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Virtually free groups are stable in permutations

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Abstract. We prove that finitely generated virtually free groups are stable in permutations. As an application, we show that almost-periodic almost-automorphisms of labelled graphs are close to periodic automorphisms.

1. Introduction

A finitely generated group G is called stable in permutations (in short P-stable) if every almost action of G on a finite set is close to an honest action (see Section 2 for definitions). As a group property, this was first defined by Arzhantseva and Păunescu [1]. For the ubiquitous class of sofic groups, the property of P-stability can be seen as a stronger form of residual finiteness [1]. Our main result is the following.

Theorem A. *Every finitely generated virtually free group is P-stable.*

It is trivially true that free groups are P-stable. But while residual finiteness is preserved under passing to finite index subgroups (or rather to any subgroup), this fact is not clear in general for P-stability.

To the best of our knowledge, Theorem A gives the first examples of P-stable groups which are not free products of P-stable amenable groups. Note that while fundamental groups of closed orientable surfaces are known to be flexibly P-stable [5], it is not clear if these groups are P-stable in the strict sense.

As a special case of Theorem A, we answer the following question of Lubotzky.

Corollary 1.1. *The modular group* $\mathrm{SL}_2(\mathbb{Z})$ *is P-stable.*

Interestingly, P-stability is not, generally speaking, preserved under direct products, for example, the groups $F_2 \times \mathbb{Z}$ are not P-stable [4]. This phenomenon is to be contrasted with the fact that the product groups $F_2 \times (\mathbb{Z}/n\mathbb{Z})$ are P-stable for all $n \in \mathbb{N}$, as follows from Theorem A. As a consequence of the P-stability of these groups, we are able to deduce the following assertion.

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Corollary B. *Fix some $d, n \in \mathbb{N}$. Let F_d be the free group of rank d and $\mathcal{G}_d$ be the family of finite labelled Schreier graphs of F_d. Then for every graph $\Gamma \in \mathcal{G}_d$ and every δ-almost automorphism α of Γ of δ-almost order n, there is a graph $\Gamma' \in \mathcal{G}_d$ on the same vertex set as Γ and $O(\delta)$-close to Γ and an automorphism α' of Γ' which is $O(\delta)$-close to α and has order n.*

More details and a precise statement of Corollary B can be found in Section 8 below.

Stable epimorphisms

Stallings theorem on ends of groups [8, 9] implies that a finitely generated group G is virtually free if and only if G is isomorphic to the fundamental group $\pi_1(\mathcal{G}, T)$ of a finite graph of groups $\mathcal{G}$ with finite vertex groups with respect to some maximal spanning tree T (see Section 3 for the definition of $\pi_1(\mathcal{G}, T)$).

Naturally associated to the graph of groups $\mathcal{G}$ and the maximal spanning tree T there is another group $\overline{\pi}_1(\mathcal{G}, T)$ admitting a quotient map $\overline{\pi}_1(\mathcal{G}, T) \to \pi_1(\mathcal{G}, T)$. This group is isomorphic to the free product of the vertex groups of $\mathcal{G}$ with the topological fundamental group of the underlying graph of $\mathcal{G}$. As finite groups are P-stable [3], it follows immediately that the group $\overline{\pi}_1(\mathcal{G}, T)$ is P-stable.

Motivated by this, we introduce a relative notion of P-stable epimorphisms, see Definition 2.1. In particular, a finitely generated group G is P-stable in the usual sense if and only if the natural epimorphism from the free group in the generators of G onto the group G is P-stable. Theorem A is thereby reduced to the following statement, to which the major part of this work is dedicated.

Theorem 1.2. *The epimorphism $\overline{\pi}_1(\mathcal{G}, T) \to \pi_1(\mathcal{G}, T)$ is P-stable.*

A detailed outline of the proof of Theorem 1.2 can be found in Section 3 below, after the necessary definitions and notations are set in place.

2. P-stable epimorphisms

Let X be a finite set. Consider the normalized Hamming distance d_X on the symmetric group $\mathrm{Sym}(X)$ given by

$$d_X(\sigma_1, \sigma_2) = \frac{1}{|X|} |\{x \in X : \sigma_1(x) \neq \sigma_2(x)\}|$$

for all pairs $\sigma_1, \sigma_2 \in \mathrm{Sym}(X)$. Note that the metric d_X is bi-invariant. Let $\overline{G}$ be a group with finite generating set S. Define a metric $d_{X,S}$ on the set $\mathrm{Hom}(\overline{G}, \mathrm{Sym}(X))$ of all group homomorphisms $\rho: \overline{G} \to \mathrm{Sym}(X)$ by

$$d_{X,S}(\rho, \rho') = \sum_{s \in S} d_X(\rho(s), \rho(s'))$$

for each pair $\rho, \rho' \in \mathrm{Hom}(\overline{G}, \mathrm{Sym}(X))$.

Let $N \lhd \overline{G}$ be a normal subgroup normally generated by some finite subset $R \subset \overline{G}$. Denote $G = \overline{G}/N$. We say that an action $\rho\colon \overline{G} \to \mathrm{Sym}(X)$ is a *δ-almost G-action* if

$$\sum_{r \in R} d_X(\rho(r), \mathrm{id}) \leq \delta.$$

This terminology is justified by the observation that ρ is an honest G-action if and only if it is a δ-almost G-action with respect to $\delta = 0$. Note that, strictly speaking, this notion depends on fixing the normal generating set R.

Definition 2.1. The epimorphism $\phi\colon \overline{G} \to G$ is *P-stable* if for every $\varepsilon > 0$, there is a $\delta > 0$ such that for every δ-almost G-action $\rho\colon \overline{G} \to \mathrm{Sym}(X)$, there is a G-action $\rho'\colon G \to \mathrm{Sym}(X)$ with $d_{X,S}(\rho, \rho' \circ \phi) < \varepsilon$.

Lemma 2.2. *The P-stability of the epimorphism $\phi\colon \overline{G} \to G$ is a well-defined notion (i.e., it is independent of the choices of the finite sets S and R).*

Proof. It is easy to see that if S_1 and S_2 are two finite generating sets for the group G, then the resulting metrics d_{X,S_1} and d_{X,S_2} on the set $\mathrm{Hom}(\overline{G}, \mathrm{Sym}(X))$ are bi-Lipschitz equivalent. A similar argument, taking into account the bi-invariance of the normalized Hamming metric d_X, shows that if R_1 and R_2 are two finite normal generating sets for the subgroup $N \lhd \overline{G}$, then there is a constant

$$C = C(R_1, C_2) > 1$$

such that

$$C^{-1} \sum_{r \in R_2} d_X(\rho(r), \mathrm{id}) \leq \sum_{r \in R_1} d_X(\rho(r), \mathrm{id}) \leq C \sum_{r \in R_2} d_X(\rho(r), \mathrm{id}).$$

The conclusion follows from these observations. ∎

Let H be any group admitting a finite generating set S and $F(S)$ be the free group in the generators S. Observe that the natural homomorphism $F(S) \to H$ is P-stable if and only if the group H is P-stable in the usual sense.

Remark 2.3. Every split epimorphism is P-stable.

The next lemma follows immediately from Definition 2.1.

Lemma 2.4. *Let $\overline{\overline{G}} \overset{\phi}{\twoheadrightarrow} \overline{G} \overset{\psi}{\twoheadrightarrow} G$ be a sequence of epimorphisms with normally finitely generated kernels. If ϕ and ψ are P-stable, then $\psi \circ \phi$ is P-stable.*

We have occasion to use Lemma 2.4 only in the following special form: if the group $\overline{G}$ is P-stable and $\phi\colon \overline{G} \twoheadrightarrow G$ is a P-stable epimorphism, then the group G is P-stable.

