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Groupoids decomposition, propagation and operator K-theory

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Abstract. In this paper, we streamline the technique of groupoids coarse decomposition for purpose of K-theory computations of groupoids crossed products. This technique was first introduced by Guoliang Yu in his proof of Novikov conjecture for groups with finite asymptotic dimension. The main tool we use for these computations is controlled operator K-theory.

Dedicated to the memory of Etienne Blanchard

Contents

1. Introduction

The concept of coarse decomposability for locally compact groupoids was introduced by several authors [3, 11, 21, 30] in order to compute K-theory of reduced C^*-algebras and of reduced crossed product algebras of locally compact groupoids. It generalizes the "cut-and-pasting" technique developed by Yu in [32] to prove the Novikov conjecture for groups with finite asymptotic dimension. The "cut-and-pasting" has been then extended by Guentner, Tessera and Yu in [8] in order to study topological rigidity of manifolds. In this work, they consider a class of finitely generated groups which satisfy a metric property called finite decomposition complexity. In particular, they proved that if the fundamental group of a closed aspherical manifold is in this class, then it satisfied the bounded

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Borel conjecture and the stable Borel conjecture. In [9], finite decomposition complexity for metric spaces has been studied in full detail. This property can be interpreted in terms of decomposition complexity of the coarse groupoid associated to the metric space, which leads naturally to extend this notion to locally compact groupoids. A first generalization was provided by Guentner, Willett and Yu in [10] in order to study the dynamical properties of finitely generated group actions on locally compact spaces with at each order of length a one step decomposition into pieces with "finite dynamic". In [11], the same authors consider the case of finitely generated group actions on locally compact spaces which, given a sequence of lengths, decompose in a finite number of steps into pieces with "finite dynamic". They give a new proof of the Baum–Connes conjecture (with trivial coefficients) for these action groupoids. This approach is of great interest since it does not involve infinite dimension analysis and can be generalized to computations in non C^*-algebraic situations (for instance to ℓ^p-crossed products as considered in [5]). The main tool used in this proof is quantitative K-theory. Quantitative K-theory was first introduced in [32] for obstruction algebras in order to prove the Novikov conjecture for groups with finite asymptotic dimension. It has been then extended in [20] to the setting of C^*-algebras equipped with a filtration arising from a length and in [6] to the general framework of C^*-algebras filtered by an abstract coarse structure which allows to replace lengths by abstract orders. A controlled Mayer–Vietoris exact sequence in quantitative K-theory associated to decomposition in "ideals at order r" which turned out to be tailored for K-theory computations under groupoid decomposability (see [6] for the extension to general filtrations) was stated in [21]. It has been applied in [3] to the Künneth formula in K-theory for groupoid C^*-algebras and crossed product algebras. Any locally compact Hausdorff groupoid is provided by a canonical order (see Definition 2.9) and loosely speaking, we consider decomposition of the set of elements of a given order of a groupoid as the union of two open subgroupoids (see Definition 2.13). Following [11], we say that a locally compact groupoid $\mathcal{G}$ has finite complexity decomposition with respect to a family $\mathcal{D}$ of open subgroupoids if starting with a given sequence of orders, then iterating the above decomposition ends up with elements belonging to $\mathcal{D}$ in a finite number of steps (see Definition 2.17). The main result of this paper is the following:

Let $\mathcal{G}$ be locally compact groupoid with finite decomposition complexity with respect to a family $\mathcal{D}$ of relatively clopen subgroupoids (see Definition 2.2) and let $f\colon A \to B$ be a homomorphism of $\mathcal{G}$-algebras. If the morphism

$$K_*(A \rtimes_r \mathcal{H}) \to K_*(B \rtimes_r \mathcal{H})$$

induced in K-theory by f is an isomorphism for any $\mathcal{H}$ in $\mathcal{D}$, then so is

$$K_*(A \rtimes_r \mathcal{G}) \to K_*(B \rtimes_r \mathcal{G}).$$

We then extend this result to morphisms induced by elements in $KK_*^{\mathcal{G}}(A, B)$. An important application of the latter result is the heredity of the Baum–Connes conjecture under the decomposition described above for second countable and locally compact Hausdorff groupoids admitting a γ-element in sense of [28]. All these stability results should be compared with those obtained by Willett in [30] using different technics.

Our prominent examples will be action groupoids of groups with finite decomposition complexity acting on its Stone–Čech compactification. Indeed these groupoids are known to be amenable. This is a consequence of [8, Theorem 4.3] and [26, Theorem A.9]. This result has been extended in [11, Theorem A.9] to étale groupoids with finite dynamical complexity. In Section 2.6, we generalize the later result and prove that a locally compact Hausdorff groupoid with finite decomposition complexity respectively to a family of open and amenable subgroupoids is amenable. In view of [27], these groupoids satisfy the Baum–Connes conjecture and hence, our approach does not provide so far new examples. But, as emphasized before, our approach is more geometric.

In the case of a groupoid with finite decomposition complexity with respect to a family $\mathcal{D}$ of relatively clopen subgroupoids satisfying the Haagerup property, it is less clear that the later property is preserved and applying once again [27], our approach might be the source of new examples. Even if so far we have no idea of what kind of groupoid can be obtained is this way, interesting new examples might also arise using [15, 16] by considering a decomposing family $\mathcal{D}$ of relatively clopen subgroupoids satisfying some hyperbolic properties (see Section 6.4 for some general perspectives).

Outline of the paper. Section 2 starts with some basic definitions concerning locally compact groupoids and their actions. Then we introduce the notion of $\mathcal{G}$-order for a locally compact groupoid $\mathcal{G}$ which can be viewed as the generalization both of a length on a group and of a distance on a proper metric space. Following the idea of [11, Definition 3.14], the notion of $\mathcal{R}$-decomposition for a $\mathcal{G}$-order $\mathcal{R}$ is then introduced. This leads to the concept of $\mathcal{D}$-decomposability of an open subgroupoid of $\mathcal{G}$ with respect to a family $\mathcal{D}$ of open subgroupoids and to finite decomposition complexity with respect to a family of open subgroupoids, generalizing finite dynamical complexity defined in [11]. We end this section with generalization of [11, Theorem A.9] and prove that a locally compact groupoid with finite decomposition complexity with respect to a family of open and amenable subgroupoids is amenable.

Section 3 is devoted to some reminders on groupoid actions on C^*-algebras and their reduced crossed product algebras.

In Section 4 is introduced the primary tool for the proof of our main theorem, the controlled Mayer–Vietoris exact sequence in quantitative K-theory associated to a groupoid decomposition. We first review from [6] the main features of quantitative K-theory for C^*-algebra filtered by an abstract coarse structure. It is pointed out that $\mathcal{G}$-orders provide such a structure on crossed product algebras of a groupoid $\mathcal{G}$. The definition of a controlled Mayer–Vietoris pair is recalled and we show that groupoid decompositions of order $\mathcal{R}$ give rise to controlled Mayer–Vietoris pairs. Eventually, we recall the statement of the controlled Mayer–Vietoris exact sequence in quantitative K-theory associated to a controlled Mayer–Vietoris pair.

The main result of the paper is proven in Section 5. Although the proof is tedious, the principle is quite simple as it is the extension of the Five lemma to the setting of controlled exact sequences. We then extend our main result to the case of the morphisms induced in K-theory by elements of $\mathcal{G}$-equivariant KK-theory. This is done by noticing that every such element is up to KK-equivalence given by an equivariant homomorphism.

In Section 6, applications to the Baum–Connes conjecture for locally compact groupoids are given. We first recall from [28] the statement of the Baum–Connes conjecture in the setting of locally compact groupoids and the definition of γ-elements. This section is ended with hereditary results of the Baum–Connes conjecture for groupoids with $\mathcal{D}$-finite decomposition complexity which admit a γ-element in sense of [28] and with a discussion concerning the range of applicability of our approach.

2. Coarse decomposition for groupoids

Coarse decomposability for locally compact Hausdorff groupoids is the generalization of the concept of decomposability for a family of metric spaces introduced in [9]. In this section, after some reminders concerning locally compact groupoids and their actions, we introduce for a locally compact Hausdorff groupoid $\mathcal{G}$ the notion of $\mathcal{G}$-orders generalizing on one hand distances on metric spaces and on the other hand lengths on groups. Following in particular ideas of [11, Section 3 and Appendix A], this allows us to define decomposition of order $\mathcal{R}$ for a subgroupoid of $\mathcal{G}$ which leads naturally to coarse decomposability and to finite decomposition complexity with respect to a set of open subgroupoids of $\mathcal{G}$.

2.1. Groupoids

We assume that the reader is familiar with the basic definition concerning groupoids. For more details, we refer to [22, 23].

A groupoid with space of units X consists of a set $\mathcal{G}$ provided with

- two maps $s: \mathcal{G} \to X$ and $r: \mathcal{G} \to X$ called the source map and the range map, respectively;
- a map $u: X \to \mathcal{G}$, $x \mapsto u_x$, called the unit map, which is a section both for s and r;
- an associative composition $\mathcal{G} \times_X \mathcal{G} \to \mathcal{G}: (\gamma, \gamma') \mapsto \gamma \cdot \gamma'$ with $\mathcal{G} \times_X \mathcal{G} = \{(\gamma, \gamma') \in \mathcal{G} \times \mathcal{G}: s(\gamma) = r(\gamma')\}$ such that $s(\gamma \cdot \gamma') = s(\gamma')$ and $r(\gamma \cdot \gamma') = r(\gamma)$ for any (γ, γ') in $\mathcal{G} \times_X \mathcal{G}$ and $\gamma \cdot u_{s(\gamma)} = u_{r(\gamma)} \cdot \gamma = \gamma$ for any γ in $\mathcal{G}$;
- an inverse map $\mathcal{G} \to \mathcal{G}$, $\gamma \mapsto \gamma^{-1}$ such that $s(\gamma^{-1}) = r(\gamma)$, $r(\gamma^{-1}) = s(\gamma)$, $\gamma \cdot \gamma^{-1} = u_{r(\gamma)}$, and $\gamma^{-1} \cdot \gamma = u_{s(\gamma)}$ for any γ in $\mathcal{G}$.

Notation 2.1. Let $\mathcal{G}$ be a groupoid with space of units X and source and range maps $s, r: \mathcal{G} \to X$.

- Let Z be a subset of $\mathcal{G}$. We set the following:
  - $Z^{-1} = \{\gamma^{-1}: \gamma \in Z\}$;

  - for any $Y \subseteq X$, $Z_Y = s^{-1}(Y) \cap Z$ and $Z^Y = r^{-1}(Y) \cap Z$;
  - for any subsets Y_1 and Y_2 of X, $Z_{Y_1}^{Y_2} = Z^{Y_2} \cap Z_{Y_1}$.
- Let Z_1 and Z_2 be subsets in $\mathcal{G}$. We set

$$Z_1 \cdot Z_2 = \{\gamma_1\gamma_2 : \gamma_1 \in Z_1,\ \gamma_2 \in Z_2 \text{ and } s(\gamma_1) = r(\gamma_2)\}.$$

A locally compact groupoid is a groupoid provided with a locally compact topology and such that the structure maps are continuous. In this paper, all the groupoids are assumed to be locally compact and Hausdorff. An open subgroupoid of $\mathcal{G}$ is a subgroupoid $\mathcal{H}$ of $\mathcal{G}$ which is open as a subset and such that the space of units is open in the space of unit of $\mathcal{G}$. Notice that the latter condition always holds if the source map of $\mathcal{G}$ is open, for instance if $\mathcal{G}$ is provided with a Haar system [28, Lemma 6.5].

Definition 2.2. Let $\mathcal{G}$ be a locally compact groupoid. A relatively clopen subgroupoid of $\mathcal{G}$ is an open subgroupoid $\mathcal{H}$ of $\mathcal{G}$ such that if Y stands for the unit space of $\mathcal{H}$, then $\mathcal{H}$ is closed in $\mathcal{G}_Y$.

Remark 2.3. Let $\mathcal{G}$ be locally compact groupoid and let $\mathcal{H}$ be a relatively clopen subgroupoid of $\mathcal{G}$ with unit space Y. Then $\mathcal{H}$ is clopen in $\mathcal{G}^Y$ and in $\mathcal{G}_Y^Y$.

The next lemma is straightforward to prove.

Lemma 2.4. *Let $\mathcal{G}$ be a locally compact groupoid and let $\mathcal{H}$ be an open subgroupoid of $\mathcal{G}$ with unit space Y. Then $\mathcal{H}$ is relatively clopen if and only if $K \cap \mathcal{H}$ is compact for any compact subset K of $\mathcal{G}_Y$.*

We recall that a locally compact groupoid with space of units X is proper if the map

$$\mathcal{G} \to X \times X, \quad \gamma \mapsto (r(\gamma), s(\gamma))$$

is proper. As a consequence of Lemma 2.4, we obtain the following corollary.

Corollary 2.5. *Let $\mathcal{G}$ be a locally compact proper groupoid. Then relatively clopen subgroupoids of $\mathcal{G}$ are proper.*

2.2. Groupoid actions

Let us recall first the definition of a (left) action of a groupoid. Let $\mathcal{G}$ be a groupoid with space of units X and source and range maps s and r and unit map u. An action of the groupoid $\mathcal{G}$ on a set Z consist of a map $p\colon Z \to X$ called the anchor map and a map

$$\mathcal{G} \times_X Z \to Z, \quad (\gamma, z) \mapsto \gamma \cdot z,$$

with $\mathcal{G} \times_X Z = \{(\gamma, z) \in \mathcal{G} \times Z : s(\gamma) = p(z)\}$ such that

(i) for any γ and γ' in $\mathcal{G}$ and z in Z such that (γ, γ') is in $\mathcal{G} \times_X \mathcal{G}$ and (γ', z) is in $\mathcal{G} \times_X Z$, then $(\gamma, \gamma' \cdot z)$ belongs to $\mathcal{G} \times_X Z$ and $\gamma \cdot (\gamma' \cdot z) = (\gamma \cdot \gamma') \cdot z$;

(ii) $u_{p(z)} \cdot z = z$ for any z in Z.

Notice that these conditions imply that $p(\gamma \cdot z) = r(\gamma)$ and $\gamma^{-1} \cdot \gamma \cdot z = z$ for any (γ, z) in $\mathcal{G} \times_X Z$. If x is an element in X, then the fiber of Z at x is $Z_x \overset{\text{def}}{=} p^{-1}(\{x\})$ and more generally, if Y is a subset of X, we set $Z_Y \overset{\text{def}}{=} p^{-1}(Y)$. If $\mathcal{G}$ is a locally compact groupoid and if Z is a locally compact space, we require the anchor map and the action map to be continuous. In this case, Z will be called a $\mathcal{G}$-space. In what follows, all $\mathcal{G}$-spaces are supposed to be Hausdorff. If Z and Z' are $\mathcal{G}$-spaces with anchor maps p_Z and $p_{Z'}$, a map $f: Z \to Z'$ is called a $\mathcal{G}$-map if f is continuous, $p_{Z'} \circ f = p_Z$ and $f(\gamma \cdot z) = \gamma \cdot f(z)$ for all (γ, z) in $\mathcal{G} \times_X Z$.

Let $\mathcal{G}$ be a groupoid with space of units X acting on a set Z with anchor map $p: Z \to X$. Then the action groupoid that corresponds to the action of $\mathcal{G}$ on Z denoted by $\mathcal{G} \ltimes Z$ is the set $\mathcal{G} \times_X Z$, with Z as space of units with source map

$$\mathcal{G} \ltimes Z \to Z, \quad (\gamma, z) \mapsto z$$

and range map

$$\mathcal{G} \ltimes Z \to Z, \quad (\gamma, z) \mapsto \gamma \cdot z,$$

unit map (u being the unit map of $\mathcal{G}$)

$$Z \to \mathcal{G} \ltimes Z, \quad z \mapsto (u_{p(z)}, z),$$

composition

$$(\mathcal{G} \ltimes Z) \times_Z (\mathcal{G} \ltimes Z), \quad (\gamma, \gamma' z) \cdot (\gamma', z) \mapsto (\gamma\gamma', z)$$

and inverse

$$\mathcal{G} \ltimes Z \to \mathcal{G} \ltimes Z, \quad (\gamma, z) \mapsto (\gamma^{-1}, \gamma \cdot z).$$

If $\mathcal{G}$ is a locally compact groupoid and Z is a $\mathcal{G}$-space, then $\mathcal{G} \ltimes Z$ is a locally compact groupoid. A $\mathcal{G}$-space Z is called proper (or the action of $\mathcal{G}$ on Z is said to be proper) if the action groupoid $\mathcal{G} \ltimes Z$ is proper.

Remark 2.6. Let $\mathcal{G}$ be a locally compact groupoid with space of units X acting on a locally compact space Y. Then

(i) a $\mathcal{G} \ltimes Y$-space is precisely a $\mathcal{G}$-space Z together with a $\mathcal{G}$-map $f: Z \to Y$;

(ii) in this case,

$$\mathcal{G} \ltimes Z \to (\mathcal{G} \ltimes Y) \ltimes Z, \quad (\gamma, z) \mapsto (\gamma, f(z), z)$$

is a groupoid isomorphism;

(iii) in consequence, a $\mathcal{G} \ltimes Y$-space Z is proper if and only if it is proper as a $\mathcal{G}$-space;

(iv) in particular, if Z is a proper $\mathcal{G}$-space, then $Z \times_X Y$ is a proper $\mathcal{G} \ltimes Y$-space with anchor map given by the projection on the second factor (here $Z \times_X Y$ stands for the fiber product over the two anchor maps).

Remark 2.7. Let $\mathcal{G}$ be locally compact groupoid and let $\mathcal{H}$ be a relatively clopen subgroupoid of $\mathcal{G}$ with unit space Y. For any left $\mathcal{G}$-space Z, the subgroupoid $\mathcal{H} \rtimes Z_Y$ is relatively clopen in $\mathcal{G} \rtimes Z$.

2.3. Induced actions

We recall now from [2, Section 2] the notion of induced action to a groupoid from a subgroupoid action. Let $\mathcal{G}$ be a locally compact groupoid with space of units X and open source and range maps, let $\mathcal{H}$ be a relatively clopen subgroupoid of $\mathcal{G}$ with space of units Y and let Z be a (left) $\mathcal{H}$-space with anchor map $p\colon Z \to Y$. Let us define on

$$\mathcal{G} \times_Y Z \stackrel{\text{def}}{=} \{(\gamma, z) \in \mathcal{G} \times Z \text{ such that } s(\gamma) = p(z)\}$$

the $\mathcal{H}$-action with anchor map $\mathcal{G} \times_Y Z \to Y$, $(\gamma, z) \mapsto p(z)$ by

$$\gamma \cdot (\gamma', z) = (\gamma'\gamma^{-1}, \gamma z)$$

for any γ in $\mathcal{H}$ and (γ', z) in $\mathcal{G} \times_Y Z$ such that $s(\gamma) = p(z)$. The $\mathcal{H}$-action defined in this way is proper and the quotient space

$$\mathcal{G} \times_{\mathcal{H}} Z \stackrel{\text{def}}{=} (\mathcal{G} \times_Y Z)/\mathcal{H}$$

is Hausdorff and locally compact. Let us denote by $[\gamma, z]$ the class in $\mathcal{G} \times_{\mathcal{H}} Z$ of an element (γ, z) in $\mathcal{G} \times_{\mathcal{H}} Z$. Then $\mathcal{G} \times_{\mathcal{H}} Z$ is provided with a $\mathcal{G}$-action with anchor map

$$\mathcal{G} \times_{\mathcal{H}} Z \to X, \quad [\gamma, z] \mapsto r(\gamma)$$

defined by $\gamma \cdot [\gamma', z] = [\gamma\gamma', z]$ for any γ in $\mathcal{G}$ and $[\gamma', z]$ in $\mathcal{G} \times_{\mathcal{H}} Z$ such that $s(\gamma) = r(\gamma')$ and is called the $\mathcal{G}$-space induced by the $\mathcal{H}$-space Z.

Proposition 2.8 ([2, Lemma 2.12]). *Let $\mathcal{G}$ be a locally compact groupoid with open source and range maps, let $\mathcal{H}$ be a relatively clopen subgroupoid of $\mathcal{G}$ and let Z be a proper $\mathcal{H}$-space. Then the induced $\mathcal{G}$-space $\mathcal{G} \times_{\mathcal{H}} Z$ is proper.*

2.4. $\mathcal{G}$-orders

Definition 2.9. Let $\mathcal{G}$ be a locally compact groupoid with space of units X. A $\mathcal{G}$-order is a subset $\mathcal{R}$ of $\mathcal{G}$ such that

- $u(s(\mathcal{R})) \subseteq \mathcal{R}$;
- $\mathcal{R}^{-1} = \mathcal{R}$ ($\mathcal{R}$ is symmetric).
- for every compact subset Y of X, then $\mathcal{R}_Y$ is compact.

Remark 2.10. Let $\mathcal{G}$ be a locally compact groupoid with unit space X.

(i) For any compact subset K of $\mathcal{G}$, then $K \cup K^{-1} \cup r(K) \cup s(K)$ is a compact $\mathcal{G}$-order and hence for any compact subset K of $\mathcal{G}$, there exists a compact $\mathcal{G}$-order $\mathcal{R}$ such that $K \subseteq \mathcal{R}$.

(ii) If $\mathcal{R}_1$ and $\mathcal{R}_2$ are $\mathcal{G}$-orders, then $\mathcal{R}_1 \cup \mathcal{R}_2$ and $\mathcal{R}_1 \cap \mathcal{R}_2$ are $\mathcal{G}$-orders.

Lemma 2.11. *Let $\mathcal{G}$ be a locally compact groupoid, then any $\mathcal{G}$-order is closed.*

Proof. Let $\mathcal{R}$ be a $\mathcal{G}$-order. Let us prove that $\mathcal{R} \cap K$ is compact for any compact subset K of $\mathcal{G}$. Let us set $Y = s(K)$. Since Y is compact, then $\mathcal{R}_Y$ is compact and hence

$$K \cap \mathcal{R} = K \cap \mathcal{R}_Y$$

is compact. ■

If $\mathcal{R}_1$ and $\mathcal{R}_2$ are two $\mathcal{G}$-orders, then

$$\mathcal{R}_1 * \mathcal{R}_2 \stackrel{\text{def}}{=} (\mathcal{R}_1 \cdot \mathcal{R}_2) \cup (\mathcal{R}_2 \cdot \mathcal{R}_1) \cup u(s(K_1)) \cup u(s(K_2))$$

is a $\mathcal{G}$-order. If $\mathcal{R}$ is a $\mathcal{G}$-order and n is an integer, then $\mathcal{R}^{*n}$ stands for $\mathcal{R} * \cdots * \mathcal{R}$ (n products). Notice that according to the first point of Definition 2.9, we have that $\mathcal{R} \subseteq \mathcal{R}^{*n}$ for every integer n. Let $\mathcal{E}_{\mathcal{G}}$ be the set of $\mathcal{G}$-orders. Then $\mathcal{E}_{\mathcal{G}}$ is a poset for the inclusion and ordered semi-group for $*$. Moreover, $\mathcal{E}_{\mathcal{G}}$ is a lattice with the infimum given by the intersection and the supremum given by the union. We denote by $\mathcal{E}_{\mathcal{G},c}$ the set of compact $\mathcal{G}$-order. Then $\mathcal{E}_{\mathcal{G},c}$ is an ordered semi-group for $*$ and a lattice for the partial order given by the inclusion, as well.

2.5. $\mathcal{R}$-decomposition of a groupoid

Remark 2.12. Let $\mathcal{G}$ be a locally compact groupoid and let $\mathcal{H}$ be a relatively clopen subgroupoid of $\mathcal{G}$.

(i) Let $\mathcal{R}$ be a $\mathcal{G}$-order, then $\mathcal{R} \cap \mathcal{H}$ is an $\mathcal{H}$-order denoted by $\mathcal{R}_{/\mathcal{H}}$.

(ii) $\mathcal{E}_{\mathcal{G}} \to \mathcal{E}_{\mathcal{H}} : \mathcal{R} \mapsto \mathcal{R}_{/\mathcal{H}}$ is a map of posets such that

$$\mathcal{R}_{1/\mathcal{H}} * \mathcal{R}_{2/\mathcal{H}} \subseteq (\mathcal{R}_1 * \mathcal{R}_2)_{/\mathcal{H}}$$

for any $\mathcal{G}$-orders $\mathcal{R}_1$ and $\mathcal{R}_2$.

Definition 2.13. Let $\mathcal{G}$ be a locally compact groupoid, and let $\mathcal{H}$ be a subgroupoid of $\mathcal{G}$ with space of units Y and let $\mathcal{R}$ be a $\mathcal{G}$-order. Then

(i) an $\mathcal{R}$-decomposition of $\mathcal{H}$ is a quadruple $(V_1, V_2, \mathcal{H}_1, \mathcal{H}_2)$ where

- V_1 and V_2 are open subsets of Y with $Y = V_1 \cup V_2$ and such that there exists a partition of unity (defined on Y) subordinated to (V_1, V_2);
- $\mathcal{H}_1$ and $\mathcal{H}_2$ are subgroupoids of $\mathcal{H}$ which are open in $\mathcal{G}$.
- $\mathcal{R}_{V_i} \cap \mathcal{H}$ is contained in $\mathcal{H}_i$ for $i = 1, 2$.

(ii) a coercive $\mathcal{R}$-decomposition of $\mathcal{H}$ is an $\mathcal{R}$-decomposition $(V_1, V_2, \mathcal{H}_1, \mathcal{H}_2)$ of $\mathcal{H}$ such that $\mathcal{H}_1$ and $\mathcal{H}_2$ are relatively clopen in $\mathcal{G}$.

Remark 2.14. If the space of units of $\mathcal{G}$ is second countable, then the existence of the partition of unity in the first item of Definition 2.13 is guaranteed.

Following the route of [11, Definition A.4], we introduce the notion of decomposability with respect to a set of open subgroupoids.

Definition 2.15. Let $\mathcal{G}$ be a locally compact groupoid and let $\mathcal{D}$ be a set of open subgroupoids of $\mathcal{G}$. A subgroupoid $\mathcal{H}$ of $\mathcal{G}$ is $\mathcal{D}$-decomposable if for every compact $\mathcal{G}$-order $\mathcal{R}$, there exists an $\mathcal{R}$-decomposition $(V_1, V_2, \mathcal{H}_1, \mathcal{H}_2)$ with $\mathcal{H}_1$ and $\mathcal{H}_2$ in $\mathcal{D}$.

For étale groupoids, this definition differs slightly from [11, Definition A.4] but can be compared under the assumption of second countability:

- $\mathcal{D}$-decomposability in the latter sense implies $\mathcal{D}$-decomposability in the former one;
- the converse is true if $\mathcal{D}$ is stable under taking open subgroupoids (see also Lemma 2.18).

