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Masur's criterion does not hold in the Thurston metric

Ivan Telpukhovskiy

Abstract. We show that there is a minimal, filling, and non-uniquely ergodic lamination λ on the seven-times punctured sphere with uniformly bounded annular projection distances. Moreover, we show that there is a geodesic in the thick part of the corresponding Teichmüller space equipped with the Thurston metric which converges to λ. This provides a counterexample to an analog of Masur's criterion for Teichmüller space equipped with the Thurston metric.

1. Introduction

The Thurston metric is an asymmetric Finsler metric on Teichmüller space that was first introduced by Thurston in [44]. The distance between marked hyperbolic surfaces X and Y is defined as the log of the infimum over the Lipschitz constants of maps from X to Y, homotopic to the identity. Thurston showed that when S has no boundary, the distance can be computed by taking the ratios of the hyperbolic lengths of the geodesic representatives of simple closed curves (s.c.c.):

$$d_{\mathrm{Th}}(X, Y) = \sup_{\alpha\text{-s.c.c.}} \log \frac{\ell_\alpha(Y)}{\ell_\alpha(X)}. \tag{1.1}$$

A class of oriented geodesics for this metric called *stretch paths* was introduced in [44]. Given a maximal geodesic lamination ν on a hyperbolic surface X, a stretch path starting from X is obtained by stretching the leaves of ν and extending this deformation to the whole surface. The stretch path is controlled by the *horocyclic foliation*, obtained by foliating the ideal triangles in the complement of ν by horocyclic arcs and endowed with the transverse measure that agrees with the hyperbolic length along the leaves of ν. That is, the projective class of the horocyclic foliation is invariant along the stretch path.

Thurston showed that there exists a geodesic between any two points in Teichmüller space equipped with this metric that is a finite concatenation of stretch path segments. In general, geodesics are not unique: the length ratio in equation (1.1) extends continuously to the compact space of projective measured laminations $\mathbb{P}\mathcal{ML}(S)$ and the supremum is usually (in a sense of the word) realized on a single point which is a simple closed curve, thus leaving freedom for a geodesic.

The following is our main theorem.

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Theorem 1.1. *There are Thurston stretch paths in a Teichmüller space with minimal, filling, but not uniquely ergodic horocyclic foliation, which stay in the thick part for the whole time.*

The theorem contributes to the study of the geometry of the Thurston metric in comparison to the better studied Teichmüller metric. Namely, our result is in contrast with a criterion for the divergence of Teichmüller geodesics in the moduli space, given by Masur.

Theorem 1.2 (Masur's criterion, [30]). *Let $\mathfrak{q}$ be a unit area quadratic differential on a Riemann surface X in the moduli space $\mathcal{M}(S)$. Suppose that the vertical foliation of $\mathfrak{q}$ is minimal but not uniquely ergodic. Then the projection of the corresponding Teichmüller geodesic X_t to the moduli space $\mathcal{M}(S)$ eventually leaves every compact set as $t \to \infty$.*

Remark 1.3. The horocyclic foliation is a natural analog of the vertical foliation in the setting of the Thurston metric, see [11, 35].

Remark 1.4. Compare Theorem 1.1 to a result of Brock and Modami in the case of the Weil–Petersson metric on Teichmüller space [10]: they show that there exist Weil–Petersson geodesics with minimal, filling, non-uniquely ergodic ending lamination, that are recurrent in the moduli space, but not contained in any compact subset. Hence our counterexample disobeys Masur's criterion even more than in their setting of the Weil–Petersson metric.

Despite being asymmetric, and in general admitting more than one geodesic between two points, the Thurston metric exhibits some similarities to the Teichmüller metric. For example, it differs from the Teichmüller metric by at most a constant in the thick part[1] and there is an analog of Minsky's product region theorem [14]; every Thurston geodesic between any two points in the thick part with bounded combinatorics is cobounded [25];[2] the shadow of a Thurston geodesic to the curve graph is a reparameterized quasi-geodesic [26].

Nevertheless, the Thurston metric is quite different from the Teichmüller metric. For one, the identity map between them is neither bi-Lipschitz [27], nor a quasi-isometry [14]. In the Teichmüller metric, whenever the vertical and the horizontal foliations of a geodesic have a large projection distance in some subsurface, the boundary of that subsurface gets short along the geodesic [39].[3] However, it follows from our construction that the endpoints of a cobounded Thurston geodesic do not necessarily have bounded combinatorics. The reason behind it is that a condition equivalent to a curve getting short along a stretch

[1] Here the constant $C(\varepsilon)$ depends on the thick part $\mathcal{T}_\varepsilon(S)$.

[2] For every $x, y \in \mathcal{T}_\varepsilon(S)$ with K-bounded combinatorics [25, Definition 2.2], every $\mathcal{G}(x, y)$ is in the $\varepsilon'(\varepsilon, K, S)$-thick part.

[3] For every $\varepsilon > 0$, there exists K such that $d_W(\mu_+, \mu_-) > K$ implies $\inf_t \ell_{\partial W}(\mathcal{G}(t)) < \varepsilon$.

path that is expressed in terms of the subsurface projections of the endpoints is more restrictive than in the case of the Teichmüller metric [40], and involves only the annular subsurface of α (see Theorem 2.10 for a precise and more general statement). This allows us to produce our counterexample by constructing a minimal, filling, non-uniquely ergodic lamination with uniformly bounded annular subsurface projections.

The construction will be done on the seven-times punctured sphere. First, in Section 3 we construct a minimal, filling, non-uniquely ergodic lamination λ using a modification of the machinery developed in [24]. Namely, we choose a partial pseudo-Anosov map τ supported on a subsurface Y homeomorphic to the three-times punctured sphere with one boundary component. We pick a finite-order homeomorphism ρ, such that the subsurface $\rho(Y)$ is disjoint from Y, and the orbit of the subsurface Y under ρ fills the surface. Then we set $\varphi_r = \tau^r \circ \rho$ and provided with a sequence of natural numbers $\{r_i\}_{n=1}^{\infty}$ and a curve γ_0, define

$$\Phi_i = \varphi_{r_1} \circ \cdots \circ \varphi_{r_i}, \quad \gamma_i = \Phi_i(\gamma_0).$$

We show that under a mild growth condition on the coefficients r_i, the sequence of curves γ_i forms a quasi-geodesic in the curve graph and converges to an ending lamination λ in the Gromov boundary. In Section 4, we introduce a Φ_i-invariant bigon track and provide matrix representations of the maps τ and $\tau \circ \rho$. In Section 5, we let γ_0 be a multicurve and produce coarse estimates for the intersection numbers between the pairs of multicurves in the sequence γ_i. In Section 6, we show that λ is non-uniquely ergodic and we find all ergodic transverse measures on λ. In Section 7, we prove that λ has uniformly bounded annular subsurface projections. Finally, in Section 8 we show that there are Thurston stretch paths whose horocyclic foliation is λ, which stay in the thick part of the Teichmüller space for the whole time.

