Groups Geom. Dyn. 17 (2023), 1073–1083
DOI 10.4171/GGD/724

Growth of pseudo-Anosov conjugacy classes in Teichmüller space

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Abstract. Athreya, Bufetov, Eskin and Mirzakhani (2012) have shown that the number of mapping class group lattice points intersecting a closed ball of radius R in Teichmüller space is asymptotic to e^{hR}, where h is the dimension of the Teichmüller space. We show for any pseudo-Anosov mapping class f, there exists a power n, such that the number of lattice points of the f^n conjugacy class intersecting a closed ball of radius R is coarsely asymptotic to $e^{\frac{h}{2}R}$.

1. Introduction

One can study a group by understanding its "growth" in various ways. Consider G acting on a metric space S by isometries, one can measure the number of orbit or lattice points of G in a ball of radius R as R goes to infinity. For example, consider $\mathbb{Z}^3$ acting on $\mathbb{R}^3$ in the standard way, the number of lattice points of $\mathbb{Z}^3$ in a ball of radius R is roughly the volume of this ball, see [8] for example. In this paper, we study mapping class groups by understanding the lattice points of pseudo-Anosov conjugacy classes in Teichmüller space.

Let M be a compact, negatively curved Riemannian manifold and let $\widetilde{M}$ denote its universal cover. The fundamental group $\pi_1(M)$ acts on $\widetilde{M}$ by isometries. Let $B_R(x)$ denote the ball of radius R in $\widetilde{M}$ centered at x. G. A. Margulis studied the growth rate of any orbit $\pi_1(M) \cdot y$ by intersecting with any metric balls $B_r(x)$.

Theorem 1.1 (Margulis [10]). *There is a function $c\colon M \times M \to \mathbb{R}^+$ so that for every $x, y \in \widetilde{M}$,*

$$|\pi_1(M) \cdot y \cap B_R(x)| \sim c(p(x), p(y))e^{hR}$$

where h equals the dimension of the manifold, which is the topological entropy of the geodesic flow on the unit tangent bundle of M.

The notation $f(R) \sim g(R)$ means $\lim_{R\to\infty} \frac{f(R)}{g(R)} = 1$.

Inspired by this result, Athreya, Bufetov, Eskin and Mirzakhani studied lattice point asymptotics in Teichmüller space. Let $S_{g,n}$ denote a closed surface of genus g with n

2020 Mathematics Subject Classification. 30F60.
Keywords. Teichmüller theory, mapping class groups.

punctures such that $3g - 3 + n > 0$, and we let $\mathrm{Mod}_{g,n}$ and $(\mathcal{T}_{g,n}, d)$ denote the corresponding mapping class group and Teichmüller space with Teichmüller metric. Then $\mathrm{Mod}_{g,n}$ acts on $\mathcal{T}_{g,n}$ by isometries. We use Mod_g, $\mathcal{T}_g$ to denote $\mathrm{Mod}_{g,0}$, $\mathcal{T}_{g,0}$ for simplicity. They showed the orbits of mapping class group have analogous asymptotics.

Theorem 1.2 (Athreya, Bufetov, Eskin, and Mirzakhani [2]). *For any $\mathcal{X}, \mathcal{Y} \in \mathcal{T}_g$, we have*

$$|\mathrm{Mod}_g \cdot \mathcal{Y} \cap B_R(\mathcal{X})| \sim e^{hR}.$$

Note in their original paper, there is a factor of $\tau(\mathcal{X})\tau(\mathcal{Y})$ in front of e^{hR}, τ is called the Hubbard–Masur function. Mirzakhani later showed that τ is a constant function, see [3]. Moreover, again we let M be a compact negatively curved Riemannian manifold and let $\widetilde{M}$ denote its universal cover, we recall the following result from Parkkonen and Paulin [13] about the lattice point asymptotics for conjugacy classes of $\pi_1(M)$.

Theorem 1.3 (Parkkonen, Paulin [13]). *Let G be a nontrivial conjugacy class of $\pi_1(M)$, for any $x \in \widetilde{M}$, we have*

$$\lim_{R \to \infty} \frac{1}{R} \ln |G \cdot \mathcal{X} \cap B_R(\mathcal{X})| = \frac{h}{2}.$$

Inspired by this result, we wish to explore the lattice point asymptotics for conjugacy classes of $\mathrm{Mod}_{g,n}$. The Nielsen–Thurston classification [14] says every element in Mod_g is one of the three types: periodic, reducible, or pseudo-Anosov. When $f \in \mathrm{Mod}_{g,n}$ is a Dehn twist, a special kind of reducible element, we prove in [6] that the lattice point growth for the conjugacy class of f is "coarsely" asymptotic to $e^{\frac{h}{2}R}$. In this paper, we are interested in pseudo-Anosov elements. Let $PA \subset \mathrm{Mod}_g$ denote the subset of pseudo-Anosov elements. Maher showed pseudo-Anosov elements are generic in the following sense.

