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Property (T) in density-type models of random groups

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Abstract. We study property (T) in the $\Gamma(n,k,d)$ model of random groups: as k tends to infinity this gives the Gromov density model, introduced in 1993. We provide bounds for property (T) in the k-angular model of random groups, i.e., the $\Gamma(n,k,d)$ model, where k is fixed and n tends to infinity. We also prove that for $d > 1/3$, a random group in the $\Gamma(n,k,d)$ model has property (T) with probability tending to 1 as k tends to infinity, strengthening the results of Żuk and Kotowski–Kotowski, who consider only groups in the $\Gamma(n,3k,d)$ model.

1. Introduction

1.1. Property (T) in random groups

Gromov proposed two models of random groups in [11] to study the notion of a 'generic' finitely presented group. There is some ambiguity in the literature between the two models, and so we provide the full definitions here.

Fix $n \geq 2$, $k \geq 3$, and $0 < d < 1$. The (*strict*) (n,k,d) *model* is obtained as followed. Let $A_n = \{a_1,\dots,a_n\}$, and let $F_n := \mathbb{F}(A_n)$ be the free group generated by A_n. Let $\mathcal{C}(n,k)$ be the set of cyclically reduced words of length k in F_n (so that $\mathcal{C}(n,k) \approx (2n-1)^k$). Uniformly randomly select a set $R \subseteq \mathcal{C}(n,k)$ of size $|R| = (2n-1)^{kd}$, and let $\Gamma := \langle A_n \mid R\rangle$. We call Γ a *random group* in the (*strict*) (n,k,d) *model*, and write $\Gamma \sim \Gamma(n,k,d)$.

If we keep n fixed and let k tend to infinity, then we obtain the *Gromov density model*, as introduced in [11], whereas if we fix k and let n tend to infinity, we obtain the k-*angular model*, as introduced in [3]. The k-angular model was first studied for $k = 3$ (the *triangular model*) by Żuk in [27] and for $k = 4$ (the *square model*) by Odrzygóźdź in [16].

The *lax* (n,k,d,f) *model* is obtained via the following procedure. Let $\mathcal{C}(n,k,f)$ be the set of cyclically reduced words of length between $k - f(k)$ and $k + f(k)$ in F_n, where $f(k) = o(k)$. Uniformly randomly select a set $R \subseteq \mathcal{C}(n,k,f)$ of size $|R| = (2n-1)^{kd}$, and let $\Gamma := \langle A_n \mid R\rangle$. We call Γ a *random group* in the *lax* (n,k,d,f) *model*, and write $\Gamma \sim \Gamma_{\mathrm{lax}}(n,k,d,f)$.

We first consider the case of the k-angular model. It is a seminal theorem of Żuk [27] (cf. [13]) that for $d > 1/3$ a random group in the triangular model has property (T) with

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probability tending to 1. As observed in [18], the case of k divisible by 3 is easier, as we may use the work of [27] and [13] to observe property (T) at densities greater than $1/3$: see [15] for the proof that $3k$-angular model has property (T) for any $d > 1/3$. This idea was in fact extended in [15] to passing from property (T) in $\Gamma(n,k,d)$ to $\Gamma(n,lk,d)$ for $l \geq 1$. For $k \geq 3$, let

$$d_k := \frac{k + (-k \bmod 3)}{3k}.$$

Here, we take the convention that $-k \bmod 3$ represents $(-k) \bmod 3$, we will always write $-(k \bmod 3)$ to represent the alternative. In particular,

$$d_k = \begin{cases} \frac{1}{3} & \text{if } k = 0 \bmod 3, \\ \frac{k+2}{3k} & \text{if } k = 1 \bmod 3, \\ \frac{k+1}{3k} & \text{if } k = 2 \bmod 3. \end{cases}$$

In the case that $k = 0 \bmod 3$, $d_k = 1/3$, which is known to be the sharp threshold for property (T) in the cases that $k = 3$ [13,27] and $k = 6$ [18]. The remaining cases are not known to be sharp.

Below, we analyse property (T) in the k-angular model. We believe this to be the first non-trivial result on property (T) in any k-angular model for any $k \geq 5$ not divisible by 3, and in fact provides a non-trivial range of densities where random k-angular groups are infinite and have property (T) for each $k \geq 8$. There is currently no known density for random k-angular groups to be infinite with property (T) for $k = 4, 5, 7$.

Theorem A. *Let $k \geq 8$, let $d > d_k$, and let $\Gamma_m \sim \Gamma(m,k,d)$. Then*

$$\lim_{m\to\infty} \mathbb{P}(\Gamma_m \text{ has property (T)}) = 1.$$

We may also consider the density model. Again, there is some ambiguity between the strict model and the lax model in the literature. Indeed, many cubulation results, such as those of [14,21], refer to groups in the strict model, whilst results on property (T) typically refer to groups in the lax model. In particular, the following result is due to Żuk [27] and Kotowski–Kotowski [13] (see [1] for finer analysis of $\Gamma(n,3,d)$ as $d \to 1/3$). There is an alternative proof of the below in [7, Corollary 12.7].

Theorem ([13,27]). *Fix $n \geq 2$, let $d > 1/3$, and let $\Gamma_k \sim \Gamma(n,3k,d)$. Then*

$$\lim_{k\to\infty} \mathbb{P}(\Gamma_k \text{ has property (T)}) = 1.$$

Note that the above results *only apply to groups whose relator length is divisible by* 3. However, these results have two important consequences: firstly, it provides an infinite number of hyperbolic torsion-free groups with property (T), since such groups are torsion-free with probability tending to 1, and the Euler characteristic of such a group is dependent only on k and d [19]. Secondly, it proves that groups in the $\Gamma_{\mathrm{lax}}(n,k,d,f)$ model have property (T), using the following argument; see [20, §I.2.c]. This step is noted in [13, p. 410].

Claim. *Fix $n \geq 2$, let $d > 0$, and let k_i be a sequence of increasing integers such that $|k_{i+1} - k_i|$ is uniformly bounded. If*

$$\lim_{k_i \to \infty} \mathbb{P}(\Gamma \sim \Gamma(n, k_i, d) \text{ has property (T)}) = 1,$$

then there exists a constant function f such that for any $d' > d$,

$$\lim_{l \to \infty} \mathbb{P}(\Gamma \sim \Gamma_{\text{lax}}(n, l, d', f) \text{ has property (T)}) = 1.$$

Proof. Let $C = \max_i |k_{i+1} - k_i|$, and choose $f = f(l)$ such that $f(l) \geq C$. For each l, choose $k_{i(l)}$ such that $l + f(l) - C \leq k_{i(l)} \leq l + f(l)$. Then for sufficiently large l, and for $\Gamma_l = \langle A_n \mid R \rangle \sim \Gamma_{\text{lax}}(n, l, d', f)$, we see that for any $d < d'' < d'$, with probability tending to 1 as l tends to infinity,

$$|R \cap \mathcal{C}(n, k_{i(l)})| \geq (2n-1)^{d'' k_{i(l)}}.$$

Hence, by choosing a random subset $R' \subseteq R \cap \mathcal{C}(n, k_{i(l)})$ of size $(2n-1)^{kd}$, and setting $\Gamma'_{i(l)} := \langle A_n \mid R' \rangle$, we see that there exists an epimorphism $\Gamma'_{i(l)} \twoheadrightarrow \Gamma_l$, and $\Gamma'_{i(l)} \sim \Gamma(n, k_{i(l)}, d)$. Since $\Gamma'_{i(l)}$ has property (T) with probability tending to 1 as $i(l)$ tends to infinity, and property (T) is preserved by epimorphisms, the result follows. ■

However, we note that the question of property (T) remains open for the strict model. If $\lim_{k \to \infty} \mathbb{P}(\Gamma \sim \Gamma(n, k, d) \text{ has property (T)}) = 1$, then we must also have that

$$\lim_{p_i \to \infty} \mathbb{P}(\Gamma \sim \Gamma(n, p_i, d) \text{ has property (T)}) = 1,$$

where p_i denotes the i-th prime. Since the results of [13, 27] do not apply in this regime, we are inspired to further analyse the question of property (T) for $\Gamma(n, k, d)$.

We now briefly explain the approach taken by [13, 27] to prove their theorem. Firstly, one takes $n \geq 2$, $d > 1/3$, and considers $\Gamma_m \sim \Gamma(m, 3, d)$. It can then be proved that

$$\lim_{m \to \infty} \mathbb{P}(\Gamma_m \text{ has property (T)}) = 1.$$

The proof of the above is very involved, and requires passing via an alternate model, the *permutation model*: we omit the definition of this model as we do not require it.

One fixes $d' > d$ and finds for each k an integer $m(k, n)$ and a surjection $\Gamma_{m(k,n)} \twoheadrightarrow \Gamma'_k$, where $\Gamma'_k \sim \Gamma(n, 3k(m, n), d')$ (technically this is a surjection onto a finite index subgroup of Γ'_k). The result then follows by preservation of property (T) under epimorphisms and taking finite index extensions.

A natural approach to extend the results to the strict model using the techniques of Żuk and Kotowski–Kotowski would be to fix $l \geq 3$, let $\Gamma_{(m,l)} \sim \Gamma(m, l, d)$, and consider $m \to \infty$. Then for each $n \geq 2$ and $k \geq 3$, find an integer $m(k, l, n)$ and

$$\Gamma'_k \sim \Gamma(n, lk, d')$$

with $\Gamma_{(m(k,l,n),l)} \twoheadrightarrow \Gamma'_{k(m,n,l)}$, as in [15]. However, if we consider the model $\Gamma(n, p_i, d')$, then we must have in the above that $lk = p_k$, where p_k is the k-th prime number, which necessarily forces $m(k,l,n) = n$, and therefore we cannot use statements of the form $\lim_{m\to\infty} \mathbb{P}(\Gamma_m$ has property (T)), as m must be bounded.

To address this, we therefore must deal with the model $\Gamma(n,k,d)$ directly. The approach is to use the works of Ballmann–Świątkowski [4] and Żuk [26] (cf. [27]), in which a spectral condition for property (T) was provided independently. This will be used to provide an alternate criterion for property (T) in terms of the first eigenvalue of a graph we define relative to Γ, $\Delta_k(\Gamma)$. A similar graph was used in the case of $k = 0 \bmod 3$ by Drutu–Mackay [7]. The bulk of this text then analyses the eigenvalues of these random graphs.

The following completes the analysis of property (T) in $\Gamma(n,k,d)$ for $d > 1/3$.

Theorem B. *Let $n \geq 2$, $d > 1/3$, and let $\Gamma_k \sim \Gamma(n,k,d)$. Then*

$$\lim_{k\to\infty} \mathbb{P}(\Gamma_k \text{ has property (T)}) = 1.$$

Note that this immediately implies for any infinite sequence, $\{k_i\}_i$, of increasing positive integers, and $\Gamma_i \sim \Gamma(n,k_i,d)$ that

$$\lim_{i\to\infty} \mathbb{P}(\Gamma_i \text{ has property (T)}),$$

so that we immediately recover the results of [13, 27].

We note that we could also consider the case of $d \to 1/3$ in a manner similar to that of [1]. For $n \geq 2$, $k \geq 3$, and $0 < p < 1$, we can define the random group model $\Gamma_p(n,k,p)$: let $\Gamma = \langle A_n \mid R\rangle$, where R is obtained by adding each word in $\mathcal{C}(n,k)$ with probability p. Since property (T) is an increasing property (one preserved by epimorphisms), it is easy to switch between $\Gamma_p(n,k,p)$ and $\Gamma(n,k,(2n-1)^k p)$ in a manner analogous to switching between the Erdös–Rényi random graph $G(m,p)$ and the random graph $G(m,M)$, since the number of relators in R is $|R| = (1+o(1))(2n-1)^k p$ almost surely, for p sufficiently large. In fact, we do analyse property (T) in $\Gamma_p(n,k,p)$ in Theorem 6.2, and then use this to prove Theorems A and B. However, we believe that the notation and constants involved in the statement of Theorem 6.2 add unnecessary complexity to the statement of Theorem B, and so we leave this to Section 6.

Indeed, random groups exhibit many interesting properties, depending on the density chosen. All of the following statements hold asymptotically almost surely, i.e., with probability tending to 1. Firstly, a random group in the density model at density $d < 1/2$ is hyperbolic and torsion-free [11] (cf. [19]). This argument also transfers to the k-angular model [3]: see [16] for the case of $k = 4$, as well as a generalisation of the argument to a wider class of diagrams. In the opposite direction to property (T), there are many results known about the lack of property (T) in various models of random groups. Groups in the density model are virtually special for $d < 1/6$ [21] and contain a free codimension-1 subgroup for $d < 5/24$ [14]. As observed in [18], this implies that for any $k \geq 3$, a random group in the k-angular model at density $d < 5/24$ does not have property (T). Groups in the triangular model are free at densities less than $1/3$ [1], groups in the square model are free at densities less than $1/4$ [16], and groups in the k-angular model are free for $d < 1/k$ [3]. Furthermore, groups in the square model are virtually special for $d < 1/3$ [8, 17] and contain a codimension-1 subgroup for $d < 3/8$ [18]. Finally, groups in the hexagonal model contain a codimension-1 subgroup for $d < 1/3$ and have property (T) for $d > 1/3$ [18].

