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The automorphism group of Rauzy diagrams

Corentin Boissy

Abstract. We give a description of the automorphism group of a Rauzy diagram as a subgroup of the symmetric group. This is based on an example that appears in some personal notes of Yoccoz that are to be published in the project "Yoccoz archives".

1. Introduction

Rauzy induction was introduced in [14] as a tool to study interval exchange maps. It is a renormalisation process that associates to an interval exchange map another one obtained as a first return map on a well-chosen subinterval. After the major work of Veech [15], the Rauzy induction became a powerful tool to study the Teichmüller geodesic flow.

A slightly different tool was used by Kerckhoff [10], Bufetov [6], and Marmi, Moussa and Yoccoz [12]. It is obtained after labelling the intervals, and keeping track of them during the renormalisation process. This small change was a significant improvement, and was used more recently to prove other important results on the Teichmüller geodesic flow, for instance, the simplicity of the Lyapunov exponents (Avila and Viana [2]), or the exponential decay of correlations (Avila, Gouezel and Yoccoz [1]). See also [9].

This labelling induces a nontrivial automorphism group on the Rauzy diagram $\mathcal{D}$ that corresponds to relabellings preserving $\mathcal{D}$. The computation of the cardinal of $\mathrm{Aut}(\mathcal{D})$ was done by the author in [4] by considering a geometric interpretation in terms of a moduli space of *labelled* translation surfaces. However, the precise description of the elements of $\mathrm{Aut}(\mathcal{D})$ as permutation elements was not done. In some personal notes (annotated by the author) that are to be published in the project "Yoccoz archives" [17], Yoccoz gives an extensive description of many "small" Rauzy classes, including the one corresponding to the stratum $\mathcal{H}(1,1,1,1)$ (see [17, Section 20]). In particular, we can find for such Rauzy class a nice description of the automorphism group. Based on this example, we propose a similar description for the automorphism group of any Rauzy class.

Structure of the paper

The paper is organised as follows:

- In Section 2, we review the general definitions and tools that are needed.

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- In Section 3, we first present a geometrical interpretation of Yoccoz's example, then generalise it to describe the automorphism group for any nonhyperelliptic Rauzy diagram (Theorem 3.2) and finally study the particular case of hyperelliptic Rauzy diagrams (Theorem 3.5).

2. Background

2.1. Combinatorial definition

Definition 2.1. Let $\mathcal{A}$ be a finite alphabet that consists of d elements. A *labelled permutation* is a pair $\pi = (\pi_t, \pi_b)$ of one-to-one maps from $\mathcal{A}$ to $\{1, \dots, d\}$. We usually represent π by

$$\pi = \begin{pmatrix} \pi_t^{-1}(1) & \pi_t^{-1}(2) & \dots & \pi_t^{-1}(d) \\ \pi_b^{-1}(1) & \pi_b^{-1}(2) & \dots & \pi_b^{-1}(d) \end{pmatrix}.$$

A *renumbering* of a labelled permutation is the composition of (π_t, π_b) by a one-to-one map f from $\mathcal{A}$ to $\mathcal{A}$. It corresponds to changing the labels without changing the underlying permutation $\pi_t \circ \pi_b^{-1}$.

A labelled permutation is irreducible if for any $k \in \{1, \dots, d-1\}$, $\pi_t^{-1}\{1, \dots, k\} \neq \pi_b^{-1}\{1, \dots, k\}$.

We define the following maps on the set of irreducible permutations, called the combinatorial Rauzy moves:

(1) $\mathcal{R}_t$: let $k = \pi_b(\pi_t^{-1}(d))$ with $k \leq d-1$. Then $\mathcal{R}_t(\pi_t, \pi_b) = (\pi'_t, \pi'_b)$, where $\pi_t = \pi'_t$ and

$$\pi_b'^{-1}(j) = \begin{cases} \pi_b^{-1}(j) & \text{if } j \leq k, \\ \pi_b^{-1}(d) & \text{if } j = k+1, \\ \pi_b^{-1}(j-1) & \text{otherwise.} \end{cases}$$

(2) $\mathcal{R}_b$: let $k = \pi_t(\pi_b^{-1}(d))$ with $k \leq d-1$. Then $\mathcal{R}_b(\pi_t, \pi_b) = (\pi'_t, \pi'_b)$, where $\pi_b = \pi'_b$ and

$$\pi_t'^{-1}(j) = \begin{cases} \pi_t^{-1}(j) & \text{if } j \leq k, \\ \pi_t^{-1}(d) & \text{if } j = k+1, \\ \pi_t^{-1}(j-1) & \text{otherwise.} \end{cases}$$

Definition 2.2. A *Rauzy class* is a minimal set of labelled permutations invariant by the combinatorial Rauzy moves.

A *Rauzy diagram* is a graph whose vertices are the elements of a Rauzy class and whose edges are the combinatorial Rauzy moves.

An automorphism of a Rauzy diagram $\mathcal{D}$ with vertices $\mathcal{R}$ is a graph automorphism of $\mathcal{D}$ that sends a "t" edge (resp. a "b" edge) to a "t" edge (resp. a "b" edge).

Proposition 2.3. *Let $\mathcal{D}$ be a Rauzy diagram and let $F\colon \mathcal{R} \to \mathcal{R}$ be an automorphism. Then, F is a relabelling, i.e., there exists a unique one-to-one map $f\colon \mathcal{A} \to \mathcal{A}$ such that for all $\pi \in \mathcal{R}$, $F(\pi) = \pi \circ f$. In particular, the automorphism group of $\mathcal{D}$ is identified with a subgroup of the permutation group $\mathfrak{S}(\mathcal{A})$.*

Proof. Note that iterating the map $\mathcal{R}_t$ starting from an element $\pi \in \mathcal{R}$ gives a loop in $\mathcal{D}$. We call it a t-loop and call similarly a b-loop the loop obtained by iterating the map $\mathcal{R}_b$. By [14], there exists an element in $\mathcal{R}$ of the form

$$\pi^0 = \begin{pmatrix} a & \dots & b \\ b & \dots & a \end{pmatrix}.$$

It corresponds to vertices in $\mathcal{D}$ whose associated t-loop and b-loop are of maximal size $d-1$. Now for $k \in \{1, \dots, d-1\}$, we denote by l_k the length of the b-loop starting from $\mathcal{R}_t^k(\pi)$. The map $k \mapsto l_k$ defines the underlying permutation $\pi_t^0 \circ (\pi_b^0)^{-1}$. Indeed, we have for all $k \in \{0, \dots, d-1\}$ that $(\pi_b^0)^{-1}(d-k) = (\pi_t^0)^{-1}(d-l_k)$.

