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Amenability and profinite completions of finitely generated groups

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Abstract. This article explores the interplay between the finite quotients of finitely generated residually finite groups and the concept of amenability. We construct a finitely generated, residually finite, amenable group A and an uncountable family of finitely generated, residually finite non-amenable groups, all of which are profinitely isomorphic to A. All of these groups are branch groups. Moreover, picking up Grothendieck's problem, the group A embeds in these groups such that the inclusion induces an isomorphism of profinite completions. In addition, we review the concept of uniform amenability, a strengthening of amenability introduced in the 70s, and we prove that uniform amenability is indeed detectable from the profinite completion.

1. Introduction

The profinite completion $\widehat{G}$ of a group G is the inverse limit of its finite quotients. If G is residually finite, then G embeds into $\widehat{G}$ and it is natural to wonder what properties of G can be detected from $\widehat{G}$. Especially a question of Grothendieck, posed in 1970 [9], sparked interest in this direction: Is an embedding $\iota\colon H \to G$ of finitely presented, residually finite groups an isomorphism if it induces an isomorphism $\hat{\iota}\colon \widehat{H} \to \widehat{G}$ of profinite completions? The answer is negative. Finitely generated counterexamples were constructed already in 1986 by Platonov and Tavgen' [20]. The finitely presented case was settled almost 20 years later by Bridson and Grunewald [6].

In this article, we explore the interplay between amenability and the profinite completion of finitely generated groups. Our interest was prompted by the following variation of Grothendieck's problem:

($*$) *Given two finitely generated residually finite groups A, G, where A is amenable. Suppose $\iota\colon A \to G$ induces an isomorphism $\hat{\iota}\colon \widehat{A} \to \widehat{G}$. Is G amenable?*

The answer is negative and we obtain the following result (a consequence of Theorem 7.6 below).

Theorem 1.1. *There are an uncountable family of pairwise non-isomorphic, residually finite 18-generator groups $(G_j)_{j\in J}$ and a residually finite 6-generator group A with embeddings $\iota_j\colon A \to G_j$ such that the following properties hold:*

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(1) $\widehat{\iota_j}\colon \widehat{A} \to \widehat{G_j}$ *is an isomorphism,*

(2) *A is amenable,*

(3) *each* G_j *contains a non-abelian free subgroup.*

In particular, amenability is not a profinite invariant of finitely generated residually finite groups.

We note that the fibre product construction used in [6, 20] is unable to provide such examples, since a fibre product $P \subseteq H \times H$ projects onto H and thus, it is non-amenable exactly if H is non-amenable (e.g., a non-elementary word hyperbolic group as in [6]). Uncountable families of finitely generated *amenable* groups with isomorphic profinite completions were constructed in [18, 21].

The groups in Theorem 1.1 are just-infinite branch groups with the congruence subgroup property. The construction is inspired by a method of Segal [22] and influenced by ideas of Nekrashevych [18]. The method is rather flexible and allows us to merge a *perfect* residually finite group (e.g., $\mathrm{SL}_n(\mathbb{Z})$) with a related amenable group in such a way that the amenable group and the merged group have isomorphic profinite completions. The first step bears similarity with the Sidki–Wilson construction of branch groups with non-abelian free subgroups [23].

Let us note that without the requirement of finite generation in $(*)$, there are obvious counterexamples. For instance, $A = \bigoplus_n \mathrm{Alt}(n)$ is amenable as a direct limit of finite groups and $G = \prod_n \mathrm{Alt}(n) = \widehat{A} = \widehat{G}$ contains a non-abelian free group. It would be interesting to have finitely presented counterexamples to $(*)$. Since our family $(G_j)_{j\in J}$ is uncountable, it is clear that most of the groups cannot be finitely presented. We were unable to verify that none of these groups admits a finite presentation.

Thinking of amenability as a concept of analytical nature (e.g., existence of a left invariant mean on $\ell^\infty(G)$), it does not seem surprising that the answer to $(*)$ is negative. From a different perspective, though, amenability is not far from being detectable on finite quotients. Kesten [15] characterized amenability in terms of the spectral radius of symmetric random walks. For a residually finite group G, a random walk can surely be studied by looking at large finite quotients of G. Trying to exploit this relation, one realizes that a *uniform* behavior of all random walks on all finite quotients can be used to deduce amenability of G.

In the 70s, Keller defined [14] a notion of *uniform amenability* by imposing that the size of an ε-Følner set for a generating set S can be uniformly bounded in terms of ε and $|S|$ (see Definition 2.1). A couple of years later the concept was independently defined by Bożejko [5]. Wysoczánski [25] showed that uniform amenability can be characterized in terms of a uniform Kesten condition. This leads to the following result.

Theorem 1.2. *Let* G_1, G_2 *be residually finite groups with* $\widehat{G}_1 \cong \widehat{G}_2$. *Then* G_1 *is uniformly amenable if and only if* G_2 *is uniformly amenable.*

As of today, the collection of finitely generated groups which are amenable but not uniformly amenable is rather small and this explains why the construction of counter-examples to $(*)$ actually requires some effort. In turn, the group A of Theorem 1.1 is a new example of an amenable group which is not uniformly amenable.

There are several ways to prove Theorem 1.2 and already Keller's results [14] point in this direction. Here we do not work out the random walk argument sketched above. Instead, we first establish new characterizations of uniform amenability in terms of a uniform isoperimetric inequality and a uniform Reiter condition and use them to give a short proof of the theorem. In addition, we give a new short proof that a uniformly amenable group satisfies a law (a result of Keller [14]). This implies that the profinite completion of a uniformly amenable group is positively finitely generated in the sense of Mann [17]. Our results on uniform amenability are discussed in Section 2.

We now give an overview of the remaining sections. In Section 3, we present the basic construction which will be applied throughout. This construction takes a perfect, self-similar subgroup $G \leq \mathrm{Aut}(T_X)$ of the automorphism group of a regular rooted tree T_X and produces – under a condition introduced by Segal [22] – a branch group $\Gamma_G^\Omega \leq \mathrm{Aut}(T_X)$. The construction depends on a certain subset Ω of the boundary of the tree. In Section 4, we show that these branch groups are just infinite and have the congruence subgroup property. We deduce that the profinite completion is always an iterated wreath product which only depends on the action of G on the first level of the tree. As a consequence, we obtain a zoo of groups with isomorphic completions and inclusions between these groups induce profinite isomorphisms. In Section 5, we use a rigidity result of Lavreniuk and Nekrashevych [16] to show that the construction (without additional assumptions) gives rise to an uncountable family of pairwise non-isomorphic groups. Finally, we discuss concrete examples in Sections 6 and 7. To obtain the amenable group A in Theorem 1.1, we apply the construction to the special affine group $\mathbb{F}_p^n \rtimes \mathrm{SL}_n(\mathbb{F}_p)$ acting on the p^n-regular rooted tree by rooted automorphisms. It follows from a result of Bartholdi, Kaimanovich and Nekrashevych [3] that the result is an amenable group (for suitable parameters Ω). On the other hand, we apply the construction to the special affine group $\mathbb{Z}^n \rtimes \mathrm{SL}_n(\mathbb{Z})$ which acts self-similarly on the p^n-regular rooted tree (obtained from the pro-p completion of $\mathbb{Z}^n$). Merging these groups with A, we obtain the family (G_j) of non-amenable groups in Theorem 1.1.

2. Uniformly amenable classes of groups

In this section, we study *uniform amenability* of groups and classes of groups. The concept was introduced by Keller in [14] and independently by Bożejko [5]. Here we establish new equivalent characterizations and use them to prove that uniform amenability is a profinite property. Let us begin with the usual characterization using a *uniform Følner condition*.

Definition 2.1. A class of groups $\mathfrak{K}$ is *uniformly amenable* if there is a function $m: \mathbb{R}_{>0} \times \mathbb{N} \to \mathbb{N}$ such that for every $\varepsilon > 0$, all $G \in \mathfrak{K}$, and every finite subset $S \subseteq G$, there is a finite set $F \subseteq G$ satisfying

(1) $|F| \leq m(\varepsilon, |S|)$,

(2) $|SF| \leq (1+\varepsilon)|F|$.

In this case, we say that $\mathfrak{K}$ is m-uniformly amenable. We say that a group G is uniformly amenable if the class consisting of G is uniformly amenable.

We will always assume that m is non-decreasing in the second argument; this can be achieved by replacing m by $m'(\varepsilon, N) = \max_{k \leq N} m(\varepsilon, k)$.

Example 2.2. (1) Let $d \in \mathbb{N}$ be given. The class $\mathfrak{Fin}_d$ of all finite groups of order at most d is uniformly amenable for the function $m(\varepsilon, N) = d$; indeed, the finite group G itself is always a suitable Følner set.

(2) The class of abelian groups is uniformly amenable.

(3) Extensions of uniformly amenable groups are uniformly amenable [5, Theorem 3]. In particular, every virtually solvable group is uniformly amenable.

(4) Direct unions of m-uniformly amenable groups are m-uniformly amenable. In particular, ascending HNN-extensions of uniformly amenable groups are uniformly amenable.

(5) Let $\mathfrak{K}$ be a class of groups such that the free group F_2 of rank 2 is residually $\mathfrak{K}$, then $\mathfrak{K}$ is not uniformly amenable (this will follow from Corollary 2.12 below). In particular, every class of groups which contains all finite symmetric groups is *not* uniformly amenable.

Example 2.3. Let G be a finite group. The direct power G^I is uniformly amenable for every set I. Given a positive integer n, the set of all n-tuples $X := G^n$ in G is finite. We enumerate the elements, say $X = \{x^{(1)}, \ldots, x^{(k)}\}$, where $k = |G|^n$. Here $x^{(i)} = (x_1^{(i)}, \ldots, x_n^{(i)})$. Consider the group G^k with the "universal" n-element subset $U = \{u_1, \ldots, u_n\}$, where $u_j = (x_j^{(1)}, x_j^{(2)}, \ldots, x_j^{(k)})$.

Let $S \subseteq G^I$ be a subset with n-elements, say $S = \{s_1, \ldots, s_n\}$. For every $i \in I$, we obtain an n-tuple $S(i) := (s_1(i), s_2(i), \ldots, s_n(i)) \in X$ and a map $t: I \to \{1, \ldots, k\}$ such that $S(i) = x^{(t(i))}$. The homomorphism $\alpha: G^k \to G^I$ defined by $\alpha(g_1, \ldots, g_k)(i) = g_{t(i)}$ maps the universal set U to S. Therefore, the subgroup generated by S is isomorphic to a subfactor of G^k, and we deduce that G^I is uniformly amenable.