Theorem 1.4 (Maher [9]). *For any $\mathcal{X}, \mathcal{Y} \in \mathcal{T}_g$, we have*

$$\frac{|PA \cdot \mathcal{Y} \cap B_R(\mathcal{X})|}{|\mathrm{Mod}_g \cdot \mathcal{Y} \cap B_R(\mathcal{X})|} \sim 1.$$

The above Theorems 1.3, 1.4, motivate us to explore the lattice point asymptotics for pseudo-Anosov conjugacy classes. For any mapping class $\phi \in \mathrm{Mod}_{g,n}$, we use $[\phi] = \{f\phi f^{-1} \mid f \in \mathrm{Mod}_{g,n}\}$ to denote its conjugacy class. For simplicity of notation, we denote $\Gamma_R(\mathcal{X}, \mathcal{Y}, \phi) = |[\phi] \cdot \mathcal{Y} \cap B_R(\mathcal{X})|$.

Let $C > 0$, we say $f(R) \overset{C}{\preceq} g(R)$ if for any $\delta > 1$, there exists a $M(\delta)$ such that $\frac{1}{\delta C} \cdot f(R) \leq g(R)$ for any $R \geq M(\delta)$. We say $f(R) \overset{C}{\sim} g(R)$ if $f(R) \overset{C}{\preceq} g(R)$ and $g(R) \overset{C}{\preceq} f(R)$, thus $f(R) \overset{1}{\sim} g(R)$ is the same as $f(R) \sim g(R)$. Accordingly, we simply write $\preceq$, $\sim$ when $C = 1$. The main results of this paper are the following.

Theorem 1.5. *Fix $S_{g,n}$ and $\varepsilon > 0$, there exists a constant $A > 0$ such that given any ε-thick pseudo-Anosov element ϕ with translation distance $\tau \geq A$ and given any $\mathcal{X}, \mathcal{Y}$ in $\mathcal{T}_{g,n}$, there exists a corresponding $G(\mathcal{X}, \mathcal{Y}, \phi)$ such that*

$$\Gamma_R(\mathcal{X}, \mathcal{Y}, \phi) \overset{G(\mathcal{X},\mathcal{Y},\phi)}{\sim} e^{\frac{h}{2}R}.$$

Corollary 1.6. *Fix $S_{g,n}$, given any pseudo-Anosov element ϕ and given any $\mathcal{X}, \mathcal{Y}$ in $\mathcal{T}_{g,n}$. There exists a power N depending on ϕ such that for any $k \geq N$, there is a corresponding $G(\mathcal{X}, \mathcal{Y}, \phi, k)$ so that the following holds:*

$$\Gamma_R(\mathcal{X}, \mathcal{Y}, \phi^k) \overset{G(\mathcal{X},\mathcal{Y},\phi,k)}{\sim} e^{\frac{h}{2}R}.$$

In parallel with Theorem 1.3 above, we note the above Theorem 1.5 and Corollary 1.6 imply the following.

Corollary 1.7. *Fix $S_{g,n}$, given any pseudo-Anosov element ϕ and given any $\mathcal{X}, \mathcal{Y}$ in $\mathcal{T}_{g,n}$, for all sufficiently large k we have*

$$\lim_{R\to\infty} \frac{1}{R} \ln \Gamma_R(\mathcal{X}, \mathcal{Y}, \phi^k) = \frac{h}{2}.$$

These results again indicate the similarity of Teichmüller spaces and hyperbolic spaces in terms of lattice point asymptotics.

2. Background

We refer the reader to [4] for the general background materials. Let $\mathrm{Homeo}^+_{g,n}$ denote the group of all the orientation-preserving homeomorphisms of $S_{g,n}$ preserving the set of punctures, and let $\mathrm{Homeo}^0_{g,n}$ denote the connected component of the identity. The mapping class group of $S_{g,n}$ is defined to be the group of isotopy classes of orientation-preserving homeomorphisms,

$$\mathrm{Mod}_{g,n} = \mathrm{Homeo}^+_{g,n}/\mathrm{Homeo}^0_{g,n} = \mathrm{Homeo}^+_{g,n}/\text{isotopy}.$$

A hyperbolic structure $\mathcal{X}$ on $S_{g,n}$ is a pair (X, ϕ) where $\phi\colon S_{g,n} \to X$ is a homeomorphism and X is a hyperbolic surface. We say two hyperbolic structures $\mathcal{X} = (X, \phi)$, $\mathcal{Y} = (Y, \psi)$ are isotopic if there is an isometry $I\colon X \to Y$ isotopic to $\psi \circ \phi^{-1}$. The Teichmüller space $\mathcal{T}_{g,n}$ is the set of hyperbolic structures on $S_{g,n}$ modulo isotopy. We let $\mathcal{X} = (X, \phi)$, $\mathcal{Y} = (Y, \psi)$ denote elements in $\mathcal{T}_{g,n}$. Given any $\mathcal{X}, \mathcal{Y} \in \mathcal{T}_{g,n}$, the Teichmüller distance between them is defined to be

$$d_{\mathcal{T}}(\mathcal{X}, \mathcal{Y}) = \frac{1}{2} \inf_{f \sim \phi\circ\psi^{-1}} \log(K_f)$$

where the infimum is over all quasi-conformal homeomorphisms f isotopic to $\phi \circ \psi^{-1}$ and K_f is the quasi-conformal dilatation of f. Equipped with the Teichmüller metric, the Teichmüller space is a complete, unique geodesic metric space.

