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Totally non-congruence Veech groups

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Abstract. Veech groups are discrete subgroups of $\mathrm{SL}(2,\mathbb{R})$ which play an important role in the theory of translation surfaces. For a special class of translation surfaces called origamis or square-tiled surfaces, their Veech groups are subgroups of finite index of $\mathrm{SL}(2,\mathbb{Z})$. We show that each stratum of the space of translation surfaces contains infinitely many origamis whose Veech group is a totally non-congruence group, i.e., it surjects to $\mathrm{SL}(2,\mathbb{Z}/n\mathbb{Z})$ for any n.

1. Introduction

Within the last thirty years, the study of *translation surfaces* has become an active field in mathematics. Their moduli spaces come equipped with a natural action of $\mathrm{SL}(2,\mathbb{R})$. It is one of the principal goals in this field to understand the orbits of this action. This study culminated in the famous breakthrough result of Eskin, Mirzakhani and Mohammadi, namely the so-called *magic wand theorem* (cf. [3,4]). The *Veech group* $\Gamma(X,\mu)$ associated to a translation surface (X,μ) plays a crucial role in this topic. $\Gamma(X,\mu)$ is the stabiliser of (X,μ) under the action of $\mathrm{SL}(2,\mathbb{R})$. It turns out to be a discrete subgroup of $\mathrm{SL}(2,\mathbb{R})$ and it carries a lot of information about the dynamical flow on the translation surface and about the Teichmüller flow defined by (X,μ). *Origamis* or *square-tiled surfaces* are a particularly important class of translation surfaces. These surfaces are tessellated by finitely many Euclidean unit squares. Their Veech groups are especially easy to handle. They are subgroups of finite index of $\mathrm{SL}(2,\mathbb{Z})$ and can be calculated explicitly from the combinatorial data which define the origami. Furthermore, the set of origamis is dense in the moduli space of translation surfaces. The action of $\mathrm{SL}(2,\mathbb{R})$ on the set of translation surfaces is restricted to an action of $\mathrm{SL}(2,\mathbb{Z})$ on origamis.

It is still an open question whether all subgroups of $\mathrm{SL}(2,\mathbb{Z})$ of finite index occur as Veech groups of origamis. A major result in this direction was achieved in [2] where it is proved that all subgroups of finite index (satisfying a slight condition) of the principal congruence group $\Gamma(2)$ occur as Veech groups, where $\Gamma(2)$ is the group of matrices which are congruent to the identity matrix modulo 2. As a result in some sense in the opposite direction, it is shown in [18] that all congruence groups (cf. below) of prime level, except five, occur as Veech groups.

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It is particularly interesting to study Veech groups of origamis that lie in the same fixed stratum, i.e., we fix the genus and the cone angles of the singularities (see below). In [9, 17], the authors succeeded to give a complete classification of the $\mathrm{SL}(2,\mathbb{Z})$-orbits of origamis in the stratum $\mathcal{H}_2(2)$ of translation surfaces of genus 2 with one singularity of angle 6π. In this case, the set of origamis with d squares decomposes, depending on d, into one or two orbits. There are only a few further classification results for certain subloci of strata (cf. [12–14]). For general strata, the classification problem is open. However, there exists a conjecture for a precise description of the orbits in each stratum by Delecroix and Lelièvre based on computer experiments.

A *congruence subgroup* Γ of $\mathrm{SL}(2,\mathbb{Z})$ is a subgroup of $\mathrm{SL}(2,\mathbb{Z})$ which is fully determined by its image in $\mathrm{SL}(2,\mathbb{Z}/n\mathbb{Z})$ for some $n \in \mathbb{N}$, i.e., it is the preimage of its image in $\mathrm{SL}(2,\mathbb{Z}/n\mathbb{Z})$ under the canonical projection $\mathrm{SL}(2,\mathbb{Z}) \to \mathrm{SL}(2,\mathbb{Z}/n\mathbb{Z})$. It turns out that such groups are rare among all finite index subgroups of $\mathrm{SL}(2,\mathbb{Z})$. Turning to Veech groups of origamis: there are several families of origamis whose Veech groups could be explicitly determined as congruence groups in [5, 6, 19]. In [20] first examples of Veech groups that are non-congruence groups were detected. Hubert and Lelièvre proved in [8] that for all but one of the origamis of genus 2 with one singularity their Veech group is a non-congruence group.

For an arbitrary subgroup Γ of $\mathrm{SL}(2,\mathbb{Z})$ of finite index, we may measure how much information we lose if we consider all its images in the finite quotient groups $\mathrm{SL}(2,\mathbb{Z}/n\mathbb{Z})$. In particular, all information is lost if for all n the image is the full group $\mathrm{SL}(2,\mathbb{Z}/n\mathbb{Z})$. In this case, we call Γ a *totally non-congruence group*. In [23] a criterion is given which detects totally non-congruence groups (cf. [23, Theorem 2]). It was further shown that in the stratum $\mathcal{H}_2(2)$ all Veech groups of origamis are totally non-congruence groups or almost totally non-congruence groups (cf. [23, Theorem 3]). Finally, it was shown that for each stratum $\mathcal{H}_{k+1}(2k)$ of translation surfaces with only one singularity of cone angle $(k+1)2\pi$ there are infinitely many origamis whose Veech group is a totally non-congruence group (cf. [23, Theorem 4]).

In this article, we generalise this statement to all strata. For this we first improve the criterion for totally non-congruence groups from [23, Theorem 2] and get the following very handy conditions which assure that a group Γ is a totally non-congruence group.

Theorem 1.1. *Let Γ be a finite index subgroup of* $\mathrm{SL}(2,\mathbb{Z})$. *Denote $e_1 = \binom{1}{0}$ and $T = \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$. Suppose that for each prime p, there exist matrices $A_1, A_2 \in \mathrm{SL}(2,\mathbb{Z})$ with the following properties:*

(A) $\forall j \in \mathbb{N}$: $A_1 e_1 \neq j \cdot A_2 e_1$ *modulo p.*

(B) *There exist $m_1, m_2 \in \mathbb{N}$ with*

$$A_1 T^{m_1} A_1^{-1} \quad \text{and} \quad A_2 T^{m_2} A_2^{-1} \quad \text{are contained in } \Gamma,$$

such that p divides neither m_1 nor m_2.

Then Γ is a totally non-congruence group.

We then describe a method to construct one-cylinder origamis in each stratum for which we have a good control over the cylinder decompositions in horizontal and vertical direction and in the diagonal direction given by the vector $v = \binom{1}{1}$. Choosing special elements of this family, we finally prove the following theorem.

Theorem 1.2. *Every stratum contains an infinite family of origamis whose Veech groups are totally non-congruence groups.*

2. Preliminaries

In this section, we give a concise introduction to translation surfaces, origamis and Veech groups suited to the purpose of this article. You can find more elaborate introductions to this topic for example in [1, 10, 15, 18, 22]. For the proof of the facts that we state here, we refer to these references.