Lemma 2.16. *Let $\mathcal{G}$ be a locally compact groupoid and let $\mathcal{H}$ be a subgroupoid of $\mathcal{G}$. Then*

(i) *if $\mathcal{D}$ is a set of open subgroupoids of $\mathcal{G}$ such that $\mathcal{H}$ is $\mathcal{D}$-decomposable, then $\mathcal{H}$ is an open subgroupoid of $\mathcal{G}$;*

(ii) *if $\mathcal{D}$ is a set of relatively clopen subgroupoids of $\mathcal{G}$ such that $\mathcal{H}$ is $\mathcal{D}$-decomposable, then $\mathcal{H}$ is a relatively clopen subgroupoid of $\mathcal{G}$.*

Proof. Let us prove the first point. Let γ be an element in $\mathcal{H}$. According to point (i) of Remark 2.10, there exists a compact $\mathcal{G}$-order $\mathcal{R}$ such that γ lies in $\mathcal{R}$. Let $(V_1, V_2, \mathcal{H}_1, \mathcal{H}_2)$ be an $\mathcal{R}$-decomposition of $\mathcal{H}$ with $\mathcal{H}_1$ and $\mathcal{H}_2$ in $\mathcal{D}$. By definition of an $\mathcal{R}$-decomposition, we see that γ belongs to $\mathcal{H}_1 \cup \mathcal{H}_2$ which is an open subset of $\mathcal{G}$ contained in $\mathcal{H}$.

For the second point, assume now that every subgroupoid in $\mathcal{D}$ is relatively clopen and let $\mathcal{H}$ be a $\mathcal{D}$-decomposable subgroupoid of $\mathcal{G}$. Let us prove that $\mathcal{H}$ is relatively clopen. Let Y be the unit space of $\mathcal{H}$. According to Lemma 2.4, it is enough to prove that $\mathcal{H} \cap K$ is compact if K is a compact subset of $\mathcal{G}_Y$. Consider a compact $\mathcal{G}$-order $\mathcal{R}$ such that $K \subseteq \mathcal{R}$ (see point (i) of Remark 2.10) and let $(V_1, V_2, \mathcal{H}_1, \mathcal{H}_2)$ be an $\mathcal{R}$-decomposition for $\mathcal{H}$. The existence of a partition of unity subordinated to (V_1, V_2) ensures that there exist two closed subsets F_1 and F_2 of Y contained in V_1 and V_2, respectively, and such that $Y = F_1 \cup F_2$. Let us set $K_1 = K \cap \mathcal{G}_{F_1}$ and $K_2 = K \cap \mathcal{G}_{F_2}$. Then K_1 and K_2 are compact subsets respectively contained in $\mathcal{G}_{V_1}$ and $\mathcal{G}_{V_2}$, and moreover, we have $K = K_1 \cup K_2$. Furthermore, since $K_1 \subseteq \mathcal{R}_{V_1}$ and $K_2 \subseteq \mathcal{R}_{V_2}$ and using the definition of an $\mathcal{R}$-decomposition, we have $\mathcal{H} \cap K_1 = \mathcal{H}_1 \cap K_1$ and $\mathcal{H} \cap K_2 = \mathcal{H}_2 \cap K_2$. Since $\mathcal{H}_1$ and $\mathcal{H}_2$ are relatively clopen subgroupoids, then $\mathcal{H}_1 \cap K_1$ and $\mathcal{H}_2 \cap K_2$ are compact and hence $\mathcal{H} \cap K$ is compact. ∎

Let $\mathcal{G}$ be a locally compact groupoid. A set $\mathcal{D}$ of open subgroupoids of $\mathcal{G}$ is closed under coarse decompositions if every $\mathcal{D}$-decomposable subgroupoid of $\mathcal{G}$ is indeed in $\mathcal{D}$. If $\mathcal{D}$ is a set of open subgroupoids of $\mathcal{G}$, let $\widehat{\mathcal{D}}$ be the smallest set of open subgroupoids of $\mathcal{G}$ closed under coarse decompositions.

Definition 2.17. Let $\mathcal{G}$ be a locally compact groupoid and let $\mathcal{D}$ be a family of open subgroupoids of $\mathcal{G}$. We say that an open subgroupoid $\mathcal{H}$ of $\mathcal{G}$ has finite decomposition complexity with respect to $\mathcal{D}$ if $\mathcal{H}$ belongs to $\widehat{\mathcal{D}}$.

Lemma 2.18. *Let $\mathcal{G}$ be a locally compact groupoid and let $\mathfrak{D}$ be a set of open subgroupoids of $\mathcal{G}$ closed under taking open subgroupoids. Then $\widehat{\mathfrak{D}}$ is closed under taking open subgroupoids.*

Proof. Let $\mathcal{D}'$ be the set of open subgroupoids $\mathcal{H}$ of $\mathcal{G}$ such that every open subgroupoid of $\mathcal{H}$ lies in $\widehat{\mathcal{D}}$. We have inclusions $\mathcal{D} \subseteq \mathcal{D}' \subseteq \widehat{\mathcal{D}}$. Let us show that $\mathcal{D}'$ is closed under coarse decompositions. Let $\mathcal{H}$ be an open subgroupoid of $\mathcal{G}$ which is $\mathcal{D}'$-decomposable and let $\mathcal{H}'$ be an open subgroupoid of $\mathcal{H}$ with unit space Y. Let $\mathcal{R}$ be a compact $\mathcal{G}$-order and let us consider an $\mathcal{R}$-decomposition $(V_1, V_2, \mathcal{H}_1, \mathcal{H}_2)$ of $\mathcal{H}$ with $\mathcal{H}_1$ in $\mathcal{H}_2$ in $\mathcal{D}'$. Then $(V_1 \cap Y, V_2 \cap Y, \mathcal{H}_1 \cap \mathcal{H}', \mathcal{H}_2 \cap \mathcal{H}')$ is an $\mathcal{R}$-decomposition of $\mathcal{H}'$ with $\mathcal{H}_1 \cap \mathcal{H}'$ and $\mathcal{H}_2 \cap \mathcal{H}'$ in $\widehat{\mathcal{D}}$. As a result, $\mathcal{H}'$ is in $\widehat{\mathcal{D}}$ for any open subgroupoid and hence $\mathcal{H}$ is in $\mathcal{D}'$. We conclude that $\widehat{\mathcal{D}} \subseteq \mathcal{D}'$ and hence $\widehat{\mathcal{D}} = \mathcal{D}'$. ∎

Lemma 2.19. *Let $\mathcal{G}$ be a locally compact groupoid and let $\mathfrak{D}$ be a set of relatively clopen subgroupoids of $\mathcal{G}$. Then*

(i) *If $\mathcal{H}$ is in $\widehat{\mathfrak{D}}$, then $\mathcal{H}$ is relatively clopen.*

(ii) *If $\mathfrak{D}$ is closed under taking relatively clopen subgroupoids, then so is $\widehat{\mathfrak{D}}$.*

Proof. To prove the first point, let us consider the set $\mathcal{D}'$ of relatively clopen subgroupoids of $\mathcal{G}$ that belongs to $\widehat{\mathcal{D}}$. Then we have inclusions $\mathcal{D} \subseteq \mathcal{D}' \subseteq \widehat{\mathcal{D}}$ and we deduce from Lemma 2.16 that $\mathcal{D}'$ is closed under coarse decompositions. Hence we have $\mathcal{D}' = \widehat{\mathcal{D}}$.

To prove the second point, we proceed as in the proof of Lemma 2.18 by considering the set of subgroupoids $\mathcal{H}$ of $\mathcal{G}$ for which every relatively clopen subgroupoid is in $\widehat{\mathcal{D}}$ and by noticing that the intersection of two relatively clopen subgroupoids is relatively clopen. ∎

Example 2.20. Let X be a metric discrete space with bounded geometry and with finite decomposition complexity in the sense of [8, Definition 2.3] and consider then $\mathcal{G}_X$ the coarse groupoid of X defined in [26, Section 3]. Then $\mathcal{G}_X$ has finite decomposition complexity with respect to the set of its compact open subgroupoids (see [11, Theorem A.7]). In the particular case of a finitely generated group Γ with finite decomposition complexity with respect to any word metric, let us denote by $|\Gamma|$ the underlying metric space structure. If we consider the action of Γ on its Stone–Čech compactification β_Γ, then the action groupoid $\Gamma \ltimes \beta_\Gamma$ is isomorphic to $\mathcal{G}_{|\Gamma|}$ (see [26, Proposition 3.4]) and hence $\Gamma \ltimes \beta_\Gamma$ has finite decomposition complexity with respect to the set of its compact open subgroupoids.

2.6. Groupoid amenability and coarse decomposition

It was shown in [8, Theorem 4.3] that a discrete metric space with bounded geometry X and finite decomposition complexity has property (A) defined in [33]. Therefore, according to [26, Theorem A.9], its coarse groupoid $\mathcal{G}_X$ is amenable. In view of this result, we prove in this subsection that amenability is closed under coarse decomposition. This generalizes [11, Theorem A.9], and indeed, a slight modification of the argument used there can be applied to our situation. We refer to [1] for a comprehensive discussion on amenable groupoids. Let us first introduce the notion of Haar system [23, Definition 2.2] that we shall also use in Section 3 to define reduced cross products of groupoids. In what follows, for a locally compact groupoid $\mathcal{G}$, $C_c(\mathcal{G})$ will denote the algebra of continuous complex valued and compactly supported functions on $\mathcal{G}$, and $C_c(\mathcal{G})^+$ will denote the set of positive functions of $C_c(\mathcal{G})$.

Definition 2.21. Let $\mathcal{G}$ be a locally compact groupoid with space of units X. A Haar system is a family $(\lambda^x)_{x\in X}$ of Radon measures on $\mathcal{G}$ such that

(i) for every x in X, the support of λ^x is $\mathcal{G}^x$;

(ii) for every f in $C_c(\mathcal{G})$, then
$$X \to \mathbb{C}, \quad x \mapsto \int_{\mathcal{G}^x} f\, d\lambda^x$$
is continuous;

(iii) for every γ in $\mathcal{G}$, we have
$$\int_{\mathcal{G}^{s(\gamma)}} f(\gamma\gamma')\, d\lambda^{s(\gamma)}(\gamma') = \int_{\mathcal{G}^{r(\gamma)}} f(\gamma')\, d\lambda^{r(\gamma)}(\gamma').$$

We shall use the following definition of amenability for a locally compact groupoid (see [1, Proposition 2.2.13]).

Definition 2.22. Let $\mathcal{G}$ be a locally compact groupoid with space of units X provided with a Haar system $(\lambda^x)_{x\in X}$. The groupoid $\mathcal{G}$ is amenable (with respect to $(\lambda^x)_{x\in X}$) if there exists a net $(g_j)_{j\in J}$ valued in $C_c(\mathcal{G})^+$ such that

(i) $\int_{\mathcal{G}^x} g_j\, d\lambda^x \leqslant 1$ for any x in X and any j in J;

(ii) $(\int_{\mathcal{G}^x} g_j\, d\lambda^x)_{j\in J}$ converges to 1 uniformly on every compact subset of X;

(iii) $(\int_{\mathcal{G}^{r(\gamma)}} |g_j(\gamma^{-1}\gamma') - g_j(\gamma')|\, d\lambda^{r(\gamma)}(\gamma'))_{j\in J}$ converges to 0 uniformly on every compact subset of $\mathcal{G}$.

Remark 2.23. This notion is called in [1] topological amenability.

The proof that amenability is closed under coarse decomposition follows from the existence of suitable almost invariant partitions of unity for groupoid decomposition (see [11, Lemma A.12]). This will require the following two preliminary lemmas.

Lemma 2.24. *Let $\mathcal{G}$ be a locally compact groupoid, and let $\mathcal{H}$ be an open subgroupoid of $\mathcal{G}$ with space of units Y and let K be a compact subset of $\mathcal{G}$. Then* $\{x \in Y$ such that $K^x \subseteq \mathcal{H}\}$ *is an open subset of Y.*

Proof. Let us show that $\{x \in Y : K^x \nsubseteq \mathcal{H}\}$ is closed in Y. Let (x_λ) be a net in Y such that $K^{x_\lambda} \nsubseteq \mathcal{H}$ for every λ converging to x in Y. Let us show that $K^x \nsubseteq \mathcal{H}$. For every λ, there exists γ_λ in K^{x_λ} such that $\gamma_\lambda \notin \mathcal{H}$. Since K is compact, we can assume passing to a subnet that (γ_λ) converges to γ in K. As $\mathcal{H}$ is open, we have $\gamma \notin \mathcal{H}$ and by continuity of r, we have $r(\gamma) = x$ and hence $K^x \nsubseteq \mathcal{H}$. ■

The next lemma should be compared with [10, Lemma 7.5].

Lemma 2.25. *Let $\mathcal{G}$ be a locally compact groupoid. Then for any compact subset K of $\mathcal{G}$ and for any integer N, there exists a compact $\mathcal{G}$-order $\mathcal{R}$ containing K for which the following is satisfied:*

- *for any open subgroupoid $\mathcal{H}$ of $\mathcal{G}$ containing K;*
- *for any $\mathcal{R}$-decomposition $(V_1, V_2, \mathcal{H}_1, \mathcal{H}_2)$ of $\mathcal{H}$,*

there exist nested sequences

$$U_i^{(0)} \subseteq U_i^{(1)} \subseteq \cdots \subseteq U_i^{(N-1)} \subseteq U_i^{(N)} \quad \textit{for } i = 1, 2$$

of open and relatively compact subsets of the space of units of $\mathcal{H}_i$ such that

(i) $s(K) \cup r(K) \subseteq U_1^{(0)} \cup U_2^{(0)}$;

(ii) $\overline{s(K^{U_i^{(n-1)}})} \cup \overline{U_i^{(n-1)}} \subseteq U_i^{(n)}$ *for $i = 1, 2$ and $n = 1, \ldots, N$;*

(iii) *for $i = 1, 2$, we have $K^{\overline{U_i^{(N)}}} \subseteq \mathcal{H}_i$.*

Proof. We can assume without loss of generality that

$$K = K^{-1}, \quad u(s(K)) \subseteq K \quad \text{and} \quad u(r(K)) \subseteq K.$$

We set $K_n = K^{*n}$ for every integer n, with $K_0 = u(s(K)) = u(r(K))$. Since $u(s(K)) = u(r(K)) \subseteq K$, we have $K_n \subseteq K_{n+1}$ for all integer n. Let $\mathcal{R}$ be a compact $\mathcal{G}$-order containing K_{N+1}, let $\mathcal{H}$ be an open subgroupoid of $\mathcal{G}$ containing K and let $(V_1, V_2, \mathcal{H}_1, \mathcal{H}_2)$ be an $\mathcal{R}$-decomposition of $\mathcal{H}$.

For $i = 1, 2$ and $n = 0, \ldots, N$, we set

$$V_i^{(n)} = \{x \in Y_i : K_{N+1-n}^x \subseteq \mathcal{H}_i\}.$$

According to Lemma 2.24, we see that $V_i^{(n)}$ is an open subset of Y_i and moreover, we have $V_i^{(n-1)} \subseteq V_i^{(n)}$ for $n = 1, \ldots, N$. By definition of an $\mathcal{R}$-decomposition, we have $V_i \subseteq V_i^{(0)}$ for $i = 1, 2$ and hence $V_1^{(0)} \cup V_2^{(0)} = Y$.

Let $U_1^{(0)}$ and $U_2^{(0)}$ be relatively compact open subsets of Y such that

- $s(K) \cup r(K) \subseteq U_1^{(0)} \cup U_2^{(0)}$;
- $\overline{U_1^{(0)}} \subseteq V_1^{(0)}$

for $i = 1, 2$. We have then

$$\overline{s(K^{U_i^{(0)}})} \subseteq s(K^{\overline{U_i^{(0)}}}) \subseteq s(K^{V_i^{(0)}}).$$

Let us show that $s(K^{V_i^{(0)}}) \subseteq V_i^{(1)}$. Let x be an element in $s(K^{V_i^{(0)}})$ and let γ be an element in K_N^x. Let us prove that γ belongs to $\mathcal{H}_i$. Let γ' be an element in $K^{V_i^{(0)}}$ such that $s(\gamma') = x = r(\gamma)$ and write $\gamma = \gamma'^{-1}\gamma'\gamma$.

- since $\gamma'\gamma$ lies in K_{N+1} and $r(\gamma'\gamma) = r(\gamma')$ belongs to $V_i^{(0)}$, we deduce from the definition of $V_i^{(0)}$ that $\gamma'\gamma$ is in $\mathcal{H}_i$;
- since we have assume that $u(s(K)) \subseteq K$, we have that $K \subseteq K_{N+1}$ and hence γ' belongs to $K_{N+1}^{r(\gamma')}$ with $r(\gamma')$ in $V_i^{(0)}$. Once again, from the definition of $V_i^{(0)}$, we deduce that γ' and hence γ'^{-1} is in $\mathcal{H}_i$.

From this we conclude that γ belongs to $\mathcal{H}_i$ and hence $s(K^{V_i^{(0)}}) \subseteq V_i^{(1)}$.

For $i = 1, 2$, let $U_i^{(1)}$ be a relatively compact open subset of $V_i^{(1)}$ such that

$$\overline{s(K^{U_i^{(0)}})} \cup \overline{U_i^{(0)}} \subseteq U_i^{(1)} \subseteq \overline{U_i^{(1)}} \subseteq V_i^{(1)}.$$

By iterating this process, we obtain a nested sequence

$$U_i^{(0)} \subseteq U_i^{(1)} \subseteq \cdots \subseteq U_i^{(N-1)} \subseteq U_i^{(N)}$$

of open and relatively compact subsets of the space of units of $\mathcal{H}_i$ such that

$$\overline{s(K^{U_i^{(n-1)}})} \cup \overline{U_i^{(n-1)}} \subseteq U_i^{(n)} \quad \text{for } i = 1, 2 \text{ and } n = 1, \dots, N.$$

Since $\overline{U_i^{(N)}} \subseteq V_i^{(N)}$, it is clear from the definition of $V_i^{(N)}$ that $K^{\overline{U_i^{(N)}}} \subseteq \mathcal{H}_i$. ∎

In view of the proof of [11, Theorem A.9], the heredity of amenability under groupoid decomposability is a consequence of the following proposition.

Proposition 2.26. *Let $\mathcal{G}$ be a locally compact groupoid. Then for any compact subset K of $\mathcal{G}$ and for any positive number ε, there exists a compact $\mathcal{G}$-order $\mathcal{R}$ containing K for which the following is satisfied:*

- *for any open subgroupoid $\mathcal{H}$ of $\mathcal{G}$ containing K;*
- *for any $\mathcal{R}$-decomposition $(V_1, V_2, \mathcal{H}_1, \mathcal{H}_2)$ of $\mathcal{H}$,*

for $i = 1, 2$ there exist continuous functions $\phi_i : Y \to [0, 1]$ such that

(i) *ϕ_i is compactly supported in the space of units of $\mathcal{H}_i$ for $i = 1, 2$;*

(ii) *$\phi_1 + \phi_2 \leqslant 1$;*

(iii) *$\phi_1(x) + \phi_2(x) = 1$ for all x in $s(K) \cup r(K)$;*

(iv) *$|\phi_i(s(\gamma)) - \phi_i(r(\gamma))| < \varepsilon$ for any γ in K and $i = 1, 2$;*

(v) *$K^{\operatorname{supp} \phi_i} \subseteq \mathcal{H}_i$ for $i = 1, 2$.*

Proof. Following the arguments of the proof of [11, Lemma A.12], we can assume without loss of generality that $K = K^{-1}$, $u(s(K)) \subseteq K$ and $u(r(K)) \subseteq K$. Let us pick an integer N

such that $\frac{6}{N} < \varepsilon$. Let $\mathcal{R}$ be a compact $\mathcal{G}$-order as in Lemma 2.25, let $\mathcal{H}$ be an open subgroupoid of $\mathcal{G}$ containing K, let $(V_1, V_2, \mathcal{H}_1, \mathcal{H}_2)$ be an $\mathcal{R}$-decomposition for $\mathcal{H}$ and let

$$U_i^{(0)} \subseteq U_i^{(1)} \subseteq \cdots \subseteq U_i^{(N-1)} \subseteq U_i^{(N)}$$

be a nested sequence of open and relatively compact subsets of the space of units of $\mathcal{H}_i$ which satisfies the conclusion of Lemma 2.25. For $n = 1, \ldots, N$ and $i = 1, 2$, let

$$\psi_i^{(n)} \colon X \to [0, 1]$$

be a continuous function compactly supported in $U_i^{(n)}$ and such that $\psi_i^{(n)}(x) = 1$ for x in $U_i^{(n-1)}$. Let us set then

$$\psi_i = \frac{1}{N} \sum_{n=1}^{N} \psi_i^{(n)} \quad \text{and} \quad \phi_i = \frac{\psi_i}{\max\{\psi_1 + \psi_2, 1\}}.$$

The two first points are clearly satisfied. For the third one, since $s(K) \cup r(K) \subseteq U_1^{(0)} \cup U_1^{(0)}$, we get that $\psi_1(x) + \psi_2(x) \geqslant 1$ for all x in $s(K) \cup r(K)$ and hence $\phi_1(x) + \phi_2(x) = 1$. The last point is a consequence of the inclusions $\operatorname{supp} \phi_i \subseteq U_i^{(N)}$ and of the third point of Lemma 2.25.

Let us prove the fourth point. For γ in K and $i = 1, 2$, let us define

$$M = M_{\gamma,i} = \min \left\{ n \text{ such that } r(\gamma) \in U_i^{(n)} \right\}$$

if $r(\gamma) \in U_i^{(N)}$ and $M = N + 1$ otherwise. We clearly have

- $\psi_i^{(n)}(r(\gamma)) = 1$ if $n \geqslant M + 1$,
- $\psi_i^{(n)}(r(\gamma)) = 0$ if $n \leqslant M - 1$.

From this we deduce that

$$\frac{N - M}{N} \leqslant \psi_i(r(\gamma)) \leqslant \frac{N + 1 - M}{N}. \tag{2.1}$$

Since $s(K^{U_i^{(M)}}) \subseteq U_i^{(M+1)}$, we get that $s(\gamma)$ is in $U_i^{(M+1)}$. Since $r(\gamma) \notin U_i^{(M-1)}$, we deduce from the inclusion

$$s(K^{U_i^{(M-2)}}) \subseteq U_i^{(M-1)}$$

that $s(\gamma^{-1}) \notin s(K^{U_i^{(M-2)}})$ and hence $\gamma^{-1} \notin K^{U_i^{(M-2)}}$. Since we have assumed that $K = K^{-1}$, then γ^{-1} is in K and hence $s(\gamma) = r(\gamma^{-1}) \notin U_i^{(M-2)}$. We obtain from that the inequality

$$\frac{N - M - 1}{N} \leqslant \psi_i(s(\gamma)) \leqslant \frac{N + 2 - M}{N}. \tag{2.2}$$

Combining equations (2.1) and (2.2), we obtain that

$$|\psi_i(r(\gamma)) - \psi_i(s(\gamma))| \leqslant \frac{2}{N}$$

for $i = 1, 2$ and γ in K. A straightforward computation leads to

$$|\phi_i(r(\gamma)) - \phi_i(s(\gamma))| \leqslant \frac{6}{N}$$

and hence we get the fourth point. ∎

We are now in position to prove that amenability is closed under $\mathcal{D}$-decomposition.

Proposition 2.27. *Let $\mathcal{G}$ be a locally compact groupoid provided with a Haar system and let $\mathcal{D}$ be a family of open and amenable subgroupoids of $\mathcal{G}$. Then any $\mathcal{D}$-decomposable subgroupoid of $\mathcal{G}$ is amenable.*

Proof. Let $\mathcal{H}$ be a $\mathcal{D}$-decomposable subgroupoid of $\mathcal{G}$ with space of units Y. Let ε be a positive number and let K and K' be respectively compact subsets of Y and $\mathcal{H}$. Let $\mathcal{R}$ be a compact $\mathcal{G}$-order as in Proposition 2.26 with respect to $\frac{\varepsilon}{6}$ and to $K' \cup u(K)$. Let $(V_1, V_2, \mathcal{H}_1, \mathcal{H}_2)$ be an $\mathcal{R}$-decomposition for $\mathcal{H}$ with $\mathcal{H}_1$ and $\mathcal{H}_2$ in $\mathcal{D}$ and, let $\phi_i : Y \to [0, 1]$ be for $i = 1, 2$ continuous compactly supported functions satisfying properties (i)–(v) of Proposition 2.26. If we set $K_i = K \cap \operatorname{supp} \phi_i$ and $K_i' = K'^{\operatorname{supp} \phi_i}$ for $i = 1, 2$, we have then

- $K_i' \subseteq \mathcal{H}_i$;
- $K = K_1 \cup K_2$;
- $K' = K_1' \cup K_2'$.

Let Y_i be for $i = 1, 2$ the space of units of $\mathcal{H}_i$. The groupoid $\mathcal{H}_i$ being amenable, there exists a function g_i in $C_c(\mathcal{H}_i)^+$ such that

(i) $\int_{\mathcal{H}_i^x} g_i \, d\lambda^x \leqslant 1$ for every x in Y_i;

(ii) $1 - \int_{\mathcal{H}_i^x} g_i \, d\lambda^x < \varepsilon$ for every x in K_i;

(iii) $\int_{\mathcal{H}_i^{r(\gamma)}} |g_i(\gamma^{-1}\gamma') - g_i(\gamma')| \, d\lambda^{r(\gamma)}(\gamma') < \frac{\varepsilon}{4}$ for every γ in K_i'.