Given any $\mathcal{X} = (X, \phi) \in \mathcal{T}_{g,n}$ and given any isotopy class γ of nontrivial simple closed curves on $S_{g,n}$, there exists a unique geodesic in the free homotopy class of $\phi(\gamma)$ on X. We define $\ell_X(\phi(\gamma))$ to be the length of this unique geodesic and define $\ell_{\mathcal{X}}(\gamma) = \ell_X(\phi(\gamma))$. A pants decomposition of the surface $S_{g,n}$ is a collection of pairwise disjoint nontrivial simple closed curves $\gamma_1, \ldots, \gamma_{3g-3+n}$ on $S_{g,n}$, together they decompose the surface $S_{g,n}$ into $2g + n - 2$ pairs of pants. Using pants decomposition and by introducing Fenchel–Nielsen coordinates, Fricke [5] showed that $\mathcal{T}_{g,n}$ is homeomorphic to $\mathbb{R}^{6g+2n-6}$.

The mapping class group acts isometrically on $\mathcal{T}_{g,n}$ by changing the marking $(f, (X, \phi)) \mapsto (X, \phi \circ f^{-1})$. This action is properly discontinuous but not cocompact. The quotient $\mathcal{M}_{g,n} = \mathcal{T}_{g,n}/\mathrm{Mod}_{g,n}$ is called the moduli space, and it is a non-compact orbifold parameterizing hyperbolic surfaces homeomorphic to $S_{g,n}$.

Given any $\varepsilon > 0$, the ε-thick part of Teichmüller space is defined to be

$$\mathcal{T}^{\varepsilon}_{g,n} = \{\mathcal{X} \in \mathcal{T}_{g,n} \mid \ell_{\mathcal{X}}(\alpha) \geq \varepsilon \text{ for any simple closed curve } \alpha \text{ on } S_{g,n}\}$$

and consequently the ε-thick part of moduli space is $\mathcal{M}^{\varepsilon}_{g,n} = \mathcal{T}^{\varepsilon}_{g,n}/\mathrm{Mod}_{g,n}$. The Mumford compactness criterion [12] says $\mathcal{M}^{\varepsilon}_{g,n}$ is compact for any $\varepsilon > 0$.

Similar to hyperbolic isometries acting on hyperbolic space, each pseudo-Anosov element $\phi \in \mathrm{Mod}_{g,n}$ acts on $\mathcal{T}_{g,n}$ by translating along its corresponding bi-infinite geodesic axis, denoted as $\mathrm{axis}(\phi)$ with translation distance denoted as $\tau(\phi)$. Moreover, we say a pseudo-Anosov element $\phi \in \mathrm{Mod}_{g,n}$ is called ε-thick if $\mathrm{axis}(\phi) \subset \mathcal{T}^{\varepsilon}_{g,n}$.

For any $r > 0$ and for every closed set $W \subset \mathcal{T}_{g,n}$, denote $\mathcal{N}_r(W)$ the r-neighborhood of W. For every closed set $C \subset \mathcal{T}_{g,n}$, the closest point projection map is defined as follows:

$$\pi_C(x) = \{y \in C \mid d(x, y) = d(x, C) = \inf_{z \in C} d(x, z)\}.$$

As one of the early works exploring negative curvature in Teichmüller space, the result below from Minsky [11] says that ε-thick geodesics in Teichmüller space satisfy the strongly contracting property.

Theorem 2.1 (Minsky [11]). *There exists a constant $A > 0$ depending on ε, $\chi(S_{g,n})$ such that if $\mathcal{L}$ is an ε-thick geodesic in $\mathcal{T}_{g,n}$, the projection $\pi_{\mathcal{L}}$ satisfies*

$$\mathrm{diam}(\pi_{\mathcal{L}}(\mathcal{X})) \leq A$$

for any $\mathcal{X} \in \mathcal{T}_{g,n}$. Moreover, if $\mathcal{X}$ satisfies $d(\mathcal{X}, \mathcal{L}) > A$, then we have

$$\mathrm{diam}(\pi_{\mathcal{L}}(\mathcal{N}_{d(\mathcal{X},L)-A}(\mathcal{X}))) \leq A.$$

For any two closed sets $A, B \subset \mathcal{T}_{g,n}$ we let $d(A, B)$ denote the minimal distance between them. For $\mathcal{L}$ a geodesic in $\mathcal{T}_{g,n}$, we let $d^{\mathcal{L}}_{\pi}(C, W) = \mathrm{diam}(\pi_{\mathcal{L}}(C) \cup \pi_{\mathcal{L}}(W))$. We can pick the constant A in Theorem 2.1 in a way so that the following holds.