Translation surfaces, origamis and strata. A *(finite) translation surface* is a surface X with an atlas μ to $\mathbb{R}^2$ such that all transition maps of the atlas μ are translations. The translation surface inherits a natural metric from the Euclidean metric in $\mathbb{R}^2$. Furthermore, we have a well-defined notion of directions since they are invariant under translations. Thus we may speak, for example, of *horizontal* and *vertical geodesics*, or more general of geodesics in direction $v \in \mathbb{R}^2$. Moreover, using local charts, we can assign to each geodesic segment a vector in the plane $\mathbb{R}^2$ which is its *development vector*. Let $\overline{X}$ be the metric completion of X. The points in $\overline{X}\backslash X$ are called *the singularities* of X. In this article, we consider the classical situation of finite *translation surfaces*, i.e., translation surfaces (X, μ) such that the metric completion is compact, the set of singularities is discrete and all singularities are cone points of finite cone angle $k2\pi$ ($k \in \mathbb{N}$). A geodesic segment between two (possibly equal) singularities which does not contain any further singularity is called a *saddle connection*. Further important geometric data of the translation surface (X, μ) are its set of closed geodesics and its set of maximal cylinders in a given direction $v \in \mathbb{R}^2$. Here a maximal cylinder is a maximal connected set of homotopic simple closed geodesics. For genus $g \geq 2$, every closed geodesic lies in a unique maximal cylinder in the direction v of the geodesic which is bounded by saddle connections, since we may move the geodesic transversely to v until we hit singularities.

Finite translation surfaces are naturally divided into strata by their type of singularities. More precisely, a finite translation surface (X, μ) is said to be *of type* $(\alpha_1, \ldots, \alpha_n)$, if $\overline{X}$ has n singularities of cone angles $(\alpha_1 + 1) \cdot 2\pi, \ldots, (\alpha_n + 1) \cdot 2\pi$. The usage of α_i instead of $\alpha_i + 1$ is related to the fact that a finite translation surface can equivalently be defined as a closed Riemann surface X together with a holomorphic differential ω. The charts of the atlas are then obtained by integrating with respect to ω, the singularities are the zeroes of ω and α_i is the order of the zero. We then define the *stratum* $\mathcal{H}_g(\alpha_1, \ldots, \alpha_n)$ as the set of all equivalence classes of translation surfaces of type $(\alpha_1, \ldots, \alpha_n)$ of genus g. Two translation surfaces (X_1, μ_1) and (X_2, μ_2) are *equivalent* if there exists a translation $f\colon X_1 \to X_2$, i.e., a homeomorphism which is a translation on each chart. We will usually write $(X,\mu) \in \mathcal{H}_g(\alpha_1,\ldots,\alpha_r)$ for the equivalence class defined by (X,μ). The set $\mathcal{H}_g(\alpha_1,\ldots,\alpha_n)$ is endowed with a topology itself. More precisely, there is a natural way to define local coordinates as manifold on a covering of it (cf. [24, Section 6.3]). Furthermore, $\mathcal{H}_g(\alpha_1,\ldots,\alpha_n)$ is endowed with a natural action of $\mathrm{SL}(2,\mathbb{R})$ as follows. For a translation surface (X,μ) and a matrix $A \in \mathrm{SL}(2,\mathbb{R})$, we define $A\cdot(X,\mu) = (X,\mu_A)$ to be the translation surface obtained from (X,μ) by composing each chart of μ with the linear map $z \mapsto A\cdot z$. It is one of the main objectives in the field to understand the orbits of this action.

There is yet another way how to define finite translation surfaces: Take finitely many polygons in the plane such that their edges come in pairs of edges of the same length and same direction. Glue for each pair its two edges by a translation. In this way, we obtain a closed surface $\bar{X}$. The points which come from the vertices of the polygons may be cone points. Removing them defines a translation surface X. If all the polygons which form the translation surface are copies of the Euclidean unit square, the translation surface is called an *origami* or a *square-tiled surface* (cf. Figure 1).

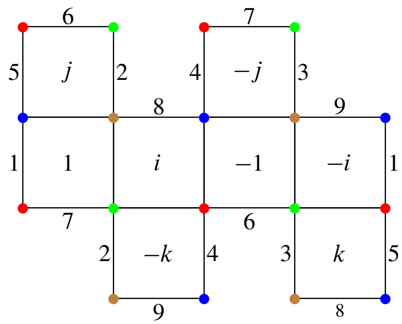

Figure 1. Gluing edges with same labels defines an origami of genus 3. This origami comes from [7].

Veech groups and the action of $\mathbf{SL(2,\mathbb{R})}$. Let (X,μ) be a finite translation surface of genus g in some stratum $\mathcal{H}_g(\alpha_1,\ldots,\alpha_n)$. The *Veech group* $\Gamma(X,\mu)$ is the stabiliser of (X,μ) for the action of $\mathrm{SL}(2,\mathbb{R})$ on $\mathcal{H}_g(\alpha_1,\ldots,\alpha_n)$. It can equivalently be defined in the following way. Consider the group $\mathrm{Aff}(X,\mu)$ of all *affine homeomorphisms* of X, i.e., homeomorphisms which are with respect to charts of the form $z \mapsto A\cdot z + b$ with $A \in \mathrm{SL}(2,\mathbb{R})$ and $b \in \mathbb{R}^2$. It turns out that since all transition maps are translations the matrix A is independent of the chosen charts. We obtain a group homomorphism $D\colon \mathrm{Aff}(X,\mu) \to \mathrm{SL}(2,\mathbb{R})$ which maps the affine homeomorphism f to the matrix A, i.e., to its derivative. The Veech group is the image of D, hence it consists of all matrices A which occur as derivative of some affine homeomorphism of the surface. It was already shown by Veech himself that $\Gamma(X,\mu)$ is a discrete subgroup of $\mathrm{SL}(2,\mathbb{R})$ (cf. [21, Proposition 2.7] or [22, Proposition 3.3] for a very nice presentation). Furthermore, two translation surfaces in the same $\mathrm{SL}(2,\mathbb{R})$-orbit have conjugated Veech groups.

Let us consider the example of the torus $\mathbb{R}^2/\mathbb{Z}^2$ endowed with the translation structure of its universal covering $\mathbb{R}^2$. Observe that the affine homeomorphisms lift to affine homeomorphisms of $\mathbb{R}^2$ which preserve the lattice $\mathbb{Z}^2$ up to a translation. And all such maps descend to the torus. Therefore, the Veech group is in this case $\mathrm{SL}(2,\mathbb{Z})$.