Definition 2.4. A class of groups $\mathfrak{K}$ satisfies a *uniform isoperimetric inequality* if there is a function $\tilde{m}: \mathbb{R}_{>0} \times \mathbb{N} \to \mathbb{N}$ such that for all $\varepsilon > 0$, every $G \in \mathfrak{K}$ and every finite symmetric subset $S \subseteq G$, there is a finite subset $E \subseteq G$ with $|E| \leq \tilde{m}(\varepsilon, |S|)$, and

$$\frac{|\partial_S E|}{|E|} \leq \varepsilon,$$

where $\partial_S E = SE \setminus E$ denotes the S-boundary of E.

Lemma 2.5. *A class of groups $\mathfrak{K}$ is uniformly amenable if and only if it satisfies a uniform isoperimetric inequality.*

Proof. Assume that $\mathfrak{K}$ is m-uniformly amenable. We define $\widetilde{m}(\varepsilon, N) := m(\varepsilon, N+1)$. Let $\varepsilon > 0$ be given, let $G \in \mathfrak{K}$ and let $S \subseteq G$ be a finite symmetric subset. We define $S^* = S \cup \{1_G\}$. Uniform amenability provides a Følner set $E \subseteq G$ with $|E| \leq m(\varepsilon, N+1)$ and

$$|S^* E| \leq (1+\varepsilon)|E|.$$

Since $S^* E = E \cup \partial_S E$, the assertion follows.

Assume conversely that $\mathfrak{K}$ satisfies a uniform isoperimetric inequality with respect to $\widetilde{m}$. We define $m(\varepsilon, N) = \max_{k \leq 2N} \widetilde{m}(\varepsilon, k)$. Let $\varepsilon > 0$, $G \in \mathfrak{K}$, and a finite set $S \subseteq G$ be given. Define $T = S \cup S^{-1}$. By assumption, there is a finite subset $E \subseteq G$ with $|E| \leq m(\varepsilon, |T|) = \max_{k \leq 2|S|} \widetilde{m}(\varepsilon, k) = m(\varepsilon, |S|)$ which satisfies

$$\frac{|\partial_T E|}{|E|} \leq \varepsilon.$$

We obtain

$$|SE| \leq |TE| \leq |E| + |\partial_T E| \leq (1+\varepsilon)|E|. \qquad \blacksquare$$

Definition 2.6. A class of groups $\mathfrak{K}$ satisfies the *uniform Reiter condition* if there is a function $r \colon \mathbb{R}_{>0} \times \mathbb{N} \to \mathbb{N}$ such that for all $\varepsilon > 0$, every $G \in \mathfrak{K}$, and every finite subset $S \subseteq G$, there is a finitely supported probability measure μ on G such that $|\mathrm{supp}(\mu)| \leq r(\varepsilon, |S|)$ and

$$\|\lambda_g^*(\mu) - \mu\|_{\ell^1} < \varepsilon \tag{2.1}$$

for all $g \in S$. Here $\lambda_g^*(\mu)$ denotes the pullback of μ with respect to the left multiplication with g, i.e., $\lambda_g^*(\mu)(A) = \mu(gA)$.

Proposition 2.7. *A class $\mathfrak{K}$ of groups is uniformly amenable if and only if it satisfies the uniform Reiter condition.*

Proof. Assume that $\mathfrak{K}$ is uniformly amenable. By Lemma 2.5, the class $\mathfrak{K}$ satisfies a uniform isoperimetric inequality with respect to a function $\widetilde{m}$. Let $\varepsilon > 0$, $G \in \mathfrak{K}$ and $S \subseteq G$ be given. Put $S^* = S \cup S^{-1}$. There is a finite subset $E \subseteq G$ with $|E| \leq \widetilde{m}(\varepsilon, 2|S|)$ for which $\frac{|\partial_{S^*} E|}{|E|} < \varepsilon$. Let μ be the uniform probability measure supported on E. Since $|\partial_{S^*} E| < \varepsilon |E|$, we have $|g^{-1} E \Delta E| < \varepsilon |E|$ for all $g \in S$ and thus

$$\|\lambda_g^*(\mu) - \mu\|_{\ell^1} = \frac{|g^{-1} E \Delta E|}{|E|} < \varepsilon.$$

Conversely, assume that $\mathfrak{K}$ satisfies the uniform Reiter condition. Let $\varepsilon' > 0$, $G \in \mathfrak{K}$ and a finite symmetric subset $S \subseteq G$ be given. Set $\varepsilon = \varepsilon'/|S|$. Using the uniform Reiter condition, we find a finitely supported probability measure μ on G with $|\mathrm{supp}(\mu)| \leq r(\varepsilon, |S|)$ which satisfies (2.1). For all $t \in [0,1]$, we define the level set $E_\mu(t) = \{g \in G \mid \mu(\{g\}) \geq t\}$.

We claim that some level set satisfies a suitable isoperimetric inequality. Summing the equality $|\lambda_g^*(\mu)(\{x\}) - \mu(\{x\})| = \int_0^1 |1_{E_\mu(t)}(gx) - 1_{E_\mu(t)}(x)|\,dt$ over all $x \in G$, we obtain

$$\|\lambda_g^*(\mu) - \mu\|_{\ell^1} = \int_0^1 \sum_{x\in G} |1_{E_\mu(t)}(gx) - 1_{E_\mu(t)}(x)|\,dt = \int_0^1 |g^{-1}E_\mu(t)\Delta E_\mu(t)|\,dt.$$

Taking the sum over all $g \in S$, we see that

$$\varepsilon' = |S|\varepsilon > \sum_{g\in S} \|\lambda_g^*(\mu) - \mu\|_{\ell^1} = \int_0^1 \sum_{g\in S} |g^{-1}E_\mu(t)\Delta E_\mu(t)|\,dt \geq \int_0^1 |\partial_S E_\mu(t)|\,dt.$$

Suppose for a contradiction that $|\partial_S E_\mu(t)| > \varepsilon'|E_\mu(t)|$ for all t. Then the last integral can be estimated by

$$\int_0^1 |\partial_S E_\mu(t)|\,dt > \varepsilon' \int_0^1 |E_\mu(t)|\,dt = \varepsilon'$$

which yields a contradiction. ∎

Remark 2.8. It was proven in [25] that uniform amenability of groups can be characterized by a uniform version of Kesten's condition on random walks. The argument given there – based on a theorem of Kaimanovich (see [12] or [13, Theorem 5.2]) – directly generalizes to classes and shows that a uniformly amenable class of groups $\mathfrak{K}$ satisfies a *uniform Kesten condition*: There is a function $\kappa\colon \mathbb{R}_{>0} \times \mathbb{N} \to \mathbb{N}$ such that for every $\varepsilon > 0$, every $G \in \mathfrak{K}$, every finitely supported symmetric probability measure μ on G and all $n \geq \kappa(\varepsilon, |\mathrm{supp}(\mu)|)$,

$$\mathbb{P}(X_{2n} = 1_G)^{\frac{1}{2n}} > 1 - \varepsilon,$$

where X_n denotes the μ-random walk on G starting at the identity 1_G.

Proposition 2.9. *Let $\mathfrak{K}$ be a uniformly amenable class of groups. The class of all quotients of groups in $\mathfrak{K}$ is uniformly amenable.*

Proof. Assume that $\mathfrak{K}$ satisfies the uniform Reiter condition for a function r. Let $G \in \mathfrak{K}$, and let $N \subseteq G$ be a normal subgroup. The canonical projection $G \to G/N$ will be denoted by π. Given $\varepsilon > 0$ and a finite subset $S \subseteq G/N$, we lift S to a finite subset S' in G, i.e., $\pi(S') = S$ and $|S'| = |S|$. There is a finitely supported probability measure μ' on G with $|\mathrm{supp}(\mu')| \leq r(\varepsilon, |S|)$ that satisfies

$$\|\lambda_g^*(\mu') - \mu'\|_{\ell^1} < \varepsilon$$

for all $g \in S'$. Let $\mu = \pi_*(\mu')$ be the pushforward measure on G/N. Clearly, the support of μ has at most as many elements as the support of μ'. Moreover,

$$\begin{aligned}\|\lambda_{gN}^*(\mu) - \mu\|_{\ell^1} &= \sum_{x\in G/N} |\mu(\{gx\}) - \mu(\{x\})| = \sum_{x\in G/N} \Big|\sum_{w\in x} \mu'(\{gw\}) - \mu'(\{w\})\Big| \\ &\leq \sum_{h\in G} |\mu'(\{gh\}) - \mu'(\{h\})| = \|\lambda_g^*(\mu') - \mu'\|_{\ell^1} < \varepsilon\end{aligned}$$

for all $g \in S'$. Therefore, the class of factor groups satisfies the uniform Reiter condition for the same function r. ∎

Theorem 2.10. *Let G be a group and let $\mathcal{F}$ be a filter base*[1] *of normal subgroups with $\cap\mathcal{F} = \{1_G\}$. The group G is uniformly amenable if and only if the class $\{G/N \mid N \in \mathcal{F}\}$ is uniformly amenable.*

Proof. Suppose G is uniformly amenable. It follows immediately from Proposition 2.9 that $\{G/N \mid N \in \mathcal{F}\}$ is uniformly amenable.

For the converse statement, assume that $\{G/N \mid N \in \mathcal{F}\}$ is uniformly amenable for a function m. Let $\varepsilon > 0$ and a finite subset $S \subseteq G$ be given. Let $\pi_N : G \to G/N$ denote the canonical projection for $N \in \mathcal{F}$.

Let C_N denote the set of finite subsets $F \subseteq G$ with $1 \in F$ and $|F| \leq m(\varepsilon, |S|)$ that satisfy $|\pi_N(SF)| < (1+\varepsilon)|F|$. By assumption, the sets C_N are non-empty and $C_N \subseteq C_M$ whenever $N \subseteq M$. Since the cardinality of sets in C_N is bounded, the set C_N is closed and hence compact in the Fell topology on the power set $\{0,1\}^G$. By compactness, there is a finite set $F \in \bigcap_{N\in\mathcal{F}}^{\infty} C_N$. By construction, $1 \in F$ and F is non-empty. Now, let $N \in \mathcal{F}$ be sufficiently small such that distinct elements in SF represent distinct cosets in G/N. Then $|SF| = |\pi_N(SF)| \leq (1+\varepsilon)|F|$ and F is a Følner set. ∎

The following result shows that uniform amenability is a profinite property and immediately implies Theorem 1.2.

Corollary 2.11. *Let H be a profinite group. If some dense subgroup $G \subseteq H$ is uniformly amenable, then H is uniformly amenable.*

Proof. Let $\mathcal{F}$ be the filter of open normal subgroups of H. Now since G is uniformly amenable, the class $\{G/(G \cap N) \mid N \in \mathcal{F}\}$ is uniformly amenable. Since G is dense in H, we have $G/(G\cap N) \cong H/N$ for all open normal subgroups $N \leq H$. The uniform amenability of H follows from Theorem 2.10. ∎

The next result is due to Keller [14, Corollary 5.9]. As it will be used in the corollary below, we include a new short proof based on the uniform Kesten condition.