Corollary 2.2 (Arzhantseva, Cashen, and Tao [1]). *Let $\mathcal{L}$ be an ε-thick geodesic in $\mathcal{T}_{g,n}$ and let $\mathcal{X}, \mathcal{Y} \in \mathcal{T}_{g,n}$ be such that $d_\pi^{\mathcal{L}}(\mathcal{X}, \mathcal{Y}) > A$, then*

$$d(\mathcal{X}, \mathcal{Y}) \geq d(\mathcal{X}, \pi_{\mathcal{L}}(\mathcal{X})) + d_\pi^{\mathcal{L}}(\mathcal{X}, \mathcal{Y}) + d(\pi_{\mathcal{L}}(\mathcal{Y}), \mathcal{Y}) - A.$$

Moreover, if $\mathcal{Y}$ happens to be on the geodesic $\mathcal{L}$, then $\pi_{\mathcal{L}}(\mathcal{Y}) = \{\mathcal{Y}\}$ and

$$d(\mathcal{X}, \mathcal{Y}) \geq d(\mathcal{X}, \pi_{\mathcal{L}}(\mathcal{X})) + d(\pi_{\mathcal{L}}(\mathcal{X}), \mathcal{Y}) - A.$$

For any pseudo-Anosov element $\phi \in \mathrm{Mod}_{g,n}$, we denote $\pi_{\mathrm{axis}(\phi)}$ as π_ϕ. Since ϕ acts by translation along its axis, it commutes with the projection map π_ϕ. That is, for any $\mathcal{X} \in \mathcal{T}_{g,n}$, we have $\pi_\phi(\phi(\mathcal{X})) = \phi(\pi_\phi(\mathcal{X}))$.

By using Theorem 2.1 and Corollary 2.2, one can show if an ε-thick pseudo-Anosov element ψ has sufficiently large translation length, then the distance it translates a point "far" from the axis is roughly twice the distance from the point to the axis. See Figure 1 for an illustration.

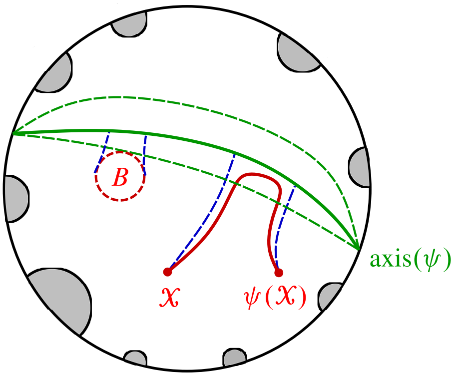

Figure 1. Shaded areas are ε-thin parts. Given an ε-thick pseudo-Anosov element ψ with $\tau(\psi) > A$, the diameter of the projection of any balls like B to axis(ψ) is bounded by A, see Theorem 2.1. The geodesic from $\mathcal{X}$ to $\psi(\mathcal{X})$ fellow travels axis(ψ), see Corollary 2.3.

Corollary 2.3. *Let ϕ be an ε-thick pseudo-Anosov element with translation distance $\tau(\phi) > A$. Then for any $\mathcal{X} \in \mathcal{T}_{g,n}$ and for any $\psi \in [\phi]$, we have*

$$2d(\mathcal{X}, \pi_\psi(\mathcal{X})) + \tau(\phi) - A \leq d(\mathcal{X}, \psi(\mathcal{X})) \leq 2d(\mathcal{X}, \pi_\psi(\mathcal{X})) + \tau(\phi) + 2A.$$

Proof. Since translation distance is invariant under conjugation, $\tau(\psi) = \tau(\phi) > A$ for any $\psi \in [\phi]$. Thus we have

$$d_\pi^\psi(\mathcal{X}, \psi(\mathcal{X})) = \mathrm{diam}(\pi_\psi(\mathcal{X}) \cup \pi_\psi(\psi(\mathcal{X}))) = \mathrm{diam}(\pi_\psi(\mathcal{X}) \cup \psi(\pi_\psi(\mathcal{X})))$$

where by Theorem 2.1

$$\tau(\phi) \leq \mathrm{diam}(\pi_\psi(\mathcal{X}) \cup \psi(\pi_\psi(\mathcal{X}))) \leq \tau(\phi) + 2\,\mathrm{diam}(\pi_\psi(\mathcal{X})) \leq \tau(\phi) + 2A.$$

Take any $\mathcal{X} \in \mathcal{T}_{g,n}$, by the triangle inequality, we have

$$\begin{aligned} d(\mathcal{X}, \psi(\mathcal{X})) &\le d(\mathcal{X}, \pi_\psi(\mathcal{X})) + d_\pi^\psi(\mathcal{X}, \psi(\mathcal{X})) + d(\psi(\mathcal{X}), \pi_\psi(\psi(\mathcal{X}))) \\ &\le 2d(\mathcal{X}, \pi_\psi(\mathcal{X})) + \tau(\phi) + 2A. \end{aligned}$$