Note that, by the definition of F, the map $k \mapsto l_k$ is the same when starting from π^0 or from $F(\pi^0)$. In particular, the underlying permutation of $F(\pi^0)$ is the same as the one of π^0 and therefore $F(\pi^0) = \pi^0 \circ f$ for some $\mathfrak{S}(\mathcal{A})$. Since we have $\mathcal{R}_\alpha(\pi \circ f) = \mathcal{R}_\alpha(\pi) \circ f$ for all π and $\alpha \in \{t, b\}$, we obtain that $F(\pi) = \pi \circ f$ for all π in the Rauzy class. ∎

2.2. Links to translation surfaces

A *translation surface* is a (real, compact, connected) genus g surface X with a translation atlas, i.e., a triple $(X, \mathcal{U}, \Sigma)$ such that Σ is a finite subset of X (whose elements are called *singularities*) and $\mathcal{U} = \{(U_i, z_i)\}$ is an atlas of $X \setminus \Sigma$ whose transition maps are translations of $\mathbb{C} \simeq \mathbb{R}^2$. We will require that for each $s \in \Sigma$, there is a neighbourhood of s isometric to a Euclidean cone whose total angle is a multiple of 2π. One can show that the holomorphic structure on $X \setminus \Sigma$ extends to X and that the holomorphic 1-form $\omega = dz_i$ extends to a holomorphic 1-form on X where Σ corresponds to the zeroes of ω and maybe some marked points. We usually call ω an *Abelian differential*. A zero of ω of order k corresponds to a singularity of angle $(k+1)2\pi$. By a slight abuse of notation, we authorise the order of a zero to be 0. In this case, it corresponds to a regular marked point.

For $g \geq 1$, we define the moduli space of Abelian differentials $\mathcal{H}_g$ as the moduli space of pairs (X, ω), where X is a genus g (compact, connected) Riemann surface and ω nonzero holomorphic 1-form defined on X. The term moduli space means that we identify the points (X, ω) and (X', ω') if there exists an analytic isomorphism $f\colon X \to X'$ such that $f^*\omega' = \omega$.

One can also see a translation surface obtained as a polygon (or a finite union of polygons) whose sides come by pairs, and for each pair, the corresponding segments are parallel and of the same length. These parallel sides are glued together by translation and we assume that this identification preserves the natural orientation of the polygons. In this context, two translation surfaces are identified in the moduli space of Abelian differentials if and only if the corresponding polygons can be obtained from each other by cutting and pasting and preserving the identifications.

The moduli space of Abelian differentials is stratified by the combinatorics of the zeroes; we usually denote by $\mathcal{H}(k_1^{n_1}, \ldots, k_r^{n_r})$ the stratum of $\mathcal{H}_g$ consisting of (classes of) pairs (X, ω) such that ω has exactly n_i zeroes of order k_i for each i. It is well known that this space is (Hausdorff) complex analytic (see for instance [13, 15, 16]). Note that in Section 3, we will use a slight variation of this notation: we denote a stratum by $\mathcal{H}(k, k_1^{\alpha_1}, \ldots, k_r^{\alpha_r})$, where $k \in \mathbb{N}$ and for $i \neq j$, $k_i \neq k_j$, but k may be in $\{k_1, \ldots, k_r\}$ (in this case, there are $n_i + 1$ zeroes of order k for i that satisfies $k = k_i$ and n_j zeroes of order k_j for all $j \neq i$).

Suppose that we have an element $\pi \in \mathcal{R}$ and a continuous datum $\zeta \in \mathbb{C}^d$, satisfying the "suspension data condition" (see, for instance, [4, 12] for details). There is a natural construction, the Veech construction (or zippered rectangle construction), that gives a translation surface $S(\pi, \zeta)$. See Figure 1. Different choices of parameter ζ give surfaces in the same connected component of a stratum in the moduli space of Abelian differentials since the set of suspension data is a connected subset of $\mathbb{C}^d$ (in fact, convex).

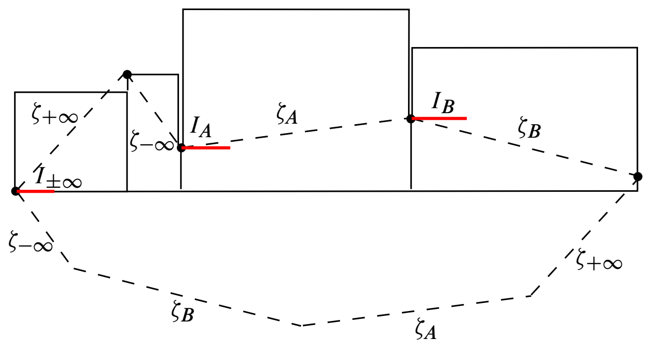

Figure 1. A framing of a surface issued from the Veech construction with alphabet $\mathcal{A} = \{+\infty, -\infty, A, B\}$.

The Veech construction with the associate Rauzy–Veech induction defines three natural invariants of a Rauzy class:

(1) The set of the degrees of the conical singularities of $S(\pi, *)$ counted with multiplicities, i.e., the stratum to which belongs to $S(\pi, *)$.

(2) When such a stratum is nonconnected, the corresponding connected component.

(3) The degree of the singularity attached on the left in the construction (denoted as the *special singularity*).

It is proven in [3] that if two labelled permutations have the same above invariant, then up to a relabelling, they are in the same Rauzy class.

In order to have a complete characterization of Rauzy classes, one needs a refinement of the invariant (1) by adding a combinatorial datum on the singularities.