Let us set then

$$g = \phi_1 \circ r \cdot g_1 + \phi_2 \circ r \cdot g_2.$$

Then g belongs to $C_c(\mathcal{G})^+$ and for any x in Y, we have

$$\begin{aligned}
\int_{\mathcal{H}^x} g \, d\lambda^x &= \int_{\mathcal{H}^x} \phi_1 \circ r \cdot g_1 \, d\lambda^x + \int_{\mathcal{H}^x} \phi_2 \circ r \cdot g_2 \, d\lambda^x \\
&= \phi_1(x) \int_{\mathcal{H}^x} g_1 \, d\lambda^x + \phi_2(x) \int_{\mathcal{H}^x} g_1 \, d\lambda^x \\
&= \phi_1(x) \int_{\mathcal{H}_1^x} g_1 \, d\lambda^x + \phi_2(x) \int_{\mathcal{H}_2^x} g_1 \, d\lambda^x \\
&\leqslant \phi_1(x) + \phi_2(x) \leqslant 1.
\end{aligned}$$

For every x in K, we have

$$\begin{aligned} 1-\int_{\mathcal{H}^x} g\,d\lambda^x &= \phi_1(x)+\phi_2(x)-\int_{\mathcal{H}^x}\phi_1\circ r\cdot g_1\,d\lambda^x-\int_{\mathcal{H}^x}\phi_2\circ r\cdot g_2\,d\lambda^x\\ &=\phi_1(x)\Big(1-\int_{\mathcal{H}^x} g_1\,d\lambda^x\Big)+\phi_2(x)\Big(1-\int_{\mathcal{H}^x} g_2\,d\lambda^x\Big)\\ &=\phi_1(x)\Big(1-\int_{\mathcal{H}_1^x} g_1\,d\lambda^x\Big)+\phi_2(x)\Big(1-\int_{\mathcal{H}_2^x} g_2\,d\lambda^x\Big)\\ &<(\phi_1(x)+\phi_2(x))\varepsilon<\varepsilon. \end{aligned}$$

For γ in K', if we set $x=r(\gamma)$ and $y=s(\gamma)$, we have

$$|\phi_i(y)-\phi_i(x)|<\frac{\varepsilon}{4}.$$

Let us show that

$$\int_{\mathcal{H}^x}|g(\gamma^{-1}\gamma')-g(\gamma')|\,d\lambda^x(\gamma')<\varepsilon.$$

We have

$$\begin{aligned} \int_{\mathcal{H}^x}|g(\gamma^{-1}\gamma')-g(\gamma')|\,d\lambda^x(\gamma') &= +\int_{\mathcal{H}^x}|\phi_1(y)g_1(\gamma^{-1}\gamma')+\phi_2(y)g_2(\gamma^{-1}\gamma')\\ &\qquad -\phi_1(x)g_1(\gamma')-\phi_2(x)g_2(\gamma')|\,d\lambda^x(\gamma')\\ &\leqslant \int_{\mathcal{H}^x}|\phi_1(y)g_1(\gamma^{-1}\gamma')-\phi_1(x)g_1(\gamma')|\,d\lambda^x(\gamma')\\ &\quad+\int_{\mathcal{H}^x}|\phi_2(y)g_2(\gamma^{-1}\gamma')-\phi_2(x)g_2(\gamma')|\,d\lambda^x(\gamma'). \end{aligned}$$

Let us give a majoration for each summand of the right-hand side. For $i=1,2$, we have

$$\begin{aligned} &\int_{\mathcal{H}^x}|\phi_i(y)g_i(\gamma^{-1}\gamma')-\phi_i(x)g_i(\gamma')|\,d\lambda^x(\gamma')\\ &\qquad\leqslant\int_{\mathcal{H}^x}|\phi_i(y)-\phi_i(x)|g_i(\gamma^{-1}\gamma')\,d\lambda^x(\gamma')\\ &\qquad\qquad+\phi_i(x)\int_{\mathcal{H}^x}|g_i(\gamma^{-1}\gamma')-g_i(\gamma')|\,d\lambda^x(\gamma')\\ &\qquad<\frac{\varepsilon}{4}\int_{\mathcal{H}^x}g_i(\gamma^{-1}\gamma')\,d\lambda^x(\gamma')+\phi_i(x)\int_{\mathcal{H}^x}|g_i(\gamma^{-1}\gamma')-g_i(\gamma')|\,d\lambda^x(\gamma')\\ &\qquad<\frac{\varepsilon}{4}\int_{\mathcal{H}^y}g_i(\gamma')\,d\lambda^y(\gamma')+\phi_i(x)\int_{\mathcal{H}^x}|g_i(\gamma^{-1}\gamma')-g_i(\gamma')|\,d\lambda^x(\gamma')\\ &\qquad<\frac{\varepsilon}{4}\int_{\mathcal{H}_i^x}g_i(\gamma')\,d\lambda^x(\gamma')+\phi_i(x)\int_{\mathcal{H}^x}|g_i(\gamma^{-1}\gamma')-g_i(\gamma')|\,d\lambda^x(\gamma')\\ &\qquad<\frac{\varepsilon}{4}+\phi_i(x)\int_{\mathcal{H}^x}|g_i(\gamma^{-1}\gamma')-g_i(\gamma')|\,d\lambda^x(\gamma'). \end{aligned}$$

If $x = r(\gamma)$ is in $\operatorname{supp}\phi_i$, then γ is in $K^{\operatorname{supp}\phi_i}$ and hence belongs to $\mathcal{H}_i$. We deduce from this that

$$\int_{\mathcal{H}^x} |g_i(\gamma^{-1}\gamma') - g_i(\gamma')|\, d\lambda^x(\gamma') = \int_{\mathcal{H}_i^x} |g_i(\gamma^{-1}\gamma') - g_i(\gamma')|\, d\lambda^x(\gamma') < \frac{\varepsilon}{4}.$$

Eventually, we obtain that

$$\int_{\mathcal{H}^x} |\phi_i(y)g_i(\gamma^{-1}\gamma') - \phi_i(x)g_i(\gamma')|\, d\lambda^x(\gamma') < \frac{\varepsilon}{2}$$

and hence that

$$\int_{\mathcal{H}^x} |g(\gamma^{-1}\gamma') - g(\gamma')|\, d\lambda^x(\gamma') < \varepsilon. \qquad \blacksquare$$

As a consequence, we obtain that amenability is closed under coarse decomposition.

Theorem 2.28. *Let $\mathcal{G}$ be a locally compact groupoid provided with a Haar system with finite decomposition complexity with respect to a family of open and amenable subgroupoids. Then $\mathcal{G}$ is amenable.*

3. Reduced crossed product of a groupoid

In this section, we review the construction of the reduced crossed-product for a groupoid action on a C^*-algebra. Some good material for this construction can be found in [17, 18].

3.1. $C(X)$-algebra

Definition 3.1. Let X be a locally compact space. A $C(X)$-algebra is a C^*-algebra A together with a morphism $\Psi\colon C_0(X) \to \mathcal{Z}(\mathcal{M}(A))$, where $\mathcal{Z}(\mathcal{M}(A))$ stands for the center of the multiplier algebra of A, such that

$$\{\Psi(f) \cdot a \colon f \in C_0(X) \text{ and } a \in A\}$$

is dense in A.

From now on, for f in $C_0(X)$ and a in A, we will denote $\Psi(f) \cdot a$ by $f \cdot a$ and omit the structure map Φ.

Let A be a $C(X)$-algebra and let us consider for x in X the ideal I_x defined as the closure of

$$\{f \cdot a \colon f \in C_0(X) \text{ and } a \in A \text{ such that } f(x) = 0\}.$$

We define the *fiber* of A at x as the quotient C^*-algebra $A_x \stackrel{\text{def}}{=} A/I_x$. For a in A, we denote by $a(x)$ the image of a under the quotient map $A \to A_x$. Then we have the following classical result [31, Proposition C.10].

Lemma 3.2. *Let X be a locally compact space and let A be a $C(X)$-algebra. Then for any a in A,*

(i) *the map $X \to \mathbb{R}$, $x \mapsto \|a(x)\|$ is upper semi-continuous and vanishing at infinity;*

(ii) $\|a\| = \sup_{\in X} \|a(x)\|$.

Let X and Y be locally compact spaces, let A be a $C(Y)$-algebra and let $f\colon X \to Y$ be a continuous map. The algebra $C_0(X, A)$ of continuous functions $\xi\colon X \to A$ vanishing at infinity is then a $C(X \times Y)$-algebra. Consider in $C_0(X, A)$ the ideal I_f defined as the closure of

$$\{h \cdot \xi\colon h \in C_0(X \times Y),\ \xi \in C_0(X, A) \text{ such that } h(x, f(x)) = 0\ \forall x \in X\}.$$

The *pull back algebra* of A by f is by definition $f^*A \stackrel{\text{def}}{=} C_0(X, A)/I_f$. Pointwise multiplication by $C_0(X)$ on $C_0(X, A)$ induces then a $C(X)$-algebra structure on f^*A. The fiber of f^*A at an element x of X is canonically isomorphic to $A_{f(x)}$, this isomorphism being induced by the map

$$C_0(X, A) \to A_{f(x)}, \quad \xi \mapsto \xi(x)(f(x)).$$

Let A and B be two $C(X)$-algebras. A morphism of C^*-algebra $\Psi\colon A \to B$ is called a morphism of $C(X)$-algebra if it is in addition $C_0(X)$-linear. It is straightforward to check that a morphism of $C(X)$-algebra $\Psi\colon A \to B$ induced for every x in X a morphism $\Psi_x\colon A_x \to B_x$. Moreover, Ψ is an isomorphism (resp. injective, surjective) if and only if Ψ_x is an isomorphism (resp. injective, surjective) for any x in X.

It is clear that if X and Y are locally compact spaces, $f\colon X \to Y$ is a continuous map and A and B are $C(Y)$-algebras, then any morphism of $C(Y)$-algebras $\Psi\colon A \to B$ gives rise to a morphism of $C(X)$-algebras

$$f^*\Psi\colon f^*A \to f * B$$

such that, up to the canonical identifications of the fibers described above,

$$(f^*\Psi)_x = \Psi_{f(x)}.$$

3.2. Groupoid actions on C^*-algebras

Groupoid actions generalize to the setting of groupoid the notion of group actions by automorphisms on a C^*-algebra.

Definition 3.3. Let $\mathcal{G}$ be a locally compact groupoid with X as space of units and let A be a $C(X)$-algebra. An action of $\mathcal{G}$ on A is given by a $C(\mathcal{G})$-isomorphism $\alpha\colon s^*A \to r^*A$ which satisfies

$$\alpha_{\gamma\gamma'} = \alpha_\gamma \circ \alpha'_\gamma$$

for any γ and γ' in $\mathcal{G}$ such that $s(\gamma) = r(\gamma')$, where

$$\alpha_\gamma\colon A_{s(\gamma)} \to A_{r(\gamma)}$$

is the morphism fiberwise induced by α at γ in $\mathcal{G}$ under the canonical isomorphisms $(s^*A)_\gamma \cong A_{s(\gamma)}$ and $(r^*A)_\gamma \cong A_{r(\gamma)}$. A $C(X)$-algebra equipped with an action of $\mathcal{G}$ will be called a $\mathcal{G}$-algebra.

In what follows, for a $\mathcal{G}$-algebra A with respect to an action $\alpha\colon s^*A \to r^*A$, we shall denote for short the morphism induced fiberwise at γ in $\mathcal{G}$ by

$$\gamma\colon A_{s(\gamma)} \mapsto A_{r(\gamma)}, \quad a \mapsto \alpha_\gamma(a).$$

Example 3.4. Let $\mathcal{G}$ be a locally compact groupoid with space of units X and let Z be a $\mathcal{G}$-space with respect to the anchor map $p_Z\colon Z \to X$.

(i) The anchor map provides a $C(X)$-algebra structure on $C(Z)$ which is acted upon by $\mathcal{G}$ in the following way. Let us define

$$s_*Z = \{(\gamma, z) \in \mathcal{G} \times Z \text{ such that } s(\gamma) = p_Z(z)\}$$

and

$$r_*Z = \{(\gamma, z) \in \mathcal{G} \times Z \text{ such that } r(\gamma) = p_Z(z)\}.$$

Then we have canonical isomorphisms $C_0(s_*Z) \cong s^*(C_0(Z))$ and $C_0(r_*Z) \cong r^*(C_0(Z))$ and under these identifications, the homeomorphism

$$r_*Z \to s_*Z, \quad (\gamma, z) \mapsto (\gamma, \gamma^{-1}z)$$

gives rise to a $C(\mathcal{G})$-isomorphism

$$\alpha\colon s^*(C_0(Z)) \xrightarrow{\cong} r^*(C_0(Z)).$$

Let γ be an element in $\mathcal{G}$. The fibers at γ of $s^*(C_0(Z))$ and $r^*(C_0(Z))$ are under the above identifications respectively $C_0(Z_{s(\gamma)})$ and $C_0(Z_{r(\gamma)})$ and α induces fiberwise at γ the isomorphism

$$C_0(Z_{s(\gamma)}) \to C_0(Z_{r(\gamma)}), \quad f \mapsto \gamma(f),$$

where $\gamma(f)(z) = f(\gamma^{-1} \cdot z)$ for any z in $Z_{r(\gamma)}$ and any f in $C_0(Z_{s(\gamma)})$.

(ii) If A is a $C(Z)$-algebra, then an action of $\mathcal{G} \ltimes Z$ on A is simply an action

$$\alpha\colon s^*A \to r^*A$$

of $\mathcal{G}$ on A which is $C(Z)$-linear, where A is viewed as a $C(X)$-algebra by using the anchor map.

Let $\mathcal{G}$ be a locally compact groupoid with space of units X and let A and B be $\mathcal{G}$-algebras. A $C(X)$-morphism $f\colon A \to B$ is a homomorphism of $\mathcal{G}$-algebras if

$$\gamma \circ f_{s(\gamma)} = f_{r(\gamma)} \circ \gamma$$

for every γ in $\mathcal{G}$.

3.3. Reduced crossed products

Let $\mathcal{G}$ be a locally compact groupoid with space of units X and let $C_c(\mathcal{G})$ be the set of complex valued and compactly supported continuous function on $\mathcal{G}$. We assume from now on that $\mathcal{G}$ is provided with a Haar system $(\lambda^x)_{x\in X}$ (see Definition 2.21). Let $L^2(\mathcal{G})$ be the $C_0(X)$-Hilbert module obtained by completion of $C_c(\mathcal{G})$ with respect to the $C_0(X)$-scalar product

$$\langle\eta,\eta'\rangle(x)=\int_{\mathcal{G}^x}\overline{\eta}(\gamma^{-1})\eta'(\gamma^{-1})\,d\lambda^x(\gamma)$$

for any η and η' in $C_c(\mathcal{G})$. An element h of $C_0(X)$ acts on $L^2(\mathcal{G})$ by multiplication by $h\circ s$.

Let A be a $\mathcal{G}$-algebra. Recall that r^*A is a $C(\mathcal{G})$-algebra and that for h in r^*A and γ in $\mathcal{G}$, $h(\gamma)\in A_{r(\gamma)}$ is the fiber evaluation of h at γ under the identification between $(r^*A)_\gamma$ and $A_{r(\gamma)}$. For h in r^*A, the support of h, denoted by $\operatorname{supp} h$, is the complementary of the largest open subset of $\mathcal{G}$ on which $\gamma\mapsto h(\gamma)$ vanishes. Let us set $C_c(X;\mathcal{G},r^*A)$ the set of elements of r^*A with compact support. In the same way, we can define $C_c(X;\mathcal{G},s^*A)$ as the set of elements of s^*A with compact support.

If A is a $\mathcal{G}$-algebra, we set

$$L^2(\mathcal{G},A)=L^2(\mathcal{G})\otimes_{C_0(X)}A.$$

Notice that $C_c(X;\mathcal{G},s^*A)$ embeds in $L^2(\mathcal{G},A)$ and for any η and η' in $C_c(X;\mathcal{G},s^*A)$, the fiber evaluation of $\langle\eta,\eta'\rangle$ at an element $x\in X$ is the element of A_x uniquely determined by

$$\langle\eta,\eta'\rangle(x)=\int_{\mathcal{G}^x}\eta^*(\gamma^{-1})\eta'(\gamma^{-1})\,d\lambda^x(\gamma).$$

Recall that $C_c(X;\mathcal{G},r^*A)$ is provided with an involutive algebra structure such that

$$f\cdot g(\gamma)=\int_{\mathcal{G}^{r(\gamma)}}f(\gamma')\gamma'(g(\gamma'^{-1}\gamma))\,d\lambda^{r(\gamma)}(\gamma')$$

and

$$f^*(\gamma)=\gamma(f(\gamma^{-1})^*)$$

for any f and g in $C_c(X;\mathcal{G},r^*A)$ and any γ in $\mathcal{G}$. Moreover, for any f in $C_c(X;\mathcal{G},r^*A)$, the map

$$C_c(X;\mathcal{G},s^*A)\to C_c(X;\mathcal{G},s^*A),\quad \xi\mapsto f\cdot\xi$$

with

$$(f\cdot\xi)(\gamma)=\int_{\mathcal{G}^{r(\gamma)}}\gamma^{-1}(f(\gamma'))\xi(\gamma'^{-1}\gamma)d\lambda^{r(\gamma)}(\gamma')$$

extends to an adjointable endomorphism of $L^2(\mathcal{G},A)$ and we obtain in this way an involutive and faithful representation of $C_c(X;\mathcal{G},r^*A)$. The reduced crossed product algebra $A\rtimes_r\mathcal{G}$ is then the closure of $C_c(X;\mathcal{G},r^*A)$ in the algebra $\mathcal{L}(L^2(\mathcal{G},A))$ of adjointable endomorphisms of $L^2(\mathcal{G},A)$.

It is clear that if A and B are two $\mathcal{G}$-algebras and $\Psi\colon A \to B$ is a homomorphism of $\mathcal{G}$-algebras, then

$$r^*\Psi(C_c(X;\mathcal{G}, r^*A)) \subseteq C_c(X;\mathcal{G}, r^*B)$$

and

$$C_c(X;\mathcal{G}, r^*A) \to C_c(X;\mathcal{G}, r^*B), \quad f \mapsto r^*\Psi(f)$$

extends to a C^*-algebra homomorphism $\Psi_{\mathcal{G}}\colon A \rtimes_r \mathcal{G} \to B \rtimes_r \mathcal{G}$.

Lemma 3.5. *Let $\mathcal{G}$ be a locally compact groupoid with space of units X provided with a Haar system. Let V be an open subset and let $\phi\colon X \to \mathbb{C}$ be a bounded and continuous function with support in V. Then there exists a bounded operator*

$$\Lambda^s_\phi\colon A \rtimes_r \mathcal{G} \to A \rtimes_r \mathcal{G}$$

such that

(i) Λ^s_ϕ *has operator norm bounded by* $\sup_{x\in X} |\phi(x)|$*;*

(ii) $\Lambda^s_\phi(h) = h \cdot \phi \circ s$ *for all h in $C_c(X;\mathcal{G}, r^*A)$.*

Proof. Let us set $M = \sup_{x\in X} |\phi(x)|$. The map

$$C_c(\mathcal{G}) \to C_c(\mathcal{G}), \quad f \mapsto f \cdot \phi \circ r$$

extends to an adjointable operator $T_\phi\colon L^2(\mathcal{G}) \to L^2(\mathcal{G})$ such that $\|T_\phi\| \leqslant M$. Then right multiplication by $T_\phi \otimes_{C_0(X)} \mathrm{Id}_A$ on $\mathcal{L}(L^2(\mathcal{G}, A))$ preserves the subalgebra $A \rtimes_r \mathcal{G}$ and hence induces a bounded operator $\Lambda^s_\phi\colon A \rtimes_r \mathcal{G} \to A \rtimes_r \mathcal{G}$ which satisfies the required conditions. ∎

Remark 3.6. In the same way, left multiplication by $T_\phi \otimes_{C_0(X)} \mathrm{Id}_A$ on $\mathcal{L}(L^2(\mathcal{G}, A))$ preserves $A \rtimes_r \mathcal{G}$ and hence induces a bounded operator $\Lambda^r_\phi\colon A \rtimes_r \mathcal{G} \to A \rtimes_r \mathcal{G}$ such that

(i) Λ^r_ϕ has operator norm bounded by $M = \sup_{x\in X} |\phi(x)|$;

(ii) $\Lambda^r_\phi(h) = h \cdot \phi \circ r$ for all h in $C_c(X;\mathcal{G}, r^*A)$;

(iii) Λ^s_ϕ and $\Lambda^r_{\phi'}$ commute for any continuous and bounded function $\phi'\colon X \to \mathbb{C}$ with support in V;

(iv) $\Lambda^r_{\bar\phi} \circ \Lambda^s_\phi(h) = \bar\phi \circ r \cdot h \cdot \phi \circ s$ for all h in $C_c(X;\mathcal{G}, r^*A)$;

(v) $\Lambda^r_{\bar\phi} \circ \Lambda^s_\phi\colon A \rtimes_r \mathcal{G} \to A \rtimes_r \mathcal{G}$ is positive with operator norm bounded by M^2.

For any open subgroupoid $\mathcal{H}$ of $\mathcal{G}$ with unit space Y, let $A_{/Y}$ be the closure of

$$\{f \cdot a\colon f \in C_0(Y) \text{ and } a \in A\}$$

in A. Then $A_{/Y}$ is an $\mathcal{H}$-algebra and moreover, the Haar system of $\mathcal{G}$ induces by restriction a Haar system on $\mathcal{H}$. We will denote the crossed product $A_{/Y} \rtimes_r \mathcal{H}$ by $A \rtimes_r \mathcal{H}$. Notice that since $\mathcal{H}$ is an open subgroupoid of $\mathcal{G}$, then $A \rtimes_r \mathcal{H}$ can be viewed as a C^*-subalgebra of $A \rtimes_r \mathcal{G}$.

4. Controlled Mayer–Vietoris exact sequence in quantitative K-theory

The concept of quantitative operator K-theory was first introduced in [32] for localization algebras in order to prove the Novikov conjecture for finitely generated groups with finite asymptotic dimension. It has been then extended in [20] to the setting of C^*-algebras equipped with a filtration arising from a length. Dell'Aiera developed in [6] quantitative K-theory in the general framework of C^*-algebras filtered by abstract coarse structure.

4.1. Review of quantitative K-theory

In this subsection, we review from [6] the main features of quantitative K-theory in the framework of C^*-algebras filtered by an abstract coarse structure.

Definition 4.1. A coarse structure $\mathcal{E}$ is an ordered abelian semi-group which is a lattice for the order. Recall that a lattice is a poset for which every pair (E, E') admits a supremum $E \vee E'$ and an infimum $E \wedge E'$.

Example 4.2. If $\mathcal{G}$ is a locally compact groupoid, then the semi-group $(\mathcal{E}_{\mathcal{G}}, *)$ of $\mathcal{G}$-orders partially ordered by the inclusion is a coarse structure with supremum and infimum respectively given by the union and the intersection. The same holds for the set $\mathcal{E}_{\mathcal{G},c}$ of compact $\mathcal{G}$-orders.

Definition 4.3. Let $\mathcal{E}$ be a coarse structure. A $\mathcal{E}$-filtered C^*-algebra A is a C^*-algebra equipped with a family $(A_E)_{E\in\mathcal{E}}$ of closed linear subspaces such that

- $A_E \subseteq A_{E'}$ if $E \leqslant E'$;
- A_E is stable by involution;
- $A_E \cdot A_{E'} \subseteq A_{E+E'}$;
- the subalgebra $\bigcup_{E\in\mathcal{E}} A_E$ is dense in A.

Elements of A_E for E in $\mathcal{E}$ are called elements with $\mathcal{E}$-propagation (less than) E. If A is unital, we also require that the unit is an element of A_E for every E in $\mathcal{E}$.

Let $\mathcal{E}$ be a coarse structure and let A and B be two $\mathcal{E}$-filtered C^*-algebras. A C^*-algebras homomorphism $\phi: A \to B$ is called $\mathcal{E}$-filtered if $\phi(A_E) \subseteq B_E$ for any E in $\mathcal{E}$.

Example 4.4. Let $\mathcal{G}$ be a locally compact groupoid provided with a Haar system and let A be a $\mathcal{G}$-algebra. For any $\mathcal{G}$-order $\mathcal{R}$, we define $A \rtimes_r \mathcal{R}$ as the closure in $A \rtimes_r \mathcal{G}$ of the set of elements g in $C_c(X; \mathcal{G}, r^*A)$ with support in $\mathcal{R}$. Then

- $(A \rtimes_r \mathcal{R})_{\mathcal{R}\in\mathcal{E}_{\mathcal{G}}}$ provides $A \rtimes_r \mathcal{G}$ with a structure of $\mathcal{E}_{\mathcal{G}}$-filtered C^*-algebra;
- $(A \rtimes_r \mathcal{R})_{\mathcal{R}\in\mathcal{E}_{\mathcal{G},c}}$ provides $A \rtimes_r \mathcal{G}$ with a structure of $\mathcal{E}_{\mathcal{G},c}$-filtered C^*-algebra;
- if $\mathcal{H}$ is an open subgroupoid of $\mathcal{G}$, then $A \rtimes_r \mathcal{H}$ is an $\mathcal{E}_{\mathcal{G}}$-filtered C^*-subalgebra of $A \rtimes_r \mathcal{G}$, i.e., $A \rtimes_r \mathcal{H}$ is filtered by $(A \rtimes_r \mathcal{H}) \cap (A \rtimes_r \mathcal{R})_{\mathcal{R}\in\mathcal{E}_{\mathcal{G}}}$;
- in the same way, $A \rtimes_r \mathcal{H}$ is an $\mathcal{E}_{\mathcal{G},c}$-filtered C^*-subalgebra of $A \rtimes_r \mathcal{G}$.

Notice that if A and B are two $\mathcal{G}$-algebras and if $\phi\colon A \to B$ is a homomorphism of $\mathcal{G}$-algebras, then the induced homomorphism $\phi_{\mathcal{G}}\colon A \rtimes_r \mathcal{G} \to B \rtimes_r \mathcal{G}$ is an $\mathcal{E}_{\mathcal{G}}$-filtered homomorphism. The same holds for $\mathcal{E}_{\mathcal{G},c}$.

Let $\mathcal{E}$ be a coarse structure and let A be an $\mathcal{E}$-filtered C^*-algebra. If A is not unital, let us denote by A^+ its unitarization, i.e.,

$$A^+ = \{(x,\lambda)\colon x \in A,\ \lambda \in \mathbb{C}\}$$

with the product

$$(x,\lambda)(x',\lambda') = (xx' + \lambda x' + \lambda' x, \lambda\lambda')$$

for all (x,λ) and (x',λ') in A^+. Then A^+ is $\mathcal{E}$-filtered with

$$A^+_E = \{(x,\lambda)\colon x \in A_E,\ \lambda \in \mathbb{C}\}$$

for any E in $\mathcal{E}$. We also define $\rho_A\colon A^+ \to \mathbb{C}$, $(x,\lambda) \mapsto \lambda$.

Let $\mathcal{E}$ be a coarse structure and let A be a unital $\mathcal{E}$-filtered C^*-algebra. For any positive number ε with $\varepsilon < \frac{1}{4}$ and any element E in $\mathcal{E}$, we call

- an element u in A an ε-E-unitary if u is in A_E, $\|u^* \cdot u - 1\| < \varepsilon$ and $\|u \cdot u^* - 1\| < \varepsilon$. The set of ε-E-unitaries on A will be denoted by $\mathrm{U}^{\varepsilon,E}(A)$.
- an element p in A an ε-E-projection if p is in A_E, $p = p^*$ and $\|p^2 - p\| < \varepsilon$. The set of ε-E-projections on A will be denoted by $\mathrm{P}^{\varepsilon,E}(A)$.

Then ε is called the control and E is called the propagation of the ε-E-projection or of the ε-E-unitary. Notice that an ε-E-unitary is invertible, and that if p is an ε-E-projection in A, then it has a spectral gap around $\frac{1}{2}$ and then gives rise by functional calculus to a projection $\kappa_0(p)$ in A such that $\|p - \kappa_0(p)\| < 2\varepsilon$. Let us first review from [20, Section 1.2] the standard properties of ε-E-projections and ε-E-projections. We have in the context of $\mathcal{E}$-filtered C^*-algebras the analog of [20, Lemma 1.7].

Lemma 4.5. *Let $\mathcal{E}$ be a coarse structure and let A be a unital $\mathcal{E}$-filtered C^*-algebra for $(A_E)_{E\in\mathcal{E}}$. Then for any ε in $(0, \frac{1}{4})$ and any E in $\mathcal{E}$ the following holds:*

(i) *If p is an ε-E-projection in A and q is a self-adjoint element of A_E such that $\|p - q\| < \frac{\varepsilon - \|p^2 - p\|}{4}$, then q is an ε-E-projection. In particular, if p is an ε-E-projection in A and if q is a self-adjoint element in A_E such that $\|p - q\| < \varepsilon$, then q is a 5ε-E-projection in A and p and q are connected by a homotopy of 5ε-E-projections.*

(ii) *If A is unital and if u is an ε-E-unitary and v is an element of A_E such that $\|u - v\| < \frac{\varepsilon - \|u^* u - 1\|}{3}$, then v is an ε-E-unitary. In particular, if u is an ε-E-unitary and v is an element of A_E such that $\|u - v\| < \varepsilon$, then v is a 4ε-E-unitary in A and u and v are connected by a homotopy of 4ε-E-unitaries.*

The next lemma can be proved in the same way as [20, Lemma 1.16].