Remark 2.5. It seems an interesting problem to look for other non-trivial instances of P-stable epimorphisms.

3. The fundamental group of a graph of groups

We recall the definition of the fundamental group of a graph of groups and, in particular, list its defining relations. This is followed by a detailed sketch of proof for our Theorem A as well as for the "relative" Theorem 1.2. Lastly, we introduce some useful asymptotic notations.

Graphs of groups

We use Serre's notation for graphs [6]. In this notation, a graph Γ consists of a set of vertices $V(\Gamma)$ and a set of edges $E(\Gamma)$. Each edge $e \in E(\Gamma)$ has an origin $o(e) \in V(\Gamma)$ and a terminus $t(e) \in V(\Gamma)$. Moreover, each edge $e \in E(\Gamma)$ admits a distinct opposite edge $\overline{e} \in E(\Gamma)$ that satisfies $\overline{\overline{e}} = e$, $o(\overline{e}) = t(e)$ and $t(\overline{e}) = o(e)$. Every pair of "oriented" edges $\{e, \overline{e}\} \subset E(\Gamma)$ represents a single "geometric" edge. An orientation of the graph Γ is a subset $\vec{E}(\Gamma) \subset E(\Gamma)$ containing exactly a single edge from each pair $\{e, \overline{e}\}$.

Definition 3.1. A *graph of groups* $\mathcal{G}$ is

$$\mathcal{G} = (\Gamma, \{G_v\}_{v \in V(\Gamma)}, \{G_e\}_{e \in E(\Gamma)}, \{i_e \colon G_e \to G_{t(e)}\}_{e \in E(\Gamma)}),$$

where Γ is a connected graph, G_v is a vertex group for all $v \in V(\Gamma)$, G_e is an edge group for all edges $e \in E(\Gamma)$ with $G_e = G_{\overline{e}}$ and $i_e \colon G_e \to G_{t(e)}$ are injective homomorphisms.

Let $\mathcal{G}$ be a graph of groups. Fix an orientation $\vec{E}(\Gamma)$ and a maximal spanning tree $T \subset \Gamma$. Consider the group $\overline{\pi}_1(\mathcal{G}, T)$ defined as the free product

$$\overline{\pi}_1(\mathcal{G}, T) = *_{v \in V(\Gamma)} G_v * F(\{s_e\}_{e \in \vec{E}(\Gamma)}),$$

where $F(\cdot)$ denotes the free group over the given basis. It will be convenient to consider the following generating set:

$$S_{\mathcal{G}} = \bigcup_{v \in V(\Gamma)} G_v \cup \{s_e\}_{e \in \vec{E}(\Gamma)}.$$

Definition 3.2. The *fundamental group* $\pi_1(\mathcal{G}, T)$ of the graph of groups $\mathcal{G}$ with respect to the subtree T is the quotient of the free product $\overline{\pi}_1(\mathcal{G}, T)$ by the normal subgroup generated by the relations

$$R_{\mathcal{G}} = \begin{cases} s_e = 1 & \forall e \in \vec{E}(T), \\ s_e^{-1} i_e(g_e) s_e = i_{\overline{e}}(g_e) & \forall e \in \vec{E}(\Gamma),\ g_e \in G_e. \end{cases}$$

Remark 3.3. The fundamental group $\pi_1(\mathcal{G}, T)$ as well as the group $\overline{\pi}_1(\mathcal{G}, T)$ are independent of the choice of maximal spanning tree T up to isomorphism [6, Chapitre I, §5].

For the remainder of the paper, we will assume that $\mathcal{G}$ is a finite graph of groups, with finite vertex groups. In particular, $S_{\mathcal{G}}$ is a finite generating set for the group $\overline{\pi}_1(\mathcal{G}, T)$.

Outline of the proof and of the paper

Note that the group $\overline{\pi}_1(\mathcal{G}, T)$ is a free product of finite groups and of a free group. As such $\overline{\pi}_1(\mathcal{G}, T)$ is easily seen to be P-stable [3]. In light of Lemma 2.4 and the remarks following it, our main result Theorem A follows immediately from Theorem 1.2 of the introduction. In other words, it suffices to show that the epimorphism $\overline{\pi}_1(\mathcal{G}, T) \to \pi_1(\mathcal{G}, T)$ is P-stable.

Towards this goal, consider some δ-almost $\pi_1(\mathcal{G}, T)$-action $\rho\colon \overline{\pi}_1(\mathcal{G}, T) \to \mathrm{Sym}(X)$. In particular, ρ restricts to actions of the finite vertex groups G_v. For ρ to factorize through the fundamental group $\pi_1(\mathcal{G}, T)$, it is necessary that for every edge $e \in \vec{E}(\Gamma)$ the two actions $\rho \circ i_e$ and $\rho \circ i_{\bar{e}}$ of the edge group G_e are isomorphic.

It is clear that the isomorphism type of an action of a finite group on a finite set is characterized by the number of occurrences of each of its finitely many transitive action types. In Section 4, we show how to represent this data using a vector in some canonical $\mathbb{Z}$-module associated to the group. The restriction maps $\rho|_{G_v} \mapsto (\rho \circ i_e)|_{G_e}$ define a $\mathbb{Z}$-linear map $\mathbf{d}_{\mathcal{G}}$ between the respective $\mathbb{Z}$-modules. The above mentioned condition (that the two actions $\rho \circ i_e$ and $\rho \circ i_{\bar{e}}$ of the edge group G_e are isomorphic) can be described as the kernel of this $\mathbb{Z}$-linear map $\mathbf{d}_{\mathcal{G}}$. Lastly, the fact that ρ is a δ-almost action of $\pi_1(\mathcal{G}, T)$ translates to having a small image under the map $\mathbf{d}_{\mathcal{G}}$.

In Section 5, we show that "$\mathbb{Z}$-linear maps are stable" in the following sense: an exact $\mathbb{Z}$-solution to a linear system of equations and inequalities can be found nearby a δ-almost solution. This is applied to the linear map $\mathbf{d}_{\mathcal{G}}$. That is, near the almost solution corresponding to ρ, there is an exact $\mathbb{Z}$-solution. Such a solution represents a collection of isomorphism types of actions, one for each vertex group of $\mathcal{G}$, that satisfies the necessary condition to be turned into an action of $\pi_1(\mathcal{G}, T)$ and which is statistically close to the collection corresponding to ρ.

Finally, in Section 6, we show how given a δ-almost action ρ, and a nearby exact $\mathbb{Z}$-solution to the corresponding linear system of equations, one can find a nearby action ρ' factoring via $\pi_1(\mathcal{G}, T)$.

We will make repeated use of the finiteness of vertex and edge groups via the following observation.

Observation 3.4. Let G be a finite group. If G acts on a finite set X and $Y \subset X$, then there exists a G-invariant subset $Y' \subseteq Y$ such that

$$|X - Y'| \leq |G||X - Y|.$$

Notations

We will need to consider inequalities involving quantities depending on the graph of groups $\mathcal{G}$ in question (such as the number of vertices or edges, the sizes of the vertex groups G_v, etc.). To avoid cumbersome formulas it would be convenient to introduce the following asymptotic notation.

We write $A \prec_{\bullet} B$ if there exists a constant $c = c(\bullet)$ such that $A \leq cB$. We omit the subscript when it is clear from the context.

4. Set of actions on finite sets

Let G be any group. Let $\mathrm{Acts}(G)$ denote the set of all actions of the group G on finite sets considered up to isomorphism. Similarly, let $\mathrm{Trans}(G)$ be the set of transitive actions of G on finite sets considered up to isomorphism.