Corollary 2.12. *Every class $\mathfrak{K}$ of uniformly amenable groups satisfies a common group law.*

Proof. It follows from the uniform Kesten condition that there is a number N such that every pair of distinct elements in any group $G \in \mathfrak{K}$ satisfies some relation of length at most N. Since there are finitely many such relations, we can form a nested commutator involving all such relations (possibly involving a new letter to make sure that the result is non-trivial); this nested commutator is a law in G. ∎

[1] $\mathcal{F}$ is a non-empty set of normal subgroups, such that for all $N, M \in \mathcal{F}$ the intersection $N \cap M$ contains an element of $\mathcal{F}$.

Corollary 2.13. *Let G be a finitely generated, uniformly amenable group. Then the profinite completion $\widehat{G}$ is positively finitely generated.*

Proof. Since G satisfies a non-trivial law u, every subquotient satisfies the same law. However, there is a finite group for which u is not a law (e.g., some large symmetric group). In particular, this finite group is not a subquotient of G and similarly is not a subquotient of $\widehat{G}$. It follows from [4, Theorem 1.1] that $\widehat{G}$ is positively finitely generated. ∎

Remark 2.14. Keller asked whether every group that satisfies a law is amenable and whether every amenable group that satisfies a law is uniformly amenable. The answer to the first question is negative, since it is known by work of Adyan [1] that free Burnside groups of large exponent are non-amenable. In fact, they are even uniformly non-amenable [19]. Zelmanov's solution of the restricted Burnside problem implies that residually finite groups (not necessarily finitely generated) of bounded exponent are uniformly amenable. From this perspective, it seems possible that the answers are affirmative for residually finite groups. A similar question has been raised by de Cornulier and Mann in [7, Question 14]: Is there a finitely generated, residually finite group which satisfies a law and is not amenable? Combining this with Keller's question leads to an appealing problem.

Question 2.15. *Is every family of finite groups which satisfies a common law uniformly amenable?*

Let us close this section by noting that in general the class of uniformly amenable groups seems to be poorly understood and it would be fruitful to have more examples. Are there uniformly amenable groups which are not elementary amenable? Are there uniformly amenable groups with intermediate growth?

3. Groups acting on rooted trees and the Ω-construction

The purpose of this section is to introduce the basic construction of groups we will frequently use. We begin by fixing some basic terminology from the theory of groups acting on rooted trees.

3.1. Groups acting on rooted trees

By a *rooted tree*, we will always mean a tree T with a distinguished vertex, called *the root of* T, which we will denote by $\emptyset$. An automorphism of T will always be assumed to fix the root of T. The group of all such automorphisms will be denoted by $\mathrm{Aut}(T)$. Accordingly, an action of a group G on a rooted tree T is an action by graph isomorphisms that fix the root of T. Let $V(T)$ denote the vertex set of T. The distance of a vertex $v \in V(T)$ to the root $\emptyset$ is called the *level* of v and will be denoted by $\mathrm{lv}(v)$. Two vertices $v, w \in V(T)$ are called *adjacent* if they are connected by an edge. In this paper, we will mostly be interested in group actions on trees that arise as Cayley graphs of free monoids. More precisely, let X be a non-empty finite set which one can think of as an alphabet. Let X^* denote the free monoid generated by X, i.e., the set of (finite) words over X with composition given by concatenation of words. Let T_X denote the Cayley graph of X^* with respect to X. Clearly, T_X is a tree, and we consider T_X as a rooted tree where the root is the empty word $\emptyset$. Note that the set X^ℓ of words of length $\ell \in \mathbb{N}$ is precisely the set of vertices of level ℓ in T_X. As every $\alpha \in \operatorname{Aut}(T_X)$ fixes the root of T_X, it follows that α preserves the level sets X^ℓ. Thus for every subgroup $G \leq \operatorname{Aut}(T_X)$, we have a natural homomorphism from G to the symmetry group $\operatorname{Sym}(X^\ell)$. If $\ell = 1$, we write $\sigma_g \in \operatorname{Sym}(X)$ to denote the image of g under this homomorphism. On the other hand, every permutation $\sigma \in \operatorname{Sym}(X)$ gives rise to an automorphism of T_X by defining $\sigma(xw) = \sigma(x)w$ for all $x \in X$ and $w \in X^*$. To simplify notation, this automorphism will be denoted by σ as well. Automorphisms obtained in this way will be called *rooted* (here we follow the terminology of [2]). Another important type of automorphism is obtained by letting the direct sum $\operatorname{Aut}(T_X)^X := \bigoplus_{x \in X} \operatorname{Aut}(T_X)$ act on T_X via $((g_x)_{x \in X}, yw) \mapsto y g_y(w)$ for all $y \in X$ and $w \in X^*$. Together with the rooted ones, these automorphisms can be used to decompose arbitrary automorphism of T_X as follows.

Definition 3.1. Let X be a finite set and let $\alpha \in \operatorname{Aut}(T_X)$. For each $x \in X$, we define the *state* of α at x as the unique automorphism $\alpha_x \in \operatorname{Aut}(T_X)$ that satisfies

$$\alpha(xw) = \sigma_\alpha(x)\alpha_x(w)$$

for every $w \in X^*$. This gives us a decomposition $\alpha = \sigma_\alpha \circ (\alpha_x)_{x \in X}$ which is called the *wreath decomposition* of α.

If the alphabet X is clear from the context, we will often just write (α_x) instead of $(\alpha_x)_{x \in X}$. Note that the wreath decomposition endows us with an isomorphism

$$\operatorname{Aut}(T_X) \to \operatorname{Sym}(X) \ltimes \operatorname{Aut}(T_X)^X, \quad \alpha \mapsto \sigma_\alpha \cdot (\alpha_x).$$

Definition 3.2. Let $G \leq \operatorname{Aut}(T_X)$ be a subgroup and let v be a vertex of T_X. The subtree of T_X whose vertex set is given by vX^* will be denoted by $(T_X)_v$. We write $\operatorname{St}_G(v)$ for the stabilizer of v in G. The *rigid stabilizer* of v in G, denoted by $\operatorname{RiSt}_G(v)$, is the subgroup of $\operatorname{St}_G(v)$ that consists of elements $g \in G$ that fix every vertex outside of $(T_X)_v$. For $\ell \in \mathbb{N}_0$, we further define the *level ℓ stabilizer subgroup*

$$\operatorname{St}_G(\ell) := \bigcap_{v \in X^\ell} \operatorname{St}_G(v)$$

and the *rigid level ℓ stabilizer subgroup*

$$\operatorname{RiSt}_G(\ell) := \Big\langle \bigcup_{v \in X^\ell} \operatorname{RiSt}_G(v) \Big\rangle$$

in G.

Let G be a group that acts on a rooted tree T. We call G a *branch group* if the index of $\mathrm{RiSt}_G(\ell)$ in G is finite for every $\ell \in \mathbb{N}$. For a subgroup $G \leq \mathrm{Aut}(T_X)$, we say that G is *self-similar* if for each $g \in G$, the elements g_x in the wreath decomposition

$$g = \sigma_g \circ (g_x)_{x \in X}$$

are contained in G.

Notation 3.3. Given a subgroup $G \leq \mathrm{Aut}(T_X)$ and a word $v \in X^*$ of length ℓ, we consider the embedding $\iota_v \colon G \to \mathrm{Aut}(T_X)$ given by

$$\iota_v(g)(uw) = \begin{cases} ug(w) & \text{if } u = v, \\ uw & \text{if } u \neq v \end{cases}$$

for every $g \in G$, $w \in X^*$ and $u \in X^\ell$.

3.2. The Ω-construction

Let us fix a non-empty finite set X and an element $o \in X$. Let $X^+ := X \setminus \{o\}$ and let $\mathcal{S}$ denote the space of infinite sequences $(\omega_n)_{n \in \mathbb{N}}$ over X^+. We consider the *left shift operator* $L \colon \mathcal{S} \to \mathcal{S}$ given by $(\omega_1, \omega_2, \omega_3, \ldots) \mapsto (\omega_2, \omega_3, \ldots)$.

Definition 3.4. Given a sequence $\omega = (\omega_n) \in \mathcal{S}$, we define the homomorphism

$$\widetilde{\cdot}^{\omega} \colon \mathrm{Aut}(T_X) \to \mathrm{Aut}(T_X), \quad \alpha \mapsto \widetilde{\alpha}^{\omega} = (\alpha_x)_{x \in X},$$

where

$$\alpha_x = \begin{cases} \widetilde{\alpha}^{L(\omega)} & \text{if } x = o, \\ \alpha & \text{if } x = \omega_1, \\ \mathrm{id} & \text{otherwise.} \end{cases}$$

If G is a subgroup of $\mathrm{Aut}(T_X)$, we write $\widetilde{G}^{\omega}$ to denote the image of G under ω. The group generated by G and $\widetilde{G}^{\omega}$ will be denoted by Γ_G^{ω}. More generally, for every non-empty subset $\Omega \subseteq \mathcal{S}$, we define Γ_G^{Ω} as the subgroup of $\mathrm{Aut}(T_X)$ that is generated by all groups Γ_G^{ω} with $\omega \in \Omega$.

Let G be a subgroup of $\mathrm{Aut}(T_X)$. Adapting a notion introduced by Segal [22], we say that G has *property H* if for all $x, y \in X$ the following hold:

- G acts transitively on the first level of T_X, i.e., on X;
- for all $x \neq y$ in X there exists $g \in \mathrm{St}_G(x)$ with $g(y) \neq y$.