Definition 2.4. Let P be a singularity of a translation surface S. A (positive) horizontal separatrix is an equivalence class of horizontal geodesics $\gamma\colon]0,a[\to S$ starting from P such that $\gamma' = 1$, where two such geodesics are equivalent if they coincide on a subinterval of the form $]0,\varepsilon[$. A negative separatrix is defined analogously with $\gamma' = -1$. In this paper, separatrices will be implicitly assumed to be positive except mentioned otherwise.

By [4], the letters in the alphabet induce a marking on the set of positive horizontal separatrices of each singularity, where each horizontal separatrix is marked by a letter and the one corresponding to the interval in the Veech construction is marked twice (by $\pi_t^{-1}(1)$ and $\pi_b^{-1}(1)$). For $\alpha \in \mathcal{A}$, we denote by I_α the separatrix marked by α.

Since all horizontal separatrices are marked in this way, we can define a permutation T of $\mathcal{A}$ by $T(\pi_b^{-1}(1)) = \pi_t^{-1}(1)$, and otherwise $I_{T(\alpha)}$ is the separatrix obtained by rotating I_α by 2π counterclockwise. The map T can be written explicitly in terms of π by the following formula:

$$T(\alpha) = \begin{cases} \pi_t^{-1}(1) & \text{if } \alpha = \pi_b^{-1}(1), \\ \pi_b^{-1}(\pi_t(\pi_b^{-1}(d)) + 1) & \text{if } \alpha = \pi_b^{-1}(\pi_t(d) + 1), \\ \pi_t^{-1}(\pi_t(\pi_b^{-1}(\pi_b(\alpha) - 1)) + 1) & \text{otherwise.} \end{cases}$$

The orbits of T are in one-to-one correspondence with the singularities of the surface: each orbit of length $k+1$ that does not contain $\pi_t^{-1}(1)$ corresponds to a singularity of degree k, while the orbit containing $\pi_t^{-1}(1)$ is of length $k+2$, where k is the degree of the singularity attached to the left. In particular, this express the invariants (1) and (3) above in terms of π (a combinatorial description of the invariant (2) can be found in [7, 8]).

We consider the corresponding moduli space $\mathcal{H}^{\mathrm{lab}}$ of *labelled* translation surfaces (i.e., translation surfaces with such markings), we see that different choices of parameters ζ define surfaces in the same connected component of $\mathcal{H}^{\mathrm{lab}}$. In [4], it is proven that two labelled permutations are in the same Rauzy class if and only if:

(1) The letters on the top left and bottom left are the same (they will be denoted by $\pm\infty$ as in [17]).

(2) The canonical cyclic order on the set of labels obtained by rotating clockwise around a singularity must be the same (for one, hence any choice of surfaces constructed from the labelled permutations), i.e., the map T.

(3) The resulting labelled translation surfaces are in the same connected component of $\mathcal{H}^{\mathrm{lab}}$.

Furthermore, once the map T and the underlying connected component of the (nonlabelled) translation surface are fixed, we have (see [4, Theorems 1.1 and 1.3]):

(1) If there are odd degree singularities, then there are exactly two such Rauzy classes.

(2) Otherwise, there is only one such Rauzy class.

One important tool of the above statement is the following proposition.

Proposition 2.5. *Let S be a translation surface, and ϕ_1, ϕ_2 be two markings of the horizontal separatrices, i.e., surjective maps from $\mathcal{A}$ to the set of horizontal positive separatrices of S and such that $\phi_i(\alpha) = \phi_i(\beta)$ if and only if $\alpha, \beta \in \{\pm\infty\}$. We assume that there is an odd degree singularity P such that*

- *For the separatrices adjacent to P, we obtain the marking ϕ_2 by rotating by 2π the marking ϕ_1.*
- *All the other marked horizontal separatrices are unchanged.*

Then (S, ϕ_1) and (S, ϕ_2) are in different connected components of $\mathcal{H}^{\mathrm{lab}}$.

This statement is mainly a key argument in [4, proof of Proposition 4.2]. We can also find a similar statement (in a slightly more general context) in [5]. We propose a sketch of this proof since it is a key argument for later in this paper.

Sketch of the proof. We construct a topological invariant on $\mathcal{H}^{\mathrm{lab}}$ similar to the well-known parity of the spin structure for translation surfaces that appears in [11].

There is an even number of odd degree singularities, and we choose once for all an ordered pairing of the set of odd degree singularities, i.e., we denote by $(P_1^-, P_1^+), \dots, (P_s^-, P_s^+)$ these singularities. We choose once for all for each $P_j^\pm$ a particular letter $\alpha_j^\pm$ that corresponds to a separatrix attached to $P_j^\pm$.

For a smooth closed curve γ in S that does not pass through any singularity, define $\mathrm{ind}(\gamma)$ to be the index of the Gauss map defined by its derivative γ'. Choose a collection of smooth simple closed curves $(\alpha_i, \beta_i)_{i\in\{1,\dots,g\}}$ representing a symplectic basis for the homology of S. For a simple curve γ joining P_j^- to P_j^+, we define $\mathrm{ind}(\gamma)$ to be the index (mod 2) of the Gauss map defined by a simple smooth path $\widetilde{\gamma}$, whose image is in a small neighbourhood of the image of γ, and such that

- $\widetilde{\gamma}$ is tangent in its starting point to the horizontal separatrix $\phi(\alpha_j^-)$ of P_j^-,
- $\widetilde{\gamma}$ is tangent in its ending point to the horizontal separatrix $\phi(\alpha_j^+)$ of P_j^+ rotated by π.

Note that since P_j^+, P_j^- are both of odd degree, their corresponding conical angles are an even multiple of 2π and hence $\mathrm{ind}(\gamma)$ does not depend on the choice of $\widetilde{\gamma}$.