Special properties of origamis. We will use three equivalent ways to describe origamis, as explained in the following. The equivalences are described in more detail in [20, Section 1]. Recall that we obtain an origami by gluing copies of the Euclidean unit square along their edges which leads to a closed surface $\overline{X}$ tiled by squares. Hence, an origami made from d unit squares is fully determined by a pair of permutations (σ_a, σ_b) as follows. We label the squares with $\{1, \ldots, d\}$, then $\sigma_a(i)$ and $\sigma_b(i)$ denote the right and the upper neighbour of the square labelled by $i \in \{1, \ldots, d\}$. The fact that the surface is connected is equivalent to the fact that the subgroup of S_d generated by the two permutations σ_a and σ_b acts transitively on the set $\{1, \ldots, d\}$. If we choose another labelling of the squares, this leads to a simultaneous conjugation of the pair of permutations (σ_a, σ_b). Altogether, we obtain an equivalence between the set of origamis up to translations and the set of pairs (σ_a, σ_b) in S_d^2 up to simultaneous conjugation. There is yet another equivalent description of origamis which we will use. Observe that the surface $\overline{X}$ comes with a covering p to the square-torus $\mathbb{T}$ obtained by gluing parallel edges of the unit square. Namely, we map each square on $\overline{X}$ to the one square forming $\mathbb{T}$ and this map is well defined with respect to the gluings. The map p is an unramified covering for all points which are not vertices. Hence if $\infty \in \mathbb{T}$ is the one point obtained from the four vertices of the unit square, then $p\colon \overline{X} \to \mathbb{T}$ is ramified at most over ∞.

For an origami (X, μ), the Veech group is always a finite index subgroup of $\mathrm{SL}(2,\mathbb{Z})$. Here we should point to a subtlety in the definition of origami. Recall that we obtain the origami by gluing copies of the Euclidean unit square along their edges. More precisely, this gives us the metric completion $\overline{X}$ of the translation surface. The singularities of the translation surface stem from the vertices of the squares. However, not every vertex has to be a singularity. Now there are two different natural ways how two define the translation surfaces X. We might either remove only the singularities of $\overline{X}$ or we might remove all points which come from a vertex. In the second case, the Veech group is indeed a subgroup of $\mathrm{SL}(2,\mathbb{Z})$ of finite index, in the first case it is commensurable to $\mathrm{SL}(2,\mathbb{Z})$. However, it turns out that for *reduced origamis* one obtains equal Veech groups for the translation surface with only singularities removed and for the surface with all vertex points removed (cf. [11, Remark 2.9]). Following [16, Section 1.2], we call an origami *reduced* if the set of development vectors of all saddle connections generate $\mathbb{Z}^2$. This is a very mild restriction since any origami O is affine equivalent to a reduced origami O', i.e., there is some matrix $A \in \mathrm{GL}(2,\mathbb{R})$ such that $O' \sim A \cdot O$ and thus their Veech groups are conjugated in $\mathrm{GL}(2,\mathbb{R})$. Here the action of $\mathrm{GL}(2,\mathbb{R})$ on translation surfaces is defined just in the same way as the action of $\mathrm{SL}(2,\mathbb{R})$. In this article, we will restrict to reduced origamis and thus all Veech groups are subgroups of $\mathrm{SL}(2,\mathbb{Z})$ of finite index.

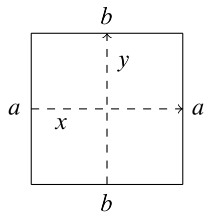

Figure 2. A torus T with the standard system of generators of the fundamental group.

Suppose that an origami is now given by the pair of permutations (σ_a, σ_b). We obtain the stratum in which the associated translation surface lives in the following way: Let $p\colon \bar{X} \to \mathbb{T}$ be the corresponding ramified cover of the torus. Let us choose a loop around the vertex of $\mathbb{T}$, namely $xyx^{-1}y^{-1}$, where x and y are the closed curves on $\mathbb{T}$ shown in Figure 2.

The connected components of the preimage of this curve are loops around the singularities. Hence the number of the connected components is the number of singularities. Furthermore, if the multiplicity of a component is k, then the corresponding singularity is of angle $2k\pi$. Hence the commutator $[\sigma_b^{-1}, \sigma_a^{-1}]$ determines the type of singularities that we obtain. More precisely, each cycle of length k in the commutator corresponds to a singularity of cone angle $k \cdot 2\pi$.

In the proof of our results, the following two facts are crucial which are described in more detail, e.g., in [23, Sections 2.2 and 2.3]:

(1) The action of $\mathrm{SL}(2,\mathbb{R})$ on translation surfaces restricts to an action of $\mathrm{SL}(2,\mathbb{Z})$ on origamis. The action can be explicitly given as described in the following. The two generators

$$S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

act on an origami given as a pair of permutations (σ_a, σ_b) in the following way:

$$S\colon (\sigma_a, \sigma_b) \mapsto (\sigma_b^{-1}, \sigma_a) \quad \text{and} \quad T\colon (\sigma_a, \sigma_b) \mapsto (\sigma_a, \sigma_b\sigma_a^{-1}).$$

(2) Suppose that the translation surface (X, μ) defined by a primitive origami O decomposes in the horizontal direction into k cylinders of height 1 and length $m_1, \ldots, m_k$ and let m be a multiple of $m_1, \ldots, m_k$. Then T^m is in the Veech group $\Gamma(O)$. Similarly, if (X, μ) decomposes in the vertical direction into l cylinders of length $m'_1, \ldots, m'_l$ and m' is a multiple of $m'_1, \ldots, m'_l$, then $\Gamma(X, \mu)$ contains $T'^{m'}$. Here

$$T' = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}.$$

If O is given by the pair of permutations (σ_a, σ_b), then the numbers $m_1, \ldots, m_k$ are precisely the cycle lengths of σ_a and $m'_1, \ldots, m'_l$ are the cycle lengths of σ_b.

3. A criterion for being a totally non-congruence group

We denote

$$T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad T' = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \quad \text{and} \quad S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}. \tag{3.1}$$

Furthermore, $p_n\colon \mathrm{SL}(2,\mathbb{Z}) \to \mathrm{SL}(2,\mathbb{Z}/n\mathbb{Z})$ is the canonical projection. We denote the images of the matrices T, T' and S in $\mathrm{SL}(2,\mathbb{Z}/n\mathbb{Z})$ also by T, T' and S. Finally, we denote by I the 2×2-identity matrix over the respective ring.

We start with a small but very useful calculation.