Lemma 3.5. *Let $G \leq \mathrm{Aut}(T_X)$ be a perfect, self-similar subgroup that satisfies property H. For every $\omega = (\omega_n)_{n \in \mathbb{N}} \in \mathcal{S}$, we have $\iota_{\omega_1}(G) \subseteq \mathrm{RiSt}_{\Gamma_G^{\omega}}(\omega_1)$.*

Proof. Since G satisfies property H, we can find some $h \in G$ with $h(\omega_1) = \omega_1$ and $h(o) \neq o$. Let $h = \sigma_h \cdot (h_x)$ be the wreath decomposition of h. Consider an arbitrary

$g \in G$ and its image $\widetilde{g}^{\omega}$ in $\widetilde{G}^{\omega}$. Recall that $\widetilde{g}^{\omega} = (g_x)$, where $g_o = \widetilde{g}^{L(\omega)}$, $g_{\omega_1} = g$ and $g_x = \mathrm{id}$ otherwise. By conjugating $\widetilde{g}^{\omega}$ with h, we obtain

$$\begin{aligned} h\widetilde{g}^{\omega}h^{-1} &= \sigma_h \cdot (h_x) \circ (g_x) \circ (h_x^{-1}) \cdot \sigma_h^{-1} = \sigma_h \cdot (h_x g_x h_x^{-1}) \cdot \sigma_h^{-1} \\ &= (h_{\sigma_h(x)} g_{\sigma_h(x)} h_{\sigma_h(x)}^{-1}). \end{aligned}$$

From the self-similarity of G, we see that

$$h_{\sigma_h(\omega_1)} g_{\sigma_h(\omega_1)} h_{\sigma_h(\omega_1)}^{-1} = h_{\omega_1} g_{\omega_1} h_{\omega_1}^{-1} = h_{\omega_1} g h_{\omega_1}^{-1}$$

is an element of G. Further, we have $h_{\sigma_h(x)} g_{\sigma_h(x)} h_{\sigma_h(x)}^{-1} = \mathrm{id}$ for $\sigma_h(x) \in X \setminus \{o, \omega_1\}$. In particular, it follows that $h_{\sigma_h(o)} g_{\sigma_h(o)} h_{\sigma_h(o)}^{-1} = \mathrm{id}$. Let $k \in G$ be a further element. Note that the commutator

$$[\widetilde{k}^{\omega}, h\widetilde{g}^{\omega}h^{-1}] = ([k_x, h_{\sigma_h(x)} g_{\sigma_h(x)} h_{\sigma_h(x)}^{-1}])$$

and that

$$[k_x, h_{\sigma_h(x)} g_{\sigma_h(x)} h_{\sigma_h(x)}^{-1}] = \mathrm{id} \quad \text{for } x \neq \omega_1.$$

On the other hand, we have $[k_{\omega_1}, h_{\sigma_h(\omega_1)} g_{\sigma_h(\omega_1)} h_{\sigma_h(\omega_1)}^{-1}] = [k, h_{\omega_1} g h_{\omega_1}^{-1}]$. Thus we see that every element of the form $\iota_{\omega_1}([k, h_{\omega_1} g h_{\omega_1}^{-1}])$ lies in $\mathrm{RiSt}_{\Gamma_G^{\omega}}(\omega_1)$. Since G is perfect and $g, k \in G$ were chosen arbitrarily, it follows that $\iota_{\omega_1}(G)$ is contained in $\mathrm{RiSt}_{\Gamma_G^{\omega}}(\omega_1)$. ∎

For every non-empty subset $\Omega \subseteq \mathcal{S}$, we consider its image

$$L(\Omega) = \{L(\omega) \mid \omega \in \Omega\} \subseteq \mathcal{S}$$

under the shift operator.

Lemma 3.6. *Let $G \leq \mathrm{Aut}(T_X)$ be a perfect, self-similar subgroup that satisfies property H. For every non-empty $\Omega \subseteq \mathcal{S}$ and every $x \in X$, we have $\mathrm{RiSt}_{\Gamma_G^{\Omega}}(x) = \iota_x(\Gamma_G^{L(\Omega)})$.*

Proof. Let $\omega = (\omega_n) \in \Omega$. By Lemma 3.5, we have $\iota_{\omega_1}(G) \subseteq \mathrm{RiSt}_{\Gamma_G^{\omega}}(\omega_1)$. Since G is self-similar and level-transitive, this implies

$$\iota_o(G) \subseteq \mathrm{RiSt}_{\Gamma_G^{\omega}}(o) \subseteq \mathrm{RiSt}_{\Gamma_G^{\Omega}}(o). \tag{3.1}$$

We observe that $\widetilde{g}^{\omega}\iota_{\omega_1}(g)^{-1} = \iota_o(\widetilde{g}^{L(\omega)})$, and hence Lemma 3.5 implies further that

$$\iota_o(\widetilde{G}^{L(\omega)}) \subseteq \mathrm{RiSt}_{\Gamma_G^{\Omega}}(o). \tag{3.2}$$

As $\omega \in \Omega$ is arbitrary, (3.1) together with (3.2) show that

$$\iota_o(\Gamma_G^{L(\Omega)}) \subseteq \mathrm{RiSt}_{\Gamma_G^{\Omega}}(o) \subseteq \mathrm{St}_{\Gamma_G^{\Omega}}(o).$$

A further application of the level-transitivity and the self-similarity of G now gives us $\iota_x(\Gamma_G^{L(\Omega)}) \subseteq \mathrm{RiSt}_{\Gamma_G^{\Omega}}(x)$ for every $x \in X$. On the other hand, each Γ_G^{ω} is generated by elements of the form $g = \sigma_g \cdot (g_x)$ with either $g_x \in \widetilde{G}^{L(\omega)}$ or $g_x \in G$. From this we see that the reverse inclusion $\mathrm{RiSt}_{\Gamma_G^{\Omega}}(x) \subseteq \iota_x(\Gamma_G^{L(\Omega)})$ is also satisfied. ∎

Corollary 3.7. *Let $G \leq \mathrm{Aut}(T_X)$ be a perfect, self-similar subgroup that satisfies property H. For every non-empty $\Omega \subseteq \mathcal{S}$ and every word $v \in X^*$ of length ℓ, the restricted stabilizer of v in Γ_G^Ω is given by $\mathrm{RiSt}_{\Gamma_G^\Omega}(v) = \iota_v(\Gamma_G^{L^\ell(\Omega)})$. Moreover, we have $\mathrm{RiSt}_{\Gamma_G^\Omega}(\ell) = \mathrm{St}_{\Gamma_G^\Omega}(\ell)$ for every $\ell \in \mathbb{N}_0$. In particular, Γ_G^Ω is a branch group and the action is level-transitive.*

Proof. The proof is by induction on the length of v. If v is the empty word, then there is nothing to show. Suppose now that the corollary holds for some $\ell \in \mathbb{N}_0$. Let $w \in X^{\ell+1}$ be a word of the form $w = vx$ with $v \in X^\ell$ and $x \in X$. From Lemma 3.6, we know that $\mathrm{RiSt}_{\Gamma_G^{L^\ell(\Omega)}}(x) = \iota_x(\Gamma_G^{L^{\ell+1}(\Omega)})$ for every $x \in X$. We obtain

$$\begin{aligned}\mathrm{RiSt}_{\Gamma_G^\Omega}(w) &= \mathrm{RiSt}_{\mathrm{RiSt}_{\Gamma_G^\Omega}(v)}(vx) = \mathrm{RiSt}_{\iota_v(\Gamma_G^{L^\ell(\Omega)})}(vx)\\ &= \iota_v(\mathrm{RiSt}_{\Gamma_G^{L^\ell(\Omega)}}(x)) = \iota_v(\iota_x(\Gamma_G^{L^{\ell+1}(\Omega)})) = \iota_w(\Gamma_G^{L^{\ell+1}(\Omega)}).\end{aligned}$$

Since Γ_G^Ω is generated by elements of the form $g = \sigma_g \cdot (g_x)$ with either $g_x \in G$ or $g_x \in \widetilde{G}^{L(\omega)}$ for some $\omega \in \Omega$, it follows that $\mathrm{St}_{\Gamma_G^\omega}(\ell)$ is contained in the group generated by all subgroups of the form $\iota_v(\Gamma_G^{L^\ell(\Omega)})$ with v of level ℓ. Together with the first part this implies $\mathrm{St}_{\Gamma_G^\Omega}(\ell) = \mathrm{RiSt}_{\Gamma_G^\Omega}(\ell)$ and since $\mathrm{St}_{\Gamma_G^\Omega}(\ell)$ is of finite index, we conclude that Γ_G^Ω is a branch group. By property H, the group G acts transitively on the first level. Since $\mathrm{RiSt}_{\Gamma_G^\Omega}(v)$ contains $\iota_v(G)$, it follows by induction that Γ_G^Ω acts transitively on every level. ∎

To finish this section, we show that the groups Γ_G^Ω act like iterated wreath products on each level. Recall that for groups G, H with actions on sets X and Y, the *permutational wreath product* $G \wr_X H$ is defined as the semidirect product $G \ltimes H^X$, where G acts on H^X by permuting the coordinates. We define the *natural action* of $G \wr_X H$ on the product set $X \times Y$ by $(g \cdot (h_x), (x, y)) \mapsto (g(x), h_x(y))$.

Given a finite set X and a subgroup $Q \leq \mathrm{Sym}(X)$, we consider the iterated permutational wreath product of Q given by

$$\wr_X^n Q = Q \wr_X (Q \wr_X (\cdots (Q \wr_X Q)) \cdots).$$

Note that the natural action of an element $\alpha \in \wr_X^n Q$ on X^n extends to a tree automorphism on T_X by setting $\alpha(vw) = \alpha(v)w$ for all $v \in X^n$ and $w \in X^*$. In the following, we will identify $\wr_X^n Q$ with its image in $\mathrm{Aut}(T_X)$ under this action.

Proposition 3.8. *Let $G \leq \mathrm{Aut}(T_X)$ be a perfect, self-similar subgroup that satisfies property H. Let $Q \leq \mathrm{Sym}(X)$ denote the image of G under the canonical action on X. Then for every non-empty subset $\Omega \subseteq \mathcal{S}$ and every $\ell \in \mathbb{N}$, the image of Γ_G^Ω in $\mathrm{Aut}(T_X)/\mathrm{St}_{\mathrm{Aut}(T_X)}(\ell)$ is given by the permutational wreath product $\wr_X^\ell Q$.*

Proof. By construction, every $g \in \Gamma_G^\Omega$ has a wreath decomposition

$$g = \sigma_g \circ (g_x), \tag{3.3}$$

where σ_g is a rooted automorphism that corresponds to an element in Q and $g_x \in \Gamma_G^{L(\Omega)}$ for every $x \in X$. Note that this implies that for every $g \in \Gamma_X^{\Omega}$ and every word $v = x_1 \dots x_\ell$ over X, its image under g is given by

$$g(v) = \sigma_1(x_1) \dots \sigma_\ell(x_\ell) \tag{3.4}$$

for some appropriate permutations $\sigma_i \in Q$, i.e., the image of Γ_G^{Ω} lies in $\wr_X^{\ell} Q$.

On the other hand, Lemma 3.6 tells us that $\mathrm{RiSt}_{\Gamma_G^{\Omega}}(x) = \iota_x(\Gamma_G^{L(\Omega)})$ for every $x \in X$. Together with (3.3), this shows that for every $\sigma \in Q$ its corresponding rooted automorphism, also denoted by σ, lies in $\Gamma_G^{L(\Omega)}$. From Corollary 3.7, it therefore follows that $\iota_v(\sigma) \in \mathrm{RiSt}_{\Gamma_G^{\Omega}}(v)$ for every $v \in X^*$ and every $\sigma \in Q$. In view of (3.4), this implies that the image of Γ_G^{Ω} in $\mathrm{Aut}(T_X)/\operatorname{St}_{\mathrm{Aut}(T_X)}(\ell)$ is given by $\wr_X^{\ell} Q$. ∎

4. The congruence subgroup property

Let T be a rooted tree. We make $\mathrm{Aut}(T)$ into a topological group by declaring the subgroups $\mathrm{St}_{\mathrm{Aut}(T)}(n)$ to be a base of open neighbourhoods of the identity. Equipped with this topology, the automorphism group $\mathrm{Aut}(T)$ is a compact, totally disconnected Hausdorff topological group, i.e., a profinite group.