Now, for a fixed choice of $(\alpha_i, \beta_i)_i$, let $\gamma_1, \dots, \gamma_s$ be a collection of simple curves, with no pairwise intersections, such that γ_j joins P_j^- to P_j^+ and each γ_j does not intersect the $(\alpha_i, \beta_i)_i$. Then, we define

$$\mathrm{Sp}(\alpha, \beta, \gamma) = \sum_{i=1}^{g} (\mathrm{ind}(\alpha_i) + 1)(\mathrm{ind}(\beta_i) + 1) + \sum_j \mathrm{ind}(\gamma_j) \bmod 2.$$

It is proven in [5] that $\mathrm{Sp}(\alpha, \beta, \gamma)$ does not depend on the choices of the curves α, β, γ (but depends on the marking and on the pairing), hence is an invariant of connected components of $\mathcal{H}^{\mathrm{lab}}$ and by construction this invariant takes two different values for (S, ϕ_1) and (S, ϕ_2). ∎

2.3. A surgery on translation surfaces

We describe the surgery called "breaking up a singularity", introduced by Kontsevich and Zorich in [11].

We start from a zero singularity P of degree $k_1 + k_2$. The neighbourhood $V_\varepsilon = \{x \in X, d(x, P) \leq \varepsilon\}$ of this conical singularity is obtained by considering $2(k_1 + k_2) + 2$ Euclidean half disks of radius ε and gluing each half side of them to another one in a cyclic order. We can break the zero of order $k_1 + k_2$ into a pair of singularities of order k_1, k_2 by changing continuously the way they are glued to each other as in Figure 2. Note that in this surgery, the metric is not modified outside V_ε. In particular, the boundary ∂V_ε is isometric to (a connected covering of) an Euclidean circle. Note that in this construction, we can "rotate" the two singularities by an angle θ by cutting the surface along ∂V_ε, rotating V_ε by an angle θ and regluing it.

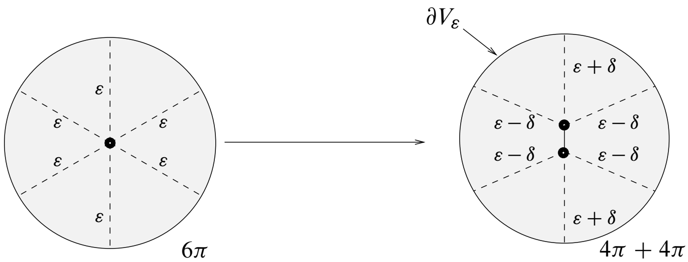

Figure 2. Local surgery that break a zero of degree $k_1 + k_2$ into two zeroes of degree k_1 and k_2, respectively.

3. Structure of automorphism groups of Rauzy diagrams

3.1. Yoccoz's example

Before stating the general result in the next section, we start with a geometrical interpretation of Yoccoz's example (see [17, Section 20.1]). It can be viewed as a simple case of the general statement. We consider the alphabet $\mathcal{A} = \{\pm\infty, 0, a_1, a_2, b_1, b_2, c_1, c_2\}$ and the following element:

$$\pi = \begin{pmatrix} -\infty & b_2 & a_2 & b_1 & a_1 & c_1 & 0 & c_2 & \infty \\ \infty & b_1 & a_2 & b_2 & a_1 & c_2 & 0 & c_1 & -\infty \end{pmatrix}.$$

The Veech construction with labels creates a translation surface in the stratum $\mathcal{H}(1, 1, 1, 1)$. Each singularity has angle 4π, hence gives two labelled separatrices where

- The doubly labelled separatrix is $\pm\infty$, and the other separatrix of the corresponding singularity is labelled "0".
- The other three pairs of marked separatrices are labelled by $\{a_1, a_2\}$, $\{b_1, b_2\}$ and $\{c_1, c_2\}$.

As described in [17], the subgroup $\mathcal{G}'$ of $\mathfrak{S}(\mathcal{A})$ preserving these pairings has order $2^3 \times 3! = 48$. The automorphism group of the Rauzy diagram $\mathcal{G}$ has order $\frac{48}{2} = 24$ and is a subgroup of $\mathcal{G}'$. Hence we want to construct a morphism ϕ from $\mathcal{G}'$ to $\{\pm 1\}$ such that $\mathcal{G} = \ker(\phi)$.

There is a natural map from $\mathcal{G}'$ onto $\mathfrak{S}(a,b,c)$ whose kernel N is identified with $\{-1,1\}^3$. For $\tau \in \mathfrak{S}(a,b,c)$, we define $\sigma(\tau) \in \mathcal{G}'$, with a slight abuse of notation, by $\sigma(\tau)(\alpha_i) = \tau(\alpha)_i$, for $\alpha \in \{a,b,c\}$. Hence the extension $1 \to N \to \mathcal{G}' \to \mathfrak{S}(a,b,c) \to 1$ is a split extension. Therefore, the group $\mathcal{G}'$ has a natural semidirect product structure $\mathcal{G}' = N \rtimes H \sim \{\pm 1\}^3 \rtimes \mathfrak{S}(a,b,c)$ with $H = \sigma(\mathfrak{S}(a,b,c))$.

Claim 3.1. *Under the above identification, $\mathcal{G}$ is the kernel of the map $f\colon ((\varepsilon_a,\varepsilon_b,\varepsilon_c),\tau) \to \varepsilon_a\varepsilon_b\varepsilon_c \operatorname{sgn}(\tau)$, where* sgn *is the signature.*

To prove the claim, we observe that

- $f(\varepsilon,\tau) = f_1(\varepsilon) f_2(\tau)$ with a morphism f_1 from $\{\pm 1\}^3$ to $\{\pm 1\}$, and a morphism f_2 from $\mathfrak{S}(a,b,c)$ to $\{\pm 1\}$, hence either identity or the signature.
- The elements $((-1,1,1),\mathrm{Id})$, $((1,-1,1),\mathrm{Id})$ and $((1,1,-1),\mathrm{Id})$ are *not* in $\mathcal{G}$. Indeed, for instance $(-1,1,1)$ corresponds to interchanging the labels a_1 and a_2 and fixing all the others. In terms of marked surfaces, it corresponds to rotating by 2π the horizontal separatrices of the "a" singularity and fixing all the other ones. This is not possible from Proposition 2.5. Hence $f_1(\varepsilon_a,\varepsilon_b,\varepsilon_c) = \varepsilon_a\varepsilon_b\varepsilon_c$.