Meanwhile we can apply the previous Corollary 2.2 and get

$$\begin{aligned} d(\mathcal{X}, \psi(\mathcal{X})) &\ge d(\mathcal{X}, \pi_\psi(\mathcal{X})) + d_\pi^\psi(\mathcal{X}, \psi(\mathcal{X})) + d(\psi(\mathcal{X}), \pi_\psi(\psi(\mathcal{X}))) - A \\ &\ge 2d(\mathcal{X}, \pi_\psi(\mathcal{X})) + \tau(\phi) - A. \end{aligned}$$

The result follows. ■

3. Proof of the main theorem

By Theorem 1.2, for any $\mathcal{X} \in \mathcal{T}_{g,n}$, we have

$$|\mathrm{Mod}_{g,n} \cdot \mathcal{X} \cap B_r(\mathcal{X})| \sim e^{hr}.$$

For any $r > 0$, define the set

$$\Omega_r(\mathcal{X}) = \{f \in \mathrm{Mod}_{g,n} \mid d(\mathcal{X}, f\mathcal{X}) \le r\}$$

and denote N the maximal order of point stabilizer subgroups in $\mathrm{Mod}_{g,n}$. Such maximum exists as shown by Kerckhoff in [7]. It follows that

$$|\mathrm{Mod}_{g,n} \cdot \mathcal{X} \cap B_r(\mathcal{X})| \le |\Omega_r(\mathcal{X})| \le N \cdot |\mathrm{Mod}_{g,n} \cdot \mathcal{X} \cap B_r(\mathcal{X})|$$

and therefore

$$e^{hr} \preceq |\Omega_r(\mathcal{X})| \preceq N \cdot e^{hr}.$$

Moreover, given any $\phi \in \mathrm{Mod}_{g,n}$, we have

$$\Gamma_r(\mathcal{X}, \mathcal{X}, \phi) \le \big|[\phi] \cap \Omega_r(\mathcal{X})\big| \le N \cdot \Gamma_r(\mathcal{X}, \mathcal{X}, \phi).$$

Rearranging the inequality above, we have

$$\frac{1}{N} \cdot \big|[\phi] \cap \Omega_r(\mathcal{X})\big| \le \Gamma_r(\mathcal{X}, \mathcal{X}, \phi) \le \big|[\phi] \cap \Omega_r(\mathcal{X})\big|. \tag{1}$$

We first prove a simplified version of the main theorem.

Theorem 3.1. *For any $S_{g,n}$ and $\varepsilon > 0$, there exists a constant $A > 0$ such that, given any ε-thick pseudo-Anosov element ϕ with translation distance $\tau \ge A$ and given any $\mathcal{X} \in$ $\mathrm{axis}(\phi)$, there exists a corresponding constant $G(\mathcal{X}, \phi) > 0$ such that*

$$\Gamma_R(\mathcal{X}, \mathcal{X}, \phi) \overset{G(\mathcal{X},\phi)}{\sim} e^{\frac{h}{2}R}.$$

Proof. Given ϕ, $\mathcal{X}$ satisfying the assumptions. For any R, define

$$P_R^+ = \left\{\psi \in [\phi] \mid d(\mathcal{X}, \pi_\psi(\mathcal{X})) \le \frac{R + A - \tau}{2}\right\},$$
$$P_R^- = \left\{\psi \in [\phi] \mid d(\mathcal{X}, \pi_\psi(\mathcal{X})) \le \frac{R - 2A - \tau}{2}\right\}.$$

Denote $\Omega_r(\mathcal{X})$ as $\Omega(r)$ for simplicity, by Corollary 2.3 we have

$$P_R^- \subset [\phi] \cap \Omega(R) \subset P_R^+. \tag{2}$$

We now work towards obtaining an upper bound for $|P_R^+|$. For any $\psi \in P_R^+$, there exists an $f \in \mathrm{Mod}_{g,n}$ such that $\psi = f\phi f^{-1}$. Since $\mathcal{X} \in \mathrm{axis}(\phi)$, $f(\mathcal{X})$ therefore lies on the axis(ψ). In particular, this means there exists a $k \in \mathbb{Z}$ such that

$$d(\psi^k \circ f(\mathcal{X}), \pi_\psi(\mathcal{X})) \le \frac{\tau}{2} \tag{3}$$

and therefore

$$d(\psi^k \circ f(\mathcal{X}), \mathcal{X}) \le d(\psi^k \circ f(\mathcal{X}), \pi_\psi(\mathcal{X})) + d(\mathcal{X}, \pi_\psi(\mathcal{X})) \le \frac{R + A}{2}.$$

See Figure 2 for an example.