Lemma 4.6. *Let $\mathcal{E}$ be a coarse structure and let A be a unital $\mathcal{E}$-filtered C^*-algebra. Then for any ε in $(0, \frac{1}{12})$ and any E in $\mathcal{E}$ the following holds:*

(i) *Let u and v be ε-E-unitaries in A, then* $\operatorname{diag}(u, v)$ *and* $\operatorname{diag}(uv, 1)$ *are homotopic as 3ε-$2E$-unitaries in $M_2(A)$.*

(ii) *Let u be an ε-E-unitary in A, then* $\operatorname{diag}(u, u^*)$ *and I_2 are homotopic as 3ε-$2E$-unitaries in $M_2(A)$.*

For any positive integer n, we set $\mathrm{U}_n^{\varepsilon,E}(A) = \mathrm{U}^{\varepsilon,E}(M_n(A))$, $\mathrm{P}_n^{\varepsilon,E}(A) = \mathrm{P}^{\varepsilon,E}(M_n(A))$. Let us consider the inclusions

$$\mathrm{P}_n^{\varepsilon,E}(A) \hookrightarrow \mathrm{P}_{n+1}^{\varepsilon,E}(A), \quad p \mapsto \begin{pmatrix} p & 0 \\ 0 & 0 \end{pmatrix}$$

and

$$\mathrm{U}_n^{\varepsilon,E}(A) \hookrightarrow \mathrm{U}_{n+1}^{\varepsilon,E}(A), \quad u \mapsto \begin{pmatrix} u & 0 \\ 0 & 1 \end{pmatrix}.$$

This allows us to define

$$\mathrm{U}_\infty^{\varepsilon,E}(A) = \bigcup_{n\in\mathbb{N}} \mathrm{U}_n^{\varepsilon,E}(A) \quad \text{and} \quad \mathrm{P}_\infty^{\varepsilon,E}(A) = \bigcup_{n\in\mathbb{N}} \mathrm{P}_n^{\varepsilon,E}(A).$$

For a unital filtered C^*-algebra A, we define the following equivalence relations on $\mathrm{P}_\infty^{\varepsilon,E}(A) \times \mathbb{N}$ and on $\mathrm{U}_\infty^{\varepsilon,E}(A)$:

- If p and q are elements of $\mathrm{P}_\infty^{\varepsilon,R}(A)$, l and l' are positive integers, $(p, l) \sim (q, l')$ if there exist a positive integer k and an element h of $\mathrm{P}_\infty^{\varepsilon,E}(A[0, 1])$ such that $h(0) = \operatorname{diag}(p, I_{k+l'})$ and $h(1) = \operatorname{diag}(q, I_{k+l})$.
- If u and v are elements of $\mathrm{U}_\infty^{\varepsilon,E}(A)$, then $u \sim v$ if there exists an element h of $\mathrm{U}_\infty^{3\varepsilon,2E}(A[0, 1])$ such that $h(0) = u$ and $h(1) = v$.

If p is an element of $\mathrm{P}_\infty^{\varepsilon,E}(A)$ and l is an integer, we denote by $[p, l]_{\varepsilon,E}$ the equivalence class of (p, l) modulo $\sim$ and if u is an element of $\mathrm{U}_\infty^{\varepsilon,E}(A)$ we denote by $[u]_{\varepsilon,E}$ its equivalence class modulo $\sim$.

Definition 4.7. Let $\mathcal{E}$ be a coarse structure, let A be an $\mathcal{E}$-filtered C^*-algebra, let E be an element of $\mathcal{E}$ and ε be positive numbers with $\varepsilon < \frac{1}{4}$. We define

(i) $K_0^{\varepsilon,E}(A) = \mathrm{P}_\infty^{\varepsilon,E}(A) \times \mathbb{N}/\sim$ if A is unital and $K_0^{\varepsilon,E}(A) = \{[p, l]_{\varepsilon,E} \in \mathrm{P}^{\varepsilon,E}(A^+) \times \mathbb{N}/\sim$ such that $\operatorname{rank} \kappa_0(\rho_A(p)) = l\}$ if A is non-unital ($\kappa_0(\rho_A(p))$ being the spectral projection associated to $\rho_A(p)$);

(ii) $K_1^{\varepsilon,E}(A) = \mathrm{U}_\infty^{\varepsilon,E}(A)/\sim$ if A is unital and $K_1^{\varepsilon,E}(A) = \mathrm{U}_\infty^{\varepsilon,E}(A^+)/\sim$ if not.

We refer to [20, Section 1.3] for the basic properties of quantitative K-theory. In particular, $K_0^{\varepsilon,E}(A)$ turns to be an abelian group, where

$$[p, l]_{\varepsilon,E} + [p', l']_{\varepsilon,E} = [\operatorname{diag}(p, p'), l + l']_{\varepsilon,E}$$

for any $[p,l]_{\varepsilon,E}$ and $[p',l']_{\varepsilon,E}$ in $K_0^{\varepsilon,E}(A)$. According to Corollary 4.6, $K_1^{\varepsilon,E}(A)$ is equipped with a structure of abelian group such that

$$[u]_{\varepsilon,E}+[u']_{\varepsilon,E}=[\operatorname{diag}(u,u')]_{\varepsilon,E}$$

for any $[u]_{\varepsilon,E}$ and $[u']_{\varepsilon,E}$ in $K_1^{\varepsilon,E}(A)$.

If $\mathcal{E}$ is a coarse structure, we have for any $\mathcal{E}$-filtered C^*-algebra A, any E, E' in $\mathcal{E}$ and any positive numbers ε and ε' with $\varepsilon\leqslant\varepsilon'<\frac{1}{4}$ and $E\leqslant E'$ natural group homomorphisms called the structure maps:

- $\iota_0^{\varepsilon,E}\colon K_0^{\varepsilon,E}(A)\to K_0(A)$, $[p,l]_{\varepsilon,E}\mapsto[\kappa_0(p)]-[I_l]$ (where $\kappa_0(p)$ is the spectral projection associated to p);
- $\iota_1^{\varepsilon,E}\colon K_1^{\varepsilon,E}(A)\to K_1(A)$, $[u]_{\varepsilon,E}\mapsto[u]$;
- $\iota_*^{\varepsilon,E}=\iota_0^{\varepsilon,E}\oplus\iota_1^{\varepsilon,E}$;
- $\iota_0^{\varepsilon,\varepsilon',E,E'}\colon K_0^{\varepsilon,E}(A)\to K_0^{\varepsilon',E'}(A)$, $[p,l]_{\varepsilon,E}\mapsto[p,l]_{\varepsilon',E'}$;
- $\iota_1^{\varepsilon,\varepsilon',E,E'}\colon K_1^{\varepsilon,E}(A)\to K_1^{\varepsilon',E'}(A)$, $[u]_{\varepsilon,E}\mapsto[u]_{\varepsilon',E'}$;
- $\iota_*^{\varepsilon,\varepsilon',E,E'}=\iota_0^{\varepsilon,\varepsilon',E,E'}\oplus\iota_1^{\varepsilon,\varepsilon',E,E'}$.

If some of the indices E, E' or ε, ε' are equal, we shall not repeat them in $\iota_*^{\varepsilon,\varepsilon',E,E'}$. In order to avoid overloading superscript in the structure maps, we shall write $\iota_*^{-,\varepsilon',E'}$ for $\iota_*^{\varepsilon,\varepsilon',E,E'}$ when ε and E in the source are implicit, $\iota_*^{\varepsilon,E,-}$ for $\iota_*^{\varepsilon,\varepsilon',E,E'}$ when ε' and E' in the range are implicit and $\iota_*^{-,-}$ where ε and E in the source and ε' and E' in the range are both implicit.

There is the equivalent of the standard form in the setting of quantitative K-theory (see [21, Lemmas 1.7 and 1.8]). First, we deal with the even case.

Lemma 4.8. *Let $\mathcal{E}$ be a coarse structure and let A be a non-unital $\mathcal{E}$-filtered C^*-algebra. Let ε be in $(0,\frac{1}{36})$ and let E be an element in $\mathcal{E}$. Then for any x in $K_0^{\varepsilon,E}(A)$, there exist*

- *two integers k and n with $k\leqslant n$;*
- *a 9ε-E-projection q in $M_n(\widetilde{A})$*

such that $\rho_A(q)=\operatorname{diag}(I_k,0)$ and $x=[q,k]_{9\varepsilon,E}$ in $K_0^{9\varepsilon,E}(A)$.

We have a similar result in the odd case.

Lemma 4.9. *Let $\mathcal{E}$ be a coarse structure and let A be a non-unital $\mathcal{E}$-filtered C^*-algebra. Let ε be in $(0,\frac{1}{84})$ and let E be an element in $\mathcal{E}$. Then*

(i) *for any x in $K_1^{\varepsilon,E}(A)$, there exists a 21ε-E-unitary u in $M_n(A^+)$ such that $\rho_A(u)=I_n$ and $\iota_1^{\varepsilon,21\varepsilon,E}(x)=[u]_{21\varepsilon,E}$ in $K_1^{21\varepsilon,E}(A)$;*

(ii) *if u and v are two ε-E-unitaries in $M_n(A^+)$ such that $\rho_A(u)=\rho_A(v)=I_n$ and $[u]_{\varepsilon,E}=[v]_{\varepsilon,E}$ in $K_1^{\varepsilon,E}(A)$, then there exist an integer k and a homotopy $(w_t)_{t\in[0,1]}$ of 21ε-E-unitaries of $M_{n+k}(A^+)$ between $\operatorname{diag}(u,I_k)$ and $\operatorname{diag}(v,I_k)$ such that $\rho_A(w_t)=I_{n+k}$ for every t in $[0,1]$.*

Let $\mathcal{E}$ be a coarse structure and let $\phi: A \to B$ be a homomorphism of $\mathcal{E}$-filtered C^*-algebras. Then ϕ preserves ε-E-projections and ε-E-unitaries and hence ϕ induces for any E in $\mathcal{E}$ and any $\varepsilon \in (0, \frac{1}{4})$ a group homomorphism

$$\phi_*^{\varepsilon,E}: K_*^{\varepsilon,E}(A) \to K_*^{\varepsilon,E}(B).$$

Moreover, quantitative K-theory is homotopy invariant with respect to homotopies which preserve $\mathcal{E}$-propagation [20, Lemma 1.26]. There is also a quantitative version of Morita equivalence [20, Proposition 1.28]. If A is an $\mathcal{E}$-filtered C^*-algebra for some coarse structure $\mathcal{E}$ and if $\mathcal{H}$ is a separable Hilbert space, then $(\mathcal{K}(\mathcal{H}) \otimes A_E)_{E\in\mathcal{E}}$ provides a structure of $\mathcal{E}$-filtered algebra for $\mathcal{K}(\mathcal{H}) \otimes A$.

Proposition 4.10. *Let $\mathcal{E}$ be a coarse structure, let A be an $\mathcal{E}$-filtered algebra and let $\mathcal{H}$ be a separable Hilbert space, then the homomorphism*

$$A \to \mathcal{K}(\mathcal{H}) \otimes A, \quad a \mapsto \begin{pmatrix} a & & \\ & 0 & \\ & & \ddots \end{pmatrix}$$

induces a ($\mathbb{Z}_2$-graded) group isomorphism (the Morita equivalence)

$$\mathcal{M}_A^{\varepsilon,E}: K_*^{\varepsilon,E}(A) \to K_*^{\varepsilon,E}(\mathcal{K}(\mathcal{H}) \otimes A)$$

for any E in $\mathcal{E}$ and any $\varepsilon \in (0, \frac{1}{4})$.

The following observation establishes a connection between quantitative K-theory and classical K-theory (see [20, Remark 1.17]).

Proposition 4.11. *Let $\mathcal{E}$ be a coarse structure.*

(i) *Let A be an $\mathcal{E}$-filtered C^*-algebra. For any positive $\varepsilon < \frac{1}{4}$ and any element y of $K_*(A)$, there exist E in $\mathcal{E}$ and an element x of $K_*^{\varepsilon,R}(A)$ such that $\iota_*^{\varepsilon,E}(x) = y$.*

(ii) *There exists a positive number $\lambda_0 > 1$ such that for any $\mathcal{E}$-filtered C^*-algebra A, any E in $\mathcal{E}$, any ε in $(0, \frac{1}{4\lambda_0})$ and any element x of $K_*^{\varepsilon,E}(A)$ for which $\iota_*^{\varepsilon,E}(x) = 0$ in $K_*(A)$, then there exists E' in $\mathcal{E}$ with $E' \geqslant E$ such that $\iota_*^{\varepsilon,\lambda_0\varepsilon,E,E'}(x) = 0$ in $K_*^{\lambda_0\varepsilon,E'}(A)$.*

Apply to $\mathcal{G}$-orders of a locally compact groupoid provided with a Haar system, we deduce the following result.

Lemma 4.12. *Let $\mathcal{G}$ be a locally compact groupoid and let A be a $\mathcal{G}$-algebra. Then*

(i) *for every ε in $(0, \frac{1}{4})$ and any y in $K_*(A \rtimes_r \mathcal{G})$, there exist a compact $\mathcal{G}$-order $\mathcal{R}$ and an element x in $K_*^{\varepsilon,\mathcal{R}}(A \rtimes_r \mathcal{G})$ such that $\iota_*^{\varepsilon,\mathcal{R}}(x) = y$;*

(ii) *there exists $\lambda_0 \geqslant 1$ such that for any ε in $(0, \frac{1}{4\lambda_0})$, any $\mathcal{G}$-order $\mathcal{R}$ and any x in $K_*^{\varepsilon,\mathcal{R}}(A \rtimes_r \mathcal{G})$ satisfying $\iota_*^{\varepsilon,\mathcal{R}}(x) = 0$ in $K_*(A \rtimes_r \mathcal{G})$, there exist a $\mathcal{G}$-order $\mathcal{R}'$ with $\mathcal{R} \subseteq \mathcal{R}'$ such that $\iota_*^{\varepsilon,\lambda_0\varepsilon,\mathcal{R},\mathcal{R}'}(x) = 0$ in $K_*^{\lambda_0\varepsilon,\mathcal{R}'}(A \rtimes_r \mathcal{G})$. The constant λ_0 depends neither on A nor on $\mathcal{G}$. Moreover, if $\mathcal{R}$ is compact, then $\mathcal{R}'$ can be chosen compact.*

Quantitative K-theory inherits many features from K-theory. In particular, there is a quantitative version of Bott periodicity and of the six-term exact sequence. As we shall not use this material in the paper, we will not go further on this point. More details can be found in [20, Sections 3 and 4].

4.2. Controlled Mayer–Vietoris pair

The concept of controlled Mayer–Vietoris pair was introduced in [21] to streamline the "cut-and-pasting" technology developed by Yu in [32] to prove the Novikov conjecture for groups with finite asymptotic dimension. It was then extended in [6] to the general setting of C^*-algebras filtered by a coarse structure. It gives rise to a controlled exact sequence that allows to compute the K-theory by letting the propagation go to infinity.

Definition 4.13. Let $\mathcal{E}$ be a coarse structure, let A be an $\mathcal{E}$-filtered C^*-algebra, let E be an element of $\mathcal{E}$ and let Δ be a closed linear subspace of A_E. Then a sub-C^*-algebra B of A is called an E-controlled Δ-neighborhood-C^*-algebra if

- B is filtered by $(B \cap A_{E'})_{E' \in \mathcal{E}}$;
- $\Delta + A_{5E} \cdot \Delta + \Delta \cdot A_{5E} + A_{5E} \cdot \Delta \cdot A_{5E} \subseteq B$.

Definition 4.14. Let $\mathcal{E}$ be a coarse structure, let A be an $\mathcal{E}$-filtered C^*-algebra, let E be an element of $\mathcal{E}$ and let c be a positive number. A completely coercive decomposition pair of order E for A is a pair (Δ_1, Δ_2) of closed linear subspaces of A_E such that for any E' in $\mathcal{E}$ with $E' \leqslant E$, for any integer n and for any x in $M_n(A_{E'})$, there exists x_1 in $M_n(\Delta_1 \cap A_{E'})$ and x_2 in $M_n(\Delta_2 \cap A_{E'})$, both with norm at most $c\|x\|$ and such that $x = x_1 + x_2$. The positive number c is called the coercivity of (Δ_1, Δ_2).

Definition 4.15. Let S_1 and S_2 be two subsets of a C^*-algebra A. The pair (S_1, S_2) is said to have complete intersection approximation property (CIA) if there exists $c > 0$ such that for any positive number ε, any integer n, any $x \in M_n(S_1)$ and any $y \in M_n(S_2)$ with $\|x - y\| < \varepsilon$, there exists $z \in M_n(S_1 \cap S_2)$ satisfying

$$\|z - x\| < c\varepsilon, \quad \|z - y\| < c\varepsilon.$$

The positive number c is called the coercivity of the pair (S_1, S_2).

Definition 4.16. Let $\mathcal{E}$ be a coarse structure, let A be an $\mathcal{E}$-filtered C^*-algebra and let E be an element in $\mathcal{E}$. An E-controlled Mayer–Vietoris pair for A is a quadruple $(\Delta_1, \Delta_2, A_{\Delta_1}, A_{\Delta_2})$ such that for some positive number c the following holds:

(i) (Δ_1, Δ_2) is a completely coercive decomposition pair for A of order E with coercivity c;

(ii) A_{Δ_i} is an E-controlled Δ_i-neighborhood-C^*-algebra for $i = 1, 2$;

(iii) the pair $(A_{\Delta_1,E'}, A_{\Delta_2,E'})$ has the CIA property with coercivity c for any E' in $\mathcal{E}$.

The positive number c is called the *coercivity* of the E-controlled Mayer–Vietoris pair $(\Delta_1, \Delta_2, A_{\Delta_1}, A_{\Delta_2})$.

Remark 4.17. In the above definition,

(i) $(\Delta_1 \cap A_{E'}, \Delta_2 \cap A_{E'}, A_{\Delta_1}, A_{\Delta_2})$ is an E'-controlled Mayer–Vietoris pair for any E' in $\mathcal{E}$ with $E' \leqslant E$ with same coercivity as $(\Delta_1, \Delta_2, A_{\Delta_1}, A_{\Delta_2})$.

(ii) $A_{\Delta_1} \cap A_{\Delta_2}$ is $\mathcal{E}$-filtered by $(A_{\Delta_1,E} \cap A_{\Delta_2,E})_{E\in\mathcal{E}}$.

(iii) If A is a unital, we will view $A^+_{\Delta_1}$ the unitarization of A_{Δ_1} as $A_{\Delta_1} + \mathbb{C} \cdot 1 \subseteq A$ and similarly for A_{Δ_2} and $A_{\Delta_1} \cap A_{\Delta_2}$.

For the purpose of rescaling the control and the propagation of an ε-E-projection or of an ε-E-unitary, we introduce the following concept of $\mathcal{E}$-control pair.

Definition 4.18. A control pair is a pair (λ, h), where

- λ is a positive number with $\lambda > 1$;
- $h\colon (0, \frac{1}{4\lambda}) \to \mathbb{N} \setminus \{0\}$, $\varepsilon \mapsto h_\varepsilon$ is a non-increasing map.

The set of control pairs is equipped with a partial order: $(\lambda, h) \leqslant (\lambda', h')$ if $\lambda \leqslant \lambda'$ and $h_\varepsilon \leqslant h'_\varepsilon$ for all ε in $(0, \frac{1}{4\lambda'})$.

The next proposition is the fundamental point to state the existence of Mayer–Vietoris controlled exact sequence (see [21, Propositions 2.11 and 2.12]).

Proposition 4.19. *For every positive number c, there exists a control pair (α, l) such that the following holds.*

Let $\mathcal{E}$ be a coarse structure, let A be a unital $\mathcal{E}$-filtered C^*-algebra, let E be an element in $\mathcal{E}$, let $(\Delta_1, \Delta_2, A_{\Delta_1}, A_{\Delta_2})$ be controlled Mayer–Vietoris pair for A of order E and coercivity c.

Then for any ε in $(0, \frac{1}{4\alpha})$ and any ε-E-unitary u in A homotopic to 1, there exist a positive integer k and two $\alpha\varepsilon$-$l_\varepsilon E$-unitaries w_1 and w_2 in $M_k(A)$ such that

- *$w_i - I_k$ is an element of the matrix algebra $M_k(A_{\Delta_i})$ for $i = 1, 2$;*
- *for $i = 1, 2$, there exists a homotopy $(w_{i,t})_{t\in[0,1]}$ of $\alpha\varepsilon$-$l_\varepsilon E$-unitaries between 1 and w_i such that $w_{i,t} - I_k \in M_k(A_{\Delta_i})$ for all t in $[0, 1]$;*
- *$\|\mathrm{diag}(u, I_{k-1}) - w_1 w_2\| < \alpha\varepsilon$.*

If A is a non-unital $\mathcal{E}$-filtered C^*-algebra, then the same result holds for u in A^+ such that $u - 1$ is in A and u is homotopic to 1 as an ε-E-unitary in A^+.

4.3. Applications to coercive decompositions of groupoids

In this subsection, we show that coercive decompositions of groupoids give rise to controlled Mayer–Vietoris pairs. In what follows, $\mathcal{G}$ is a locally compact groupoid equipped with a Haar system and A is a $\mathcal{G}$-algebra in the sense of Definition 3.3.

For any $\mathcal{G}$-order $\mathcal{R}$ and any open subset V of the unit space of $\mathcal{G}$, we define $A \rtimes_r \mathcal{R}_V$ as the closure of the set of elements h in $C_c(X; \mathcal{G}, r^*A)$ with support in $\mathcal{R}_V$.

Lemma 4.20. *Let $\mathcal{G}$ be a locally compact groupoid with unit space X provided with a Haar system and let A be a $\mathcal{G}$-algebra. Let V_1 and V_2 be open subsets of X with $X = V_1 \cup V_2$ and such that there exists a partition of unity subordinated to (V_1, V_2). Then $(A \rtimes_r \mathcal{R}_{V_1}, A \rtimes_r \mathcal{R}_{V_2})$ is for any $\mathcal{G}$-order $\mathcal{R}$ a completely coercive $\mathcal{R}$-decomposition pair for $A \rtimes_r \mathcal{G}$ with coercivity* 1.

Proof. Let (ϕ_1, ϕ_2) be a partition of unity for X subordinated to (V_1, V_2). Let us consider the bounded operator

$$\Lambda^s_{\phi_i}\colon A \rtimes_r \mathcal{G} \to A \rtimes_r \mathcal{G}, \quad i = 1, 2$$

of Lemma 3.5 and for any x in $A \rtimes_r \mathcal{R}$, let us set $x_i = \Lambda^s_{\phi_i}(x)$. According to Lemma 3.5, we have $x = x_1 + x_2$, $\|x_i\| \leqslant 1$ and x_i lies in $A \rtimes_r \mathcal{R}_{V_i}$ for $i = 1, 2$. Replacing A by $M_n(A)$, we get the complete coercivity. ∎

Notice that if $\mathcal{R}'$ is a $\mathcal{G}$-order with $\mathcal{R} \subseteq \mathcal{R}'$ and V is an open subset of the unit space of $\mathcal{G}$, then $A \rtimes_r \mathcal{R}_V \subseteq A \rtimes_r \mathcal{R}'_V$.

Lemma 4.21. *Let $\mathcal{G}$ be a locally compact groupoid with unit space X provided with a Haar system. Let $\mathcal{R}$ and $\mathcal{R}'$ be $\mathcal{G}$-orders such that $\mathcal{R}' \subseteq \mathcal{R}$ and let V be an open subset of X. Then*

$$A \rtimes_r \mathcal{R}'_V = A \rtimes_r \mathcal{R}_V \cap A \rtimes_r \mathcal{R}'.$$

Proof. We clearly have $A \rtimes_r \mathcal{R}'_V \subseteq A \rtimes_r \mathcal{R}_V \cap A \rtimes_r \mathcal{R}'$. Conversely, let x be an element in $A \rtimes_r \mathcal{R}_V \cap A \rtimes_r \mathcal{R}'$. Then there exist two sequences $(h_n)_{n\in\mathbb{N}}$ and $(h'_n)_{n\in\mathbb{N}}$ in $C_c(X; \mathcal{G}, r^*A)$ with support respectively in $\mathcal{R}_V$ and in $\mathcal{R}'$ converging to x. Let us set $K_n = s(\operatorname{supp} h_n)$ for any integer n and let $\phi_n\colon X \to [0, 1]$ be continuous compactly supported in V and such that $\phi_n(x) = 1$ for any x in K_n. According to Lemma 3.5, we see that

$$(h_n - h'_n \cdot \phi_n \circ s)_{n\in\mathbb{N}} = ((h_n - h'_n) \cdot \phi_n \circ s)_{n\in\mathbb{N}} = (\Lambda^s_{\phi_n}(h_n - h'_n))_{n\in\mathbb{N}}$$

converges to zero in $A \rtimes_r \mathcal{G}$ and hence $(h'_n \cdot \phi_n \circ s)_{n\in\mathbb{N}}$ is a sequence of elements in $C_c(X; \mathcal{G}, r^*A)$ with support in $\mathcal{R}'_V$ converging to x. ∎

As a consequence, we obtain the following corollary.

Corollary 4.22. *Under the assumption of Lemma 4.20, then $(A \rtimes_r \mathcal{R}_{V_1}, A \rtimes_r \mathcal{R}_{V_2})$ is for any $\mathcal{G}$-order $\mathcal{R}$ a completely coercive $\mathcal{R}$-decomposition pair for $A \rtimes_r \mathcal{G}$ with coercivity* 1.

Proof. Since for every integer n and for $\mathcal{G}$-order $\mathcal{R}'$ such that $\mathcal{R}' \subseteq \mathcal{R}$, we have

$$((A \rtimes_r \mathcal{R}') \otimes M_n(\mathbb{C})) \cap ((A \rtimes_r \mathcal{R}_{V_i}) \otimes M_n(\mathbb{C})) = (A \otimes M_n(\mathbb{C})) \rtimes_r \mathcal{R}'_{V_i}$$

for $i = 1, 2$, the result is a consequence of Lemma 4.20. ∎

Lemma 4.23. *Let $\mathcal{G}$ be a locally compact groupoid provided with a Haar system and let $\mathcal{H}$ be a relatively clopen subgroupoid of $\mathcal{G}$ with unit space Y. Then for any compactly supported continuous function $\phi: Y \to \mathbb{C}$ and for any $\mathcal{G}$-algebra A, there exists a completely positive continuous linear map $\Upsilon_\phi: A \rtimes_r \mathcal{G} \to A \rtimes_r \mathcal{H}$ such that*

(i) $\Upsilon_\phi(f) = \phi \circ r \cdot f_{/\mathcal{H}} \circ \bar{\phi} \circ s$, *for any f in $C_c(X; \mathcal{G}, r^*A)$, where $f_{/\mathcal{H}}: \mathcal{H} \to \mathbb{C}$ is the restriction of f to $\mathcal{H}$.*

(ii) Υ_ϕ *is completely bounded in norm by* $\sup_{y\in Y} |\phi(y)|^2$.