Recall that the *profinite completion* of a residually finite group G is defined as the inverse limit

$$\widehat{G} := \varprojlim_{N \trianglelefteq_f G} G/N$$

of the system of all normal subgroups of finite index in G. If G is a subgroup of $\mathrm{Aut}(T)$, we can further consider its *tree completion* $\overline{G}$: the closure of G in $\mathrm{Aut}(T)$ with respect to the profinite topology. In particular, $\overline{G}$ is a profinite group and $\overline{G} \cong \varprojlim_n G/\operatorname{St}_G(n)$. In this case, the universal property of the profinite completion gives rise to a canonical homomorphism

$$\mathrm{res}_T^G \colon \widehat{G} \to \overline{G}.$$

The homomorphism res_T^G allows us to extend the action of G on T to an action of $\widehat{G}$ on T. Since G is dense in both $\widehat{G}$ and $\overline{G}$, the map res_T is always surjective. The goal of this section is to formulate sufficient conditions under which res_T is injective.

Definition 4.1. Let T be a rooted tree. A subgroup $G \leq \mathrm{Aut}(T)$ satisfies the *congruence subgroup property* (CSP) if $\mathrm{res}_T^G \colon \widehat{G} \to \overline{G}$ is an isomorphism.

Remark 4.2. From the definition, it directly follows that a subgroup $G \leq \mathrm{Aut}(T)$ satisfies the congruence subgroup property if and only if for every normal subgroup $N \trianglelefteq G$ of finite index there is a number $n \in \mathbb{N}$ such that $\operatorname{St}_G(n)$ is contained in N.

The following very useful observation was extracted by Segal [22, Lemma 4] from the proof of [8, Theorem 4].

Lemma 4.3. *Let T be a rooted tree and let $G \leq \mathrm{Aut}(T)$ be a subgroup that acts level-transitively on T. Then for every non-trivial normal subgroup $N \trianglelefteq G$, there is some $n \in \mathbb{N}$ with $\mathrm{RiSt}_G(n)' \leq N$, where $\mathrm{RiSt}_G(n)'$ denotes the commutator subgroup of $\mathrm{RiSt}_G(n)$.*

Recall that an infinite group G is called *just infinite* if every proper quotient of G is finite.

Corollary 4.4. *Let T be a rooted tree and let $G \leq \mathrm{Aut}(T)$ be a subgroup that acts level-transitively on T. Suppose that every rigid stabilizer $\mathrm{RiSt}_G(v)$ is perfect and that the groups $\mathrm{St}_G(n)$ and $\mathrm{RiSt}_G(n)$ coincide for every $n \in \mathbb{N}$. Then G is just infinite and satisfies the* CSP.

Proof. Let N be a non-trivial normal subgroup of G. From Lemma 4.3, we know that there is some n with $\mathrm{RiSt}_G(n)' \leq N$. Since the rigid stabilizers are perfect, it follows that

$$\mathrm{RiSt}_G(v) = \mathrm{RiSt}_G(v)' \leq \mathrm{RiSt}_G(n)'$$

for every vertex v of level n in T. On the other hand, $\mathrm{St}_G(n) = \mathrm{RiSt}_G(n)$ is generated by the level n rigid vertex stabilizers $\mathrm{RiSt}_G(v)$. Thus we obtain $\mathrm{St}_G(n) = \mathrm{RiSt}_G(n)' \leq N$, which proves the claim. ∎

This result can be applied to the groups Γ_G^Ω defined in the previous section.

Theorem 4.5. *Let $G \leq \mathrm{Aut}(T_X)$ be a perfect, self-similar subgroup that satisfies property H. Then for every non-empty subset $\Omega \subseteq \mathcal{S}$, the group Γ_G^Ω is just infinite and satisfies the congruence subgroup property.*

Proof. From Corollary 3.7, we know that the groups $\mathrm{RiSt}_{\Gamma_G^\Omega}(\ell)$ and $\mathrm{St}_{\Gamma_G^\Omega}(\ell)$ coincide for every $\ell \in \mathbb{N}$. As each rigid stabilizer $\mathrm{RiSt}_{\Gamma_G^\Omega}(v)$ is generated by isomorphic copies of the perfect group G, it follows that $\mathrm{RiSt}_{\Gamma_G^\Omega}(v)$ is perfect itself. Now the claim follows from Corollary 4.4. ∎

As a consequence of Theorem 4.5, we see that the action of $\widehat{\Gamma_G^\Omega}$ on T_X is a faithful extension of the action of Γ_G^Ω on T_X and that $\widehat{\Gamma_G^\Omega}$ is isomorphic to $\overline{\Gamma_G^\Omega} \leq \mathrm{Aut}(T_X)$. In the following, it will be important for us to observe that under the assumptions of Theorem 4.5 the tree completion $\overline{\Gamma_G^\Omega}$ does not depend on Ω.

In fact, the tree completion is always an *iterated wreath product*. Let X be a finite set and let $Q \leq \mathrm{Sym}(X)$. Consider the inverse limit $\wr_X^\infty Q := \varprojlim_n \wr_X^n Q$ of the iterated wreath products, where the projection $\wr_X^{n+1} Q \to \wr_X^n Q$ is given by restricting the natural action of $\wr_X^n Q$ on X^{n+1} to the first n coordinates. Then the iterated wreath product $\wr_X^\infty Q$ acts on T_X, and we identify $\wr_X^\infty Q$ with its image in $\mathrm{Aut}(T_X)$ under this action. We note that this is a closed subgroup of $\mathrm{Aut}(T_X)$.

Since a closed subgroup of $\mathrm{Aut}(T_X)$ is uniquely determined by its actions on all finite levels of the tree, the following result is a direct consequence of Theorem 4.5 and Proposition 3.8.

Corollary 4.6. *Let $G \leq \mathrm{Aut}(T_X)$ be a perfect, self-similar subgroup that satisfies H. Let $Q \leq \mathrm{Sym}(X)$ denote the image of G under the canonical action on X. For every non-empty subset $\Omega \subseteq \mathcal{S}$, the canonical map $\mathrm{res}_{T_X}^{\Gamma_G^\Omega}$ defines an isomorphism from $\widehat{\Gamma_G^\Omega}$ onto $\wr_X^\infty Q \leq \mathrm{Aut}(T_X)$.*

Corollary 4.7. *Let $G, H \leq \mathrm{Aut}(T_X)$ be perfect, self-similar subgroups that satisfy H. If the images of G and H in $\mathrm{Sym}(X)$ coincide, then the profinite completions $\widehat{\Gamma_G^\Omega}$ and $\widehat{\Gamma_H^{\Omega'}}$ are isomorphic for all non-empty subsets $\Omega, \Omega' \subseteq \mathcal{S}$. If moreover G is a subgroup of H and $\Omega \subseteq \Omega'$, then Γ_G^Ω is a subgroup of $\Gamma_H^{\Omega'}$ and the inclusion map j induces an isomorphism*

$$\hat{j}\colon \widehat{\Gamma_G^\Omega} \to \widehat{\Gamma_H^{\Omega'}}.$$

Proof. The first assertion follows immediately from Corollary 4.6. Assume that $G \leq H$ and $\Omega \subseteq \Omega'$. By definition $\Gamma_G^\Omega \subseteq \Gamma_H^{\Omega'}$. We observe that the following diagram commutes:

$$\begin{array}{ccccc}
\Gamma_G^\Omega & \xrightarrow{i} & \Gamma_H^{\Omega'} & \longrightarrow & \mathrm{Aut}(T_X) \\
\downarrow & & \downarrow & & \| \\
\widehat{\Gamma_G^\Omega} & \xrightarrow{\hat{i}} & \widehat{\Gamma_H^{\Omega'}} & \xrightarrow{\mathrm{res}_{T_X}^{\Gamma_H^{\Omega'}}} & \mathrm{Aut}(T_X),
\end{array}$$

and we deduce that $\mathrm{res}_{T_X}^{\Gamma_G^\Omega} = \mathrm{res}_{T_X}^{\Gamma_H^{\Omega'}} \circ \hat{i}$. Now it follows from Corollary 4.6 that $\hat{i}$ is an isomorphism. ∎

5. Uncountably many groups up to isomorphism

The aim of this section is to prove that – under mild assumptions on G – the family of groups Γ_G^Ω where Ω runs through the non-empty subsets $\Omega \subseteq \mathcal{S}$ contains uncountably many isomorphism types of groups.

Let G be a group that acts via two homomorphisms $\varphi_1, \varphi_2\colon G \to \mathrm{Aut}(T)$ on a rooted tree T. We say that the actions are *conjugated* if there is an automorphism $\gamma \in \mathrm{Aut}(T)$ such that $\varphi_2(g) = \gamma\varphi_1(g)\gamma^{-1}$ for every $g \in G$.

Definition 5.1. Let T be a tree. We say that a subgroup $G \leq \mathrm{Aut}(T)$ is *rigid* if every automorphism α of G is induced by a conjugation of T. More precisely, this means that there is some $\gamma \in \mathrm{Aut}(T)$ with $\alpha(g) = \gamma g \gamma^{-1}$ for every $g \in G$.

The following result is a special case of [16, Proposition 8.1].

Proposition 5.2. *Let T be a rooted tree and let $G \leq \mathrm{Aut}(T)$ be a branch group. Suppose that for every vertex v the rigid stabilizer $\mathrm{RiSt}_G(v)$ acts level-transitively on the subtree T_v. Then G is rigid in $\mathrm{Aut}(T)$.*

Recall that we write $\widehat{G}$ to denote the profinite completion of a residually finite group G.

Lemma 5.3. *Let T be a rooted tree and let $G_1, G_2 \leq \mathrm{Aut}(T)$ be two branch groups whose restricted stabilizers $\mathrm{RiSt}_{G_i}(v)$ act level-transitively on T_v for every vertex v. Suppose that G_1 and G_2 satisfy the congruence subgroup property and that $\overline{G_1} = \overline{G_2} \subseteq \mathrm{Aut}(T)$. Then every isomorphism between G_1 and G_2 is induced by a conjugation in* $\mathrm{Aut}(T)$.