We claim one can define an injective map from $P_R^+ \to \Omega(\frac{R+A}{2})$ by sending ψ to $\psi^k f$. Indeed, if there is any another $\eta \in P_R^+, \eta \ne \psi$, $\eta = h\phi h^{-1}$ for some $h \in \mathrm{Mod}_{g,n}$, then $h(\mathcal{X}) \in \mathrm{axis}(\eta)$ and there exists an $m \in \mathbb{Z}$ such that

$$d(\eta^m \circ h(\mathcal{X}), \pi_\eta(\mathcal{X})) \le \frac{\tau}{2} \quad \text{and} \quad d(\eta^m \circ h(\mathcal{X}), \mathcal{X}) \le \frac{R + A}{2}.$$

We claim in this case $\psi^k f \ne \eta^m h$. Indeed, suppose they are equal, then

$$\psi = \psi^k \psi \psi^{-k} = \psi^k f\phi f^{-1} \psi^{-k} = \eta^m h\phi h^{-1} \eta^{-m} = \eta^m \eta \eta^{-m} = \eta.$$

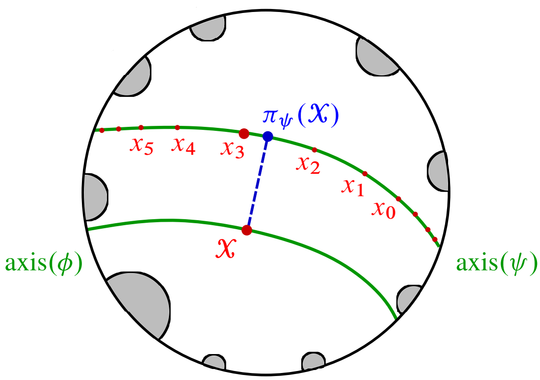

Figure 2. Each x_i denotes $\psi^i \circ f(x)$ and the distance between any two adjacent x_i is τ. The injective map maps $\mathcal{X}$ to x_3 since x_3 is the closest point to $\pi_\psi(\mathcal{X})$ in $\{x_i\}_{i \in \mathbb{Z}}$.

However, this contradicts $\psi \neq \eta$. This means for R large, we can inject P_R^+ into $\Omega(\frac{R+A}{2})$, so that

$$|P_R^+| \le \left|\Omega\left(\frac{R+A}{2}\right)\right| \preceq e^{\frac{hA}{2}} \cdot e^{\frac{hR}{2}}. \tag{4}$$

To obtain the lower bound for $|P_R^-|$, we define $\mathcal{A}_R = \{\mathrm{axis}(\psi) \mid \psi \in P_R^-\}$. This gives us a surjective map $F\colon P_R^- \to \mathcal{A}_R, \psi \mapsto \mathrm{axis}(\psi)$. By the definitions of $\mathcal{A}_R$ and P_R^-, each $\Theta \in \mathcal{A}_R$ has the form $\Theta = \mathrm{axis}(\psi)$ for some $\psi = f\phi f^{-1} \in P_R^-$, and this f can be chosen so that $f \in \Omega(\frac{R-2A}{2})$ by applying (3) to P_R^- instead. Thus each $\Theta \in \mathcal{A}_R$ can be written as $\mathrm{axis}(f\phi f^{-1})$ for some $f \in \Omega(\frac{R-2A}{2})$. For any $L < \frac{R-2A-\tau}{2}$, we define

$$\mathcal{A}_R^L = \{\Theta \in \mathcal{A}_R \mid d(\mathcal{X}, \pi_\Theta(\mathcal{X})) > \frac{R-2A-\tau}{2} - L\}$$

so that $\mathcal{A}_R^L \subset \mathcal{A}_R$. For each $\Theta \in \mathcal{A}_R$, we denote

$$H(\Theta) = \{f \in \Omega(\frac{R-2A}{2}) \mid \mathrm{axis}(f\phi f^{-1}) = \Theta\},$$

which is a subset of $\Omega(\frac{R-2A}{2})$.

By Corollary 2.2, for any $\Theta \in \mathcal{A}_R^L$, there are at most $\frac{2(L+A)}{\tau} + 2$ many $f \in H(\Theta)$ satisfying $\mathrm{axis}(f\phi f^{-1}) = \Theta$ since

$$d(\mathcal{X}, \pi_\Theta(\mathcal{X})) \in \left(\frac{R-2A-\tau}{2} - L, \frac{R-2A-\tau}{2}\right].$$

In the example of Figure 3, there are six such f for this Θ. This means

$$|\mathcal{A}_R^L| \ge \frac{\tau}{2(L+A+\tau)} \cdot \sum_{\Theta \in \mathcal{A}_R^L} |H(\Theta)|. \tag{5}$$

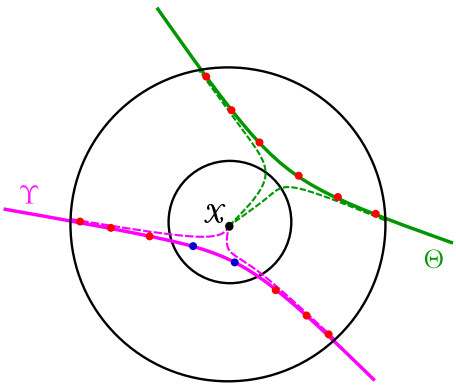

Figure 3. Θ is of type (a) and Υ is of type (c). The lengths of Θ and Υ intersecting $B_{\frac{R-2A-\tau}{2}}$ can be approximated by Corollary 2.2, which is shown as the dotted geodesic segments.