It remains to prove that $f_2 \neq \mathrm{Id}$. In this particular example, this can be done by exhibiting a particular path in the Rauzy diagram: we start from π and apply the moves $\mathcal{R}_b^4 \circ \mathcal{R}_t^3 \circ \mathcal{R}_b^4 \circ \mathcal{R}_t^3$ and we obtain the element

$$\pi' = \begin{pmatrix} -\infty & a_2 & b_1 & a_1 & b_2 & c_1 & 0 & c_2 & \infty \\ \infty & a_1 & b_1 & a_2 & b_2 & c_2 & 0 & c_1 & -\infty \end{pmatrix}.$$

We see that π' is also obtained from π by a relabelling and it gives the element $((-1,1,1),(a\ b)) \in \mathcal{G}$. Hence $f_2((a\ b)) = -1$.

This combinatorial approach is not easily generalisable for any Rauzy class. In Lemma 3.4, we use a geometric construction to get an element $g = ((\varepsilon_a,\varepsilon_b,1),(a\ b)) \in \mathcal{G}$ (hence with $\varepsilon_a\varepsilon_b f_2((a\ b)) = 1$) such that $g^2 = ((-1,-1,1),\mathrm{Id}) \neq 1_{\mathcal{G}}$. But a computation gives $g^2 = ((\varepsilon_a\varepsilon_b,\varepsilon_a\varepsilon_b,1),\mathrm{Id})$. This implies $f_2((a\ b)) = -1$, hence $f_2 = \operatorname{sgn}$.

3.2. Nonhyperelliptic case

Let $\mathcal{D}$ be a nonhyperelliptic Rauzy diagram whose associate stratum is $\mathcal{H}(k,k_1^{n_1},\dots,k_r^{n_r})$, and we assume that the degree of the special (left) singularity is k. From [4] the order of the group $\mathrm{Aut}(\mathcal{D})$ is

$$\varepsilon \prod_{i=1}^{r} n_i!(k_i+1)^{n_i},$$

where $\varepsilon = 1$ if all k_i are even and $\frac{1}{2}$ otherwise. We have defined in Section 2.2 a one-to-one map $T: \mathcal{A} \to \mathcal{A}$.

We consider the orbits in $\mathcal{A}$ for the action of T. Denote by *special orbit* the one containing $\pm\infty$ and by *regular orbits* the other ones. For $i \in \{1, \ldots, r\}$, there are n_i regular orbits of length $k_i + 1$. We denote Θ_i the set of regular orbits of length $k_i + 1$, $\Theta_i = \{\Theta_{i,j},\ j \in \{1, \ldots, n_i\}\}$, and we chose for each (i, j) an element $\alpha_{i,j} \in \Theta_{i,j}$.

Note that the special orbit corresponds to the special singularity (of order k) described in Section 2.2 and the regular orbits correspond to the other singularities, hence Θ_i corresponds to the (nonspecial) singularities of degree k_i.

An element $\sigma \in \mathrm{Aut}(\mathcal{D}) \subset \mathfrak{S}(\mathcal{A})$ satisfies

$$\sigma \circ T = T \circ \sigma. \tag{3.1}$$

In particular, σ induces a permutation on the set of (regular) orbits of the same size and it fixes all the elements of the special orbit.

This property defines a subgroup $\mathcal{G}'$ of $\mathfrak{S}(\mathcal{A})$ of order $\prod_{i=1}^{r} n_i!(k_i + 1)^{n_i}$. If all singularities are of even order, then $\mathcal{G}' = \mathrm{Aut}(\mathcal{D})$ and there is nothing to do. If there are singularities of odd order, then $\mathcal{G} = \mathrm{Aut}(\mathcal{D})$ is a subgroup of order 2 of $\mathcal{G}'$. We will define a nontrivial homomorphism ϕ for which $\mathcal{G}$ is the kernel.

First, we observe that $\mathcal{G}' \simeq \prod_{i=1}^{r} \mathcal{G}'_i$, where $\mathcal{G}'_i$ are elements of $\mathcal{G}'$ whose supports are in $\mathrm{Supp}(\Theta_i) = \coprod_j \Theta_{i,j}$. Note that $\mathrm{Supp}(\Theta_i)$ corresponds to the set of labels that correspond to a horizontal separatrix attached to a nonspecial singularity of degree k_i.

For each $i \in \{1, \ldots, r\}$, there is a natural map from $\mathcal{G}'_i$ onto the group $\mathfrak{S}(\{1, \ldots, n_i\})$, whose kernel N_i is identified with $U_{k_i+1}^{n_i}$, where U_p denotes the group of complex p-roots of unity. For $\tau \in \mathfrak{S}(\{1, \ldots, n_i\})$, we define $\sigma_i(\tau) \in \mathcal{G}'_i$ by $\sigma_i(\tau)(\alpha_{i,j}) = \alpha_{i,\tau(j)}$, and extend on $\mathrm{Supp}(\Theta_i)$ by using equation (3.1). Hence the extension

$$1 \to N_i \to \mathcal{G}'_i \to \mathfrak{S}(\{1, \ldots, n_i\}) \to 1$$

is a split extension. Therefore, the group $\mathcal{G}'_i$ has a natural semidirect product structure $\mathcal{G}'_i = N_i \rtimes H_i \sim U_{k_i+1}^{n_i} \rtimes \mathfrak{S}(\{1, \ldots, n_i\})$ with $H_i = \sigma(\mathfrak{S}(\{1, \ldots, n_i\}))$.

The group $\mathcal{G}$ is a subgroup of index 2 of $\mathcal{G}'$. Hence, it is the kernel of a homomorphism $\phi: \mathcal{G}' \to \{\pm 1\}$ that we write, with a slight abuse of notation, as $\prod_i \phi_i$, where $\phi_i: \mathcal{G}'_i \to \{\pm 1\}$ is a homomorphism.