Every action $\rho: G \to \mathrm{Sym}(X)$ on some finite set X can be decomposed into a disjoint union of its finitely many orbits $O_1, \dots, O_n \subseteq X$. The restriction

$$\rho \restriction_{O_i}: G \to \mathrm{Sym}(O_i)$$

of ρ to each orbit O_i is transitive for all $i = 1, \dots, n$. The isomorphism class of the action ρ is determined by counting the isomorphism classes of its restricted actions $\rho \restriction_{O_i}$ with multiplicity.

This observation enables us to identify the set of actions $\mathrm{Acts}(G)$ with a non-negative cone in the free $\mathbb{Z}$-module Λ_G with basis $\mathrm{Trans}(G)$, namely

$$\Lambda_G = \bigoplus_{\rho \in \mathrm{Trans}(G)} \mathbb{Z}\rho.$$

More precisely, given an action $\rho \in \mathrm{Acts}(G)$, we define

$$\rho^\sharp = \sum_{O \in G \backslash X} \rho \restriction_O \in \Lambda_G.$$

The correspondence $\rho \mapsto \rho^\sharp$ is injective and its image $\mathrm{Acts}(G)^\sharp$ in Λ_G is the non-negative cone

$$\Lambda_G^+ := \{(\lambda_\rho)_{\rho \in \mathrm{Trans}(G)} : \lambda_\rho \geq 0\}.$$

We observe that the correspondence $\rho \to \rho^\sharp$ is additive in the following sense: any two actions $\rho_1, \rho_2 \in \mathrm{Acts}(G)$ with $\rho_i: G \to \mathrm{Sym}(X_i)$ for $i \in \{1, 2\}$ satisfy

$$(\rho_1 \coprod \rho_2)^\sharp = \rho_1^\sharp + \rho_2^\sharp,$$

where $\rho_1 \coprod \rho_2 : G \to \mathrm{Sym}(X_1 \coprod X_2)$ is the disjoint union of ρ_1 and ρ_2.

We find it convenient to introduce a norm $\|\cdot\|_G$ on the $\mathbb{Z}$-module Λ_G by

$$\|\lambda\|_G = \sum_{\substack{\rho \in \mathrm{Trans}(G) \\ \rho: G \to \mathrm{Sym}(X_\rho)}} |\lambda_\rho| \cdot |X_\rho| \quad \forall \lambda = (\lambda_\rho) \in \Lambda_G.$$

This norm is chosen in such a way that every action $\rho \in \mathrm{Acts}(G)$ with $\rho: G \to \mathrm{Sym}(X)$ satisfies $\|\rho^\sharp\|_G = |X|$.

Remark 4.1. The module Λ_G can be equipped with a multiplication given by $\rho \otimes \rho'$. The resulting ring is known as the Burnside ring of the group G [2, 7]. However, we will not make use of the Burnside ring structure in the rest of the paper.

Pullback on set of actions

Let H be any group admitting a homomorphism $i\colon H \to G$. There is a pullback map i^* on the corresponding sets of isomorphism classes of actions on finite sets given by

$$i^*\colon \mathrm{Acts}(G) \to \mathrm{Acts}(H), \quad i^*\rho = \rho \circ i \quad \forall \rho \in \mathrm{Acts}(G).$$

Allowing for a slight abuse of notation, we also let i^* denote the resulting $\mathbb{Z}$-linear map $i^*\colon \Lambda_G \to \Lambda_H$ defined in terms of the basis by

$$i^*(\rho^\sharp) = (\rho \circ i)^\sharp, \quad \rho \in \mathrm{Trans}(G).$$

Observation 4.2. Let $\phi\colon H \to \mathrm{Sym}(X)$ be a group action such that $\phi^\sharp = i^*(\lambda)$ for some $\lambda \in \Lambda_G$. Then there exists a group action $\rho\colon G \to \mathrm{Sym}(X)$ satisfying $\rho^\sharp = \lambda$ and $\rho \circ i = \phi$.

Set of actions for a graph of groups

We extend notions introduced above to the setting of graphs of groups. Recall that

$$\mathcal{G} = (\Gamma, \{G_v\}_{v \in V(\Gamma)}, \{G_e\}_{e \in E(\Gamma)}, \{i_e\colon G_e \to G_{t(e)}\}_{e \in E(\Gamma)})$$

is a finite graph of groups with finite vertex groups. We define the $\mathbb{Z}$-modules

$$\Lambda_V = \bigoplus_{v \in V(\Gamma)} \Lambda_{G_v} \quad \text{and} \quad \Lambda_E = \bigoplus_{e \in \vec{E}(\Gamma)} \Lambda_{G_e}$$

and the respective positive cones

$$\Lambda_V^+ = \bigoplus_{v \in V(\Gamma)} \Lambda_{G_v}^+ \quad \text{and} \quad \Lambda_E^+ = \bigoplus_{e \in \vec{E}(\Gamma)} \Lambda_{G_e}^+.$$

It will be convenient to consider the $\mathbb{Z}$-modules with the norms

$$\|\cdot\|_V = \frac{1}{|V(\Gamma)|} \sum_{v \in V(\Gamma)} \|\cdot\|_{G_v}, \quad \|\cdot\|_E = \frac{1}{|\vec{E}(\Gamma)|} \sum_{e \in \vec{E}(\Gamma)} \|\cdot\|_{G_e},$$

where $\|\cdot\|_{G_v}$ and $\|\cdot\|_{G_e}$ are the norms defined on the $\mathbb{Z}$-modules Λ_{G_v} and Λ_{G_e} as above.

Let $\mathbf{d}_{\mathcal{G}}\colon \Lambda_V \to \Lambda_E$ be the $\mathbb{Z}$-linear map defined on each direct summand Λ_{G_v} of the $\mathbb{Z}$-module Λ_V by

$$(\mathbf{d}_{\mathcal{G}})|_{\Lambda_{G_v}} = \sum_{e\colon t(e)=v} i_e^* - \sum_{e\colon o(e)=v} i_{\bar{e}}^*.$$

In other words, the image of the vector $\lambda = (\lambda_v)_v \in \Lambda_V$ under the $\mathbb{Z}$-linear map $\mathbf{d}_{\mathcal{G}}$ in each coordinate $e \in \vec{E}(\Gamma)$ is given by

$$(\mathbf{d}_{\mathcal{G}}(\lambda))_e = i_e^*(\lambda_{t(e)}) - i_{\bar{e}}^*(\lambda_{o(e)}).$$

Actions of the group $\overline{\pi}_1(\mathcal{G}, T)$ on finite sets

Let X be a fixed finite set. Given an action $\rho\colon \overline{\pi}_1(\mathcal{G}, T) \to \mathrm{Sym}(X)$, we denote (abusing our previous notations)

$$\rho^\sharp \in \Lambda_V, \quad (\rho^\sharp)_v = (\rho|_{G_v})^\sharp \quad \forall v \in V.$$

Note that $\|\rho^\sharp\|_V = |X|$. Moreover, the vector $\rho^\sharp$ depends only on the restrictions of the action ρ to the vertex groups G_v's but not to the free factor $F(\{s_e\})$.

Proposition 4.3. *If the action* $\rho\colon \overline{\pi}_1(\mathcal{G}, T) \to \mathrm{Sym}(X)$ *factors through* $\pi_1(\mathcal{G}, T)$*, then* $\rho^\sharp \in \ker \mathbf{d}_{\mathcal{G}}$.