For any element $f \in \Omega(\frac{R-2A-\tau}{2})$, let us denote $\Theta_f = \operatorname{axis}(f\phi f^{-1})$, then each f is exactly one of the following types.

(a) Θ_f never enters $B_{\frac{R-2A-\tau}{2}-L}(\mathcal{X})$.

(b) Θ_f enters $B_{\frac{R-2A-\tau}{2}-L}(\mathcal{X})$ and $d(\mathcal{X}, f(\mathcal{X})) \le \frac{R-2A-\tau}{2} - L$.

(c) Θ_f enters $B_{\frac{R-2A-\tau}{2}-L}(\mathcal{X})$ and $d(\mathcal{X}, f(\mathcal{X})) > \frac{R-2A-\tau}{2} - L$.

The union of type (a) elements is $\bigsqcup_{\Theta \in \mathcal{A}_R^L} H(\Theta)$, and the union of type (b) elements is $\Omega(\frac{R-2A-\tau}{2} - L) \subset \Omega(\frac{R-2A}{2} - L)$. By Corollary 2.2, we notice there are at most $\frac{2(L+A)}{\tau}$ many type (c) elements that can share the same axis, and the number of axes going through $B_{\frac{R-2A-\tau}{2}-L}(\mathcal{X})$ is bounded by $|\Omega(\frac{R-2A}{2} - L)|$. In the example of Figure 3, there are six f satisfying type (c) conditions sharing the axis Υ. Notice there are two f that realize $\Upsilon = \Theta_f$ but not satisfy the type (c) assumption. Since type (a), (b), (c) elements compose $\Omega(\frac{R-2A-\tau}{2})$, we have

$$\sum_{\Theta \in \mathcal{A}_R^L} |H(\Theta)| \ge \Big|\Omega\Big(\frac{R-2A-\tau}{2}\Big)\Big| - \Big(1 + \frac{2(L+A)}{\tau}\Big) \cdot \Big|\Omega\Big(\frac{R-2A}{2} - L\Big)\Big|.$$

Moreover, we let L be a constant satisfying $e^{hL} > 2 \cdot e^{h\frac{\tau}{2}} \cdot N(1 + \frac{2(L+A)}{\tau})$, then

$$\sum_{\Theta \in \mathcal{A}_R^L} |H(\Theta)| \succeq e^{\frac{h(R-2A-\tau)}{2}} - \Big(1 + \frac{2(L+A)}{\tau}\Big) \cdot N \cdot e^{\frac{h(R-2A)}{2} - hL} \tag{6}$$

$$= e^{\frac{h}{2}R} \cdot e^{-hA} \cdot \left(\frac{1}{e^{h\frac{\tau}{2}}} - \frac{N \cdot \big(1 + \frac{2(L+A)}{\tau}\big)}{e^{hL}}\right) \ge e^{\frac{h}{2}R} \cdot \frac{1}{2e^{h(\frac{\tau}{2}+A)}}.$$

Thus, to construct the lower bound for $|P_R^-|$, we let L be a constant satisfying $e^{hL} > 2 \cdot e^{h\frac{\tau}{2}} \cdot N(1 + \frac{2(L+A)}{\tau})$. Applying formulas (5), (6) from above, for R large we have

$$|P_R^-| \ge |\mathcal{A}_R| \ge |\mathcal{A}_R^L| \ge \frac{\tau}{2(L+A+\tau)} \cdot \sum_{\Theta \in \mathcal{A}_R^L} |H(\Theta)| \tag{7}$$

$$\succeq e^{\frac{h}{2}R} \cdot \frac{\tau}{2(L+A+\tau)e^{hA}} \cdot \frac{1}{2e^{h(\frac{\tau}{2}+A)}}.$$

Finally, combining formulas (1), (2), (7) we have

$$\big|[\phi] \cdot \mathcal{X} \cap B_R(\mathcal{X})\big| \ge \frac{1}{N} \cdot \big|[\phi] \cap \Omega(R)\big| \ge \frac{1}{N} \cdot |P_R^-| \succeq G_L(\mathcal{X}, \phi) \cdot e^{\frac{h}{2}R}$$

where

$$G_L(\mathcal{X}, \phi) = \frac{\tau}{2N(L+A+\tau)e^{hA}} \cdot \frac{1}{2e^{h(\frac{\tau}{2}+A)}}.$$

And combining formulas (1), (2), (4) we have

$$\big|[\phi] \cdot \mathcal{X} \cap B_R(\mathcal{X})\big| \le \big|[\phi] \cap \Omega(R)\big| \le |P_R^+| \preceq G_U(\mathcal{X}, \phi) \cdot e^{\frac{h}{2}R}$$

where

$$G_U(\mathcal{X}, \phi) = Ne^{\frac{hA}{2}}.$$

Recall $f(R) \overset{A}{\preceq} g(R)$ is the same as $f(R) \overset{1}{\preceq} Ag(R)$. Thus we have

$$e^{\frac{h}{2}R} \overset{G_L^{-1}(\mathcal{X},\phi)}{\preceq} \left|[\phi]\cdot\mathcal{X}\cap B_R(\mathcal{X})\right| \overset{G_U(\mathcal{X},\phi)}{\preceq} e^{\frac{h}{2}R}.$$

This means by setting

$$G(\mathcal{X},\phi) = \max\{G_L^{-1}(\mathcal{X},\phi), G_U(\mathcal{X},\phi)\},$$

we obtain the desired result. ∎

Now we are ready to prove the general case.

