity. Therefore, we have*

$$\mathrm{Stab}_{\mathrm{Out}^0(W_n)}(\mathcal{S}) = \mathrm{Stab}^0_{\mathrm{Out}^0(W_n)}(\mathcal{S}).$$

Proof. The underlying graph $\overline{W_n \backslash S}$ of $W_n \backslash S$ is a tree. Moreover, since S is a free splitting, if L is the set of leaves of $\overline{W_n \backslash S}$, then the set $\{[G_v]\}_{v \in L}$ is a free factor system of W_n. Note that, as $\{[x_1], \dots, [x_n]\}$ is a free factor system of W_n which is minimal for inclusion, for every $i \in \{1, \dots, n\}$, there exists one $v \in VS$ such that $x_i \in G_v$. Since S is a free splitting, for every $i \in \{1, \dots, n\}$, the element x_i is contained in a unique vertex group. Moreover, for every $v \in L$, there exist $k \in \{0, \dots, n-1\}$ and $\{i_1, \dots, i_k\} \subseteq \{1, \dots, n\}$ such that G_v is isomorphic to W_k and $\{[x_{i_1}] \cap G_v, \dots, [x_{i_k}] \cap G_v\}$ is a free factor system of G_v. As $f \in \mathrm{Out}^0(W_n)$, and as f fixes $\mathcal{S}$, it follows that, for every $v \in L$, we have $f([G_v]) = [G_v]$. Hence the graph automorphism $\widehat{f}$ of $\overline{W_n \backslash S}$ induced by f acts as the identity on L. As any graph automorphism of a finite tree is determined by its action on the set of leaves, it follows that $\widehat{f} = \mathrm{id}$. This concludes the proof. ∎

Remark 4.3. The subgroup $\mathrm{Out}^0(W_n)$ of $\mathrm{Out}(W_n)$ is our (weak) analogue of the subgroup $\mathrm{IA}_N(\mathbb{Z}/3\mathbb{Z})$ of $\mathrm{Out}(F_N)$, which is defined as the kernel of the natural homomorphism $\mathrm{Out}(F_N) \to \mathrm{GL}(N, \mathbb{Z}/3\mathbb{Z})$. Indeed, the group $\mathrm{IA}_N(\mathbb{Z}/3\mathbb{Z})$ satisfies a statement similar to Lemma 4.2, but it has the additional property that if $\phi \in \mathrm{IA}_N(\mathbb{Z}/3\mathbb{Z})$ has a periodic orbit in the free splitting graph of F_N, then the cardinality of this orbit is equal to 1. In the context of $\mathrm{Out}^0(W_n)$, we do not know if $\mathrm{Out}^0(W_n)$ contains a torsion free finite index subgroup which satisfies this property.

The next lemma relates the stabilizer of a free factor of W_n isomorphic to W_{n-1} and the stabilizer of a W_{n-1}-star.

Lemma 4.4. *Let $n \geq 3$. Let A be a free factor of W_n isomorphic to W_{n-1}. Then, up to W_n-equivariant homeomorphism, there exists a unique free splitting S in which A is elliptic. In particular, if $f \in \mathrm{Out}(W_n)$ is such that $f([A]) = [A]$, then f fixes the equivalence class of S.*

Proof. By definition of a free factor, there exists a free splitting S of W_n such that A is elliptic in S. This proves the existence. We now prove the uniqueness statement. We may assume that $\{x_1, \dots, x_{n-1}\}$ is a standard generating set of A and $x_n \in W_n$ is such that

$$W_n = A * \langle x_n \rangle.$$

Then, the free factor system $\mathcal{F} = \{[A], [\langle x_n \rangle]\}$ is a sporadic free factor system which contains $[A]$. Let $\mathcal{F}'$ be a free factor system of W_n which contains $[A]$. Since the free factor system $\{[\langle x_1 \rangle], \dots, [\langle x_n \rangle]\}$ is the minimal element of the set of free factor systems of W_n, we see that there exists $[B] \in \mathcal{F}'$ such that $x_n \in B$. As $\mathcal{F}'$ contains $[A]$ and as $W_n = A * \langle x_n \rangle$, it follows that $W_n = A * B$ and that $B \subseteq \langle x_n \rangle$. Therefore,

$$[B] = [\langle x_n \rangle] \quad \text{and} \quad \mathcal{F}' = \{[A], [\langle x_n \rangle]\}.$$

We deduce that $\mathcal{F}$ is the unique nontrivial free factor system which contains $[A]$. But the spine $K(W_n, \mathcal{F})$ of the outer space relative to $\mathcal{F}$ is reduced to a point, i.e., it is reduced to a unique equivalence class of free splittings. This proves the uniqueness statement. ∎

Remark 4.5. In the context of $\mathrm{Out}(F_N)$, the analogue of the splitting given by Lemma 4.4 is the following one. Let $[A]$ be the conjugacy class of a free factor of F_N isomorphic to F_{N-1}. Then the canonical splitting associated with A is the splitting corresponding to the HNN extension $F_N = A*$ over the trivial group. However, there does not exist a natural choice (up to conjugacy) of an element $g \in F_N$ such that $\{[A], [\langle g \rangle]\}$ is a free factor system of F_N.

Let S be a splitting with exactly one orbit of edges, whose stabilizer is root-closed and isomorphic to $\mathbb{Z}$. Then the group of twists of S is isomorphic to $\mathbb{Z}$ by a result of Levitt (see [28, Proposition 3.1]). The next proposition is similar to a result in the case of the outer automorphism group of a free group (see [6] and [25, Lemma 2.7]). Recall that an element $w \in W_n$ is *root-closed* if there does not exist $w_0 \in W_n$ and an integer $n \geq 2$ such that $w = w_0^n$.

Lemma 4.6. *Let $n \geq 3$. Let A be a free factor of W_n isomorphic to W_{n-1} and let $w \in A$ be a root-closed element of infinite order. Let $x \in W_n$ be such that $W_n = A * \langle x \rangle$. Let $\mathcal{S}$ be the equivalence class of a splitting S whose associated amalgamated decomposition of W_n is the following:*

$$W_n = A *_{\langle w \rangle} (\langle w \rangle * \langle x \rangle).$$

Let D be a nontrivial twist about S. Let $\mathcal{R}$ be the equivalence class of a free splitting R of W_n such that $D(\mathcal{R}) = \mathcal{R}$. Let R' and S' be metric representatives of R and S, let $\mathcal{R}'$ and $\mathcal{S}'$ be their W_n-equivariant isometry classes and let $[\mathcal{R}']$ and $[\mathcal{S}']$ be their homothety classes.

(1) *In $\mathbb{P}\overline{\mathcal{O}(W_n)}$, there exists an increasing function $\psi\colon \mathbb{N} \to \mathbb{N}$ such that*

$$\lim_{n\to\infty} D^{\psi(n)}([\mathcal{R}']) = [\mathcal{S}'].$$

(2) *The splittings S and R are compatible.*

Proof. We prove the first part. As $\mathbb{P}\overline{\mathcal{O}(W_n)}$ is compact, up to passing to a subsequence, there exist a sequence $(\lambda_n)_{n\in\mathbb{N}} \in (\mathbb{R}_+^*)^{\mathbb{N}}$ and a W_n-equivariant isometry class $\mathcal{T}$ of an $\mathbb{R}$-tree T such that

$$\lim_{n\to\infty} \lambda_n D^n(\mathcal{R}') = \mathcal{T}.$$

Since translation length functions are continuous for the Gromov–Hausdorff topology (see [31]), for every $g \in W_n$, we have

$$\lim_{n\to\infty} \lambda_n \|g\|_{D^n(\mathcal{R}')} = \|g\|_{\mathcal{T}},$$

where $\|g\|_{\mathcal{T}}$ is the translation length of g in T. Hence, for every $g \in W_n$, the limit $\lim_{n\to\infty} \lambda_n \|g\|_{D^n(\mathcal{R}')}$ is finite. But as D has infinite order, we have $\lim_{n\to\infty} \lambda_n = 0$. As there exists a representative $\phi \in \mathrm{Aut}(W_n)$ of D such that $\phi_A = \mathrm{id}_A$, for every $g \in A$, we have

$$\lim_{n\to\infty} \lambda_n \|g\|_{D^n(\mathcal{R}')} = \lim_{n\to\infty} \lambda_n \|g\|_{\mathcal{R}'} = 0.$$

Hence every element of A fixes a point in T. As A is finitely generated, this implies that A fixes a point in T (see for instance [8, Section 3]). Similarly, we see that $\langle w\rangle * \langle x\rangle$ fixes a point in T. As $W_n = A * \langle x\rangle$, we see that A and $\langle w\rangle * \langle x\rangle$ cannot fix the same point in T. Let U be the free splitting of W_n associated with the free factor decomposition $W_n = A * \langle x\rangle$. Let v_0 be the vertex of U fixed by A, let v_1 be the vertex fixed by x and let v_2 be the vertex fixed by wxw^{-1}. Let e_1 be the edge between v_0 and v_1 and e_2 be the edge between v_0 and v_2. The arguments above show that we have a canonical W_n-equivariant morphism from U to T. This morphism is obtained by a fold of the edges e_1 and e_2 of U and this fold is extended W_n-equivariantly. Since w is root-closed, there is no other edge of U that can be folded as otherwise the stabilizer of an edge of T would not be cyclic. Therefore, the $\mathbb{R}$-tree T is simplicial and the decomposition of W_n associated with $W_n\backslash T$ is

$$W_n = A *_{\langle w\rangle} (\langle w\rangle * \langle x\rangle).$$

Hence $\mathcal{T} = \mathcal{S}'$ and the first statement follows.

Let us prove the second statement. For every $n \in \mathbb{N}$, the equivalence classes $\lambda_n D^n(\mathcal{R})$ and $\mathcal{R}$ have compatible representatives. But as $\lim_{n\to\infty} \lambda_n D^n(\mathcal{R}) = \mathcal{S}$, it follows from [19, Corollary A.12] that, in the limit, the splittings S and R are compatible. ∎

Lemma 4.7. *Let $n \geq 3$ and let $\mathcal{S}$ be the equivalence class of a W_{n-1}-star S. Let T be the group of twists of $\mathcal{S}$ and let $f \in T$ be an element of infinite order. Let $\mathcal{R}$ be the equivalence class of a W_{n-1}-star R such that $f(\mathcal{R}) = \mathcal{R}$. Then S and R are compatible.*

Proof. Let

$$W_n = A * \langle x_n \rangle$$

be a free factor decomposition of W_n associated with S and let $z_f \in A$ be the twistor of f. Let z be a root-closed element of A such that there exists $m \geq 1$ with $z^m = z_f$. Let $h \in T$ be the twist about z. We see that $h^m = f$. Let S' be the splitting associated with the following amalgamated decomposition of W_n:

$$W_n = A *_{\langle z \rangle} (\langle x_n \rangle * \langle z \rangle).$$

Let $\mathcal{S}'$ be the equivalence class of S'. Let T' be the group of twists of $\mathcal{S}'$. Since A is isomorphic to W_{n-1} and since z is root-closed, we see that $C_A(z) = \langle z \rangle$. Therefore, T' is isomorphic to $\mathbb{Z}$ and a generator of T' is h. As $f(\mathcal{R}) = \mathcal{R}$, Lemma 4.6 implies that S' and R are compatible. Let U be a common refinement of S' and R whose number of orbits of edges is minimal. Since both S' and R are one-edge splittings and are different, the splitting U has 2 orbits of edges. It follows that $W_n \backslash U$ is obtained from $W_n \backslash S'$ by blowing-up an edge at one of the two vertices of $W_n \backslash S'$. Let $\widetilde{v}$ be the vertex of S' whose stabilizer is A and let v be its image in $W_n \backslash S'$. Let $\widetilde{w}$ be the vertex of S' fixed by $\langle x_n \rangle * \langle z \rangle$ and let w be its image in $W_n \backslash S'$.

Claim 4.8. *Either $\mathcal{S} = \mathcal{R}$ or the splitting $W_n \backslash U$ is obtained from $W_n \backslash S'$ by blowing-up an edge at v.*

Proof. Suppose that $W_n \backslash U$ is obtained from $W_n \backslash S'$ by blowing-up an edge at w. Then, since the group G_w associated with w is $\langle x_n \rangle * \langle z \rangle$ and since z must fix an edge of U, we see that a free splitting of G_w, such that z fixes a vertex, is a $(G_w, \{\langle z \rangle, \langle x_n \rangle\})$-free splitting. But $(G_w, \{\langle z \rangle, \langle x_n \rangle\})$ has exactly one such equivalence class of one-edge free splitting: the one with vertex stabilizers conjugated with $\langle z \rangle$ and $\langle x_n \rangle$. This implies that $\mathcal{R} = \mathcal{S}$. The claim follows. ∎

Suppose that $\mathcal{R} \neq \mathcal{S}$. The claim implies that the amalgamated decomposition of W_n associated with U is

$$W_n = B * C *_{\langle z \rangle} (\langle z \rangle * \langle x_n \rangle),$$

where B and C are free factors of W_n such that $A = B * C$ and $z \in C$. Let U' be a refinement of U whose associated amalgamated decomposition of W_n is

$$W_n = B * C *_{\langle z \rangle} \langle z \rangle * \langle x_n \rangle,$$

that is, z and x_n fix distinct points in U'. Then, since $A = B * C$, the splitting U' is a refinement of S. This concludes the proof of Lemma 4.7. ∎

Proposition 4.9. *Let $n \geq 3$. Let S be a W_{n-1}-star and let $f \in \mathrm{Out}(W_n)$ be a twist about the unique edge of $W_n \backslash S$. Let $g \in \mathrm{Out}^0(W_n)$ be such that $g \in C_{Out(W_n)}(f)$. Then $g(\mathcal{S}) = \mathcal{S}$.*

Proof. Let

$$W_n = \langle x_1, \dots, x_{n-1} \rangle * \langle x_n \rangle$$

be the free factor decomposition associated with S and let $\mathcal{S}$ be the equivalence class of S. By Lemma 4.4, in order to prove that $g(\mathcal{S}) = \mathcal{S}$, it suffices to show that g preserves the conjugacy class of $A = \langle x_1, \dots, x_{n-1} \rangle$. Let $\widetilde{f}$ be a representative of f such that $\widetilde{f}|_A = \mathrm{id}_A$. Let $\widetilde{g}$ be a representative of g. Suppose towards a contradiction that $\widetilde{g}$ does not preserve the conjugacy class of A. By hypothesis, there exists $I \in \mathrm{Inn}(W_n)$ such that $\widetilde{f} \circ \widetilde{g} = I \circ \widetilde{g} \circ \widetilde{f}$. Thus,

$$\widetilde{f} \circ \widetilde{g}(A) = I \circ \widetilde{g} \circ \widetilde{f}(A) = I \circ \widetilde{g}(A).$$

Therefore, f preserves the conjugacy class of $\widetilde{g}(A)$. By Lemma 4.4, f fixes the unique equivalence class $\mathcal{R}$ of the W_{n-1}-star R associated with $\widetilde{g}(A)$. By Lemma 4.7, the splittings S and R are compatible. Since we suppose that $\widetilde{g}(A) \notin [A]$, there exists a common refinement S' of S and R which is a W_{n-2}-star. Thus, there exists $y_n \in W_n$ such that the free factor decomposition associated with S' is

$$W_n = \langle x_n \rangle * B * \langle y_n \rangle,$$

where B is such that $A = B * \langle y_n \rangle$ and $B * \langle x_n \rangle$ is a conjugate of $\widetilde{g}(A)$. Up to changing the representative $\widetilde{g}(A)$, we may suppose that $\widetilde{g}(A) = B * \langle x_n \rangle$. This implies that $x_n \in \widetilde{g}(A)$, that is $\widetilde{g}^{-1}(x_n) \in A$. But, since $A = \langle x_1, \dots, x_{n-1} \rangle$, we see that $[\widetilde{g}^{-1}(x_n)] \in \{[x_1], \dots, [x_{n-1}]\}$. This contradicts the fact that $g \in \mathrm{Out}^0(W_n)$. ■

Combining Lemma 4.7 and Proposition 4.9, we have the following corollary.

Corollary 4.10. *Let $n \geq 3$. Let $\mathcal{S}$ and $\mathcal{R}$ be two distinct W_n-equivariant homeomorphism classes of two W_{n-1}-stars S and R. Let f and g be twists about respectively S and R such that f and g commute. Then S and R are compatible.*

Proof. Let $k \geq 1$ be such that $g^k \in \mathrm{Out}^0(W_n)$. By Proposition 4.9, since g^k and f commute, we have $g^k(\mathcal{S}) = \mathcal{S}$. Since g^k is a twist about $\mathcal{R}$, by Lemma 4.7, we have that S and R are compatible. ■

Let $\mathcal{S}$ be the equivalence class of a W_{n-1}-star S and let

$$W_n = \langle x_1, \dots, x_{n-1} \rangle * \langle x_n \rangle$$

be the free factor decomposition of W_n associated with S. Let $A = \langle x_1, \dots, x_{n-1} \rangle$. Let $f \in \mathrm{Stab}_{\mathrm{Out}(W_n)}(\mathcal{S})$. Then any representative of f sends A to a conjugate of itself. Let $\widetilde{f}'$ be a representative of f such that $\widetilde{f}'(A) = A$. Since the vertices in S fixed by A

and x_n are adjacent, and since the stabilizer of every vertex in S adjacent to the vertex fixed by A is a conjugate of $\langle x_n\rangle$ by an element of A, we see that $\widetilde{f}'(x_n) = xx_nx^{-1}$ with $x \in A$. Therefore, there exists a representative $\widetilde{f}$ of f such that $\widetilde{f}(A) = A$ and $\widetilde{f}(x_n) = x_n$. The automorphism $\widetilde{f}$ is the unique representative of f such that $\widetilde{f}(A) = A$ and $\widetilde{f}(x_n) = x_n$.

We have a similar result for W_{n-2}-stars. Indeed, let $\mathcal{S}'$ be the equivalence class of a W_{n-2}-star S' and let

$$W_n = \langle x_1\rangle * \langle x_2, \dots, x_{n-1}\rangle * \langle x_n\rangle$$

be the free factor decomposition of W_n associated with S' and let $B = \langle x_2, \dots, x_{n-1}\rangle$. Let $f \in \mathrm{Stab}_{\mathrm{Out}^0(W_n)}(\mathcal{S}')$. A similar argument as in the case of a W_{n-1}-star shows that there exists a representative $\widetilde{f}$ of f such that $\widetilde{f}(B) = B$ and $\widetilde{f}(x_n) = x_n$.

Lemma 4.11. *Let $n \geq 4$. Let $\mathcal{S}$ be the W_n-equivariant homeomorphism class of a W_{n-1}-star S. Let T be the group of twists of $\mathcal{S}$. Let $\mathcal{S}'$ be the W_n-equivariant homeomorphism class of a W_{n-2}-star S' which refines S. Let e be the edge of $W_n\backslash S'$ such that a representative of $\mathcal{S}$ is obtained from $W_n\backslash S'$ by collapsing the edge distinct from e. Let T' be the group of twists of S' about the edge e. Then $T \cap \mathrm{Stab}_{\mathrm{Out}^0(W_n)}(\mathcal{S}') \subseteq T'$.*

Proof. Let

$$W_n = \langle x_1\rangle * \langle x_2, \dots, x_{n-1}\rangle * \langle x_n\rangle$$

be the free factor decomposition of W_n induced by S' and let $A = \langle x_2, \dots, x_{n-1}\rangle$. Let

$$W_n = B * \langle y_n\rangle$$

be the free factor decomposition associated with S. Up to changing the representative S, we may suppose that $B = \langle x_1, \dots, x_{n-1}\rangle$ and that $y_n = x_n$. Let $f \in T \cap \mathrm{Stab}_{\mathrm{Out}^0(W_n)}(\mathcal{S}')$. Let $\widetilde{f}$ be the representative of f such that $\widetilde{f}(B) = B$ and $\widetilde{f}(x_n) = x_n$, which exists since $f \in \mathrm{Stab}_{\mathrm{Out}(W_n)}(\mathcal{S})$. Since $f \in T$, there exists $g \in B$ such that $\widetilde{f}|_B$ is the global conjugation by g. Let $\widetilde{f}'$ be a representative of f such that $\widetilde{f}'(A) = A$ and $\widetilde{f}'(x_n) = x_n$, which exists since $f \in \mathrm{Stab}_{\mathrm{Out}^0(W_n)}(\mathcal{S}')$. Since the centralizer in W_n of x_n is $\langle x_n\rangle$ and since A is malnormal in W_n, we see that $\widetilde{f} = \widetilde{f}'$. Hence $\widetilde{f}(A) = A$, and, since A is malnormal, we see that $g \in A$. Therefore, $f \in T'$, which concludes the proof. ■

Lemma 4.12. *Let $n \geq 3$. Let $\mathcal{S}$ be the equivalence class of a W_{n-1}-star S and let*

$$W_n = \langle x_1, \dots, x_{n-1}\rangle * \langle x_n\rangle$$

be the free factor decomposition associated with S. Let $A = \langle x_1, \dots, x_{n-1}\rangle$. Let T be the group of twists of S. For $f \in T$, let $z_f \in A$ be the twistor of f. Let $g \in \mathrm{Stab}(\mathcal{S})$ and let $\widetilde{g}$ be a representative of g such that $\widetilde{g}(A) = A$ and $\widetilde{g}(x_n) = x_n$. Then $g \in C_{\mathrm{Out}(W_n)}(\langle f\rangle)$ if and only if $\widetilde{g}(z_f) = z_f$.