2. Background

2.1. Notation

We adopt the following notation. Given two quantities (or functions) A and B, we write $A \asymp_{K,C} B$ if $\frac{1}{K}B - C \leqslant A \leqslant KB + C$. Further, unless explicitly stated, by the following notation we will mean that there are universal constants $K \geqslant 1$, $C \geqslant 0$ such that

- $A \overset{+}{\prec} B$ means $A \leqslant B + C$.
- $A \overset{*}{\prec} B$ means $A \leqslant KB$.
- $A \overset{+}{\asymp} B$ means $A - C \leqslant B \leqslant A + C$.
- $A \overset{*}{\asymp} B$ means $\frac{1}{K}B \leqslant A \leqslant KB$.
- $A \underset{+}{\overset{*}{\prec}} B$ means $A \leqslant KB + C$.
- $A \underset{+}{\overset{*}{\asymp}} B$ means $\frac{1}{K}B - C \leqslant A \leqslant KB + C$.

2.2. Curves and markings

Let $S = S_{g,n}$ be the oriented surface of genus $g \geqslant 0$ with $n \geqslant 0$ punctures and with negative Euler characteristic. A simple closed curve on S is called *essential* if it does not bound a disk or a punctured disk. We will call a *curve* on S the free homotopy class of an essential simple closed curve on S. Given two curves α and β on S, we will denote the minimal geometric intersection number between their representatives by $i(\alpha, \beta)$. A *multicurve* is a collection of pairwise disjoint curves on S. A *pants decomposition* $\mathcal{P}$ on S is a maximal multicurve on S, i.e., whose complement in S is a disjoint union of three-times punctured spheres. A collection of curves Γ is called *filling* if for any curve β on S: $i(\alpha, \beta) > 0$ for some $\alpha \in \Gamma$. A *marking* μ on S is a filling collection of curves. The intersection number between two collections of curves Γ and Γ' is defined as

$$i(\Gamma, \Gamma') = \sum_{\gamma \in \Gamma, \gamma' \in \Gamma'} i(\gamma, \gamma').$$

2.3. Curve graph

The *curve graph* $\mathcal{C}(S)$ of a surface S is a graph whose vertex set $\mathcal{C}_0(S)$ is the set of all curves on S. Two vertices α and β are connected by an edge if the underlying curves realize the minimal possible geometric intersection number for two curves on S. This means that $i(\alpha, \beta) = 0$, i.e., the curves are disjoint, unless S is one of the exceptional surfaces: if S is the punctured torus, then $i(\alpha, \beta) = 1$, and if S is the four-times punctured sphere, then $i(\alpha, \beta) = 2$. The curve graph is the 1-skeleton of the curve complex, introduced by Harvey in [20]. The metric d_S on the curve graph is induced by letting each edge have unit length. Masur and Minsky showed in [31] that the curve graph is Gromov hyperbolic using Teichmüller theory.

Theorem 2.1 ([31]). *The curve graph $\mathcal{C}(S)$ is Gromov hyperbolic.*

Later, Bowditch gave another proof of this result and showed that the hyperbolicity constant of $\mathcal{C}(S_{g,n})$ is bounded above by a function that is logarithmic in $g + n$ [6]. It was then shown that the hyperbolicity constant is uniformly bounded independently by Bowditch [7], Aougab [1], Hensel, Przytycki and Webb [22], Clay, Rafi and Schleimer [15].

Although the compact annulus $\mathcal{A}$ is not a surface of a negative Euler characteristic, it is crucial for us to consider it and we separately define its curve graph. Let the vertices of $\mathcal{C}(\mathcal{A})$ be the arcs connecting two boundary components of $\mathcal{A}$, up to homotopies that fix the endpoints. Two vertices are connected by an edge of length 1 if the underlying arcs have representatives with disjoint interiors. It is easy to check that $\mathcal{C}(\mathcal{A})$ is quasi-isometric to $\mathbb{Z}$ with the standard metric, hence also Gromov hyperbolic (see [32, Section 2.4] for more details).

2.4. Measured laminations and measured foliations

We denote the space of *geodesic laminations* on S equipped with the Hausdorff topology by $\mathcal{GL}(S)$. For the background on geodesic laminations, we refer to [13, Chapter 4]. We fix some definitions. A geodesic lamination is *minimal* if it does not contain any proper sublaminations. A geodesic lamination is *maximal* if it is not contained in any lamination as a proper subset. A geodesic lamination is *filling* if the connected components of its complement are open disks or once punctured open disks. A geodesic lamination is *chain-recurrent* if it is in the closure of the set of multicurves in $\mathcal{GL}(S)$.

We denote the space of *measured laminations* on S equipped with the weak* topology by $\mathcal{ML}(S)$. For the background on measured laminations, we refer to [29, Chapter 8]. The *stump* of a geodesic lamination is its maximal sublamination that admits a transverse measure of full support. We note that a minimal, filling geodesic lamination admits a transverse measure of full support. A geodesic lamination is *uniquely ergodic* if it supports a unique transverse measure up to scaling. Otherwise, it is *non-uniquely ergodic*.

We denote the space of *projective measured laminations* on S equipped with the quotient topology of $\mathcal{ML}(S) \setminus \{0\}$ by $\mathbb{P}\mathcal{ML}(S)$. For a non-zero measured lamination $\eta \in \mathcal{ML}(S)$, we denote its projective class by $[\eta] \in \mathbb{P}\mathcal{ML}(S)$. The intersection number $i(\cdot,\cdot)$ extends continuously to the space of measured laminations (for a further extension to the space of geodesic currents see [29, Chapter 8]). We say that the intersection number between two projective measured laminations equals zero if it holds for every pair of their representatives in $\mathcal{ML}(S)$.