Lemma 3.1. *Let $A, B \in \mathrm{GL}(2,\mathbb{Z}/n\mathbb{Z})$ with $A \cdot \binom{1}{0} = B \cdot \binom{1}{0}$. Then we have that*

$$ATA^{-1} = BT^{\det(B)/\det(A)}B^{-1}.$$

Observe for the statement in Lemma 3.1 that T^a with $a \in \mathbb{Z}/n\mathbb{Z}$ gives a well-defined matrix in $\mathrm{GL}(2,\mathbb{Z}/n\mathbb{Z})$ and we have for any $A \in \mathrm{GL}(2,\mathbb{Z}/n\mathbb{Z})$ that

$$AT^aA^{-1} = (ATA^{-1})^a.$$

Proof of Lemma 3.1. Suppose first that $A\binom{1}{0} = B\binom{1}{0} = \binom{1}{0}$. Hence we can write

$$A = \begin{pmatrix} 1 & x \\ 0 & \det(A) \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 1 & y \\ 0 & \det(B) \end{pmatrix}$$

with $x, y \in \mathbb{Z}/n\mathbb{Z}$. A short calculation gives

$$ATA^{-1} = \begin{pmatrix} 1 & \det(A)^{-1} \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad BTB^{-1} = \begin{pmatrix} 1 & \det(B)^{-1} \\ 0 & 1 \end{pmatrix}.$$

Thus the claim holds in this case. In the general situation, we consider the two matrices $A^{-1}B$ and I satisfying $A^{-1}B\binom{1}{0} = I\binom{1}{0} = \binom{1}{0}$ and obtain from the preceding consideration

$$T = (A^{-1}B)T^{\det(A^{-1}B)}(B^{-1}A) = A^{-1}BT^{\det(B)/\det(A)}B^{-1}A,$$

which implies the claim. ∎

We now deduct from Lemma 3.1 a criterion whether two conjugates of T generate the full group $\mathrm{SL}(2,\mathbb{Z}/p^r\mathbb{Z})$.

Lemma 3.2. *Let p be prime and $r \in \mathbb{N}$. Let Γ be a subgroup of $\mathrm{SL}(2,\mathbb{Z}/p^r\mathbb{Z})$. Suppose that Γ contains $A_1TA_1^{-1}$ and $A_2TA_2^{-1}$ with $A_1, A_2 \in \mathrm{SL}(2,\mathbb{Z}/p^r\mathbb{Z})$ such that*

$$\forall m \in \mathbb{N}, \quad mA_1e_1 \neq A_2e_1 \bmod p, \quad \textit{where } e_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \in (\mathbb{Z}/p^r\mathbb{Z})^2.$$

Then $\Gamma = \mathrm{SL}(2,\mathbb{Z}/p^r\mathbb{Z})$.

Proof. By conjugation, we may assume that $A_1 = I$ is the identity matrix. Consider the vector $\binom{a}{c} = A_2 \cdot e_1$. By assumption, c is not divisible by p, hence c is in the multiplicative group $(\mathbb{Z}/p^r\mathbb{Z})^\times$. Consider the following matrix $B \in \mathrm{GL}(2, \mathbb{Z}/p^r\mathbb{Z})$ and its inverse B^{-1}:

$$B = \begin{pmatrix} 1 & a \\ 0 & c \end{pmatrix} \quad \text{and} \quad B^{-1} = c^{-1} \cdot \begin{pmatrix} c & -a \\ 0 & 1 \end{pmatrix}.$$

It follows directly from the definition of B that

$$B^{-1}e_1 = e_1 \quad \text{and} \quad B^{-1}A_2e_1 = e_2 = Se_1, \quad \text{where } e_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

Hence we obtain from Lemma 3.1,

$$(B^{-1}TB)^{\det(B^{-1})} = T \quad \text{and} \quad (B^{-1}A_2TA_2^{-1}B)^{\det(B^{-1})} = STS^{-1} = T'^{-1}. \tag{3.2}$$

It follows that

$$\mathrm{SL}(2, \mathbb{Z}/p^r\mathbb{Z}) = \langle T, T' \rangle \subseteq B^{-1}\Gamma B \subseteq \mathrm{SL}(2, \mathbb{Z}/p^r\mathbb{Z}).$$

Hence we have $B^{-1}\Gamma B = \mathrm{SL}(2, \mathbb{Z}/p^r\mathbb{Z})$ and thus $\Gamma = \mathrm{SL}(2, \mathbb{Z}/p^r\mathbb{Z})$. Here it is crucial that $\mathrm{SL}(2, \mathbb{Z}/p^r\mathbb{Z})$ is a normal subgroup in $\mathrm{GL}(2, \mathbb{Z}/p^r\mathbb{Z})$. ■

Lemma 3.2 is the main ingredient that we need to prove Theorem 1.1, which provides us with a criterion for whether a group is a totally non-congruence group.

Theorem 1.1. *Let Γ be a finite index subgroup of $\mathrm{SL}(2, \mathbb{Z})$. Denote $e_1 = \binom{1}{0}$. Suppose that for each prime p, there exist matrices $A_1, A_2 \in \mathrm{SL}(2, \mathbb{Z})$ with the following properties:*

(A) $\forall j \in \mathbb{N}$: $A_1e_1 \neq j \cdot A_2e_1$ *modulo* p.

(B) *There exist $m_1, m_2 \in \mathbb{N}$ with*

$$A_1T^{m_1}A_1^{-1} \quad \textit{and} \quad A_2T^{m_2}A_2^{-1} \quad \textit{are contained in } \Gamma,$$

such that p divides neither m_1 nor m_2.

Then Γ is a totally non-congruence group.

Proof. We have to show that $\mathrm{pr}_n(\Gamma) = \mathrm{SL}(2, \mathbb{Z}/n\mathbb{Z})$ for all $n \in \mathbb{N}$.

Let $n = p_1^{r_1} \cdots p_k^{r_k}$ be the prime factorisation of n. We thus have by the Chinese remainder theorem

$$\mathrm{SL}(2, \mathbb{Z}/n\mathbb{Z}) = \mathrm{SL}(2, \mathbb{Z}/{p_1}^{r_1}\mathbb{Z}) \times \cdots \times \mathrm{SL}(2, \mathbb{Z}/{p_k}^{r_k}\mathbb{Z}).$$

We show that $\forall i \in \{1, \ldots, k\}$,

$$\mathrm{pr}_n(\Gamma) \supseteq \{I\} \times \cdots \times \{I\} \times \mathrm{SL}(2, \mathbb{Z}/{p_i}^{r_i}\mathbb{Z}) \times \{I\} \times \cdots \times \{I\}. \tag{3.3}$$

For $p = p_i$, we decompose $n = p^r \cdot n_2$ with $\gcd(p, n_2) = 1$. Choose m_1, m_2 such that they satisfy assumptions (A) and (B) with respect to p. In particular, m_1 and m_2 are coprime to p. By Bézout's identity, we find $a, b \in \mathbb{Z}$ with $1 = a \cdot p^r + b \cdot m_1 m_2 n_2$.