Proof. By Proposition 2.5 (2), the group $\mathrm{Stab}(\mathcal{S})$ is isomorphic to $\mathrm{Aut}(A)$. The isomorphism $\mathrm{Stab}(\mathcal{S}) \to \mathrm{Aut}(A)$ is defined by sending $f \in \mathrm{Stab}(\mathcal{S})$ to its representative $\widetilde{f}$ such that $\widetilde{f}(A) = A$ and $\widetilde{f}(x_n) = x_n$. In particular, for every $h_1, h_2 \in \mathrm{Out}(W_n) \cap \mathrm{Stab}(\mathcal{S})$, we see that h_1 and h_2 commute if and only if there exist representatives $\widetilde{h}_1$ and $\widetilde{h}_2$ of h_1 and h_2, respectively, such that $\widetilde{h}_1(A) = A$, $\widetilde{h}_2(A) = A$, $\widetilde{h}_1(x_n) = \widetilde{h}_2(x_n) = x_n$ and $\widetilde{h}_1 \circ \widetilde{h}_2 = \widetilde{h}_2 \circ \widetilde{h}_1$. Moreover, Proposition 2.5 (2) identifies the group of twists T with the group $\mathrm{Inn}(A)$. For $a \in A$, let ad_a be the inner automorphism of A induced by a. Since, for every $h \in \mathrm{Aut}(A)$ and every $a \in A$, we have $h\, \mathrm{ad}_a\, h^{-1} = \mathrm{ad}_{h(a)}$, we see that h commutes with ad_a if and only if $h(a) = a$. Hence $g \in C_{\mathrm{Out}(W_n)}(\langle f \rangle)$ if and only if $\widetilde{g}(z_f) = z_f$. ■

5. Direct products of nonabelian free groups in Out(W_n)

Following [25, Section 6], we define the *product rank* of a group H, denoted by $\mathrm{rk}_{\mathrm{prod}}(H)$, to be the maximal integer k such that a direct product of k nonabelian free groups embeds in H. Note that, if H' is a finite index subgroup of H, then $\mathrm{rk}_{\mathrm{prod}}(H') = \mathrm{rk}_{\mathrm{prod}}(H)$. Moreover, if $\phi\colon H \to \mathbb{Z}$ is a homomorphism, then $\mathrm{rk}_{\mathrm{prod}}(\ker(\phi)) = \mathrm{rk}_{\mathrm{prod}}(H)$. The aim of this section is to prove the following theorem.

Theorem 5.1. *The groups* $\mathrm{Aut}(W_n)$ *and* $\mathrm{Out}(W_n)$ *satisfy the following properties:*

(1) *For every* $n \geq 3$*, we have* $\mathrm{rk}_{\mathrm{prod}}(\mathrm{Aut}(W_n)) = n - 2$.

(2) *For every* $n \geq 4$*, we have* $\mathrm{rk}_{\mathrm{prod}}(\mathrm{Out}(W_n)) = n - 3$.

(3) *Suppose that* $n \geq 5$*. If* H *is a subgroup of* $\mathrm{Out}(W_n)$ *isomorphic to a direct product of* $n-3$ *nonabelian free groups, then* H *has a subgroup* H' *isomorphic to a direct product of* $n-3$ *nonabelian free groups which virtually fixes the* W_n*-equivariant homeomorphism class of a* W_{n-1}*-star. In addition,* H *does not virtually fix the* W_n*-equivariant homeomorphism class of any one-edge free splitting that is not a* W_{n-1}*-star.*

We first recall an estimate regarding product ranks and group extensions due to Horbez and Wade.

Lemma 5.2 ([25, Lemma 6.3]). *Let* $1 \to N \to G \to Q \to 1$ *be a short exact sequence of groups. Then* $\mathrm{rk}_{\mathrm{prod}}(G) \leq \mathrm{rk}_{\mathrm{prod}}(N) + \mathrm{rk}_{\mathrm{prod}}(Q)$.

In order to compute the product rank of $\mathrm{Out}(W_n)$, we take advantage of its action on the Gromov hyperbolic free factor complex. We recall a general result concerning actions of direct products on a hyperbolic space.

Lemma 5.3 ([25, Proposition 4.2 and Lemma 4.4]). *Let* X *be a Gromov hyperbolic space, and let* H *be a group acting by isometries on* X*. Assume that* H *contains a normal subgroup* K *isomorphic to a direct product* $K = \prod_{i=1}^{k} K_i$.

If there exists $j \in \{1, \ldots, k\}$ *such that* K_j *contains a loxodromic element, then* $\prod_{i \neq j} K_i$ *has a finite orbit in* $\partial_\infty X$.

If there exist two distinct $i, j \in \{1, \dots, k\}$ such that both K_i and K_j contain a loxodromic element, then H has a finite orbit in $\partial_\infty X$.

If, for every $j \in \{1, \dots, k\}$, the group K_j does not contain a loxodromic element, then either K has a finite orbit in $\partial_\infty X$ or H has bounded orbits in X.

Let $\mathcal{F}$ be a free factor system of W_n. Recall that $\mathcal{O}(W_n, \mathcal{F})$ is the outer space of W_n relative to $\mathcal{F}$. Given $T \in \overline{\mathcal{O}(W_n, \mathcal{F})}$, let $[T]$ be the homothety class of T. The *homothetic stabilizer* $\mathrm{Stab}([T])$ is the stabilizer of $[T]$ for the action of $\mathrm{Out}(W_n, \mathcal{F})$ on $\mathbb{P}\overline{\mathcal{O}(W_n, \mathcal{F})}$. Equivalently, $\Phi \in \mathrm{Out}(W_n, \mathcal{F})$ lies in $\mathrm{Stab}([T])$ if there exists a lift $\widetilde{\Phi} \in \mathrm{Aut}(W_n, \mathcal{F})$ of Φ and a homothety $I_{\widetilde{\Phi}}: T \to T$ such that, for all $g \in W_n$ and $x \in T$, we have

$$I_{\widetilde{\Phi}}(gx) = \widetilde{\Phi}(g) I_{\widetilde{\Phi}}(x).$$

The scaling factor of $I_{\widetilde{\Phi}}$ does not depend on the choice of a representative of Φ, and we denote it by $\lambda_T(\Phi)$. This gives a homomorphism

$$\mathrm{Stab}([T]) \to \mathbb{R}_+^*, \quad \Phi \mapsto \lambda_T(\Phi).$$

The kernel of this morphism is called the *isometric stabilizer of T* and is denoted by $\mathrm{Stab}^{\mathrm{is}}(T)$. It is the stabilizer of T for the action of $\mathrm{Out}(W_n, \mathcal{F})$ on $\overline{\mathcal{O}(W_n, \mathcal{F})}$.

Lemma 5.4 ([16, Lemma 6.1]). *Let $n \geq 3$. Let $\mathcal{F}$ be a nonsporadic free factor system of W_n. For every $T \in \overline{\mathcal{O}(W_n, \mathcal{F})}$, the image of the morphism λ_T is a cyclic (maybe trivial) subgroup of $\mathbb{R}_+^*$.*

We will also use a theorem due to Guirardel and Horbez which assigns to every nonempty collection of free splittings whose elementwise stabilizer is infinite, a canonical (not necessarily free) splitting.

Theorem 5.5 ([15, Theorem 6.12]). *Let $n \geq 3$. There exists an* $\mathrm{Out}(W_n)$*-equivariant map which assigns to every nonempty collection $\mathcal{C}$ of free splittings of W_n whose elementwise* $\mathrm{Out}(W_n)$*-stabilizer is infinite, a nontrivial splitting $U_{\mathcal{C}}$ of W_n whose set of vertices $VU_{\mathcal{C}}$ has a W_n-invariant partition $VU_{\mathcal{C}} = V_1 \amalg V_2$ with the following properties:*

(1) *For every vertex $v \in V_1$, the following holds:*
   (a) *either some edge incident to v has trivial stabilizer, or the set of stabilizers of edges incident to v induces a nontrivial free factor system of the vertex stabilizer G_v,*
   (b) *there exists a finite index subgroup H_0 of the elementwise stabilizer of the collection $\mathcal{C}$ such that every outer automorphism in H_0 has a representative in* $\mathrm{Aut}(W_n)$ *which restricts to the identity on G_v.*

(2) *The collection of all conjugacy classes of stabilizers of vertices in V_2 is a free factor system of W_n.*

Finally, we state a proposition due to Guirardel and Horbez concerning the isometric stabilizer of an arational tree.

Proposition 5.6 ([16, Proposition 6.5]). *Let $n \geq 3$. Let $\mathcal{F}$ be a nonsporadic free factor system of W_n, and let T be an arational $(W_n, \mathcal{F})$-tree. Let H be a subgroup of* $\mathrm{Out}(W_n, \mathcal{F})$ *which is virtually contained in* $\mathrm{Stab}^{\mathrm{is}}(T)$. *Then H has a finite index subgroup H' which fixes infinitely many $(W_n, \mathcal{F})$-free splittings, and in particular, H fixes the conjugacy class of a proper $(W_n, \mathcal{F})$-free factor.*

Note that the statement of Proposition 5.6 in [16] only mentions that H' fixes one $(W_n, \mathcal{F})$-free splitting, but the proof uses an arbitrary free splitting of W_n, so that one can construct infinitely many pairwise distinct free splittings fixed by H' by varying the chosen free splitting of W_n.

Proof of Theorem 5.1. The proof is inspired by [25, Theorem 6.1] due to Horbez and Wade and [23, Theorem 4.3] due to Hensel, Horbez and Wade.

We first prove that if $n \geq 4$, then $\mathrm{rk}_{\mathrm{prod}}(\mathrm{Out}(W_n)) \geq n-3$ and that, if $n \geq 3$, then $\mathrm{rk}_{\mathrm{prod}}(\mathrm{Aut}(W_n)) \geq n-2$. Pick a standard generating set $\{x_1, \dots, x_n\}$ of W_n. Then the group H generated by $\{x_1x_2, x_2x_3\}$ is a nonabelian free group (see [30, Theorem A]).

Suppose first that $n \geq 4$. For $i \in \{4, \dots, n\}$ and $h \in H$, let $F_{i,h}$ be the automorphism sending x_i to hx_ih^{-1} and, for $j \neq i$, fixing x_j. Then, for all distinct $i, j \in \{4, \dots, n\}$ and for every $g, h \in H$, the automorphisms $F_{i,g}$ and $F_{j,h}$ commute, giving a direct product of $n-3$ nonabelian free groups in $\mathrm{Out}(W_n)$. Moreover, for all $g, h \in H$, and every $i \in \{4, \dots, n\}$, the inner automorphism ad_g commutes with $F_{i,h}$, which yields a direct product of $n-2$ nonabelian free groups in $\mathrm{Aut}(W_n)$. In the case where $n = 3$, the group $\mathrm{Aut}(W_3)$ contains the subgroup $\langle \mathrm{ad}_h \rangle_{h \in H}$, which is a nonabelian free group.

We now prove that, if $n \geq 3$, then $\mathrm{rk}_{\mathrm{prod}}(\mathrm{Aut}(W_n)) \leq n-2$, if $n = 3$, then we have $\mathrm{rk}_{\mathrm{prod}}(\mathrm{Out}(W_n)) = 1$ and if $n \geq 4$, then $\mathrm{rk}_{\mathrm{prod}}(\mathrm{Out}(W_n)) \leq n-3$. The proof is by induction on n. The base case where $n = 3$ follows from the fact that the group $\mathrm{Aut}(W_3)$ is isomorphic to $\mathrm{Aut}(F_2)$ (see [36, Lemma 2.3]) and the fact that the group $\mathrm{Aut}(F_2)$ does not contain a direct product of two nonabelian free groups (see [25, Lemma 6.2]). Moreover, by [12, Proposition 2.2], the group $\mathrm{Out}(W_3)$ is isomorphic to $\mathrm{PGL}(2, \mathbb{Z})$ which is virtually free.

Let $k \geq \max\{n-3, 2\}$ and let $H = H_1 \times H_1 \times \cdots \times H_k$ be a subgroup of $\mathrm{Out}(W_n)$ isomorphic to a direct product of k nonabelian free groups. Note that $k = n-3$ if $n \geq 5$ and $k = 2$ if $n = 4$. We prove that there exists a subgroup K of H isomorphic to a direct product of k nonabelian free groups which virtually fixes a one-edge free splitting of W_n. Let $\mathcal{F}$ be a maximal H-periodic free factor system. If $\mathcal{F}$ is sporadic, then H virtually fixes a one-edge free splitting, so we are done. Therefore, we may suppose that $\mathcal{F}$ is nonsporadic. As $\mathcal{F}$ is supposed to be maximal, by Proposition 2.2, the group H acts on $\mathrm{FF}(W_n, \mathcal{F})$ with unbounded orbits. Lemma 5.3 implies that, after possibly reordering the factors, the group $H' = H_1 \times H_2 \times \cdots \times H_{k-1}$ has a finite orbit in $\partial_\infty \mathrm{FF}(W_n, \mathcal{F})$. By Lemma 2.4, the group H' virtually fixes the homothety class $[T]$ of an arational $(W_n, \mathcal{F})$-tree T.

Let H_0 be a normal subgroup of finite index in H' that is contained in $\mathrm{Stab}([T])$.

Claim 5.7. *The group H contains a subgroup isomorphic to a direct product of k nonabelian free groups, which fixes the equivalence class of a one-edge free splitting.*

Proof. By Lemma 5.4, the homomorphism $\lambda_T|_{H_0}$ from H_0 to $\mathbb{R}_+^*$ given by the scaling factor has cyclic image. As H_0 contains a direct product of $k-1$ nonabelian free groups, so does $P = \ker(\lambda_T|_{H_0})$ (see the beginning of Section 5). In particular, the intersection of P with every direct factor H_i of H' is a nonabelian free group. As P is contained in the isometric stabilizer of T, Proposition 5.6 implies that P contains a finite index subgroup P_0 which fixes infinitely many $(W_n, \mathcal{F})$-free splittings.

Let $\mathcal{C}$ be the (nonempty) collection of all $(W_n, \mathcal{F})$-free splittings fixed by the infinite group P_0, let $U_{\mathcal{C}}$ be the splitting provided by Theorem 5.5, and let $\mathcal{U}_{\mathcal{C}}$ be its equivalence class. Since P_0 commutes with H_k, the equivalence class $\mathcal{U}_{\mathcal{C}}$ is $(P_0 \times H_k)$-invariant.

Suppose first that the splitting $U_{\mathcal{C}}$ contains an edge $e \in EU_{\mathcal{C}}$ with trivial stabilizer. Let U' be the splitting obtained from $U_{\mathcal{C}}$ by collapsing every edge of $U_{\mathcal{C}}$ that is not contained in the orbit of e, and let $\mathcal{U}'$ be its equivalence class. Then $\mathcal{U}'$ is the equivalence class of a one-edge free splitting virtually fixed by $P_0 \times H_k$. Since P_0 contains a direct product of $k-1$ nonabelian free groups, the claim follows.

Thus, we can suppose that all edge stabilizers of $U_{\mathcal{C}}$ are nontrivial. We show that this leads to a contradiction. Let $VU_{\mathcal{C}} = V_1 \amalg V_2$ be the partition of $VU_{\mathcal{C}}$ given by Theorem 5.5. Let P' be a finite index subgroup of P_0 which acts trivially on the quotient $W_n \backslash U_{\mathcal{C}}$. We claim that the intersection of P' with the group of twists of $U_{\mathcal{C}}$ is trivial. Indeed, let e be an oriented edge of $U_{\mathcal{C}}$. As every subgroup of W_n with nontrivial centralizer is cyclic, if the edge stabilizer G_e of e is not cyclic, the group of twists around this edge is trivial. Thus, oriented edges with nontrivial group of twists have cyclic stabilizers. But twists about edges with cyclic stabilizers are central in a finite index subgroup of $\mathrm{Stab}^0(U_{\mathcal{C}})$ by Lemma 2.7. Let P'' be a finite index subgroup of P'. Then the intersection of P'' with every direct factor H_i of H' is a nonabelian free group. Therefore, every element of P'' is contained in a nonabelian free subgroup of P''. In particular, the center of every finite index subgroup of P' is trivial. Thus, we see that the intersection of P' with the group of twists is trivial. By Remark 2.6, up to passing to a further finite index subgroup of P', we may suppose that the intersection of P' with the group of bitwists is trivial.

By Proposition 2.5 (1) and Remark 2.6, the fact that the intersection of P' with the group of bitwists is trivial implies that we have an injective homomorphism

$$P' \to \prod_{v \in W_n \backslash VU_{\mathcal{C}}} \mathrm{Out}(G_v).$$

By Theorem 5.5 (1) (b), for every vertex $v \in V_1$, the homomorphism $P' \to \mathrm{Out}(G_v)$ has finite image. Therefore, up to passing to a finite index subgroup of P', we have an injective map

$$P' \to \prod_{v \in W_n \backslash V_2} \mathrm{Out}(G_v).$$

By Theorem 5.5 (2), for every $v \in V_2$, the vertex stabilizer G_v is an element of a free factor system of W_n. Therefore, there exists k such that G_v is isomorphic to W_k. By Lemma 5.2, we have

$$n-4 \leq k-1 = \mathrm{rk}_{\mathrm{prod}}(P') \leq \sum_{v \in W_n \backslash V_2} \mathrm{rk}_{\mathrm{prod}}(\mathrm{Out}(G_v)).$$

By induction, we see that, if $|W_n \backslash V_2| \geq 2$, then

$$\sum_{v \in W_n \backslash V_2} \mathrm{rk}_{\mathrm{prod}}(\mathrm{Out}(G_v)) \leq n-6,$$

which leads to a contradiction. Thus $|W_n \backslash V_2| = 1$. Let $v \in W_n \backslash V_2$. Then there exists $\ell \in \{1, \dots, n-1\}$ such that G_v is isomorphic to W_ℓ. If $\ell \leq n-2$, then

$$\mathrm{rk}_{\mathrm{prod}}(\mathrm{Out}(G_v)) \leq n-5,$$

which leads to a contradiction. If $\ell = n-1$, then the free factor system $\mathcal{F}$ contains a free factor isomorphic to W_{n-1} and is therefore a sporadic free factor system, which leads to a contradiction. ∎

Therefore, we see that there exists a subgroup K of H isomorphic to a direct product of k nonabelian free groups such that K fixes the W_n-equivariant homeomorphism class of a one-edge-free splitting $\mathcal{S}$. We now prove that $\mathcal{S}$ is the equivalence class of a W_{n-1}-star. Let S be a representative of $\mathcal{S}$, let v_1 and v_2 be the vertices of the underlying graph of $W_n \backslash S$ and, for $i \in \{1, 2\}$, let k_i be such that W_{k_i} is isomorphic to G_{v_i}. Let K_0 be the finite index subgroup of K which acts as the identity on $W_n \backslash S$. Then $K_0 \subseteq \mathrm{Stab}^0(\mathcal{S})$. By Proposition 2.5 (2), the group $\mathrm{Stab}^0(\mathcal{S})$ is isomorphic to $\mathrm{Aut}(W_{k_1}) \times \mathrm{Aut}(W_{k_2})$. Suppose towards a contradiction that, for every $i \in \{1, 2\}$, we have that $k_i \neq 1$. Suppose first that, for every $i \in \{1, 2\}$, we have $k_i \geq 3$. Then, by Lemma 5.2, we see that

$$\begin{aligned} k = \mathrm{rk}_{\mathrm{prod}}(K_0) &\leq \mathrm{rk}_{\mathrm{prod}}(\mathrm{Aut}(W_{k_1})) + \mathrm{rk}_{\mathrm{prod}}(\mathrm{Aut}(W_{k_2})) \\ &\leq k_1 - 2 + k_2 - 2 = n-4, \end{aligned}$$

where the second inequality comes from the induction hypothesis. If there exists $i \in \{1, 2\}$ such that $k_i = 2$, then, as $\mathrm{Aut}(W_2)$ is virtually cyclic (it is isomorphic to W_2 by [35, Lemma 1.4.2]), we see that

$$k = \mathrm{rk}_{\mathrm{prod}}(K_0) \leq \mathrm{rk}_{\mathrm{prod}}(\mathrm{Aut}(W_{k_1})) + \mathrm{rk}_{\mathrm{prod}}(\mathrm{Aut}(W_{k_2})) \leq k_1 - 2 \leq n-4.$$

In both cases, we have a contradiction as $k \geq n-3$ when $k \geq 5$ and $k = n-2$ when $n = 4$. Thus, there exists $i \in \{1, 2\}$ such that $k_i = 1$. This shows that S is a W_{n-1}-star. In particular, when $k = n-3$, that is, when $n \geq 5$, this proves Theorem 5.1 (3).