(iii) Υ_ϕ *maps $A \rtimes_r \mathcal{R}$ to $A \rtimes_r \mathcal{R}_{/\mathcal{H}}$ for any $\mathcal{G}$-order $\mathcal{R}$.*

Proof. Let us denote by $\lambda = (\lambda^x)_{x\in X}$ the Haar system for $\mathcal{G}$. Then the restriction of λ to $\mathcal{H}$ is a Haar system for $\mathcal{H}$ that we shall denote by $\lambda_{/\mathcal{H}} = (\lambda^y_{/\mathcal{H}})_{y\in Y}$. Using the inclusion $C_c(\mathcal{H}) \hookrightarrow C_c(\mathcal{G})$, we see that $L^2(\mathcal{H})$ can be viewed as a $C_0(X)$-Hilbert submodule of $L^2(\mathcal{G})$ and therefore $L^2(\mathcal{H}, A)$ is a right A-Hilbert submodule of $L^2(\mathcal{G}, A)$. Since $\mathcal{H}$ is clopen in $\mathcal{G}^Y$, we get that $\phi \circ r: \mathcal{H} \to \mathbb{C}$ extends to a continuous function $\psi: \mathcal{G} \to \mathbb{C}$ defined by $\psi(\gamma) = \phi \circ r(\gamma)$ if γ is in $\mathcal{H}$ and $\psi(\gamma) = 0$ else. We have $\operatorname{supp}\psi \subseteq \mathcal{H}$ and $|\psi(\gamma)| \leqslant M$ for any γ in $\mathcal{H}$ with $M = \sup_{y\in Y} |\phi(y)|$. Define $T_\psi: L^2(\mathcal{G}) \to L^2(\mathcal{G})$ as the unique bounded operator extending the map

$$C_c(\mathcal{G}) \to C_c(\mathcal{G}), \quad \xi \mapsto \psi\xi.$$

Then T_ψ has operator norm bounded by M and $\operatorname{Im} T_\psi \subseteq L^2(\mathcal{H})$. In consequence $T_\psi \otimes \mathrm{Id}_A$ maps $L^2(\mathcal{G}, A)$ to $L^2(\mathcal{H}, A)$. Consider the map

$$\Upsilon_\phi: A \rtimes_r \mathcal{G} \to \mathcal{L}(L^2(\mathcal{H}, A)), \quad x \mapsto T_{\bar{\psi}} \cdot x \cdot T_\psi.$$

Since $T^*_\psi = T_{\bar{\psi}}$, we deduce that Υ_ϕ is a positive operator with norm bounded by M^2. By replacing the C^*-algebra A in the above formula by $M_n(A)$ for any positive integer n, we obtain the complete positivity and boundedness statements. Moreover, for any f in $C_c(X; \mathcal{G}, r^*A)$, any ξ in $C_c(Y; \mathcal{H}, s^*A_{/Y})$ and any γ in $\mathcal{H}$, we have

$$\begin{aligned}
(\Upsilon_\phi(f)\cdot\xi)(\gamma) &= \bar{\psi}(\gamma) \int_{\mathcal{G}^{r(\gamma)}} \gamma^{-1}(f(\gamma'))\psi(\gamma'^{-1}\gamma)\xi(\gamma'^{-1}\gamma)\, d\lambda^{r(\gamma)}(\gamma') \\
&= \bar{\psi}(\gamma) \int_{\mathcal{H}^{r(\gamma)}} \gamma^{-1}(f(\gamma'))\psi(\gamma'^{-1}\gamma)\xi(\gamma'^{-1}\gamma)\, d\lambda^{r(\gamma)}_{/\mathcal{H}}(\gamma') \\
&= \bar{\phi} \circ r(\gamma) \int_{\mathcal{H}^{r(\gamma)}} \gamma^{-1}(f(\gamma'))\phi \circ s(\gamma')\xi(\gamma'^{-1}\gamma)\, d\lambda^{r(\gamma)}_{/\mathcal{H}}(\gamma') \\
&= \int_{\mathcal{H}^{r(\gamma)}} \gamma^{-1}(g(\gamma'))\xi(\gamma'^{-1}\gamma)\, d\lambda^{r(\gamma)}_{/\mathcal{H}}(\gamma')
\end{aligned}$$

with

$$g(\gamma) = \bar{\phi} \circ r(\gamma) \cdot \phi \circ s(\gamma) \cdot f(\gamma) = \bar{\psi}(\gamma) \cdot \psi(\gamma^{-1}) \cdot f(\gamma) \tag{4.1}$$

for any γ in $\mathcal{H}$. Moreover, g has support in $\operatorname{supp}\psi \cap \operatorname{supp} f \cap \operatorname{supp}^{-1}\psi$. Since $\operatorname{supp}\psi$ is closed in $\mathcal{G}$ and contained in $\mathcal{H}$, we deduce that g is in $C_c(Y; \mathcal{H}, r^*A)$. Hence Υ_ϕ maps

$C_c(X;\mathcal{G},r^*A)$ to $A\rtimes_r\mathcal{H}$ and by continuity maps $A\rtimes_r\mathcal{G}$ to $A\rtimes_r\mathcal{H}$. It is then clear that Υ_ϕ satisfies the required conditions. ∎

Remark 4.24. (i) According to equation (4.1) and since $\mathcal{H}$ is open in $\mathcal{G}$, we see that if f is in $C_c(X;\mathcal{G},r^*A)$, then $\Upsilon_\phi(f)$ is supported in $\mathcal{H}\cap\operatorname{supp} f$.

(ii) Let B be any C^*-algebra and let us consider the spatial tensor product $A\rtimes_r\mathcal{G}\otimes B$. Since Υ_ϕ is completely positive and completely bounded, we deduce that there is a well-defined positive and bounded map (with the same bound as the complete bound of Υ_ϕ)

$$\Upsilon_\phi\otimes\mathrm{Id}_B\colon A\rtimes_r\mathcal{G}\otimes B\to A\rtimes_r\mathcal{G}\otimes B$$

defined on elementary tensor by

$$(\Upsilon_\phi\otimes\mathrm{Id}_B)(x\otimes b)=\Upsilon_\phi(x)\otimes b$$

for any x in $A\rtimes_r\mathcal{G}$ and any b in B.

Corollary 4.25. *Let $\mathcal{H}$ be a relatively clopen subgroupoid of a locally compact groupoid $\mathcal{G}$, let $\mathcal{R}$ be a $\mathcal{G}$-order and let V be an open subset of X. Then we have*

$$(A\rtimes_r\mathcal{H})\cap(A\rtimes_r\mathcal{R})=A\rtimes_r\mathcal{R}_{/\mathcal{H}}$$

for any $\mathcal{G}$-algebra A.

Proof. We clearly have $A\rtimes_r\mathcal{R}_{/\mathcal{H}}\subseteq(A\rtimes_r\mathcal{H})\cap(A\rtimes_r\mathcal{R})$. Conversely, let x be an element in $(A\rtimes_r\mathcal{H})\cap(A\rtimes_r\mathcal{R})$. Then there exist two sequences $(h_n)_{n\in\mathbb{N}}$ and $(h'_n)_{n\in\mathbb{N}}$ in $C_c(X;\mathcal{G},r^*A)$ with support respectively in $\mathcal{H}$ and in $\mathcal{R}$ converging to x. Let us set $K_n=s(\operatorname{supp} h_n)\cup r(\operatorname{supp} h_n)$ for any integer n and let $\phi_n\colon X\to[0,1]$ be a continuous function compactly supported in the unit space of $\mathcal{H}$ and such that $\phi_n(x)=1$ for any x in K_n. According to Lemma 4.23, we see that

$$(h_n-\Upsilon_{\phi_n}(h'_n))_{n\in\mathbb{N}}=(\Upsilon_{\phi_n}(h_n-h'_n))_{n\in\mathbb{N}}$$

converges to zero in $A\rtimes_r\mathcal{G}$. In view of the first point of Remark 4.24, we deduce that $\Upsilon_{\phi_n}(h'_n)$ has compact support in $\mathcal{R}\cap\mathcal{H}=\mathcal{R}_{/\mathcal{H}}$, and thus $(\Upsilon_{\phi_n}(h'_n))_{n\in\mathbb{N}}$ is a sequence in $C_c(X;\mathcal{G},r^*A)$ with support in $\mathcal{R}_{/\mathcal{H}}$ converging to x and hence x belongs to $A\rtimes_r\mathcal{R}_{/\mathcal{H}}$. ∎

Corollary 4.26. *Let $\mathcal{G}$ be a locally compact groupoid with unit space X provided with a Haar system. Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be relatively clopen subgroupoids of $\mathcal{G}$. Then the following holds:*

(i) $A\rtimes_r(\mathcal{H}_1\cap\mathcal{H}_2)=(A\rtimes_r\mathcal{H}_1)\cap(A\rtimes_r\mathcal{H}_2)$*;*

(ii) $A\rtimes_r\mathcal{R}_{/\mathcal{H}_1\cap\mathcal{H}_2}=(A\rtimes_r\mathcal{R}_{/\mathcal{H}_1})\cap(A\rtimes_r\mathcal{R}_{/\mathcal{H}_2})$ *for any $\mathcal{G}$-order $\mathcal{R}$.*

Proof. Let us prove the first point. We clearly have

$$A\rtimes_r(\mathcal{H}_1\cap\mathcal{H}_2)\subseteq(A\rtimes_r\mathcal{H}_1)\cap(A\rtimes_r\mathcal{H}_2).$$

Conversely, let x be an element in $(A \rtimes_r \mathcal{H}_1) \cap (A \rtimes_r \mathcal{H}_2)$. Then there exist two sequences $(h_n)_{n\in\mathbb{N}}$ and $(h'_n)_{n\in\mathbb{N}}$ in $C_c(X;\mathcal{G}, r^*A)$ with support respectively in $\mathcal{H}_1$ and $\mathcal{H}_2$ converging to x. Let us set $K_n = s(\operatorname{supp} h_n) \cup r(\operatorname{supp} h_n)$ for any integer n and let $\phi_n\colon X \to [0,1]$ be a continuous function compactly supported in the unit space of $\mathcal{H}_1$ and such that $\phi_n(x) = 1$ for any x in K_n. According to Lemma 4.23, we see that

$$(h_n - \Upsilon_{\phi_n}(h'_n))_{n\in\mathbb{N}} = (\Upsilon_{\phi_n}(h_n - h'_n))_{n\in\mathbb{N}}$$

converges to zero in $A \rtimes_r \mathcal{G}$. In view of the first point of Remark 4.24, we deduce that $\Upsilon_{\phi_n}(h'_n)$ has compact support in $\mathcal{H}_1 \cap \mathcal{H}_2$, and thus that $(\Upsilon_{\phi_n}(h'_n))_{n\in\mathbb{N}}$ is a sequence in $C_c(X;\mathcal{G}, r^*A)$ with support in $\mathcal{H}_1 \cap \mathcal{H}_2$ converging to x and hence x belongs to $A \rtimes_r (\mathcal{H}_1 \cap \mathcal{H}_2)$. To prove the second point, let us observe that according to Corollary 4.25 we have

$$A \rtimes_r \mathcal{R}_{/\mathcal{H}_1\cap\mathcal{H}_2} = A \rtimes_r \mathcal{R} \cap A \rtimes_r (\mathcal{H}_1 \cap \mathcal{H}_2).$$

The result is then a consequence of the first point. ∎

Remark 4.27. The proof of the first point only requires $\mathcal{H}_1$ to be relatively clopen.

Theorem 4.28. *Let $\mathcal{G}$ be a locally compact groupoid provided with a Haar system, let A be a $\mathcal{G}$-algebra and let $\mathcal{R}$ and $\mathcal{R}'$ be $\mathcal{G}$-orders such that $\mathcal{R}^{*6} \subseteq \mathcal{R}'$. Assume that $(V_1, V_2, \mathcal{H}_1, \mathcal{H}_2)$ is a coercive $\mathcal{R}'$-decomposition for $\mathcal{G}$. Then*

$$(A \rtimes_r \mathcal{R}_{V_1}, A \rtimes_r \mathcal{R}_{V_2}, A \rtimes_r \mathcal{H}_1, A \rtimes_r \mathcal{H}_2)$$

is an $\mathcal{R}$-controlled Mayer–Vietoris pair with coercivity 2.

Proof. According to Corollary 4.22, $(A \rtimes_r \mathcal{R}_{V_1}, A \rtimes_r \mathcal{R}_{V_2})$ is a completely coercive $\mathcal{R}$-decomposition pair for $A \rtimes_r \mathcal{G}$ with coercivity 1. Let us prove that $A \rtimes_r \mathcal{H}_i$ is for $i = 1, 2$ an $\mathcal{R}$-controlled $A \rtimes_r \mathcal{R}_{V_i}$-neighborhood-$C^*$-algebra. By Lemma 4.25, we see that the C^*-algebra $A \rtimes_r \mathcal{H}_i$ is filtered by

$$((A \rtimes_r \mathcal{H}_i) \cap (A \rtimes_r \mathcal{R}))_{\mathcal{R}\in\mathcal{E}_\mathcal{G}} = (A \rtimes_r \mathcal{R}_{/\mathcal{H}_i})_{\mathcal{R}\in\mathcal{E}_\mathcal{G}}.$$

Since $\mathcal{R}^{*6} \subseteq \mathcal{R}'$, $\mathcal{R}'_{V_i} \subseteq \mathcal{H}_i$ and $\mathcal{H}_i$ is a subgroupoid of $\mathcal{G}$, we see that

- $\mathcal{R}_{V_i} \subseteq \mathcal{H}_i$;
- $\mathcal{R}^{*5} \cdot \mathcal{R}_{V_i} \subseteq \mathcal{H}_i$;
- $\mathcal{R}_{V_i} \cdot \mathcal{R}^{*5} \subseteq \mathcal{H}_i$;
- $\mathcal{R}^{*5} \cdot \mathcal{R}_{V_i} \cdot \mathcal{R}^{*5} \subseteq \mathcal{H}_i$

and hence

- $A \rtimes_r \mathcal{R}_{V_i} \subseteq A \rtimes_r \mathcal{H}_i$;
- $A \rtimes_r \mathcal{R}_{V_i} \cdot A \rtimes_r \mathcal{R}^{*5} \subseteq A \rtimes_r \mathcal{H}_i$;
- $A \rtimes_r \mathcal{R}^{*5} \cdot A \rtimes_r \mathcal{R}_{V_i} \subseteq A \rtimes_r \mathcal{H}_i$;
- $A \rtimes_r \mathcal{R}^{*5} \cdot A \rtimes_r \mathcal{R}_{V_i} \cdot A \rtimes_r \mathcal{R}^{*5} \subseteq A \rtimes_r \mathcal{H}_i$.

This proves that $A \rtimes_r \mathcal{H}_i$ is an $\mathcal{R}$-controlled $A \rtimes_r \mathcal{R}_{V_i}$-neighborhood-$C^*$-algebra.

Let us prove that $(A \rtimes_r \mathcal{H}_1, A \rtimes_r \mathcal{H}_2)$ satisfies the CIA property with coercivity 2. Up to replacing A by $A \otimes M_n(\mathbb{C})$, it is enough to show that for every positive number and any x_1 in $A \rtimes_r \mathcal{R}_{/\mathcal{H}_1}$ and x_2 in $A \rtimes_r \mathcal{R}_{/\mathcal{H}_2}$ such that $\|x_1 - x_2\| < \varepsilon$, there exists z in $(A \rtimes_r \mathcal{R}_{/\mathcal{H}_1}) \cap (A \rtimes_r \mathcal{R}_{/\mathcal{H}_2})$ such that $\|z - x_1\| < \varepsilon$. Notice that in view of Corollary 4.26, we have

$$(A \rtimes_r \mathcal{R}_{/\mathcal{H}_1}) \cap (A \rtimes_r \mathcal{R}_{/\mathcal{H}_2}) = A \rtimes_r \mathcal{R}_{/(\mathcal{H}_1 \cap \mathcal{H}_2)}.$$

Set $\alpha = \varepsilon - \|x_1 - x_2\|$ and let h be an element in $C_c(X; \mathcal{G}, r^*A)$ with support included in $\mathcal{R}_{/\mathcal{H}_1}$ and such that $\|x_1 - h\| < \frac{\alpha}{2}$. Let $\phi\colon \mathcal{G} \to [0,1]$ be a continuous function compactly supported in the space of unit of $\mathcal{H}_1$ and such that $\phi(x) = 1$ for all x in $r(\operatorname{supp} h) \cup s(\operatorname{supp} h)$. According to Lemma 4.23, we see that $\Upsilon_\phi(h) = h$, $\Upsilon_\phi(x_2)$ belongs to $A \rtimes_r \mathcal{R}_{/\mathcal{H}_1}$ and

$$\begin{aligned}
\|x_1 - \Upsilon_\phi(x_2)\| &< \|x_1 - h\| + \|h - \Upsilon_\phi(x_2)\| \\
&< \frac{\alpha}{2} + \|\Upsilon_\phi(h) - \Upsilon_\phi(x_2)\| < \frac{\alpha}{2} + \|h - x_2\| \\
&< \frac{\alpha}{2} + \|h - x_1\| + \|x_1 - x_2\| < \frac{\alpha}{2} + \frac{\alpha}{2} + \|x_1 - x_2\| < \varepsilon.
\end{aligned}$$

But x_2 is a limit of elements of $C_c(X; \mathcal{G}, r^*A)$ with support in $\mathcal{R}_{/\mathcal{H}_2}$ and hence according to the first point of Remark 4.24, $\Upsilon_\phi(x_2)$ is also a limit of element of $C_c(\mathcal{G}, r^*A)$ with support in $\mathcal{R}_{/\mathcal{H}_2}$ and therefore $\Upsilon_\phi(x_2)$ belongs to $A \rtimes_r \mathcal{R}_{/\mathcal{H}_2}$. ∎

Remark 4.29. In view of the second point of Remark 4.24, we can show in the same way that under assumptions of Lemma 4.20, then $(A \rtimes_r \mathcal{R}_{V_1}, A \rtimes_r \mathcal{R}_{V_2}, A \rtimes_r \mathcal{H}_1, A \rtimes_r \mathcal{H}_2)$ is an $\mathcal{R}$-controlled nuclear Mayer–Vietoris pair in the sense of [21, Definition 4.8] for $A \rtimes_r \mathcal{G}$ with coercivity 2.

4.4. The Mayer–Vietoris controlled exact sequence

An $\mathcal{R}$-controlled Mayer–Vietoris pair gives rise to a controlled six-term exact sequence that computes the quantitative K-theory up to the order of the pair and up to rescaling by a control pair. In view of Theorem 4.28, it turns out that this controlled Mayer–Vietoris six-term exact sequence is a powerful tool for K-theory computations in the setting of coercive decompositions for groupoids.

Notation 4.30. Let $\mathcal{E}$ be a coarse structure, let A be an $\mathcal{E}$-filtered C^*-algebra, let E be an element in $\mathcal{E}$ and let $(\Delta_1, \Delta_2, A_{\Delta_1}, A_{\Delta_2})$ be an E-controlled Mayer–Vietoris pair for A. We denote by

- $\jmath_{\Delta_1}\colon A_{\Delta_1} \to A$;
- $\jmath_{\Delta_2}\colon A_{\Delta_2} \to A$;
- $\jmath_{\Delta_1,\Delta_2}\colon A_{\Delta_1} \cap A_{\Delta_2} \to A_{\Delta_1}$;
- $\jmath_{\Delta_2,\Delta_1}\colon A_{\Delta_1} \cap A_{\Delta_2} \to A_{\Delta_2}$

the obvious inclusion maps.

Proposition 4.31. *For every positive number c, there exists a control pair (α, l) such that for any coarse structure $\mathcal{E}$, any $\mathcal{E}$-filtered C^*-algebra A, any E in $\mathcal{E}$ and any E-controlled Mayer–Vietoris pair*

$$(\Delta_1, \Delta_2, A_{\Delta_1}, A_{\Delta_2})$$

for A with coercivity c, the following holds:

For any ε in $(0, \frac{1}{4\alpha})$, any E' in $\mathcal{E}$ such that $l_\varepsilon \cdot E' \leqslant E$, any y_1 in $K_*^{\varepsilon,E'}(A_{\Delta_1})$ and any y_2 in $K_*^{\varepsilon,E'}(A_{\Delta_2})$ such that

$$\jmath_{\Delta_1,*}^{\varepsilon,E'}(y_1) = \jmath_{\Delta_2,*}^{\varepsilon,E'}(y_2) \quad \text{in } K_*^{\varepsilon,E'}(A),$$

there exists an element x in $K_*^{\alpha\varepsilon,l_\varepsilon E'}(A_{\Delta_1} \cap A_{\Delta_2})$ such that

$$\jmath_{\Delta_1,\Delta_2,*}^{\alpha\varepsilon,l_\varepsilon E'}(x) = \iota_*^{-,\alpha\varepsilon,l_\varepsilon E'}(y_1) \quad \text{in } K_*^{\alpha\varepsilon,l_\varepsilon E'}(A_{\Delta_1}),$$

$$\jmath_{\Delta_2,\Delta_1,*}^{\alpha\varepsilon,l_\varepsilon E'}(x) = \iota_*^{-,\alpha\varepsilon,l_\varepsilon E'}(y_2) \quad \text{in } K_*^{\alpha\varepsilon,l_\varepsilon E'}(A_{\Delta_2}).$$

In other words, this means that the composition

$$K_*^{\bullet,\bullet}(A_{\Delta_1} \cap A_{\Delta_2}) \xrightarrow{(\jmath_{\Delta_1,\Delta_2,*}^{\bullet,\bullet},\, \jmath_{\Delta_2,\Delta_1,*}^{\bullet,\bullet})} K_*^{\bullet,\bullet}(A_{\Delta_1}) \oplus K_*^{\bullet,\bullet}(A_{\Delta_2}) \xrightarrow{(\jmath_{\Delta_1,*}^{\bullet,\bullet} - \jmath_{\Delta_2,*}^{\bullet,\bullet})} K_*^{\bullet,\bullet}(A)$$

is "exact at order E, up to rescaling by (α, l)" (see [21, Proposition 3.2] for a proof of this proposition). We shall see later on that this composition fits at order E into a controlled six-term exact sequence (called in [3, Theorem 8.4] E-controlled Mayer–Vietoris exact sequence).

We introduce first the quantitative boundary map of this controlled Mayer–Vietoris exact sequence (see [21, Lemma 3.3]).

Lemma 4.32. *For every positive number c, there exists a control pair (λ, k) such that the following holds:*

Let $\mathcal{E}$ be a coarse structure, let A be a unital $\mathcal{E}$-filtered C^*-algebra, let E be an element in $\mathcal{E}$ and let $(\Delta_1, \Delta_2, A_{\Delta_1}, A_{\Delta_2})$ be an E-controlled Mayer–Vietoris pair for A with coercivity c. Let E' be an element in $\mathcal{E}$ such that $2 \cdot E' \leqslant E$, let ε be in $(0, \frac{1}{4\lambda^3})$, let m and n be integers and let u be in $U_n^{\varepsilon,E'}(A)$, let v be in $U_m^{\varepsilon,E'}(A)$ and let w_1, w_2 be ε-E'-unitaries in $M_{n+m}(A)$ such that

- *$w_i - I_{n+m}$ is an element in the matrix algebra $M_{n+m}(A_{\Delta_i})$ for $i = 1, 2$;*
- *$\|\mathrm{diag}(u, v) - w_1 w_2\| < \varepsilon$.*

Then,

(i) *there exists a $\lambda\varepsilon$-$k_\varepsilon E'$-projection q in $M_{n+m}(A)$ such that*
   - *$q - \mathrm{diag}(I_n, 0)$ is an element in the matrix algebra $M_{n+m}(A_{\Delta_1} \cap A_{\Delta_2})$;*
   - *$\|q - w_1^* \,\mathrm{diag}(I_n, 0) w_1\| < \lambda\varepsilon$;*
   - *$\|q - w_2 \,\mathrm{diag}(I_n, 0) w_2^*\| < \lambda\varepsilon$.*

(ii) *if q and q' are two $\lambda\varepsilon$-$k_\varepsilon E'$-projections in $M_{n+m}(A)$ that satisfy the first point, then*

$$[q,n]_{\lambda^2\varepsilon,k_\varepsilon E'} = [q',n]_{\lambda^2\varepsilon,k_\varepsilon E'}$$

in $K_0^{\lambda^2\varepsilon,k_\varepsilon E'}(A_{\Delta_1} \cap A_{\Delta_2})$.

(iii) *Let (w_1, w_2) and (w_1', w_2') be two pairs of ε-E'-unitaries in $M_{n+m}^{\varepsilon,E'}(A)$ satisfying the assumption of the lemma and let q and q' be $\lambda\varepsilon$-$k_\varepsilon \cdot E'$-projections in $M_{n+m}(A)$ that satisfy the first point relatively to respectively (w_1, w_2) and (w_1', w_2'), then*

$$[q,n]_{\lambda^3\varepsilon,2k_\varepsilon E'} = [q',n]_{\lambda^3\varepsilon,2k_\varepsilon E'}$$

in $K_0^{\lambda^3\varepsilon,2k_\varepsilon E'}(A_{\Delta_1} \cap A_{\Delta_2})$.

Remark 4.33. We have a similar statement in the non-unital case with u in $U_n^{\varepsilon,E'}(A^+)$ and v in $U_m^{\varepsilon,E'}(A^+)$ such that $u - I_n$ and $v - I_m$ have coefficients in A.

We recall now the definition of the quantitative boundary map associated to a controlled Mayer–Vietoris pair. Let $\mathcal{E}$ be a coarse structure, let A be an $\mathcal{E}$-filtered C^*-algebra and let $(\Delta_1, \Delta_2, A_{\Delta_1}, A_{\Delta_2})$ be an E-controlled Mayer–Vietoris pair for A with coercivity c. Assume first that A is unital.

Let (α, l) be a control pair as is Proposition 4.19. For any ε in $(0, \frac{1}{4\alpha})$, any E' in $\mathcal{E}$ such that $2E' \leqslant E$ and any ε-E'-unitary u in $M_n(A)$, let v be an ε-E'-unitary in some $M_m(A)$ such that $\operatorname{diag}(u, v)$ is homotopic to I_{n+m} as a 3ε-$2E'$-unitary in $M_{n+m}(A)$, we can take for instance $v = u^*$ (see Lemma 4.6). According to Proposition 4.19 and up to replacing v by $\operatorname{diag}(v, I_k)$ for some integer k, there exist two $3\alpha\varepsilon$-$2l_{3\varepsilon}E'$-unitaries w_1 and w_2 in $M_{n+m}(A)$ such that

- $w_i - I_{n+m}$ is an element in the matrix algebra $M_{n+m}(A_{\Delta_i})$ for $i = 1, 2$;
- for $i = 1, 2$, there exists a homotopy $(w_{i,t})_{t\in[0,1]}$ of $3\alpha\varepsilon$-$2l_{3\varepsilon}E'$-unitaries between 1 and w_i such that $w_{i,t} - I_{n+m}$ is an element in the matrix algebra $M_{n+m}(A_{\Delta_i})$ for all t in $[0, 1]$.
- $\|\operatorname{diag}(u, v) - w_1 w_2\| < 3\alpha\varepsilon$.