1.2. Structure of the paper and some notation

The idea of the proof is the following: for a finitely presented group Γ, we find a graph $\Delta(\Gamma)$, and using [4, 26], we prove that if $\lambda_1(\Delta(\Gamma)) > 1/2$, then Γ has property (T). This graph loosely corresponds to the 'link of depth $k/3$' of the presentation complex for Γ. For random groups, this graph $\Delta(\Gamma)$ can be written as the union of a graph Σ_2 and two bipartite graphs Σ_1, Σ_3. If we allowed all freely reduced words as relators, then these graphs would have the marginal distributions of Erdös–Rényi random graphs. Since we restrict to only having cyclically reduced words as relators, these graphs will not allow some edges, and so will have the marginal distributions of *reduced random graphs*. We need to analyse the eigenvalues of these graphs, and then prove the union of these graphs has high eigenvalue with large probability.

The paper is structured as follows. Sections 2 introduces some relevant graph theoretic definitions, and in Section 3, we provide a spectral criterion for property (T), related to the graph Δ_k. Sections 4 and 5 are more geared towards graph theory, and allow us to analyse the eigenvalues of specific random graphs. In Section 6, we apply these results to prove the main theorems of this paper.

We now briefly discuss some notation and assumptions. We are dealing with asymptotics, and so we frequently arrive at situations where m is some parameter tending to infinity that is required to be an integer: if m is not integer, we will implicitly replace it by $\lfloor m \rfloor$. Since we are dealing with asymptotics, this does not affect any of our arguments.

Definition 1.1. Suppose that $m_1 : \mathbb{N} \to \mathbb{N}$ is a function such that $m_1(n) \to \infty$ as $n \to \infty$. We will write $m_2 = m_2(m_1(n))$ to mean that $m_2(n) = f(m_1(n))$ for some function f, and furthermore, $f(m_1(n)) \to \infty$ as $n \to \infty$. In particular, m_2 only depends on m_1, and tends to infinity as m_1 tends to infinity.

The following are standard.

Definition 1.2. Let $f, g : \mathbb{N} \to \mathbb{R}_+$ be two functions. We write

(1) $f = o(g)$ if $f(m)/g(m) \to 0$ as $m \to \infty$,

(2) $f = O(g)$ if there exists a constant $N \geq 0$ and $M \geq 1$ such that $f(m) \leq Ng(m)$ for all $m \geq M$,

(3) $f = \Omega(g)$ if $g = o(f)$.

We write $f = o_m(g)$ etc. to indicate that the variable name is m.

Note that typically we will deal with functions $m_2 = m_2(m_1)$, and $f = f(m_1, m_2)$. We will write $f = o_{m_1}(g)$ etc. to mean that the function $f'(m_1) = f(m_1, m_2(m_1)) = o_{m_1}(g'(m_1))$, where $g'(m_1) = g(m_1, m_2(m_1))$.

Definition 1.3. Let $\mathcal{M}(m)$ be some model of random groups (or graphs) depending on a parameter m, and let $\mathcal{P}$ be a property of groups (or graphs). We say that $\mathcal{P}$ holds *asymptotically almost surely with m (a.a.s. (m))* if

$$\lim_{m\to\infty} \mathbb{P}(G \sim \mathcal{M}(m) \text{ has } \mathcal{P}) = 1.$$

Again, typically we will have deal with cases where $m_2 = m_2(m_1)$ is fixed, $\mathcal{M}(m_1, m_2)$ is some model of random groups (or graphs) depending on parameters m_1 and m_2, and $\mathcal{P}$ is a property of groups (or graphs). We say that $\mathcal{P}$ holds *asymptotically almost surely with* (m_1) (*a.a.s.* (m_1)) if

$$\lim_{m_1\to\infty} \mathbb{P}(G \sim \mathcal{M}(m_1, m_2(m_1)) \text{ has } \mathcal{P}) = 1.$$

Typically, we will only use the above in proofs or in the statements of auxiliary technical lemmas.

Finally, we will often be working with bipartite graphs: the vertex partition of a bipartite graph G will always be written $V(G) = V_1(G) \sqcup V_2(G)$.

2. Graphs and eigenvalues

In this short section, we provide some definitions that will be central throughout.

Definition 2.1. A *multiset* M is a pair $M = (A, \mu_M)$, where A is a set and $\mu\colon A \to \mathbb{N}$ is a set function. We call A the *universe* of M and μ_M its *multiplicity*. Typically, in an abuse of notation, we write $M = (M, \mu_M)$ to be a multiset, where we view M as both the underlying universe, and the multiset.

Let $M = (A, \mu)$, $N = (B, \nu)$ be multisets. The *sum* of M and N is the multiset defined by

$$M \uplus N := (A \cup B, \mu + \nu),$$

where we extend $\mu|_{B\setminus A} := 0$, $\nu|_{A\setminus B} := 0$.

Definition 2.2. A graph is a pair $G = (V, E)$, where V is the *set of vertices*, and $E = (E, \mu_E)$ is a multiset. We typically refer to E as the *set of edges*, which consists of unordered pairs of the form $\{u, v\}$, for $u, v \in V$. Note that here we allow pairs $\{u, u\}$. An edge $\{u, v\}$ is said to *join* the vertices u and v. We refer to $\mu_E(\{u, v\})$ as the *number of edges* joining u and v.

Definition 2.3. Let $G = (V, E)$, $G' = (V', E')$. The *union* of G and G' is the graph $G \cup G' := (V \cup V', E \uplus E')$.

Let $G = (V, E)$ be a graph with vertex set $V = \{v_1, \dots, v_m\}$. The *adjacency matrix* of G, $A(G)$, is the $m \times m$ matrix with $A(G)_{i,j}$ defined to be the number of edges between v_i and v_j, i.e., $A(G)_{i,j} = \mu_E(\{v_i, v_j\})$. The *degree matrix* of G, $D(G)$, is the diagonal matrix with entries $D(G)_{i,i} = \deg(v_i) := \sum_{v_j} \mu_E(\{v_i, v_j\})$. The *Laplacian* of G, $L(G)$, is defined by

$$L(G) = I - D^{-1/2} A D^{-1/2}.$$

We note that $L(G)$ is symmetric positive semi-definite, with eigenvalues

$$0 \leq \lambda_0(L(G)) \leq \lambda_1(L(G)) \leq \cdots \leq \lambda_{m-1}(L(G)) \leq 2.$$

For $i = 1, \dots, m$, we define $\lambda_i(G) := \lambda_i(L(G))$.

We note the following lemma, commonly known as Weyl's inequality, which will also be of frequent use. If A is a symmetric real $m \times m$ matrix, then A has real eigenvalues, which we order by $\lambda_0(A) \leq \lambda_1(A) \leq \cdots \leq \lambda_{m-1}(A)$. We define the reverse ordering of eigenvalues $\mu_1(A) \geq \mu_2(A) \geq \cdots \geq \mu_m(A)$, i.e., $\mu_i(A) = \lambda_{m-i}(A)$.

Lemma 2.4 (Weyl's inequality [25]). *Let A and B be symmetric $m \times m$ real matrices. For $i = 1, \dots, m$, $\mu_i(A) + \mu_m(B) \leq \mu_i(A + B) \leq \mu_i(A) + \mu_1(B)$.*

We also make the following remarks.

Remark 2.5. Let M be a symmetric $n \times n$ matrix. For $i = 1, \dots, n$,

$$\mu_i(-M) = -\mu_{n+1-i}(M).$$

This follows as $\{\mu_i(-M) : 1 \leq i \leq n\} = \{-\mu_i(M) : 1 \leq i \leq n\}$, and $\mu_1(M) \geq \mu_2(M) \geq \cdots \geq \mu_n(M)$, so that $-\mu_1(M) \leq -\mu_2(M) \leq \cdots \leq -\mu_m(M)$.

Remark 2.6. Let G be a graph. For $i = 0, \dots, |V(G)| - 1$,

$$\lambda_i(G) = 1 - \mu_{i+1}(D(G)^{-1/2} A(G) D(G)^{-1/2})$$

since $L(G) = I - D(G)^{-1/2} A(G) D(G)^{-1/2}$, so that

$$\{\lambda_i(L(G)) : 0 \leq i \leq |V(G)| - 1\} = \{1 - \mu_j(D(G)^{-1/2} A(G) D(G)^{-1/2}) : 1 \leq j \leq |V(G)|\},$$

and

$$\begin{aligned} 1 - \mu_1(D(G)^{-1/2} A(G) D(G)^{-1/2}) &\leq 1 - \mu_2(D(G)^{-1/2} A(G) D(G)^{-1/2}) \leq \cdots \\ &\leq 1 - \mu_{|V(G)|}(D(G)^{-1/2} A(G) D(G)^{-1/2}). \end{aligned}$$

3. A spectral criterion for property (T)

In this section, we deduce a spectral criterion for property (T): we first remind the reader of some of the relevant definitions. We focus only on finitely generated discrete groups: for a further exposition the reader should see, for example, [5].

Let Γ be a finitely generated group with finite generating set S, let $\mathcal{H}$ be a Hilbert space, and let $\pi\colon\Gamma\to\mathcal{U}(\mathcal{H})$ be a unitary representation of Γ. We say that π has *almost-invariant vectors* if for every $\varepsilon>0$, there is some non-zero $u_\varepsilon\in\mathcal{H}$ such that for every $s\in S$, $\|\pi(s)u_\varepsilon-u_\varepsilon\|<\varepsilon\|u_\varepsilon\|$.

Definition 3.1. We say that Γ has *property (T)* if for every Hilbert space $\mathcal{H}$, and for every unitary representation $\pi\colon\Gamma\to\mathcal{U}(\mathcal{H})$ with almost-invariant vectors, there exists a non-zero invariant vector for π.

It is standard that the choice of generating set does not matter. We now note the following well-known results concerning property (T): for proofs see, for example, [5]. We will use these results implicitly throughout.

Lemma 3.2. *Let Γ be a finitely generated group, and let H be a finite index subgroup of Γ. Then Γ has property (T) if and only if H has property (T).*

Lemma 3.3. *Let Γ be a finitely generated group with property (T) and let Γ' be a homomorphic image of Γ. Then Γ' has property (T).*

3.1. A spectral criterion for property (T)

Now, let $\Gamma=\langle A_n\mid R\rangle$ be a finite presentation of a group. Let R_k be the set of words in R of length k. Define the graph $\Delta_3(A_n\mid R)$ by

$$V(\Delta_3(A_n\mid R))=A_n\sqcup A_n^{-1}$$

and for each relator $r=r_1r_2r_3\in R_3$ add the edges

$$(r_1,r_3^{-1}),\quad (r_2,r_1^{-1}),\quad (r_3,r_2^{-1}).$$

The use of this graph is the following, proved independently by [26] (cf. [27]) and [4]. The result is often stated for a model of Δ_3 without multiple edges, and is often known as *Żuk's criterion for property (T)*.

Theorem 3.4 ([4, 26]). *Let $\Gamma=\langle A_n\mid R\rangle$ be a finite presentation. If $\lambda_1(\Delta_3(A_n\mid R))>1/2$, then Γ has property (T).*

We now apply this to recover an alternate spectral criterion for property (T). However, before we introduce the graph Δ_k, we first note a result regarding finite index subgroups of free groups. For the free group $F_n:=\mathbb{F}(A_n)$ and $l\geq 1$, we define $\mathcal{W}(n,l)$ to be the set of freely reduced words of length l in F_n. We now prove that these sets always generate finite index subgroups of F_n.

Lemma 3.5. *Let $l\geq 1$. Then $[F_n:\langle\mathcal{W}(n,l)\rangle]<\infty$.*

(In fact, it is easily seen that $[F_n:\langle\mathcal{W}(n,l)\rangle]\leq 2$.)

Proof. Note that $\mathcal{W}(n,l) = S_l(F_n)$, the sphere of radius l in F_n. Hence

$$[F_n : \langle \mathcal{W}(n,l) \rangle] \leq |B_{F_n}(\mathrm{id}, l-1)| = 2n(2n-1)^{l-2},$$

since $F_n = B_{F_n}(\mathrm{id}, l-1)\langle S_l(F_n)\rangle$. ■

We now introduce the graph to which our spectral criterion will apply.

Definition 3.6. Let $G = \langle A_n \mid R \rangle$ be a finite presentation of a group and let $k \geq 3$. We define the graph $\Delta_k(A_n \mid R)$, as follows, depending on k mod 3:

- $k = 0$ mod 3. Let $V(\Delta_k(A_n \mid R)) = \mathcal{W}(n, k/3)$. For each relator $r = r_1 \ldots r_k \in R_k$, write $r = r_x r_y r_z$ with $r_x, r_y, r_z \in \mathcal{W}(n, k/3)$, and add the edges

$$(r_x, r_z^{-1}), \quad (r_y, r_x^{-1}), \quad (r_z, r_y^{-1}).$$

- $k = 1$ mod 3. Let $\Delta_k(A_n \mid R)$ be the graph with

$$V(\Delta_k(A_n \mid R)) = \mathcal{W}\Big(n, \frac{k-1}{3}\Big) \sqcup \mathcal{W}\Big(n, \frac{k+2}{3}\Big).$$

For each relator $r = r_1 \ldots r_k \in R_k$, write $r = r_x r_y r_z$ with $r_x, r_y \in \mathcal{W}(n, (k-1)/3)$ and $r_z \in \mathcal{W}(n, (k+2)/3)$, and add the edges

$$(r_x, r_z^{-1}), \quad (r_y, r_x^{-1}), \quad (r_z, r_y^{-1}).$$

- $k = 2$ mod 3. Let $\Delta_k(A_n \mid R)$ be the graph with

$$V(\Delta_k(A_n \mid R)) = \mathcal{W}\Big(n, \frac{k-2}{3}\Big) \sqcup \mathcal{W}\Big(n, \frac{k+1}{3}\Big).$$

For each relator $r = r_1 \ldots r_k \in R_k$, write $r = r_x r_y r_z$ with $r_x, r_y \in \mathcal{W}(n, (k+1)/3)$ and $r_z \in \mathcal{W}(n, (k-2)/3)$, and add the edges

$$(r_x, r_z^{-1}), \quad (r_y, r_x^{-1}), \quad (r_z, r_y^{-1}).$$

We can prove the following result.