Since $K_0 \subseteq \mathrm{Stab}^0(\mathcal{S})$, Proposition 2.5 (2) implies that

$$k = \mathrm{rk}_{\mathrm{prod}}(K_0) \leq \mathrm{rk}_{\mathrm{prod}}(\mathrm{Aut}(W_{n-1})) = n-1-2 = n-3.$$

When $n = 4$, then $k = 2 = n - 2$. Therefore, we have a contradiction in this case. This shows that, for all $n \geq 4$, the product rank of $\mathrm{Out}(W_n)$ is equal to $n - 3$. This concludes the proof of Theorem 5.1 (2).

It remains to prove that, if $n \geq 4$, we have $\mathrm{rk}_{\mathrm{prod}}(\mathrm{Aut}(W_n)) \leq n - 2$. We have the following short exact sequence

$$1 \to W_n \to \mathrm{Aut}(W_n) \to \mathrm{Out}(W_n) \to 1.$$

By Lemma 5.2, as W_n is virtually free, we see that

$$\mathrm{rk}_{\mathrm{prod}}(\mathrm{Aut}(W_n)) \leq \mathrm{rk}_{\mathrm{prod}}(W_n) + \mathrm{rk}_{\mathrm{prod}}(\mathrm{Out}(W_n)) = 1 + n - 3 = n - 2.$$

This concludes the proof of Theorem 5.1 (1). ∎

6. Subgroups of stabilizers of W_{n-1}-stars

In the next two sections, we prove an algebraic characterization of stabilizers of equivalence classes of W_{n-2}-stars. In this section, we take advantage of properties satisfied by stabilizers of equivalence classes of W_{n-2}-stars which are sufficiently rigid to show that a subgroup H of $\mathrm{Out}(W_n)$ which satisfies these properties virtually fixes a W_{n-1}-star. In the next section, we will take advantage of the fact that stabilizers of equivalence classes of compatible W_{n-2}-stars have large intersections to give a characterization of stabilizers of equivalence classes of W_{n-2}-stars.

Let Γ be a finite index subgroup of the group $\mathrm{Out}^0(W_n)$ (defined after Theorem 4.1). We introduce the following algebraic property for a subgroup $H \subseteq \Gamma$.

$(P_{W_{n-2}})$ The group H satisfies the following three properties:

(1) The group H contains a normal subgroup isomorphic to a direct product $K_1 \times K_2$ of two normal subgroups such that each one contains a nonabelian finitely generated normal free subgroup of finite index and such that for every $i \in \{1, 2\}$, for every nontrivial normal subgroup P of a finite index subgroup K_i' of K_i, and for every finite index subgroup P' of P, the group $C_{\mathrm{Out}^0(W_n)}(P')$ contains K_{i+1} as a finite index subgroup (where indices are taken modulo 2).

(2) The group H contains a direct product of $n - 3$ nonabelian free groups.

(3) The group H contains a subgroup isomorphic to $\mathbb{Z}^{n-2}$.

Remark 6.1. (1) Notice that property $(P_{W_{n-2}})$ is closed under taking finite index subgroups.

(2) Hypothesis $(P_{W_{n-2}})$ (1) implies that, if for every $i \in \{1, 2\}$, the group P_i is a finite index subgroup of a nontrivial normal subgroup of a finite index subgroup of K_i, the centralizer in $\mathrm{Out}^0(W_n)$ of $P_1 \times P_2$ is finite.

We first prove that the stabilizer in Γ of the equivalence class of a W_{n-2}-star satisfies $(P_{W_{n-2}})$. We then show that a group satisfying $(P_{W_{n-2}})$ virtually fixes the equivalence class of a W_{n-1}-star.

6.1. Properties of $\mathcal{Z}_{RC}$-factors

In order to prove that the stabilizer in Γ of the equivalence class of a W_{n-2}-star satisfies $(P_{W_{n-2}})$, we first need some background concerning $\mathcal{Z}_{RC}$-splittings. Let G be a finitely generated group. A *$\mathcal{Z}_{RC}$-splitting* of G is a splitting of G such that every edge stabilizer is either trivial or isomorphic to $\mathbb{Z}$ and root-closed. A *$\mathcal{Z}_{RC}$-factor* of G is a subgroup of G which arises as a vertex stabilizer of a $\mathcal{Z}_{RC}$-splitting of G. Note that since edge stabilizers are root-closed, so are the vertex stabilizers.

We now describe a finite index subgroup of W_n that we will use in the proof of Proposition 6.3. Let $\mathbb{F}$ be the kernel of the homomorphism $W_n \to F$ which sends every generator of a standard generating set of W_n to the nontrivial element of F. Remark that $\mathbb{F}$ does not depend on the choice of the basis. Indeed, if $\{x_1, \dots, x_n\}$ is a standard generating set of W_n, and if x is an element of W_n of order 2, there exists $i \in \{1, \dots, n\}$ and $g \in W_n$ such that $x = gx_ig^{-1}$. We have the following result due to Mühlherr.

Lemma 6.2 ([30, Theorem A]). *The group $\mathbb{F}$ is a nonabelian free group of rank $n-1$ which is a characteristic subgroup of W_n. Moreover, the natural restriction homomorphism*

$$\mathrm{Aut}(W_n) \to \mathrm{Aut}(\mathbb{F})$$

is injective.

We now outline here some properties of $\mathcal{Z}_{RC}$-factors (see, e.g., [25, Proposition 7.3]).

Proposition 6.3. *Let $n \geq 3$. The $\mathcal{Z}_{RC}$-factors of W_n satisfy the following properties:*

(1) *Let H be a finitely generated subgroup of W_n which is not virtually cyclic. There exists $g \in H$ which is not contained in any proper $\mathcal{Z}_{RC}$-factor of H.*

(2) *There exists $C \in \mathbb{N}^*$ such that, for every strictly ascending chain $G_1 \subsetneq \cdots \subsetneq G_k$ of $\mathcal{Z}_{RC}$-factors of W_n, one has $k \leq C$.*

(3) *If a subgroup K of W_n is not contained in any proper $\mathcal{Z}_{RC}$-factor of W_n and if P is either a finite index subgroup of K or a nontrivial normal subgroup of K, then P is not contained in any proper $\mathcal{Z}_{RC}$-factor of W_n.*

(4) *A subgroup K of W_n is contained in a proper $\mathcal{Z}_{RC}$-factor of W_n if and only if every element of K is contained in a proper $\mathcal{Z}_{RC}$-factor of W_n.*

Proof. The first assertion is a consequence of [11, Lemma 4.3] due to Genevois and Horbez.

For the second assertion, let $G_1 \subsetneq \cdots \subsetneq G_k$ be a sequence of strictly ascending $\mathcal{Z}_{RC}$-factors. Then, since $\mathcal{Z}_{RC}$-factors are root-closed, for every $i \geq 3$ the group G_i is not cyclic. Thus, as we want an upper bound on the number of subgroups of such a sequence, we may suppose that for every $i \in \{1, \dots, n\}$, the group G_i is not cyclic. We claim that, for every $i \in \{1, \dots, k\}$, there exists $\phi_i \in \mathrm{Aut}(W_n)$ such that $\mathrm{Fix}(\phi_i) = G_i$. Indeed, let S_i be a $\mathcal{Z}_{RC}$-splitting of W_n such that there exists $v \in VS_i$ whose stabilizer is equal to G_i. Up to collapsing edges, we may suppose that every vertex of S_i has nontrivial stabilizer. Let $e_1, \dots, e_\ell$ be the edges with origin v which are in pairwise distinct orbits. Let $F_0 \subseteq \{e_1, \dots, e_\ell\}$ be the subset made of all edges with nontrivial stabilizer. By the definition of a $\mathcal{Z}_{RC}$-splitting, for every $e_s \in F_0$, the group G_{e_s} is cyclic. For every $e_s \in F_0$, let z_s be a generator of G_{e_s}. For every $e_{s'} \in \{e_1, \dots, e_\ell\} \backslash F_0$, let $z_{s'} \in G_i$ be such that, if $w_{s'}$ is the endpoint of $e_{s'}$ distinct from v, we have $z_{s'} G_{w_{s'}} z_{s'}^{-1} \neq G_{w_{s'}}$. Let $\phi_i = D_{e_1,z_1} \circ \dots \circ D_{e_\ell,z_\ell}$ be a multitwist about every edge with origin v. Then, as the centralizer of an infinite cyclic subgroup of W_n is infinite cyclic, we have $\mathrm{Fix}(\phi_i) = G_i$. Therefore, in order to prove the second assertion, it suffices to prove that there exists $C \in \mathbb{N}^*$ such that for every strictly ascending chain $\mathrm{Fix}(\phi_1) \subsetneq \dots \subsetneq \mathrm{Fix}(\phi_k)$ of fixed points sets of automorphisms of W_n, one has $k \leq C$.

Let $\mathbb{F}$ be the characteristic subgroup of W_n given by Lemma 6.2 and let

$$\Phi\colon \mathrm{Aut}(W_n) \to \mathrm{Aut}(\mathbb{F})$$

be the natural injective homomorphism given by restriction. Then

$$\mathrm{Fix}(\Phi(\phi_1)) \subseteq \dots \subseteq \mathrm{Fix}(\Phi(\phi_k))$$

is an ascending chain of fixed points sets.

Claim 6.4. *For every $i \in \{2, \dots, k-1\}$, the set $\{\mathrm{Fix}(\Phi(\phi_{i-1})), \mathrm{Fix}(\Phi(\phi_i)), \mathrm{Fix}(\Phi(\phi_{i+1}))\}$ contains at least 2 elements.*

Proof. Suppose towards a contradiction that

$$|\{\mathrm{Fix}(\Phi(\phi_{i-1})), \mathrm{Fix}(\Phi(\phi_i)), \mathrm{Fix}(\Phi(\phi_{i+1}))\}| = 1.$$

As $\mathrm{Fix}(\phi_{i-1}) \subsetneq \mathrm{Fix}(\phi_i)$ and $\mathrm{Fix}(\Phi(\phi_{i-1})) = \mathrm{Fix}(\Phi(\phi_i))$, there exists $a \in W_n \backslash \mathbb{F}$ such that $\phi_i(a) = a$ and $\phi_{i-1}(a) \neq a$. Since the index of $\mathbb{F}$ is equal to 2, we see that $\phi_{i-1}(a^2) = a^2$. Therefore, $\phi_{i-1}(a)^2 = a^2$ and $\phi_{i-1}(a)$ is a square root of a^2. If a^2 has infinite order, its only square root is a. This implies that $\phi_{i-1}(a) = a$, a contradiction. Thus, we can assume that a has order 2 and, up to changing the basis $\{x_1, \dots, x_n\}$, we may suppose that $a = x_1$.

As the index of $\mathbb{F}$ is equal to 2, we have $W_n = \mathbb{F} \amalg x_1\mathbb{F}$. Let $x \in \mathrm{Fix}(\phi_{i+1}) \backslash \mathbb{F}$. Then there exists $y \in \mathbb{F}$ such that $x = x_1 y$. As $x_1 \in \mathrm{Fix}(\phi_i)$ and $\mathrm{Fix}(\phi_i) \subsetneq \mathrm{Fix}(\phi_{i+1})$, we have that $\phi_{i+1}(x_1) = x_1$. Hence $\phi_{i+1}(y) = y$. As $y \in \mathbb{F}$ and $\mathrm{Fix}(\Phi(\phi_i)) = \mathrm{Fix}(\Phi(\phi_{i+1}))$, we see that

$$\phi_i(y) = y \quad \text{and} \quad \phi_i(x) = \phi_i(x_1 y) = x_1 y = x.$$

Therefore, we have that $\mathrm{Fix}(\phi_i) = \mathrm{Fix}(\phi_{i+1})$, which is a contradiction. Then the claim follows. ∎

From Claim 6.4, we have that the length of the strictly ascending chain associated with $\mathrm{Fix}(\Phi(\phi_1)) \subseteq \cdots \subseteq \mathrm{Fix}(\Phi(\phi_k))$ is at least equal to $\frac{k}{2}$. But any strictly ascending chain of fixed subgroups in a free group on $n-1$ generators has length at most $2(n-1)$ (see [29, Theorem 4.1]). Therefore, there exists C which depends only on n such that $k \leq C$. The second assertion of Proposition 6.3 follows.

We now prove the third assertion. Let P and K be as in Proposition 6.3 (3). If K is a virtually infinite cyclic group, then K is either isomorphic to $\mathbb{Z}$ or to W_2. Let a be a generator of the subgroup of K isomorphic to $\mathbb{Z}$ and root-closed in K. Since $\langle a \rangle$ is a finite index subgroup of K and since K is not contained in any proper $\mathcal{Z}_{RC}$-factor of W_n, then neither is a. Remark that any nontrivial normal subgroup of K intersects the subgroup $\langle a \rangle$ nontrivially. Therefore, if P is contained in a proper $\mathcal{Z}_{RC}$-factor of W_n, then a is elliptic in a $\mathcal{Z}_{RC}$-splitting. This contradicts the fact that a is not contained in any proper $\mathcal{Z}_{RC}$-factor of W_n.

So we can assume that K is not virtually cyclic. As every finite index subgroup contains a nontrivial normal subgroup of K, we may assume that P is a nontrivial normal subgroup of K. Notice that P is necessarily noncyclic. Suppose towards a contradiction that P is contained in a $\mathcal{Z}_{RC}$-factor. Then there exists a $\mathcal{Z}_{RC}$-splitting S of W_n such that P is elliptic in S. Since edge stabilizers are cyclic, the group P fixes a unique vertex x of S. But, as P is normal in K, for every $k \in K$, we have that kx is also fixed by P, hence we have $kx = x$. Therefore, x is fixed by K, which contradicts the fact that K is not contained in any proper $\mathcal{Z}_{RC}$-factor.

We finally prove Proposition 6.3 (4). Suppose that K is contained in a proper $\mathcal{Z}_{RC}$-factor. Then it is clear that every element of K is contained in a proper $\mathcal{Z}_{RC}$-factor.

Conversely, assume that K is not contained in any proper $\mathcal{Z}_{RC}$-factor of W_n. Let us prove that there exists $g \in K$ such that g is not contained in any proper $\mathcal{Z}_{RC}$-factor. By Proposition 6.3 (2), there exists a bound on the length of an increasing chain of $\mathcal{Z}_{RC}$-factors of W_n. Therefore, the group K contains a finitely generated subgroup K' which is not contained in any proper $\mathcal{Z}_{RC}$-factor. By Proposition 6.3 (1), there exists $g \in K'$ such that g is not contained in a proper $\mathcal{Z}_{RC}$-factor of K'. Let S be a $\mathcal{Z}_{RC}$-splitting of W_n. As K' is not contained in any proper $\mathcal{Z}_{RC}$-factor of W_n, the group K' has a well-defined, nontrivial minimal subtree $S_{K'}$ with respect to the action of K' on S. As S is a $\mathcal{Z}_{RC}$-splitting of W_n, the splitting $S_{K'}$ is a $\mathcal{Z}_{RC}$-splitting of K'. Since g is not contained in any proper $\mathcal{Z}_{RC}$-factor of K', it follows that g is a hyperbolic isometry of $S_{K'}$ and is not elliptic in S. As S is arbitrary, it follows that g is not contained in any $\mathcal{Z}_{RC}$-factor of W_n. ∎

Proper $\mathcal{Z}_{RC}$-factors appear naturally when studying stabilizers of conjugacy classes of elements as shown by the following theorem. Recall that, if $\mathcal{H} = \{H_1, \ldots, H_k\}$ is a finite family of finitely generated subgroups of W_n, the group $\mathrm{Out}(W_n, \mathcal{H}^{(t)})$ is the subgroup of $\mathrm{Out}(W_n)$ consisting of all outer automorphisms $\phi \in \mathrm{Out}(W_n)$ such that, for every $i \in \{1, \ldots, k\}$, there exists a representative $\widetilde{\phi}_i \in \mathrm{Aut}(W_n)$ of ϕ such that $\widetilde{\phi}_i(H_i) = H_i$ and $\widetilde{\phi}_i|_{H_i} = \mathrm{id}_{H_i}$.

Theorem 6.5 ([18, Theorem 7.14]). *Let $n \geq 3$, $g \in W_n$. Then the subgroup $\mathrm{Out}(W_n, \langle g \rangle)$ of outer automorphisms which preserve $\langle g \rangle$ up to conjugacy is infinite if and only if g is contained in a proper $\mathcal{Z}_{RC}$-factor of W_n.*

More generally, Let G be a finitely generated Gromov hyperbolic group. If $\mathcal{H}$ is a finite family of finitely generated subgroups of G, then the group $\mathrm{Out}(G, \mathcal{H}^{(t)})$ is infinite if and only if there exists a nontrivial $\mathcal{Z}_{RC}$-splitting S of G such that every subgroup of $\mathcal{H}$ fixes a vertex of S.

6.2. Stabilizers of W_{n-2}-stars satisfy $(P_{W_{n-2}})$

Lemma 6.6. *Let $n \geq 5$ and let Γ be a finite index subgroup of $\mathrm{Out}^0(W_n)$. Let $\mathcal{S}$ be the equivalence class of a W_{n-2}-star S. Let e_1 and e_2 be the two edges of $W_n \backslash S$ and, for $i \in \{1, 2\}$, let T_i' be the group of twists about e_i in $\mathrm{Stab}_\Gamma(\mathcal{S})$. Let $i \in \{1, 2\}$, let T_i be a finite index subgroup of T_i' and let P' be a finite index subgroup of a nontrivial normal subgroup of T_i. Then for every finite index subgroup P_0 of P', the group P_0 fixes exactly one equivalence class of W_{n-2}-stars.*

Proof. Let

$$W_n = \langle x_1 \rangle * \langle x_3, \dots, x_n \rangle * \langle x_2 \rangle$$

be a free factor decomposition associated with $W_n \backslash S$ and $A = \langle x_3, \dots, x_n \rangle$. Up to exchanging the roles of e_1 and e_2, we may suppose that P' is contained in the group of twists of the equivalence class of the W_{n-1}-star S_1 whose associated free factor decomposition of W_n is, up to global conjugation:

$$W_n = \langle x_1 \rangle * \langle x_2, x_3, \dots, x_n \rangle.$$

Let $B = \langle x_2, x_3, \dots, x_n \rangle$ and let $\mathcal{S}_1$ be the equivalence class of S_1. Finally, let $\mathcal{S}_2$ be the equivalence class of the W_{n-1}-star S_2 whose associated free factor decomposition of W_n is, up to global conjugation:

$$W_n = \langle x_2 \rangle * \langle x_1, x_3, \dots, x_n \rangle.$$

Let $C = \langle x_1, x_3, \dots, x_n \rangle = A * \langle x_1 \rangle$.