Let (λ, k) be the control pair of Lemma 4.32 (recall that (λ, k) depends only on the coercivity c). Then if ε is in $(0, \frac{1}{12\alpha\lambda^3})$, there exists a $3\alpha\lambda\varepsilon$-$2l_{3\varepsilon}k_{3\alpha\varepsilon}E'$-projection q in $M_{n+m}(A)$ such that

- $q - \operatorname{diag}(I_n, 0)$ is an element in the matrix algebra
$$M_{n+m}(A_{\Delta_1 \cap A_{\Delta_2}});$$
- $\|q - w_1^* \operatorname{diag}(I_n, 0) w_1\| < 3\alpha\lambda\varepsilon$;
- $\|q - w_2 \operatorname{diag}(I_n, 0) w_2^*\| < 3\alpha\lambda\varepsilon$.

In view of the second and the third points of Lemma 4.32, the class $[q,n]_{3\alpha\lambda^3\varepsilon,4l_{3\varepsilon}k_{3\alpha\varepsilon}E'}$ in

$$K_0^{3\alpha\lambda^3\varepsilon,4l_{3\varepsilon}k_{3\alpha\varepsilon}E'}(A_{\Delta_1} \cap A_{\Delta_2})$$

does not depend on the choice of w_1, w_2 or q. Set then $\alpha_c = 3\alpha\lambda^3$ and

$$k_c\colon \left(0, \frac{1}{4\alpha_c}\right) \to \mathbb{N} \setminus \{0\}, \quad \varepsilon \mapsto 4l_{3\varepsilon}k_{3\alpha\varepsilon}$$

and define $\partial^{\varepsilon,E'}_{\Delta_1,\Delta_2,1}([u]_{\varepsilon,E'}) = [q,n]_{\alpha_c\varepsilon,k_cE'}$. Then for any ε in $(0,\frac{1}{4\alpha_c})$ and any E' in $\mathcal{E}$ such that $k_{c,\varepsilon}E' \leqslant E$, the morphism

$$\partial^{\varepsilon,E'}_{\Delta_1,\Delta_2,1}\colon K_1^{\varepsilon,E'}(A) \to K_0^{\alpha_c\varepsilon,k_cE'}(A_{\Delta_1} \cap A_{\Delta_2})$$

is well defined.

In the non-unital case $\partial^{\varepsilon,s}_{\Delta_1,\Delta_2,1}$ is defined similarly by noticing that in view of Lemma 4.9 and up to rescaling ε, every element x in $K_1^{\varepsilon,E}(A)$ is the class of an ε-E-unitary u in some $M_n(A^+)$ such that $u - I_n$ has coefficients in A. It is straightforward to check that $\partial^{\bullet,\bullet}_{\Delta_1,\Delta_2,1}$ is compatible with the structure morphisms, i.e.,

$$\iota_*^{-,\alpha_c\varepsilon'',k_{c,\varepsilon''}E''} \circ \partial^{\varepsilon,E'}_{\Delta_1,\Delta_2,1} = \partial^{\varepsilon'',E''}_{\Delta_1,\Delta_2,1} \circ \iota_*^{\varepsilon',E',-}$$

for any ε' and ε'' in $(0,\frac{1}{4\alpha_c})$ and any E' and E'' in $\mathcal{E}$ with $E' \leqslant E''$ and $k_{c,\varepsilon'}E' \leqslant k_{c,\varepsilon''}E'' \leqslant E$.

In the even case, the quantitative boundary map associated to a controlled Mayer–Vietoris pair is defined by using controlled Bott periodicity [6, Section 2]. Up to rescaling the control pair (α_c, k_c), we obtain for any ε in $(0,\frac{1}{4\alpha_c})$ and any E' in $\mathcal{E}$ such that $k_{c,\varepsilon}E' \leqslant E$, the morphism

$$\partial^{\varepsilon,E'}_{\Delta_1,\Delta_2,0}\colon K_0^{\varepsilon,E'}(A) \to K_1^{\alpha_c\varepsilon,k_{c,\varepsilon}E'}(A_{\Delta_1} \cap A_{\Delta_2}).$$

We set then

$$\partial^{\varepsilon,E'}_{\Delta_1,\Delta_2,*} = \partial^{\varepsilon,E'}_{\Delta_1,\Delta_2,0} \oplus \partial^{\varepsilon,E'}_{\Delta_1,\Delta_2,1}.$$

Then

$$\partial^{\varepsilon,E'}_{\Delta_1,\Delta_2,*}\colon K_*^{\varepsilon,E'}(A) \to K_{*+1}^{\alpha_c\varepsilon,k_cE'}(A_{\Delta_1} \cap A_{\Delta_2})$$

is a morphism of degree 1 compatible with the structure morphisms called the ε-E'-quantitative Mayer–Vietoris boundary map.

Notice that the quantitative boundary map associated to an E-controlled Mayer–Vietoris pair is natural in the following sense: let $\mathcal{E}$ be a coarse structure, let A and B be $\mathcal{E}$-filtered C^*-algebras, let E be an element in $\mathcal{E}$, let $(\Delta_1, \Delta_2, A_{\Delta_1}, A_{\Delta_2})$ and $(\Delta'_1, \Delta'_2, B_{\Delta'_1}, B_{\Delta'_2})$ be respectively E-controlled Mayer–Vietoris pairs for A and B with coercivity c and let $f\colon A \to B$ be a homomorphism of $\mathcal{E}$-filtered C^*-algebras such that $f(\Delta_1) \subseteq \Delta'_1$, $f(\Delta_2) \subseteq \Delta'_2$, $f(A_{\Delta_1}) \subseteq B_{\Delta'_1}$ and $f(A_{\Delta_2}) \subseteq B_{\Delta'_2}$. Then we have

$$f^{\alpha_c\varepsilon,k_{c,\varepsilon}E'}_{/A_{\Delta_1}\cap A_{\Delta_2},*} \circ \partial^{\varepsilon,E'}_{\Delta_1,\Delta_2,*} = \partial^{\varepsilon,E'}_{\Delta'_1,\Delta'_2,*} \circ f_*^{\varepsilon,E'} \tag{4.2}$$

for any ε in $(0, \frac{1}{4\alpha_c})$ and any E' in $\mathcal{E}$ with $k_{c,\varepsilon}E' \leqslant E$, where

$$f_{/A_{\Delta_1}\cap A_{\Delta_2}}\colon A_{\Delta_1}\cap A_{\Delta_2} \to B_{\Delta'_1}\cap B_{\Delta'_2}$$

is the restriction of f to $A_{\Delta_1}\cap A_{\Delta_2}$.

We now investigate the controlled exactness at the domain for the quantitative boundary map associated to a controlled Mayer–Vietoris pair. We start with the following lemma which will play a key role in the proof of the main theorem (see [21, Lemma 3.5] for a proof).

Lemma 4.34. *There exists a control pair (λ, l) such that the following statement is satisfied:*

- *for any coarse structure $\mathcal{E}$, any unital $\mathcal{E}$-filtered C^*-algebra A and any subalgebras A_1 and A_2 of A such that A_1, A_2 and $A_1\cap A_2$ are filtered by $(A_1\cap A_E)_{E\in\mathcal{E}}$, $(A_2\cap A_E)_{E\in\mathcal{E}}$ and $(A_1\cap A_2\cap A_E)_{E\in\mathcal{E}}$, respectively;*
- *for any positive number ε with $\varepsilon < \frac{1}{4\lambda}$, any E in $\mathcal{E}$, any integers n and m and any ε-E-unitaries u_1 in $M_n(A)$ and u_2 in $M_m(A)$;*
- *for any ε-E-unitaries v_1 and v_2 in $M_{n+m}(A_1^+)$ and $M_{n+m}(A_2^+)$, respectively, such that*
  - $\|\mathrm{diag}(u_1, u_2) - v_1v_2\| < \varepsilon$;
  - *there exists an ε-E-projection q in $M_{n+m}(A)$ such that $q - \mathrm{diag}(I_n, 0)$ is in $M_{n+m}(A_1\cap A_2)$, $\|q - v_1^*\,\mathrm{diag}(I_n,0)v_1\| < \varepsilon$ and $[q,n]_{\varepsilon,E} = 0$ in $K_0^{\varepsilon,E}(A_1\cap A_2)$.*

Then there exist an integer k and $\lambda\varepsilon$-$l_\varepsilon E$-unitaries w_1 and w_2 respectively in $M_{n+k}(A_1^+)$ and $M_{n+k}(A_2^+)$ such that

$$\|\mathrm{diag}(u_1, I_k) - \mathrm{diag}(w_1w_2)\| < \lambda\varepsilon.$$

Moreover, if $v_i - I_{n+k}$ lies in $M_{n+k}(A_i)$ for $i = 1, 2$, then w_1 and w_2 can be chosen such that $w_i - I_{n+k}$ lies in $M_{n+m}(A_i)$ for $i = 1, 2$.

As a consequence, we deduce the controlled exactness at the domain for the quantitative boundary map associated to a controlled Mayer–Vietoris pair (see [21, Corollary 3.6]). Moreover, this controlled exactness is persistent at any order in the sense that the vanishing may occur (up to compose with the structure maps) at any order (and not just at the order of the Mayer–Vietoris pair).

Corollary 4.35. *For any positive number c, there exists a control pair (λ, l) such that the following statement is satisfied:*

- *for any coarse structure $\mathcal{E}$ and any $\mathcal{E}$-filtered C^*-algebra A;*
- *for any E in $\mathcal{E}$ and any E-controlled Mayer–Vietoris pair $(\Delta_1, \Delta_2, A_{\Delta_1}, A_{\Delta_2})$ for A with coercivity c;*
- *for any positive numbers ε' and ε'' with $0 < \alpha_c\varepsilon' \leqslant \varepsilon'' < \frac{1}{4\lambda}$ and any E' and E'' in $\mathcal{E}$ with $k_{c,\varepsilon'}E' \leqslant E$ and $k_{c,\varepsilon'}E' \leqslant E''$.*

Then for any y in $K_*^{\varepsilon,E'}(A)$ such that

$$\iota_*^{-,\varepsilon'',E''} \circ \partial_{\Delta_1,\Delta_2,*}^{\varepsilon',E'}(y) = 0$$

in $K_{*+1}^{\varepsilon'',E''}(A_{\Delta_1} \cap A_{\Delta_2})$, there exist x_1 in $K_*^{\lambda\varepsilon'',l_{\varepsilon''}E''}(A_{\Delta_1})$ and x_2 in $K_*^{\lambda\varepsilon'',l_{\varepsilon''}E''}(A_{\Delta_2})$ such that

$$\iota_*^{-,\lambda\varepsilon'',l_{\varepsilon''}E''}(y) = \jmath_{\Delta_1,*}^{\lambda\varepsilon'',l_{\varepsilon''}E''}(x_1) - \jmath_{\Delta_2,*}^{\lambda\varepsilon'',l_{\varepsilon''}E''}(x_2).$$

We now investigate the controlled exactness at the codomain of the quantitative boundary map associated to a controlled Mayer–Vietoris pair. We start with the following lemma which will play a key role in the proof of the main theorem (see [21, Lemma 3.8] for the proof).

Lemma 4.36. *There exists a control pair (λ, h) such that the following holds:*

- *Let $\mathcal{E}$ be a coarse structure, let A be a unital $\mathcal{E}$-filtered C^*-algebra and let A_1 and A_2 be subalgebras of A such that A_1, A_2 and $A_1 \cap A_2$ are filtered by $(A_1 \cap A_E)_{E\in\mathcal{E}}$, $(A_2 \cap A_E)_{E\in\mathcal{E}}$ and $(A_1 \cap A_2 \cap A_E)_{E\in\mathcal{E}}$, respectively;*
- *let ε be in $(0, \frac{1}{4\lambda})$ and let E be in $\mathcal{E}$;*
- *let n and N be positive integers such that $n \leqslant N$, and let p be an ε-E-projection in $M_N((A_1 \cap A_2)^+)$ such that $\rho_{A_1\cap A_2}(p) = \mathrm{diag}(I_n, 0)$.*

  Assume that
- *p is homotopic to $\mathrm{diag}(I_n, 0)$ as an ε-E-projection in $M_N(A_1^+)$;*
- *p is homotopic to $\mathrm{diag}(I_n, 0)$ as an ε-E-projection in $M_N(A_2^+)$.*

Then there exist an integer N' with $N' \geqslant N$ and four $\lambda\varepsilon$-$h_\varepsilon E$-unitaries w_1 and w_2 in $M_{N'}(A)$, u in $M_n(A)$ and v in $M_{N'-n}(A)$ such that

- *$w_i - I_{N'}$ is an element in $M_{N'}(A_i)$ for $i = 1, 2$;*
- *$\|w_1^* \mathrm{diag}(I_n, 0)w_1 - \mathrm{diag}(p, 0)\| < \lambda\varepsilon$ and $\|w_2 \mathrm{diag}(I_n, 0)w_2^* - \mathrm{diag}(p, 0)\| < \lambda\varepsilon$.*
- *for $i = 1, 2$, w_i is connected to $I_{N'}$ by a homotopy of $\lambda\varepsilon$-$h_\varepsilon E$-unitaries $(w_{i,t})_{t\in[0,1]}$ in $M_{N'}(A)$ such that $w_{i,t} - I_{N'}$ is in $M_{N'}(A_i)$ for all t in $[0, 1]$.*
- *$\|\mathrm{diag}(u, v) - w_1w_2\| < \lambda\varepsilon$.*

As a consequence, we deduce the controlled exactness at the codomain for the quantitative boundary map associated to a controlled Mayer–Vietoris pair (see [21, Proposition 3.9]).

Proposition 4.37. *For every positive number c, there exists a control pair (α, l) such that for any coarse structure $\mathcal{E}$, any $\mathcal{E}$-filtered C^*-algebra A, any E in $\mathcal{E}$ and any E-controlled Mayer–Vietoris pair $(\Delta_1, \Delta_2, A_{\Delta_1}, A_{\Delta_2})$ for A with coercivity c, the following holds: for any ε in $(0, \frac{1}{4\lambda\alpha_c})$ and any E' in $\mathcal{E}$ with $k_{c,\lambda\varepsilon}l_\varepsilon E' \leqslant E$, any y in $K_*^{\varepsilon,E'}(A_{\Delta_1} \cap A_{\Delta_2})$ such that*

$$\jmath_{\Delta_1,\Delta_2,*}^{\varepsilon,E'}(y) = 0 \quad \textit{in } K_*^{\varepsilon,E'}(A_{\Delta_1})$$

and

$$j^{\varepsilon,E'}_{\Delta_2,\Delta_1,*}(y)=0 \quad \text{in } K^{\varepsilon,E'}_*(A_{\Delta_2}),$$

there exists an element x in $K^{\lambda\varepsilon,l_\varepsilon E'}_{*+1}(A)$ such that

$$\partial^{\lambda\varepsilon,l_\varepsilon E'}_{\Delta_1,\Delta_2,*}(x)=\iota_*^{-,\alpha_c\lambda\varepsilon,k_{c,\lambda\varepsilon}l_\varepsilon E'}(y) \quad \text{in } K^{\alpha_c\lambda\varepsilon,k_{c,\lambda\varepsilon}l_\varepsilon E'}_*(A_{\Delta_1}\cap A_{\Delta_2}).$$

Example 4.38. With notations of Theorem 4.28, we denote by

$$\begin{aligned}
j_{1,2,A}&\colon A\rtimes_r(\mathcal{H}_1\cap\mathcal{H}_2)\hookrightarrow A\rtimes_r\mathcal{H}_1,\\
j_{2,1,A}&\colon A\rtimes_r(\mathcal{H}_1\cap\mathcal{H}_2)\hookrightarrow A\rtimes_r\mathcal{H}_2,\\
j_{1,A}&\colon A\rtimes_r\mathcal{H}_1\hookrightarrow A\rtimes_r\mathcal{G},\\
j_{2,A}&\colon A\rtimes_r\mathcal{H}_2\hookrightarrow A\rtimes_r\mathcal{G}
\end{aligned}$$

the obvious inclusions; for any ε in $(0,1)$ and any $\mathcal{R}_0$ in $\mathcal{E}$ such that $k_{c,\varepsilon}\mathcal{R}_0\subseteq\mathcal{R}$, we denote by

$$\partial^{\varepsilon,\mathcal{R}_0}_{\mathcal{H}_1,\mathcal{H}_2,A,*}=\partial^{\varepsilon,\mathcal{R}_0}_{A\rtimes_r\mathcal{H}_1,A\rtimes_r\mathcal{H}_2,*}\colon K_1^{\varepsilon,\mathcal{R}_0}(A\rtimes_r\mathcal{G})\to K_0^{\alpha_c\varepsilon,k_{c,\varepsilon}\mathcal{R}_0}(A\rtimes_r(\mathcal{H}_1\cap\mathcal{H}_2))$$

the ε-$\mathcal{R}_0$-quantitative Mayer–Vietoris boundary map associated to the $\mathcal{R}_0$-controlled Mayer–Vietoris pair

$$(A\rtimes_r\mathcal{R}_{V_1},A\rtimes_r\mathcal{R}_{V_2},A\rtimes_r\mathcal{H}_1,A\rtimes_r\mathcal{H}_2).$$

5. Statement of the main result

Recall from Section 3.3 that any homomorphism of $\mathcal{G}$-algebras $f\colon A\to B$ induces a homomorphism of C^*-algebras

$$f_{\mathcal{G},*}\colon A\rtimes_r\mathcal{G}\to B\rtimes_r\mathcal{G}.$$

Theorem 5.1. *Let $\mathcal{G}$ be a locally compact groupoid provided with a Haar system and let $f\colon A\to B$ be a homomorphism of $\mathcal{G}$-algebras. Let us assume that there exists a subset $\mathcal{D}$ of relatively clopen groupoids of $\mathcal{G}$, closed under taking relatively clopen subgroupoids and such that*

(i) *$f_{\mathcal{H},*}\colon K_*(A\rtimes_r\mathcal{H})\to K_*(B\rtimes_r\mathcal{H})$ is an isomorphism for any $\mathcal{H}$ in $\mathcal{D}$*

(ii) *$\mathcal{G}$ has finite decomposition complexity with respect to $\mathcal{D}$.*

Then $f_{\mathcal{G},*}\colon K_*(A\rtimes_r\mathcal{G})\to K_*(B\rtimes_r\mathcal{G})$ is an isomorphism.

Theorem 5.1 is a consequence of the following result.

Lemma 5.2. *Let $\mathcal{G}$ be a locally compact groupoid provided with a Haar system, let $\mathcal{H}$ be a relatively clopen subgroupoid of $\mathcal{G}$, let $f\colon A\to B$ be a homomorphism of $\mathcal{G}$-algebras. Let us assume that there exists a subset $\mathcal{D}$ of relatively clopen groupoids of $\mathcal{G}$, closed under taking relatively clopen subgroupoids and such that*

(i) $f_{\mathcal{H}',*}\colon K_*(A \rtimes_r \mathcal{H}') \to K_*(B \rtimes_r \mathcal{H}')$ *is an isomorphism for any* $\mathcal{H}'$ *in* $\mathcal{D}$.

(ii) $\mathcal{H}$ *is* $\mathcal{D}$*-decomposable.*

Then $f_{\mathcal{H},*}\colon K_*(A \rtimes_r \mathcal{H}) \to K_*(B \rtimes_r \mathcal{H})$ *is an isomorphism.*

Proof. By Bott periodicity, this amounts to prove that

$$f_{\mathcal{H},*}\colon K_1(A \rtimes_r \mathcal{H}) \to K_1(B \rtimes_r \mathcal{H})$$

is an isomorphism. Let

$$\widetilde{f_{\mathcal{H}}}\colon \widetilde{A \rtimes_r \mathcal{H}} \to \widetilde{B \rtimes_r \mathcal{H}}$$

be the unitalization of $f_{\mathcal{H}}$ with $\widetilde{f_{\mathcal{H}}}$, $\widetilde{A \rtimes_r \mathcal{H}}$ and $\widetilde{B \rtimes_r \mathcal{H}}$ respectively equal to $f_{\mathcal{H}}$, $A \rtimes_r \mathcal{H}$ and $B \rtimes_r \mathcal{H}$ if $f_{\mathcal{H}}$ is already a morphism of unital C^*-algebras, and $f_{\mathcal{H}}^+$, $(A \rtimes_r \mathcal{H})^+$ and $(B \rtimes_r \mathcal{H})^+$ otherwise. Let us fix a control pair (λ, l) such that

- $\lambda \geqslant \lambda_0$, where λ_0 is the constant of Lemma 4.12;
- (λ, l) is larger than
  - the control pair corresponding to the quantitative boundary map associated to a coarse Mayer–Vietoris pair with coercivity $c = 2$ (see Section 4.4),
  - the control pairs of Proposition 4.19 and of Lemmas 4.34 and 4.36.

We proceed by using a quantitative version of the Five lemma.

5.1. Injectivity part

Let x be an element in $K_1(A \rtimes_r \mathcal{H})$ such that $f_{\mathcal{H},*}(x) = 0$ in $K_1(B \rtimes_r \mathcal{H})$. Let us show then that $x = 0$. We divide the proof into five steps.

Step I. Let us fix a positive number ε in $(0, \frac{1}{256\lambda^5})$. According to Lemma 4.12, there exist, up to stabilization, a compact $\mathcal{G}$-order $\mathcal{R}_0$ and an ε-$\mathcal{R}_0$-unitary in $1 + A \rtimes_r \mathcal{H}$ such that

- $\iota_*^{\varepsilon,\mathcal{R}_0}([u]_{\varepsilon,\mathcal{R}_0}) = x$;
- $[\widetilde{f_{\mathcal{H}}}(u)]_{\varepsilon,\mathcal{R}_0} = 0$ in $K_1^{\varepsilon,\mathcal{R}_0}(B \rtimes_r H)$.

Let $(V_1, V_2, \mathcal{H}_1, \mathcal{H}_2)$ be a $(6l_\varepsilon \cdot \mathcal{R}_0)$-decomposition of $\mathcal{H}$ with $\mathcal{H}_1$ and $\mathcal{H}_2$ in $\mathcal{D}$. In view of Theorem 4.28, we see that

$$(A \rtimes_r \mathcal{R}_{V_1}, A \rtimes_r \mathcal{R}_{V_2}, A \rtimes_r \mathcal{H}_1, A \rtimes_r \mathcal{H}_2)$$

is an $l_\varepsilon \cdot \mathcal{R}_0$-controlled Mayer–Vietoris pair relatively to $A \rtimes_r \mathcal{H}$ and

$$(B \rtimes_r \mathcal{R}_{V_1}, B \rtimes_r \mathcal{R}_{V_2}, B \rtimes_r \mathcal{H}_1, B \rtimes_r \mathcal{H}_2)$$

is an $l_\varepsilon \cdot \mathcal{R}_0$-controlled Mayer–Vietoris pair relatively to $B \rtimes_r \mathcal{H}$. For the sake of simplicity, we rescale (α_c, k_c) to be equal to (λ, l).

According to Proposition 4.19 applied with coercivity $c = 2$, there exist, up to stabilization, two $\lambda\varepsilon$-$(l_\varepsilon \cdot \mathcal{R}_0)$-unitaries v_1 and v_2 in $\widetilde{B \rtimes_r \mathcal{H}}$ such that

- $v_i - 1$ is in $B \rtimes_r \mathcal{H}_i$ for $i = 1, 2$;
- v_i is homotopic to 1 as a $\lambda\varepsilon$-$(l_\varepsilon \cdot \mathcal{R}_0)$-unitary in $\widetilde{B \rtimes_r \mathcal{H}_i}$ for $i = 1, 2$;
- $\|\widetilde{f_{\mathcal{H}}}(u) - v_1 v_2\| \leqslant \lambda\varepsilon$.

Step II. By naturality of the quantitative Mayer–Vietoris boundary map (see equation (4.2) of Section 4.4), we have

$$\begin{aligned} f^{\lambda\varepsilon, l_\varepsilon\cdot\mathcal{R}_0}_{\mathcal{H}_1\cap\mathcal{H}_2,*} \circ \partial^{\varepsilon,\mathcal{R}_0}_{\mathcal{H}_1,\mathcal{H}_2,A,*}([u]_{\varepsilon,\mathcal{R}_0}) &= \partial^{\varepsilon,\mathcal{R}_0}_{\mathcal{H}_1,\mathcal{H}_2,B,*} \circ f^{\varepsilon,\mathcal{R}_0}_{\mathcal{H},*}([u]_{\varepsilon,\mathcal{R}_0}) \\ &= \partial^{\varepsilon,\mathcal{R}_0}_{\mathcal{H}_1,\mathcal{H}_2,B,*}([\widetilde{f_{\mathcal{H}}}(u)]_{\varepsilon,\mathcal{R}_0}) \\ &= 0. \end{aligned}$$

In particular,

$$\begin{aligned} f_{\mathcal{H}_1\cap\mathcal{H}_2,*} \circ \iota^{\lambda\varepsilon, l_\varepsilon\cdot\mathcal{R}_0}_{*} \circ \partial^{\varepsilon,\mathcal{R}_0}_{\mathcal{H}_1,\mathcal{H}_2,A,*}([u]_{\varepsilon,\mathcal{R}_0}) &= \iota^{\lambda\varepsilon, l_\varepsilon\cdot\mathcal{R}_0}_{*} \circ f^{\lambda\varepsilon, l_\varepsilon\cdot\mathcal{R}_0}_{\mathcal{H}_1,\mathcal{H}_2,A,*} \circ \partial^{\varepsilon,\mathcal{R}_0}_{\mathcal{H}_1,\mathcal{H}_2,A,*}([u]_{\varepsilon,\mathcal{R}_0}) \\ &= 0, \end{aligned}$$

and since $f_{\mathcal{H}_1\cap\mathcal{H}_2,*}$ is injective, we deduce from Lemma 4.12 that there exists a compact $\mathcal{G}$-order $\mathcal{R}$ containing $l_\varepsilon \cdot \mathcal{R}_0$ such that

$$\iota^{-,\lambda^2\varepsilon,\mathcal{R}}_{*} \circ \partial^{\varepsilon,\mathcal{R}_0}_{\mathcal{H}_1,\mathcal{H}_2,A,*}([u]_{\varepsilon,\mathcal{R}_0}) = 0.$$

Step III. According to Lemma 4.34, up to stabilization and up to replacing $\mathcal{R}$ by $l_{\lambda^2\varepsilon} \cdot \mathcal{R}$, there exist two $\lambda^3\varepsilon$-$\mathcal{R}$-unitaries w_1 and w_2 in $\widetilde{A \rtimes_r \mathcal{H}}$ such that

- $w_i - 1$ is in $A \rtimes_r \mathcal{H}_i$ for $i = 1, 2$;
- $\|u - w_1 w_2\| < \lambda^3\varepsilon$.