Proof. Define $\bar{G} := \overline{G_1} = \overline{G_2}$. Suppose that $f\colon G_1 \to G_2$ is an isomorphism and let $\hat{f}\colon \widehat{G_1} \to \widehat{G_2}$ be the corresponding isomorphism on the profinite completions. By the congruence subgroup property, the restrictions $\mathrm{res}_T^{G_i}\colon \widehat{G_i} \to \overline{G_i}$ are isomorphisms between the profinite completions and the tree completions. The homomorphism $f_0 := \mathrm{res}_T^{G_2} \circ \hat{f} \circ (\mathrm{res}_T^{G_1})^{-1}$ is thus an automorphism of $\bar{G}$, i.e., the following diagram commutes:

$$\begin{array}{ccc} \widehat{G_1} & \xrightarrow{\hat{f}} & \widehat{G_2} \\ {\scriptstyle \mathrm{res}_T^{G_1}}\downarrow & & \downarrow{\scriptstyle \mathrm{res}_T^{G_2}} \\ \bar{G} & \xrightarrow{f_0} & \bar{G}. \end{array}$$

Since the rigid stabilizers of $\bar{G}$ contain those of G_1 (and G_2), we can therefore apply Proposition 5.2 to deduce that there is some $\gamma \in \mathrm{Aut}(T)$ with $f_0(g) = \gamma g \gamma^{-1}$ for all $g \in \bar{G}$. For every $g \in G_1 \subseteq \bar{G}$, we therefore obtain $f(g) = f_0(g) = \gamma g \gamma^{-1}$. ∎

Definition 5.4. Let X be a finite alphabet and let T_X be the corresponding $|X|$-regular rooted tree with vertex set X^*. Given a tree automorphism $g \in \mathrm{Aut}(T_X)$ and a number $\ell \in \mathbb{N}$, we consider the subset $\mathrm{Fix}_\ell(g) \subseteq X^n$ of vertices of level ℓ that are fixed by g. The *support volume* of g is defined as

$$\mathrm{vol}(g) := \lim_{\ell \to \infty} \frac{|X^\ell \setminus \mathrm{Fix}_\ell(g)|}{|X^\ell|}.$$

Given a tree automorphism $g \in \mathrm{Aut}(T)$ and a vertex v with $g(v) \neq v$, it follows that no descendant of v is fixed by g. Thus $\frac{|X^\ell \setminus \mathrm{Fix}_\ell(g)|}{|X^\ell|}$ is a non-decreasing sequence of numbers that are bounded above by 1. In particular, this tells us that the limit $\mathrm{vol}(g) = \lim_{\ell \to \infty} \frac{|X^\ell \setminus \mathrm{Fix}_\ell(g)|}{|X^\ell|}$ indeed exists. In fact, the support volume measures the set of elements in the boundary of T_X which are moved by g. The support volume is invariant under conjugation. Let $\alpha \in \mathrm{Aut}(T)$ be an automorphism. Then $\mathrm{Fix}_\ell(\alpha g \alpha^{-1}) = \alpha(\mathrm{Fix}_\ell(g))$ and hence $\mathrm{vol}(g) = \mathrm{vol}(\alpha g \alpha^{-1})$.

We return to the construction introduced in Section 3.2. Let X be a non-empty finite set with an element $o \in X$ and define $X^+ = X \setminus \{o\}$. Recall that $\mathcal{S} := (X^+)^\infty$.

Theorem 5.5. *Let X be a finite set with $|X| \geq 3$. Let $G \leq \mathrm{Aut}(T_X)$ be a non-trivial subgroup. For every $\omega \in \mathcal{S}$, the set of real numbers*

$$\{\mathrm{vol}(g) \mid g \in \Gamma_G^{\{\omega,\omega'\}},\ \omega' \in \mathcal{S}\} \subseteq [0,1]$$

is uncountable.

Proof. Let $t \in G$ be an element, which acts non-trivially on T_X; in particular, $\mathrm{vol}(t) > 0$. Since $|X| \geq 3$, we can pick an element $z_n \in X \setminus \{o, \omega_n\}$ for every $n \in \mathbb{N}$. Let $S \subseteq \mathbb{N}$ be a set of natural numbers. We define $\omega' = \omega'(S)$ by

$$\omega'_n = \begin{cases} \omega_n & \text{if } n \notin S, \\ z_n & \text{if } n \in S. \end{cases}$$

Consider the element $g = (\tilde{t}^{\omega})^{-1}\tilde{t}^{\omega'} \in \Gamma_G^{\{\omega,\omega'\}}$. Then g acts like t^{-1} on $(T_X)_{o^{n-1}\omega_n}$ and like t on $(T_X)_{o^{n-1}z_n}$ for every $n \in S$ and acts trivially on all vertices not contained in one of these subtrees. We obtain

$$\mathrm{vol}(g) = \sum_{n \in S} \frac{2\,\mathrm{vol}(t)}{|X|^n} = 2\,\mathrm{vol}(t) \sum_{n \in S} |X|^{-n}$$

and observe that this number uniquely determines the set S. Indeed, since $|X| \geq 3$ the first non-zero term dominates the series. This completes the proof of the theorem, using that there are uncountably many subsets $S \subseteq \mathbb{N}$. ∎

Corollary 5.6. *Let $G \leq \mathrm{Aut}(T_X)$ be a countable, perfect, self-similar subgroup that satisfies property H. For every $\omega \in \mathcal{S}$, the family of groups $(\Gamma_G^{\{\omega,\omega'\}})_{\omega' \in \mathcal{S}}$ contains uncountably many distinct isomorphism types.*

Proof. Recall that by Corollary 3.7, the groups Γ_G^{Ω} are branch groups and the rigid stabilizers act level-transitively. By Theorem 4.5, these groups have the congruence subgroup property and by Corollary 4.6, the closure of Γ_G^{Ω} in $\mathrm{Aut}(T_X)$ does not depend on Ω. We conclude using Lemma 5.3 that every isomorphism between two of the groups Γ_G^{Ω} is induced by a conjugation in $\mathrm{Aut}(T_X)$. In particular, this means that isomorphisms between these groups preserve the support volume of elements.

We note that G is perfect and acts transitively on X, hence we must have $|X| \geq 5$. Theorem 5.5 therefore shows that the set of support volumes of elements in the groups $\Gamma_G^{\{\omega,\omega'\}}$ is uncountable. However, G is countable and so the groups $\Gamma_G^{\{\omega,\omega'\}}$ are countably generated and thus countable. In conclusion, each isomorphism type contributes at most countably many numbers to the uncountable set of support volumes and consequently uncountably many isomorphisms types have to occur. ∎

In the next section, we will discuss a concrete example of a group G where a similar argument can be used to show that the number of isomorphism types in the family $(\Gamma_G^{\omega})_{\omega \in \mathcal{S}}$ is uncountable.

Remark 5.7. We briefly return to Grothendieck's question. If $G \leq \mathrm{Aut}(T)$ is a finitely generated group which satisfies the assumptions of Corollary 5.6, then the groups $(\Gamma_G^{\Omega})_{\Omega \subseteq \mathcal{S}}$ where Ω runs in the finite subsets of $\mathcal{S}$ form an uncountable directed system of finitely generated residually finite groups in which every inclusion induces an isomorphism between profinite completions (see Corollary 4.7).

6. Matrix groups acting on trees

Given a prime number p and a natural number n, we consider the set $\mathcal{A}_{p,n} := \{0, \dots, p-1\}^n$ which takes the role of the alphabet (called X in the previous sections). Let $\mathcal{A}_{p,n}^*$ denote the free monoid generated by $\mathcal{A}_{p,n}$, i.e., the set of (finite) words over $\mathcal{A}_{p,n}$. Let $T_{p,n}$ denote the Cayley graph of $\mathcal{A}_{p,n}^*$ with respect to $\mathcal{A}_{p,n}$. Clearly, $T_{p,n}$ is a tree whose boundary $\partial T_{p,n}$ can be identified with the set $\mathcal{A}_{p,n}^\infty$ of infinite sequences over $\mathcal{A}_{p,n}$. The element $0 = (0, 0, \dots, 0) \in \mathcal{A}_{p,n}$ is the distinguished element, and we write $\mathcal{S}_{p,n}$ to denote the space of infinite sequences over $\mathcal{A}_{p,n} \setminus \{0\}$.

Definition 6.1. Given a commutative, unital ring R and a natural number $n \in \mathbb{N}$, we write $\mathrm{SAff}_n(R)$ to denote the group of affine transformations of R^n whose linear part lies in $\mathrm{SL}_n(R)$. We note that $\mathrm{SAff}_n(R) \cong R^n \rtimes \mathrm{SL}_n(R)$.

It is a well-known fact that $\mathrm{SL}_n(\mathbb{Z})$ and $\mathrm{SL}_n(\mathbb{F}_p)$ are perfect for $n \geq 3$ (see, for example, [10, §1.2.15] and [24, p. 46]). In the following, we need an affine version of this observation.

Lemma 6.2. *The groups* $\mathrm{SAff}_n(\mathbb{Z})$ *and* $\mathrm{SAff}_n(\mathbb{F}_p)$ *are perfect for* $n \geq 3$.

Proof. For every $v \in \mathbb{Z}^n$ let T_v, denote the translation by v. Since $\mathrm{SL}_n(\mathbb{Z})$ is perfect (for $n \geq 3$), it suffices to show that every translation T_{e_i} by a standard unit vector $e_i \in \mathbb{Z}^n$ can be written as a commutator in $\mathrm{SAff}_n(\mathbb{Z})$. To see this, we consider the elementary matrices $E_{i,j} \in \mathrm{SL}_n(\mathbb{Z})$ for $1 \leq i < j \leq n$ which are defined by $E_{i,j} \cdot e_j = e_i + e_j$ and $E_{i,j} \cdot e_k = e_k$ for $k \neq j$. Then

$$[T_{-e_j}, E_{i,j}] = T_{-e_j} E_{i,j} T_{-e_j}^{-1} E_{i,j}^{-1} = T_{-e_j} + T_{E_{i,j} \cdot e_j} = T_{-e_j} + T_{e_i + e_j} = T_{e_i},$$

and the result for $\mathrm{SAff}(\mathbb{Z})$ follows. The same argument applies to $\mathrm{SAff}(\mathbb{F}_p)$. ∎

The set $\mathcal{A}_{p,n}^\infty$ can be identified with $\mathbb{Z}_p^n$ via $(x_n) \mapsto \sum_{n=0}^\infty p^i x_i$ and similarly the ℓ-th level of the tree can be identified with $(\mathbb{Z}/p^\ell\mathbb{Z})^n$. In view of this, the natural action of $\mathrm{SAff}_n(\mathbb{Z}_p)$ on $\mathbb{Z}_p^n$ induces an action on $T_{p,n}$. In fact, the action on the ℓ-th level factors through $\mathrm{SAff}_n(\mathbb{Z}/p^\ell\mathbb{Z})$.

Lemma 6.3. *The subgroup* $\mathrm{SAff}_n(\mathbb{Z}) \leq \mathrm{Aut}(T_{p,n})$ *is self-similar and satisfies property H.*

Proof. Let $A \in \mathrm{SL}_n(\mathbb{Z})$, let $b \in \mathbb{Z}^n$, and let $g \in \mathrm{SAff}_n(\mathbb{Z})$ be the element defined by $g(v) = Av + b$. Let $u \in \mathbb{Z}_p$ be an element of the form $u = x + pw$ with $x \in \mathcal{A}_{p,n}$ and $w \in \mathbb{Z}_p$. Let further $x' \in \mathcal{A}_{p,n}$ and $b' \in \mathbb{Z}_p$ be such that $Ax + b = x' + pb'$. Then we have

$$g(u) = A(x + pw) + b = Ax + b + pAw = x' + p(Aw + b'),$$

which tells us that g_x is given by $g_x(w) = Aw + b'$. As $Ax + b = x' + pb'$ implies $b' \in \mathbb{Z}$ and $g_x \in \mathrm{SAff}_n(\mathbb{Z})$, we deduce that $\mathrm{SAff}(\mathbb{Z})$ is self-similar.