Lemma 3.7. *Let* $\Gamma = \langle A_n \mid R \rangle$ *be a finite presentation and let* $k \geq 3$*. If* $\lambda_1(\Delta_k(A_n \mid R)) > 1/2$*, then* Γ *has property (T).*

We note that this lemma is not particularly effective when given a specific finite presentation of a group: for the above spectral condition to hold, we heuristically require $|R| \gg (2n-1)^{(k+(-k \bmod 3))/3}$. However, this is exactly the regime we consider for random groups.

Proof. We prove this for $k = 2$ mod 3; the other cases are similar. First, for ease, let $\Gamma' = \langle A_n \mid R_k \rangle$. Since Γ is a homomorphic image of Γ', it suffices to prove that Γ' has property (T). Let $\phi\colon F_n \twoheadrightarrow \Gamma'$ be the canonical epimorphism induced by the choice of presentation for Γ'. Let $\mathcal{W} = \mathcal{W}(n, (k-2)/3) \sqcup \mathcal{W}(n, (k+1)/3)$, $W = \phi(\mathcal{W})$, and let $H = \langle W \rangle_{\Gamma'}$: by Lemma 3.5, we have that $[\Gamma' : H] < \infty$.

For each $r \in R_k$, let us write $r = r_x r_y r_z$, where $r_x, r_y \in \mathcal{W}(n, (k+1)/3)$ and $r_z \in \mathcal{W}(n, (k-2)/3)$. Let $T = \{r_x r_y r_z : r \in R_k\}$ and let

$$\widetilde{\Gamma} := \mathbb{F}(\mathcal{W})/\langle\langle T \rangle\rangle = \langle \mathcal{W} \mid T \rangle.$$

It is clear that there is a surjective homomorphism $\psi : \widetilde{\Gamma} \twoheadrightarrow H$, so that Γ' is virtually a homomorphic image of $\widetilde{\Gamma}$. Next, we note that $\Delta_k(A_n \mid R) \cong \Delta_3(\mathcal{W} \mid T)$. By Theorem 3.4, if $\lambda_1(\Delta_k(A_n \mid R)) = \lambda_1(\Delta_3(\mathcal{W} \mid T)) > 1/2$, then $\widetilde{\Gamma}$ has property (T). Since property (T) is preserved under epimorphisms and passing to finite index extensions, it follows that if $\lambda_1(\Delta_k(A_n \mid R)) > 1/2$, then Γ has property (T). ■

4. The spectral theory of almost regular graphs, Erdös–Rényi random graphs, and the unions of regular graphs

In this section, we analyse the spectral theory of almost regular graphs, as well as some results on the eigenvalues of Erdös–Rényi random graphs. We also prove a result concerning the eigenvalues of the union of a well-connected graph and two bipartite graphs. We first note the following lemma.

Lemma 4.1. *Let G be a graph. Then $\max_i |\mu_i(A(G))| \leq \max_{v \in V(G)} \deg(v)$. If G is bipartite, then*

$$\max_i |\mu_i(A(G))| \leq \max_{\substack{v \in V_1(G) \\ w \in V_2(G)}} \sqrt{\deg(v)\deg(w)}.$$

Proof. The first result follows as $\|A(G)\|_\infty = \max_{v \in V(G)} \deg(v)$, and it is standard that $\|A(G)\|_\infty$ is an upper bound for the absolute values of the eigenvalues of $A(G)$.

The second inequality follows from, e.g., [12, (3.7.2)], as follows. In this case, we have

$$A(G) = \begin{pmatrix} 0 & B \\ B^\top & 0 \end{pmatrix}$$

for some matrix B. By definition, the set of eigenvalues of A are the set of *singular values* of B, $\{\sigma_j(B)\}_j$. Therefore, $\max_i |\lambda_i(A(G))| = \max_i |\sigma_i(B)|$. By [12, (3.7.2)],

$$\max_i |\sigma_i(B)| \leq \sqrt{\|B\|_\infty \|B\|_1} = \max_{\substack{v \in V_1(G) \\ w \in V_2(G)}} \sqrt{\deg(v)\deg(w)}. \qquad ■$$

4.1. The spectra of almost regular graphs

We now analyse the spectra of almost regular graphs. These definitions are standard in graph theory and appear, e.g., in [13].

Definition 4.2 (Almost regular graphs). Let $\{G_m\}_{m=1}^{\infty}$ be a collection of graphs. We say that the graphs G_m are *almost d_m-regular* if for every G_m its minimum and maximum degree are $(1+o_m(1))d_m$.

Definition 4.3 (Almost regular bipartite graphs). Let $\{G_m\}_{m=1}^{\infty}$ be a collection of bipartite graphs. We say that the graphs G_m are *almost $(d_m^{(1)}, d_m^{(2)})$-regular* if for every G_m the minimum and maximum degree of vertices in $V_1(G_m)$ are $(1+o_m(1))d_m^{(1)}$ and the minimum and maximum degree of vertices in $V_2(G_m)$ are $(1+o_m(1))d_m^{(2)}$.

We note the following results.

Lemma 4.4 ([13, Lemma 4.4]). *Let $d_m \to \infty$ and let G_m be almost d_m-regular. Then*

$$\frac{1}{d_m}\mu_2(A(G_m)) = (1+o_m(1))(1-\lambda_1(G_m)).$$

In particular, if $\mu_2(A(G_m)) = o_m(d_m)$, then $\lambda_1(G_m) = 1 - o_m(1)$.

Lemma 4.5 ([13, Lemma 4.5]). *Let G_m be an almost d_m-regular graph, and let G'_m be a graph on the same vertex set whose maximum degree is $o_m(d_m)$. Then*

(i) *$G_m \cup G'_m$ is almost d_m-regular,*

(ii) *$\lambda_1(G_m) = \lambda_1(G_m \cup G'_m) + o_m(1)$.*

Again, recall that $\lambda_1(G) = 1 - \mu_2(D^{-1/2}AD^{-1/2})$. We now prove the corresponding result for bipartite graphs: our proofs differ from [13], and rely on Weyl's inequality.

Lemma 4.6. *Let $d_m^{(1)}, d_m^{(2)} \to \infty$, and let G_m be almost $(d_m^{(1)}, d_m^{(2)})$-regular. For $i = 1, \dots, |V(G_m)|$,*

$$\frac{1}{\sqrt{d_m^{(1)} d_m^{(2)}}}\mu_i(A(G_m)) = \mu_i(D^{-1/2}(G_m)A(G_m)D^{-1/2}(G_m)) + o_m(1).$$

In particular, if $\mu_2(A(G_m)) = o_m(\sqrt{d_m^{(1)} d_m^{(2)}})$, then $\lambda_1(G_m) = 1 - o_m(1)$.

Proof. As G_m is almost $(d_m^{(1)}, d_m^{(2)})$-regular, we see that for

$$A = A(G_m) = \begin{pmatrix} 0 & A_1 \\ A_1^{\intercal} & 0 \end{pmatrix}, \quad D = D(G_m),$$

there exists a matrix K with norm $\|K\|_{\infty} = o_m(1)$ such that

$$\frac{1}{\sqrt{d_m^{(1)} d_m^{(2)}}} A = D^{-1/2}AD^{-1/2} + K.$$

Since $|\mu_i(K)| \le \|K\|_{\infty} = o_m(1)$ for all i, the first statement of the lemma follows easily by Weyl's inequality. The second statement follows from Remark 2.6. ∎

Lemma 4.7. *Let $d_m^{(1)}, d_m^{(2)} \to \infty$, and let G_m be almost $(d_m^{(1)}, d_m^{(2)})$-regular. Let G'_m be a bipartite graph on the same vertex set as G_m with the same vertex partitions such that the maximum degree of $v \in V_i(G'_m)$ is $o_m(d_m^{(i)})$. Then*

(i) *$G_m \cup G'_m$ is almost $(d_m^{(1)}, d_m^{(2)})$-regular,*

(ii) *$\lambda_1(G_m) = \lambda_1(G_m \cup G'_m) + o_m(1)$.*

Proof. Part (i) is immediate. For part (ii), we see that $A(G_m \cup G'_m) = A(G_m) + A(G'_m)$: since the maximum degree of a vertex $v \in V_i(G'_m) = V_i(G_m)$ is $o(d_m^{(i)})$, we have by Lemma 4.1 that

$$\max_i |\mu_i(A(G'_m))| \leq o_m\Big(\sqrt{d_m^{(1)} d_m^{(2)}}\Big),$$

and hence

$$\max_i \Big|\mu_i\Big(\frac{1}{\sqrt{d_m^{(1)} d_m^{(2)}}} A(G'_m)\Big)\Big| = o_m(1).$$

By Weyl's inequality,

$$\begin{aligned} &\mu_2\Big(\frac{1}{\sqrt{d_m^{(1)} d_m^{(2)}}}(A(G_m) + A(G'_m))\Big) \\ &\quad \leq \mu_2\Big(\frac{1}{\sqrt{d_m^{(1)} d_m^{(2)}}} A(G_m)\Big) + \mu_1\Big(\frac{1}{\sqrt{d_m^{(1)} d_m^{(2)}}} A(G'_m)\Big) \\ &\quad = \mu_2\Big(\frac{1}{\sqrt{d_m^{(1)} d_m^{(2)}}} A(G_m)\Big) + o_m(1). \end{aligned}$$

Similarly,

$$\mu_2\Big(\frac{1}{\sqrt{d_m^{(1)} d_m^{(2)}}}(A(G_m) + A(G'_m))\Big) \geq \mu_2\Big(\frac{1}{\sqrt{d_m^{(1)} d_m^{(2)}}} A(G_m)\Big) + o_m(1),$$

and the result follows by Remark 2.6 and Lemma 4.6. ■

4.2. Almost regularity of Erdös–Rényi random graphs and their eigenvalues

In this section, we introduce some models of random graphs, and then prove they are almost regular.

Definition 4.8 (*Erdös–Rényi random graph*). Let $m \geq 1, 0 < p := p(m) < 1$. The Erdös–Rényi random graph $G(m, p)$ is the random graph model with vertex set $\{u_1, \ldots, u_m\}$ and edge set obtained by adding each edge $\{u_i, u_j\}$ independently with probability p. For a random graph G, we write $G \sim G(m, p)$ to indicate that the distribution of G is that of $G(m, p)$.

Definition 4.9 (*Erdös–Rényi random bipartite graph*). Let $m_1, m_2 \geq 1$ and let $0 < p := p(m_1, m_2) < 1$. The Erdös–Rényi random bipartite graph $G(m_1, m_2, p)$ is the random bipartite graph model with vertex set $V_1 = \{u_1, \ldots, u_{m_1}\}$, $V_2 = \{v_1, \ldots, v_{m_2}\}$, and edge set obtained by adding each edge $\{u_i, v_j\}$ independently with probability p.

Given a model of random graphs $\mathcal{M}$, and a random matrix M, we write $M \sim A(\mathcal{M})$ to indicate that the distribution of M is the same as that obtained by sampling a graph $G \sim \mathcal{M}$ and then taking its adjacency matrix.

We now analyse the regularity of random bipartite graphs. For this we will use the Chernoff bounds: for $X \sim \mathrm{Bin}(n, p)$ and $\delta \in [0, 1]$,

$$\mathbb{P}(|X - np| \geq \delta np) \leq 2\exp\Big(-\frac{np\delta^2}{3}\Big).$$

Lemma 4.10. *Let $m_2 = m_2(m_1)$ and $p = p(m_1, m_2) = p(m_1)$ be such that*

$$\min\{m_1, m_2\}p = \Omega_{m_1}(\log \max\{m_1, m_2\}).$$

Then a.a.s. (m_1) $G(m_1, m_2, p)$ is almost $(m_2 p, m_1 p)$-regular.