Proof. The Λ_{G_e}-coordinate of the image of the vector $\rho^\sharp$ under the $\mathbb{Z}$-linear map $\mathbf{d}_{\mathcal{G}}$ for any fixed oriented edge $e \in \vec{E}(\Gamma)$ is given by

$$(\mathbf{d}_{\mathcal{G}}(\rho^\sharp))_e = i_e^*((\rho|_{G_{t(e)}})^\sharp) - i_{\bar{e}}^*((\rho|_{G_{o(e)}})^\sharp) = (\rho \circ i_e)^\sharp - (\rho \circ i_{\bar{e}})^\sharp \in \Lambda_{G_e}.$$

The two actions $\rho \circ i_e$ and $\rho \circ i_{\bar{e}}$ of the edge group G_e on the finite set X are conjugate via the permutation $\rho(s_e)$. Therefore, $(\rho \circ i_e)^\sharp = (\rho \circ i_{\bar{e}})^\sharp$ so that the Λ_{G_e}-coordinate in question vanishes. This concludes the proof as the oriented edge $e \in \vec{E}(\Gamma)$ was arbitrary. ∎

We remark that the converse of Proposition 4.3 is also true, in the sense that if a vector $\lambda \in \Lambda_V^+$ is in $\ker \mathbf{d}_{\mathcal{G}}$, then there exists a finite set Y with $\|\lambda\|_V = |Y|$ and some action $\rho\colon \pi_1(\mathcal{G}, T) \to \mathrm{Sym}(Y)$ such that $\rho^\sharp = \lambda$. We will need a much sharper version of this fact proved in Proposition 6.1 below.

Proposition 4.4. *Let* $\rho\colon \overline{\pi}_1(\mathcal{G}, T) \to \mathrm{Sym}(X)$ *be an action. If* ρ *is a* δ*-almost* $\pi_1(\mathcal{G}, T)$*-action, then*

$$\|\mathbf{d}_{\mathcal{G}}(\rho^\sharp)\|_E \prec_{\mathcal{G}} \delta \|\rho^\sharp\|_V.$$

Proof. Fix an oriented edge $e \in \vec{E}(\Gamma)$ with $t(e) = u$ and $o(e) = v$. For each group element $g \in G_e$, consider the subset

$$X_g = \{x \in X : \rho(s_e^{-1} i_e(g) s_e)(x) = \rho(i_{\bar{e}}(g))(x)\}.$$

The assumption that ρ is a δ-almost $\pi_1(\mathcal{G}, T)$-action implies that $|X_g| \geq (1 - \delta)|X|$. Denote $X_e = \bigcap_{g \in G_e} X_g$ so that $|X_e| \geq (1 - \delta|G_e|)|X|$ and

$$\rho(s_e^{-1} i_e(g) s_e)(x) = \rho(i_{\bar{e}}(g))(x) \quad \forall x \in X_e,\ g \in G_e.$$

According to Observation 3.4, there is some $i_{\bar{e}}(G_e)$-invariant subset $Y_e \subset X_e$ satisfying $|Y_e| \geq (1 - \delta|G_e|^2)|X|$. Note that the set $\rho(s_e)(Y_e)$ is $i_e(G_e)$-invariant. Moreover, the two actions $(\rho \circ i_e) \restriction_{\rho(s_e)(Y_e)}$ and $(\rho \circ i_{\bar{e}}) \restriction_{Y_e}$ of the group G_e are isomorphic (via conjugation by the permutation $\rho(s_e)$).

To simplify our notations, let $\rho_u = \rho|_{G_u}$ and $\rho_v = \rho|_{G_v}$ for the remainder of this proof. The previous paragraph implies that

$$((\rho_u \circ i_e) \restriction_{\rho(s_e)(Y_e)})^\sharp = ((\rho_v \circ i_{\bar{e}}) \restriction_{Y_e})^\sharp.$$

The norm of the coordinate of the vector $\mathbf{d}_{\mathcal{G}}(\rho^\sharp)$ corresponding to the edge e is given by

$$\begin{aligned}
\|(\mathbf{d}_{\mathcal{G}}(\rho^\sharp))_e\|_{G_e} &= \|i_e^*(\rho_u^\sharp) - i_{\bar{e}}^*(\rho_v^\sharp)\|_{G_e} \\
&= \|((\rho_u \circ i_e) \restriction_{\rho(s_e)(Y_e)})^\sharp + ((\rho_u \circ i_e) \restriction_{X-\rho(s_e)(Y_e)})^\sharp \\
&\quad - ((\rho_v \circ i_{\bar{e}}) \restriction_{Y_e})^\sharp - ((\rho_v \circ i_{\bar{e}}) \restriction_{X-Y_e})^\sharp\|_{G_e} \\
&\le \|i_e^*(\rho_u \restriction_{X-\rho(s_e)(Y_e)})^\sharp)\|_{G_e} + \|i_{\bar{e}}^*(\rho_v \restriction_{X-Y_e})^\sharp)\|_{G_e} \\
&= |X - s_e(Y_e)| + |X - Y_e| \le 2\delta|G_e|^2|X|.
\end{aligned}$$

Averaging the above estimate over all oriented edges $e \in \vec{E}(\Gamma)$ gives

$$\|\mathbf{d}_{\mathcal{G}}(\rho^\sharp)\|_E \le c\delta|X| = c\delta\|\rho^\sharp\|_V$$

with respect to the constant $c = 2\max_{e \in \vec{E}(\Gamma)} |G_e|^2$. This constant depends only on the graph of groups $\mathcal{G}$. ∎

5. Linear algebra and cones

This section is, formally speaking, independent of the rest of the paper. Its goal is to show that "$\mathbb{Z}$-linear maps are stable", in the sense that an approximate solution to a system of linear equations and inequalities must be close to an exact $\mathbb{Z}$-solution (see Lemma 5.3 below for a precise statement).

Let Λ_1 and Λ_2 be a pair of finitely generated free $\mathbb{Z}$-modules. Let $\mathbf{d}\colon \Lambda_1 \to \Lambda_2$ be a $\mathbb{Z}$-linear map.

Let $V_i = \Lambda_i \otimes \mathbb{R}$ be the $\mathbb{R}$-vector spaces obtained by extending scalars from Λ_i and $\|\cdot\|_i$ be norms on V_i for $i = 1, 2$. By abuse of notation, we continue using $\mathbf{d}\colon V_1 \to V_2$ to denote the $\mathbb{R}$-linear extension of $\mathbf{d}\colon \Lambda_1 \to \Lambda_2$. We will make essential use of the fact that $\mathbf{d}\colon V_1 \to V_2$ is defined over $\mathbb{Q}$. Denote $K = \ker \mathbf{d}$ so that K is a $\mathbb{Q}$-subspace of the $\mathbb{R}$-vector space V_1.

Assume that $C \subset V_1$ is a closed positive cone[1] defined by finitely many inequalities over $\mathbb{Q}$ and satisfying $\mathrm{Span}(C) = V_1$. Denote $\Lambda_1^+ = C \cap \Lambda_1$ so that the subset Λ_1^+ is closed under addition.

Lemma 5.1. *For all $v \in C$, there exists $v'' \in C \cap K$ such that $\|v - v''\|_1 \prec \|\mathbf{d}v\|_2$.*

[1] A *positive cone* is a subset C of a real vector space satisfying $C \cap -C = \{0\}$, $C + C \subset C$ and $\lambda C \subset C$ for any $\lambda \ge 0$.

We point out that the intersection $C \cap K$ is non-empty for it contains the zero vector $0 \in V_1$. Lemma 5.1 does not require the assumption that the subspace K, the linear map $\mathbf{d}$ and the positive cone C are all defined over $\mathbb{Q}$. We do need however the assumption that C is defined by finitely many inequalities.