We claim that the only equivalence classes of W_{n-1}-stars fixed by any finite index subgroup of P' are $\mathcal{S}_1$ and $\mathcal{S}_2$. Indeed, fix $i \in \{1, 2\}$. The group T_i is isomorphic to a finite index subgroup N of W_{n-2}. By Proposition 6.3 (3) applied with $K = W_{n-2}$ and $P = N$, as $n \geq 5$, the group N is not contained in any proper $\mathcal{Z}_{RC}$-free factor of W_{n-2}. By Proposition 6.3 (4), there exists $g \in N$ such that W_{n-2} is freely indecomposable relative to g. Hence there exists $g \in A$ such that A is freely indecomposable relative to g and P' contains the twist about e_1 whose twistor is g. Note that this twist can be seen as a twist about the W_{n-1}-star S_1. Let $\mathcal{S}_1'$ be the equivalence class of the one-edge cyclic splitting S_1' whose associated amalgamated decomposition of W_n is, up to global conjugation:

$$W_n = (\langle x_1 \rangle * \langle g \rangle) *_{\langle g \rangle} B.$$

Let $\mathcal{S}_3$ be the equivalence class of a W_{n-1}-star S_3 fixed by some finite index subgroup of P' and distinct from $\mathcal{S}_1$. Let

$$W_n = \langle y \rangle * D$$

be the free factor decomposition associated with S_3. We claim that $\mathcal{S}_3 = \mathcal{S}_2$. As P' contains the twist about g, by Lemma 4.7, the splitting S_3 is compatible with S_1'. Let U be a two-edge refinement of S_1' and S_3. Then U is obtained from S_3 by blowing-up an edge at vertices whose stabilizers are conjugate to D. Moreover, U is obtained from S_1' by blowing-up an edge at vertices whose stabilizers are conjugate to B or by blowing-up an edge at the vertices whose stabilizers are conjugate to $\langle x_1 \rangle * \langle g \rangle$. But, the second case can only occur when $\mathcal{S}_3 = \mathcal{S}_1$ (see Claim 4.8). Therefore, we may suppose that U is obtained from S_1' by blowing up an edge at vertices whose stabilizers are conjugate to B. Thus, up to applying a global conjugation, we may assume that $\langle x_1 \rangle * \langle g \rangle \subseteq D$. But, as g is not contained in any proper $\mathcal{Z}_{RC}$-factor of A and as $A \cap D$ is a free factor of A, we see that $A \cap D = A$. Hence $A * \langle x_1 \rangle \subseteq D$, and, as $A * \langle x_1 \rangle$ is isomorphic to W_{n-1}, we have in fact $A * \langle x_1 \rangle = D$. It follows that $C = D$ and, by Lemma 4.4, we see that $\mathcal{S}_2 = \mathcal{S}_3$. Thus the only equivalence classes of W_{n-1}-stars fixed by finite index subgroups of P' are $\mathcal{S}_1$ and $\mathcal{S}_2$.

Therefore, the only equivalence classes of W_{n-2}-stars fixed by finite index subgroups of P' are the equivalence classes of the W_{n-2}-stars which refine S_1 and S_2. As S_1 and S_2 are refined by a unique (up to W_n-equivariant homeomorphism) W_{n-2}-star by Theorem 3.7, we conclude that $\mathcal{S}$ is the only equivalence class of W_{n-2}-star fixed by finite index subgroups of P'. This completes the proof. ■

Proposition 6.7. *Let $n \geq 5$ and let Γ be a finite index subgroup of* $\mathrm{Out}^0(W_n)$. *Let $\mathcal{S}$ be the equivalence class of a W_{n-2}-star S. Then* $\mathrm{Stab}_\Gamma(\mathcal{S})$ *satisfies* $(P_{W_{n-2}})$. *Moreover, we can choose for the subgroup $K_1 \times K_2$ of property* $(P_{W_{n-2}})$ *(1) the direct product of the groups of twists of $\mathcal{S}$ about the two edges of S.*

Proof. The fact that $\mathrm{Stab}_\Gamma(\mathcal{S})$ satisfies $(P_{W_{n-2}})$ (2) follows from the fact that $\mathrm{Stab}_\Gamma(\mathcal{S})$ contains the stabilizer in Γ of the equivalence class of a W_3-star obtained from S by blowing-up $n-5$ edges at the center of $W_n \backslash S$. Indeed, Proposition 2.5 (3) ensures that the group of twists of a W_3-star is isomorphic to a direct product of $n-3$ copies of W_3.

The fact that $\mathrm{Stab}_\Gamma(\mathcal{S})$ satisfies $(P_{W_{n-2}})$ (3) follows from the fact that $\mathrm{Stab}_\Gamma(\mathcal{S})$ contains the stabilizer in Γ of the equivalence class of a W_2-star obtained from S by blowing-up $n-4$ edges at the center of $W_n \backslash S$. Indeed, the group of twists of a W_2-star is isomorphic to a direct product of $n-2$ copies of W_2 by Proposition 2.5 (3).

Let us now prove that $\mathrm{Stab}_\Gamma(\mathcal{S})$ satisfies $(P_{W_{n-2}})$ (1). Let T' be the group of twists of $\mathcal{S}$ and let $T = T' \cap \Gamma$. The group T is normal in $\mathrm{Stab}_\Gamma(\mathcal{S})$ since $\Gamma \subseteq \mathrm{Out}^0(W_n)$. By Proposition 2.5 (3), the group T' is isomorphic to $T_1' \times T_2'$, where, for $i \in \{1, 2\}$, T_i' is the group of twists in $\mathrm{Out}(W_n)$ about one edge of $W_n \backslash S$. For $i \in \{1, 2\}$, let $T_i = T_i' \cap \Gamma$. For every $i \in \{1, 2\}$, the group T_i is a normal subgroup of $\mathrm{Stab}_\Gamma(\mathcal{S})$ and the group $T_1 \times T_2$ is a normal subgroup of $\mathrm{Stab}_\Gamma(\mathcal{S})$. Let $T_1^{(2)}$ be a finite index subgroup of T_1 and let P'

be a finite index subgroup of a nontrivial normal subgroup of $T_1^{(2)}$. We prove that the centralizer of P' in Γ contains T_2 as a finite index subgroup. This will conclude the proof of the proposition by symmetry of T_1 and T_2. By Lemma 6.6, the equivalence class $\mathcal{S}$ is the only equivalence class of W_{n-2}-star fixed by every finite index subgroup of P'. Hence $C_\Gamma(P')$ fixes $\mathcal{S}$.

Let H be a finite index subgroup of $C_\Gamma(P')$ which fixes $\mathcal{S}$. Let

$$W_n = \langle x_1 \rangle * \langle x_3, \dots, x_n \rangle * \langle x_2 \rangle$$

be a free factor decomposition associated with $W_n \backslash S$ and $A = \langle x_3, \dots, x_n \rangle$. By Proposition 2.5 (1), the kernel of the natural homomorphism $H \to \mathrm{Out}(A)$ is isomorphic to $H \cap T$. We claim that the image of H in $\mathrm{Out}(A)$ is finite. Indeed, as P' is a finite index subgroup of a nontrivial normal subgroup of a finite index subgroup of T_1 and as T_1 is isomorphic to a finite index subgroup of W_{n-2}, we see that P' is isomorphic to a finite index subgroup N of a nontrivial normal subgroup of a finite index subgroup of W_{n-2}. By Proposition 6.3 (3), N is not contained in any proper $\mathcal{Z}_{RC}$-factor of W_{n-2}. By Proposition 6.3 (4), there exists $g \in N$ such that g is not contained in any proper $\mathcal{Z}_{RC}$-factor of W_{n-2}. Thus, there exists $g \in A$ such that g is not contained in any proper $\mathcal{Z}_{RC}$-factor of A and the twist about g is contained in P'. As H commutes with the twist about g, Lemma 4.12 implies that H preserves the conjugacy class of g. Hence, by Theorem 6.5, the image of H in $\mathrm{Out}(A)$ is finite.

Thus, $H \cap T$ has finite index in H and in $C_\Gamma(P')$. But, as H commutes with $P' \subseteq T_1$, and as T_1 is virtually a nonabelian free group, the intersection $H \cap T_2$ has finite index in $H \cap T$, hence has finite index in $C_\Gamma(P')$. This completes the proof. ∎

6.3. Groups satisfying $(P_{W_{n-2}})$ and stabilizers of W_{n-1}-stars

We prove in this section that if H is a subgroup of $\mathrm{Out}(W_n)$ which satisfies $(P_{W_{n-2}})$, then H virtually fixes the equivalence class of a W_{n-1}-star. We first recall a general lemma.

Lemma 6.8. *Let G be a group and let N be a finitely generated normal subgroup of G. Let $n \in \mathbb{N}^*$.*

(1) *There exist only finitely many subgroups of N of index equal to n.*

(2) *For every finite index subgroup N' of N there exists a finite index subgroup G' of G such that N' is a normal subgroup of G'.*

Proof. Assertion (1) is well known, we only prove assertion (2). Let N' be a subgroup of N of index n and let $g \in G$. As N is a normal subgroup of G, the automorphism $\mathrm{ad}_g \colon G \to G$ induces an automorphism $\mathrm{ad}_g|_N \colon N \to N$ by restriction. Therefore, ad_g permutes the subgroups of index n in N. Since there exists a finite number of subgroups of index n in N by the first assertion, we see that there exists a finite index subgroup G' of G such that, for every $g \in G'$, we have $\mathrm{ad}_g(N') = N'$. Therefore, N' is a normal subgroup of G'. This concludes the proof. ∎

Lemma 6.9. *Let $n \geq 5$. Let H be a subgroup of* $\mathrm{Out}^0(W_n)$ *satisfying* $(P_{W_{n-2}})$. *Let* $K_1 \times K_2$ *be a normal subgroup of H given by* $(P_{W_{n-2}})$ (1). *Then one of the following holds:*

(1) *For every $i \in \{1,2\}$, the group K_i does not virtually fix the equivalence class of a free splitting.*

(2) *The group H virtually fixes the equivalence class of a one-edge free splitting.*

Proof. Suppose that there exists $i \in \{1,2\}$ such that K_i virtually fixes the equivalence class of a free splitting. Up to reordering, we may assume that $i = 1$. Let K_1' be a finite index subgroup of K_1 which fixes the equivalence class of a free splitting, and let $\mathcal{C}$ be the set of all equivalence classes of free splittings fixed by K_1'. Since K_1 is a finitely generated normal subgroup of H, by Lemma 6.8 (2), there exists a finite index subgroup H_0 of H such that K_1' is a normal subgroup of H_0. In particular, the set $\mathcal{C}$ is preserved by H_0.

Suppose first that the set $\mathcal{C}$ is finite. Then the set $\mathcal{C}$ is virtually fixed pointwise by H_0. Hence the group H virtually fixes the equivalence class of a free splitting.

So we may assume that the set $\mathcal{C}$ is infinite. Let $U_{\mathcal{C}}$ be the splitting provided by Theorem 5.5, and let $\mathcal{U}_{\mathcal{C}}$ be its equivalence class. By the equivariance property in Theorem 5.5 the equivalence class $\mathcal{U}_{\mathcal{C}}$ is H_0-invariant. Suppose first that the splitting $U_{\mathcal{C}}$ contains an edge $e \in EU_{\mathcal{C}}$ with trivial stabilizer. Let U' be the splitting obtained from $U_{\mathcal{C}}$ by collapsing every edge of $U_{\mathcal{C}}$ that are not contained in the orbit of e, and let $\mathcal{U}'$ be its equivalence class. Then $\mathcal{U}'$ is the equivalence class of a one-edge free splitting virtually fixed by H.

Thus, we may assume that all edge stabilizers of $U_{\mathcal{C}}$ are nontrivial. We show that this leads to a contradiction. Let H' be the subgroup of finite index in H_0 which acts trivially on $W_n \backslash U_{\mathcal{C}}$. We claim that the intersection of H' with the group of twists of $U_{\mathcal{C}}$ is finite. Indeed, let e be an oriented edge of $U_{\mathcal{C}}$. As W_n is virtually free, if the edge stabilizer G_e of e is not cyclic, the group of twists about this edge is trivial. Thus, as we suppose that all edge stabilizers are nontrivial, oriented edges with nontrivial group of twists have cyclic stabilizers. But by Lemma 2.7, twists about edges with cyclic stabilizers are central in a finite index subgroup of $\mathrm{Stab}^0(U_{\mathcal{C}})$. Note that Remark 6.1 (2) implies that the center of every finite index subgroup of H' is finite. Therefore, the intersection of H' with the group of twists is finite. By Remark 2.6, the intersection of H' with the group of bitwists is finite. Thus, up to passing to a finite index subgroup, we may suppose that the map

$$H' \to \prod_{v \in V(W_n \backslash U_{\mathcal{C}})} \mathrm{Out}(G_v)$$

given by the action on the vertex groups is injective.

Let $VU_{\mathcal{C}} = V_1 \amalg V_2$ be the partition of $VU_{\mathcal{C}}$ given by Theorem 5.5 and, for every $i \in \{1,2\}$, let H_i be the subgroup of H' made of all automorphisms whose image in $\prod_{v \in W_n \backslash V_i} \mathrm{Out}(G_v)$ is trivial. Then H_1 and H_2 centralize each other and, according to Theorem 5.5 (1) (b), the group $H_1 \cap K_1'$ is a finite index subgroup of K_1'. Thus H_2 centralizes a finite index subgroup of K_1'. We prove that $\mathrm{rk}_{\mathrm{prod}}(H_2) \geq 2$, which will contradict the fact that the centralizer of every finite index subgroup of K_1' is virtually free.

By Theorem 5.5 (2), the set of all conjugacy classes of groups G_v with $v \in V_2$ is a free factor system of W_n. In particular, for every $v \in V_2$, there exists $k_v \in \{0, \dots, n-1\}$ such that G_v is isomorphic to W_{k_v}. Suppose first that $|W_n \backslash V_2| \geq 3$. In this case, by Theorem 5.1 (2) and since $\mathrm{rk}_{\mathrm{prod}}(\mathrm{Out}(W_3)) = 1$ and $\mathrm{rk}_{\mathrm{prod}}(\mathrm{Out}(W_2)) = 0$, for all $v \in V_2$, we have $\mathrm{rk}_{\mathrm{prod}}(\mathrm{Out}(W_{k_v})) \leq k_v - 2$. Hence

$$\mathrm{rk}_{\mathrm{prod}}\Big(\prod_{v \in W_n \backslash V_2} \mathrm{Out}(G_v)\Big) \leq n - 6.$$

Since $\mathrm{rk}_{\mathrm{prod}}(H') = n-3$, using Lemma 5.2, we see that $\mathrm{rk}_{\mathrm{prod}}(H_2) \geq 3$. This leads to a contradiction. Suppose now that $|W_n \backslash V_2| = 2$ and let $v_1, v_2 \in W_n \backslash V_2$ be distinct. Then for every $i \in \{1,2\}$ there exists $k_i \in \{1, \dots, n-1\}$ such that G_{v_i} is isomorphic to W_{k_i}. If $W_n = W_{k_1} * W_{k_2}$, then the group H' virtually fixes the equivalence class of the one-edge free splitting determined by this free factor decomposition of W_n. So we may assume that $W_n \neq W_{k_1} * W_{k_2}$. This implies that $k_1 + k_2 \leq n-1$. Hence

$$\mathrm{rk}_{\mathrm{prod}}\Big(\prod_{v \in W_n \backslash V_2} \mathrm{Out}(G_v)\Big) \leq n - 5.$$

Since $\mathrm{rk}_{\mathrm{prod}}(H') = n-3$, using Lemma 5.2, we see that $\mathrm{rk}_{\mathrm{prod}}(H_2) \geq 2$. This leads to a contradiction. Suppose now that $|W_n \backslash V_2| = 1$, and let $v \in W_n \backslash V_2$. Then there exists $k \in \{1, \dots, n-1\}$ such that G_v is isomorphic to W_k. Suppose first that $k \leq n-2$. Then according to Theorem 5.1 (2), and since $\mathrm{rk}_{\mathrm{prod}}(\mathrm{Out}(W_3)) = 1$, $\mathrm{rk}_{\mathrm{prod}}(\mathrm{Out}(W_1)) = 0$ and $\mathrm{rk}_{\mathrm{prod}}(\mathrm{Out}(W_2)) = 0$, if $n \neq 5$, we have

$$\mathrm{rk}_{\mathrm{prod}}(\mathrm{Out}(W_k)) \leq n - 5.$$

Thus, by Lemma 5.2, we see that $\mathrm{rk}_{\mathrm{prod}}(H_2) \geq 2$. When $n = 5$, the case where $k = 3$ and $\mathrm{rk}_{\mathrm{prod}}(\mathrm{Out}(W_k)) = 1 = n - 4$ can occur. But by property $(P_{W_{n-2}})$ (3), the group H' contains a subgroup isomorphic to $\mathbb{Z}^3$. Since $\mathrm{Out}(W_3)$ is virtually free, the group H_2 contains a subgroup isomorphic to $\mathbb{Z}^2$. This contradicts the fact that the centralizer of every finite index subgroup of K_1' is virtually nonabelian free. Hence we have $k = n-1$. But then, by Lemma 4.4, the group H' (and hence the group H) virtually fixes the equivalence class of a W_{n-1}-star. This concludes the proof. ∎

Lemma 6.10. *Let $n \geq 5$. Let $\mathcal{F}$ be a nonsporadic free factor system. Let H be a subgroup of* $\mathrm{Out}^0(W_n) \cap \mathrm{Out}(W_n, \mathcal{F})$ *containing a direct product of $n-3$ nonabelian free groups. Then H cannot contain a finite index subgroup which fixes the homothety class of a $(W_n, \mathcal{F})$-arational tree.*

Proof. Suppose towards a contradiction that H has a finite index subgroup which fixes the equivalence class of a $(W_n, \mathcal{F})$-arational tree. Up to passing to a finite index subgroup, we may suppose that H itself fixes the homothety class of a $(W_n, \mathcal{F})$-arational tree. By Lemma 5.4, there exists a homomorphism from H to $\mathbb{Z}$ whose kernel K' is exactly the isometric stabilizer of a $(W_n, \mathcal{F})$-arational tree. Note that K' contains a direct product of $n-3$ nonabelian free groups as it is the kernel of a homomorphism from H to $\mathbb{Z}$. By Proposition 5.6, there exists a finite index subgroup K of K' such that K fixes infinitely many equivalence classes of free splittings. Let $\mathcal{C}$ be the collection of all equivalence classes of free splittings fixed by K.

We claim that $\mathcal{C}$ is in fact finite, which will lead to a contradiction. Since $K \subseteq \mathrm{Out}^0(W_n)$, Lemma 4.2 implies that if $\mathcal{S}$ is the equivalence class of a free splitting S fixed by K, then the group K fixes the equivalence class of every one-edge free splitting onto which S collapses. By Theorem 3.7, if $\mathcal{S}$ is the equivalence class of a free splitting S, then $\mathcal{S}$ is determined by the finite set of equivalence classes of one-edge free splittings onto which S collapses. Therefore, it suffices to show that K can only fix finitely many equivalence classes of one-edge free splittings. Let $\mathcal{S}$ be the equivalence class of a one-edge free splitting fixed by K. Since K contains a direct product of $n-3$ nonabelian free groups, Theorem 5.1 (3) implies that S is a W_{n-1}-star. Let

$$W_n = \langle x_1, \dots, x_{n-1} \rangle * \langle x_n \rangle$$

be a free factor decomposition associated with S and let $A = \langle x_1, \dots, x_{n-1} \rangle$. By Proposition 2.5 (1), the kernel of the natural homomorphism $K \to \mathrm{Out}(A)$ is the intersection of K with the group of twists T of $\mathcal{S}$. By Theorem 5.1 (2), the product rank of $\mathrm{Out}(A)$ is equal to $n-4$. Since K contains a direct product of $n-3$ nonabelian free groups, we see that $K \cap T$ is infinite. Therefore, for every equivalence class $\mathcal{S}$ of a W_{n-1}-star S fixed by K, the group K contains an infinite twist about $\mathcal{S}$.

Let $\mathcal{S}$ and $\mathcal{S}'$ be two distinct equivalence classes of W_{n-1}-stars fixed by K. Let S be a representative of $\mathcal{S}$ and let S' be a representative of $\mathcal{S}'$. We claim that S and S' are compatible. Indeed, by the above, there exists $f \in K$ of infinite order such that f is a twist about $\mathcal{S}$. Since f fixes $\mathcal{S}'$, Lemma 4.7 implies that S and S' are compatible. Therefore, for all distinct equivalence classes $\mathcal{S}$ and $\mathcal{S}'$ of one-edge free splittings fixed by K, there exist $S \in \mathcal{S}$ and $S' \in \mathcal{S}'$ such that S and S' are compatible. By Theorem 3.7, this is only possible when $\mathcal{C}$ is finite. This leads to a contradiction since K must fix infinitely many equivalence classes of free splittings. This concludes the proof. ∎

Proposition 6.11. *Let $n \geq 5$. Let H be a subgroup of $\mathrm{Out}^0(W_n)$ satisfying $(P_{W_{n-2}})$. Then H virtually fixes the equivalence class of a W_{n-1}-star.*

Proof. The proof is inspired by [25, Proposition 8.2] and [23, Proposition 6.5]. We prove that H virtually fixes the equivalence class of a one-edge free splitting. Since H contains a direct product of $n-3$ nonabelian free groups, we will then conclude by Theorem 5.1 (3). Suppose towards a contradiction that H does not virtually fix the equivalence class of a one-edge free splitting. Let $\mathcal{F}$ be a maximal H-periodic free factor system. We can assume that $\mathcal{F}$ is nonsporadic, otherwise H virtually fixes the equivalence class of a one-edge free splitting and we are done. As $\mathcal{F}$ is maximal, by Proposition 2.2, the group H acts with unbounded orbits on $\mathrm{FF}(W_n, \mathcal{F})$.