Consider the subspace of $\mathbb{P}\mathcal{ML}(S)$ consisting of projective measured laminations with minimal and filling support. Consider the quotient of this subspace by identifying the laminations that have the same support. The resulting space equipped with the quotient subspace topology is the space of *ending laminations* $\mathcal{EL}(S)$. Alternatively, the topology of $\mathcal{EL}(S)$ can be described as follows: a sequence $\{\nu_i\}$ of minimal, filling geodesic laminations converges to $\nu \in \mathcal{EL}(S)$ if every limit point of $\{\nu_i\}$ in $\mathcal{GL}(S)$ contains ν as a sublamination. We refer to [19] for more details. Klarreich proved the following assertion.

Theorem 2.2 ([23]). *The Gromov boundary of the curve graph $\mathcal{C}(S)$ is homeomorphic to the space of ending laminations $\mathcal{EL}(S)$. If a sequence of curves $\{\nu_i\}$ is a quasi-geodesic in $\mathcal{C}(S)$ that converges to $\nu \in \mathcal{EL}(S)$, then any limit point of $\{\nu_i\}$ in $\mathbb{P}\mathcal{ML}(S)$ projects to ν under the forgetful map.*

We denote the space of *measured foliations* on S equipped with the weak* topology by $\mathcal{MF}(S)$. For the background on measured foliations, we refer to [18]. The spaces $\mathcal{MF}(S)$ and $\mathcal{ML}(S)$ are canonically identified, and sometimes we will not distinguish between measured laminations and measured foliations; similarly for their projectivizations $\mathbb{P}\mathcal{ML}(S)$ and $\mathbb{P}\mathcal{MF}(S)$.

2.5. Teichmüller space and Thurston boundary

A *marked hyperbolic surface* is a complete finite-area Riemannian surface of constant curvature -1 with a fixed homeomorphism from the underlying topological surface S. Two marked hyperbolic surfaces X and Y are called equivalent if there is an isometry between X and Y in the correct homotopy class. The collection of equivalence classes of marked hyperbolic surfaces is called the *Teichmüller space* $\mathcal{T}(S)$ of the surface S. By $\ell_\alpha(X)$ we denote the hyperbolic length of the unique geodesic representative of the curve α on the surface X. For $\varepsilon > 0$, the *ε-thick part* $\mathcal{T}_\varepsilon(S)$ of the Teichmüller space is the set of all marked hyperbolic surfaces with no curves shorter than ε. A *Bers constant* of S is a number $B(S)$ such that for every $X \in \mathcal{T}(S)$, there exists a pants decomposition on X such that the length of each curve in it is at most $B(S)$. We recall that the Teichmüller space can be compactified via the *Thurston boundary* homeomorphic to $\mathbb{P}\mathcal{ML}(S)$ so that the compactification is homeomorphic to the closed ball of dimension $6g - 6 + 2n$. For the details of the construction using the space of geodesic currents in the case of a closed surface, we refer to [29, Chapter 8].

2.6. Mapping class group

The *mapping class group* of a surface S is the group of the isotopy classes of orientation-preserving self-homeomorphisms of S. The mapping class group acts continuously on the space of projective measured laminations $\mathbb{P}\mathcal{ML}(S)$. A non-periodic element of the mapping class group that has no invariant multicurves is called *pseudo-Anosov*. A pseudo-Anosov mapping class has exactly two fixed points in $\mathbb{P}\mathcal{ML}(S)$ that represent a pair of transverse measured laminations that are minimal, filling and uniquely ergodic. Moreover, given a pseudo-Anosov mapping class Ψ, there is a number $\lambda_\Psi > 1$ such that

$$\Psi(\nu^u) = \lambda_\Psi \nu^u, \quad \Psi(\nu^s) = \lambda_\Psi^{-1} \nu^s. \tag{2.1}$$

The (classes of the) laminations $\nu^{u,s}$ in equation (2.1) are called the *unstable* and *stable* laminations of Ψ, respectively. We refer to [17, 18] for more background on pseudo-Anosov homeomorphisms.

2.7. Subsurface projections

By a *subsurface* $Y \subset S$ we mean the isotopy class of a proper, closed, connected, embedded subsurface, such that its boundary consists of curves on S and its punctures agree with those of S. Whenever we talk about curves or laminations on Y, we think of the boundary components of Y as punctures. We allow Y to be an annular subsurface, whose core curve is a curve on S. We assume Y is not a three-times punctured sphere.

The subsurface projection is a map $\pi_Y : \mathcal{GL}(S) \to 2^{\mathcal{C}_0(Y)}$ from the space of geodesic laminations on S to the power set of the vertex set of the curve graph of Y. Equip S with a hyperbolic metric. Let $\widetilde{Y}$ be the Gromov compactification of the cover of S corresponding to the subgroup $\pi_1(Y)$ of $\pi_1(S)$ with the hyperbolic metric pulled back from S. There is a natural homeomorphism from to $\widetilde{Y}$ to Y, allowing to identify the curve graphs $\mathcal{C}(\widetilde{Y})$ and $\mathcal{C}(Y)$. For any geodesic lamination ν on S, let $\widetilde{\nu}$ be the closure of the complete preimage of ν in $\widetilde{Y}$. Suppose that $Y \subset S$ is a non-annular subsurface. An arc $\beta \in \widetilde{\nu}$ is *essential* if no component of $\widetilde{Y} \setminus \beta$ has closure which is a disk. For each essential arc $\beta \in \widetilde{\nu}$, let $\mathcal{N}_\beta$ be a regular neighborhood of $\beta \cup \partial\widetilde{Y}$. Define $\pi_Y(\nu)$ to be the union of all curves which are either curve components of $\widetilde{\nu}$ or curve components of $\partial\mathcal{N}_\beta$, where β is an essential arc in $\widetilde{\nu}$. If $Y \subset S$ is an annular subsurface, define $\pi_Y(\nu)$ to be the union of all arcs β in $\widetilde{\nu}$ that connect two boundary components of $\widetilde{Y}$.