Theorem 3.2. *Let $\mathcal{D}$ be a nonhyperelliptic Rauzy diagram and let $\mathcal{G}$ be its automorphism group. We denote the corresponding stratum by $\mathcal{H}(k, k_1^{n_1}, \ldots, k_r^{n_r})$ with $k_i \neq k_j$ if $i \neq j$, where the degree of the special (left) singularity is k. Then under the above notations and identifications, we have $\mathcal{G} = \ker(\phi)$ with*

$$\phi\colon \mathcal{G}' \simeq \prod_i \mathcal{G}'_i \to \{\pm 1\},$$

$$(g_1, \ldots, g_r) \mapsto \prod_i \phi_i(g_i),$$

where ϕ_i satisfies the following:

- *If k_i is even, then $\phi_i = 1$.*
- *Otherwise, let $2p_i = k_i + 1$, then*

$$\phi_i((\zeta_1, \dots, \zeta_{n_i}), \tau_i) = (\zeta_1 \dots \zeta_{n_i})^{p_i} \operatorname{sgn}(\tau_i),$$

where $\operatorname{sgn}\colon \mathfrak{S}(\{1, \dots, n_i\}) \to \{\pm 1\}$ *is the signature.*

Remark 3.3. When $n_i = 1$, the group $\mathfrak{S}(\{1, \dots, n_i\})$ is the trivial group and sgn is the trivial map.

Proof. We notice that the homomorphism ϕ_i from $U_{k_i+1}^{n_i} \rtimes \mathfrak{S}(\{1, \dots, n_i\})$ to $\{\pm 1\}$ is of the form $\phi_i((\zeta_1, \dots, \zeta_{n_i}), \tau_i) = f_{i,1}(\zeta_1) \cdots f_{i,n_i}(\zeta_{n_i}) h_i(\tau_i)$. If k_i is even, then necessarily the $f_{i,k}$ are trivial, and if k_i is odd, the $f_{i,k}$ are either trivial or satisfy $f_{i,k}(\zeta) = \zeta^{\frac{k_i+1}{2}}$. The homomorphism h_i is either trivial or the signature.

If k_i is even, it remains to prove h_i is trivial. When $n_i = 1$, the group $\mathfrak{S}(\{1, \dots, n_i\})$ is trivial and hence h_i is trivial. If $n_i > 1$, then from Lemma 3.4, there is an element in $\mathcal{G} \cap \mathcal{G}_i$ of the form $((\zeta_1, \zeta_2, 1, \dots, 1), (1\ 2))$ and therefore $h_i((1\ 2)) = 1$, hence h_i is the trivial map.

If k_i is odd, we first prove that the homomorphisms $f_{i,k}$ are nontrivial. Let $\zeta = \exp(\frac{2i\pi}{2p_i})$. The element $((1, \dots, 1, \zeta, 1, \dots, 1), \mathrm{Id})$ is *not* in $\mathcal{G}$ since it corresponds to rotating the separatrices adjacent to an odd degree singularity by 2π, and preserving all the other separatrices and this is not possible by Proposition 2.5. Hence, we have $\phi_i((1, \dots, 1, \zeta, 1, \dots, 1), \mathrm{Id}) = -1 = f_{i,k}(\zeta) = \zeta^{p_i}$. Hence

$$\phi_i((\zeta_1, \dots, \zeta_{n_i}), \mathrm{Id}) = (\zeta_1 \dots \zeta_{n_i})^{p_i}.$$

Now we prove that h_i is the signature. As above, the case $n_i = 1$ is trivial. If $n_i > 1$, let g_i be the element given by Lemma 3.4. A simple computation gives

$$g_i^2 = ((\zeta_1\zeta_2, \zeta_1\zeta_2, 1, \dots, 1), \mathrm{Id}),$$

hence

$$g_i^{2p_i} = ((\zeta_1^{p_i}\zeta_2^{p_i}, \zeta_1^{p_i}\zeta_2^{p_i}, 1, \dots, 1), \mathrm{Id}) \neq 1.$$

But $\phi_i(g_i) = (\zeta_1\zeta_2)^{p_i} h_i((1\ 2))$, hence $h_i((1\ 2)) = -1$. ∎

Lemma 3.4. *For each i such that $n_i > 1$, there exists an element $g_i \in \mathcal{G} \cap \mathcal{G}_i$ of the form $((\zeta_1, \zeta_2, 1, \dots, 1), (1\ 2))$ such that g_i^2 is of order $k_i + 1$.*

Proof. We have defined in Section 2.2 the moduli space of translation surfaces with labelled separatrices $\mathcal{H}^{\mathrm{lab}}$. We also define the moduli space $\mathcal{H}^{\mathrm{sing}}$ of translation surfaces with marked singularities, i.e., we give a name to each singularity (see [4] for a precise definition). There are canonical coverings $\mathcal{H}^{\mathrm{lab}} \to \mathcal{H}^{\mathrm{sing}}$ and $\mathcal{H}^{\mathrm{sing}} \to \mathcal{H}$. For any connected component $\mathcal{C}$ of $\mathcal{H}$, the preimage in $\mathcal{H}^{\mathrm{sing}}$ is connected (once a proper condition on the label is fixed).

We start from a labelled permutation π and choose a suspension datum ζ to obtain an element S in $\mathcal{H}^{\text{lab}}(k, k_1^{n_1}, \dots, k_r^{n_r})$. We denote by "1" and "2" the singularities corresponding to the interchanged orbits (for the map T) in the transposition $(1\ 2) \in \mathfrak{S}(\{1, \dots, n_i\})$.

We describe a path in $\mathcal{H}^{\text{lab}}(k, k_1^{n_1}, \dots, k_r^{n_r})$ that will induce the required element g_i. Denote by $\mathcal{C}$ the underlying connected component of the (usual) moduli space of Abelian differentials.

From [11], there exists a surface $S_0 \in \mathcal{C}$ obtained after breaking up a singularity of order $2k_i$ into a pair of singularities of order k_i. Considering S, S_0 as elements in $\mathcal{H}^{\text{sing}}$, we can assume that the pairs of singularities in S_0 after the breaking up procedure are "1" and "2", and there exists a path γ joining S to S_0. With the notations of Section 2.3, we cut the surface along ∂V_ε, rotate the disk by an angle θ, and we get a family of surfaces (S_θ). Since $S_{\pi(2k_i+1)} = S_0$ in $\mathcal{H}$, by composing with γ^{-1}, we obtain a closed path in $\mathcal{C}$. Considering the lift of this path in $\mathcal{H}^{\text{lab}}$, we see that

- The two singularities "1" and "2" have been interchanged.
- All the other singularities, and the corresponding labels on horizontal separatrices are fixed, since the surgery (cutting on a circle, rotating, pasting) does not change the metric outside a neighbourhood of the two singularities.