Proof. First note $m_2 p \geq \omega \log m_1$ and $m_1 p \geq \omega \log m_2$ for some $\omega \to \infty$ as $m_1 \to \infty$. Let $G \sim G(m_1, m_2, p)$. Let $v \in V_1(G)$, $w \in V_2(G)$. Note that $\mathbb{E}(\deg(v)) = m_2 p$, $\mathbb{E}(\deg(w)) = m_1 p$, and $\mathrm{Var}(\deg(v)) = m_2 p(1-p)$, $\mathrm{Var}(\deg(w)) = m_1 p(1-p)$. Let $\varepsilon = \omega^{-1/3}$. By the Chernoff bounds, for a fixed vertex v in V_1,

$$\mathbb{P}(|\deg(v) - m_2 p| \geq \varepsilon m_2 p) \leq 2\exp\Big(-\frac{\varepsilon^2 m_2 p}{3}\Big).$$

Hence the probability that there exists a vertex in V_1 with degree too large or small is

$$\begin{aligned}&\mathbb{P}(\exists\, v \in V_1 \colon |\deg(v) - m_2 p| \geq \varepsilon m_2 p)\\ &\quad \leq 2m_1 \exp\Big(-\frac{\varepsilon^2 m_2 p}{3}\Big) \leq 2m_1 \exp\Big(-\frac{\omega^{1/3}\log m_1}{3}\Big) = 2m_1^{-\Omega_{m_1}(1)}.\end{aligned}$$

Similarly,

$$\begin{aligned}&\mathbb{P}(\exists\, w \in V_2 \colon |\deg(w) - m_1 p| \geq \varepsilon m_1 p)\\ &\quad \leq 2m_2 \exp\Big(-\frac{\varepsilon^2 m_1 p}{3}\Big) \leq 2m_2 \exp\Big(-\frac{\omega^{1/3}\log m_2}{3}\Big) = 2m_2^{-\Omega_{m_1}(1)}.\end{aligned}$$ ∎

Therefore, we immediately see the following.

Lemma 4.11. *Let $m_2 = m_2(m_1)$ and $p = p(m_1)$ be such that*

$$\min\{m_1, m_2\}p = \Omega_{m_1}(\log \max\{m_1, m_2\}).$$

Then a.a.s. (m_1)

$$\mu_1(A(G(m_1, m_2, p))) \leq [1 + o_{m_1}(1)]p\sqrt{m_1 m_2}.$$

Proof. According to Lemma 4.10, a.a.s. (m_1) the maximum degree of a vertex in V_1 is $(1 + o_{m_1}(1))m_2 p$, and the maximum degree of a vertex in V_2 is $(1 + o_{m_1}(1))m_1 p$. By Lemma 4.1,

$$\max_i |\mu_i(A(G(m_1, m_2 p)))| \leq \max_{\substack{v \in V_1(G) \\ w \in V_2(G)}} \sqrt{\deg(v)\deg(w)} \leq [1 + o_{m_1}(1)]\sqrt{m_1 m_2 p^2}$$

with probability tending to 1 as m_1 tends to infinity. ∎

Similarly, we can deduce that the Erdös–Rényi random graph is almost regular.

Lemma 4.12. *Let $m \geq 1$ and $p = p(m)$ be such that $mp = \Omega_m(\log m)$. Then a.a.s. (m) $G(m, p)$ is almost mp-regular.*

We now note some results on the eigenvalues of Erdös–Rényi random graphs. The eigenvalues of $G(m, p)$ were first analysed in [10]: we use the following result, due to [10] (an extension to a more general model can be found in [6]).

Theorem 4.13 ([10, Theorem 1]). *Let $p > 0$ be such that $mp = \Omega_m(\log^6(m))$, and let $G \sim G(m, p)$. Then a.a.s. (m),*

$$\max_{i \neq 1} |\mu_i(A(G))| \leq 2[1 + o_m(1)]\sqrt{mp},$$
$$\max_{i \neq 0} |1 - \lambda_i(G)| = o_m(1).$$

The eigenvalues of the bipartite version, $G(m_1, m_2, p)$ were analysed far more recently: see, e.g., [2, Theorem A].

Theorem 4.14. *Let $m_1 \geq 1$, $m_2 = m_2(m_1)$, and let $p = p(m_1)$ be such that $m_1 p = \Omega(\log^6 m_1)$, $m_2 p = \Omega(\log^6 m_2)$. Let $G \sim G(m_1, m_2, p)$. Then with probability tending to 1 as m_1 tends to infinity:*

$$\max_{i \neq 0, m_1+m_2-1} |1 - \lambda_i(G)| = o_{m_1}(1).$$

4.3. Spectra of unions of regular graphs

The purpose of this subsection is to analyse the spectral distribution of unions of three graphs with relatively high first eigenvalue. This is already known when all three graphs share the same vertex set.

Lemma 4.15 ([27, p. 665]). *Let G_1, G_2, G_3 be d-regular graphs on the same vertex set, and suppose $\lambda_1(G_i) > 1 - c$ for each i. Then*

$$\lambda_1(G_1 \cup G_2 \cup G_3) \geq 1 - c.$$

We now wish to extend this to the case where the graphs are relatively well connected, and they do not share the same vertex set. We first recall (a partial consequence of) the Courant–Fischer theorem as follows.

Theorem 4.16 (Courant–Fischer theorem). *Let M be a symmetric $m \times m$ matrix with first eigenvalue $\mu_1(M)$ and corresponding eigenvector $\underline{e}$. Then*

$$\mu_2(M) = \max_{\substack{\underline{\boldsymbol{x}} \perp \underline{e} \\ \|\underline{\boldsymbol{x}}\|=1}} \langle M\underline{\boldsymbol{x}}, \underline{\boldsymbol{x}}\rangle = \max_{\underline{\boldsymbol{x}} \perp \underline{e}} \frac{\langle M\underline{\boldsymbol{x}}, \underline{\boldsymbol{x}}\rangle}{\langle \underline{\boldsymbol{x}}, \underline{\boldsymbol{x}}\rangle}.$$

Hence we can prove the following (recall that for a bipartite graph G, $V_1(G)$ and $V_2(G)$ are the vertex partitions of G).

Lemma 4.17. *Let G_1, G_2, G_3 be graphs such that*

(i) *G_2, G_3 are bipartite, $V(G_1) = V_1(G_2) = V_1(G_3)$, $V_2(G_2) = V_2(G_3)$,*

(ii) *G_1 is $2d_1$-regular, and G_2, G_3 are (d_1, d_2)-regular,*

(iii) *for $i = 1, 2, 3$, there exists $0 \le c_i < 1$ with $\lambda_1(G_i) \ge 1 - c_i$.*

Then

$$\lambda_1(G_1 \cup G_2 \cup G_3) \ge 1 - \frac{\sqrt{2}c_1 + c_2 + c_3}{2\sqrt{2}}.$$

Proof. Let $\underline{\mathbf{1}}_l$ be the all 1's vector with l entries, and let $G = G_1 \cup G_2 \cup G_3$. Let $V_1 = V(G_1) = V_1(G_2) = V_1(G_3)$ and $V_2 = V_2(G_2) = V_2(G_3)$. Let $m_1 = |V_1|$ and $m_2 = |V_2|$. For $i = 1, 2, 3$, let $\Lambda_i = D_i^{-1/2} A_i D_i^{-1/2}$, where $D_i = D(G_i)$, $A_i = A(G_i)$ (here we view G_1 as a graph on $V_1 \sqcup V_2$). Let $D = D(G)$, $A = A(G)$, and consider $\Lambda = D(G)^{-1/2} A(G) D(G)^{-1/2}$, so that

$$\Lambda = \frac{1}{2}\Lambda_1 + \frac{1}{2\sqrt{2}}\Lambda_2 + \frac{1}{2\sqrt{2}}\Lambda_3,$$

where each of Λ, Λ_1, Λ_2, Λ_3 is symmetric and hence self-adjoint. We remark again that $\mu_2(\Lambda_i) = 1 - \lambda_1(G_i)$. We also note that $m_2 d_2 = m_1 d_1$, so that $d_2 = m_1 d_1 / m_2$.

Now, we consider the first eigenvalues of the matrices Λ and Λ_i. The eigenvector corresponding to $\mu_1(\Lambda) = 1$ is

$$D^{1/2}\underline{\mathbf{1}}_{m_1+m_2} = \begin{pmatrix} 2\sqrt{d_1}\underline{\mathbf{1}}_{m_1} \\ \sqrt{2d_2}\underline{\mathbf{1}}_{m_2} \end{pmatrix}.$$

The eigenvector corresponding to $\mu_1(\Lambda_2) = 1$ and $\mu_1(\Lambda_3) = 1$ is

$$D_2^{1/2}\underline{\mathbf{1}}_{m_1+m_2} = D_3^{1/2}\underline{\mathbf{1}}_{m_1+m_2} = \begin{pmatrix} \sqrt{d_1}\underline{\mathbf{1}}_{m_1} \\ \sqrt{d_2}\underline{\mathbf{1}}_{m_2} \end{pmatrix}.$$

The eigenvector corresponding to $\mu_1(\Lambda_1) = 1$ is

$$D_1^{1/2}\underline{\mathbf{1}}_{m_1+m_2} = \begin{pmatrix} \sqrt{2d_1}\underline{\mathbf{1}}_{m_1} \\ 0 \end{pmatrix}.$$

Let $\underline{\boldsymbol{\phi}}$ be a vector with $\|\underline{\boldsymbol{\phi}}\| = 1$, $\underline{\boldsymbol{\phi}} \cdot D^{1/2}\underline{\mathbf{1}}_{m_1+m_2} = 0$, and $\mu_2(\Lambda) = \langle \Lambda\underline{\boldsymbol{\phi}}, \underline{\boldsymbol{\phi}}\rangle$, which exists by the Courant–Fischer theorem. We may write

$$\underline{\boldsymbol{\phi}} = \begin{pmatrix} \alpha\underline{\mathbf{1}}_{m_1} + \underline{\boldsymbol{u}} \\ \beta\underline{\mathbf{1}}_{m_2} + \underline{\boldsymbol{v}} \end{pmatrix},$$

where $\underline{\boldsymbol{u}} \cdot \underline{\mathbf{1}}_{m_1} = \underline{\boldsymbol{v}} \cdot \underline{\mathbf{1}}_{m_2} = 0$. As

$$\underline{\boldsymbol{\phi}} \cdot D^{1/2}\underline{\mathbf{1}}_{m_1+m_2} = 2\sqrt{d_1}\alpha m_1 + \sqrt{2d_2}\beta m_2 = 2\sqrt{d_1}\alpha m_1 + \sqrt{\frac{2d_1 m_1}{m_2}}\beta m_2,$$

we see

$$\beta = \frac{-\sqrt{2m_1}\alpha}{\sqrt{m_2}}.$$

Let

$$\underline{\boldsymbol{\phi_1}} = \begin{pmatrix} \alpha\underline{\mathbf{1}}_{m_1} \\ \beta\underline{\mathbf{1}}_{m_2} \end{pmatrix}, \quad \underline{\boldsymbol{\phi_2}} = \begin{pmatrix} \underline{\boldsymbol{u}} \\ \underline{\boldsymbol{v}} \end{pmatrix},$$

so that $\underline{\boldsymbol{\phi_1}} \cdot D^{1/2}\underline{\mathbf{1}}_{m_1+m_2} = \underline{\boldsymbol{\phi_2}} \cdot D^{1/2}\underline{\mathbf{1}}_{m_1+m_2} = 0$. Write $\gamma = \|\underline{\boldsymbol{\phi_1}}\|^2$ with $\|\underline{\boldsymbol{\phi_2}}\|^2 = 1-\gamma$. Note that

$$\gamma = \alpha^2 m_1 + \beta^2 m_2 = 3\alpha^2 m_1,$$

so that $3\alpha^2 m_1 \le 1$. We now calculate

$$\langle \Lambda_1\underline{\boldsymbol{\phi_1}}, \underline{\boldsymbol{\phi_1}}\rangle = \begin{pmatrix} \alpha\underline{\mathbf{1}}_{m_1} \\ 0 \end{pmatrix} \cdot \begin{pmatrix} \alpha\underline{\mathbf{1}}_{m_1} \\ \beta\underline{\mathbf{1}}_{m_2} \end{pmatrix} = \alpha^2 m_1.$$

Furthermore,

$$\langle \Lambda_1\underline{\boldsymbol{\phi_1}}, \underline{\boldsymbol{\phi_2}}\rangle = \begin{pmatrix} \alpha\underline{\mathbf{1}}_{m_1} \\ 0 \end{pmatrix} \cdot \begin{pmatrix} \underline{\boldsymbol{u}} \\ \underline{\boldsymbol{v}} \end{pmatrix} = \alpha\underline{\mathbf{1}}_{m_1} \cdot \underline{\boldsymbol{u}} = 0.$$

Since Λ_1 is self-adjoint, $\langle \underline{\boldsymbol{\phi_1}}, \Lambda_1\underline{\boldsymbol{\phi_2}}\rangle = \langle \Lambda_1\underline{\boldsymbol{\phi_1}}, \underline{\boldsymbol{\phi_2}}\rangle = 0$. Also, since $\underline{\boldsymbol{u}} \cdot D_1^{1/2}\underline{\mathbf{1}}_{m_1} = 0$, we have by the Courant–Fischer theorem,

$$\langle \Lambda_1\underline{\boldsymbol{\phi_2}}, \underline{\boldsymbol{\phi_2}}\rangle = \langle \Lambda_1'\underline{\boldsymbol{u}}, \underline{\boldsymbol{u}}\rangle \le \mu_2(\Lambda_1)\|\underline{\boldsymbol{u}}\|^2 = c_1\|\underline{\boldsymbol{u}}\|^2 \le c_1\|\underline{\boldsymbol{\phi_2}}\|^2 = c_1(1-\gamma),$$

where Λ_1' is $D(G_1)^{-1/2}A(G_1)D(G_1)^{-1/2}$, with G_1 considered as a graph on the vertex set V_1.