The action of $\mathrm{SAff}(\mathbb{Z})$ on the first level $A_{p,n}$ factors through $\mathrm{SAff}(\mathbb{F}_p)$. In fact, this is the natural action of $\mathrm{SAff}(\mathbb{F}_p)$ on $\mathbb{F}_p^n$. Since this action is 2-transitive, it clearly satisfies property H. ■

Definition 6.4. Let $\omega = (\omega_n)_{n\in\mathbb{N}} \in \mathcal{S}_{p,n}$. For $g \in \mathrm{SAff}_n(\mathbb{Z})$, we define the map $\tilde{g}^\omega : \mathbb{Z}_p^n \to \mathbb{Z}_p^n$ by

$$\tilde{g}^\omega(u) = \begin{cases} p^{\ell-1}\omega_\ell + p^\ell g(v) & \text{if } u = p^{\ell-1}\omega_\ell + p^\ell v \text{ for some } \ell \in \mathbb{N} \text{ and } v \in \mathbb{Z}_p^n, \\ u & \text{if } u \not\equiv p^{\ell-1}\omega_\ell \bmod p^\ell \text{ for every } \ell \in \mathbb{N}. \end{cases}$$

Further, we define $\widetilde{\mathrm{SAff}_n(\mathbb{Z})}^\omega := \{\tilde{g}^\omega \mid g \in \mathrm{SAff}_n(\mathbb{Z})\}$.

From this definition, one can easily see that $\widetilde{\mathrm{SAff}_n(\mathbb{Z})}^\omega$ is a group and that

$$\tilde{\cdot}^\omega : \mathrm{SAff}_n(\mathbb{Z}) \to \widetilde{\mathrm{SAff}_n(\mathbb{Z})}^\omega, \quad g \mapsto \tilde{g}^\omega$$

is a group isomorphism. The elements $\tilde{g}^\omega$ can also be defined recursively with the *left shift operator*

$$L : \mathcal{A}_{p,n}^\infty \to \mathcal{A}_{p,n}^\infty, \quad (x_1, x_2, x_3, \ldots) \mapsto (x_2, x_3, x_4, \ldots).$$

Indeed, given a sequence $\omega = (\omega_n)_{n\in\mathbb{N}} \in \mathcal{S}_{p,n}$ and an element $g \in \mathrm{SAff}_n(\mathbb{Z})$, then we can write

$$\tilde{g}^\omega = (g_x)_{x\in\mathcal{A}_{p,n}},$$

where

$$g_x = \begin{cases} \tilde{g}^{S(\omega)} & \text{if } x = 0, \\ g & \text{if } x = \omega_1, \\ \mathrm{id} & \text{otherwise.} \end{cases}$$

This is exactly the formula used in Definition 3.4. For every subset $\omega \in \mathcal{S}_{p,n}$, we define the subgroup $\Gamma_{p,n}^\omega \leq \mathrm{Aut}(T_{p,n})$ to be the group generated by $\mathrm{SAff}_n(\mathbb{Z})$ and $\widetilde{\mathrm{SAff}_n(\mathbb{Z})}^\omega$. Recall that for a set $\Omega \subseteq \mathcal{S}_{p,n}$, we define the subgroup $\Gamma_{p,n}^\Omega \leq \mathrm{Aut}(T_{p,n})$ to be generated by the groups $\Gamma_{p,n}^\omega$ with $\omega \in \Omega$. Lemmas 6.2 and 6.3 allow us to use the results developed in the foregoing sections. In particular, we obtain the following result.

Corollary 6.5. *Let $n \geq 3$ and let $\Omega, \Omega_1, \Omega_2 \subseteq \mathcal{S}_{p,n}$ be non-empty subsets.*

(1) *Then $\Gamma_{p,n}^\Omega$ is a level-transitive, just infinite branch group which contains a non-abelian free group and satisfies the congruence subgroup property. The profinite completion is isomorphic to the closure of $\Gamma_{p,n}^\Omega$ in $\mathrm{Aut}(T_{p,n})$ and does not depend on Ω.*

(2) *If $\Gamma_{p,n}^{\Omega_1}$ and $\Gamma_{p,n}^{\Omega_2}$ are isomorphic, then they are already conjugated in $\mathrm{Aut}(T_{p,n})$.*

(3) *For $\Omega_1 \subseteq \Omega_2$, the inclusion $\Gamma_{p,n}^{\Omega_1} \to \Gamma_{p,n}^{\Omega_2}$ induces an isomorphism between the profinite completions.*

Proof. It is well known that $\mathrm{SL}_n(\mathbb{Z})$ contains non-abelian free subgroups. By Corollary 3.7, the groups $\Gamma^{\Omega}_{p,n}$ are branch groups and the rigid stabilizers act level-transitively. By Theorem 4.5, these groups have the congruence subgroup property and by Corollary 4.6, the closure of $\Gamma^{\Omega}_{p,n}$ in $\mathrm{Aut}(T_{p,n})$ is isomorphic to the profinite completion and does not depend on Ω. Lemma 5.3 shows that every isomorphism between two of the groups is induced by a conjugation in $\mathrm{Aut}(T_{p,n})$. The third assertion follows from Corollary 4.7. ∎

It follows from Corollary 5.6 that the number of isomorphism types among the groups $\Gamma^{\{\omega,\omega'\}}_{p,n}$ is uncountable. A variation of the argument shows that we can also find uncountably many groups up to isomorphism in the family $(\Gamma^{\omega}_{p,n})_{\omega\in\mathcal{S}_{p,n}}$. In particular, most of these groups do not admit a finite presentation.

Proposition 6.6. *For every $n \geq 3$ and every prime p, there are uncountably many isomorphism classes of groups of the form $\Gamma^{\omega}_{p,n}$.*

Proof. If two of the groups $\Gamma^{\omega}_{p,n}$ are isomorphic, then they are conjugated in $\mathrm{Aut}(T_{p,n})$ (see Corollary 6.5) and since conjugation in $\mathrm{Aut}(T_{p,n})$ preserves support volumes, we deduce that for isomorphic groups the sets

$$\mathrm{vol}(\Gamma^{\omega}_{p,n}) = \{\mathrm{vol}(g) \mid g \in \Gamma^{\omega}_{p,n}\}$$

coincide. We note that the groups $\Gamma^{\omega}_{p,n}$ are finitely generated and thus $\mathrm{vol}(\Gamma^{\omega}_{p,n})$ is a countable set. In particular, it is sufficient to prove – following Theorem 5.5 – that the set $\bigcup_{\omega\in\mathcal{S}_{p,n}} \mathrm{vol}(\Gamma^{\omega}_{p,n})$ is uncountable.

Let $e_1, e_2, \dots, e_n$ denote the standard basis of $\mathbb{Z}^n$. Consider the elementary matrix $A = E_{1,2} \in \mathrm{SL}_n(\mathbb{Z})$ with $Ae_1 = e_1$ and $Ae_2 = e_1 + e_2$. Let $T = T_{e_1} \in \mathrm{SAff}_n(\mathbb{Z})$ be the translation with the first standard basis vector. For every subset $S \subseteq \mathbb{N}$, we define $\omega = \omega(S) \in \mathcal{S}_{p,n}$ such that

$$\omega_i = \begin{cases} e_1 & \text{if } i \notin S, \\ e_2 & \text{if } i \in S. \end{cases}$$

Let now $\omega' = A\omega$. Using the formula given in Definition 6.4, it is readily checked that $A\widetilde{T}^{\omega}A^{-1} = \widetilde{T}^{\omega'}$. We consider the commutator $g = [A, \widetilde{T}^{\omega}] \in \Gamma^{\omega}_{p,n}$ and observe that

$$g = [A, \widetilde{T}^{\omega}] = A\widetilde{T}^{\omega}A^{-1}(\widetilde{T}^{\omega})^{-1} = \widetilde{T}^{\omega'}(\widetilde{T}^{\omega})^{-1}.$$

In particular, g acts non-trivially exactly on the boundary points $x \in \mathbb{Z}_p^n$ congruent to $p^i e_2$ or $p^i(e_1 + e_2)$ with $i \in S$. We obtain

$$\mathrm{vol}(g) = \sum_{i\in S} \frac{2}{p^{ni}} = 2\sum_{i\in S} p^{-ni}$$

and observe that this number uniquely determines the set S. Since there are uncountably many subsets $S \subseteq \mathbb{N}$, this completes the proof. ∎

7. Amenable groups acting on trees

Let X be a finite set, let $o \in X$ and let $X^+ := X \setminus \{o\}$. Let $\mathcal{S}$ denote the set of infinite sequences over X^+. Our goal in this section is to introduce amenable groups that have the same profinite completions as $\Gamma^{\Omega}_{p,n}$ for $n \geq 3$. To this end, we introduce automatic automorphisms of T_X. Recall that for every vertex $v \in T_X$, we write $(T_X)_v$ to denote the subtree of T_X whose vertex set is given by vX^*. For $\alpha \in \mathrm{Aut}(T_X)$, we have $\alpha((T_X)_v) = (T_X)_{\alpha(v)}$. Thus we can define the *state of* α *at* v as the unique automorphism α_v of T_X that satisfies $\alpha(vw) = \alpha(v)\alpha_v(w)$ for every $w \in X^*$. The set of all states of α will be denoted by $S(\alpha) := \{\alpha_v \in \mathrm{Aut}(T_X) \mid v \in X^*\}$.

Definition 7.1. An automorphism α of T_X is called *automatic* if $S(\alpha)$ is finite.

Example 7.2. Let σ be a rooted automorphism of T_X and let $\omega = (\omega_\ell)_{\ell\in\mathbb{N}} \in \mathcal{S}$. Consider the automorphism $\widetilde{\alpha}^{\omega}$ of T_X. For $v \in X^*$, we have

$$\widetilde{\alpha}^{\omega}_v = \begin{cases} \widetilde{\alpha}^{L^{\ell}(\omega)} & \text{if } v = o^{\ell} \text{ for some } \ell \in \mathbb{N}_0, \\ \alpha & \text{if } v = o^{\ell}\omega_{\ell} \text{ for some } \ell \in \mathbb{N}_0, \\ \mathrm{id} & \text{otherwise.} \end{cases}$$

Thus the set of states of $\widetilde{\alpha}^{\omega}$ is finite if and only if $\{L^{\ell}(\omega) \in \mathcal{S} \mid \ell \in \mathbb{N}\}$ is a finite subset of $\mathcal{S}$. From this we see that $\widetilde{\alpha}^{\omega}$ is automatic if and only if there is some $N \in \mathbb{N}$ such that $L^N(\omega)$ is periodic.