We say that a lamination ν intersects the subsurface Y *essentially* if $\pi_Y(\nu)$ is non-empty. The projection distance between two laminations $\nu, \nu' \in \mathcal{GL}(S)$ that intersect Y essentially is

$$d_Y(\nu, \nu') = \operatorname{diam}_{\mathcal{C}(Y)}(\pi_Y(\nu) \cup \pi_Y(\nu')).$$

If Y is an annular subsurface with the core curve α, we will write $d_\alpha(\nu, \nu')$ instead of $d_Y(\nu, \nu')$ for convenience (when the quantity makes sense). More generally, if Γ is a collection of laminations, we define $\pi_Y(\Gamma) = \bigcup_{\nu\in\Gamma} \pi_Y(\nu)$ and denote by $d_Y(\Gamma)$ the quantity $\operatorname{diam}_{\mathcal{C}(Y)}(\pi_Y(\Gamma))$. We say that a collection of laminations Γ intersects the subsurface Y *essentially* if $\pi_Y(\Gamma)$ is non-empty. Similarly, if Γ, Γ' are collections of laminations that intersect Y essentially, we define $d_Y(\Gamma, \Gamma') = \operatorname{diam}_{\mathcal{C}(Y)}(\pi_Y(\Gamma) \cup \pi_Y(\Gamma'))$. A collection of subsurfaces Γ is called *filling* if for any $\nu \in \mathcal{GL}(S)$ there is $Y \in \Gamma$ such that $\pi_Y(\nu)$ is non-empty.

The following lemma provides an upper bound for a subsurface projection distance in terms of intersection numbers.

Lemma 2.3 ([21, Lemma 2.1], [32, Section 2.4]). *If $Y \subset S$ is a subsurface or $Y = S$, and α, β are curves on S that intersect Y essentially, then*

$$d_Y(\alpha, \beta) \leqslant 2i(\alpha, \beta) + 2.$$

If Y is an annular subsurface, the above bound holds with multiplicative and additive factors 1.

We state the *bounded* geodesic image theorem proved by Masur and Minsky in [32].

Theorem 2.4 ([32]). *Given a surface S, there is a constant $M = M(S)$ such that whenever Y is a subsurface and $g = \{\gamma_i\}_{i\in I}$ is a geodesic in $\mathcal{C}(S)$ such that γ_i intersects Y essentially for all $i \in I$, then $d_Y(g) \leqslant M$.*

Later, Webb proved that the value of M can be chosen to be independent of the surface [45]. We state a corollary of Theorem 2.4, which follows from the stability of quasi-geodesics in Gromov hyperbolic spaces [8, Chapter III.H, Theorem 1.7].

Corollary 2.5. *Given $k \geqslant 1$, $c \geqslant 0$ and a surface S, there exists a constant $A = A(k, c, S)$ such that the following holds. Let $\{\gamma_i\}_{i\in I}$ be a (k, c)-quasi-geodesic in $\mathcal{C}(S)$ which is also*

1-*Lipschitz, and let* Y *be a subsurface of* S. *If every* γ_i *intersects* Y *essentially, then for every* $i, j \in I$,

$$d_Y(\gamma_i, \gamma_j) \leqslant A.$$

We say that two subsurfaces Y, Z are *overlapping* if the multicurve ∂Y intersects Z essentially and the multicurve ∂Z intersects Y essentially. The following relationship between subsurface projection distances was found in [2], and an elementary proof with explicit constants was later obtained in [28].

Theorem 2.6 (Behrstock inequality). *If* $Y, Z \subset S$ *are overlapping subsurfaces and* α *is a lamination that intersects both of them essentially, then*

$$d_Y(\alpha, \partial Z) \geqslant 10 \;\Rightarrow\; d_Z(\alpha, \partial Y) \leqslant 4.$$

We also state a useful lemma on the convergence of the projection distances (we note that the definition of the projection distance in [10] is slightly different from ours, but this only results in a bounded change of the additive error compared to their statement).

Lemma 2.7 ([10, Lemma 2.7]). *Suppose that a sequence of curves* $\{\nu_i\}$ *converges to a lamination* ν *in the Hausdorff topology on* $\mathcal{GL}(S)$. *Let* Y *be a subsurface, so that* ν *intersects* Y *essentially. Then for any geodesic lamination* ν' *that intersects* Y *essentially, we have*

$$d_Y(\nu_i, \nu') \stackrel{+}{\asymp}_8 d_Y(\nu, \nu')$$

for all i *sufficiently large.*

Finally, we state the following proposition.

Proposition 2.8 ([34, pp. 121–122]). *Let* ν *be the unstable or stable lamination of a pseudo-Anosov map* Ψ *on a surface* S *and let* Γ *be a collection of curves on* S. *Then there is a constant* $C_{\Psi,\Gamma} > 0$ *such that if* $Y \subset S$ *is a subsurface such that* Γ *intersects* Y *essentially, then*

$$d_Y(\nu, \Gamma) \leqslant C_{\Psi,\Gamma}.$$

2.8. Relative twisting

In Section 2.7, the projection distances between laminations for the annular subsurfaces were defined. Here we extend the definition to allow us to compute projection distances between a lamination and a point in Teichmüller space, and between two points in Teichmüller space. We refer to any of these quantities as the relative twisting around a curve α.

Suppose α is a curve, X is a point in Teichmüller space and ν is a geodesic lamination on S. Suppose that ν intersects α essentially. Consider the Gromov compactification of the annular cover X_α that corresponds to the cyclic subgroup $\langle\alpha\rangle$ in the fundamental group $\pi_1(S)$, with the hyperbolic metric pulled back from X. Consider the complete preimage $\widetilde{\nu}$ of ν in X_α. Let $\alpha^\perp$ be a geodesic arc in X_α that is perpendicular to the geodesic in the homotopy class of the core curve. Define $d_\alpha(X, \nu)$ to be the maximal distance between $\widetilde{\omega}$ and $\alpha^{\perp}$ in $\mathcal{C}(X_\alpha)$, where $\widetilde{\omega}$ is any arc of $\widetilde{\nu}$ that connects two boundary components of X_α and $\alpha^{\perp}$ is any perpendicular. We refer to [33, Section 3] for another way to define the twisting of a lamination around a curve in a hyperbolic surface using the projection of lifts in the universal cover. We note that the quantity in their definition differs from ours by at most 2.