Let $K_1 \times K_2$ be a normal subgroup of H given by $(P_{W_{n-2}})$ (1). Suppose first that neither K_1 nor K_2 contains a loxodromic element on $\mathrm{FF}(W_n, \mathcal{F})$. As H has unbounded orbits on $\mathrm{FF}(W_n, \mathcal{F})$, Lemma 5.3 implies that $K_1 \times K_2$ has a finite orbit in $\partial_\infty \mathrm{FF}(W_n, \mathcal{F})$.

By Lemma 2.4, there exists a finite index subgroup $K_1' \times K_2'$ of $K_1 \times K_2$ such that $K_1' \times K_2'$ fixes the homothety class of an arational $(W_n, \mathcal{F})$-tree T. Since $K_1 \times K_2$ does not contain a loxodromic element, $K_1' \times K_2'$ fixes T up to isometry, not just homothety (see, e.g., [16, Proposition 6.2]). By Proposition 5.6, the group $K_1' \times K_2'$ virtually fixes infinitely many equivalence classes of $(W_n, \mathcal{F})$-free splittings. By Lemma 6.9, the group H virtually fixes the equivalence class of a one-edge free splitting of W_n.

So we may suppose that there exists a loxodromic element $\Phi \in K_1 \times K_2$. First suppose that there exists a unique $i \in \{1, 2\}$ such that the group K_i contains a loxodromic element Φ_i. We may assume, up to reordering, that only K_2 contains a loxodromic element Φ. Therefore, by Lemma 5.3, the group K_1 virtually fixes a point in $\partial_\infty \mathrm{FF}(W_n, \mathcal{F})$. By Lemma 2.4, the group K_1 virtually fixes the homothetic class an arational $(W_n, \mathcal{F})$-tree T. Let K_1' be a normal subgroup of K_1 of finite index that is contained in $\mathrm{Stab}([T])$. As K_1' does not contain any loxodromic element, as in the above step, K_1' fixes T up to isometry. By Proposition 5.6, the group K_1' fixes the equivalence class of a free splitting relative to $\mathcal{F}$. By Lemma 6.9, the group H virtually fixes the equivalence class of a one-edge free splitting of W_n.

Now suppose that for every $i \in \{1, 2\}$, the group K_i contains a loxodromic element. By Lemma 5.3, the whole group H virtually fixes a point in $\partial_\infty \mathrm{FF}(W_n, \mathcal{F})$. By Lemma 2.4, the group H virtually fixes the homothety class of an arational tree. This contradicts Lemma 6.10.

Hence in all cases, the group H virtually fixes the equivalence class $\mathcal{S}$ of a one-edge free splitting S. By Theorem 5.1 (3), since H contains a direct product of $n-3$ nonabelian free groups, the group H virtually fixes the equivalence class of a W_{n-1}-star. ■

We now prove a proposition which gives a sufficient condition for equivalence classes of W_{n-1}-stars provided by Proposition 6.11 to be compatible. We first need the following result due to Krstić and Vogtmann.

Proposition 6.12 ([27, Corollary 10.2]). *Let $n \geq 3$. The virtual cohomological dimension of $\mathrm{Out}(W_n)$ is equal to $n-2$. In particular, the maximal rank of a free abelian subgroup of $\mathrm{Out}(W_n)$ is equal to $n-2$.*

Proposition 6.13. *Let $n \geq 5$ and let Γ be a subgroup of $\mathrm{Out}^0(W_n)$ of finite index. Let $k \in \mathbb{N}^*$ and let $H_1, \dots, H_k$ be subgroups of Γ which satisfy $(P_{W_{n-2}})$ and such that the intersection $\bigcap_{i=1}^k H_i$ contains a subgroup H isomorphic to $\mathbb{Z}^{n-2}$. For $i \in \{1, \dots, k\}$, let $\mathcal{S}_i$ be the equivalence class of a W_{n-1}-star S_i which is virtually fixed by H_i. Then, for every $i, j \in \{1, \dots, k\}$, the W_{n-1}-stars S_i and S_j are compatible.*

Proof. Let $i, j \in \{1, \dots, k\}$ be distinct integers. Let H' be a finite index subgroup of H contained in $\mathrm{Stab}_\Gamma(\mathcal{S}_i) \cap \mathrm{Stab}_\Gamma(\mathcal{S}_j)$. Let A_i and A_j be the vertex groups isomorphic to W_{n-1} of $W_n \backslash S_i$ and $W_n \backslash S_j$, respectively (well-defined up to conjugation). By Proposition 6.12, the rank of a maximal abelian subgroup of $\mathrm{Out}(W_{n-1})$ is equal to $n-3$. Therefore, the kernel of the homomorphisms $H' \to \mathrm{Out}(A_i)$ and $H' \to \mathrm{Out}(A_j)$ given by the action on the vertex group contains an element of infinite order. Let $f_i \in \ker(H' \to \mathrm{Out}(A_i))$ and $f_j \in \ker(H' \to \mathrm{Out}(A_i))$ be infinite order elements. By Proposition 2.5 (1), f_i and f_j are twists about S_i and S_j, respectively. As f_i and f_j commute, by Corollary 4.10, S_i and S_j are compatible. This concludes the proof. ∎

7. Algebraic characterization of stabilizers of W_{n-2}-stars

In this section, we give an algebraic characterization of stabilizers of W_{n-2}-stars. By the previous section, we know that groups which satisfy $(P_{W_{n-2}})$ virtually stabilize equivalence classes of W_{n-1}-stars, and we have given an algebraic criterion to show that these W_{n-1}-stars are compatible. In order to prove that a group H which satisfies $(P_{W_{n-2}})$ virtually stabilizes the equivalence class of a W_{n-2}-star, we study the intersection of a normal subgroup $K_1 \times K_2$ of H given by $(P_{W_{n-2}})$ (1) with the group of twists of the equivalence class of a W_{n-1}-star virtually fixed by H.

7.1. Groups of twists in groups satisfying $(P_{W_{n-2}})$

We start this section with a lemma which gives a sufficient condition for a group H satisfying $(P_{W_{n-2}})$ to be the stabilizer of a W_{n-2}-star.

Lemma 7.1. *Let $n \geq 5$ and let Γ be a subgroup of finite index of* $\mathrm{Out}^0(W_n)$. *Let H be a subgroup of Γ which satisfies $(P_{W_{n-2}})$ and let $K_1 \times K_2$ be a normal subgroup of H given by $(P_{W_{n-2}})$* (1). *Let $\mathcal{S}_1$ be the equivalence class of a W_{n-1}-star S_1 virtually fixed by H and let T_1 be the group of twists of $\mathcal{S}_1$.*

Suppose that $T_1 \cap K_1$ is infinite and that there exists an equivalence class $\mathcal{S}_2$ of a W_{n-1}-star S_2 such that the intersection of K_2 with the group of twists T_2 of $\mathcal{S}_2$ is infinite. Then S_1 and S_2 are compatible and H virtually fixes the equivalence class $\mathcal{S}$ of the W_{n-2}-star which refines S_1 and S_2. Moreover, $\mathcal{S}$ is the unique equivalence class of a W_{n-2}-star virtually fixed by H. Finally, the groups $T_1 \cap \mathrm{Stab}_\Gamma(\mathcal{S})$ and K_1 (resp. $T_2 \cap \mathrm{Stab}_\Gamma(\mathcal{S})$ and K_2) are commensurable.

Proof. For $i \in \{1,2\}$, let $f_i \in T_i \cap K_i$ be of infinite order. First remark that, as f_1 and f_2 generate a free abelian group of order 2, we have $T_1 \neq T_2$ because the group of twists of a W_{n-1}-star is virtually a nonabelian free group. Hence we have $\mathcal{S}_1 \neq \mathcal{S}_2$. As K_1 commutes with f_2, Proposition 4.9 shows that K_1 fixes $\mathcal{S}_2$. As K_1 contains a twist of $\mathcal{S}_1$, Lemma 4.7 shows that S_1 and S_2 are compatible.

Let S be a W_{n-2}-star which refines S_1 and S_2, let $\mathcal{S}$ be its equivalence class and let T be the group of twists of $\mathcal{S}$ in Γ. Then T contains a finite index normal subgroup isomorphic to $K_1^{\mathcal{S}_1} \times K_2^{\mathcal{S}_2}$, where $K_1^{\mathcal{S}_1}$ and $K_2^{\mathcal{S}_2}$ are virtually nonabelian free groups.

By Proposition 6.7, we can choose $K_1^{\mathcal{S}_1} \times K_2^{\mathcal{S}_2}$ such that $K_1^{\mathcal{S}_1} \times K_2^{\mathcal{S}_2}$ is a group satisfying property $(P_{W_{n-2}})$ (1). Moreover, up to reordering, $K_1^{\mathcal{S}_1} \subseteq T_1$ and $K_2^{\mathcal{S}_2} \subseteq T_2$. Since K_1 fixes both $\mathcal{S}_1$ and $\mathcal{S}_2$, we see that K_1 fixes $\mathcal{S}$. Therefore, by Proposition 2.5 (1), we have a homomorphism $\Phi\colon K_1 \to \mathrm{Out}(W_{n-2})$ whose kernel is exactly $K_1 \cap T$. By Lemma 4.11, we see that $T_1 \cap \mathrm{Stab}_\Gamma(\mathcal{S}) \cap K_1^{\mathcal{S}_1}$ is a finite index subgroup of $T_1 \cap \mathrm{Stab}_\Gamma(\mathcal{S})$. As $K_1 \cap T_1$ is infinite, so is $K_1 \cap K_1^{\mathcal{S}_1}$. Let

$$P = \ker(\Phi) \cap K_1^{\mathcal{S}_1} = K_1 \cap K_1^{\mathcal{S}_1}.$$

Then, since $K_1 \subseteq \mathrm{Out}^0(W_n)$, the group $K_1^{\mathcal{S}_1} \cap K_1$ is a normal subgroup of K_1. Therefore, P is a nontrivial normal subgroup of K_1. By property $(P_{W_{n-2}})$ (1), we see that K_2 is a finite index subgroup of $C_\Gamma(P)$. But P is centralized by $K_2^{\mathcal{S}_2}$ since $P \subseteq K_1^{\mathcal{S}_1}$. Hence $K_2^{\mathcal{S}_2} \cap K_2$ is a finite index subgroup of $K_2^{\mathcal{S}_2}$. As $K_1^{\mathcal{S}_1}$ is a finite index subgroup of the centralizer of $K_2^{\mathcal{S}_2}$ by property $(P_{W_{n-2}})$ (1), and as K_1 is a finite index subgroup of the centralizer of K_2, we see that $K_1^{\mathcal{S}_1} \cap K_1$ has finite index in K_1 and therefore P has finite index in K_1. Let

$$W_n = \langle x_1 \rangle * \langle x_3, \dots, x_n \rangle * \langle x_2 \rangle$$

be the free factor decomposition of W_n induced by S and let $A = \langle x_3, \dots, x_n \rangle$. Then, up to reordering, for every $f \in P$, there exist $z_f \in A$ and a representative F of f such that F sends x_1 to $z_f x_1 z_f^{-1}$, and, for every $i \neq 1$, fixes x_i.

Claim 7.2. *The only equivalence classes of W_{n-1}-stars which are virtually fixed by K_1 are $\mathcal{S}_1$ and $\mathcal{S}_2$.*

Proof. Let $\mathcal{S}_3$ be the equivalence class of a W_{n-1}-star S_3 virtually fixed by K_1. Suppose towards a contradiction that $\mathcal{S}_3$ is distinct from both $\mathcal{S}_1$ and $\mathcal{S}_2$. Let $K_1' = K_1 \cap \mathrm{Stab}_\Gamma(\mathcal{S}_3)$ and $P' = P \cap \mathrm{Stab}_\Gamma(\mathcal{S}_3)$. Then, as P is an infinite subgroup of the group of twists of $\mathcal{S}_1$, and as P' is a finite index subgroup of P, we see that P' is an infinite subgroup of the group of twists of $\mathcal{S}_1$. By Lemma 4.7, we see that S_1 and S_3 are compatible. Let S' be a W_{n-2}-star that refines S_1 and S_3 and let $\mathcal{S}'$ be its equivalence class. Let

$$W_n = \langle y_1 \rangle * \langle y_3, \dots, y_n \rangle * \langle y_2 \rangle$$

be the free factor decomposition of W_n induced by S' and let $B = \langle y_3, \dots, y_n \rangle$. Since S is a refinement of S_1, we may suppose that $B * \langle y_2 \rangle = A * \langle x_2 \rangle$ and that y_1 is a conjugate of x_1 by an element of $B * \langle y_2 \rangle$. Up to applying a global conjugation, we may also suppose that $y_1 = x_1$ and that $B * \langle y_2 \rangle = A * \langle x_2 \rangle$.

Let T' be the group of twists of $\mathcal{S}'$. Then T' contains a finite index normal subgroup isomorphic to $P_1' \times P_2'$, where both P_1' and P_2' are virtually nonabelian free subgroups of T' which correspond to the groups of twists about the two edges of $W_n \backslash S'$. Then, as P' is a group of twists of $\mathcal{S}_1$, and as P' fixes $\mathcal{S}'$, by Lemma 4.11, up to reordering, the group P' is contained in P_1'.

Let $f' \in P_1'$, let F' be the representative of f' which acts as the identity on $B * \langle y_2 \rangle$ and let $z_{f'} \in B$ be the twistor of F'. Then F' acts as the identity on $A * \langle x_2 \rangle$ and

$F'(x_1) = z_{f'} x_1 z_{f'}^{-1}$. Recall that for every $\psi \in P'$, there exists a unique $z_\psi \in A$ and a unique representative Ψ of ψ such that Ψ sends x_1 to $z_\psi x_1 z_\psi^{-1}$, and, for every $i \neq 1$, fixes x_i. Thus, a necessary condition for f' to be in P' is that $z_{f'} \in A \cap B$.

But as A and B are free factors of W_n, the group $A \cap B$ is a free factor of B. To see this, let U be a free splitting of W_n such that A is a vertex stabilizer of U and let U_B be the minimal subtree of B in U. Then, as U is a free splitting of W_n, we see that U_B is a free splitting of B. But then, as A is a vertex stabilizer in U, we see that $A \cap B$ is a vertex stabilizer in U_B. Therefore, $A \cap B$ is a free factor of B. Thus one can find a W_{n-3}-star $S^{(2)}$ which refines S' and such that, for every $f' \in P'$, the twistor $z_{f'}$ fixes a vertex of $S^{(2)}$. Indeed, one can equivariantly blow up an edge e at the vertex of S' whose stabilizer is B such that the stabilizer of one of the endpoints of e is a subgroup C isomorphic to W_{n-3} with $A \cap B \subseteq C$. Therefore, we may also assume that $S^{(2)}$ is a W_{n-3}-star. Let $\mathcal{S}^{(2)}$ be the equivalence class of $S^{(2)}$. By Proposition 2.5 (3), the group of twists of $\mathcal{S}^{(2)}$ is isomorphic to a direct product W_{n-3}^3 of three infinite groups, where each factor is a group of twists about an edge of $W_n \backslash S^{(2)}$. This implies that P' is contained in exactly one of the three factors isomorphic to W_{n-3}. It follows that the centralizer of P' contains two elements which generate a free abelian group of order 2. This contradicts the fact that the centralizer of P' is virtually a nonabelian free group by $(P_{W_{n-2}})$ (1). The claim follows. ■

The claim above then implies, as K_1 is a normal subgroup of H, that H virtually fixes $\mathcal{S}_2$. As H virtually fixes $\mathcal{S}_1$, we see that H virtually fixes the equivalence class $\mathcal{S}$. Moreover, the above claim shows that $\mathcal{S}$ is the unique equivalence class of a W_{n-2}-star virtually fixed by K_1, and hence virtually fixed by H.

We finally prove that K_1 and $T_1 \cap \mathrm{Stab}_\Gamma(\mathcal{S})$ (resp. K_2 and $T_2 \cap \mathrm{Stab}_\Gamma(\mathcal{S})$) are commensurable. By Lemma 4.11, for every $i \in \{1,2\}$ we see that $K_i^{\mathcal{S}_i} \cap T_i \cap \mathrm{Stab}_\Gamma(\mathcal{S})$ is a finite index subgroup of $T_i \cap \mathrm{Stab}_\Gamma(\mathcal{S})$. Moreover, for every $i \in \{1,2\}$ and every $f \in K_i^{\mathcal{S}_i}$, the twist f of $\mathcal{S}$ is also a twist of $\mathcal{S}_i$. Hence we have $K_i^{\mathcal{S}_i} \subseteq T_i \cap \mathrm{Stab}_\Gamma(\mathcal{S})$. Therefore, for every $i \in \{1,2\}$, the groups $K_i^{\mathcal{S}_i}$ and $T_i \cap \mathrm{Stab}_\Gamma(\mathcal{S})$ are commensurable. Hence it suffices to show that, for every $i \in \{1,2\}$, the groups K_i and $K_i^{\mathcal{S}_i}$ are commensurable.

Recall that $K_2^{\mathcal{S}_2} \cap K_2$ is a finite index subgroup of $K_2^{\mathcal{S}_2}$ and that $K_1^{\mathcal{S}_1} \cap K_1$ has finite index in K_1. Since H virtually fixes $\mathcal{S}$, and since $K_2^{\mathcal{S}_2}$ is a normal subgroup of $\mathrm{Stab}_\Gamma(\mathcal{S})$, we see that $K_2^{\mathcal{S}_2} \cap K_2$ is a normal subgroup of a finite index subgroup of K_2. We know that $K_2^{\mathcal{S}_2} \cap K_2$ commutes with $K_1^{\mathcal{S}_1}$ because $K_1^{\mathcal{S}_1}$ and $K_2^{\mathcal{S}_2}$ commute with each other. Thus, by property $(P_{W_{n-2}})$ (1) applied to $K_1 \times K_2$, the centralizer of $K_2^{\mathcal{S}_2} \cap K_2$ contains K_1 as a finite index subgroup. This shows that $K_1 \cap K_1^{\mathcal{S}_1}$ is a finite index subgroup of $K_1^{\mathcal{S}_1}$. Hence K_1 and $K_1^{\mathcal{S}_1}$ are commensurable. By property $(P_{W_{n-2}})$ (1) applied to $K_1^{\mathcal{S}_1} \times K_2^{\mathcal{S}_2}$, the centralizer of a finite index subgroup of $K_1^{\mathcal{S}_1}$ contains $K_2^{\mathcal{S}_2}$ as a finite index subgroup. Moreover, the centralizer of a finite index subgroup of K_1 contains K_2 as a finite index subgroup. Hence the centralizer of $K_1 \cap K_1^{\mathcal{S}_1}$ contains both K_2 and $K_2^{\mathcal{S}_2}$ as finite index subgroups. Thus K_2 and $K_2^{\mathcal{S}_2}$ are commensurable. This completes the proof of Lemma 7.1. ■

Lemma 7.1 suggests that in order to show that a group H which satisfies $(P_{W_{n-2}})$ is in fact virtually the stabilizer of the equivalence class of a W_{n-2}-star, it suffices to study the intersections of H with groups of twists. A first step towards such a result is the following lemma.

Lemma 7.3. *Let $n \geq 5$ and let Γ be a subgroup of $\mathrm{Out}^0(W_n)$ of finite index. Let H be a subgroup of Γ satisfying $(P_{W_{n-2}})$ and let $K_1 \times K_2$ be a normal subgroup of H given by $(P_{W_{n-2}})$ (1). Let $\mathcal{S}$ be the equivalence class of a W_{n-1}-star S virtually fixed by H and let T be the group of twists of $\mathcal{S}$ contained in Γ.*

There exists a unique $i \in \{1,2\}$ such that $K_i \cap T$ is infinite. Moreover, $H \cap T \cap K_i$ has finite index in $H \cap T$.

Proof. Up to passing to a finite index subgroup of H, we may suppose that H fixes $\mathcal{S}$. The uniqueness assertion follows from the fact that T is virtually a nonabelian free group and that $K_1 \times K_2$ is a direct product. Therefore, up to reordering, we may suppose that $K_1 \cap T$ is finite. Since $\mathrm{Out}(W_n)$ is virtually torsion free by [17, Corollary 5.3], there exists a finite index subgroup K_1' of K_1 such that $K_1' \cap T$ is trivial. Since K_1 is a finitely generated normal subgroup of H, Lemma 6.8 implies that there exists a finite index subgroup H' of H such that K_1' is a normal subgroup of H'. Therefore, we may suppose that $K_1 \cap T$ is trivial. By Proposition 2.5 (1), the natural homomorphism $K_1 \to \mathrm{Out}(W_{n-1})$ given by the action on the vertex groups is injective.