Definition 7.3. An automorphism $\alpha \in \mathrm{Aut}(T_X)$ is called *bounded* if there is some $C \geq 0$ such that

$$|\{v \in X^{\ell} \mid \alpha_v \neq \mathrm{id}\}| \leq C$$

for all $\ell \in \mathbb{N}_0$.

Example 7.4. Let σ be a rooted automorphism of T_X. Then $\widetilde{\alpha}^{\omega}$ is clearly bounded for every choice of $\omega \in \mathcal{S}$.

It can be easily seen that the set of all bounded automatic automorphisms of T_X forms a group. In [3, Theorem 1.2], Bartholdi, Kaimanovich and Nekrashevych proved that this group is amenable. As subgroups of amenable groups are amenable, it follows that every subgroup of $\mathrm{Aut}(T_X)$ that is generated by bounded automatic automorphisms is amenable. In view of Examples 7.2 and 7.4, we therefore obtain the following.

Proposition 7.5. *Let* $G \leq \mathrm{Aut}(T_X)$ *be a group of rooted automorphisms and let* $\Omega \subseteq \mathcal{S}$ *be a non-empty subset. Suppose that every* $\omega \in \Omega$ *is eventually periodic, i.e., there is an* $N_\omega \in \mathbb{N}$ *such that* $L^{N_\omega}(\omega)$ *is periodic. Then* Γ^{Ω}_G *is amenable.*

Now we are able to proof Theorem 1.1. Let p be a prime and let $n \geq 3$ be a natural number. Consider the natural action of $\mathrm{SAff}_n(\mathbb{F}_p)$ on $\mathcal{A}_{p,n} := \{0,\ldots,p-1\}^n$. Let $\mathcal{A}^+_{p,n}$

denote the complement of $0 := (0, 0, \ldots, 0)$ in $\mathcal{A}_{p,n}$ and let $\mathcal{S}_{p,n}$ denote the set of sequences in $\mathcal{A}_{p,n}^{+}$. For every non-empty set $\Omega \subseteq \mathcal{S}_{p,n}$, we write $A_{p,n}^{\Omega} := \Gamma_{\mathrm{SAff}_n(\mathbb{F}_p)}^{\Omega}$, where $\mathrm{SAff}_n(\mathbb{F}_p)$ is identified with the corresponding group of rooted automorphisms of $T_{p,n} := T_{\mathcal{A}_{p,n}}$. Let further $G_{p,n}$ denote the subgroup of $\mathrm{Aut}(T_{p,n})$ that is generated by the canonical actions of $\mathrm{SAff}_n(\mathbb{F}_p)$ and $\mathrm{SAff}_n(\mathbb{Z})$ on $T_{p,n}$. Let $M_{p,n}^{\Omega} := \Gamma_{G_{p,n}}^{\Omega}$ be the corresponding Ω-group. Equivalently, $M_{p,n}$ is the subgroup of $\mathrm{Aut}(T_{p,n})$ that is generated by $A_{p,n}^{\Omega}$ and $\Gamma_{p,n}^{\Omega}$.

Theorem 7.6. *Let $n \geq 3$ be a natural number, let p be a prime, let $\omega \in \mathcal{S}_{p,n}$ be eventually periodic, and let $\Omega \subseteq \mathcal{S}_{p,n}$ be a finite subset. Then the following hold:*

(1) *$A_{p,n}^{\omega}$ is a finitely generated amenable group.*

(2) *$M_{p,n}^{\Omega}$ is finitely generated and contains a non-abelian free group.*

(3) *If $\omega \in \Omega$, then the inclusion $\iota\colon A_{p,n}^{\omega} \to M_{p,n}^{\Omega}$ induces an isomorphism $\hat{\iota}\colon \widehat{A_{p,n}^{\omega}} \to \widehat{M_{p,n}^{\Omega}}$ of profinite completions.*

(4) *The family $(M_{p,n}^{\{\omega,\omega'\}})_{\omega' \in \mathcal{S}_{p,n}}$ contains uncountably many pairwise non-isomorphic groups.*

Proof. The first assertion follows from Proposition 7.5. Since Ω is finite, it follows that $M_{p,n}^{\Omega}$ is finitely generated. Since $M_{p,n}^{\Omega}$ contains $\Gamma_{p,n}^{\Omega}$, it contains a non-abelian free subgroup by Corollary 6.5.

To prove the third assertion, we verify the assumptions of Corollary 4.7. First we observe that the groups $\mathrm{SAff}_n(\mathbb{F}_p)$ and $\langle \mathrm{SAff}_n(\mathbb{F}_p) \cup \mathrm{SAff}_n(\mathbb{Z}) \rangle$ are perfect (see Lemma 6.2). In addition, these groups are self-similar and satisfy property H; to see this, one can use the argument given in Lemma 6.3.

Finally, it follows from Corollary 5.6 that the family $(M_{p,n}^{\{\omega,\omega'\}})_{\omega' \in \mathcal{S}_{p,n}}$ of subgroups contains uncountably many pairwise non-isomorphic groups ∎

In order to deduce Theorem 1.1 from Theorem 7.6, it remains to determine the number of generators. It is known that $\mathrm{SL}_n(\mathbb{Z})$ and $\mathrm{SL}_n(\mathbb{F}_p)$ are 2-generated (see [11]), and so $\mathrm{SAff}_n(\mathbb{Z})$ and $\mathrm{SAff}_n(\mathbb{F}_p)$ can be generated by 3 elements. Since $A_{p,n}^{\omega}$ is generated by two copies of $\mathrm{SAff}_n(\mathbb{F}_p)$, it is 6-generated. Similarly, the group $G_{p,n}$ is 6-generated and so $M_{p,n}^{\{\omega,\omega'\}}$ – which is generated by three copies of $G_{p,n}$ – can be generated using 18 elements.

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References

[1] S. I. Adyan, Random walks on free periodic groups. *Math. USSR Izv.* **21** (1983), no. 3, 425–434 Zbl 0528.60011 MR 682486

[2] L. Bartholdi, R. I. Grigorchuk, and Z. Šunić, Branch groups. In *Handbook of algebra, Vol. 3*, pp. 989–1112, Handb. Algebr. 3, Elsevier, Amsterdam, 2003 Zbl 1140.20306 MR 2035113

[3] L. Bartholdi, V. A. Kaimanovich, and V. V. Nekrashevych, On amenability of automata groups. *Duke Math. J.* **154** (2010), no. 3, 575–598 Zbl 1268.20026 MR 2730578

[4] A. V. Borovik, L. Pyber, and A. Shalev, Maximal subgroups in finite and profinite groups. *Trans. Amer. Math. Soc.* **348** (1996), no. 9, 3745–3761 Zbl 0866.20018 MR 1360222

[5] M. Bożejko, Uniformly amenable discrete groups. *Math. Ann.* **251** (1980), no. 1, 1–6 Zbl 0422.43001 MR 583820

[6] M. R. Bridson and F. J. Grunewald, Grothendieck's problems concerning profinite completions and representations of groups. *Ann. of Math. (2)* **160** (2004), no. 1, 359–373 Zbl 1083.20023 MR 2119723

[7] Y. de Cornulier and A. Mann, Some residually finite groups satisfying laws. In *Geometric group theory*, pp. 45–50, Trends Math., Birkhäuser, Basel, 2007 Zbl 1147.20025 MR 2395788

[8] R. I. Grigorchuk, Just infinite branch groups. In *New horizons in pro-p groups*, pp. 121–179, Progr. Math. 184, Birkhäuser, Boston, MA, 2000 Zbl 0982.20024 MR 1765119

[9] A. Grothendieck, Représentations linéaires et compactification profinie des groupes discrets. *Manuscripta Math.* **2** (1970), 375–396 Zbl 0239.20065 MR 262386

[10] A. J. Hahn and O. T. O'Meara, *The classical groups and K-theory*. Grundlehren Math. Wiss. 291, Springer, Berlin, 1989 Zbl 0683.20033 MR 1007302

[11] L. K. Hua and I. Reiner, On the generators of the symplectic modular group. *Trans. Amer. Math. Soc.* **65** (1949), 415–426 Zbl 0034.30503 MR 29942

[12] V. A. Kaimanovich, The spectral measure of transition operator and harmonic functions connected with random walks on discrete groups. *Zap. Nauchn. Semin. Leningr. Otd. Mat. Inst. Steklova (LOMI)* **97** (1980), 102–109 Zbl 0452.60013 MR 602365

[13] V. A. Kaĭmanovich and A. M. Vershik, Random walks on discrete groups: boundary and entropy. *Ann. Probab.* **11** (1983), no. 3, 457–490 Zbl 0641.60009 MR 704539

[14] G. Keller, Amenable groups and varieties of groups. *Illinois J. Math.* **16** (1972), 257–269 Zbl 0231.22006 MR 296141

[15] H. Kesten, Symmetric random walks on groups. *Trans. Amer. Math. Soc.* **92** (1959), 336–354 Zbl 0092.33503 MR 109367

[16] Y. Lavreniuk and V. Nekrashevych, Rigidity of branch groups acting on rooted trees. *Geom. Dedicata* **89** (2002), 159–179 Zbl 0993.05050 MR 1890957

[17] A. Mann, Positively finitely generated groups. *Forum Math.* **8** (1996), no. 4, 429–459 Zbl 0852.20019 MR 1393323

[18] V. Nekrashevych, An uncountable family of 3-generated groups with isomorphic profinite completions. *Internat. J. Algebra Comput.* **24** (2014), no. 1, 33–46 Zbl 1297.20029 MR 3189664

[19] D. V. Osin, Uniform non-amenability of free Burnside groups. *Arch. Math. (Basel)* **88** (2007), no. 5, 403–412 Zbl 1173.43002 MR 2316885

[20] V. P. Platonov and O. I. Tavgen', On Grothendieck's problem on profinite completions of groups. *Sov. Math. Dokl.* **33** (1986), no. 5, 822–825 Zbl 0614.20016 MR 852649

[21] L. Pyber, Groups of intermediate subgroup growth and a problem of Grothendieck. *Duke Math. J.* **121** (2004), no. 1, 169–188 Zbl 1057.20019 MR 2031168

[22] D. Segal, The finite images of finitely generated groups. *Proc. London Math. Soc. (3)* **82** (2001), no. 3, 597–613 Zbl 1022.20011 MR 1816690

[23] S. Sidki and J. S. Wilson, Free subgroups of branch groups. *Arch. Math. (Basel)* **80** (2003), no. 5, 458–463 Zbl 1044.20012 MR 1995624

[24] R. A. Wilson, *The finite simple groups*. Grad. Texts in Math. 251, Springer, London, 2009 Zbl 1203.20012 MR 2562037

[25] J. Wysoczánski, On uniformly amenable groups. *Proc. Amer. Math. Soc.* **102** (1988), no. 4, 933–938 Zbl 0651.43001 MR 934870

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