In particular, according to the first point of Lemma 4.6, we have

$$[u]_{3\lambda^3\varepsilon,2\cdot\mathcal{R}} = J^{3\lambda^3\varepsilon,2\cdot\mathcal{R}}_{1,A,*}([w_1]_{3\lambda^3\varepsilon,2\cdot\mathcal{R}}) + J^{3\lambda^3\varepsilon,2\cdot\mathcal{R}}_{2,A,*}([w_2]_{3\lambda^3\varepsilon,2\cdot\mathcal{R}}) \tag{5.1}$$

in $K_1^{3\lambda^3\varepsilon,2\cdot\mathcal{R}}(A \rtimes_r \mathcal{H})$. Moreover, we have

$$\|v_1 v_2 - \widetilde{f_{\mathcal{H}}}(w_1)\widetilde{f_{\mathcal{H}}}(w_2)\| < 2\lambda^3\varepsilon$$

and, in consequence,

$$\|v_1^* \widetilde{f_{\mathcal{H}}}(w_1) - v_2 \widetilde{f_{\mathcal{H}}}(w_2^*)\| < 8\lambda^3\varepsilon.$$

The CIA-condition with coercivity $c = 2$ implies that, up to replacing $\mathcal{R}$ by $2 \cdot \mathcal{R}$, there exists an element v in $1 + B \rtimes_r \mathcal{R}_{/\mathcal{H}_1\cap\mathcal{H}_2}$ such that

$$\|v - v_1^* \widetilde{f_{\mathcal{H}}}(w_1)\| < 16\lambda^3\varepsilon \quad \text{and} \quad \|v - v_2 \widetilde{f_{\mathcal{H}}}^*(w_2)\| < 16\lambda^3\varepsilon.$$

In view of the second point of Lemma 4.5, v is a $\lambda'\varepsilon$-$\mathcal{R}$-unitary with $\lambda' = 64\lambda^3$. Moreover, v is homotopic to $v_1^* \widetilde{f_{\mathcal{H}}}(w_1)$ as a $\lambda'\varepsilon$-$\mathcal{R}$-unitary in $B \rtimes_r \mathcal{H}_1$ and homotopic to

$v_2\widetilde{f_{\mathcal{H}}}^*(w_2)$ as a $\lambda'\varepsilon$-$\mathcal{R}$-unitary in $\widetilde{B \rtimes_r \mathcal{H}_2}$. By surjectivity of $f_{\mathcal{H}_1\cap\mathcal{H}_2,*}$ and in view of Lemma 4.12, there exist a compact $\mathcal{G}$-order $\mathcal{R}'$ containing $\mathcal{R}$ and an element z in $K_1^{\lambda\lambda'\varepsilon,\mathcal{R}'}(A \rtimes_r (\mathcal{H}_1 \cap \mathcal{H}_2))$ such that

$$f_{\mathcal{H}_1\cap\mathcal{H}_2,*}^{\lambda\lambda'\varepsilon,\mathcal{R}'}(z) = [v]_{\lambda\lambda'\varepsilon,\mathcal{R}'}$$

in $K_1^{\lambda\lambda'\varepsilon,\mathcal{R}'}(B \rtimes_r (\mathcal{H}_1 \cap \mathcal{H}_2))$.

Step IV. Let us set

$$z_1 = J_{1,2,A,*}^{\lambda\lambda'\varepsilon,\mathcal{R}'}(z) \quad \text{and} \quad z_2 = J_{2,1,A,*}^{\lambda\lambda'\varepsilon,\mathcal{R}'}(z).$$

We deduce from the discussion at the end of the previous step that

$$\begin{aligned} f_{\mathcal{H}_1,*} \circ \iota_*^{\lambda\lambda'\varepsilon,\mathcal{R}'}(z_1) &= \iota_*^{\lambda\lambda'\varepsilon,\mathcal{R}'} \circ f_{\mathcal{H}_1,*}^{\lambda\lambda'\varepsilon,\mathcal{R}'}(z_1) \\ &= \iota_*^{\lambda\lambda'\varepsilon,\mathcal{R}'}([v_1^*\widetilde{f_{\mathcal{H}_1}}(w_1)]_{\lambda\lambda'\varepsilon,\mathcal{R}'}) \\ &= \iota_*^{\lambda\lambda'\varepsilon,\mathcal{R}'}([\widetilde{f_{\mathcal{H}_1}}(w_1)]_{\lambda\lambda'\varepsilon,\mathcal{R}'}) \\ &= f_{\mathcal{H}_1,*} \circ \iota_*^{\lambda\lambda'\varepsilon,\mathcal{R}'}([w_1]_{\lambda\lambda'\varepsilon,\mathcal{R}'}), \end{aligned}$$

where the third equality holds because v_1 is homotopic to 1 as a $\lambda\varepsilon$-$l_\varepsilon \cdot \mathcal{R}_0$-unitary in $\widetilde{B \rtimes_r \mathcal{H}_1}$. Since $f_{\mathcal{H}_1,*}$ is one-to-one, we get that

$$\iota_*^{\lambda\lambda'\varepsilon,\mathcal{R}'}(z_1) = \iota_*^{\lambda\lambda'\varepsilon,\mathcal{R}'}([w_1]_{\lambda\lambda'\varepsilon,\mathcal{R}'})$$

and similarly,

$$\iota_*^{\lambda\lambda'\varepsilon,\mathcal{R}'}(z_2) = -\iota_*^{\lambda\lambda'\varepsilon,\mathcal{R}'}([w_2]_{\lambda\lambda'\varepsilon,\mathcal{R}'}).$$

Step V. According to Lemma 4.12, there exists a compact $\mathcal{G}$-order $\mathcal{R}''$ containing $\mathcal{R}'$ and such that

$$\iota_*^{-,\lambda^2\lambda'\varepsilon,\mathcal{R}''}(z_1) = [w_1]_{\lambda^2\lambda'\varepsilon,\mathcal{R}''} \quad \text{and} \quad \iota_*^{-,\lambda^2\lambda'\varepsilon,\mathcal{R}''}(z_2) = -[w_2]_{\lambda^2\lambda'\varepsilon,\mathcal{R}''}.$$

From equation (5.1), we deduce

$$\begin{aligned} [u]_{\lambda^2\lambda'\varepsilon,\mathcal{R}''} &= J_{1,A,*}^{\lambda^2\lambda'\varepsilon,\mathcal{R}''} \circ \iota_*^{-,-}(z_1) - J_{2,A,*}^{\lambda^2\lambda'\varepsilon,\mathcal{R}''} \circ \iota_*^{-,-}(z_2) \\ &= \iota_*^{-,\lambda^2\lambda'\varepsilon,\mathcal{R}''} \circ J_{1,A,*}^{\lambda^2\lambda'\varepsilon,\mathcal{R}''}(z_1) - \iota_*^{-,\lambda^2\lambda'\varepsilon,\mathcal{R}''} \circ J_{2,A,*}^{\lambda^2\lambda'\varepsilon,\mathcal{R}''}(z_2) \\ &= \iota_*^{-,\lambda^2\lambda'\varepsilon,\mathcal{R}''} \circ (J_{1,A,*}^{\lambda^2\lambda'\varepsilon,\mathcal{R}''} \circ J_{1,2,A,*}^{\lambda\lambda'\varepsilon,\mathcal{R}'} - J_{2,A,*}^{\lambda^2\lambda'\varepsilon,\mathcal{R}''} \circ J_{2,1,A,*}^{\lambda_0\lambda'\varepsilon,\mathcal{R}'})(z) \\ &= 0 \end{aligned}$$

and hence $x = \iota_*^{\lambda^2\lambda'\varepsilon,\mathcal{R}''}[u]_{\lambda^2\lambda'\varepsilon,\mathcal{R}''} = 0$.

5.2. Surjectivity part

Let us set $\alpha_0 = 567\lambda^5$. In view of Lemma 4.12, let us prove that for every ε in $(0, \frac{1}{4\alpha_0})$, any $\mathcal{G}$-order $\mathcal{R}_0$ and any y in $K_1^{\varepsilon,\mathcal{R}_0}(B \rtimes_r G)$, there exist a compact $\mathcal{G}$-order $\mathcal{R}_1$ containing $\mathcal{R}_0$ and an element x in $K_1^{\alpha_0\varepsilon,\mathcal{R}_1}(A \rtimes_r G)$ such that $f_{\mathcal{H},*}^{\varepsilon,\mathcal{R}_1}(x) = \iota_*^{\varepsilon,\alpha_0\varepsilon,\mathcal{R}_0,\mathcal{R}_1}(y)$ in $K_1^{\alpha_0\varepsilon,\mathcal{R}_1}(B \rtimes_r G)$. We divide this proof into four steps.

Step I. Up to replacing of ε by 21ε and in view of the first point of Lemma 4.9, we can assume that there exists an ε-$\mathcal{R}_0$-unitary u' in $I_{N'} + M_{N'}(B \rtimes_r \mathcal{H})$ for some integer N' such that $y = [u']_{\varepsilon,\mathcal{R}_0}$. Let $(V_1, V_2, \mathcal{H}_1, \mathcal{H}_2)$ be a $(6l_\varepsilon \cdot \mathcal{R}_0)$-decomposition for $\mathcal{H}$ with $\mathcal{H}_1$ and $\mathcal{H}_2$ in $\mathcal{D}$. Since $f_{\mathcal{H}_1 \cap \mathcal{H}_2,*}$ is onto and according to Lemma 4.12, there exist a compact $\mathcal{G}$-order $\mathcal{R}$ containing $l_\varepsilon \cdot \mathcal{R}_0$ and an element $x_{1,2}$ in $K_0^{\lambda^2\varepsilon,\mathcal{R}}(A \rtimes_r (\mathcal{H}_1 \cap \mathcal{H}_2))$ such that

$$f^{\lambda^2\varepsilon,\mathcal{R}}_{\mathcal{H}_1\cap\mathcal{H}_2,*}(x_{1,2}) = \iota_*^{-,\lambda^2\varepsilon,\mathcal{R}} \circ \partial^{\varepsilon,\mathcal{R}_0}_{\mathcal{H}_1,\mathcal{H}_2,B,*}(y).$$

Let us set

$$\begin{aligned} x_1 &= j^{\lambda^2\varepsilon,\mathcal{R}}_{1,2,A,*}(x_{1,2}) \quad \text{in } K_0^{\lambda^2\varepsilon,\mathcal{R}}(A \rtimes_r \mathcal{H}_1),\\ x_2 &= j^{\lambda^2\varepsilon,\mathcal{R}}_{2,1,A,*}(x_{1,2}) \quad \text{in } K_0^{\lambda^2\varepsilon,\mathcal{R}}(A \rtimes_r \mathcal{H}_2). \end{aligned}$$

Then we have

$$\begin{aligned} f_{\mathcal{H}_1,*} \circ \iota_*^{\lambda^2\varepsilon,\mathcal{R}}(x_1) &= \iota_*^{\lambda^2\varepsilon,\mathcal{R}} \circ f^{\lambda^2\varepsilon,\mathcal{R}}_{\mathcal{H}_1,*}(x_1)\\ &= \iota_*^{\lambda^2\varepsilon,\mathcal{R}} \circ f^{\lambda^2\varepsilon,\mathcal{R}}_{\mathcal{H}_1,*} \circ j^{\lambda^2\varepsilon,\mathcal{R}}_{1,2,A,*}(x_{1,2})\\ &= \iota_*^{\lambda^2\varepsilon,\mathcal{R}} \circ j^{\lambda^2\varepsilon,\mathcal{R}}_{1,2,B,*} \circ f^{\lambda^2\varepsilon,\mathcal{R}}_{\mathcal{H}_1\cap\mathcal{H}_2,*}(x_{1,2})\\ &= \iota_*^{\lambda^2\varepsilon,\mathcal{R}} \circ j^{\lambda^2\varepsilon,\mathcal{R}}_{1,2,B,*} \circ \iota_*^{-,-} \circ \partial^{\varepsilon,\mathcal{R}_0}_{\mathcal{H}_1,\mathcal{H}_2,B,*}(y)\\ &= \iota_*^{\lambda^2\varepsilon,\mathcal{R}} \circ \iota_*^{-,-} \circ j^{\lambda_\varepsilon,l_\varepsilon\mathcal{R}_0}_{1,2,B,*} \circ \partial^{\varepsilon,\mathcal{R}_0}_{\mathcal{H}_1,\mathcal{H}_2,B,*}(y)\\ &= 0 \end{aligned}$$

and in the same way $f_{\mathcal{H}_2,*} \circ \iota_*^{\lambda^2\varepsilon,\mathcal{R}}(x_2) = 0$. Since $f_{\mathcal{H}_1,*}$ and $f_{\mathcal{H}_2,*}$ are one-to-one, we deduce that $\iota_*^{\lambda^2\varepsilon,\mathcal{R}}(x_1) = 0$ and $\iota_*^{\lambda^2\varepsilon,\mathcal{R}}(x_2) = 0$. According to Lemma 4.12, there exists a compact $\mathcal{G}$-order $\mathcal{R}'$ containing $\mathcal{R}$ and such that

$$j^{\lambda^3\varepsilon,\mathcal{R}'}_{1,2,A,*} \circ \iota_*^{\lambda^2\varepsilon,\mathcal{R},-}(x_{1,2}) = 0 \quad \text{and} \quad j^{\lambda^3\varepsilon,\mathcal{R}'}_{2,1,A,*} \circ \iota_*^{\lambda^2\varepsilon,\mathcal{R},-}(x_{1,2}) = 0.$$

Step II. In view of Lemma 4.8, there exists for some positive integers n and N with $n \leqslant N$ a $9\lambda^2\varepsilon$-$\mathcal{R}$-projection in $\operatorname{diag}(I_n, 0) + M_N(A \rtimes_r \mathcal{H})$ such that

$$\iota_*^{\lambda^2\varepsilon,9\lambda^2\varepsilon,\mathcal{R}}(x_{1,2}) = [p,n]_{9\lambda^2\varepsilon,\mathcal{R}}$$

in $K_0^{9\lambda^2\varepsilon,\mathcal{R}}(A \rtimes_r (\mathcal{H}_1 \cap \mathcal{H}_2))$. According to Lemma 4.36 and up to stabilization, there exist four $9\lambda^3\varepsilon$-$l_{9\lambda^2\varepsilon}\mathcal{R}$-unitaries, v_1 and v_2 in $I_N + M_N(A \rtimes_r \mathcal{H})$, u_1 in $M_n(\widetilde{A \rtimes_r \mathcal{H}})$ and u_2 in $M_{N-n}(\widetilde{A \rtimes_r \mathcal{H}})$ such that

- $\|v_1^* \operatorname{diag}(I_n, 0) v_1 - p\| < 9\lambda^3\varepsilon$;
- $\|v_2 \operatorname{diag}(I_n, 0) v_2^* - p\| < 9\lambda^3\varepsilon$;
- $\|\operatorname{diag}(u_1, u_2) - v_1 v_2\| < 9\lambda^3\varepsilon$;
- for $i = 1, 2$ v_i is connected to I_N by a homotopy of $9\lambda^3\varepsilon$-$l_{9\lambda^2\varepsilon}\mathcal{R}$-unitaries in $I_N + M_N(A \rtimes_r \mathcal{H}_i)$.

Step III. By construction of the controlled Mayer–Vietoris boundary applied to $-y = \widetilde{[u'^*]_{\varepsilon,\mathcal{R}_0}}$, there exist (up to stabilization) two $\lambda\varepsilon$-l_ε-unitaries w_1 and w_2 in $M_{2N'}(B \rtimes_r \mathcal{H})$ and a $\lambda\varepsilon$-l_ε-projection q in $\mathrm{diag}(I_{N'}, 0) + M_{2N'}(B \rtimes_r \mathcal{H})$ such that

- $w_i - I_{2N'}$ lies in $M_{2N'}(B \rtimes_r \mathcal{H}_i)$ for $i = 1, 2$;
- $\|\mathrm{diag}(u'^*, u') - w_1 w_2\| < \lambda\varepsilon$;
- $\|w_1^* \,\mathrm{diag}(I_{N'}, 0) w_1 - q\| < \lambda\varepsilon$ and $\|w_2 \,\mathrm{diag}(I_{N'}, 0) w_2^* - q\| < \lambda\varepsilon$;
- $-\partial^{\varepsilon,\mathcal{R}_0}_{\mathcal{H}_1,\mathcal{H}_2,B,*}(z) = [q, N']_{\lambda\varepsilon, l_\varepsilon \cdot \mathcal{R}_0}$.

Step IV. If we apply Lemma 4.34 to $\mathrm{diag}(\widetilde{f_{\mathcal{H}}(u_1)}, u'^*)$, $\mathrm{diag}(\widetilde{f_{\mathcal{H}}(u_2)}, u')$ and to the matrices obtained from $\mathrm{diag}(\widetilde{f_{\mathcal{H}}}(v_1), w_1)$, $\mathrm{diag}(\widetilde{f_{\mathcal{H}}}(v_2), w_2)$ and $\mathrm{diag}(\widetilde{f_{\mathcal{H}}}(p), q)$ by flipping the coordinates $n+1, \dots, N$ and $N+1, \dots, N+N'$, we see that, up to replacing $\mathcal{R}'$ by $2l_{9\lambda^3\varepsilon}\mathcal{R}'$, there exists for some integer N'' and for $i = 1, 2$ a $9\lambda^4\varepsilon$-$\mathcal{R}'$-unitary v'_i in $I_{N''} + M_{N''}(B \rtimes_r \mathcal{H}_i)$ such that

$$[v'_1]_{27\lambda^4\varepsilon,\mathcal{R}'} + [v'_2]_{27\lambda^4\varepsilon,\mathcal{R}'} = [\widetilde{f_{\mathcal{H}}}(u_1)]_{27\lambda^4\varepsilon,\mathcal{R}'} - \iota_*^{\varepsilon,27\lambda^4\varepsilon,\mathcal{R}_0,\mathcal{R}'}(y)$$

(we have also used the first point of Corollary 4.6).

Since $f_{\mathcal{H}_1,*}$ and $f_{\mathcal{H}_2,*}$ are onto and according to Lemma 4.12, there exist a compact $\mathcal{G}$-order $\mathcal{R}_1$ containing $\mathcal{R}'$ and for $i = 1, 2$ an element x_i in $K_1^{27\lambda^5\varepsilon,\mathcal{R}_1}(A \rtimes_r \mathcal{H}_i)$ such that $f^{27\lambda^5\varepsilon,\mathcal{R}_1}_{\mathcal{H}_i,*}(x_i) = -[v'_i]_{27\lambda^5\varepsilon,\mathcal{R}_1}$ in $K_1^{27\lambda^5\varepsilon,\mathcal{R}_1}(B \rtimes_r \mathcal{H}_i)$. Then we have

$$\iota_*^{\varepsilon,27\lambda^5\varepsilon,\mathcal{R}_0,\mathcal{R}_1}(y) = f^{27\lambda^5\varepsilon,\mathcal{R}_1}_{\mathcal{H},*}(\jmath^{27\lambda^5\varepsilon,\mathcal{R}_1}_{1,A,*}(x_1) + \jmath^{27\lambda^5\varepsilon,\mathcal{R}_1}_{2,A,*}(x_2) + [u_1]_{27\lambda^5\varepsilon,\mathcal{R}_1}),$$

and hence $f_{\mathcal{H},*}$ is onto. ∎

5.3. Extension to Kasparov product

The aim of this section is to extend Theorem 5.1 to morphisms induced in K-theory by right Kasparov product (under second countability assumption for groupoids and separability assumption for $\mathcal{G}$-algebras). Indeed, this a consequence of the following standard fact which says that up to KK-equivalence, a Kasparov element is equivalent to a C^*-algebra homomorphism (see [19] for an approach via triangulated categories). Useful material for groupoid equivariant KK-theory can be found in [17, 18].

Lemma 5.3. *Let $\mathcal{G}$ be a second countable and locally compact groupoid provided with a Haar system, let A and B be separable $\mathcal{G}$-algebras and let z be an element in $KK_*^{\mathcal{G}}(A, B)$. Then there exist*

- *A' and B' two separable $\mathcal{G}$-algebras;*
- *$f : A' \to B'$ a homomorphism of $\mathcal{G}$-algebras;*
- *α in $KK_*^{\mathcal{G}}(A, A')$ and β in $KK_*^{\mathcal{G}}(B', B)$ invertible elements,*

such that

$$z = f_*(\alpha) \otimes_{B'} \beta.$$

Proof. Let us first prove the result for z in $KK_1^{\mathcal{G}}(A,B)$. The imprimitivity $\mathcal{K}(\mathcal{L}^2(\mathcal{G},A))$-$A$-bimodule $\mathcal{L}^2(\mathcal{G},A)$ gives rise to an invertible element $[\mathcal{M}]$ in $KK_*^{\mathcal{G}}(\mathcal{K}(\mathcal{L}^2(\mathcal{G},A)),A)$ and hence, this reduces to proving the result for $[\mathcal{M}]\otimes_A z$. According to [15, Appendix, Lemma 3.5] (see also [2, Section 5]), this amounts to prove the result for any element z in $KK_1^{\mathcal{G}}(A,B)$ that can be represented by a Kasparov K-cycle $(\mathcal{E},\pi,F)$ such that $F\colon\mathcal{E}\to\mathcal{E}$ is $\mathcal{G}$-equivariant. Up to adding a degenerate Kasparov K-cycle, we can assume without loss of generality that the linear space generated by $\{\langle\xi,\nu\rangle\colon\xi$ and ν in $\mathcal{E}\}$ is dense in B. Let us set $P=\frac{1}{2}(F+\mathrm{Id}_{\mathcal{E}})$ and let us consider the $\mathcal{G}$-algebra

$$E_P=\{(a,T)\text{ in }A\oplus\mathcal{L}(\mathcal{E})\text{ such that }T-P\cdot\pi(a)\cdot P\text{ belongs to }\mathcal{K}(\mathcal{E})\}.$$

Then the projection on the first factor of E_P gives rise to an extension of $\mathcal{G}$-algebra

$$0\to\mathcal{K}(\mathcal{E})\to E_P\to A\to 0 \tag{5.2}$$

semi-split by

$$A\to E_P,\quad a\mapsto(a,P\cdot\pi(a)\cdot P).$$

Let $[M]$ be the element in $KK_*^{\mathcal{G}}(\mathcal{K}(\mathcal{E}),B)$ corresponding to the $\mathcal{K}(\mathcal{E})$-B-imprimitivity bimodule $\mathcal{E}$. Then $[M]$ is invertible and $z\otimes_B[M]^{-1}$ is the class in $KK_1^{\mathcal{G}}(A,\mathcal{K}(\mathcal{E}))$ of the semi-split extension (5.2). Hence this amounts to prove that the lemma holds for the class $[\partial_{I,A}]$ in $KK_1^{\mathcal{G}}(A/I,I)$ of any semi-split extension $0\to I\to A\to A/I\to 0$. We proceed by using the mapping cone. For B that is a $\mathcal{G}$-algebra, let us set

$$\begin{aligned}B(0,1]&=\{f\colon[0,1]\to\mathbb{C}\text{ continuous such that }f(0)=0\},\\ B(0,1)&=\{f\colon[0,1]\to\mathbb{C}\text{ continuous such that }f(0)=f(1)=0\},\end{aligned}$$

and let us consider the class $[\partial_B]$ in $KK_1^{\mathcal{G}}(B,B(0,1))$ of the semi-split extension of $\mathcal{G}$-algebras

$$0\to B(0,1)\to B(0,1]\xrightarrow{\mathrm{ev}_1}B\to 0,$$

where $\mathrm{ev}_1\colon B(0,1]\to B$ is the evaluation at 1. Recall that $[\partial_B]$ is invertible. For a semi-split extension

$$0\to I\to A\xrightarrow{q}A/I\to 0,$$

we define the mapping cone algebra of A by

$$C_q=\{(x,f)\in A\oplus A/I(0,1]\text{ such that }f(1)=q(x)\}.$$

Let us consider the morphisms of $\mathcal{G}$-algebras

$$e_q\colon I\to C_q,\quad x\mapsto(x,0)$$

and

$$\phi_q\colon A/I(0,1)\to C_q,\quad f\mapsto(0,f).$$

According to [25, pp. 195–196], the element $[e_q]$ in $KK_*^{\mathcal{G}}(I, C_q)$ induced by e_q is invertible and moreover,

$$e_{q,*}[\partial_{I,A}] = \phi_{q,*}[\partial_{A/I}].$$

Hence we have

$$[\partial_{I,A}] = \phi_{q,*}[\partial_{A/I}] \otimes_{C_q} [e_q]^{-1}.$$

Since $[\partial_{A/I}]$ is an invertible element in $KK_1^{\mathcal{G}}(A/I, A/I(0,1))$, we deduce that the conclusion of the lemma holds for $[\partial_{I,A}]$.

Let us prove now that the result holds in the even case. Let z be an element in $KK_0^{\mathcal{G}}(A, B)$. Noticing that $[\partial_A]$ is invertible in $KK_1^{\mathcal{G}}(A, A(0,1))$ and applying the odd case to $[\partial_A]^{-1} \otimes_A z$, we deduce the result for z. ∎

As a consequence, we can extend Theorem 5.1 to KK-elements. Recall that for any second countable and locally compact groupoid $\mathcal{G}$ provided with a Haar system

$$J_{\mathcal{G}} \colon KK_*^{\mathcal{G}}(\bullet, \bullet) \to KK_*(\bullet \rtimes_r \mathcal{G}, \bullet \rtimes_r \mathcal{G})$$

stands for the Kasparov transformation. For any $\mathcal{G}$-algebras A and B and for any element z in $KK_*^{\mathcal{G}}(A, B)$, we denote by

$$\otimes J_{\mathcal{G}}(z) \colon K_*(A \rtimes_r \mathcal{G}) \to K_*(B \rtimes_r \mathcal{G})$$

the morphism induced by Kasparov right multiplication by $J_{\mathcal{G}}(z)$ (see [17] for the description of this transformation in the setting of groupoids). For z in $KK_*^{\mathcal{G}}(A, B)$ and for $\mathcal{H}$ that is a relatively clopen subgroupoid of $\mathcal{G}$, we also denote by $z_{/\mathcal{H}}$ the restriction of z to $\mathcal{H}$, i.e., the image of z under the morphism

$$KK_*^{\mathcal{G}}(A, B) \to KK_*^{\mathcal{H}}(A_{/Y}, B_{/Y})$$

corresponding under functoriality in the groupoids to the inclusion $\mathcal{H} \hookrightarrow \mathcal{G}$ (see [18, Section 7.1]).

Corollary 5.4. *Let $\mathcal{G}$ be a second countable and locally compact groupoid provided with a Haar system, let A and B be separable $\mathcal{G}$-algebras and let z be an element in $KK_*^{\mathcal{G}}(A, B)$. Let us assume that there exists a subset $\mathfrak{D}$ of relatively clopen groupoids of $\mathcal{G}$, closed under taking relatively clopen subgroupoids and such that*

- (i) *$\mathcal{G}$ has finite decomposition complexity with respect to $\mathfrak{D}$;*
- (ii) *for any $\mathcal{H}$ in $\mathfrak{D}$, the morphism $\otimes J_{\mathcal{H}}(z_{/\mathcal{H}}) \colon K_*(A \rtimes_r \mathcal{H}) \to K_*(B \rtimes_r \mathcal{H})$ is an isomorphism.*

Then $\otimes J_{\mathcal{G}}(z) \colon K_*(A \rtimes_r \mathcal{G}) \to K_*(B \rtimes_r \mathcal{G})$ is an isomorphism.