We claim that $H \cap T$ is infinite. Indeed, consider the natural homomorphism $\Phi\colon H \to \mathrm{Out}(W_{n-1})$. The rank of a maximal free abelian subgroup of $\mathrm{Out}(W_{n-1})$ is equal to $n-3$ by Proposition 6.12. As H contains a subgroup isomorphic to $\mathbb{Z}^{n-2}$ by $(P_{W_{n-2}})$ (3), the kernel of $H \to \mathrm{Out}(W_{n-1})$ is infinite. But, by Proposition 2.5 (1), this is precisely $H \cap T$. Therefore, $H \cap T$ is infinite.

We now prove that $H \cap T \cap K_2$ has finite index in $H \cap T$. This will conclude the proof as $H \cap T$ is infinite. Let $K = \Phi^{-1}(\Phi(K_2))$. Note that $H \cap T \subseteq K$. Then, as K_2 is normal in H, we see that K is a normal subgroup of H which contains $H \cap T$ and K_2. We claim that $K \cap K_1$ is finite. Indeed, suppose towards a contradiction that there exists $f \in K \cap K_1$ of infinite order. Then, as the homomorphism

$$\Phi|_{K_1}\colon K_1 \to \mathrm{Out}(W_{n-1})$$

is injective, the element $\Phi(f)$ has infinite order. By definition of K, we see that

$$\Phi(f) \in \Phi(K_1) \cap \Phi(K_2).$$

But, as the homomorphism $\Phi|_{K_1}\colon K_1 \to \mathrm{Out}(W_{n-1})$ is injective, and as K_1 is virtually a nonabelian free group, there exists $g \in K_1$ of infinite order such that $\Phi(g)$ does not commute with $\Phi(f)$. Since $\Phi(f) \in \Phi(K_2)$, this contradicts the fact that K_1 and K_2 commute with each other. Hence $K \cap K_1$ is finite.

The groups K and K_1 are two normal subgroups of H with finite intersection. Let $K_1^{(2)}$ be a finite index normal subgroup of K_1 such that $K \cap K_1^{(2)} = \{1\}$. Since K_1 is finitely generated, by Lemma 6.8 (2), there exists a finite index subgroup $H^{(2)}$ of H such that $K_1^{(2)}$ is a normal subgroup of $H^{(2)}$. Hence $K_1^{(2)}$ and $K \cap H^{(2)}$ are normal subgroups of $H^{(2)}$ with trivial intersection. Therefore, $K \cap H^{(2)} \subseteq C_\Gamma(K_1^{(2)})$. But, property $(P_{W_{n-2}})$ (1) implies that K and K_2 are commensurable. Since K contains $H \cap T$, we see that $K_2 \cap T$ and $H \cap T$ are commensurable. This concludes the proof. ∎

7.2. Groups satisfying $(P_{W_{n-2}})$ and stabilizers of W_{n-2}-stars

In this section, we prove that a subgroup of $\mathrm{Out}^0(W_n)$ which satisfies $(P_{W_{n-2}})$ virtually fixes the equivalence class of a W_{n-2}-star. We first prove a series of properties for elements of $\mathrm{Out}(W_n)$.

Lemma 7.4. *Let $n \geq 3$. Let $w \in W_n$ be a root-closed element of infinite order. Let $\mathcal{S}$ be the equivalence class of a splitting S whose associated amalgamated decomposition of W_n is*

$$W_n = A *_{\langle w \rangle} B,$$

where A and B are subgroups of W_n containing w. Let D be a nontrivial twist about S. Let $h \in W_n$. Then D preserves the conjugacy class of h if and only if there exists $h' \in W_n$ such that $h' \in [h]$ and $h' \in A \cup B$.

Proof. It is clear that D preserves the conjugacy classes of elements in A and B. Conversely, let $h \in W_n$ be such that $D([h]) = [h]$. Let R be a Grushko splitting of W_n. Let R' and S' be metric representatives of R and S, let $\mathcal{R}'$ and $\mathcal{S}'$ be their W_n-equivariant isometry classes and let $[\mathcal{R}']$ and $[\mathcal{S}']$ be their homothety classes. As $\mathbb{P}\overline{\mathcal{O}(W_n)}$ is compact, up to passing to a subsequence, there exists a sequence $(\lambda_n)_{n\in\mathbb{N}} \in (\mathbb{R}_+^*)^{\mathbb{N}}$ and a W_n-equivariant isometry class $\mathcal{T}$ of an $\mathbb{R}$-tree T such that $\lim_{n\to\infty} \lambda_n D^n(\mathcal{R}') = \mathcal{T}$. Since translation length functions are continuous for the Gromov–Hausdorff topology (see [31]), for every $g \in W_n$, we have

$$\lim_{n\to\infty} \lambda_n \|g\|_{D^n(\mathcal{R}')} = \|g\|_{\mathcal{T}},$$

where $\|g\|_{\mathcal{T}}$ is the translation length of g in T. Hence, for every $g \in W_n$, the limit $\lim_{n\to\infty} \lambda_n \|g\|_{D^n(\mathcal{R}')}$ is finite. But as there exists $g' \in W_n$ such that $\|g'\|_{D^n(R')}$ tends to infinity as n goes to infinity, we have $\lim_{n\to\infty} \lambda_n = 0$. As there exists a representative $\phi \in \mathrm{Aut}(W_n)$ of D such that $\phi_A = \mathrm{id}_A$, for every $g \in A$, we have

$$\lim_{n\to\infty} \lambda_n \|g\|_{D^n(\mathcal{R}')} = \lim_{n\to\infty} \lambda_n \|g\|_{\mathcal{R}'} = 0.$$

Hence every element of A fixes a point in T. As A is finitely generated, this implies that A fixes a point in T (see for instance [8, Section 3]). Similarly, we see that the groups B and $\langle h \rangle$ fix points in T. As $W_n = \langle A, B \rangle$, we see that A and B cannot fix the same point in T. Thus, there exists a natural W_n-equivariant application $\Psi: S' \to T$. Let us prove that Ψ is an isometry. It suffices to prove that Ψ is a local isometry, that is, it suffices to prove that the application Ψ does not fold edges. By W_n-equivariance and symmetry, it suffices to prove that, If e and e' are two distinct edges of S' whose origin is the vertex fixed by A, then $\Psi(e) \neq \Psi(e')$. Suppose towards a contradiction that $\Psi(e) = \Psi(e')$. Then $\langle G_e, G_{e'} \rangle$ fixes $\Psi(e)$. Note that G_e and $G_{e'}$ are isomorphic to $\mathbb{Z}$. Moreover, since w is root-closed, we neither have $G_e \subseteq G_{e'}$ nor $G_{e'} \subseteq G_e$. Since G_e is a malnormal subgroup of W_n and since $G_{e'}$ is a nontrivial conjugate of G_e, we see that $G_e \cap G_{e'} = \{1\}$. Hence $\langle G_e, G_{e'} \rangle$ is a nonabelian free group which fixes an arc in T. But arc stabilizers in T are cyclic, a contradiction. Hence Ψ is an isometric embedding and, by minimality of T, the application Ψ is a W_n-equivariant isometry. Therefore, as h fixes a point in T, it also fixes a point in S'. Therefore, h is contained in a conjugate of A or B. ∎

For the next proposition, recall the definition of the subgroup $\mathbb{F}$ of W_n from Lemma 6.2.

Proposition 7.5. *Let $n \geq 3$. Let $(H_N)_{N \in \mathbb{N}^*}$ be an increasing sequence of subgroups of $\mathbb{F}$. There exists an integer n_0 such that for every $N \geq n_0$, we have*

$$\mathrm{Out}(W_n, H_N^{(t)}) = \mathrm{Out}(W_n, H_{n_0}^{(t)}).$$

Proof. We show that the result is a consequence of a similar result in the context of the automorphism group of a nonabelian free group due to Martino and Ventura [29, Corollary 4.2]. Since $\mathbb{F}$ is a nonabelian free group, we may suppose that, for every $N \in \mathbb{N}^*$, the group H_N is a nonabelian free group. Hence for every $N \in \mathbb{N}^*$, we have $C_{W_n}(H_N) = \{1\}$. Therefore, for every $N \in \mathbb{N}^*$ and every $\phi \in \mathrm{Out}(W_n, H_N^{(t)})$, there exists a unique representative $\Phi \in \mathrm{Aut}(W_n)$ of ϕ such that $\Phi(H_N) = H_N$ and $\Phi|_{H_N} = \mathrm{id}_{H_N}$. This implies that, for every $N \in \mathbb{N}^*$, we have an injective homomorphism $\mathrm{Out}(W_n, H_N^{(t)}) \hookrightarrow \mathrm{Aut}(W_n, H_N)$, where $\mathrm{Aut}(W_n, H_N)$ is the group of automorphisms of W_n which fix every element of H_N. Therefore, it suffices to prove the result for $\mathrm{Aut}(W_n, H_N)$. Since there exists an injective homomorphism $\mathrm{Aut}(W_n) \to \mathrm{Aut}(\mathbb{F})$ and since, for every $N \in \mathbb{N}^*$, we have $H_N \subseteq \mathbb{F}$, it suffices to prove that there exists $n_0 \in \mathbb{N}^*$ such that, for every $N \geq n_0$, we have $\mathrm{Aut}(\mathbb{F}, H_N) = \mathrm{Aut}(\mathbb{F}, H_{n_0})$. We then conclude using [29, Corollary 4.2]. ∎

We now recall a theorem due to Guirardel and Levitt which provides a canonical splitting for a relative one-ended hyperbolic group (recall that a group G is *one-ended relative to a family of subgroups* $\mathcal{H}$ if G does not have a one-edge splitting with finite edge stabilizers such that every subgroup of $\mathcal{H}$ fixes a point).

Theorem 7.6 ([19, Theorem 9.14]). *Let G be a hyperbolic group and let $\mathcal{H}$ be a family of subgroups such that G is one-ended relative to $\mathcal{H}$. There exists a splitting S of G such that*

(1) *Every edge stabilizer is virtually infinite cyclic.*

(2) *For every $H \in \mathcal{H}$, the group H is elliptic in S.*

(3) *The tree S is invariant under all automorphisms of G preserving $\mathcal{H}$. Moreover, S is compatible with every splitting S' with virtually cyclic edge stabilizers and such that for every $H \in \mathcal{H}$, the group H is elliptic in S'.*

(4) *Let $H \in \mathcal{H}$ be such that H is virtually a (possibly not finitely generated) non-abelian free group, and let v be the vertex of S fixed by H. Let* Inc_v *be the finite set of representatives of all conjugacy classes of groups associated with edges in S which are incident to v. Then the group* $\mathrm{Out}(G_v, \{H, \mathrm{Inc}_v\}^{(t)})$ *is finite.*

Assertion (4) is a bit stronger than what is stated in [19], hence we add some explanations.

Proof of assertion (4) *of Theorem* 7.6. Let S, H and v be as in assertion (4). By, for instance, [32, Proposition 2.5], the set Inc_v is finite. Note that every group in Inc_v is virtually cyclic by assertion (1). Thus, the set Inc_v is a finite set of finitely generated groups. Up to taking finite index subgroups, we may suppose that $H \subseteq \mathbb{F}$ and that for every subgroup H' in Inc_v, the group H' is contained in $\mathbb{F}$. Suppose towards a contradiction that $\mathrm{Out}(G_v, \{H, \mathrm{Inc}_v\}^{(t)})$ is infinite. Suppose first, following the terminology of [19], that the vertex v is *rigid*. By Proposition 7.5, there exists a finitely generated subgroup K of H such that

$$\mathrm{Out}(W_n, H^{(t)}) = \mathrm{Out}(W_n, K^{(t)}).$$

By [18, Theorem 7.14], there exists a one-edge splitting U of G_v whose edge stabilizer is isomorphic to $\mathbb{Z}$ such that K and every group in Inc_v are elliptic in U. Since v is a rigid vertex, there exists $h \in H$ such that h acts loxodromically on U. Since every group in Inc_v fixes a point in U, one can blow up the splitting U at the vertex v of S. This gives a refinement S' of S. Let D' be a nontrivial infinite twist of U. Then D' induces a twist D of S'. By Lemma 7.4, the element D fixes the conjugacy class of K but does not fix the conjugacy class of h. This contradicts $\mathrm{Out}(W_n, H^{(t)}) = \mathrm{Out}(W_n, K^{(t)})$.

So we may suppose, following the terminology of [19], that the vertex v is *flexible*. By [19, Theorem 9.14 (2)], as H is virtually a nonabelian free group, the vertex v is a *QH vertex* (see [19, Definition 5.13]). But the definition of a QH vertex implies, as H is contained in $\mathcal{H}$, that the group H must be virtually contained in a boundary subgroup of the fundamental group of the orbifold associated with G_v. Thus the group H must be virtually cyclic, a contradiction. ∎

We also need some results about splittings over virtually cyclic groups, whose generalization to virtually free groups is due to Cashen.

Theorem 7.7 ([5, Theorem 1.2]). *Let G_1 and G_2 be finitely generated virtually nonabelian free groups, and let C be a virtually cyclic group which is a proper subgroup of both G_1 and G_2. Then $G_1 *_C G_2$ is virtually a nonabelian free group if and only if there exists $i \in \{1, 2\}$ such that G_i has a splitting with finite edge stabilizers such that C is a vertex stabilizer.*

Corollary 7.8. *Let $n \geq 3$ and let G_1, G_2 be subgroups of W_n such that $W_n = G_1 *_C G_2$ is a nontrivial amalgamated product of W_n, where C is isomorphic to W_2 and G_1 and G_2 are not virtually cyclic.*

(1) *There exists* $i \in \{1,2\}$ *such that* C *is a free factor of* G_i. *Moreover, if* $j \in \{1,2\} \backslash \{i\}$, *then* G_j *is a free factor of* W_n.

(2) *There exist* $3 \leq k_1, k_2 \leq n-1$ *such that* $k_1 + k_2 = n+2$ *and, for every* $i \in \{1,2\}$, *the group* G_i *is isomorphic to* W_{k_i}. *In particular,* $n \geq 4$.

Proof. (1) By Lemma 6.2, the subgroup $\mathbb{F}$ of W_n is a nonabelian free group of finite index. Since both G_1 and G_2 are not virtually cyclic, the intersections $G_1 \cap \mathbb{F}$ and $G_2 \cap \mathbb{F}$ are finite index subgroups of G_1 and G_2 which are nonabelian free groups. Hence G_1 and G_2 are virtually nonabelian free groups. Moreover, since W_n and C are finitely generated, so are G_1 and G_2. By Theorem 7.7, up to exchanging the roles of G_1 and G_2, we may suppose that G_1 has a splitting S such that every edge stabilizer is finite and C is the stabilizer of a vertex $v \in VS$. Note that, since the finite subgroups of W_n are all isomorphic to F, every edge stabilizer of S is either trivial or isomorphic to F. Since every element of W_n of order 2 is a conjugate of an element in a standard generating set of W_n, every nontrivial edge stabilizer is a free factor of both of its endpoint stabilizers. Let V_1 be the set of vertices of S distinct from v and fixed by a subgroup of C isomorphic to F. Therefore, for every $w \in V_1$, there exists a subgroup A_w of G_w and an element $x_w \in C$ of order 2 such that $G_w = A_w * \langle x_w \rangle$. Let S_0 be a splitting of W_n obtained from S by blowing-up, at every vertex $w \in V_1$, the free splitting $A_w * \langle x_w \rangle$ and by attaching the edge fixed by x_w to its corresponding fixed point. Let S' be the splitting of W_n obtained from S_0 by collapsing every edge with nontrivial stabilizer. Then the stabilizer in G_1 of every edge of S' adjacent to the vertex fixed by C has trivial stabilizer. Thus, C is a free factor of G_1 and there exists $H_1 \subset G_1$ such that $G_1 = H_1 * C$. This proves the first assertion of (1). The second assertion of (1) follows from the fact that

$$W_n = G_1 *_C G_2 = (H_1 * C) *_C G_2 = H_1 * G_2.$$

Hence H_1 and G_2 are free factors of W_n.

(2) Therefore, there exist $h_1, k_2 \in \{1, \dots, n-2\}$ with $h_1 + k_2 = n$ such that H_1 is isomorphic to W_{h_1} and G_2 is isomorphic to W_{k_2}. Thus G_1 is isomorphic to W_{h_1+2}. Set $k_1 = h_1 + 2$. Since the amalgamated product is nontrivial and since G_1 and G_2 are not virtually cyclic, we have $3 \leq k_1, k_2 \leq n-1$. This proves (2). ∎

Lemma 7.9. *Let* $n \geq 4$ *and let* S *be a splitting of* W_n. *Let* $\mathcal{S}$ *be its equivalence class. Let* v_1 *and* v_2 *be adjacent vertices of* S *and let* e *be the edge between* v_1 *and* v_2. *Suppose that* G_e *is isomorphic to* W_2. *Let* $f \in \mathrm{Stab}_{\mathrm{Out}(W_n)}(\mathcal{S})$ *be such that*

(1) *the graph automorphism of* $W_n \backslash S$ *induced by* f *is trivial;*

(2) *the natural homomorphisms* $\langle f \rangle \to \mathrm{Out}(G_{v_1}, G_e)$ *and* $\langle f \rangle \to \mathrm{Out}(G_{v_2}, G_e)$ *are trivial.*

Then f *has a representative which acts as the identity on* $\langle G_{v_1}, G_{v_2} \rangle$.

Proof. By (2), the outer automorphism f has two representatives F_1 and F_2 such that for every $i \in \{1,2\}$, we have $F_i(G_{v_i}) = G_{v_i}$ and $F_i|_{G_{v_i}} = \mathrm{id}_{G_{v_i}}$. Note that $G_{v_1} \cap G_{v_2} = G_e$.

Hence F_1 and F_2 act as the identity on G_e. Therefore, F_1 and F_2 differ by an inner automorphism ad_z with $z \in C_{W_n}(G_e)$. However, since G_e is isomorphic to W_2, we have $C_{W_n}(G_e) = \{e\}$. Hence $F_1 = F_2$. This concludes the proof. ■

Proposition 7.10. *Let $n \geq 5$ and let Γ be a finite index subgroup of $\mathrm{Out}^0(W_n)$. Let H be a subgroup of Γ which satisfies $(P_{W_{n-2}})$. Then H virtually stabilizes the equivalence class of a W_{n-2}-star. Moreover, this equivalence class is unique.*

Proof. By Proposition 6.11, the group H virtually fixes the equivalence class $\mathcal{S}$ of a W_{n-1}-star S. Let $W_n = A * \langle x_n \rangle$ be the free factor decomposition of W_n induced by S. Up to passing to a finite index subgroup, we may suppose that H fixes $\mathcal{S}$. Let T be the group of twists of $\mathcal{S}$ contained in Γ. By Proposition 2.5 (2), the group $\mathrm{Stab}(\mathcal{S})$ is isomorphic to $\mathrm{Aut}(A)$ and the group of twists of $\mathcal{S}$ is identified with the inner automorphism group of A.

Let $K_1 \times K_2$ be a normal subgroup of H given by property $(P_{W_{n-2}})$ (1). According to Lemma 7.3, up to exchanging the roles of K_1 and K_2, we may assume that $K_1 \cap T$ is infinite, that $H \cap T \cap K_1$ is a finite index subgroup of $H \cap T$ and that $K_2 \cap T$ is finite. Up to passing to a finite index subgroup of H, we may assume that $K_2 \cap T = \{1\}$. In particular, the natural homomorphism $\phi\colon K_2 \to \mathrm{Out}(A)$ is injective. Let $K \subseteq A$ be the group of twistors associated with twists contained in K_1. Note that to every splitting S_0 of A such that K fixes a unique vertex of S_0, one can deduce a splitting S_0' of W_n such that K fixes a point of S_0'. Indeed, by blowing-up the splitting S_0 at the vertex v of S whose associated group is A, and by attaching the edges of S adjacent to v to the vertex fixed by K, we obtain a splitting S_0' of W_n such that K fixes a point of S_0'. Let $\mathcal{S}_0'$ be the equivalence class of S_0'. We claim that the group $K_1 \cap T$ fixes $\mathcal{S}_0'$. Indeed, let e_0 be the edge of S_0' adjacent to the vertex v_0 fixed by K and the vertex fixed by $\langle x_n \rangle$. Since the stabilizer of e_0 is trivial, Proposition 2.5 implies that the group of twists about e_0 at the vertex v_0 contains all the twists whose twistor is an element of K. Hence $K_1 \cap T$ fixes $\mathcal{S}_0'$.

We now construct a one-edge free splitting S_0 of A such that K fixes a vertex of S_0. By the above discussion, this will give a two-edge free splitting of W_n such that K fixes a vertex of this splitting which is not a leaf and whose equivalence class is fixed by $K_1 \cap T$. We distinguish between three cases, according to whether A is one-ended relative to K and according to the edge stabilizers of a splitting of A relative to K.