Proof of Lemma 5.1. We argue by induction on the $\mathbb{R}$-dimension of V_1. The base case where $\dim_{\mathbb{R}} V_1 = 0$ is trivial.

Consider the $\mathbb{R}$-subspace $\mathbf{d}(V_1)$ of the $\mathbb{R}$-vector space equipped with two different norms, namely the restriction of norm $\|\cdot\|_2$ coming from V_2, and the quotient norm $\|\cdot\|_1'$ defined by

$$\|\mathbf{d}v\|_1' := \inf_{w \in K} \|v - w\|_1 \quad \forall v \in V_1.$$

Since any two norms on a finite-dimensional $\mathbb{R}$-vector space are bi-Lipschitz equivalent, there is some constant $c > 0$ such that $\|\mathbf{d}v\|_1' \leq c\|\mathbf{d}v\|_2$ for all $v \in V_1$.

Fix some vector $v \in C$. The infimum appearing in the definition of the quotient norm $\|\mathbf{d}v\|_1'$ is attained at some vector $w \in K$, hence

$$\|\mathbf{d}v\|_1' = \|v - w\|_1 \leq c\|\mathbf{d}v\|_2.$$

If $w \in C$, then we are done by choosing $v' = w \in C \cap K$. Otherwise, let u be the closest point to w along the closed segment $[w, v] \subseteq V_1$ and belonging to the closed cone C. Then, since u is on the segment $[w, v]$,

$$\|v - u\|_1 \leq \|v - w\|_1 = \|\mathbf{d}v\|_1' \leq c\|\mathbf{d}v\|_2,$$

and since $\mathbf{d}u$ is on the segment $[0, \mathbf{d}v]$,

$$\|\mathbf{d}u\|_2 \leq \|\mathbf{d}v\|_2.$$

Since the point u lies on the boundary of the positive cone C, it belongs to some proper face $D \subset C$ spanning a $\mathbb{Q}$-subspace $U_1 = \operatorname{Span}_{\mathbb{R}}(D) \subset V_1$ of strictly lower dimension. By the induction hypothesis there exists some constant c_D, such that for $u \in D$ there exists a point $v' \in D \cap K$ with $\|u - v'\|_1 \leq c_D\|\mathbf{d}u\|_2$. Hence,

$$\|v - v'\|_1 \leq \|v - u\|_1 + \|u - v'\|_1 \leq (c + c_D)\|\mathbf{d}v\|_2 \leq c_1\|\mathbf{d}v\|_2,$$

where $c_1 = c + \max_{D \subset C} c_D$ and the maximum is taken over the finite set of proper faces of the positive cone C. ∎

Recall that K denotes the kernel of the linear map $\mathbf{d}$ regarded as a $\mathbb{Q}$-subspace of the $\mathbb{R}$-vector space V_1.

Lemma 5.2. *There are constants $c_1, A > 0$ such that if $v \in C$, then there is a vector $\lambda \in \Lambda_1^+ \cap K$ satisfying $\|v - \lambda\|_1 \leq c_1\|\mathbf{d}v\|_2 + A$.*

Proof. Let $v \in C$ be any vector. By Lemma 5.1, there exists a vector $v'' \in C \cap K$ such that $\|v - v''\|_1 \leq c_1 \|\mathbf{d}v\|_2$ for some constant $c_1 > 0$ independent of v.

Note that $C \cap K$ is a positive cone defined over $\mathbb{Q}$. Let $U = \mathrm{Span}_{\mathbb{R}}(C \cap K) \subseteq V_1$ so that $C \cap U = C \cap K$ is a closed cone with a non-empty interior in the vector subspace U. Denote $B_A = \{w \in V_1 \colon \|w\|_1 \leq A\}$. Therefore, $C \cap U \cap B_A$ contains in its interior a ball in U of an arbitrary large radius, provided the radius $A > 0$ is sufficiently large. Since $\Lambda_1 \cap U$ is a lattice in the $\mathbb{R}$-vector space U, the set $C \cap U \cap B_A$ surjects onto $U/(U \cap \Lambda_1)$ for all $A > 0$ sufficiently large. Fix any sufficiently large such $A > 0$.

Since $v'' \in C \cap K$, the translated set $v'' + C \cap U \cap B_A \subset C \cap K$ also surjects onto $U/(U \cap \Lambda_1)$. In particular, this set admits a point $\lambda \in \Lambda_1 \cap C \cap K = \Lambda_1^+ \cap K$. We conclude that $\|v - \lambda\|_1 \leq \|v - v''\|_1 + \|v'' - \lambda\|_1 \leq c_1 \|\mathbf{d}v\|_2 + A$ as required. ∎

Lemma 5.3. *For any vector $\lambda \in \Lambda_1^+$, there is another vector $\lambda' \in \Lambda_1^+ \cap K$ satisfying $\|\lambda - \lambda'\|_1 \prec \|\mathbf{d}\lambda\|_2$ and $\|\lambda'\|_1 \leq \|\lambda\|_1$.*

Proof. Let the vector $\lambda \in \Lambda_1^+$ be fixed. If $\lambda \in K = \ker \mathbf{d}$, then there is nothing to prove, for we may simply take $\lambda' = \lambda \in \Lambda_1^+ \cap K$. Assume therefore that $\lambda \notin K$.

Since the linear map $\mathbf{d}$ is defined over $\mathbb{Q}$, there is a constant $M > 0$ such that

$$\|\mathbf{d}\lambda\|_2 \geq M$$

for every vector $\lambda \in \Lambda_1 - K$. Denote

$$\theta = \frac{c_1 \|\mathbf{d}\lambda\|_2 + A}{\|\lambda\|_1},$$

where the constants c_1 and A are as in Lemma 5.2.

If $\theta \geq 1$, then we may take $\lambda' = 0$. This vector λ' satisfies $0 = \|\lambda'\|_1 \leq \|\lambda\|_1$ and

$$\|\lambda - \lambda'\|_1 \leq \theta \|\lambda\|_1 = c_1 \|\mathbf{d}\lambda\|_2 + A \leq (c_1 + \frac{A}{M}) \|\mathbf{d}\lambda\|_2$$

as desired (the constants c_1, A and M are all independent of λ).

Finally, assume that $0 < \theta < 1$. Apply Lemma 5.2 to the vector $v = (1 - \theta)\lambda$. This gives a new vector $\lambda' \in \Lambda_1^+ \cap K$ with

$$\|v - \lambda'\|_1 \leq c_1 \|\mathbf{d}v\|_2 + A \leq c_1 \|\mathbf{d}\lambda\|_2 + A.$$

Therefore,

$$\|\lambda'\|_1 \leq \|v\|_1 + \|v - \lambda'\|_1 \leq (1 - \theta)\|\lambda\|_1 + c_1 \|\mathbf{d}\lambda\|_2 + A = \|\lambda\|_1.$$

This verifies the second condition. As for the first condition, we have

$$\begin{aligned} \|\lambda - \lambda'\|_1 &\leq \|\lambda - v\|_1 + \|v - \lambda'\|_1 \leq \theta \|\lambda\|_1 + c_1 \|\mathbf{d}\lambda\|_2 + A \leq 2c_1 \|\mathbf{d}\lambda\|_2 + 2A \\ &\leq 2\Big(c_1 + \frac{A}{M}\Big) \|\mathbf{d}\lambda\|_2. \end{aligned}$$

This concludes the proof, noting as above that the constants c_1, A and M are all independent of the vector λ. ∎

6. From linear algebra back to actions

We show that any δ-almost $\pi_1(\mathcal{G}, T)$-action ρ whose isomorphism type $\rho^\sharp$ is compatible with some $\pi_1(\mathcal{G}, T)$-action, can be corrected to such an action. More precisely, we establish the following result, using without further mention all the notations introduced in Sections 2, 3 and 4.