We then have for $K = bm_1 m_2 n_2$ that

$$\Gamma \ni A_1 T^K A_1^{-1} = A_1 \begin{pmatrix} 1 & bm_1 m_2 n_2 \\ 0 & 1 \end{pmatrix} A_1^{-1}.$$

Furthermore, we have

$$\begin{aligned} A_1 T^K A_1^{-1} &\equiv A_1 \cdot \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} A_1^{-1} = A_1 T A_1^{-1} \bmod p^r, \\ A_1 T^K A_1^{-1} &\equiv I \bmod n_2. \end{aligned}$$

Hence the group $\mathrm{pr}_n(\Gamma)$ contains

$$\begin{aligned} \mathrm{pr}_n(A_1 T^K A_1^{-1}) &= (A_1 T A_1^{-1}, I) \in \mathrm{SL}(2, \mathbb{Z}/p^r\mathbb{Z}) \times \mathrm{SL}(2, \mathbb{Z}/n_2\mathbb{Z}) \\ &= \mathrm{SL}(2, \mathbb{Z}/n\mathbb{Z}). \end{aligned}$$

Similarly, we obtain that

$$\begin{aligned} \mathrm{pr}_n(\Gamma) \ni \mathrm{pr}_n(A_2 T^K A_2^{-1}) &= (A_2 T A_2^{-1}, I) \in \mathrm{SL}(2, \mathbb{Z}/p^r\mathbb{Z}) \times \mathrm{SL}(2, \mathbb{Z}/n_2\mathbb{Z}) \\ &= \mathrm{SL}(2, \mathbb{Z}/n\mathbb{Z}). \end{aligned}$$

It follows from Lemma 3.2 that

$$\mathrm{pr}_n(\Gamma) \supseteq \mathrm{SL}(2, \mathbb{Z}/p^r\mathbb{Z}) \times \{I\}.$$

This implies the claim. ∎

Theorem 1.1 is a generalisation of [23, Theorem 2] which we restate adapted to our context in Corollary 3.3.

Corollary 3.3 ([23, Theorem 2]). *Let Γ be a finite index subgroup of $\mathrm{SL}(2, \mathbb{Z})$. Suppose there exist matrices $C_1, C_2 \in \mathrm{SL}(2, \mathbb{Z})$ and $m_1, m_1', m_2, m_2' \in \mathbb{N}$ with*

$$\Gamma \ni C_1 T^{m_1} C_1^{-1}, C_1 T'^{m_1'} C_1^{-1} \quad \textit{and} \quad \Gamma \ni C_2 T^{m_2} C_2^{-1}, C_2 T'^{m_2'} C_2^{-1}$$

such that $\gcd(m_1 m_1', m_2 m_2') = 1$. Then Γ is a totally non-congruence group.

Proof. We show that the assumptions of Theorem 1.1 are fulfilled. Let p be prime. If p does not divide $m_1 m_1'$, then we may choose $A_1 = C_1$, $A_2 = C_1 S$. Denote $e_1 = \binom{1}{0}$ and $e_2 = \binom{0}{1}$. Since $e_1 \neq j \cdot e_2 \mod p$ for all $j \in \mathbb{N}$, we have that $A_1 e_1 \neq j \cdot A_2 e_1 = j \cdot A_1 e_2 \mod p$. Thus in this case the assumptions are satisfied. If p divides $m_1 m_1'$, then it does not divide $m_2 m_2'$ and we can use the same arguments with C_2 instead of C_1. ∎

4. Nice one-cylinder origamis

In this section, we give explicit examples for one-cylinder origamis in each stratum. The following examples will provide building blocks for them.

Example 4.1. In the following, we construct special one-cylinder origamis in $\mathcal{H}(\alpha)$ with α even and in $\mathcal{H}(\alpha_1, \alpha_2)$ with α_1, α_2 odd.

(i) *A family of origamis in* $\mathcal{H}(\alpha)$. Let $\alpha = 2k$ be an even number. Define the origami $O(\alpha)$ with $N = 3k + 1 = \frac{3}{2}\alpha + 1$ squares by the following permutations (cf. Figure 3):

$$\sigma_a(\alpha) = (1, \ldots, N),$$
$$\sigma_b(\alpha) = (1, 2, 3) \circ (4, 5, 6) \circ \cdots \circ (3(k-1)+1, 3(k-1)+2, 3(k-1)+3).$$

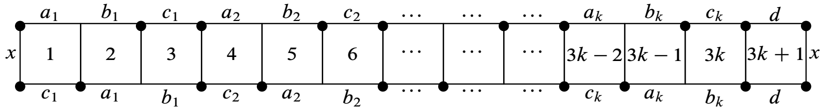

Figure 3. The origami $O(\alpha)$ from Example 4.1 in $\mathcal{H}(\alpha)$.

Observe that we obtain the commutator

$$[\sigma_b^{-1}, \sigma_a^{-1}] = (3, 6, 9, \ldots, 3(k-1), 3k, 3k-1, 3k-4, 3k-7, \ldots, 8, 5, 2, N).$$

In particular, the commutator consists of one cycle of length $2k + 1$. Hence the origami has one singularity with cone angle $(2k+1) \cdot 2\pi = (\alpha + 1) \cdot 2\pi$ and thus lies in $\mathcal{H}(\alpha)$.

We now define for arbitrary $l \geq 1$ the one-cylinder origami $O(\alpha; l)$ in $\mathcal{H}(\alpha)$ as a deformation of $O(\alpha)$ in the following way (cf. Figure 4).

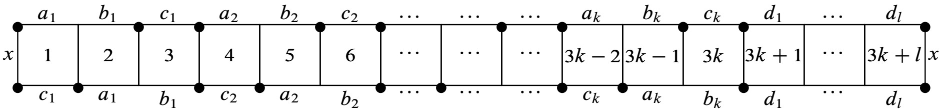

Figure 4. The origami $O(\alpha; l)$ from Example 4.1 in $\mathcal{H}(\alpha)$.

The origami $O(\alpha; l)$ has $N' = N + l - 1 = \frac{3}{2}\alpha + l$ squares and is defined by the permutations

$$\sigma_a(\alpha; l) = (1, \ldots, N'),$$
$$\sigma_b(\alpha; l) = \sigma_b(\alpha) = (1, 2, 3) \circ (4, 5, 6) \circ \cdots$$
$$\circ\, (3(k-1)+1, 3(k-1)+2, 3(k-1)+3).$$

Observe that $O(\alpha; l)$ has again one singularity and lies in $\mathcal{H}(\alpha)$.

(ii) *A family of origamis in* $\mathcal{H}(\alpha_1, \alpha_2)$ (cf. Figure 5). Let

$$\alpha_1 = 2k_1 + 1, \quad \alpha_2 = 2k_2 + 1$$

be odd numbers. Define $O(\alpha_1, \alpha_2)$ with $N = 3(k_1 + k_2) + 6 = \frac{3}{2}(\alpha_1 + \alpha_2) + 3$ squares by the following permutations:

$$\sigma_a(\alpha_1, \alpha_2) = (1, \dots, N), \quad \sigma_b(\alpha_1, \alpha_2) = \sigma_1 \circ \sigma_2 \circ \sigma_3,$$

where

$$\begin{aligned}
\sigma_1 &= (1, 2, 3) \circ (4, 5, 6) \circ \cdots \circ (3k_1 - 2, 3k_1 - 1, 3k_1),\\
\sigma_2 &= (3k_1 + 1, 3k_1 + 5, 3k_1 + 2, 3k_1 + 3, 3k_1 + 4),\\
\sigma_3 &= (3k_1 + 6, 3k_1 + 7, 3k_1 + 8) \circ (3k_1 + 9, 3k_1 + 10, 3k_1 + 11) \circ \cdots\\
&\quad \circ (3(k_1 + k_2) + 3, 3(k_1 + k_2) + 4, 3(k_1 + k_2) + 5).
\end{aligned}$$