Case 1. There exists a free splitting S_0 of A such that K fixes a vertex of S_0.

In particular, the corresponding splitting S_0' of W_n constructed above is a free splitting of W_n. We claim that the splitting S_0' has two orbits of edges. Indeed, suppose that S_0' has k orbits of edges, with $k \geq 3$. Then, S_0' is obtained from S by blowing-up at least two orbits of edges at v. Therefore, the group of twistors K is contained in a free factor B of W_n isomorphic to W_{n-3}. Let B' be a free factor of W_n isomorphic to W_2 such that

$$W_n = \langle x_n \rangle * B * B'$$

and let R be the free splitting associated with this decomposition. Then the equivalence class $\mathcal{R}$ of R is a free splitting of W_n fixed by $K_1 \cap T$. But by Proposition 2.5 (3), the group of twists of $\mathcal{R}$ is isomorphic to $B \times B \times W_2$. Moreover, the group $K_1 \cap T$ is contained in one of the factors of $B \times B \times W_2$ isomorphic to B. Therefore, the centralizer of $K_1 \cap T$ contains a free abelian group of rank 2. Since $K_1 \cap T$ is a normal subgroup of K_1, this contradicts the fact that the centralizer of $K_1 \cap T$ is virtually a nonabelian free group by property $(P_{W_{n-2}})$ (1). Therefore, the splitting S_0' is a two-edge free splitting.

Case 2. There exists a splitting S_0 of A such that K fixes a vertex of S_0 and such that one of the edge stabilizers of S_0 is finite.

Let S_0' be the corresponding splitting of W_n constructed in the above discussion. If S_0 has an edge e' with trivial stabilizer, then by collapsing every orbit of edges of S_0 except the one containing e', we obtain a splitting S_1 of A such that K fixes a vertex of K. Then the corresponding splitting S_1' of W_n is a free splitting. Thus, we can apply case 1.

Therefore, we may assume that every edge stabilizer of S_0 is infinite or a nontrivial finite subgroup of W_n. By collapsing every edge of S_0 with infinite stabilizer and by collapsing all but one orbit of edges with finite edge stabilizer, we may suppose that S_0 is a one-edge splitting such that every edge stabilizer of S_0 is a nontrivial finite subgroup of W_n. Every finite subgroup of W_n is isomorphic to F and is in fact a free factor of W_n. We claim that we can construct a splitting X_0 of A which contains an edge with trivial stabilizer and such that K fixes a vertex of X_0. Indeed, let x_0 be the vertex of S_0 fixed by K, let f_0 be an edge adjacent to x_0 and let x_1 be the vertex of f_0 distinct from v_0. Let G_{x_0} be the stabilizer of x_0, let G_{x_1} be the stabilizer of x_1 and let G_{f_0} be the stabilizer of f_0. Note that, since there does not exist HNN extensions in W_n, the groups G_{x_0} and G_{x_1} are not conjugate in W_n. The group G_{f_0} is a free factor of both G_{x_0} and G_{x_1}. Thus, there exists a free factor A' of G_{x_1} such that $G_{x_1} = G_{f_0} * A'$. Let U be the splitting of A such that the underlying tree of $W_n \backslash U$ is the same one as the underlying tree of $W_n \backslash S_0$, such that the stabilizer of every vertex which is not in the orbit of x_1 is the same one as the stabilizer of the corresponding vertex in S_0 and the stabilizer of x_1 is A'. Then the edge f_0 has trivial stabilizer in U and K fixes a vertex of U. This proves the claim. Therefore, case 2 is a consequence of case 1.

Case 3. The group A is one-ended relative to K.

We prove that this assumption leads to a contradiction. By Theorem 7.6, there exists a canonical splitting S_0 of A whose edge stabilizers are virtually infinite cyclic, such that K fixes a point of S_0 and such that every automorphism of A preserving K fixes the equivalence class of S_0. Let S_0' be the corresponding splitting of W_n, and let $\mathcal{S}_0'$ be its equivalence class. Recall that the group $K_1 \cap T$ is a normal subgroup of H contained in $\mathrm{Inn}(A)$. Let $k \in K$ and let $f \in H$. Let F be a representative of f which fixes x_n and which preserves A. As $K_1 \cap T$ is a normal subgroup of H, there exists $k' \in K$ such that

$$F \circ \mathrm{ad}_k \circ F^{-1} = \mathrm{ad}_{F(k)} = \mathrm{ad}_{k'}.$$

Since the center of A is trivial, we have $F(k) = k'$. Hence the group H viewed as a subset of $\mathrm{Aut}(A)$ preserves K. Thus H preserves $\mathcal{S}_0'$.

Let v_0 be the vertex of S_0' fixed by K and let e_0 be the edge of S_0' between v_0 and the point fixed by $\langle x_n \rangle$. By construction, the stabilizer of every edge of S_0' which is not in the orbit of e_0 is virtually cyclic, that is, it is isomorphic either to $\mathbb{Z}$ or to W_2. By Lemma 2.7, a twist about an edge whose stabilizer is isomorphic to $\mathbb{Z}$ is central in a finite index subgroup of $\mathrm{Stab}_{\mathrm{Out}(W_n)}(\mathcal{S}_0')$. Since any finite index subgroup of H has finite center by Remark 6.1 (2), we see that the stabilizer of every edge of S_0' which is not in the orbit of e_0 is isomorphic to W_2. Therefore, Remark 2.6 implies that the group of bitwists about every edge of S_0' which is not in the orbit of e_0 is trivial. Thus, the group of bitwists T_0 of $\mathcal{S}_0'$ is reduced to the group of twists about e_0.

Let $\overline{W_n \backslash S_0'}$ be the graph associated with $W_n \backslash S_0'$. For every vertex $v \in V(\overline{W_n \backslash S_0'})$, let Inc_v be the set containing the conjugacy class of the edge group of every edge adjacent to v (seen as a subgroup of G_v). Let $\overline{v}_0$ be the image of v_0 in $\overline{W_n \backslash S_0'}$ and let $\overline{e}_0$ be the image of e_0 in $\overline{W_n \backslash S_0'}$. By Proposition 2.5 and Remark 2.6, up to taking a finite index subgroup of H, we have a natural homomorphism

$$\Psi\colon H \to \mathrm{Out}(G_{v_0}, \{K, \mathrm{Inc}_{\overline{v}_0}\}) \times \prod_{v \in V(\overline{W_n \backslash S_0'}), v \neq \overline{v}_0} \mathrm{Out}(G_v, \mathrm{Inc}_v),$$

whose kernel is $T_0 \cap H$. Note that every edge stabilizer is isomorphic to W_2, hence the outer automorphism group of every edge stabilizer is finite. Thus, up to taking a finite index subgroup of H, we may suppose that the image of Ψ is contained in

$$\mathrm{Out}(G_{v_0}, \{K, \mathrm{Inc}_{\overline{v}_0}^{(t)}\}) \times \prod_{v \in V(\overline{W_n \backslash S_0'}), v \neq \overline{v}_0} \mathrm{Out}(G_v, \mathrm{Inc}_v^{(t)}).$$

Recall that $K_2 \cap T_0 = \{1\}$, hence $\Psi|_{K_2}$ is injective. Moreover, as K_2 commutes with K_1, the group K_2 is contained in $\mathrm{Out}(W_n, K^{(t)})$. Recall that Theorem 7.6 (4) implies that the group $\mathrm{Out}(G_{v_0}, \{K, \mathrm{Inc}_{\overline{v}_0}\}^{(t)})$ is finite. Since $\Psi|_{K_2}$ is injective, this implies that

$$\prod_{v \in V(\overline{W_n \backslash S_0'}), v \neq \overline{v}_0} \mathrm{Out}(G_v, \mathrm{Inc}_v^{(t)})$$

is infinite. Since the graph $\overline{W_n \backslash S_0'}$ is finite, there exists $v \in V(\overline{W_n \backslash S_0'})$ such that $v \neq \overline{v}_0$ and $\mathrm{Out}(G_v, \mathrm{Inc}_v^{(t)})$ is infinite.

Suppose first that there exist two distinct vertices v and w of $\overline{W_n \backslash S_0'}$ such that $v, w \neq \overline{v}_0$ and both $\mathrm{Out}(G_v, \mathrm{Inc}_v^{(t)})$ and $\mathrm{Out}(G_w, \mathrm{Inc}_v^{(t)})$ are infinite. Since G_v and G_w are subgroups of W_n whose outer automorphism groups are infinite, they are virtually nonabelian free groups. Thus, we can apply Theorem 6.5 to both (G_v, Inc_v) and (G_w, Inc_w) to show that there exist a $\mathcal{Z}_{RC}$-splitting U_v of G_v and U_w of G_w such that every group in Inc_v fixes a point in U_v and every group in Inc_w fixes a point in U_w. One can then blow up the splittings U_v and U_w at the vertices v and w of $\overline{W_n \backslash S_0'}$ and attach the edges adjacent to v and w in $\overline{W_n \backslash S_0'}$ to the points fixed by their corresponding edge groups in U_v and U_w. This gives a refinement S_1 of S_0'. Let $\mathcal{S}_1$ be the equivalence class of S_1. Note that, since the group of twists about the edge e_0 of S'_0 is contained in the group of twists of $\mathcal{S}_1$, the group $K_1 \cap T$ fixes $\mathcal{S}_1$. Note that the stabilizer of an edge in U_v or U_w is either finite or isomorphic to $\mathbb{Z}$. If there exists an edge in U_v or U_w with a finite edge stabilizer, as v and w come from vertices in S_0, we can apply case 2 to conclude. Suppose that every edge stabilizer of U_v and U_w is isomorphic to $\mathbb{Z}$. By Lemma 2.7, a twist about an edge whose stabilizer is isomorphic to $\mathbb{Z}$ is central in a finite index subgroup of $\mathrm{Stab}_{\mathrm{Out}(W_n)}(\mathcal{S}_1)$. Hence $K_1 \cap T$ has a finite index subgroup which is centralized by a free abelian group of rank 2. This contradicts property $(P_{W_{n-2}})$ (1).

Suppose now that there exists a unique vertex $v \in V(\overline{W_n \backslash S'_0})$ such that $v \neq \overline{v}_0$ and $\mathrm{Out}(G_v, \mathrm{Inc}_v^{(t)})$ is infinite. Recall that the image of the homomorphism $\Psi|_{K_2}$ is contained in

$$\mathrm{Out}(G_{v_0}, \{K, \mathrm{Inc}_{\overline{v}_0}\}^{(t)}) \times \prod_{w \in V(W_n \backslash S'_0), w \neq \overline{v}_0} \mathrm{Out}(G_w, \mathrm{Inc}_w^{(t)}).$$

In particular, as $\mathrm{Out}(G_{v_0}, \{K, \mathrm{Inc}_{\overline{v}_0}\}^{(t)})$ is finite, up to taking a finite index subgroup of K_2, we may suppose that the image of $\Psi|_{K_2}$ is contained in $\mathrm{Out}(G_v, \mathrm{Inc}_v^{(t)})$.

Claim 7.11. *Let $f \in K_2$ and let X be a connected subgraph of $\overline{W_n \backslash S'_0}$ such that every vertex of X is distinct from v and such that the group associated with every edge of X is isomorphic to W_2. Then f has a representative which acts as the identity on $\langle G_w \rangle_{w \in VX}$.*

Proof. We prove the result by induction on the number m of edges of X. If X is reduced to a vertex, then the conclusion is immediate. Suppose that $|EX| = m \geq 1$. Let w_1 and w_2 by two adjacent vertices in VX such that w_1 is a leaf of X. Let e' be the edge in X between w_1 and w_2. Let X' be the graph obtained from X be removing w_1 and e'. The graph X' is a connected subgraph of $\overline{W_n \backslash S'_0}$ which satisfies the hypothesis of the lemma and such that $|EX'| = m - 1$. By the induction hypothesis, the element f has a representative which acts as the identity on $\langle G_w \rangle_{w \in VX'}$. Let $W_n \backslash S'_2$ be the graph of groups obtained from $W_n \backslash S'_0$ by collapsing X' and let $p \colon W_n \backslash S'_0 \to W_n \backslash S'_2$ be the natural projection. Since f has a representative which acts as the identity on $\langle G_w \rangle_{w \in VX'}$, the element f fixes the equivalence class of $W_n \backslash S'_2$. Note that the group associated with $p(w_2)$ is $\langle G_w \rangle_{w \in VX'}$ and that the group associated with $p(w_1)$ is G_{w_1}. Moreover, the group associated with $p(e')$ is $G_{e'}$, in particular, it is isomorphic to W_2. Thus for every $i \in \{1, 2\}$, the outer automorphism f has a representative F_i such that $F_i(G_{p(w_i)}) = G_{p(w_i)}$ and $F_i|_{G_{p(w_i)}} = \mathrm{id}_{G_{p(w_i)}}$. Thus, by Lemma 7.9 applied to $W_n \backslash S'_2$, the outer automorphism f has a representative which acts as the identity on

$$\langle G_{p(w_1)}, G_{p(w_2)} \rangle = \langle G_w \rangle_{w \in VX}.$$

The claim follows. ∎

Let e' be the edge adjacent to v in $\overline{W_n \backslash S'_0}$ which is contained in the path between v and $\overline{v}_0$ and let $\widetilde{e}'$ be a lift of e' in S'_0. Note that v is contained in the same connected component of $\overline{W_n\backslash S_0'}\backslash\mathring{\overline{e}}_0$ as $\overline{v}_0$. Thus the edges e' and e_0 are not in the same orbit. Moreover, the stabilizer of e' is isomorphic to W_2. Let S_1' be the splitting of W_n obtained from S_0' by collapsing every edge of S_0' which is not in the orbit of $\widetilde{e}'$ and e_0. Let $\mathcal{S}_1'$ be the equivalence class of S_1'. Then S_1' has two orbits of edges. Let v_2 be the vertex of S_1' fixed by K and let v_1 be the vertex of S_1' adjacent to v_2 which is fixed by a conjugate of G_v. Let e be the edge adjacent to v_1 and v_2. Note that, up to taking a finite index subgroup of K_2, the group K_2 fixes $\mathcal{S}_1'$. Thus, by Proposition 2.5 and Remark 2.6, we have a natural homomorphism $\Phi\colon K_2 \to \mathrm{Out}(G_{v_1}, G_e) \times \mathrm{Out}(G_{v_2}, G_e)$ whose kernel is contained in $K_2 \cap T_0 = \{1\}$. Moreover, the claim applied to the connected component of $\overline{W_n\backslash S_0'}\backslash(\mathring{\overline{e}} \cup \mathring{\overline{e}}_0)$ containing $\overline{v}_0$ shows that every element $f \in K_2$ has a representative which acts as the identity on G_{v_2}. Hence $\Phi(K_2)$ is isomorphic to K_2 and is contained in $\mathrm{Out}(G_{v_1}, G_e)$. We also see that, as K is virtually a nonabelian free group, its centralizer in W_n is trivial. Hence every element $f \in K_2$ has a unique representative which acts as the identity on K. Let $f \in K_2$. Recall that $W_n = A * \langle x_n\rangle$. Then the representative of f which preserves A and fixes x_n must fix K by Lemma 4.12 (since $f \in K_2$ centralizes K_1). As f has a representative which acts as the identity on G_{v_2} and as $K \subseteq G_{v_2}$, we see that f has a representative which acts as the identity on $G_{v_2} * \langle x_n\rangle$.

Note that the group $G_{v_1} *_{G_e} G_{v_2}$ is a splitting of A such that G_e is isomorphic to W_2. Moreover, as K fixes v_2, the group G_{v_2} is not virtually cyclic. Since the group $\mathrm{Out}(G_v)$ is infinite, the group G_v is not virtually cyclic. Hence the group G_{v_1} is not virtually cyclic. Therefore, we may apply Corollary 7.8 to $G_{v_1} *_{G_e} G_{v_2}$: there exist $k_1, k_2 \geq 3$ such that for every $i \in \{1,2\}$, the group G_{v_i} is isomorphic to W_{k_i}. Moreover, there exist $i \in \{1,2\}$ and $j \in \{1,2\}\backslash\{i\}$ such that W_{k_i} is a free factor of W_{k_i} and G_e is a free factor of W_{k_j}.

Suppose first that W_{k_1} is a free factor of A and that G_e is a free factor of W_{k_2}. Let B be such that $W_{k_2} = G_e * B$. Then B is a free factor of A since

$$A = G_{v_1} *_{G_e} G_{v_2} = G_{v_1} *_{G_e} (G_e * B) = G_{v_1} * B.$$

Since $k_2 \geq 3$, the group B is nontrivial. Let z be an infinite order element of G_e. Let F_1 be the automorphism of W_n which acts as a global conjugation by z on B and which fixes x_n and G_{v_1} (recall that as $W_n = B * G_{v_1} * \langle x_n\rangle$, the automorphism F_1 is uniquely determined). Let F_2 be the automorphism of W_n which acts as a global conjugation by z on A and which fixes x_n. Then $\langle [F_1], [F_2]\rangle$ is a subgroup of $\mathrm{Out}(W_n)$ isomorphic to a free abelian group of rank 2. Recall that every element of K_2 has a representative which acts as the identity on $G_{v_2} * \langle x_n\rangle$. Since $[F_1]$ and $[F_2]$ have representatives whose support is contained in $G_{v_2} * \langle x_n\rangle$, the group $\langle [F_1], [F_2]\rangle$ is contained in $C_{\mathrm{Out}(W_n)}(K_2)$. This contradicts property $(P_{W_{n-2}})$ (1) which says that the centralizer of K_2 is virtually a nonabelian free group.

Suppose now that W_{k_2} is a free factor of A and that G_e is a free factor of W_{k_1}. Let B be such that $W_{k_1} = G_e * B$. As before, the group B is a free factor of A and $A = B * G_{v_2}$. But K is contained in G_{v_2}. This contradicts the fact that A is one-ended relative to K. The conclusion in case 3 follows.

Therefore, we have constructed a free splitting S'_0 of W_n which is a two-edge free splitting fixed by $K_1 \cap T$. Moreover, the construction of the splitting is such that the vertex of the underlying graph of $W_n \backslash S'_0$ whose associated group contains K is not a leaf. We now prove that S'_0 is a W_{n-2}-star. Let C be the vertex stabilizer of S'_0 containing K, and let C' be a vertex stabilizer of S'_0 which is not a conjugate of C nor $\langle x_n \rangle$. Then C' is the vertex group of a leaf of the underlying graph of $W_n \backslash S'_0$. By Proposition 2.5 (3), the group of twists of $\mathcal{S}'_0$ is isomorphic to $C \times C \times C'/Z(C')$. Since the centralizer of $K \cap T_1$ is virtually a nonabelian free group by property $(P_{W_{n-2}})$ (1), we conclude that $C'/Z(C')$ is finite. Hence C' is isomorphic to F and S'_0 is a W_{n-2}-star.

We now prove that H virtually fixes $\mathcal{S}'_0$. By Proposition 2.5 (3), the group of twists of $\mathcal{S}'_0$ is isomorphic to $W_{n-2} \times W_{n-2}$. By Lemma 4.11, the group $K_1 \cap T$ is contained in one of the factors isomorphic to W_{n-2} of the group of twists of $\mathcal{S}'_0$. Therefore, $K_1 \cap T$ is centralized by the other factor of the group of twists of $\mathcal{S}'_0$. Since the centralizer of $K_1 \cap T$ contains K_2 as a finite index subgroup, the group K_2 contains a twist f of infinite order about the edge e of S'_0 which does not collapse onto S. This twist is a twist about a W_{n-1}-star obtained from S'_0 by collapsing the orbit of edges which does not contain e. By Lemma 7.1, the group H virtually fixes $\mathcal{S}'_0$. Moreover, K_1 is commensurable with $T \cap \mathrm{Stab}(\mathcal{S}'_0)$, that is K_1 is commensurable with the group of twists about one edge of $\mathcal{S}'_0$. Lemma 6.6 then implies that K_1 virtually fixes a unique equivalence class of W_{n-2}-stars. Therefore, since K_1 is a normal subgroup of H, we see that H virtually fixes a unique equivalence class of W_{n-2}-stars. This concludes the proof. ∎

Proposition 7.12. *Let $n \geq 5$ and let Γ be a finite index subgroup of $\mathrm{Out}^0(W_n)$. Let $\Psi \in \mathrm{Comm}(\Gamma)$. Then for every equivalence class $\mathcal{S}$ of W_{n-2}-stars, there exists a unique equivalence class $\mathcal{S}'$ of W_{n-2}-stars such that $\Psi([\mathrm{Stab}_\Gamma(\mathcal{S})]) = [\mathrm{Stab}_\Gamma(\mathcal{S}')]$.*

Proof. The uniqueness statement follows from Lemma 6.6 which shows that the stabilizers in finite index subgroups of $\mathrm{Out}(W_n)$ of two distinct equivalence classes of W_{n-2}-stars are not commensurable.