We now perform the same calculations for Λ_2. Firstly, for some matrix B_2,

$$\Lambda_2\underline{\boldsymbol{\phi_1}} = \frac{1}{\sqrt{d_1 d_2}}\begin{pmatrix} 0 & B_2 \\ B_2^{\intercal} & 0 \end{pmatrix}\begin{pmatrix} \alpha\underline{\mathbf{1}}_{m_1} \\ \beta\underline{\mathbf{1}}_{m_2} \end{pmatrix} = \begin{pmatrix} \beta\sqrt{\frac{d_1}{d_2}}\underline{\mathbf{1}}_{m_1} \\ \alpha\sqrt{\frac{d_2}{d_1}}\underline{\mathbf{1}}_{m_2} \end{pmatrix} = \begin{pmatrix} \beta\sqrt{\frac{m_2}{m_1}}\underline{\mathbf{1}}_{m_1} \\ \alpha\sqrt{\frac{m_1}{m_2}}\underline{\mathbf{1}}_{m_2} \end{pmatrix},$$

so that

$$\langle \Lambda_2\underline{\boldsymbol{\phi_1}}, \underline{\boldsymbol{\phi_1}}\rangle = \begin{pmatrix} \beta\sqrt{\frac{m_2}{m_1}}\underline{\mathbf{1}}_{m_1} \\ \alpha\sqrt{\frac{m_1}{m_2}}\underline{\mathbf{1}}_{m_2} \end{pmatrix} \cdot \begin{pmatrix} \alpha\underline{\mathbf{1}}_{m_1} \\ \beta\underline{\mathbf{1}}_{m_2} \end{pmatrix} = 2\alpha\beta\sqrt{m_1 m_2}.$$

Next, by the Courant–Fischer theorem, $\langle \Lambda_2 \underline{\boldsymbol{\phi_2}}, \underline{\boldsymbol{\phi_2}} \rangle \leq c_2 \|\underline{\underline{\boldsymbol{\phi_2}}}\|^2 = c_2(1-\gamma)$ (since $\underline{\boldsymbol{\phi_2}} \cdot D_2^{1/2} \underline{\mathbf{1}}_{m_1+m_2} = 0$). Furthermore,

$$\langle \Lambda_2 \underline{\boldsymbol{\phi_1}}, \underline{\boldsymbol{\phi_2}} \rangle = \begin{pmatrix} \beta \sqrt{\frac{m_2}{m_1}} \underline{\mathbf{1}}_{m_1} \\ \alpha \sqrt{\frac{m_1}{m_2}} \underline{\mathbf{1}}_{m_2} \end{pmatrix} \cdot \begin{pmatrix} \underline{\boldsymbol{u}} \\ \underline{\boldsymbol{v}} \end{pmatrix} = \beta \sqrt{\frac{m_2}{m_1}} \underline{\mathbf{1}}_{m_1} \cdot \underline{\boldsymbol{u}} + \alpha \sqrt{\frac{m_1}{m_2}} \underline{\mathbf{1}}_{m_2} \cdot \underline{\boldsymbol{v}} = 0,$$

since $\underline{\mathbf{1}}_{m_1} \cdot \underline{\boldsymbol{u}} = \underline{\mathbf{1}}_{m_2} \cdot \underline{\boldsymbol{v}} = 0$. Finally, since Λ_2 is symmetric and hence self-adjoint, we see that

$$\langle \Lambda_2 \underline{\boldsymbol{\phi_2}}, \underline{\boldsymbol{\phi_1}} \rangle = \langle \underline{\boldsymbol{\phi_2}}, \Lambda_2 \underline{\boldsymbol{\phi_1}} \rangle = 0.$$

We can perform similar calculations for Λ_3. Putting this all together, we have

$$\begin{aligned} \langle \Lambda_1 \underline{\boldsymbol{\phi}}, \underline{\boldsymbol{\phi}} \rangle &\leq \alpha^2 m_1 + c_1(1-\gamma), \\ \langle \Lambda_2 \underline{\boldsymbol{\phi}}, \underline{\boldsymbol{\phi}} \rangle &\leq 2\alpha\beta\sqrt{m_1 m_2} + c_2(1-\gamma), \\ \langle \Lambda_3 \underline{\boldsymbol{\phi}}, \underline{\boldsymbol{\phi}} \rangle &\leq 2\alpha\beta\sqrt{m_1 m_2} + c_3(1-\gamma). \end{aligned}$$

We calculate

$$\frac{1}{\sqrt{2}} \alpha\beta\sqrt{m_1 m_2} = -\alpha^2 \sqrt{\frac{m_1}{m_2}} \sqrt{m_1 m_2} = -\alpha^2 m_1.$$

Therefore,

$$\begin{aligned} \langle \Lambda \underline{\boldsymbol{\phi}}, \underline{\boldsymbol{\phi}} \rangle &= \frac{1}{2} \langle \Lambda_1 \underline{\boldsymbol{\phi}}, \underline{\boldsymbol{\phi}} \rangle + \frac{1}{2\sqrt{2}} \langle \Lambda_2 \underline{\boldsymbol{\phi}}, \underline{\boldsymbol{\phi}} \rangle + \frac{1}{2\sqrt{2}} \langle \Lambda_3 \underline{\boldsymbol{\phi}}, \underline{\boldsymbol{\phi}} \rangle \\ &\leq \frac{1}{2} c_1(1-\gamma) + \frac{1}{2}\alpha^2 m_1 + \frac{1}{\sqrt{2}} \alpha\beta\sqrt{m_1 m_2} + \frac{1}{2\sqrt{2}} c_2(1-\gamma) \\ &\quad + \frac{1}{\sqrt{2}} \alpha\beta\sqrt{m_1 m_2} + \frac{1}{2\sqrt{2}} c_3(1-\gamma) \\ &= \frac{1}{2}\alpha^2 m_1 - 2\alpha^2 m_1 + \frac{1-\gamma}{2\sqrt{2}} (\sqrt{2} c_1 + c_2 + c_3) \\ &= \frac{-3}{2}\alpha^2 m_1 + \frac{1-\gamma}{2\sqrt{2}} (\sqrt{2} c_1 + c_2 + c_3) \\ &= -\frac{1}{2}\gamma + \frac{1-\gamma}{2\sqrt{2}} (\sqrt{2} c_1 + c_2 + c_3) \leq \frac{\sqrt{2} c_1 + c_2 + c_3}{2\sqrt{2}}, \end{aligned}$$

since $0 \leq \gamma \leq 1$.

Since $\underline{\boldsymbol{\phi}}$ was chosen with $\mu_2(\Lambda) = \langle \Lambda \underline{\boldsymbol{\phi}}, \underline{\boldsymbol{\phi}} \rangle$, we see that $\mu_2(\Lambda) \leq (\sqrt{2} c_1 + c_2 + c_3)/(2\sqrt{2})$, and hence

$$\lambda_1(G) = 1 - \mu_2(\Lambda) \geq 1 - \frac{\sqrt{2} c_1 + c_2 + c_3}{2\sqrt{2}}. \qquad \blacksquare$$

Lemma 4.18. *Let G_i, c_i be as above. Suppose $c_1 = \varepsilon$, $c_2 = c_3 = \varepsilon + 1/3$ for some $\varepsilon < 1/100$. Then*

$$\lambda_1(G_1 \cup G_2 \cup G_3) \geq \frac{3}{4}.$$

Proof. We may apply Lemma 4.17 to deduce that

$$\lambda_1(G_1 \cup G_2 \cup G_3) \geq 1 - \frac{(\sqrt{2}+2)\varepsilon + 2/3}{2\sqrt{2}} \geq 1 - \frac{2/3 + (2+\sqrt{2})/100}{2\sqrt{2}} \geq \frac{3}{4}. \quad \blacksquare$$

5. The spectrum of reduced random graphs

We have almost understood the spectral distribution of $\Delta_k(A_n \mid R)$ for $\langle A_n \mid R\rangle$ in the $\Gamma(n,k,d)$ model. However, there is one small complication which arises from the fact that we insist upon using cyclically reduced words as relators: the random graphs $\Delta_k(A_n \mid R)$ will not allow edges between certain types of words. Therefore, we need to introduce a slightly altered model of random graphs.

Some of the results contained within this section are already known. Indeed, [7, Sections 11 and 12] provides far more general results concerning the eigenvalues of reduced random graphs: we provide alternate proofs of the results we require (again we stress that the results of [7] are far more general than the results we obtain) as the proofs provide an introduction to the proof strategies of alternate results we require that are not covered by [7]. We indicate in the text the results already known.

5.1. Reduced random graphs

Definition 5.1. Fix $n, l \geq 1$, and let $0 < p < 1$. For $i = 1, \dots, n$, let $a_{i+n} := a_i^{-1}$, and for $i = 1, \dots, 2n$, let

$$S_i = \{w_1 \dots w_l \in \mathcal{W}(n,l) \colon w_1 = a_i\} = \{(w_1 \dots w_l)^{-1} \in \mathcal{W}(n,l) \colon w_l = a_i^{-1}\}.$$

For $v \in \mathcal{W}(n,l)$, let $i(v)$ be the unique integer such that $v \in S_{i(v)}$. The *reduced random graph* $\mathfrak{Red}(n,l,p)$ is the random graph with vertex set $\mathcal{W}(n,l)$, and edge set constructed as follows.

Let $i = 1, \dots, 2n$. For each pair of vertices $v, w \in \mathcal{W}(n,l)$, add (each of) the directed edges:

- (v,w) labelled by $i(v)$ with probability $p(v,w)$,
- (w,v), labelled by $i(w)$ with probability $p(w,v)$, where

$$p(s,t) = \begin{cases} p & \text{if } i(s) \neq i(t), \\ 0 & \text{if } i(s) = i(t). \end{cases}$$

Note that $|\mathcal{W}(n,l)| = 2n(2n-1)^{l-1}$. Furthermore, we can break $\mathfrak{Red}(n,l,p)$ into a union of graphs $\mathfrak{R}_i$, where for $i = 1, \dots, 2n$, each $\mathfrak{R}_i$ is a bipartite graph with vertex set $V_1 = S_i$, $V_2 = \mathcal{W}(n,l) \setminus S_i$, and each edge is added with probability p. Note that $\mathfrak{R}_i \sim G((2n-1)^{l-1}, (2n-1)^l, p)$; therefore, for large p, a.a.s. each graph $\mathfrak{R}_i$ is almost $((2n-1)^{l-1}p, (2n-1)^l p)$-regular. Hence for large p a.a.s. the graph $\mathfrak{Red}(n,l,p)$ is almost $2(2n-1)^l p$-regular. Next we prove the following.

Lemma 5.2. *Let $n, l \geq 1$, let p be such that $(2n-1)^l p = \Omega_l(\log^6 (2n-1)^l)$, and let $G \sim \mathfrak{Red}(n,l,p)$. There exists a random graph*

$$G' \sim G(2n(2n-1)^{l-1}, 2p - p^2)$$

such that a.a.s. (l),

$$\mu_1(A(G) - A(G')) \leq O_l\big(\max\big\{l, (2n-1)^l p^2, \sqrt{(2n-1)^{l-1} p}\big\}\big).$$

Proof. Let Σ_i be the random graph with vertex set S_i and each edge added with probability $2p(1-p)$, so that $\Sigma_i \sim G((2n-1)^{l-1}, 2p-p^2)$. By our assumptions on p, we see by Theorem 4.13 that a.a.s. (l) for all i (there are $2n$ such i, so we take the intersection of the $2n$ events)

$$\max_{j \neq 1} |\mu_j(A(\Sigma_i))| \leq O_l(\sqrt{(2n-1)^{l-1} p}).$$

Let $H = \bigcup_i (\mathfrak{R}_i \cup \Sigma_i)$. The probability that at least one edge connects two vertices $v, w \in S_i$ is $2p - p^2$. If $v \in S_i$ and $w \in S_j$ for $i \neq j$, the probability that at least one edge connects v and w is $1 - (1-p)^2 = 2p - p^2$. Hence, by collapsing duplicate edges in H, we obtain $G' \sim G(2n(2n-1)^{l-1}, 2p - p^2)$. Next, note that

$$A(G') = A(G) + \sum A_i + K,$$

where K takes into account the double edges obtained from the unions, and A_i is the adjacency matrix of the graph G_i which has vertex set $V(G)$ and edge set $E(\Sigma_i)$. Since the edge sets of each Σ_i are pairwise disjoint, one can easily see that $\mu_1(-\sum A_i) = \max_i \mu_1(-A_i)$.

The matrix K is the adjacency matrix of a random graph where edges are added with probability 0 or p^2. Using the Chernoff bounds for the degrees, we can see that if $(2n-1)^l p^2 = \Omega_l(l)$, then a.a.s. (l) $\|K\|_\infty = O_l((2n-1)^l p^2)$. Otherwise, we may deduce that $\|K\|_\infty = O_l(\log(2n-1)^l) = O_l(l)$.