In this case, we obtain the commutator

$$\begin{aligned}
[\sigma_b^{-1}, \sigma_a^{-1}] = (3, 6, 9, \dots, 3k_1, 3k_1 + 3, 3k_1 - 1, 3k_1 - 4, 3k_1 - 7, \dots, 5, 2, N)&\\
\circ (3k_1 + 1, 3k_1 + 5, 3k_1 + 8, 3k_1 + 11, \dots, N - 1,&\\
N - 2, N - 5, N - 8, \dots, 3k_1 + 5 + 2).&
\end{aligned}$$

In particular, it consists of two cycles of length

$$2k_1 + 2 = \alpha_1 + 1 \quad \text{and} \quad 2k_2 + 2 = \alpha_2 + 1.$$

Hence $O(\alpha_1, \alpha_2)$ lies in $\mathcal{H}(\alpha_1, \alpha_2)$. Similarly to (i), we define for $l \geq 1$ the origami $O(\alpha_1, \alpha_2; l)$ in $\mathcal{H}(\alpha_1, \alpha_2)$ with $N' = 3(k_1 + k_2) + 5 + l = \frac{3}{2}(\alpha_1 + \alpha_2) + 2 + l$ squares by the two permutations (cf. Figure 5)

$$\sigma_a(\alpha_1, \alpha_2; l) = (1, \dots, N'), \quad \sigma_b(\alpha_1, \alpha_2; l) = \sigma_b(\alpha_1, \alpha_2).$$

We may now construct one-cylinder origamis in a general stratum $\mathcal{H}(\alpha_1, \dots, \alpha_k)$ by cutting and pasting the origamis from Example 4.1 as described in the following. We assume that the numbers $\alpha_1, \dots, \alpha_k$ are ordered such that the first part consists of even

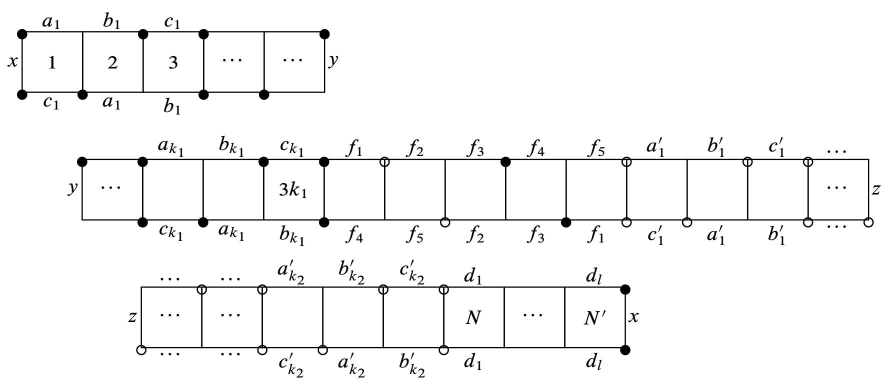

Figure 5. The origami $O(\alpha_1, \alpha_2; l)$ from Example 4.1.

numbers and the second part of odd numbers. Recall that $\alpha_1 + 1, \dots, \alpha_k + 1$ are the cycle lengths of the commutator $[\sigma_b^{-1}, \sigma_a^{-1}]$. Since the commutator is an even permutation, the number of odd α_i is even.

Lemma 4.2. *Let $\alpha_1, \dots, \alpha_p$ be even, $\alpha_{p+1}, \dots, \alpha_{p+2q}$ be odd numbers. Let further l be a natural number. We obtain a one-cylinder origami O in $\mathcal{H}(\alpha_1, \dots, \alpha_{p+2q})$ with*

$$L = \frac{3}{2}(\alpha_1 + \cdots + \alpha_{p+2q}) + p + 3q + l - 1$$

squares as follows (cf. Figure 6). If $q \neq 0$, we take the origamis

$$O(\alpha_1), \dots, O(\alpha_p), O(\alpha_{p+1}, \alpha_{p+2}), \dots, O(\alpha_{p+2q-3}, \alpha_{p+2q-2})$$

and

$$O(\alpha_{p+2q-1}, \alpha_{p+2q}; l)$$

defined in Example 4.1. We cut them along the left vertical edge of their first square which is equal to the right vertical edge of their last square. We then glue them in the stated order along these slits. If $q = 0$, we take the origamis $O(\alpha_1), \dots, O(\alpha_{p-1}), O(\alpha_p; l)$ and do the same procedure.

This means the origami O is defined by the two permutations (σ_a, σ_b) given as follows: If $q \neq 0$, we have

$$\begin{aligned} \sigma_a &= (1, \dots, L), \\ \sigma_b &= \widehat{\sigma}_b(\alpha_1) \circ \cdots \circ \widehat{\sigma}_b(\alpha_p) \circ \widehat{\sigma}(\alpha_{p+1}, \alpha_{p+2}) \circ \cdots \circ \widehat{\sigma}(\alpha_{p+2q-3}, \alpha_{p+2q-2}) \\ &\quad \circ \widehat{\sigma}(\alpha_{p+2q-1}, \alpha_{p+2q}; l). \end{aligned} \tag{4.1}$$

Here $\widehat{\sigma}_b(\alpha_i)$, $\widehat{\sigma}_b(\alpha_i, \alpha_{i+1})$ and $\widehat{\sigma}(\alpha_{p+2q-1}, \alpha_{p+2q}; l)$ are conjugates of permutations $\sigma_b(\alpha_i)$, $\sigma_b(\alpha_i, \alpha_{i+1})$ and $\sigma(\alpha_{p+2q-1}, \alpha_{p+2q}; l)$, respectively, which shift the labels of $O(\alpha_i)$, $O(\alpha_i, \alpha_{i+1})$ and $O(\alpha_{p+2q-1}, \alpha_{p+2q}; l)$ by the sum of the lengths of the origamis before them. More precisely, we define these permutations in the following way. Let $s_i = \frac{3}{2}\alpha_i + 1$ if $i \leq p$ and $s_i = \frac{3}{2}\alpha_i + \frac{3}{2}$ if $p + 1 \leq i \leq p + 2q - 1$. Then $O(\alpha_i)$ is of length s_i for $i \leq p$ and $O(\alpha_i, \alpha_{i+1})$ is of length $s_i + s_{i+1}$ for $p + 1 \leq i \leq p + 2q - 3$. Define $S_i = \sum_{j=1}^{i-1} s_j$. Let furthermore $\mathrm{sh}(a)\colon \mathbb{N} \to \mathbb{N}$ be the map $n \mapsto n + a$. Then

$$\begin{aligned} \widehat{\sigma}_b(\alpha_i) &= \mathrm{sh}(S_i) \circ \sigma_b(\alpha_i) \circ \mathrm{sh}(S_i)^{-1}, \\ \widehat{\sigma}_b(\alpha_i, \alpha_{i+1}) &= \mathrm{sh}(S_i) \circ \sigma_b(\alpha_i, \alpha_{i+1}) \circ \mathrm{sh}(S_i)^{-1}, \\ \widehat{\sigma}_b(\alpha_{p+2q-1}, \alpha_{p+2q}; l) &= \mathrm{sh}(S_{p+2q-1}) \circ \sigma_b(\alpha_{p+2q-1}, \alpha_{p+2q}; l) \circ \mathrm{sh}(S_{p+2q-1})^{-1}. \end{aligned}$$