Proof. Let A' and B' be separable $\mathcal{G}$-algebras, let $f \colon A' \to B'$ be a morphism of $\mathcal{G}$-algebras and let α in $KK_*^{\mathcal{G}}(A, A')$ and β in $KK_*^{\mathcal{G}}(B', B)$ be invertible elements as in Lemma 5.3. Then

$$\otimes J_{\mathcal{G}}(z) \colon K_*(A \rtimes_r \mathcal{G}) \to K_*(B \rtimes_r \mathcal{G})$$

is an isomorphism if and only if $f_{\mathcal{G},*}\colon K_*(A' \rtimes_r \mathcal{G}) \to K_*(B' \rtimes_r \mathcal{G})$ is an isomorphism, and in the same way

$$\otimes J_{\mathcal{H}}(z_{/\mathcal{H}})\colon K_*(A \rtimes_r \mathcal{H}) \to K_*(B \rtimes_r \mathcal{H})$$

is an isomorphism if and only if $f_{\mathcal{H},*}\colon K_*(A' \rtimes_r \mathcal{H}) \to K_*(B' \rtimes_r \mathcal{H})$ is an isomorphism.

Then the corollary is the consequence of Theorem 5.1 applied to $f\colon A' \to B'$. ∎

Using the same argument as in the proof of [15, Proposition A.5.1], we also have the following consequence of Lemma 5.3.

Corollary 5.5. *Let $\mathcal{G}$ be a second countable and locally compact groupoid provided with a Haar system, let A, B and D be $\mathcal{G}$-algebras with A and B separable and let z be an element in $KK_*^{\mathcal{G}}(A, B)$. Let us assume that there exists a subset $\mathcal{D}$ of relatively clopen groupoids of $\mathcal{G}$, closed under taking relatively clopen subgroupoids and such that*

(i) *$\mathcal{G}$ has finite decomposition complexity with respect to $\mathcal{D}$.*

(ii) *for any subgroupoid $\mathcal{H}$, the morphism*

$$\otimes J_{\mathcal{H}}(\tau_D(z)_{/\mathcal{H}})\colon K_*((A \otimes D) \rtimes_r \mathcal{H}) \to K_*((B \otimes D) \rtimes_r \mathcal{H})$$

is an isomorphism.

Then $\otimes J_{\mathcal{G}}(\tau_D(z))\colon K_*((A \otimes D) \rtimes_r \mathcal{G}) \to K_*((A \otimes B) \rtimes_r \mathcal{G})$ is an isomorphism.

6. Application to the Baum–Connes conjecture

In this section, we show that for groupoids that admit a γ-element in the sense of [28], the Baum–Connes conjecture is closed under coarse decomposability.

6.1. The Baum–Connes conjecture for groupoids

Let us recall the statement of the Baum–Connes conjecture for groupoids. Let $\mathcal{G}$ be a locally compact groupoid provided with a Haar system, let A be a $\mathcal{G}$-algebra and let $A \rtimes_r \mathcal{G}$ be the reduced crossed product of A by $\mathcal{G}$ (with respect to the given Haar system). Then the Baum–Connes conjecture for the pair $(A, \mathcal{G})$ is the claim that the assembly map

$$\mu_{A,\mathcal{G}}\colon K_*^{\mathrm{top}}(\mathcal{G}, A) \to K_*(A \rtimes_r \mathcal{G})$$

is an isomorphism, the left-hand side being the topological K-theory for the groupoid $\mathcal{G}$ with coefficients in A, defined as the inductive limit

$$\lim_X KK_*^{\mathcal{G}}(C_0(X), A),$$

where X runs through $\mathcal{G}$-compact subsets of the universal example for proper actions of $\mathcal{G}$ (see [28, Section 5.1] for a complete description of the Baum–Connes conjecture in the setting of groupoids). Although the conjecture holds for a large class of pairs $(A, \mathcal{G})$, (e.g., if $\mathcal{G}$ is an amenable groupoid [27]), counterexamples have been given by Higson, Lafforgue and Skandalis in [12].

6.2. The case of groupoids admitting a γ-element

The concept of γ-element was introduced by Kasparov in [13] in order to prove the Novikov conjecture for discrete subgroups of almost connected groups. He showed that for an almost connected group G acting on a C^*-algebra A strongly continuously by automorphisms, the image of the Baum–Connes assembly map is the range of γ acting on $K_*(A \rtimes_r G)$ as an idempotent. The notion of γ-element was extended to groupoid by Tu in [28, Proposition 5.20 and Remark 5.21], where he developed an abstract setting for such an element.

Definition 6.1. A second countable locally compact groupoid $\mathcal{G}$ admits a γ-element if there exist an element γ in $KK_*^{\mathcal{G}}(C_0(X), C_0(X))$, a proper $\mathcal{G}$-space Z, a $\mathcal{G} \ltimes Z$-algebra A, an element η in $KK_*^{\mathcal{G}}(C_0(X), A)$ and an element D in $KK_*^{\mathcal{G}}(C_0(X), A)$ such that

- $\gamma = \eta \otimes_A D$;
- $p_Z^* \gamma = 1$ in $K_*^{\mathcal{G} \ltimes Z}(C_0(Z), C_0(Z))$ for every proper $\mathcal{G}$-space Z, where $p_Z\colon \mathcal{G} \ltimes Z \to \mathcal{G}$ is the forgetful map.

Such an element, if it exists, is unique and is called a γ-element. As in the case of the γ-element of Kasparov, a γ-element is an idempotent of $KK_*^{\mathcal{G}}(C_0(X), C_0(X))$ and acts as an idempotent on $K_*(A \rtimes_r \mathcal{G})$. This idempotent is given by right Kasparov product by $J_{\mathcal{G}}(\tau_A(\gamma))$, where $\tau_A(\gamma) \in KK_*^{\mathcal{G}}(A, A)$ is obtained by tensorization over $C(X)$ by A. Moreover, it is related to the Baum–Connes conjecture in the following way (see [28, Proposition 5.23]):

Proposition 6.2. *Let $\mathcal{G}$ be a second countable locally compact groupoid provided with a Haar system admitting a γ-element and let A be a $\mathcal{G}$-algebra. Then the following assertions are equivalent:*

(i) $\mu_{A,\mathcal{G}}\colon K_*^{\text{top}}(\mathcal{G}, A) \to K_*(A \rtimes_r \mathcal{G})$ *is an isomorphism;*

(ii) $J_{\mathcal{G}}(\tau_A(\gamma))$ *acts as the identity by right Kasparov product on* $K_*(A \rtimes_r \mathcal{G})$.

Remark 6.3. Since $J_{\mathcal{G}}(\tau_A(\gamma))$ is an idempotent, it acts as the identity by right Kasparov product on $K_*(A \rtimes_r \mathcal{G})$ if and only if it acts as an isomorphism.

The restriction of a γ-element to a relatively clopen subgroupoid is a γ-element.

Lemma 6.4. *Let $\mathcal{G}$ be a second countable and locally compact groupoid and let $\mathcal{H}$ be a relatively clopen subgroupoid of $\mathcal{G}$. If $\mathcal{G}$ admits a γ-element, then the restriction of γ to $\mathcal{H}$ is a γ-element for $\mathcal{H}$.*

Proof. Let us denote respectively by X and Y the space of units of $\mathcal{G}$ and $\mathcal{H}$. Let Z be a proper $\mathcal{G}$-space, let A be a $\mathcal{G} \rtimes Z$-algebra, let η be an element in $KK_*^{\mathcal{G}}(C_0(X), A)$ and let D be an element in $KK_*^{\mathcal{G}}(A, C_0(X))$ as in Definition 6.1. According to Remark 2.7 and Corollary 2.5, the proper action of $\mathcal{G}$ on Z restricts to a proper action of $\mathcal{H}$ on Z_Y. Let $A_{/Z_Y}$ be the restriction of A to Z_Y. According to the second point of Example 3.4,

$A_{/Z_Y}$ is an $\mathcal{H} \rtimes Z_Y$-algebra. Let $\gamma_{/\mathcal{H}}$ in $KK_*^{\mathcal{H}}(C_0(Y), C_0(Y))$, $\eta_{/\mathcal{H}}$ in $KK_*^{\mathcal{H}}(C_0(Y), A_{/Z_Y})$ and $D_{/\mathcal{H}}$ in $KK_*^{\mathcal{H}}(A_{/Z_Y}, C_0(Y))$ be respectively the restriction of γ, η and D to $\mathcal{H}$ (i.e., induced by functoriality in the groupoids by the inclusion $\mathcal{H} \hookrightarrow \mathcal{G}$). According to [18, Proposition 7.2 (b)], the restriction respects Kasparov products and hence we deduce that $\gamma_{/\mathcal{H}} = \eta_{/\mathcal{H}} \otimes_{A_{/Z_Y}} D_{/\mathcal{H}}$. Let us check the second point of the definition of a γ-element. Let Z' be a proper $\mathcal{H}$-space, let $Z'' = \mathcal{G} \times_{\mathcal{H}} Z'$ be the proper induced $\mathcal{G}$-space (see Section 2.3) and let us recall that $p_{Z''}: \mathcal{G} \ltimes Z'' \to \mathcal{G}$ stands for the forgetful map. We have by definition of a γ-element that $p_{Z''}^*\gamma = 1$ in $K_*^{Z'' \rtimes \mathcal{G}}(C_0(Z''), C_0(Z''))$. We have an obvious inclusion of groupoids

$$\mathcal{H} \rtimes Z' \hookrightarrow \mathcal{G} \rtimes Z'', \quad (\gamma, z) \mapsto (\gamma, [u_{p_{Z'}(z)}, z])$$

which pulls back $p_{Z''}^*\gamma$ to $p_{Z'}^*\gamma_{/\mathcal{H}}$ (using [18, Proposition 7.2 (a)]) and hence $p_{Z'}^*\gamma_{/\mathcal{H}} = 1$ in $K_*^{\mathcal{H} \rtimes Z'}(C_0(Z'), C_0(Z'))$. We conclude that $\gamma_{/\mathcal{H}}$ is a γ-element for $\mathcal{H}$. ∎

Remark 6.5. As a consequence and using induced algebras [2, Section 3], we can prove that if $\mathcal{G}$ is a second countable and locally compact groupoid which admits a γ-element and satisfies the Baum–Connes conjecture with coefficients, then any relatively clopen subgroupoid of $\mathcal{G}$ satisfies the Baum–Connes conjecture with coefficients.

An action groupoid of a groupoid with a γ-element has a γ-element.

Lemma 6.6. *Let $\mathcal{G}$ be a second countable and locally compact groupoid and let Y be a second countable and locally compact (left) $\mathcal{G}$-space. If $\mathcal{G}$ admits a γ-element, then the action groupoid $\mathcal{G} \ltimes Y$ admits a γ-element.*

Proof. Let us denote by X the space of units of $\mathcal{G}$ and let $q_Y: Y \to X$ be the anchor map for the $\mathcal{G}$-action on Y. Let Z be a proper $\mathcal{G}$-space, let A be a $\mathcal{G} \ltimes Z$-algebra, let η be an element in $KK_*^{\mathcal{G}}(C_0(X), A)$ and let D be an element in $KK_*^{\mathcal{G}}(A, C_0(X))$ as in Definition 6.1. Let $p_Y: \mathcal{G} \rtimes Y \to \mathcal{G}$ be the forgetful map with respect to the $\mathcal{G}$-action on Y. According to the fourth point of Remark 2.6, we see that $Z \times_X Y$ is a proper $\mathcal{G} \rtimes Y$-space with anchor map $q_{Z \times_X Y}: Z \times_X Y \to Y$ given by the projection on the second factor. Consider then the elements $\gamma_Y = p_Y^*\gamma$ in $KK_*^{\mathcal{G} \ltimes Y}(C_0(Y), C_0(Y))$, $\eta_Y = p_Y^*\eta$ in $KK_*^{\mathcal{G} \ltimes Y}(C_0(Y), q_Y^*A)$ and $D_Y = p_Y^*D$ in $KK_*^{\mathcal{G} \ltimes Y}(q_Y^*A, C_0(Y))$. Using the second point of Example 3.4, we see that $q_Y^*A = A \otimes_{C_0(X)} C_0(Y)$ is a $(\mathcal{G} \ltimes Y) \ltimes (Z \times_X Y)$-algebra and since p_Y^* preserves Kasparov products (see [18, Proposition 7.2 (b)]), we have

$$\gamma_Y = \eta_Y \otimes_{q_Y^*A} D_Y.$$

Let us check now the second condition of Definition 6.1. Let Z' be a proper $\mathcal{G} \ltimes Y$-space. According to the first and to the third point of Remark 2.6, we see that Z' is a proper $\mathcal{G}$-space equipped with a $\mathcal{G}$-map $Z' \to Y$. Let

$$p_{Z'}: (\mathcal{G} \ltimes Y) \ltimes Z' \to \mathcal{G} \ltimes Y$$

be the forgetful map with respect to the $\mathcal{G} \ltimes Y$-action on Z'. Then we have

$$p_{Z'}^* \gamma_Y = p_{Z'}^*(p_Y^* \gamma) = (p_Y \circ p_{Z'})^* \gamma.$$

But under the identification between $(\mathcal{G} \ltimes Y) \ltimes Z'$ and $\mathcal{G} \ltimes Z'$ of the second point of Remark 2.6, then

$$p_Y \circ p_{Z'} \colon (\mathcal{G} \ltimes Y) \ltimes Z' \to \mathcal{G}$$

corresponds to the forgetful map $\mathcal{G} \ltimes Z' \to \mathcal{G}$ with respect to the proper $\mathcal{G}$-action on Z'. From this we deduce that $p_{Z'}^* \gamma_Y = 1$ in $KK_*^{(\mathcal{G} \ltimes Y) \ltimes Z'}(C_0(Z'), C_0(Z'))$ and hence γ_Y is a γ-element for $\mathcal{G} \ltimes Y$. ∎

Example 6.7. The following examples of second countable and locally compact groups G are known to have a γ-element:

(i) if G acts properly on a simply connected manifold with non-positive sectional curvature [13];

(ii) if G is (a closed subgroup of) an almost connected group [13];

(iii) if G is a group acting properly on an Euclidean buildings [14];

(iv) if G is a countable discrete group that coarsely embeds into a Hilbert space (see [7, Theorem 9.2] and [29]).

For any action of such a group G on a second countable locally compact space X, the action groupoid $G \ltimes X$ has a γ-element.

6.3. The Baum–Connes conjecture and coarse decomposability

Theorem 6.8. *Let $\mathcal{G}$ be a second countable and locally compact groupoid provided with a Haar system which moreover admits a γ-element in sense of* [28, Proposition 5.20] *and let A be a $\mathcal{G}$-algebra. Assume that there exists a subset $\mathcal{D}$ of relatively clopen subgroupoids of $\mathcal{G}$, closed under taking relatively clopen subgroupoids such that*

(i) *every groupoid in $\mathcal{D}$ satisfies the Baum–Connes conjecture with coefficients in A;*

(ii) *$\mathcal{G}$ has finite decomposition complexity with respect to $\mathcal{D}$.*

Then $\mathcal{G}$ satisfies the Baum–Connes conjecture with coefficients in A.

Proof. According to Proposition 6.2 and Remark 6.3, this amounts to prove that the action of $J_{\mathcal{G}}(\tau_A(\gamma))$ by right Kasparov product on $K_*(A \rtimes_r \mathcal{G})$ is an isomorphism. If A is separable, this is the consequence of Corollary 5.4 applied to $\tau_A(\gamma)$ and of Lemma 6.4, by noticing that $\tau_A(\gamma)_{/\mathcal{H}} = \tau_A(\gamma_{/\mathcal{H}})$ for any relatively clopen subgroupoids $\mathcal{H}$ in $\mathcal{D}$. If A is not separable, this is a consequence of Corollary 5.5. ∎

We end this subsection with an application to the Baum–Connes conjecture with coefficients.

Corollary 6.9. *Let $\mathcal{G}$ be a second countable and locally compact groupoid provided with a Haar system and which moreover admits a γ-element in sense of* [28, Proposition 5.20]. *Assume that there exists a subset $\mathcal{D}$ of relatively clopen subgroupoids of $\mathcal{G}$ such that*

(i) *every groupoid in $\mathcal{D}$ satisfies the Baum–Connes conjecture with coefficients;*

(ii) *$\mathcal{G}$ has finite decomposition complexity with respect to $\mathcal{D}$.*

Then $\mathcal{G}$ satisfies the Baum–Connes conjecture with coefficients.

Proof. Let $\mathcal{D}'$ be the set of all relatively clopen subgroupoids of elements of $\mathcal{D}$. According to Remark 6.5, any groupoid $\mathcal{H}$ in $\mathcal{D}'$ satisfies the Baum–Connes conjecture with coefficients. Since $\mathcal{G}$ has finite $\mathcal{D}$-complexity, it has finite $\mathcal{D}'$-complexity. Then the result is a consequence of Theorem 6.8. ∎

6.4. Perspectives and open questions

We end this paper with a discussion about the range of applicability of Theorem 6.8. Even if we are asking more questions than giving answers, the idea is to inspire further works on the quest of new examples of groupoids satisfying the Baum–Connes conjecture. As we have seen in Section 2.6, groupoid amenability is closed under coarse decomposition and hence if we start with a family $\mathcal{D}$ of amenable subgroupoids (which satisfies the Baum–Connes conjecture with coefficients by [27]), then Theorem 6.8 does not bring any new example of groupoid satisfying the Baum–Connes conjecture. The situation might be different for groupoid which have the Haagerup property. This property was introduced in [27, Section 3] in terms of affine and proper action on a continuous field of real and affine Hilbert spaces. We shall use an equivalent definition using functions conditionally of negative type (see [24, Proposition 2.13]).

Definition 6.10. Let $\mathcal{G}$ be a locally compact groupoid with space of units X. A function $\Psi\colon \mathcal{G} \to \mathbb{R}$ is conditionally of negative type if

(i) $\Psi(u(x)) = 0$ for every x in X;

(ii) $\Psi(\gamma) = \Psi(\gamma^{-1})$ for any γ in $\mathcal{G}$;

(iii) for any positive integer n, any x in X, any $\gamma_1, \dots, \gamma_n$ in $\mathcal{G}^x$ and any real numbers $\lambda_1, \dots, \lambda_n$ such that $\lambda_1 + \cdots + \lambda_n = 0$,

$$\sum_{1 \leqslant i,j \leqslant n} \lambda_i \lambda_j \Psi(\gamma_i^{-1}\gamma_j) \leqslant 0.$$

We are now in position to give the definition of the Haagerup property for groupoids in terms of functions conditionally of negative type.

Definition 6.11. A locally compact groupoid $\mathcal{G}$ has the Haagerup property if $\mathcal{G}$ admits a conditionally of negative type function $\Psi\colon \mathcal{G} \to \mathbb{R}$ which is locally proper, i.e., for any compact subset K of $\mathcal{G}$, the restriction of Ψ to $\mathcal{G}_K^K$ is proper.

According to [27, Proposition 3.8 and Théorème 0.1], any locally compact and second countable groupoid with the Haagerup property satisfies the Baum–Connes conjecture with coefficients. In consequence, under the assumptions of Corollary 6.9, if $\mathcal{G}$ has finite decomposition complexity with respect to a family $\mathcal{D}$ of relatively clopen subgroupoids which have the Haagerup property, then $\mathcal{G}$ satisfies the Baum–Connes conjecture with coefficients.

Question 6.12. Can we obtain, using the coarse decomposition, groupoids which do not have the Haagerup property and provide in this way new examples of groupoids satisfying the Baum–Connes conjecture with coefficients?

Another source of inspiration to provide genuinely new examples of groupoids satisfying the Baum–Connes conjecture is the work of Lafforgue on hyperbolic groups and groupoids [15, 16].

Let $\mathcal{D}$ be a family of relatively clopen subgroupoids of a second countable and locally compact groupoid $\mathcal{G}$. Assume that every groupoid in $\mathcal{H}$ is a relatively clopen subgroupoid of a finitely generated group action groupoid $\Gamma \ltimes X$, where Γ is a Gromov hyperbolic group acting on a second countable and locally compact space X. According to [16, Théorème 0.4], to [4, Corollary 0.2] and to Remark 6.5, any groupoid in $\mathcal{D}$ satisfies the Baum–Connes conjecture with coefficients. Hence, under the assumptions of Corollary 6.9, if $\mathcal{G}$ has finite decomposition complexity with respect to $\mathcal{D}$, then $\mathcal{G}$ satisfies the Baum–Connes conjecture with coefficients.

According to [15, Corollaire 4.0.2], we have a similar result for the Baum–Connes conjecture with commutative coefficients if $\mathcal{D}$ is a family of relatively clopen subgroupoids of Poincaré groupoids of foliations with compact base space and which admit a longitudinal Riemannian metric of negative sectional curvature.

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References

[1] C. Anantharaman-Delaroche and J. Renault, *Amenable groupoids*. Monogr. Linguist. Math. 36, L'Enseignement Mathématique, Geneva, 2000 Zbl 1016.46042 MR 1799683

[2] C. Bönicke, A going-down principle for ample groupoids and the Baum–Connes conjecture. *Adv. Math.* **372** (2020), article no. 107314 Zbl 1451.19016 MR 4128575

[3] C. Bönicke and C. Dell'Aiera, Going-down functors and the Künneth formula for crossed products by étale groupoids. *Trans. Amer. Math. Soc.* **372** (2019), no. 11, 8159–8194 Zbl 1434.46043 MR 4029694

[4] J. Chabert, S. Echterhoff, and H. Oyono-Oyono, Shapiro's lemma for topological K-theory of groups. *Comment. Math. Helv.* **78** (2003), no. 1, 203–225 Zbl 1030.19001 MR 1966758

[5] Y. C. Chung, Quantitative K-theory for Banach algebras. *J. Funct. Anal.* **274** (2018), no. 1, 278–340 Zbl 1386.19011 MR 3718054

[6] C. Dell'Aiera, Controlled K-theory for groupoids & applications to coarse geometry. *J. Funct. Anal.* **275** (2018), no. 7, 1756–1807 Zbl 1405.46051 MR 3832008

[7] H. Emerson and R. Meyer, Dualizing the coarse assembly map. *J. Inst. Math. Jussieu* **5** (2006), no. 2, 161–186 Zbl 1092.19005 MR 2225040

[8] E. Guentner, R. Tessera, and G. Yu, A notion of geometric complexity and its application to topological rigidity. *Invent. Math.* **189** (2012), no. 2, 315–357 Zbl 1257.57028 MR 2947546

[9] E. Guentner, R. Tessera, and G. Yu, Discrete groups with finite decomposition complexity. *Groups Geom. Dyn.* **7** (2013), no. 2, 377–402 Zbl 1272.52041 MR 3054574

[10] E. Guentner, R. Willett, and G. Yu, Dynamic asymptotic dimension: relation to dynamics, topology, coarse geometry, and C^*-algebras. *Math. Ann.* **367** (2017), no. 1–2, 785–829 Zbl 1380.37018 MR 3606454

[11] E. Guentner, R. Willett, and G. Yu, Dynamical complexity and controlled operator K-theory. 2022, arXiv:1609.02093

[12] N. Higson, V. Lafforgue, and G. Skandalis, Counterexamples to the Baum–Connes conjecture. *Geom. Funct. Anal.* **12** (2002), no. 2, 330–354 Zbl 1014.46043 MR 1911663

[13] G. G. Kasparov, Equivariant KK-theory and the Novikov conjecture. *Invent. Math.* **91** (1988), no. 1, 147–201 Zbl 0647.46053 MR 918241

[14] G. G. Kasparov and G. Skandalis, Groups acting on buildings, operator K-theory, and Novikov's conjecture. *K-Theory* **4** (1991), no. 4, 303–337 Zbl 0738.46035 MR 1115824

[15] V. Lafforgue, K-théorie bivariante pour les algèbres de Banach, groupoïdes et conjecture de Baum–Connes (avec un appendice d'Hervé Oyono–Oyono). *J. Inst. Math. Jussieu* **6** (2007), no. 3, 415–451 Zbl 1135.19002 MR 2329760

[16] V. Lafforgue, La conjecture de Baum–Connes à coefficients pour les groupes hyperboliques. *J. Noncommut. Geom.* **6** (2012), no. 1, 1–197 Zbl 1328.19010 MR 2874956

[17] P.-Y. Le Gall, *Théorie de Kasparov équivariante et groupoïdes*. Ph.D. thesis, 1994, Université Paris VII

[18] P.-Y. Le Gall, Théorie de Kasparov équivariante et groupoïdes. I. *K-Theory* **16** (1999), no. 4, 361–390 Zbl 0932.19004 MR 1686846

[19] R. Meyer and R. Nest, The Baum–Connes conjecture via localisation of categories. *Topology* **45** (2006), no. 2, 209–259 Zbl 1092.19004 MR 2193334

[20] H. Oyono-Oyono and G. Yu, On quantitative operator K-theory. *Ann. Inst. Fourier* **65** (2015), no. 2, 605–674 Zbl 1329.19009 MR 3449163

[21] H. Oyono-Oyono and G. Yu, Quantitative K-theory and the Künneth formula for operator algebras. *J. Funct. Anal.* **277** (2019), no. 7, 2003–2091 Zbl 1475.19004 MR 3989138

[22] A. L. T. Paterson, *Groupoids, inverse semigroups, and their operator algebras*. Progr. Math. 170, Birkhäuser, Boston, MA, 1999 Zbl 0913.22001 MR 1724106

[23] J. Renault, *A groupoid approach to C^*-algebras*. Lecture Notes in Math. 793, Springer, Berlin, 1980 Zbl 0433.46049 MR 584266

[24] J. Renault, Groupoid cocycles and derivations. *Ann. Funct. Anal.* **3** (2012), no. 2, 1–20 Zbl 1257.22005 MR 2948385

[25] G. Skandalis, Exact sequences for the Kasparov groups of graded algebras. *Canad. J. Math.* **37** (1985), no. 2, 193–216 Zbl 0603.46064 MR 780621

[26] G. Skandalis, J. L. Tu, and G. Yu, The coarse Baum–Connes conjecture and groupoids. *Topology* **41** (2002), no. 4, 807–834 Zbl 1033.19003 MR 1905840

[27] J.-L. Tu, La conjecture de Baum–Connes pour les feuilletages moyennables. *K-Theory* **17** (1999), no. 3, 215–264 Zbl 0939.19001 MR 1703305

[28] J. L. Tu, La conjecture de Novikov pour les feuilletages hyperboliques. *K-Theory* **16** (1999), no. 2, 129–184 Zbl 0932.19005 MR 1671260

[29] J.-L. Tu, The gamma element for groups which admit a uniform embedding into Hilbert space. In *Recent advances in operator theory, operator algebras, and their applications*, pp. 271–286, Oper. Theory Adv. Appl. 153, Birkhäuser, Basel, 2005 Zbl 1074.46049 MR 2105483

[30] R. Willett, Approximate ideal structures and K-theory. *New York J. Math.* **27** (2021), 1–52 Zbl 1467.46071 MR 4195416

[31] D. P. Williams, *Crossed products of C^*-algebras*. Math. Surveys Monogr. 134, American Mathematical Society, Providence, RI, 2007 Zbl 1119.46002 MR 2288954

[32] G. Yu, The Novikov conjecture for groups with finite asymptotic dimension. *Ann. of Math. (2)* **147** (1998), no. 2, 325–355 Zbl 0911.19001 MR 1626745

[33] G. Yu, The coarse Baum–Connes conjecture for spaces which admit a uniform embedding into Hilbert space. *Invent. Math.* **139** (2000), no. 1, 201–240 Zbl 0956.19004 MR 1728880

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