Hence by Weyl's inequality,

$$\begin{aligned}
\mu_1(A(G) - A(G')) &= \mu_1\big(-K - \sum A_i\big) \leq \mu_1(-K) + \mu_1\big(-\sum A_i\big) \\
&= O_l\big(\max\big\{\|K\|_\infty, \mu_1\big(-\sum A_i\big)\big\}\big) \\
&= O_l\big(\max\big\{\|K\|_\infty, \max_i\{\mu_1(-A_i)\}\big\}\big) \\
&\leq O_l\big(\max\big\{l, (2n-1)^l p^2, \sqrt{(2n-1)^{l-1} p}\big\}\big).
\end{aligned}$$

■

Similarly, we define the following.

Definition 5.3. Fix $n \geq 1$, $l \geq 3$, $0 < p < 1$. Let $a_{i+n} := a_i^{-1}$. For $i = 1, \ldots, 2n$, let

$$\begin{aligned}
S_i' &= \{w_1 \ldots w_l \in \mathcal{W}(n,l) \colon w_1 = a_i\}, \\
T_i' &= \big\{(w_1 \ldots w_{l+1})^{-1} \in \mathcal{W}(n, l+1) \colon w_{l+1} = a_i^{-1}\big\}.
\end{aligned}$$

The *reduced random bipartite graph* $\mathfrak{BRed}(n, l, p)$ is the random graph with vertex set $V_1 = \mathcal{W}(n, l)$, $V_2 = \mathcal{W}(n, l+1)$, and for each $v \in S_i'$ and vertex $w \in V_2 - T_i'$, the edge (v, w) is added with probability p. The graph $\mathfrak{BR}_i$ is the random bipartite graph obtained as a subgraph with vertex set $V_1 = S_i'$ and $V_2 = \mathcal{W}(n, l+1) \setminus T_i'$.

Again, for large p the graph $\mathfrak{BRed}(n, l, p)$ is almost $((2n-1)^{l+1}p, (2n-1)^l p)$-regular. We can approximate this graph by an Erdös–Rényi random bipartite graph, similarly to the case of $\mathfrak{Red}(n, l, p)$.

Lemma 5.4. *Let $G \sim \mathfrak{BRed}(n, l, p)$, where $(2n-1)^l p = \Omega_l(\log(2n-1)^l)$. There exists a random graph $G' \sim G(2n(2n-1)^{l-1}, 2n(2n-1)^l, p)$ such that a.a.s. (l),*

$$\mu_1(A(G) - A(G')) \leq (1 + o_l(1))(2n-1)^{l-1/2} p.$$

Proof. This follows similarly to the proof of Lemma 5.2 for $\mathfrak{Red}(n, l, p)$.

For $i = 1, \ldots, 2n$, let Σ_i be the random graph with vertex set $V_1 = S_i$, $V_2 = T_i$ and each edge added with probability p, so that $\Sigma_i \sim G((2n-1)^{l-1}, (2n-1)^l, p)$.

Then

$$G' = G \cup \bigcup_i \Sigma_i \sim G(2n(2n-1)^{l-1}, 2n(2n-1)^l, p).$$

We see that $\mu_1(A(G) - A(G')) = \mu_1(-\sum_i A_i)$, where A_i is the adjacency matrix of the graph with vertex set $V(G)$ and edge set $E(\Sigma_i)$. Since the edge sets of the Σ_i are pairwise disjoint (and the graphs are bipartite, so their spectrum is symmetric around 0), we see that

$$\mu_1\Big(-\sum_i A_i\Big) \leq \max_i \mu_1(-A_i) = \max_i \mu_1(A_i) \leq (1 + o_l(1))(2n-1)^{l-1/2} p,$$

by Lemmas 4.1 and 4.10. ∎

We may analyse the eigenvalues of reduced random graphs as follows.

Lemma 5.5 ([7, Theorems 11.8 and 11.9]). *Let $n \geq 2$, and p be such that $p = o_l(1)$ and $(2n-1)^l p = \Omega_l(l^6)$. Let $G \sim \mathfrak{Red}(n, l, p)$. Then a.a.s. (l) $\lambda_1(G) \geq 1 - o_l(1)$.*

Proof. Let G' be the graph from Lemma 5.2, so that $G' \sim G(2n(2n-1)^{l-1}, 2p - p^2)$ and

$$\mu_1(A(G) - A(G')) \leq O_l\Big(\max\Big\{l, (2n-1)^l p^2, \sqrt{(2n-1)^{l-1} p}\Big\}\Big).$$

Let $D' = D(G')$ and $A' = A(G')$. Note that G is almost $2(2n-1)^l p$-regular, and hence,

$$\begin{aligned}
&\mu_1(D^{-1/2}(A - A')D^{-1/2}) \\
&\quad \leq O_l\Big(\frac{1 + o_l(1)}{(2n-1)^l p} \max\Big\{l, (2n-1)^l p^2, \sqrt{(2n-1)^{l-1} p}\Big\}\Big) = o_l(1).
\end{aligned}$$

Next, by our assumption on p,

$$2n(2n-1)^l p = \Omega_l(l^6) = \Omega_l(\log^6 2n(2n-1)^{l-1}),$$

so that by Theorem 4.13, a.a.s. (l),

$$\mu_2(D'^{-1/2}A'D'^{-1/2}) = o_l(1).$$

Next,

$$D(G)^{-1/2}AD(G)^{-1/2} = \frac{(2-p)n}{2n-1}D'^{-1/2}A'D^{-1/2} + K,$$

where $\|K\|_\infty = o_l(1)$. Hence $\mu_1(K) = o_l(1)$. Therefore, by Theorem 4.13 and Weyl's inequality, a.a.s. (l)

$$\begin{aligned}
\mu_2(D^{-1/2}AD^{-1/2}) &= \mu_2(D^{-1/2}A'D^{-1/2} + D^{-1/2}AD^{-1/2} - D^{-1/2}A'D^{-1/2}) \\
&\le \mu_2(D^{-1/2}A'D^{-1/2}) + \mu_1(D^{-1/2}(A-A')D^{-1/2}) \\
&= \mu_2\Big(\frac{(2-p)n}{2n-1}D'^{-1/2}A'D'^{-1/2} + K\Big) + o_l(1) \\
&\le \frac{(2-p)n}{2n-1}\mu_2(D'^{-1/2}A'D'^{-1/2}) + \mu_1(K) + o_l(1) \\
&\le \frac{(2-p)n}{2n-1}\mu_2(D'^{-1/2}A'D'^{-1/2}) + o_l(1) = o_l(1).
\end{aligned}$$

■

The result follows by Remark 2.6.

Lemma 5.6. *Let $n \ge 2$, and p be such that $p = o_l(1)$ and $(2n-1)^l p = \Omega_l(l^6)$. Let $G \sim \mathfrak{BRed}(n,l,p)$. Then a.a.s. (l)*

$$\lambda_1(G) \ge 1 - \frac{1}{2n-1} - o_l(1).$$

We note that we cannot prove that the above bound is sharp, but it is sufficient for our needs.

Proof. Let G' be the graph from Lemma 5.4 such that

$$G' \sim G(2n(2n-1)^{l-1}, 2n(2n-1)^l, p),$$

and

$$\mu_1(A(G) - A(G')) \le (1 + o_l(1))(2n-1)^{l-1/2}.$$

By Lemma 5.4,

$$\mu_1(D^{-1/2}(A-A')D^{-1/2}) \le [1 + o_l(1)]\frac{1}{2n-1}.$$

Next,

$$D(G)^{-1/2}A'D^{-1/2} = \frac{2n}{2n-1}D'^{-1/2}A'D'^{-1/2} + K,$$

where

$$K = \begin{pmatrix} 0 & H \\ H^{\mathsf{T}} & 0 \end{pmatrix} \quad \text{and} \quad \sqrt{\|H\|_\infty \|H\|_1} = o_l(1).$$

Hence $\mu_1(K) = o_l(1)$. Therefore, by Theorem 4.14, and using Remark 2.6 and Weyl's inequalities similarly to the proof of Lemma 5.5,

$$\begin{aligned} \mu_2(D^{-1/2}AD^{-1/2}) &= \mu_2(D^{-1/2}A'D^{-1/2} + D^{-1/2}AD^{-1/2} - D^{-1/2}A'D^{-1/2}) \\ &\leq \mu_2(D^{-1/2}A'D^{-1/2}) + \mu_1(D^{-1/2}(A-A')D^{-1/2}) \\ &\leq \mu_2\Big(\frac{2n}{2n-1}D'^{-1/2}A'D'^{-1/2} + K\Big) + \frac{1}{2n-1} + o_l(1) \\ &\leq \frac{2n}{2n-1}\mu_2(D'^{-1/2}A'D'^{-1/2}) + \mu_1(K) + \frac{1}{2n-1} + o_l(1) \\ &= \frac{1}{2n-1} + o_l(1). \end{aligned}$$

The result follows by Remark 2.6. ∎

5.2. Regular subgraphs of random graphs

We now need an auxiliary result concerning regular subgraphs of random graphs. Recall that a subgraph H of G is *spanning* if $V(H) = V(G)$. We first note the following.

Theorem 5.7 ([23]). *Suppose $mp = \omega(m)\log(m)$ for some $\omega(m) \to \infty$. Let $\delta \geq \omega^{-\theta}$ for some $0 < \theta < 1/2$, and let $G \sim G(m,p)$. Then a.a.s. (m), G contains a $(1-\delta)mp$-regular spanning subgraph.*

We wish to prove the analogue for random bipartite graphs. We do this similarly to [9, Theorem 1.4], which proves the result in the regime $m_1 = m_2$.

Theorem 5.8 ([9, Theorem 1.4]). *Let $m \geq 1$ and $p = p(m) > 0$ be such that $mp = \omega(m)\log m$ for some $\omega \to \infty$ as $m \to \infty$. Let $\delta \geq \omega^{-\theta}$ for some $\theta < 1/2$, and $G \sim G(m,m,p)$. Then a.a.s. (m) G contains a $((1-\delta)mp, (1-\delta)mp)$-regular spanning subgraph.*

In the k-angular model, we have $m_1 = m_2/n$, where $n \to \infty$, so we need to extend the above to a more general setting. We will use the following theorem, commonly known as the Ore–Reyser theorem: see, for example, [22] or Tutte [24]. Recall that for a graph G, and disjoint sets $A, B \subseteq V(G)$, we define $e_G(A,B)$ to be the number of edges in G between the sets A and B.

Theorem 5.9 (Ore–Reyser theorem). *Let G be a bipartite graph and let $d_1, d_2 \geq 0$. The graph G contains a (d_1, d_2)-regular spanning subgraph if and only if $d_1|V_1| = d_2|V_2|$, and for all $A \subseteq V_1$ and $B \subseteq V_2$, $d_1|A| \leq e_G(A,B) + d_2(|V_2| - |B|)$.*

Using the above, we can prove the following: this follows almost identically to the proof of [9, Theorem 1.4], with very minor changes.

Theorem 5.10. *Let $m_2 = m_2(m_1) \geq m_1$, and let $p = p(m_1) > 0$ be such that $m_1 p = \omega(m_1)\log m_2$ for some $\omega \to \infty$ as $m_1 \to \infty$. Let $\delta \geq \omega^{-\theta}$ for some $\theta < 1/2$, and $G \sim G(m_1, m_2, p)$. Then a.a.s. (m_1) G contains a $((1-\delta)m_2 p, (1-\delta)m_1 p)$-regular spanning subgraph with probability greater than $1 - m_2^{-\Omega_{m_1}(1)}$.*

Again, the proof of this follows extremely similarly to the proof of [9, Theorem 1.4]; we include it for completeness.

Proof of Theorem 5.10. Let $d_1 = (1-\delta)m_2 p$ and $d_2 = (1-\delta)m_1 p$. We wish to prove that a.a.s. (m_1) for all $A \subseteq V_1$ and $B \subseteq V_2$:

$$0 \leq e_G(A,B) + d_2(m_2 - |B|) - d_1|A| = e_G(A,B) + d_1(m_1 - |A| - m_1|B|/m_2).$$

If we are able to prove this, then we may conclude the desired result by the Ore–Reyser theorem. Note that if $|A| + m_1|B|/m_2 \leq m_1$, then we are immediately finished. Let us suppose otherwise; we now analyse different cases.