If $q = 0$, we similarly have

$$\sigma_a = (1, \dots, L) \quad \textit{and} \quad \sigma_b = \widehat{\sigma}_b(\alpha_1) \circ \cdots \circ \widehat{\sigma}_b(\alpha_{p-1}) \circ \widehat{\sigma}_b(\alpha_p; l)$$

with $\widehat{\sigma}_b(\alpha_1), \dots, \widehat{\sigma}_b(\alpha_{p-1})$ and $\widehat{\sigma}_b(\alpha_p; l)$ defined as conjugates of $\sigma_b(\alpha_1), \dots, \sigma_b(\alpha_{p-1})$ and $\sigma_b(\alpha_p; l)$ with the suitable shifts similarly to case $q \neq 0$.

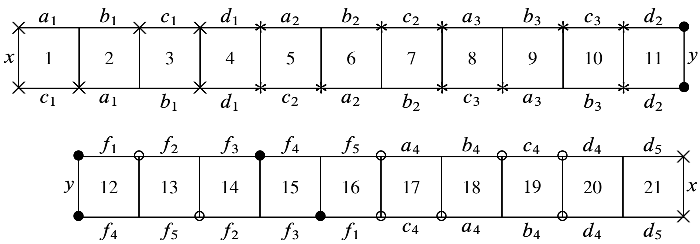

Figure 6. The origami in $\mathcal{H}(2,4,1,3)$ with $l = 2$ obtained from the construction in Lemma 4.2.

Figure 6 shows the origami in $\mathcal{H}(2,4,1,3)$ obtained by this construction with $l = 2$.

Proof. Assume first that $q \neq 0$. You can directly check from the definition of O and Example 4.1 that each building block $O(\alpha_i)$ contributes one singularity of order α_i to the surface. Furthermore, each $O(\alpha_i, \alpha_{i+1})$ contributes two singularities of order α_i and α_{i+1}. $O(\alpha_{p+2q-1}, \alpha_{p+2q}; l)$ also contributes two singularities of order α_{p+2q-1} and α_{p+2q}. Finally, the numbers of squares of the origamis $O(\alpha_1), \dots, O(\alpha_{p+2q-1}, \alpha_{p+2q}; l)$ add up to the number L of squares of the constructed origami O. Thus we obtain

$$\begin{aligned} L &= \frac{3}{2}\alpha_1 + 1 + \cdots + \frac{3}{2}\alpha_p + 1 + \frac{3}{2}(\alpha_{p+1} + \alpha_{p+2}) \\ &\quad + 3 + \cdots + \frac{3}{2}(\alpha_{p+2q-1} + \alpha_{p+2q}) + 3 + l - 1 \\ &= \frac{3}{2}(\alpha_1 + \cdots + \alpha_{p+2q}) + p + 3q + l - 1. \end{aligned}$$

The proof works similarly if $q = 0$. ∎

In the following, we consider cylinder decompositions in different directions of the origamis constructed in Lemma 4.2. Based on this, we obtain parabolic elements in the Veech groups of these origamis.

Lemma 4.3. *Let Γ be the Veech group of the origami $O = O(l)$ with $L = \frac{3}{2}(\alpha_1 + \cdots + \alpha_{p+2q}) + p + 3q + l - 1$ squares constructed in Lemma 4.2. Then Γ contains the parabolic matrices T^L, T'^{15} and $T''^{2(L-4q)}$ with T and T' defined in (3.1) and $T'' = T'TT'^{-1}$.*

Proof. It follows from its definition that O consists of one horizontal cylinder which has length L and height 1. Thus the Veech group contains the matrix T^L. Furthermore, since all cycles of σ_b are of length 1, 3 or 5, we have that O decomposes into vertical cylinders of height 1 and length 1, 3 or 5. Hence T'^{15} is contained in Γ. Finally, the origami $T'^{-1} \cdot O$ is given by the two permutations $(\sigma_b\sigma_a, \sigma_b)$ (cf. [23, Section 2.2]). We will show below that $\sigma_b\sigma_a$ consists of one cycle of length $L - 4q$ and further cycles of length 2. Hence $T'^{-1} \cdot O$ composes into horizontal cylinders of length $L - 4q$ and of length 2. Therefore, $T^{2(L-4q)} \in \Gamma(T'^{-1}O) = T'^{-1}\Gamma T'$ and thus $T'T^{2(L-4q)}T'^{-1} = T''^{2(L-4q)} \in \Gamma$. This finishes the claim.

Let us now show that the permutation $\sigma_b\sigma_a$ is of the desired form. We assume that $q \neq 0$. The case $q = 0$ works in the same way. Recall that O consists of the origamis $O(\alpha_1), \dots, O(\alpha_p)$, $O(\alpha_{p+1}, \alpha_{p+2}), \dots, O(\alpha_{p+2q-3}, \alpha_{p+2q-2})$, $O(\alpha_{p+2q-1}, \alpha_{p+2q}; l)$ which are glued in a row along slits. We label the squares of O from left to right by $1, \dots, L \in \mathbb{Z}/L\mathbb{Z}$. Let us consider how the permutation $\sigma_b\sigma_a$ acts on the labels of the squares.

Recall the definition of S_i and s_i in Lemma 4.2. The origamis $O(\alpha_i)$ are then of length s_i and the origamis $O(\alpha_i, \alpha_{i+1})$ are of length $s_i + s_{i+1}$. Let us consider the squares belonging to the origami $O(\alpha_i)$ ($i \in \{1, \dots, p\}$). The first square of the origami $O(\alpha_i)$ is labelled by $S_i + 1$ and the last one is labelled by $S_i + s_i$. Observe (cf. Figure 3) that the permutation $\sigma_b\sigma_a$ acts in the following way:

$$\begin{aligned} S_i &\mapsto S_i + 2 \mapsto S_i + 1 \mapsto S_i + 3 \mapsto S_i + 5 \mapsto S_i + 4 \mapsto S_i + 6 \mapsto \cdots \\ &\mapsto S_i + s_i - 2 \mapsto S_i + s_i - 3 \mapsto S_i + s_i - 1 \mapsto S_i + s_i = S_{i+1}. \end{aligned}$$

In particular, all squares of the origamis $O(\alpha_1), \dots, O(\alpha_p)$, i.e., all squares labelled by $1, 2, \dots, S_{p+1}$, lie in the same orbit.