We now prove the existence statement. Let $f\colon \Gamma_1 \to \Gamma_2$ be an isomorphism between finite index subgroups of Γ that represents Ψ. By Proposition 6.7, the group $\mathrm{Stab}_{\Gamma_1}(\mathcal{S})$ satisfies $(P_{W_{n-2}})$. As f is an isomorphism, we deduce that $f(\mathrm{Stab}_{\Gamma_1}(\mathcal{S}))$ also satisfies $(P_{W_{n-2}})$. Proposition 7.10 implies that there exists a unique equivalence class of W_{n-2}-stars $\mathcal{S}'$ such that $f(\mathrm{Stab}_{\Gamma_1}(\mathcal{S})) \subseteq \mathrm{Stab}_{\Gamma_2}(\mathcal{S}')$, where the inclusion holds up to a finite index subgroup. Applying the same argument with f^{-1}, we see that there exists an equivalence class $\mathcal{S}''$ of a W_{n-2}-star such that

$$\mathrm{Stab}_{\Gamma_1}(\mathcal{S}) \subseteq f^{-1}(\mathrm{Stab}_{\Gamma_2}(\mathcal{S}')) \subseteq \mathrm{Stab}_{\Gamma_1}(\mathcal{S}''),$$

where the inclusion holds up to a finite index subgroup. Lemma 6.6 then implies that $\mathcal{S}$ is the unique equivalence class of W_{n-2}-stars virtually fixed by $\mathrm{Stab}_{\Gamma_1}(\mathcal{S})$. Therefore, we see that $\mathcal{S} = \mathcal{S}''$ and we have equality everywhere. This completes the proof. ∎

8. Algebraic characterization of compatibility of W_{n-2}-stars and conclusion

8.1. Algebraic characterization of compatibility of W_{n-2}-stars

In this section, we give an algebraic characterization of the fact that two equivalence classes of W_{n-2}-stars have both a common collapse and a common refinement. This will imply that $\mathrm{Comm}(\mathrm{Out}(W_n))$ preserves the set of pairs of commensurability classes of stabilizers of adjacent pairs in the graph X_n introduced in Definition 3.2 (2).

Let $n \geq 5$ and let Γ be a finite index subgroup of $\mathrm{Out}^0(W_n)$. We consider the following properties of a pair (H_1, H_2) of subgroups of Γ:

(P_{comp}) The pair (H_1, H_2) satisfies the following properties:

(1) For every $i \in \{1,2\}$, the group H_i satisfies ($P_{W_{n-2}}$).

(2) For every normal subgroups $K_1^{(1)} \times K_2^{(1)}$ of H_1 and $K_1^{(2)} \times K_2^{(2)}$ of H_2 given by ($P_{W_{n-2}}$) (1), there exist $i, j \in \{1,2\}$ such that $K_i^{(1)} \cap K_j^{(2)}$ is infinite.

(3) The group $H_1 \cap H_2$ contains a subgroup isomorphic to $\mathbb{Z}^{n-2}$.

Proposition 8.1. *Let $n \geq 5$ and let Γ be a finite index subgroup of* $\mathrm{Out}^0(W_n)$. *Let $\mathcal{S}_1$ and $\mathcal{S}_2$ be two distinct equivalence classes of W_{n-2}-stars S_1 and S_2 and, for every $i \in \{1,2\}$, let $H_i = \mathrm{Stab}_\Gamma(\mathcal{S}_i)$. Then S_1 and S_2 have a refinement S which is a W_{n-3}-star if and only if (H_1, H_2) satisfies property (P_{comp}).*

Proof. We first assume that S_1 and S_2 have a common refinement S which is a W_{n-3}-star. Let $\mathcal{S}$ be the equivalence class of S. Let us prove that (H_1, H_2) satisfies (P_{comp}). By Proposition 6.7, for every $i \in \{1,2\}$, the group H_i satisfies ($P_{W_{n-2}}$). This proves that the pair (H_1, H_2) satisfies (P_{comp}) (1).

Let us check property (P_{comp}) (2). For every $i \in \{1,2\}$, let $T_1^{(i)} \times T_2^{(i)}$ be the group of twists of $\mathcal{S}_i$ and let $K_1^{(i)} = T_1^{(i)} \cap \Gamma$ and $K_2^{(i)} = T_2^{(i)} \cap \Gamma$. By Proposition 6.7, for every $i \in \{1,2\}$, the group $K_1^{(i)} \times K_2^{(i)}$ satisfies ($P_{W_{n-2}}$) (1) and Lemma 7.1 implies that every normal subgroup of H_i given by ($P_{W_{n-2}}$) (1) is commensurable with $K_1^{(i)} \times K_2^{(i)}$. Thus it suffices to check (P_{comp}) (2) for $K_1^{(1)} \times K_2^{(1)}$ and $K_1^{(2)} \times K_2^{(2)}$. The group of twists of $\mathcal{S}$ is isomorphic to a direct product $A_1 \times A_2 \times A_3$ of three copies of W_{n-3}. Since $n \geq 5$, we have $n-3 \geq 2$ and W_{n-3} is infinite. Since S is a common refinement of S_1 and S_2 and since S has 3 orbits of edges, there exists a W_{n-1}-star S_0 which is a common collapse of S_1 and S_2. Moreover, there exists $k \in \{1,2,3\}$ such that A_k is contained in the group of twists of S_0. Therefore, for every $i \in \{1,2\}$, there exists $j \in \{1,2\}$ such that the group A_k is contained in $T_j^{(i)}$. Thus, there exist $i, j \in \{1,2\}$ such that $A_k \cap \Gamma \subseteq K_i^{(1)} \cap K_j^{(2)}$. In particular, $K_i^{(1)} \cap K_j^{(2)}$ is infinite. This shows (P_{comp}) (2).

Finally, since $n \geq 5$, the W_{n-2}-stars S_1 and S_2 have a common refinement which is a W_2-star (take any W_2-star which refines S). Since the group of twists of a W_2-star contains a subgroup isomorphic to $\mathbb{Z}^{n-2}$ by Proposition 2.5 (3), this shows (P_{comp}) (3).

Conversely, suppose that (H_1, H_2) satisfies (P_{comp}). For $i \in \{1,2\}$, let $K_1^{(i)} \times K_2^{(i)}$ be the direct product of the groups of twists in Γ about the two edges of $\mathcal{S}_i$. Then for every $i \in \{1,2\}$, the group $(H_i \cap K_1^{(i)}) \times (H_i \cap K_2^{(i)})$ satisfies $(P_{W_{n-2}})$ (1) by Proposition 6.7. Hence property (P_{comp}) (2) implies that there exist $i, j \in \{1,2\}$ such that

$$(H_1 \cap K_i^{(1)}) \cap (H_2 \cap K_j^{(2)})$$

is infinite. For $i \in \{1,2\}$, let $S_1^{(i)}$ and $S_2^{(i)}$ be the two distinct W_{n-1}-stars on which S_i collapses. By Proposition 6.13, since $H_1 \cap H_2$ fixes pointwise the set $\{\mathcal{S}_1^{(1)}, \mathcal{S}_2^{(1)}, \mathcal{S}_1^{(2)}, \mathcal{S}_2^{(2)}\}$, and since $H_1 \cap H_2$ contains a subgroup isomorphic to $\mathbb{Z}^{n-2}$ by (P_{comp}) (3), the W_{n-1}-stars $S_1^{(1)}$, $S_2^{(1)}$, $S_1^{(2)}$ and $S_2^{(2)}$ are pairwise compatible. Hence S_1 and S_2 have a common refinement S which is either a W_{n-3}-star or a W_{n-4}-star. Since the groups of twists of $\mathcal{S}_1$ and $\mathcal{S}_2$ have infinite intersection, the refinement S cannot be a W_{n-4}-star since otherwise the W_{n-1}-stars $S_1^{(1)}$, $S_2^{(1)}$, $S_1^{(2)}$ and $S_2^{(2)}$ would be pairwise nonequivalent and hence their groups of twists would have trivial intersection. Thus S is a W_{n-3}-star. ∎

8.2. Conclusion

In this last section, we complete the proof of our main theorem.

Theorem 8.2. *Let $n \geq 5$ and let Γ be a finite index subgroup of* $\mathrm{Out}^0(W_n)$. *Then any isomorphism $f\colon H_1 \to H_2$ between two finite index subgroups of Γ is given by conjugation by an element of* $\mathrm{Out}(W_n)$ *and the natural map*

$$\mathrm{Out}(W_n) \to \mathrm{Comm}(\mathrm{Out}(W_n))$$

is an isomorphism.

Proof. Suppose that $\mathcal{S}$ and $\mathcal{S}'$ are two distinct equivalence classes of W_{n-2}-stars. Then $\mathrm{Stab}_\Gamma(\mathcal{S})$ and $\mathrm{Stab}_\Gamma(\mathcal{S}')$ are not commensurable by Lemma 6.6. Proposition 7.12 shows that the collection $\mathcal{I}$ of all commensurability classes of Γ-stabilizers of equivalence classes of W_{n-2}-stars is $\mathrm{Comm}(\Gamma)$-invariant. Proposition 8.1 shows that the collection $\mathcal{J}$ of all pairs $([\mathrm{Stab}_\Gamma(\mathcal{S})], [\mathrm{Stab}_\Gamma(\mathcal{S}')])$ is also $\mathrm{Comm}(\Gamma)$-invariant. Since the natural homomorphism $\mathrm{Out}(W_n) \to \mathrm{Aut}(X_n)$ is an isomorphism by Theorem 3.3, the conclusion follows from Proposition 2.1 and the fact that $\mathrm{Comm}(\Gamma)$ is isomorphic to $\mathrm{Comm}(\mathrm{Out}(W_n))$ since Γ has finite index in $\mathrm{Out}(W_n)$. ∎

A. Rigidity of the graph of W_{n-1}-stars

The *graph of W_{n-1}-stars*, denoted by Y_n, is the graph whose vertices are the W_n-equivariant homeomorphism classes of W_{n-1}-stars, where two equivalence classes $\mathcal{S}$ and $\mathcal{S}'$ are joined by an edge if there exist $S \in \mathcal{S}$ and $S' \in \mathcal{S}'$ such that S and S' are compatible. This graph arises naturally in the study of $\mathrm{Out}(W_n)$ as it is isomorphic to the full subgraph of the free splitting graph $\bar{K}_n$ of W_n whose vertices are equivalence classes of W_k-stars, with k varying in $\{0, \dots, n-1\}$. As $\mathrm{Aut}(W_n)$ acts on $\bar{K}_n$ by precomposition of the marking, we have an induced action of $\mathrm{Aut}(W_n)$ on Y_n. As $\mathrm{Inn}(W_n)$ acts trivially on Y_n, the action of $\mathrm{Aut}(W_n)$ induces an action of $\mathrm{Out}(W_n)$. We denote by $\mathrm{Aut}(Y_n)$ the group of graph automorphisms of Y_n. In this section we prove the following theorem.

Theorem A.1. *Let $n \geq 4$. The natural homomorphism*

$$\mathrm{Out}(W_n) \to \mathrm{Aut}(Y_n)$$

is an isomorphism.

In order to prove this theorem, we take advantage of the action of $\mathrm{Out}(W_n)$ on the graph of $\{0\}$-stars and F-stars L_n. The strategy in order to prove Theorem A.1 is to construct an injective homomorphism $\Phi\colon \mathrm{Aut}(Y_n) \to \mathrm{Aut}(L_n)$ such that every automorphism in the image preserves the set of $\{0\}$-stars and the set of F-stars.

The homomorphism $\Phi\colon \mathrm{Aut}(Y_n) \to \mathrm{Aut}(L_n)$ is defined as follows. Let $f \in \mathrm{Aut}(Y_n)$. Let $\mathcal{S}$ be the equivalence class of a $\{0\}$-star and let S be a representative of $\mathcal{S}$. By Theorem 3.7, there exist exactly n W_{n-1}-stars $S_1, \dots, S_n$ refined by S. Moreover, these W_{n-1}-stars are pairwise compatible. For $i \in \{1, \dots, n\}$, let $\mathcal{S}_i$ be the equivalence class of S_i. Since f is an automorphism of Y_n, $f(\mathcal{S}_1), \dots, f(\mathcal{S}_n)$ are pairwise adjacent in Y_n. Let $S_1', \dots, S_n'$ be representatives of respectively $f(\mathcal{S}_1), \dots, f(\mathcal{S}_n)$ that are pairwise compatible. Then Theorem 3.7 implies that there exists a unique common refinement S' of $S_1', \dots, S_n'$ with exactly n edges. Since, for every $i \in \{1, \dots, n\}$, the splitting S_i' is a W_{n-1}-star, the splitting S' is necessarily a $\{0\}$-star. Let $\mathcal{S}'$ be the equivalence class of S'. We then define $\Phi(f)(\mathcal{S}) = \mathcal{S}'$. If $\mathcal{T}$ is an F-star, we define $\Phi(f)(\mathcal{T})$ similarly.

Lemma A.2. *Let $n \geq 4$. Let $f \in \mathrm{Aut}(Y_n)$. Let $\Phi(f)$ be as above.*

(1) *The map $\Phi(f)\colon VL_n \to VL_n$ induces a graph automorphism $\widetilde{\Phi}(f)\colon L_n \to L_n$.*

(2) *If $\widetilde{\Phi}(f) = \mathrm{id}_{L_n}$, then $f = \mathrm{id}_{Y_n}$.*

Proof. We prove the first statement. As $\Phi(f) \circ \Phi(f^{-1}) = \Phi(f \circ f^{-1}) = \mathrm{id}$, we see that $\Phi(f)$ is a bijection. Let $\mathcal{S}$ be the equivalence class of a $\{0\}$-star and let $\mathcal{T}$ be the equivalence class of an F-star. Suppose that $\mathcal{S}$ and $\mathcal{T}$ are adjacent in L_n. We prove that $\Phi(f)(\mathcal{S})$ and $\Phi(f)(\mathcal{T})$ are adjacent in L_n. Applying the same result to f^{-1}, this will prove that $\mathcal{S}$ and $\mathcal{T}$ are adjacent in L_n if and only if $\Phi(f)(\mathcal{S})$ and $\Phi(f)(\mathcal{T})$ are adjacent in L_n, and this will conclude the proof. Let S and T be representatives of $\mathcal{S}$ and $\mathcal{T}$, respectively. Let $S_1, \dots, S_n$ be the n W_{n-1}-stars refined by S, and let $T_1, \dots, T_{n-1}$ be the $n-1$ W_{n-1}-stars refined by T. As S refines T, and as S refines exactly n W_{n-1}-stars by Theorem 3.7, up to reordering, we can suppose that, for every $i \in \{1, \dots, n-1\}$, we have $S_i = T_i$. For $i \in \{1, \dots, n\}$, let $\mathcal{S}_i$ be the equivalence class of S_i, and let S_i' be a representative of $\Phi(f)(\mathcal{S}_i)$ such that for distinct $i, j \in \{1, \dots, n\}$, S_i and S_j are compatible. Then, by Theorem 3.7, a representative T' of $\Phi(f)(\mathcal{T})$ is the unique (up to W_n-equivariant homomorphism) F-star such that, for every $j \in \{1, \dots, n-1\}$, T' is compatible with S'_j. Moreover, a representative S' of $\Phi(f)(\mathcal{S})$ is the unique $\{0\}$-star such that, for every $i \in \{1,\dots,n\}$, S' is compatible with S'_i. For $i \in \{1,\dots,n\}$, let x_i be the preimage by the marking of $W_n \backslash S'_i$ (well-defined up to global conjugation) of the generator of the vertex group isomorphic to F (which exists since S'_i is a W_{n-1}-star). Then the preimages by the marking of $W_n \backslash T'$ of the generators of the groups associated with the $n-1$ leaves of the underlying graph of $W_n \backslash T'$ are $x_1,\dots,x_{n-1}$ and the preimage by the marking of $W_n \backslash T'$ of the generator of the group associated with the center of the underlying graph of $W_n \backslash T'$ is x_n. Moreover, the preimages by the marking of $W_n \backslash S'$ of the generators of the groups associated with the n leaves of the underlying graph of $W_n \backslash S'$ are $x_1,\dots,x_n$. Let v_n be the leaf of the underlying graph of $W_n \backslash S'$ such that the preimage by the marking of $W_n \backslash S'$ of the generator of the group associated with v_n is x_n. Then T' is obtained from S' by contracting the edge adjacent to v_n. Thus $\Phi(f)(\mathcal{S})$ and $\Phi(f)(\mathcal{T})$ are adjacent in L_n.

The proof of the second statement is identical to the proof of [13, Lemma 5.4]. We add the proof for completeness as the statement of [13, Lemma 5.4] is about automorphisms of $\bar{K}_n$. Let $\mathcal{S} \in VY_n$ and let S be a representative of $\mathcal{S}$. We prove that $f(\mathcal{S}) = \mathcal{S}$. Let

$$W_n = \langle x_1,\dots,x_{n-1}\rangle * \langle x_n\rangle$$

be the free factor decomposition of W_n induced by S. Let S' be a representative of $f(\mathcal{S})$. Let $\mathcal{X}$ be the equivalence class of the F-star X represented in Figure 4 on the left.

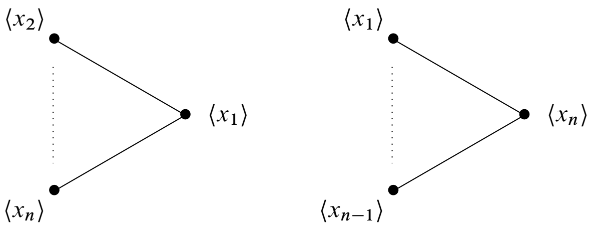

Figure 4. The F-stars X and X' from the proof of Lemma A.2.

Since $\Phi(f)(\mathcal{X}) = \mathcal{X}$, the free splitting S' is a W_{n-1}-star obtained from X by collapsing $n-1$ edges. But if T is a W_{n-1}-star obtained from X by collapsing $n-1$ edges, then there exists $i \in \{1,\dots,n\}$ such that the free factor decomposition of W_n induced by T is

$$W_n = \langle x_1,\dots,\widehat{x}_i,\dots,x_n\rangle * \langle x_i\rangle.$$

For $i \in \{1,\dots,n\}$, we will denote by T_i the W_{n-1}-star with associated free factor decomposition $\langle x_1,\dots,\widehat{x}_i,\dots,x_n\rangle * \langle x_i\rangle$, and by $\mathcal{T}_i$ its equivalence class. For $i \neq n$, the free splitting T_i is a collapse of the F-star X' depicted in Figure 4 on the right, whereas S is not a collapse of X'.

Let $\mathcal{X}'$ be the equivalence class of X'. Since $\Phi(f)(\mathcal{X}') = \mathcal{X}'$, there does not exist a representative of $f(\mathcal{S})$ that is obtained from a representative of $\mathcal{X}'$ by collapsing a forest. Thus, for all $i \neq n$, we have $f(\mathcal{S}) \neq \mathcal{T}_i$. Hence, as $\mathcal{S} = \mathcal{T}_n$, we conclude that $f(\mathcal{S}) = \mathcal{S}$. ∎

Proof of Theorem A.1. Let $n \geq 4$. We first prove injectivity. By Theorem 3.5, the homomorphism $\mathrm{Out}(W_n) \to \mathrm{Aut}(L_n)$ is injective. Moreover, the homomorphism $\mathrm{Out}(W_n) \to \mathrm{Aut}(L_n)$ factors through $\mathrm{Out}(W_n) \to \mathrm{Aut}(Y_n) \to \mathrm{Aut}(L_n)$. Therefore, we deduce the injectivity of $\mathrm{Out}(W_n) \to \mathrm{Aut}(Y_n)$. We now prove surjectivity. Let $f \in \mathrm{Aut}(Y_n)$. By Lemma A.2 (1), we have a homomorphism $\Phi\colon \mathrm{Aut}(Y_n) \to \mathrm{Aut}(L_n)$ whose image consists in automorphisms preserving the set of $\{0\}$-stars and the set of F-stars. By Theorem 3.5, the automorphism $\Phi(f)$ is induced by an element $\gamma \in \mathrm{Out}(W_n)$. Since the homomorphism $\mathrm{Aut}(Y_n) \to \mathrm{Aut}(L_n)$ is injective by Lemma A.2 (2), f is induced by γ. This concludes the proof. ∎

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Yassine Guerch
Laboratoire de mathématique d'Orsay, Université Paris-Saclay, 307 Rue Michel Magat, 91400 Orsay, France; yassine.guerch@ens-lyon.fr