Proposition 6.1. *Let $\rho: \overline{\pi}_1(\mathcal{G}, T) \to \mathrm{Sym}(X)$ be a δ-almost $\pi_1(\mathcal{G}, T)$-action with $\lambda = \rho^\sharp$. Let $\lambda' \in \Lambda_V^+$ be any vector with $\|\lambda'\|_V = \|\lambda\|_V$. If*

(a) $\lambda' \in \ker \mathbf{d}_{\mathcal{G}}$,

(b) $\|\lambda - \lambda'\|_V \leq \delta \|\lambda\|_V$,

then there is a group action $\rho': \pi_1(\mathcal{G}, T) \to \mathrm{Sym}(X)$ satisfying

(i) $\lambda' = (\rho')^\sharp$,

(ii) $d_{X, S_{\mathcal{G}}}(\rho, \rho') \prec_{\mathcal{G}} \delta$.

We precede the proof of Proposition 6.1 with an analogous statement in the simpler context of a single group homomorphism.

Lemma 6.2. *Let $i: H \to G$ be a homomorphism of finite groups. Let $\phi: H \to \mathrm{Sym}(X)$ and $\rho: G \to \mathrm{Sym}(X)$ be a pair of group actions. Denote $\lambda = \rho^\sharp$. If $\lambda' \in \Lambda_G^+$ and $\delta > 0$ are such that*

(a) $d_{X,H}(\rho \circ i, \phi) \leq \delta$,

(b) $i^*(\lambda') = \phi^\sharp$,

(c) $\|\lambda - \lambda'\|_G \leq \delta \|\lambda\|_G$,

then there exists a group action $\rho': G \to \mathrm{Sym}(X)$ satisfying

(i) $\rho' \circ i = \phi$,

(ii) $\lambda' = (\rho')^\sharp$,

(iii) $d_{X,G}(\rho, \rho') \prec_{G,H} \delta$.

Note that the "small" action ϕ of the group H is not being changed, rather the "large" action ρ is being replaced with a new action ρ' compatible with ϕ.

Proof of Lemma 6.2. We combine assumption (a) and Observation 3.4 applied with respect to the finite group H in order to find a $\phi(H)$-invariant subset $X_0 \subset X$ satisfying $\phi \restriction_{X_0} = (\rho \circ i) \restriction_{X_0}$ and $|X_0| \geq (1 - \delta |H|)|X|$. By applying Observation 3.4 a second time with respect to the finite group G, we find a $\rho(G)$-invariant subset $X_1 \subset X_0$ satisfying $|X_1| \geq (1 - \delta |H||G|)|X|$.

Consider the vector $\lambda_1 = (\rho \restriction_{X_1})^\sharp \in \Lambda_G^+$. Let $\mu_1 \in \Lambda_G^+$ be the component-wise minimum of the two vectors λ' and λ_1, i.e., μ_1 is the vector given by

$$(\mu_1)_\chi = \min\{(\lambda')_\chi, (\lambda_1)_\chi\} \quad \forall \chi \in \mathrm{Trans}(G).$$

The previous paragraph implies that $\|\lambda - \lambda_1\|_G \le \delta |H||G||X|$. Therefore, assumption (c) gives

$$\begin{aligned}\max\{\|\lambda' - \mu_1\|_G, \|\lambda_1 - \mu_1\|_G\} &\le \|\lambda' - \lambda_1\|_G \le \|\lambda' - \lambda\|_G + \|\lambda - \lambda_1\|_G \\ &\le \delta|X| + \delta|H||G||X| = \delta(1 + |H||G|)|X|.\end{aligned}$$

Let $Y_1 \subset X_1$ be any $\rho(G)$-invariant subset satisfying $\mu_1 = (\rho \upharpoonright_{Y_1})^\sharp$. Write $\mu_2 = \lambda' - \mu_1 \in \Lambda_G^+$ and $Y_2 = X - Y_1$ so that $\lambda' = \mu_1 + \mu_2$ and $X = Y_1 \coprod Y_2$. It will not be the case in general that $(\rho \upharpoonright Y_2)^\sharp$ coincides with μ_2. However $|Y_2| = \|\mu_2\|_G$ and, in particular, the size of the subset Y_2 is bounded by

$$|Y_2| = \|\mu_2\|_G \le \delta(1 + |H||G|)|X|.$$

We define a new action $\rho' \colon G \to \mathrm{Sym}(X)$ as follows. To begin with, the restriction of ρ' to the $\rho(G)$-invariant subset Y_1 coincides with ρ, namely $\rho' \upharpoonright_{Y_1} = \rho \upharpoonright_{Y_1}$. As $i^*(\lambda') = \phi^\sharp$ by assumption (b) and as $i^*(\mu_1) = (\phi \upharpoonright_{Y_1})^\sharp$ by the choice of the subset Y_1, we have $i^*(\mu_2) = (\phi \upharpoonright_{Y_2})^\sharp$. It remains to define the action ρ' on the $\rho(G)$-invariant complement $Y_2 = X - Y_1$. Taking into account Observation 4.2, we let $\rho' \upharpoonright_{Y_2}$ be an arbitrary action satisfying $(\rho' \circ i) \upharpoonright_{Y_2} = \phi \upharpoonright_{Y_2}$ and $(\rho' \upharpoonright_{Y_2})^\sharp = \mu_2$. This completes the definition of the new action ρ'.

Statements (i) and (ii) of the lemma hold true since $\rho' \circ i = \phi$ and

$$\rho'^\sharp = (\rho' \upharpoonright_{Y_1})^\sharp + (\rho' \upharpoonright_{Y_2})^\sharp = \mu_1 + \mu_2 = \lambda'.$$

To verify statement (iii), we compute

$$\begin{aligned} d_X(\rho(g), \rho'(g)) &= \frac{|Y_1|}{|X|} d_{Y_1}(\rho(g) \upharpoonright_{Y_1}, \rho'(g) \upharpoonright_{Y_1}) + \frac{|Y_2|}{|X|} d_{Y_2}(\rho(g) \upharpoonright_{Y_2}, \rho'(g) \upharpoonright_{Y_2}) \\ &\le \frac{|Y_2|}{|X|} \le (1 + |H||G|)\delta \le 2|H||G|\delta \end{aligned}$$

for all elements $g \in G$. Therefore, $d_{X,G}(\rho, \rho') \le 2|H||G|^2\delta$ as required. ■

We are now in a position to prove the main result of Section 6.

Proof of Proposition 6.1. We define the new action $\rho' \colon \pi_1(\mathcal{G}, T) \to \mathrm{Sym}(X)$ of the fundamental group $\pi_1(\mathcal{G}, T)$ by specifying it on the finite generating set $S_\mathcal{G}$. This is done in two steps: first we define ρ' on the vertex groups G_v and then on the generators of the form s_e.

Step 1. *Defining ρ' on G_v for all $v \in V(\Gamma)$.* Fix an arbitrary base vertex v_0 in $V(\Gamma)$. We define the vertex group actions $\rho'|_{G_v}$ by induction on the distance in the spanning tree T of the vertex v from the base vertex v_0 such that

(1) $(\rho'|_{G_v})^\sharp = \lambda'_v$,

(2) $d_{X,G_v}(\rho|_{G_v}, \rho'|_{G_v}) \prec \delta$,

(3) $\rho|_{G_{t(e)}} \circ i_e = \rho|_{G_{o(e)}} \circ i_{\bar{e}}$ for the unique edge $e \in E(T)$ such that $t(e) = v$ and such that the unique path in the tree T from v_0 to v passes through $o(e)$.