Lastly, we define $d_\alpha(X, Y)$, where X, Y are two points in Teichmüller space. Let S_α be the compactification of the annular cover that corresponds to α. Let X_α, Y_α be the compactified covers with the hyperbolic metrics defined as before. Using the first metric, construct a geodesic arc $\alpha_X^{\perp}$, perpendicular to the geodesic in the homotopy class of the core curve. Similarly, construct a geodesic arc $\alpha_Y^{\perp}$. Define $d_\alpha(X, Y)$ to be the maximal distance between $\alpha_X^{\perp}$ and $\alpha_Y^{\perp}$ in $\mathcal{C}(S_\alpha)$, over all possible choices of the perpendiculars.

2.9. Thurston metric on Teichmüller space

We assume that S has no boundary. For a background on the Thurston metric, we refer to [37, 44], while here we mention the necessary notions and state the results that we will use.

In [44], Thurston showed that the best Lipschitz constant is realized by a homeomorphism from X to Y. Moreover, there is a unique largest chain-recurrent lamination $\Lambda(X, Y)$, called the *maximally stretched lamination*, such that any map from X to Y realizing the infimum in equation (1.1), multiplies the arc length along the lamination by the factor of $e^{d_{\mathrm{Th}}(X,Y)}$. Generically, $\Lambda(X, Y)$ is a curve (see [44, Section 10]).

For a maximal lamination ν, Thurston constructed a homeomorphism $\mathcal{F}_\nu\colon \mathcal{T}(S) \to \mathcal{MF}(\nu)$, where $\mathcal{MF}(\nu)$ is the subspace of measured foliations transverse to ν and standard near the cusps (the latter means that every puncture has a neighborhood in which the leaves are homotopic to that puncture and the transverse measure of a (non-compact) arc going out to a cusp is infinite). The image of a point X in the Teichmüller space under $\mathcal{F}_\nu$ is the horocyclic foliation of the pair (X, ν). The space $\mathcal{MF}(\nu)$ has a natural cone structure given by the *shearing coordinates* which produce an embedding $s_\nu\colon \mathcal{T}(S) \to \mathbb{R}^{\dim \mathcal{T}(S)}$ such that the image is an open convex cone. We refer to [5, 43] for the details of the construction. We assume that ν is not an ideal triangulation of S. The stretch paths form open rays from the origin in the image of s_ν. Namely, given any X in Teichmüller space $\mathcal{T}(S)$, a maximal lamination ν, and $t \in \mathbb{R}$, we let $\mathrm{stretch}(X, \nu, t)$ be a unique point in $\mathcal{T}(S)$ such that

$$s_\nu(\mathrm{stretch}(X, \nu, t)) = e^t s_\nu(X).$$

Every stretch path converges to the projective class of the horocyclic foliation in the Thurston boundary as $t \to \infty$ (see [36, Theorem 5.1]). Every stretch path such that the stump of ν is uniquely ergodic converges to the projective class of the stump of ν as $t \to -\infty$ [42]. We summarize these results in one theorem.

Theorem 2.9 ([36, 42]). *Suppose that ν is a maximal lamination on S that is not an ideal triangulation. The stretch path* $\mathrm{stretch}(X, \nu, t)$ *converges to the projective class of the horocyclic foliation $[\mathcal{F}_\nu(X)]$ in the Thurston boundary as $t \to \infty$. Every stretch path* $\mathrm{stretch}(X, \nu, t)$ *such that* $\mathrm{stump}(\nu)$ *is uniquely ergodic converges to the projective class of the stump* $[\mathrm{stump}(\nu)]$ *in the Thurston boundary as $t \to -\infty$.*

2.10. Twisting parameter along a Thurston geodesic

We introduce the notions necessary to state Theorem 2.10. We say that a curve α *interacts* with a lamination ν if α is a leaf of ν or if ν intersects α essentially. We call $[a, b]$ the ε-*active interval* for α along a Thurston geodesic $\mathcal{G}(t)$ if $[a, b]$ is the maximal interval such that $\ell_\alpha(a) = \ell_\alpha(b) = \varepsilon$. We use the notation $\mathrm{Log}(x) = \max(1, \log(x))$. Denote $X_t = \mathcal{G}(t)$.

Theorem 2.10 ([16, Theorem 3.1]). *There exists a constant $\varepsilon_0 > 0$ such that the following statement holds. Let $X, Y \in \mathcal{T}_{\varepsilon_0}(S)$ and α be a curve that interacts with $\Lambda(X, Y)$. Let $\mathcal{G}$ be any geodesic from X to Y and $\ell_\alpha = \min_t \ell_\alpha(t)$. Then*

$$d_\alpha(X, Y) \overset{*}{\underset{+}{\asymp}} \frac{1}{\ell_\alpha} \mathrm{Log} \frac{1}{\ell_\alpha}.$$

If $\ell_\alpha < \varepsilon_0$, then $d_\alpha(X, Y) \overset{+}{\asymp} d_\alpha(X_a, X_b)$, where $[a, b]$ is the ε_0-active interval for α. Further, for all sufficiently small ℓ_α, the relative twisting $d_\alpha(X_t, \Lambda(X, Y))$ is uniformly bounded for all $t \leqslant a$ and $\ell_\alpha(t) \overset{*}{\asymp} e^{t-b}\ell_\alpha(b)$ for all $t \geqslant b$. All errors in this statement depend only on ε_0.

Remark 2.11. We note that the statement of Theorem 2.10 remains true if the condition $X, Y \in \mathcal{T}_{\varepsilon_0}(S)$ is replaced by the weaker condition $\ell_\alpha(X), \ell_\alpha(Y) \geqslant \varepsilon_0$. The proof is identical. This will be crucial for us to make Corollary 8.3.

3. Construction of the lamination

In this section, we construct a quasi-geodesic $\{\alpha_i\}$ in the curve graph of the seven-times punctured sphere $S_{0,7}$ converging to the ending lamination λ in the Gromov boundary. We thank the referee for suggesting simpler proofs.

3.1. Alpha sequence

Denote by $S = S_{0,7}$ the seven-times punctured sphere, obtained by doubling of a regular heptagon on the plane along its boundary. Consider four curves α_0, α_1, α_2, α_3 on S as shown in Figure 1.