Proof of Theorem 1.5. Take any $\mathcal{X},\mathcal{Y} \in \mathcal{T}_{g,n}$, let $\mathcal{X}'$ be a point in $\pi_\phi(\mathcal{X})$ and let D be the maximum between $\operatorname{diam}(\mathcal{X}\cup\pi_\phi(\mathcal{X}))$ and $\operatorname{diam}(\pi_\phi(\mathcal{X})\cup\mathcal{Y})$. Since the mapping class group is acting by isometries, we have

$$\left|[\phi]\cdot\mathcal{Y}\cap B_R(\mathcal{X})\right| \geq \left|[\phi]\cdot\mathcal{X}'\cap B_{R-D}(\mathcal{X})\right| \geq \left|[\phi]\cdot\mathcal{X}'\cap B_{R-2D}(\mathcal{X}')\right|,$$
$$\left|[\phi]\cdot\mathcal{Y}\cap B_R(\mathcal{X})\right| \leq \left|[\phi]\cdot\mathcal{X}'\cap B_{R+D}(\mathcal{X})\right| \leq \left|[\phi]\cdot\mathcal{X}'\cap B_{R+2D}(\mathcal{X}')\right|.$$

By applying these inequalities and by applying Theorem 3.1 to ϕ and $\mathcal{X}'$, without loss of generality, we get the desired result by setting $G(\mathcal{X},\mathcal{Y},\phi) = G(\mathcal{X}',\phi)\cdot e^{hD}$. ∎

Proof of Corollary 1.6. Given ϕ, we pick ε so that $\operatorname{axis}(\phi)$ is in $\mathcal{T}^{\varepsilon}_{g,k}$. Since $\tau(\phi^k) = k\cdot\tau(\phi)$ for any pseudo-Anosov element ϕ, there exists a $N(\phi)$ such that $\tau(\phi^k)\geq A$ for any $k\geq N(\phi)$. We now can apply Theorem 1.5, and the corresponding error constant G depends on $\mathcal{X},\mathcal{Y},\phi,k$. ∎

Proof of Corollary 1.7. Assuming the conditions, we can apply Corollary 1.6. This means for any $k\geq N$ and for any $\delta>1$, there exists a $M(\delta)$ such that

$$\frac{1}{\delta G(\mathcal{X},\mathcal{Y},\phi,k)}\cdot e^{\frac{h}{2}R} \leq \Gamma_R(\mathcal{X},\mathcal{Y},\phi^k) \leq \delta G(\mathcal{X},\mathcal{Y},\phi,k)\cdot e^{\frac{h}{2}R}$$

for any $R\geq M(\delta)$. Let $\varepsilon>0$, one can pick $\delta>0$ and pick $M(\varepsilon)\geq M(\delta)$ so that

$$\delta G(\mathcal{X},\mathcal{Y},\phi,k) \leq e^{\varepsilon\frac{h}{2}R},$$
$$e^{-\varepsilon\frac{h}{2}R} \leq \frac{1}{\delta G(\mathcal{X},\mathcal{Y},\phi,k)},$$

for any $R\geq M(\varepsilon)$. This implies for any $\varepsilon>0$, we have

$$e^{(1-\varepsilon)\frac{h}{2}R} \leq \Gamma_R(\mathcal{X},\mathcal{Y},\phi^k) \leq e^{(1+\varepsilon)\frac{h}{2}R},$$
$$(1-\varepsilon)\frac{h}{2}R \leq \ln\Gamma_R(\mathcal{X},\mathcal{Y},\phi^k) \leq (1+\varepsilon)\frac{h}{2}R,$$
$$(1-\varepsilon)\frac{h}{2} \leq \frac{1}{R}\ln\Gamma_R(\mathcal{X},\mathcal{Y},\phi^k) \leq (1+\varepsilon)\frac{h}{2},$$

whenever $R\geq M(\varepsilon)$. That is,

$$\lim_{R\to\infty}\frac{1}{R}\ln\Gamma_R(\mathcal{X},\mathcal{Y},\phi^k) = \frac{h}{2}.$$

This finishes the proof. ∎

Acknowledgments. I would like to thank my advisor, Spencer Dowdall, for suggesting this project, and for his guidance and consistent support throughout.

Funding. The author was partially supported by NSF grant DMS-2102018.

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Received 4 June 2021; revised 8 July 2022.

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