To begin, let $n_1 := m_1/\log\log m_1$. We may now assume that $|A| + m_1|B|/m_2 > m_1$. Suppose first that $|A| \leq n_1$, then $(m_2(m_1 - |A|)/m_1) + 1 \leq |B| \leq m_2$. Note that $e_G(A,B)$ has the distribution $\mathrm{Bin}(|A||B|, p)$. We may apply the Chernoff bounds to deduce that

$$\mathbb{P}(e_G(A,B) \leq (1-\delta)|A||B|p) \leq \exp\Big(-\frac{\delta^2|A||B|p}{2}\Big).$$

For

$$|A| = a \leq n_1, \quad |B| = b \geq \frac{m_2(m_1 - a)}{m_1},$$

and m_1 sufficiently large, this is bounded above by

$$\exp\Big(\frac{-\delta^2 a m_2(m_1 - a)p/m_1}{2}\Big) \leq \exp\Big(a m_2 p \frac{-\delta^2(m_1 - n_1)}{2m_1}\Big) \leq \exp\Big(-\delta^2\frac{m_2 a p}{4}\Big).$$

Therefore, the probability that there exist such sets with $e_G(A,B) \leq (1-\delta)|A||B|p$ is bounded above by

$$\sum_{a=1}^{n_1}\sum_{b=m_2\frac{m_1-a}{m_1}+1}^{m_2}\binom{m_1}{a}\binom{m_2}{b}e^{-\delta^2 m_2 a p/4} = \sum_{a=1}^{n_1}\sum_{b=1}^{m_2 a/m_1}\binom{m_1}{a}\binom{m_2}{b}e^{-\delta^2 m_2 a p/4}$$

$$\Big(\text{using } \binom{m_2}{b} \leq \binom{m_2}{m_2 a/m_1} \text{ for } b \leq m_2 a/m_1\Big) \quad \leq \sum_{a=1}^{n_1}\frac{m_2 a}{m_1}\binom{m_1}{a}\binom{m_2}{m_2 a/m_1}e^{-\delta^2 m_2 a p/4}$$

$$\Big(\text{using } \binom{m_1}{a} \leq \binom{m_2}{m_2 a/m_1} \text{ as } m_2/m_1 \geq 1\Big) \quad \leq \sum_{a=1}^{n_1}\frac{m_2 a}{m_1}\binom{m_2}{m_2 a/m_1}^2 e^{-\delta^2\frac{m_2}{m_1}a m_1 p/4}$$

$$\leq m_2\sum_{a=1}^{n_1}\Big(\frac{m_2^2 e^2}{m_2^2 a^2/m_1^2}e^{-\Omega(\log m_2)}\Big)^{\frac{a m_2}{m_1}}$$

$$= m_2^{-\Omega_{m_1}(1)},$$

since $\delta^2 m_1 p \geq \omega^{1-2\theta} \log m_2$ for some $\theta < 1/2$. The case is similar for $|B| \leq n_2 := m_2/\log\log m_2$. Next we may assume that $|A| \geq n_1$ and that $|B| \geq n_2$. First assume that $|A| \leq m_1|B|/m_2$, so that $|B| \geq m_2/2$. The probability that there exist such A, B with $e_G(A, B) \leq (1-\delta)|A||B|p$ is bounded above by

$$\begin{aligned}\sum_{a=n_1}^{m_1}\sum_{b=m_2/2}^{m_2}\binom{m_1}{a}\binom{m_2}{b}e^{-\delta^2 abp/2} &\leq \sum_{a=n_1}^{m_1}\sum_{b=m_2/2}^{m_2}\binom{m_1}{a}\binom{m_2}{b}e^{-\delta^2 n_1 m_2 p/4}\\ &\leq 2^{m_1+m_2}e^{-\delta^2 m_1 m_2 p/(4\log\log m_1)}\\ &\leq m_2^{-\Omega_{m_1}(1)},\end{aligned}$$

since $\delta^2 m_1 p/\log\log m_1 \geq \omega^{1-2\theta}\log m_2/\log\log m_1 = \Omega_{m_1}(1)$. Analogously, if $|A| \geq m_1|B|/m_2$, the probability that there exist A, B with $e_G(A, B) \leq (1-\delta)|A||B|p$ is bounded above by

$$\begin{aligned}\sum_{b=n_2}^{m_2}\sum_{a=m_1/2}^{m_1}\binom{m_1}{a}\binom{m_2}{b}e^{-\delta^2 abp/2} &\leq \sum_{b=n_2}^{m_2}\sum_{b=m_1/2}^{m_1}\binom{m_1}{a}\binom{m_2}{b}e^{-\delta^2 n_2 m_1 p/4}\\ &\leq 2^{m_1+m_2}e^{-\delta^2 m_1 m_2 p/(4\log\log m_2)}\\ &\leq m_2^{-\Omega_{m_1}(1)},\end{aligned}$$

since $\delta^2 m_1 p = \Omega_{m_1}(\log m_2)$.

Now, consider $A \subseteq V_1$, $B \subseteq V_2$. If $|A| + m_1|B|/m_2 \leq m_1$, then it is immediate that

$$0 \leq e_G(A, B) + d_1\Big(m_1 - |A| - \frac{m_1|B|}{m_2}\Big).$$

Otherwise, we have proved that a.a.s. (m_1) $e_G(A, B) \geq (1-\delta)|A||B|p$, so that a.a.s. (m_1)

$$\begin{aligned}&e_G(A, B) + d_1\Big(m_1 - |A| - \frac{m_1|B|}{m_2}\Big)\\ &\quad\geq (1-\delta)|A||B|p + (1-\delta)m_2 p\Big(m_1 - |A| - \frac{m_1|B|}{m_2}\Big)\\ &\quad= (1-\delta)|A||B|p + (1-\delta)m_1 m_2 p - (1-\delta)|A|m_2 p - (1-\delta)m_1|B|p\\ &\quad= (1-\delta)p(|A||B| + m_1 m_2 - |A|m_1 - |B|m_2)\\ &\quad= (1-\delta)p(m_1 - |A|)(m_2 - |B|) \geq 0,\end{aligned}$$

since $|A| \leq m_1$ and $|B| \leq m_2$. The result now follows by the Ore–Reyser theorem. ■

5.3. Regular subgraphs in reduced random graphs

Finally, we need to address the issue of vertex degrees: in order to use Lemmas 4.15 and 4.17, we need our graphs to be regular, and to have large eigenvalue. Therefore, we need to show that $\mathfrak{Red}(n, l, p)$, $\mathfrak{BRed}(n, l, p)$ contain regular spanning subgraphs with large first eigenvalue.

Lemma 5.11. *Let $n \geq 2$, and let p be such that $(2n-1)^l p = \Omega_l(\log^6 (2n-1)^{l+1}) = \Omega_l(l^6)$ and $p = o_l(1)$. Let $G_1 \sim \mathfrak{Red}(n, l, p)$ and $G_2 \sim \mathfrak{BRed}(n, l, p)$. There exists $\varepsilon = \varepsilon(p) = o_l(1)$ such that for all $o_l(1) = \delta \geq \varepsilon$, a.a.s. (l) there exist spanning subgraphs $H_i \leq G_i$ such that*

(i) *H_1 is $2(1-\delta)(2n-1)^l p$-regular graph, with $\lambda_1(H_1) \geq 1 - o_l(1)$,*

(ii) *H_2 is $((1-\delta)(2n-1)^{l+1} p, (1-\delta)(2n-1)^l p)$-regular graph, with $\lambda_1(H_2) \geq 1 - 1/(2n-1) + o_l(1)$.*

Proof. The first parts of (i) and (ii), i.e., the existence of the regular subgraphs, follow from [23] and Theorem 5.10. In particular, for such a random graph G_1, and for $i = 1, \ldots, n$, the random graph $\mathfrak{R}_i$ contains a $((1-\delta)(2n-1)^l p, (1-\delta)(2n-1)^{l-1} p)$-regular spanning subgraph SP_i with probability at least

$$1 - (2n-1)^{-l\omega(l)}$$

for some $\omega = \Omega_l(1)$. Therefore, the probability that all the graphs $\mathfrak{R}_i$ contain a spanning subgraph is at least

$$(1 - (2n-1)^{-l\omega(l)})^n = 1 - o_l(1).$$

Taking $H_1 = \bigcup_i SP_i$, the result on regular subgraphs follows.

By [13, Lemma 4.5] and Lemma 4.7, $\lambda_1(H_i) = \lambda_1(G_i) + o_l(1)$, since the G_i is formed from H_i by the addition of graphs of suitably small degrees. The result follows by Lemmas 5.5 and 5.6. ∎

Similarly, we can prove the following.

Lemma 5.12. *Let $n \geq 2$, $l \geq 5$. Let p be such that $(2n-1)^l p = \Omega_n(\log^6 (2n-1)^{l+1}) = \Omega_n(\log^6(2n-1))$ and $p = o_n(1)$. Let $G_1 \sim \mathfrak{Red}(n, l, p)$ and $G_2 \sim \mathfrak{BRed}(n, l, p)$. There exists $\varepsilon = \varepsilon(p) = o_n(1)$ such that for all $o_n(1) = \delta \geq \varepsilon$, a.a.s. (n) there exist spanning subgraphs $H_i \leq G_i$ such that*

(i) *H_1 is $2(1-\delta)(2n-1)^l p$-regular graph, with $\lambda_1(H_1) \geq 1 - o_n(1)$,*

(ii) *H_2 is $((1-\delta)(2n-1)^{l+1} p, (1-\delta)(2n-1)^l p)$-regular graph, with $\lambda_1(H_2) \geq 1 - 1/(2n-1) + o_n(1)$.*

Proof. This is extremely similar to the previous lemma.

The first parts of (i) and (ii), i.e., the existence of the regular subgraphs, follow from [23] and Theorem 5.10. In particular, for such a random graph G_1, and for $i = 1, \ldots, n$, the random graph $\mathfrak{R}_i$ contains a $((1-\delta)(2n-1)^l p, (1-\delta)(2n-1)^{l-1} p)$-regular spanning subgraph SP_i with probability at least

$$1 - (2n-1)^{-l\omega(n)}$$

for some $\omega = \Omega_n(1)$. Therefore, the probability that all the graphs $\mathfrak{R}_i$ contain a spanning subgraph is at least

$$(1 - (2n-1)^{-l\omega(n)})^n = 1 - o_n(1).$$

Taking $H_1 = \bigcup_i SP_i$, the result on regular subgraphs follows.

In the case of growing n, the graphs $\mathfrak{R}(n, l, p)$ and $\mathfrak{BRed}(n, l, p)$ have a very small proportion of disallowed edges so have eigenvalues extremely close to those of an (bipartite) Erdös–Rényi random graph. The result then follows from Theorems 4.13 and 4.14. ■

6. Property (T) in random quotients of free groups

Finally, we may prove Theorems A and B. We in fact provide the full proof for Theorem B, as this is the harder of the two theorems to prove, and indicate how to alter the proof of this theorem in order to prove Theorem A. However, we first define a slightly different model of random groups.

Definition 6.1. Let $n \geq 2$, $k \geq 3$, and let $0 < p = p(n,k) < 1$. The random group model $\Gamma_p(n,k,p)$ is the model obtained as following. We let $\Gamma = \langle A_n \mid R \rangle$, where R is obtained by adding each word in $\mathcal{C}(n,k)$ with probability p.

We in fact prove the following theorem.

Theorem 6.2. *Let $n \geq 2$, and let p be such that*

$$(2n-1)^{k/3} p = \Omega_k(k^6).$$

Let $\Gamma_k \sim \Gamma_p(n,k,p)$. Then

$$\lim_{k\to\infty} \mathbb{P}(\Gamma_k \text{ has property (T)}) = 1.$$

Assuming this, we may prove Theorem B.

Proof of Theorem B. Fix $n \geq 2$ and $d > 1/3$. Choose $1/3 < d' < d$, and let

$$\Gamma'_k = \langle A_n \mid R' \rangle \sim \Gamma_p(n,k,(2n-1)^{kd'-k}).$$

It is easily seen that a.a.s. (k)

$$|R'| = (1 + o_k(1))(2n-1)^{kd'}.$$

Choose a random subset R with $R' \subseteq R \subseteq \mathcal{W}(n,k)$ and $|R| = (2n-1)^{kd}$, and let $\Gamma_k = \langle A_n \mid R \rangle$. Then $\Gamma_k \sim \Gamma(n,k,d)$, and there is a clear epimorphism $\Gamma'_k \twoheadrightarrow \Gamma_k$. Since property (T) is preserved under epimorphisms, the result follows by Theorem 6.2. ■

Let Γ be a random group in the $\Gamma_p(n,k,p)$ model. We consider the three cases.

Case $k = 0$ mod 3. Let $l_k = L_k = k/3$. We may define the graphs Σ_1, Σ_2, Σ_3, where

$$V(\Sigma_1) = V(\Sigma_2) = V(\Sigma_3) = \mathcal{W}(n,k/3),$$

and for each relator $r = r_x r_y r_z$ with $r_x, r_y, r_z \in \mathcal{W}(n,k/3)$, we add the edge (r_x, r_z^{-1}) to Σ_1, (r_y, r_x^{-1}) to Σ_2 and (r_z, r_y^{-1}) to Σ_3.

Case $k = 1 \bmod 3$. Let $l_k = (k-1)/3$ and $L_k = (k+2)/3$. Again, we may write each relator $r = r_x r_y r_z$ for $r_x, r_y \in \mathcal{W}(n,(k-1)/3)$ and $r_z \in \mathcal{W}(n,(k+2)/3)$. We again split the graph $\Delta_k(A_n \mid R)$ into Σ_1, Σ_2, Σ_3, where

$$V(\Sigma_1) = V(\Sigma_3) = \mathcal{W}\Big(n, \frac{k-1}{3}\Big) \sqcup \mathcal{W}\Big(n, \frac{k+2}{3}\Big),$$

and $V(\Sigma_2) = \mathcal{W}(n,(k-1)/3)$. For each relator $r = r_x r_y r_z$, we add the edge (r_x, r_z^{-1}) to Σ_1, (r_y, r_x^{-1}) to Σ_2, and (r_z, r_y^{-1}) to Σ_3.