Let ρ be the finite order homeomorphism of S which is realized by the counterclockwise rotation along the angle of $\frac{6\pi}{7}$. In other words, the map ρ rotates S by 3 ‘clicks’ counterclockwise. Let Y_0, Y_1, Y_2, Y_3 be the subsurfaces of S with the boundary curves

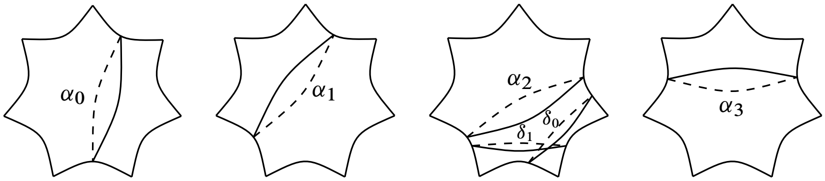

Figure 1. The curves $\alpha_0, \alpha_1, \alpha_2, \alpha_3$ and δ_0, δ_1 on S.

α_0, α_1, α_2, α_3, respectively, and with 3 punctures each. Denote by τ the partial pseudo-Anosov map on S supported on the subsurface Y_2 and obtained as the composition of two half-twists $\tau = H_{\delta_1}^{-1} \circ H_{\delta_0}$ (the core curves are shown in Figure 1).

For any $n \in \mathbb{N}$, let $\varphi_n = \tau^n \circ \rho$. Let $\{r_n\}_{n=1}^{\infty}$ be a sequence of natural numbers. We will impose certain conditions on $\{r_n\}$ later in this section and also in Section 5. Set

$$\Phi_i = \varphi_{r_1}\varphi_{r_2} \dots \varphi_{r_{i-1}}\varphi_{r_i}. \tag{3.1}$$

Define the curves $\alpha_i = \Phi_i(\alpha_0)$ for every $i \in \mathbb{N}$. Denote by Y_i the subsurface with the boundary curve α_i and 3 punctures.

Observe that for any $a, b, c \in \mathbb{N}$,

$$\begin{aligned} \alpha_1 &= \varphi_c(\alpha_0), \quad & \alpha_2 &= \varphi_b\varphi_c(\alpha_0), \quad & \alpha_3 &= \varphi_a\varphi_b\varphi_c(\alpha_0); \\ Y_1 &= \varphi_c(Y_0), \quad & Y_2 &= \varphi_b\varphi_c(Y_0), \quad & Y_3 &= \varphi_a\varphi_b\varphi_c(Y_0). \end{aligned} \tag{3.2}$$

In particular, for $i = 1, 2, 3$ we have that $\Phi_i(\alpha_0) = \alpha_i$, $\Phi_i(Y_0) = Y_i$.

We begin with the observations on the sizes of the subsurface projections between the curves in the sequence $\{\alpha_i\}$.

Claim 3.1. *There is a constant $c > 0$, so that for every $i \geqslant 2$*

$$d_{Y_i}(\alpha_{i-2}, \alpha_{i+2}) \geqslant c r_{i-1} - 1.$$

Proof. First we expand the expression using equation (3.1), then simplify it by applying equation (3.2) and using the fact that the mapping class group acts on the curve graph by isometries, and then apply the triangle inequality:

$$\begin{aligned} d_{Y_i}(\alpha_{i-2}, \alpha_{i+2}) &= d_{\Phi_{i-2}\varphi_{r_{i-1}}\varphi_{r_i}(Y_0)}(\Phi_{i-2}(\alpha_0), \Phi_{i-2}\varphi_{r_{i-1}}\varphi_{r_i}\varphi_{r_{i+1}}\varphi_{r_{i+2}}(\alpha_0)) \\ &= d_{Y_2}(\alpha_0, \varphi_{r_{i-1}}(\alpha_3)) = d_{Y_2}(\alpha_0, \tau^{r_{i-1}}(\rho\alpha_3)) \\ &\geqslant d_{Y_2}(\alpha_0, \tau^{r_{i-1}}(\alpha_0)) - d_{Y_2}(\tau^{r_{i-1}}(\alpha_0), \tau^{r_{i-1}}(\rho\alpha_3)) \\ &= d_{Y_2}(\alpha_0, \tau^{r_{i-1}}(\alpha_0)) - d_{Y_2}(\alpha_0, \rho\alpha_3) \\ &= d_{Y_2}(\alpha_0, \tau^{r_{i-1}}(\alpha_0)) - 1. \end{aligned}$$

Since the mapping class τ restricts to a pseudo-Anosov map on the surface Y_2, by [31, Proposition 3.6] we have $d_{Y_2}(\alpha_0, \tau^n(\alpha_0)) \geqslant cn$ for some $c > 0$, so the result follows. ∎

From now on, we will assume that the sequence $\{r_n\}$ satisfies $r_n \geqslant R$ for all $n \in \mathbb{N}$, where $R > 0$ is such that $cR - 9 \geqslant M + 10$, where M is the constant from Theorem 2.4.

Lemma 3.2. *For every $i < j < k$ with $j - i \geqslant 2$, $k - j \geqslant 2$, the curves α_i, α_k intersect Y_j essentially and*

$$d_{Y_j}(\alpha_i, \alpha_k) \geqslant c r_{j-1} - 9.$$

Proof. The proof is by induction on $n = k - i$.

Base: $n = 4$. It follows from Claim 3.1.

Step. Suppose that $k - i = n + 1$. We show that the curve α_i intersects the subsurface Y_j essentially. If $j - i < 4$, it follows from equation (3.2) together with $i(\alpha_0, \alpha_2) > 0$, $i(\alpha_0, \alpha_3) > 0$. If $j - i \geqslant 4$, then applying the induction hypothesis to the triple $i < j - 2 < j$, we obtain $d_{Y_{j-2}}(\alpha_i, \alpha_j) \geqslant c r_{j-3} - 9 \geqslant M + 10$. If $i(\alpha_i, \alpha_j) = 0$, then since the subsurface projection distance for the disjoint curves is at most 2 (see [32, Lemma 2.2]), we have $d_{Y_{j-2}}(\alpha_i, \alpha_j) \leqslant 2$, contradiction. Therefore, $i(\alpha_i, \alpha_j) \neq 0$ and hence the curve α_i intersects Y_j essentially.