Let us now consider the origamis $O(\alpha_i, \alpha_{i+1})$ ($i - p$ odd, $1 \leq i \leq 2q - 3$). The first square of $O(\alpha_i, \alpha_{i+1})$ is labelled by $S_i + 1$, and the last one by $S_i + s_i + s_{i+1}$. Observe that $\sigma_b\sigma_a$ acts in the following way (cf. Figure 5).

Denote $k_i = \frac{\alpha_i - 1}{2}$ and $k_{i+1} = \frac{\alpha_{i+1}-1}{2}$, then

$$\begin{aligned} S_i &\mapsto S_i + 2 \mapsto S_i + 1 \mapsto S_i + 3 \mapsto S_i + 5 \mapsto S_i + 4 \mapsto S_i + 6 \mapsto \cdots \\ &\mapsto S_i + 3k_i - 1 \mapsto S_i + 3k_i - 2 \mapsto S_i + 3k_i \mapsto S_i + 3k_i + 5 \\ &\mapsto S_i + 3k_i + 7 \mapsto S_i + 3k_i + 6 \mapsto S_i + 3k_i + 8 \mapsto \cdots \\ &\mapsto S_i + 3(k_i + k_{i+1}) + 4 \mapsto S_i + 3(k_i + k_{i+1}) + 3 \mapsto S_i + 3(k_i + k_{i+1}) + 5 \\ &\mapsto S_i + 3(k_i + k_{i+1}) + 6. \end{aligned}$$

The remaining squares of $O(\alpha_i, \alpha_{i+1})$ which do not belong to this orbit are $S_i + 3k_i + 1$, $S_i + 3k_i + 2$, $S_i + 3k_i + 3$ and $S_i + 3k_i + 4$. They form two cycles $(S_i + 3k_i + 1, S_i + 3k_i + 3)$ and $(S_i + 3k_i + 2, S_i + 3k_i + 4)$ of length two.

Similarly, the permutation $\sigma_b\sigma_a$ acts on the squares of $O(\alpha_{p+2q-1}, \alpha_{p+2q}; l)$ as described in the following. Denote $i = p + 2q - 1$, then

$$\begin{aligned} S_i &\mapsto S_i + 2 \mapsto S_i + 1 \mapsto S_i + 3 \mapsto S_i + 5 \mapsto S_i + 4 \mapsto S_i + 6 \mapsto \cdots \\ &\mapsto S_i + 3k_i - 1 \mapsto S_i + 3k_i - 2 \mapsto S_i + 3k_i \mapsto S_i + 3k_i + 5 \\ &\mapsto S_i + 3k_i + 7 \mapsto S_i + 3k_i + 6 \mapsto S_i + 3k_i + 8 \mapsto \cdots \\ &\mapsto S_i + 3(k_i + k_{i+1}) + 4 \mapsto S_i + 3(k_i + k_{i+1}) + 3 \mapsto S_i + 3(k_i + k_{i+1}) + 5 \\ &\mapsto S_i + 3(k_i + k_{i+1}) + 6 \mapsto \cdots \mapsto S_i + 3(k_i + k_{i+1}) + 5 + l \end{aligned}$$

and by two cycles $(S_i + 3k_i + 1, S_i + 3k_i + 3)$ and $(S_i + 3k_i + 2, S_i + 3k_i + 4)$.

Altogether, we obtain for the permutation $\sigma_b\sigma_a$ one long cycle containing all squares except the squares $S_i + 3k_i + 1$, $S_i + 3k_i + 2$, $S_i + 3k_i + 3$ and $S_i + 3k_i + 4$ with $i - p$ odd and $p + 1 \leq i \leq p + 2q$. This cycle has length $L - 4q$. Furthermore, we obtain $2q$ cycles of length 2. Hence $\sigma_b\sigma_a$ has the form which we claimed. ∎

We are now able to obtain explicit origamis in each stratum whose Veech groups are totally non-congruence groups.

Proposition 4.4. *Let $\alpha_1, \ldots, \alpha_p$ be even, $\alpha_{p+1}, \ldots, \alpha_{p+2q}$ be odd numbers. Recall that in Lemma 4.2 we constructed an origami O in $\mathcal{H}(\alpha_1, \ldots, \alpha_{p+2q})$ with L squares, where*

$$L = \frac{3}{2}(\alpha_1 + \cdots + \alpha_{p+2q}) + p + 3q + l - 1.$$

Choose $l \in \mathbb{N}$ such that

(i) $\gcd(L, 30q) = 1$,

(ii) 3 *and* 5 *do not divide* $L - 4q$.

Then the Veech group $\Gamma = \Gamma(O)$ of O is a totally non-congruence group.

Proof. We know from Lemma 4.3 that the matrices T^L, T'^{15} and $T''^{2(L-4q)}$ with $T'' = T'TT'^{-1} = \left(\begin{smallmatrix} 0 & 1 \\ -1 & 2 \end{smallmatrix}\right)$ are contained in Γ. We apply Theorem 1.1. Observe firstly that each pair (A_1, A_2) of two matrices in $\{T, T', T''\}$ satisfies property (A) in Theorem 1.1 for any prime p. We distinguish now three cases. Suppose as first case that p is neither a divisor of L nor of 15. Then we choose $A_1 = T$, $A_2 = T'$, $m_1 = L$ and $m_2 = 15$. By the assumption on p, we have that p does neither divide m_1 nor m_2. As second case, we consider that p divides L. Then we choose $A_1 = T'$, $A_2 = T''$, $m_1 = 15$ and $m_2 = 2(L - 4q)$. Now, p does not divide m_1 by (i). Furthermore, it follows from (i) that p does not divide $4q$. Thus since it is a divisor of L, it does not divide $m_2 = L - 4q$. In the remaining case, namely $p = 3$ or $p = 5$, we choose $A_1 = T$, $A_2 = T''$, $m_1 = L$ and $m_2 = 2(L - 4q)$. In this case, p does neither divide m_1 (by (i)) nor m_2 (by (ii)). Hence, in all three cases, we obtain that also property (B) in Theorem 1.1 holds. This finishes the proof. ∎

In particular, Proposition 4.4 defines in each stratum an infinite family of origamis.

Theorem 1.2. *Every stratum contains an infinite family of origamis whose Veech groups are totally non-congruence groups.*

Proof. The theorem directly follows from Proposition 4.4. Namely, we can choose l for example such that L is a prime with $L > 4q$ which satisfies the following conditions:

$$L \equiv \begin{cases} 4q + 1 \bmod 3 & \text{if 3 does not divide } 4q + 1, \\ 4q + 2 \bmod 3 & \text{otherwise,} \end{cases}$$

$$L \equiv \begin{cases} 4q + 1 \bmod 5 & \text{if 5 does not divide } 4q + 1, \\ 4q + 2 \bmod 5 & \text{otherwise.} \end{cases}$$

By Dirichlet's theorem on arithmetic progressions, there are infinitely many primes which satisfy these conditions. ∎

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