Base of the induction. We apply Lemma 6.2 with the following data: the group G is the vertex group G_{v_0} with the action $\rho|_{G_{v_0}}$ on X, the group H and the homomorphism $i\colon H \to G$ are trivial, and the vector $\lambda' \in \Lambda_G^+$ is the coordinate $\lambda'_{v_0} \in \Lambda^+_{G_{v_0}}$. This results in a new action $\rho'|_{G_{v_0}}$ of the base vertex group G_{v_0} satisfying $d_{X,G_{v_0}}(\rho|_{G_{v_0}}, \rho'|_{G_{v_0}}) \prec \delta$ and $(\rho'|_{G_{v_0}})^\sharp = \lambda'_{v_0}$.

Induction step. Let $v \in V(\Gamma)$ be a vertex at distance $n \in \mathbb{N}$ from the base vertex v_0 in the tree T and $e \in E(T)$ be the unique edge such that $t(e) = v$ and $o(e)$ is at distance $n - 1$ from the base vertex v_0. Denote $u = o(e)$. By the induction hypothesis, the vertex group action $\rho'|_{G_u}$ has been defined and satisfies $(\rho'|_{G_u})^\sharp = \lambda'_u$.

We apply Lemma 6.2 with the following data: the group G is the vertex group G_v, the group H is the edge group G_e, the homomorphism $i\colon H \to G$ is the map i_e, the action ϕ of the group H is $(\rho'|_{G_u}) \circ i_{\bar{e}}$, the action ρ of the group G is $\rho|_{G_v}$ and lastly, the vector $\lambda' \in \Lambda_G^+$ is the coordinate $\lambda'_v \in \Lambda^+_{G_v}$.

We proceed to verify the assumptions of Lemma 6.2. The induction hypothesis combined with the assumption that $\lambda' \in \ker \mathbf{d}_{\mathcal{G}}$ imply

$$i^*(\lambda'_v) = i_e^*(\lambda'_v) = i_{\bar{e}}^*(\lambda'_u) = i_{\bar{e}}^*((\rho'|_{G_u})^\sharp) = (\rho' \circ i_{\bar{e}})^\sharp = \phi^\sharp.$$

By the triangle inequality, the two actions $\rho \circ i$ and ϕ of the edge group G_e satisfy

$$\begin{aligned} d_{X,G_e}(\rho \circ i, \phi) &= d_{X,G_e}(\rho \circ i_e, \rho' \circ i_{\bar{e}}) \\ &\le d_{X,G_e}(\rho \circ i_e, \rho(s_e) \cdot (\rho \circ i_e)) \\ &\quad + d_{X,G_e}(\rho(s_e) \cdot (\rho \circ i_e), \rho(s_e) \cdot (\rho \circ i_e) \cdot \rho(s_e)^{-1}) \\ &\quad + d_{X,G_e}(\rho(s_e) \cdot (\rho \circ i_e) \cdot \rho(s_e)^{-1}, \rho \circ i_{\bar{e}}) \\ &\quad + d_{X,G_e}(\rho \circ i_{\bar{e}}, \rho' \circ i_{\bar{e}}). \end{aligned}$$

The normalized Hamming metric d_X is bi-invariant so that the first and second summands are both less than $d_{X,G_e}(\rho(s_e), \mathrm{id}) < \delta$. The third summand is also less than δ as ρ is a δ-almost $\pi_1(\mathcal{G}, T)$-action and taking into account the corresponding relation in $R_{\mathcal{G}}$. Lastly, the fourth summand satisfies

$$d_{X,G_e}(\rho \circ i_{\bar{e}}, \rho' \circ i_{\bar{e}}) \prec \delta$$

by the induction hypothesis. We conclude that

$$d_{X,G_e}(\rho \circ i, \phi) \prec \delta.$$

Having verified all of the assumptions for Lemma 6.2, we get a new action $\rho'|_{G_v}$ of the vertex group G_v such that $\rho' \circ i_e = \rho' \circ i_{\bar{e}}$ on the edge group G_e, $d_{X,G_v}(\rho|_{G_v}, \rho'|_{G_v}) \prec \delta$ and $(\rho'|_{G_v})^\sharp = \lambda'_v$. This completes the step of the induction.

Proceed with the induction until the new action ρ' is defined on all vertex groups.

Step 2. *Defining* ρ' *on the generators* s_e *for all* $e \in \vec{E}(\Gamma)$. Let $e \in \vec{E}(\Gamma)$ be a directed edge with $o(e) = u$ and $t(e) = v$.

Assume that $e \in E(T)$. Define $\rho'(s_e) = \mathrm{id}$. Recall that the action ρ' of the edge group G_e satisfies $\rho' \circ i_e = \rho' \circ i_{\bar{e}}$ by step 1. Therefore,

$$\rho'(i_e(g_e)s_e)(x) = \rho'(s_e i_{\bar{e}}(g_e))(x)$$

for all points $x \in X$ and all elements $g_e \in G_e$. Moreover, since ρ is a δ-almost $\pi_1(\mathcal{G}, T)$-action, we have $d_X(\rho(s_e), \rho'(s_e)) \leq \delta$.

Assume that $e \in E(\Gamma) - E(T)$. According to Observation 3.4, there exists a $\rho'|_{i_e(G_e)}$-invariant subset $X_e \subseteq X$ such that $|X - X_e| \prec \delta |X|$ and such that the following conditions are satisfied for all points $x \in X_e$ and all elements $g_e \in G_e$:

$$\begin{aligned} \rho(i_e(g_e)s_e)(x) &= \rho(s_e i_{\bar{e}}(g_e))(x), \\ \rho(i_{\bar{e}}(g_e))(x) &= \rho'(i_{\bar{e}}(g_e))(x), \\ \rho(i_e(g_e))(\rho(s_e)x) &= \rho'(i_e(g_e))(\rho(s_e)x). \end{aligned}$$

Define the restriction $\rho'(s_e) \restriction_{X_e}$ of the new action to be the same as $\rho(s_e) \restriction_{X_e}$. The above conditions imply that the permutation $\rho'(s_e)$ satisfies

$$\rho'(i_e(g_e)s_e)(x) = \rho'(s_e i_{\bar{e}}(g_e))(x)$$

for all points $x \in X_e$ and all edge group elements $g_e \in G_e$.

It remains to define the permutation $\rho'(s_e)$ on the complement $X - X_e$ and verify the above relation for all points $x \in X - X_e$. The two actions $\rho' \circ i_e$ and $\rho' \circ i_{\bar{e}}$ of the edge group G_e are conjugate as $\lambda' \in \ker \mathbf{d}_{\mathcal{G}}$ and $\rho'^{\sharp} = \lambda'$. Since the permutation $\rho(s_e)$ conjugates $(\rho' \circ i_e) \restriction_{X_e}$ to $(\rho' \circ i_{\bar{e}}) \restriction_{\rho(s_e)X_e}$, we know that their complements must be conjugate as well. Define the restriction $\rho'(s_e) \restriction_{X - X_e}$ to be an arbitrary bijection from $X - X_e$ to $X - \rho(s_e)X_e$ implementing this isomorphism of actions. Note that $d_X(\rho(s_e), \rho'(s_e)) \prec \delta$. This concludes the definition of the permutation $\rho'(s_e)$ for this particular oriented edge e.