Next, if $j - i \geqslant 4$, then since $d_{Y_{j-2}}(\alpha_i, \alpha_j) \geqslant M + 10 \geqslant 10$, by Theorem 2.6, we have $d_{Y_j}(\alpha_i, \alpha_{j-2}) \leqslant 4$. If $j - i < 4$, then the curves α_i, α_{j-2} are disjoint and $d_{Y_j}(\alpha_i, \alpha_{j-2}) \leqslant 2$. The same argument applied to the triple $\{j, j+2, k\}$ shows that the curve α_k intersects Y_j essentially and that $d_{Y_j}(\alpha_k, \alpha_{j+2}) \leqslant 4$. Then from the triangle inequality together with Claim 3.1, we obtain

$$\begin{aligned} d_{Y_j}(\alpha_i, \alpha_k) &\geqslant d_{Y_j}(\alpha_{j-2}, \alpha_{j+2}) - d_{Y_j}(\alpha_{j-2}, \alpha_i) - d_{Y_j}(\alpha_{j+2}, \alpha_k) \\ &\geqslant c r_{j-1} - 1 - 4 - 4 = c r_{j-1} - 9. \end{aligned}$$ ■

Next, we prove the main result of the section.

Proposition 3.3. *The path $\{\alpha_i\}$ is a quasi-geodesic in the curve graph $\mathcal{C}(S)$.*

Proof. We prove that if $k - i \geqslant 7d - 4$ for $d \in \mathbb{N}$, then $d_S(\alpha_i, \alpha_k) \geqslant d$. Let $\mathcal{G}$ be a geodesic between α_i and α_k in the curve graph. By Lemma 3.2 and Theorem 2.4, for each $j \in \{i+2, \ldots, k-2\}$ there exists a curve v in $\mathcal{G}$ such that v does not intersect the subsurface Y_j essentially. We show that if a curve v does not intersect Y_j and $Y_{j'}$ essentially for $j, j' \in \{i+2, \ldots, k-2\}$, then $|j - j'| < 7$. Assume on the contrary that $|j - j'| \geqslant 7$. Observe that for every $p \in \mathbb{N}$, the subsurfaces $\{Y_p, Y_{p+1}, Y_{p+2}, Y_{p+3}\}$ fill S. Indeed, by equation (3.2) it is sufficient to consider the case $p = 0$, which easily follows from Figure 1. This observation allows us to find $m \in \mathbb{N}$ with $j + 2 \leqslant m \leqslant j' - 2$, such that the curve v intersects Y_m essentially. From Lemma 3.2, we know that $d_{Y_m}(\alpha_j, \alpha_{j'}) \geqslant c r_{m-1} - 9 \geqslant 10$. On the other hand, since $i(v, \alpha_j) = i(v, \alpha_{j'}) = 0$, by the triangle inequality we have

$$d_{Y_m}(\alpha_j, \alpha_{j'}) \leqslant d_{Y_m}(\alpha_j, v) + d_{Y_m}(v, \alpha_{j'}) \leqslant 2 + 2 = 4,$$

contradiction.

For each $j \in \{i+2, \ldots, k-2\}$, map the curve α_j to some vertex in $\mathcal{G}$ that does not intersect Y_j essentially. We have shown that this map is at most 7-to-1. Also by Lemma 3.2, it omits the endpoints of $\mathcal{G}$, therefore if $k - i \geqslant 7d - 4$, then $|\{i+2, \ldots, k-2\}| \geqslant 7d - 7$ and $d_S(\alpha_i, \alpha_k) \geqslant d$. It follows that path $\{\alpha_i\}$ is a quasi-geodesic. ■

We obtain an immediate corollary from Theorem 2.2.

Corollary 3.4. *There is an ending lamination λ on S representing a point in the Gromov boundary of $\mathcal{C}(S)$ such that*

$$\lim_{i\to\infty} \alpha_i = \lambda.$$

Furthermore, every limit point of $\{\alpha_i\}$ in $\mathbb{P}\mathcal{ML}(S)$ defines a projective class of transverse measures on λ.

In the remainder of the section, we prove more claims about the sequence $\{\alpha_i\}$ that will be used in Section 7.

Lemma 3.5. *For every $i < j$ with $j - i \geqslant 5$, the curves α_i, α_j fill S.*

Proof. The triples $i < i+2 < j$, $i < i+3 < j$ satisfy the conditions of Lemma 3.2. Hence

$$d_{Y_{i+2}}(\alpha_i, \alpha_j), d_{Y_{i+3}}(\alpha_i, \alpha_j) \geqslant M + 10.$$

If α_i, α_j are disjoint, then $d_{Y_{i+2}}(\alpha_i, \alpha_j) \leqslant 2$, contradiction. If we have that $d_S(\alpha_i, \alpha_j) = 2$, let $\{\alpha_i, \alpha', \alpha_j\}$ be a geodesic in the curve graph between α_i and α_j. By Theorem 2.4, the curve α' does not intersect Y_{i+2} and Y_{i+3} essentially. A curve that does not intersect Y_{i+2} and Y_{i+3} essentially is either α_{i+2} or α_{i+3}: indeed, by equation (3.2) it is enough to consider the case $i = 0$, which follows from Figure 1. Equation (3.2) also gives $d_S(\alpha_i, \alpha_{i+2}) = d_S(\alpha_i, \alpha_{i+3}) = 2 > 1$, contradiction. Therefore, the curves α_i, α_j fill S. ■

Remark 3.6. There is a constant $R' > 0$ such that if $r_n \geqslant R'$ for all $n \in \mathbb{N}$, then the sequence of subsurfaces $\{Y_i\}$ satisfies the conditions of [9, Theorem 4.1] for $m = 2$, $n = 3$ (in their notation). This gives another proof of Proposition 3.3.

Next, we prove the following claim.

Claim 3.7. *For each $i \in \mathbb{N}$, there is a unique curve β_i on S such that*

$$i(\beta_i, \alpha_i) = i(\beta_i, \alpha_{i+4}) = 0.$$

Further, β_i is disjoint from α_{i+1}, α_{i+2} and α_{i+3}.