Case $k = 2 \bmod 3$. Let $l_k = (k+1)/3$ and $L_k = (k-2)/3$. Again, we may write each relator $r = r_x r_y r_z$ for $r_x, r_y \in \mathcal{W}(n,(k+1)/3)$ and $r_z \in \mathcal{W}(n,(k-2)/3)$. We again split the graph $\Delta_k(A_n \mid R)$ into Σ_1, Σ_2, Σ_3, where

$$V(\Sigma_1) = V(\Sigma_3) = \mathcal{W}\Big(n, \frac{k-2}{3}\Big) \sqcup \mathcal{W}\Big(n, \frac{k+1}{3}\Big),$$

and $V(\Sigma_2) = \mathcal{W}(n,(k+1)/3)$. For each relator $r = r_x r_y r_z$, we add the edge (r_x, r_z^{-1}) to Σ_1, (r_y, r_x^{-1}) to Σ_2, and (r_z, r_y^{-1}) to Σ_3.

Next we show there are not too many double edges in the graphs Σ_i, similarly to [1].

Lemma 6.3. *Let $n \geq 2$, and let p be such that*

(i) $(2n-1)^{2k-L_k} p^3 = o_k(1)$,

(ii) $(2n-1)^{2k+l_k} p^4 = o_k(1)$.

Let $\Gamma_k \sim \Gamma_p(n,k,p)$, and let Σ_i be described as above. For $i = 1,2,3$ a.a.s. (k), there is no pair of vertices u, v with at least three edges between them in Σ_i, and the set of double edges in Σ_i forms a matching, i.e., the endpoints of the double edges are all distinct.

Note that for $1/3 < d < 5/12$, $p(d) = (2n-1)^{kd-k}$ satisfies the above conditions.

Proof. We prove this for $i = 1$. Throughout, note that $k = 2l_k + L_k$. The probability, $\mathbb{P}_3$, that there exists a pair of vertices u, v with at least three edges between u and v is bounded above by

$$\mathbb{P}_3 \leq O_k((2n-1)^{l_k+L_k}(2n-1)^{3l_k} p^3) = o_k(1).$$

The probability, $\mathbb{P}_{\text{doub}}$, that there are vertices u, v, w with double edges between u and v and u and w is bounded by

$$\begin{aligned}\mathbb{P}_{\text{doub}} &= O_k((2n-1)^{l_k}(2n-1)^{2L_k}(2n-1)^{4l_k} p^4) \\ &= O_k((2n-1)^{2k+l_k} p^4) = o_k(1).\end{aligned}$$ ■

This is sufficient to prove our main theorem.

Proof of Theorem 6.2. Let $\Gamma_k = \langle A_n \mid R\rangle \sim \Gamma_p(n,k,p)$, and consider $\Delta_k := \Delta_k(A_n \mid R)$. Since property (T) is preserved by epimorphisms, we may assume that $p \leq (2n-1)^{kd-k}$

for some $d < 4/9$: for any $1/3 < d < 4/9$, $p(n,k,d) = (2n-1)^{kd-k}$ satisfies the conditions of Lemma 6.3 and the conditions of Theorem 6.2.

As above, we may write $\Delta_k = \Sigma_1 \cup \Sigma_2 \cup \Sigma_3$. Now, after collapsing edges, we find Σ_1', Σ_3' with the marginal distribution (up to perturbing p to $(1+o_l(1))p$) of

$$\begin{cases} \mathfrak{Red}(n, k/3, (2n-1)^{k/3}p), & k = 0 \bmod 3, \\ \mathfrak{BRed}(n, (k-1)/3, (2n-1)^{(k-1)/3}p), & k = 1 \bmod 3, \\ \mathfrak{BRed}(n, (k-2)/3, (2n-1)^{(k+1)/3}p), & k = 2 \bmod 3. \end{cases}$$

Similarly, by collapsing double edges we find Σ_2' with the marginal distribution of

$$\begin{cases} \mathfrak{Red}(n, k/3, (2n-1)^{k/3}p), & k = 0 \bmod 3, \\ \mathfrak{Red}(n, (k-1)/3, (2n-1)^{(k+2)/3}p), & k = 1 \bmod 3, \\ \mathfrak{Red}(n, (k+1)/3, (2n-1)^{(k-2)/3}p), & k = 2 \bmod 3. \end{cases}$$

Furthermore, letting $\Sigma' = \Sigma_1' \cup \Sigma_2' \cup \Sigma_3'$, then as usual we can see that

$$\mu_1(D(\Sigma')^{-1/2}[A(\Delta_k) - A(\Sigma')]D(\Sigma')^{-1/2}) = o_k(1).$$

According to Lemma 5.11, there exists some $\delta = o_k(1)$ such that a.a.s. (k): Σ_2' has a $2(1-\delta)d_2$-regular spanning subgraph, Π_2, with $\lambda_1(\Pi_2) > 1 - o_k(1)$; if $k \neq 0 \bmod 3$, then Σ_1', Σ_3' contain $((1-\delta)d_1, (1-\delta)d_2)$-regular spanning subgraphs Π_1, Π_3, with $\lambda_1(\Pi_1), \lambda_1(\Pi_3) \geq 1 - 1/(2n-1) + o_k(1)$; and if $k = 0 \bmod 3$, then Σ_1', Σ_3' contain $2(1-\delta)d$-regular spanning subgraphs Π_1, Π_3, with $\lambda_1(\Pi_1), \lambda_1(\Pi_3) \geq 1 - o_k(1)$.

As $n \geq 2$, we may apply Lemmas 4.15 and 4.18 to deduce that a.a.s. (k)

$$\lambda_1(\Pi_1 \cup \Pi_2 \cup \Pi_3) > \frac{3}{4}.$$

We see that

$$\begin{aligned} \mu_1(A(\Sigma_1' \cup \Sigma_2' \cup \Sigma_3') - A(\Pi_1 \cup \Pi_2 \cup \Pi_3)) &\leq \frac{\delta + o_k(1)}{1-\delta} \|A(\Pi_1 \cup \Pi_2 \cup \Pi_3)\|_\infty \\ &= o_k(1) \|A(\Pi_1 \cup \Pi_2 \cup \Pi_3)\|_\infty. \end{aligned}$$

Hence, letting $\Pi = \Pi_1 \cup \Pi_2 \cup \Pi_3$, we see that a.a.s. (k):

$$\begin{aligned} \lambda_1(\Sigma') &= +1 - \mu_2(D^{-1/2}(\Sigma')A(\Sigma')D^{-1/2}(\Sigma')) \\ &= 1 - \mu_2(D^{-1/2}(\Sigma')[A(\Pi) + A(\Sigma') - A(\Pi)]D^{-1/2}(\Sigma')) \\ &\geq 1 - \mu_2(D^{-1/2}(\Sigma')A(\Pi)D^{-1/2}(\Sigma')) \\ &\quad - \mu_1(D^{-1/2}(\Sigma')(A(\Sigma') - A(\Pi))D^{-1/2}(\Sigma')) \\ &= 1 - \left(\frac{1}{1-\delta} + o_k(1)\right)\mu_2(D^{-1/2}(\Pi)A(\Pi)D^{-1/2}(\Pi)) \\ &\quad - \left(\frac{1}{1-\delta} + o_k(1)\right)\mu_1(D^{-1/2}(\Pi)(A(\Sigma') - A(\Pi))D^{-1/2}(\Pi)) \\ &\geq 1 - \frac{1}{4}\left(\frac{1}{1-\delta} + o_k(1)\right) - \left(\frac{1}{1-\delta} + o_k(1)\right)\frac{\delta}{1-\delta} = \frac{3[1 + o_k(1)]}{4}. \end{aligned}$$

Since $\lambda_1(\Delta_k) = \lambda_1(\Sigma') + o_k(1)$, it follows by Lemma 3.7 that a.a.s. (k) Γ_k has property (T). However, as property (T) is preserved under epimorphisms, it follows immediately that a.a.s. (k) a random group $\Gamma_k \sim \Gamma_p(n,k,p)$ has property (T) for any p with

$$(2n-1)^{2k/3} p = \Omega_k(k). \qquad \blacksquare$$

To prove Theorem A, we wish to prove the corresponding result for the k-angular model: the approach to achieve this is similar.

Lemma 6.4. *Let $n \geq 2$, and let p be such that there exists $M \geq 1$ with*

(i) $(2n-1)^{(M+1)l_k+L_k} p^M = o_n(1)$,

(ii) $(2n-1)^{2l_k+ML_k} p^M = o_n(1)$,

(iii) $(2n-1)^{(2M+1)l_k+ML_k} p^{2M} = o_n(1)$,

(iv) $(2n-1)^{3Ml_k+L_k} p^{2M} = o_n(1)$,

(v) $(2n-1)^{(M+1)l_k+2ML_k} p^{2M} = o_n(1)$.

Let $\Gamma_k \sim \Gamma_p(n,k,p)$, and let Σ_i be described as above. For $i = 1,2,3$ a.a.s. (n) in Σ_i, there is no pair of vertices u, v with at least M edges between them, and no vertex is connected to more than M other vertices by double edges.

Proof. We first prove this for $i = 1,3$. Throughout, note that $k = 2l_k + L_k$. The probability, $\mathbb{P}_{M,1}$, that there exists a pair of vertices u, v with at least M edges between u and v is bounded above by

$$\mathbb{P}_{M,1} \leq O_n((2n-1)^{l_k+L_k}(2n-1)^{Ml_k} p^M) = O_n((2n-1)^{(M+1)l_k+L_k} p^M) = o_n(1).$$

The probability, $\mathbb{P}_{\text{doub},1}$, that there are vertices $u \in V_1$ and $v_1,\dots,v_M \in V_2$ with double edges between u and each v_i is bounded by

$$\begin{aligned}\mathbb{P}_{\text{doub},1} &= O_n((2n-1)^{l_k}(2n-1)^{ML_k}(2n-1)^{2Ml_k} p^{2M})\\ &= O_n((2n-1)^{(2M+1)l_k+ML_k} p^{2M}) = o_n(1).\end{aligned}$$

The probability, $\mathbb{P}'_{\text{doub},1}$, that there are vertices $u \in V_2$ and $v_1,\dots,v_M \in V_1$ with double edges between u and each v_i is bounded by

$$\begin{aligned}\mathbb{P}'_{\text{doub},1} &= O_n((2n-1)^{Ml_k}(2n-1)^{L_k}(2n-1)^{2Ml_k} p^{2M})\\ &= O_n((2n-1)^{L_k+3Ml_k} p^{2M}) = o_n(1).\end{aligned}$$

Let us now switch to Σ_2. Then the probability, $\mathbb{P}_{M,2}$, that there exists a pair of vertices u, v with at least M edges between u and v is bounded above by

$$\mathbb{P}_{M,2} \leq O_n((2n-1)^{2l_k}(2n-1)^{ML_k} p^M) = o_n(1).$$

Finally, the probability, $\mathbb{P}_{\text{doub},2}$, that there are vertices u and $v_1,\dots,v_M$ with double edges between u and each v_i is bounded by

$$\mathbb{P}_{\text{doub},2} = O_n((2n-1)^{(M+1)l_k}(2n-1)^{2ML_k} p^{2M}) = o_n(1). \qquad \blacksquare$$

Remark 6.5. Let $d > 0$ and $p_d = (2n-1)^{kd-k}$. Then p_d satisfies the conditions above for some M if, respectively,

(i) $l_k + kd - k < 0$, so that $d < (l_k + L_k)/k$,

(ii) $L_k + kd - k < 0$, so that $d < 2l_k/k$,

(iii) $2l_k + L_k + 2kd - 2k < 0$, i.e., $d < 1/2$ since $2l_k + L_k = k$,

(iv) $3l_k + 2kd - 2k < 0$, so that $d < (k + L_k - l_k)/2k$,

(v) $l_k + 2L_k + 2kd - 2k < 0$, so that $d < (k + l_k - L_k)/2k$.

This reduces to $d < (k-1)/2k$. For $k \geq 8$, this is satisfied whenever $d < 7/16$. For $k \geq 8$, we have $d_k \leq 5/12 < 7/16$, and so we can find d satisfying the requirements of the above lemma and Theorem 6.6.

We can now observe the following.

Theorem 6.6. *Let $n \geq 2$, $k \geq 8$. Let p be such that*

$$(2n-1)^{2l_k} p = \Omega_n(\log(2n-1)^{L_k}), \quad \text{and} \quad (2n-1)^{l_k+L_k} p = \Omega_n(\log(2n-1)^{l_k}).$$

Let $\Gamma_k \sim \Gamma_p(n,k,p)$. Then $\lim_{n\to\infty} \mathbb{P}(\Gamma_k$ has property (T)$) = 1$.

We remark that for $d > d_k$ and $p = (2n-1)^{kd-k}$ the above is satisfied.

Outline of proof of Theorem 6.6. This follows similarly to the proof of Theorem 6.2. We may assume that p also satisfies the requirements of Lemma 6.4 for some M. The main replacement is in the fact that the Σ_i have a very small proportion of disallowed edges so can be treated as having the marginal distribution of an (bipartite) Erdös–Rényi random graph. We then find regular spanning subgraphs using Lemma 5.12, and repeat the above argument, using Lemma 6.4 in place of Lemma 6.3. This guarantees us that by collapsing double edges, we remove at most M^2 edges adjacent to each vertex, and the argument follows similarly. ■

We then apply the above to prove Theorem A, as in the case for the density model.

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