A bound on $d_{X,S_{\mathcal{G}}}(\rho, \rho')$. The $\pi_1(\mathcal{G}, T)$-action ρ' has been constructed in steps 1 and 2. It was specified in terms of the finite generating set $S_{\mathcal{G}}$ while making sure that the defining relations $R_{\mathcal{G}}$ of the fundamental group $\pi_1(\mathcal{G}, T)$ hold true. It follows from the construction that $\rho'^{\sharp} = \lambda'$. To conclude the proof it remains to bound the normalized Hamming distance $d_{X,S_{\mathcal{G}}}(\rho, \rho')$. Namely,

$$\begin{aligned} d_{X,S_{\mathcal{G}}}(\rho, \rho') &= \sum_{\sigma \in S_{\mathcal{G}}} d_X(\rho(\sigma), \rho'(\sigma)) \\ &= \sum_{v \in V(G)} \sum_{g \in G_v} d_X(\rho(g), \rho'(g))d + \sum_{e \in E(T)} d_X(\rho(s_e), \rho'(s_e)) \\ &\quad + \sum_{e \in E(\Gamma) - E(T)} d_X(\rho(s_e), \rho'(s_e)) \prec \delta \end{aligned}$$

as required. ∎

7. Proof of the main theorem

We are ready to show that the epimorphism $\overline{\pi}_1(\mathcal{G}, T) \to \pi_1(\mathcal{G}, T)$ is P-stable.

Proof of Theorem 1.2. Let X be a finite set admitting a δ-almost $\pi_1(\mathcal{G}, T)$-action

$$\rho\colon \overline{\pi}_1(\mathcal{G}, T) \to \mathrm{Sym}(X).$$

Denote $\lambda = \rho^\sharp$. We know by Proposition 4.4 that

$$\|\mathbf{d}_{\mathcal{G}}(\lambda)\|_E \prec_{\mathcal{G}} \delta \|\lambda\|_V.$$

Lemma 5.3 allows us to find a vector $\lambda'' \in \Lambda_V^+ \cap \ker \mathbf{d}_{\mathcal{G}}$ such that

$$\|\lambda'' - \lambda\|_V \prec \delta \|\lambda\|_V \quad \text{and} \quad \|\lambda''\|_V \leq \|\lambda\|_V.$$

We will make an auxiliary use of the action of the fundamental group $\pi_1(\mathcal{G}, T)$ on a singleton. Denote this action by s. By Proposition 4.3, we know that $s^\sharp \in \ker \mathbf{d}_{\mathcal{G}}$. Moreover, $\|s^\sharp\|_V = 1$. Let

$$\lambda' = \lambda'' + (\|\lambda\|_V - \|\lambda''\|_V) s^\sharp.$$

It is clear that $\lambda' \in \ker \mathbf{d}_{\mathcal{G}}$, $\|\lambda'\|_V = \|\lambda\|_V = \|\rho^\sharp\|_V = |X|$ and

$$\|\lambda' - \lambda\|_V \leq \|\lambda - \lambda''\|_V + \|\lambda' - \lambda''\|_V \prec \delta |X|.$$

To conclude the proof, we apply Proposition 6.1 and obtain the desired action

$$\rho'\colon \pi_1(\mathcal{G}, T) \to \mathrm{Sym}(X)$$

satisfying $(\rho')^\sharp = \lambda'$ and $d_{X, S_{\mathcal{G}}}(\rho, \rho') \prec \delta$. ■

Remark 7.1. It follows from the proof that one can take $\delta \prec \varepsilon$ for the P-stability of $\pi_1(\mathcal{G}, T)$.

The derivation of Theorem A from the above Theorem 1.2 is immediate and has been discussed in Section 3.

8. Graph automorphisms of finite order

Fix some $d \in \mathbb{N}$ and let $F_d = F(s_1, \ldots, s_d)$ be the free group of rank d.

A finite Schreier graph A of the group F_d is a finite directed graph edge-labelled by the generators $s_1, \ldots, s_d$ such that every vertex has exactly one incoming and one outgoing edge of each label. Let $\vec{E}(A)$ be the directed edges of A. We indicate the labelling using a function $c = c_A\colon \vec{E}(A) \to \{s_1, \ldots, s_d\}$.

A *weak δ-almost-automorphism* α of the Schreier graph A is a pair of bijections $\alpha\colon V(A) \to V(A)$ and $\alpha\colon \vec{E}(A) \to \vec{E}(A)$ (we use the same letter for both by abuse of notation) such that for all directed edges $e \in \vec{E}(A)$ except for a subset of size $\delta|\vec{E}(A)|$, we have

$$c(\alpha(e)) = c(e), \quad o(\alpha(e)) = \alpha(o(e)) \quad \text{and} \quad t(\alpha(e)) = \alpha(t(e)).$$

A *δ-almost-automorphism* α is a weak δ-almost-automorphism that moreover satisfies the first two conditions, namely $c(\alpha(e)) = c(e)$ and $o(\alpha(e)) = \alpha(o(e))$, for all directed edges $e \in \vec{E}(A)$.

Observation 8.1. Let α be a weak δ-almost-automorphism of the finite Schreier graph A. Up to changing α on at most $O(\delta|\vec{E}(A)|)$ edges, we can make α into a δ-almost-automorphism.

Fix some integer $n \in \mathbb{N}$.

Definition 8.2. A (weak) δ-almost-automorphism α has *δ-almost order n* if the condition $\alpha^n(v) = v$ holds true for all $v \in V(A)$ except for a subset of size $\delta|V(A)|$.

Given an action $\rho\colon F_d * \langle a \rangle \to \mathrm{Sym}(X)$ on some finite set X, we denote by A_ρ the Schreier graph of the restricted action $\rho|_{F_d}\colon F_d \to \mathrm{Sym}(X)$. Let α_ρ denote the bijection on the vertices of the Schreier graph A_ρ defined by $\alpha_\rho = \rho(a)$. Moreover, by abuse of notation, let α_ρ denote the bijection of the directed edges of A_ρ defined for every $e \in \vec{E}(A)$ by $\alpha_\rho(e) = e'$, where e' is the unique edge satisfying $c(e) = c(e')$ and $o(e') = o(\alpha_\rho(e))$.

Observation 8.3. If $\rho\colon F_d * \langle a \rangle \to \mathrm{Sym}(X)$ is a δ-almost $(F_d \times \mathbb{Z})$-action (resp. δ-almost $(F_d \times (\mathbb{Z}/n\mathbb{Z}))$-action) on some finite set X, then α_ρ is a δ-almost-automorphism of the Schreier graph A_ρ (resp. of δ-almost-order n).

Vice versa, if A is a finite Schreier graph of the group F_d and α is δ-almost-automorphism (resp. of δ-almost-order n) of the free group F_d, then there exists a δ-almost $(F_d \times \mathbb{Z})$-action (resp. δ-almost $(F_d \times (\mathbb{Z}/n\mathbb{Z}))$-action)

$$\rho\colon F_d * \langle a \rangle \to \mathrm{Sym}(X)$$

such that $A = A_\rho$ and $\alpha = \alpha_\rho$.

Theorem A applied to the virtually free group $F_d \times (\mathbb{Z}/n\mathbb{Z})$ and combined with the above observations immediately gives the following corollary.

Corollary 8.4. *Let A be a finite Schreier graph of the free group F_d and let α be a weak δ-almost automorphism of δ-almost order n. Then there exist a Schreier graph A' of the group F_d with $V(A) = V(A')$, and an automorphism α' of A' of order n such that the graphs A and A' differ on at most $O(\delta|\vec{E}|)$ edges, and the automorphisms α and α' differ on at most $O(\delta|V|)$ vertices.*

Note that Corollary 8.4 is false without requiring that α has δ-almost order n since $F_d \times \mathbb{Z}$ is not P-stable by [4].

We end this paper with the following related question.

Question 8.5. Is the conclusion of Corollary 8.4 true in the setting of general d-regular graphs and graph automorphisms (rather than Schreier graphs of F_d)?

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