Hence the resulting element g_i in $\text{Aut}(\mathcal{D})$ is in $\mathcal{G}_i'$ and of the form $((\zeta_1, \zeta_2, 1, \dots, 1), (1\ 2))$.

Now we look at g_i^2. It corresponds to the following path: consider γ, then the path (S_θ), for $\theta \in [0, \dots, 2\pi(2k_i + 1)]$, then γ^{-1}. Keeping track of the marked horizontal separatrices, we see at the end that the marked horizontal separatrices for "1", "2" have changed by an angle $-2\pi(2k_i + 1) \bmod 2\pi(k_i + 1)$, hence 2π. So we have

$$g_i^2 = ((\zeta, \zeta, 1, \dots, 1), \text{Id})$$

with $\zeta = \exp(\frac{2i\pi}{k_i+1})$. Hence g_i^2 is of order $k_i + 1$. ∎

3.3. Hyperelliptic case

A hyperelliptic component of the strata of the moduli space of Abelian differentials is a component that consists only of hyperelliptic translation surfaces. From [11], these are components of the strata (without marked points) $\mathcal{H}(2g-2)$ and $\mathcal{H}(g-1, g-1)$ (for $g \geq 2$).

In terms of Rauzy diagrams, a hyperelliptic connected component without marked points corresponds to the Rauzy diagram generated by the element

$$\pi = \begin{pmatrix} 1 & 2 & \dots & n \\ n & n-1 & \dots & 1 \end{pmatrix}.$$

The stratum is $\mathcal{H}(n-2)$ if n is even and $\mathcal{H}(\frac{n-3}{2}, \frac{n-3}{2})$ if n is odd.

In this case, the Rauzy group is trivial.

We can however obtain nontrivial groups if we have marked points. A Rauzy diagram is said to be hyperelliptic if the corresponding connected component of the moduli space of Abelian differential consists only of hyperelliptic surfaces (i.e., we obtain one of the above hyperelliptic connected components after removing marked points).

We have the following statement.

Theorem 3.5. *Let $\mathcal{D}$ be a hyperelliptic Rauzy diagram and let $\mathcal{G}$ be its automorphism group.*

(1) *If the corresponding stratum is $\mathcal{H}(2g-2, 0^n)$ or $\mathcal{H}(g-1, g-1, 0^n)$ and the special singularity is of nonzero degree, then*

$$\mathcal{G} \simeq \mathfrak{S}(\{1, \dots, n\})$$

and an element of $\mathcal{G}$ corresponds to a permutation of the marked points.

(2) *If the corresponding stratum is $\mathcal{H}(2g-2, 0^n)$ and the special singularity is of zero degree, then*

$$\mathcal{G} \simeq U_{2g-1} \times \mathfrak{S}(\{1, \dots, n-1\}),$$

where the elements of the form $(1, p)$ correspond to permutations of the marked points, while the element (ζ, Id) with $\zeta = \exp(\frac{2i\pi}{2g-1})$ corresponds to the map T restricted to its orbit of length $2g-1$ (equivalently, rotating by 2π the separatrices adjacent to the singularity of degree $2g-2$).

(3) *If the corresponding stratum is $\mathcal{H}(g-1, g-1, 0^n)$ and the special singularity is of zero degree, then*

$$\mathcal{G} \simeq U_{2g} \times \mathfrak{S}(\{1, \dots, n-1\}).$$

The elements of the form $(1, p)$ correspond to permutations of the marked points, while the element $\tau \in \mathfrak{S}(A)$ corresponding to (ζ, Id) with $\zeta = \exp(\frac{2i\pi}{2g})$ is defined in the following way: Consider a surface S in $\mathcal{H}^{\mathrm{lab}}$ constructed from any permutation of the Rauzy diagram, choose a pair $(a_0, b_0) \in \mathcal{A}^2$ such that

- *the corresponding separatrices I_{a_0}, I_{b_0} of S are attached to the two singularities of order $g-1$, respectively,*
- *the angle between $\iota(I_{a_0})$ and I_{b_0} is π, where ι is the hyperelliptic involution (note that ι interchanges the two singularities of order $g-1$, hence the angle between $\iota(I_{a_0})$ and I_{b_0} is well defined).*

For $i \in \{1, \dots, g-1\}$, we denote $a_i = T^i(a_0)$ and $b_i = T^i(b_0)$. Then $\tau\colon \mathcal{A} \to \mathcal{A}$ satisfies $\tau(a_i) = b_i$, $\tau(b_i) = a_{i+1}$ and $\tau(c) = c$ if $c \notin \{a_0, \dots, a_{g-1}, b_0, \dots, b_{g-1}\}$.

Proof. As in the previous sections, we consider the orbits of the map T. Nonspecial singularities of degree $k \geq 0$ correspond to orbits of length $k+1$. An element $\sigma \in \mathrm{Aut}(\mathcal{D})$ satisfies $\sigma \circ T = T \circ \sigma$ and fixes pointwise the elements of the special orbit. This defines a subgroup $\mathcal{G}'$ of $\mathfrak{S}(A)$ and $\mathcal{G} = \mathrm{Aut}(\mathcal{D})$ is a subgroup of $\mathcal{G}'$. Note that, as before, for an element π in the Rauzy class and an element $\sigma \in \mathcal{G}'$, we have $\sigma \in \mathrm{Aut}(\mathcal{D})$ if and only if π and $\pi \circ \sigma$ define elements in the same connected component of $\mathcal{H}^{\mathrm{lab}}$.

We first consider the case $\mathcal{H}(2g-2, 0^n)$. If the special singularity is of degree $2g-2$, the nonspecial T orbits are singletons and $\mathcal{G}' \simeq \mathfrak{S}(\{1, \dots, n\})$, and any element of $\mathcal{G}'$ is in $\mathrm{Aut}(\mathcal{D})$ since the corresponding surfaces are in the same connected component of $\mathcal{H}^{\mathrm{lab}}$. If the degree of the special singularity is zero, then $\mathcal{G}' \simeq U_{2g-1} \times \mathfrak{S}(\{1, \dots, n-1\})$. Elements in $\mathcal{G}'$ of the form $(1, p)$ correspond to permutations of the marked points and are in $\mathrm{Aut}(\mathcal{D})$. The element of the form (ζ, Id) corresponds to rotating by 2π the separatrices adjacent to the singularity of degree $2g-2$ and preserving all the other separatrices. It is obtained by continuously rotating (clockwise) the surface (and keeping the separatrices horizontal). After rotating by 2π the nonmarked translation surface is the same as before, and the separatrices attached to marked points have not changed (since there is only one horizontal separatrix attached to a marked point), while the separatrices attached to the singularity of degree $2g-2$ have rotated by 2π (counterclockwise).

Now we consider the case $\mathcal{H}(g-1, g-1, 0^n)$ with the special singularity of degree $g-1$. The group $\mathcal{G}'$ is isomorphic to $U_g \times \mathfrak{S}(\{1, \dots, n\})$. This time, we prove that $\mathrm{Aut}(\mathcal{D})$ corresponds to elements of the form $(1, p)$ for $p \in \mathfrak{S}(\{1, \dots, n\})$ and is therefore a subgroup of $\mathcal{G}'$ of order g. Clearly, these elements are in $\mathrm{Aut}(\mathcal{D})$ by the same argument as in the previous paragraph. We check that they are the only elements. Consider a surface S in $\mathcal{H}^{\mathrm{lab}}$ constructed from a permutation π in the Rauzy class. Denote by P, Q the singularities of degree $g-1$ with P being the special one. Consider a separatrix I_a attached to Q, marked by the label $a \in \mathcal{A}$. The image of P by the hyperelliptic involution ι is Q, hence the image by ι of the separatrix I_∞ is a negative horizontal separatrix attached to Q. Hence, the angle between I_a and $\iota(I_\infty)$ is constant in a connected component of $\mathcal{H}^{\mathrm{lab}}$. Therefore, any element of $\mathrm{Aut}(\mathcal{D})$ must fix a, and therefore it must fix pointwise the associated T-orbit.

The last case is $\mathcal{H}(g-1, g-1, 0^n)$ with the special singularity of degree 0. The group $\mathcal{G}'$ is isomorphic to $\mathcal{G}_0 \times \mathfrak{S}_{n-1}$, where elements of the form $(1, p)$ correspond to permutations of the marked points and are in $\mathrm{Aut}(\mathcal{D})$. The group $\mathcal{G}_0$ corresponds to elements of $\mathcal{G}'$ that fix pointwise the marked points. It is isomorphic to $U_g^2 \rtimes \mathfrak{S}_2$ (see the previous section) hence is of cardinal $2g^2$. We show that elements of the form $(s, 1)$ in $\mathrm{Aut}(\mathcal{D})$ correspond to a cyclic subgroup of $\mathcal{G}_0$ of order $2g$. Denote by P, Q the singularities of degree $g-1$ such that I_{a_0} is attached to P and I_{b_0} is attached to Q. Denote by $a_1, \dots, a_{g-1}$ the labels of the other marked horizontal separatrices attached to P (taken counterclockwise, in particular we have $a_i = T^i(a_0)$), and similarly $b_1, \dots, b_{g-1}$ for the labels of the horizontal separatrices attached to Q. As in the proof of Lemma 3.4, there is a continuous path in $\mathcal{H}^{\mathrm{lab}}$ that interchanges P and Q and does not change any other singularity. Then, by continuously rotating the surface, we finally obtain a path joining the initial surface $S_0 \in \mathcal{H}^{\mathrm{lab}}$ to a surface S_1 such that the underlying surface without marking is the same and the separatrix I_{a_0} of S_0 has been replaced by the separatrix I_{b_0} in S_1. Since the angle between $\iota(I_{a_0})$ and I_{b_0} is constant in a connected component of $\mathcal{H}^{\mathrm{lab}}$, it is still π, and therefore the angle between $\iota(I_{b_0})$ and I_{a_0} is also π, since ι is an isometric involution. Hence, the separatrix I_{b_g} of S_0 has been replaced by I_{a_0} in S_1. Therefore, we obtain the map $\tau \in \mathrm{Aut}(\mathcal{A})$ defined by $\tau(a_i) = b_i$, $\tau(b_i) = a_{i+1}$

and $\tau(c) = c$ if $c \notin \{a_0, \dots, b_g\}$. The element τ is of order $2g$ and we obtain the required element. ∎

The practical difficulty of the description of τ in case 3 is that for a given π in the Rauzy diagram, it is not algorithmically clear, to our knowledge, how to find the pair (a_0, b_0). We give a particular case where it is easy.

Proposition 3.6. *Let π be in the hyperelliptic Rauzy class corresponding to the stratum $\mathcal{H}(g-1, g-1, 0^n)$ with the special singularity of degree* 0*. We assume that*

- *There is $k \in \{1, \dots, d-1\}$ such that the corresponding left and right singularities in the Veech construction of $a_0 = \pi_t^{-1}(k)$ are the two singularities of degree $g-1$.*
- *The permutation π' obtained by removing a_0 is irreducible.*

Then, denoting $b_0 = \pi_t^{-1}(k+1)$, the pair (a_0, b_0) satisfies the hypothesis of case 3 *of Theorem* 3.5.

Proof. We start from a suspension datum ζ' for the permutation π', and we deduce a suspension datum ζ for π by setting $\zeta_{a_0} = \varepsilon$ and $\zeta_\alpha = \zeta'_\alpha$ otherwise. Then for enough small ε, the surface corresponding to (π, ζ) admits a unique (horizontal) smallest saddle connection of length ε. Since the hyperelliptic involution ι is an isometry it preserves γ, and since ι interchanges the two singularities of degree $g-1$, it changes the orientation of γ. Therefore, we obtain the angle π between $\iota(I_{a_0})$ and I_{b_0} as required. ∎

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Corentin Boissy
Institut de Mathématiques de Toulouse, Université Toulouse 3, 118 route de Narbonne, 31062 Toulouse, France; corentin.boissy@math.univ-toulouse.fr