Proof. First we show that there is a unique curve β_0 on S that is disjoint from α_0, α_4 and does not intersect Y_2 essentially. A curve that is disjoint from α_0 that does not intersect Y_2

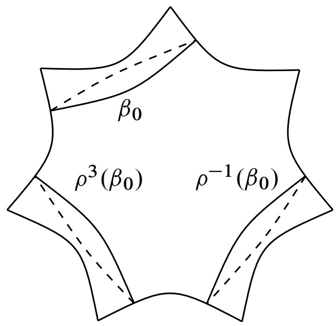

Figure 2. The curves β_0, $\rho^{-1}(\beta_0)$, $\rho^3(\beta_0)$ on S.

essentially is either α_1 or any curve in Y_1. The curve α_4 intersects α_1 essentially and $Y_1 \setminus (\alpha_1 \cup \alpha_4)$ has two connected components: the twice punctured sphere and the three-times punctured sphere. Hence β_0 is the curve in the second component that does not bound a punctured disk in S. The curve β_0 is shown in Figure 2. Define the curves $\beta_i = \Phi_i(\beta_0)$ for $i \in \mathbb{N}$.

Next, any curve β disjoint from α_i, α_{i+4} cannot intersect Y_{i+2} essentially, otherwise we have

$$d_{Y_{i+2}}(\alpha_i, \alpha_{i+4}) \leqslant d_{Y_{i+2}}(\alpha_i, \beta) + d_{Y_{i+2}}(\beta, \alpha_{i+4}) \leqslant 2 + 2 = 4,$$

which contradicts the lower bound of Claim 3.1: $d_{Y_{i+2}}(\alpha_i, \alpha_{i+4}) \geqslant c r_{i+1} - 1 \geqslant M + 10$. Then by equation (3.2), the curve $\Phi_i^{-1}(\beta)$ is disjoint from α_0 and $\varphi_{r_{i+1}}(\alpha_3)$ and does not intersect Y_2 essentially. By an argument as above, we obtain $\Phi_i^{-1}(\beta) = \beta_0$. Therefore, $\beta = \beta_i$. Finally, the curve β_0 is disjoint from the curves $\alpha_1, \alpha_2, \alpha_3$, hence by equation (3.2) β_i is disjoint from $\alpha_{i+1}, \alpha_{i+2}, \alpha_{i+3}$. ■

Claim 3.8. *For each $i \in \mathbb{N}$, there are exactly three curves on S that are disjoint from α_i and α_{i+3}. Further, only one of them intersects both α_{i-1} and α_{i+4} essentially, and this curve intersects both α_{i+1} and α_{i+2} essentially.*

Proof. By applying the homeomorphism Φ_i^{-1} and using equation (3.2), the problem reduces to finding the curves disjoint from α_0 and α_3. It follows from Figure 1 that these curves are $\rho^{-1}(\beta_0)$, β_0 and $\rho^3(\beta_0)$, which are shown in Figure 2. By Claim 3.7, the curves $\Phi_i(\rho^{-1}(\beta_0)) = \beta_{i-1}$, $\Phi_i(\beta_0) = \beta_i$ do not intersect essentially either α_{i-1} or α_{i+4}. By Claim 3.7, the curve $\Phi_i(\rho^3(\beta_0))$ intersects both α_{i-1} and α_{i+4} essentially. Further, the curve $\rho^3(\beta_0)$ intersects α_1 and α_2 essentially, hence by equation (3.2) $\Phi_i(\rho^3(\beta_0))$ intersects both α_{i+1} and α_{i+2} essentially. ■

Claim 3.9. *If a curve on S is disjoint from α_i and α_{i+2} for some $i \in \mathbb{N}$, then it is also disjoint from α_{i+1}.*

Proof. By equation (3.2), it is sufficient to consider the case $i = 0$. Notice that the curve α_1 is a boundary component of a unique subsurface which is filled by the curves α_0 and α_2. Therefore, a curve on S that is disjoint from α_0 and α_2 is also disjoint from α_1, which proves the claim. ∎

Claim 3.10. *For each $i \in \mathbb{N}$, there is no curve on S that is disjoint from α_{i+1}, α_{i+2} and intersects α_i, α_{i+3} essentially.*

Proof. By equation (3.2), it is sufficient to consider the case $i = 0$. If a curve γ on S is disjoint from α_1 and α_2, then one of the following holds: $\gamma = \alpha_1$, $\gamma = \alpha_2$, $\gamma \subset Y_1$, $\gamma \subset Y_2$. If $\gamma = \alpha_1$ or $\gamma \subset Y_1$, then γ is disjoint from α_0, if $\gamma = \alpha_2$ or $\gamma \subset Y_2$, then γ is disjoint from α_3, so the result follows. ∎

We have the following corollary.

Corollary 3.11. *If a curve γ on S is disjoint from some curves in the sequence $\{\alpha_i\}$, then one of the following holds: γ is disjoint from 5 consecutive curves, γ is disjoint from two curves α_i, α_j with $j - i = 3$, γ is disjoint from 3 consecutive curves or γ is disjoint from 1 curve.*

Proof. Let $\ell \in \mathbb{N}$ be the smallest index so that γ is disjoint from α_ℓ and $r \geqslant \ell$ be the largest index so that γ is disjoint from α_r. By Lemma 3.5, we have $r - \ell \leqslant 4$. If $r - \ell = 4$, then by Claim 3.7, γ is disjoint from 5 consecutive curves. If $r - \ell = 3$, then by Claim 3.8, γ is disjoint only from α_ℓ and α_r. If $r - \ell = 2$, then by Claim 3.9, γ is disjoint from 3 consecutive curves. The case $r - \ell = 1$ is impossible by Claim 3.10. If $r - \ell = 0$, then γ is disjoint from 1 curve in $\{\alpha_i\}$. ∎

4. Invariant bigon track

In this section, we introduce a maximal birecurrent bigon track on S that is invariant under the homeomorphisms Φ_i defined in equation (3.1). We refer the reader to [38] for more details on train tracks and specifically to [38, §3.4] for more details on bigon tracks. The bigon track T is shown in Figure 3.

The complement to T in S consists of 7 punctured monogons, 3 trigons and one bigon. The shaded region in Figure 4 shows a part of the bigon in the complement of T.

Let $V(T)$ be the convex cone consisting of all non-negative real assignments of weights to the branches of T that satisfy the switch conditions. Pick the ordered subset of 9 branches of T as in Figure 3. Notice that every non-negative assignment of weights to the chosen branches can be uniquely promoted to a vector in $V(T)$. Denote by e_i the vector in $V(T)$ that assigns the weight 1 to the i-th branch ($i = 1, \dots, 9$) and the weight 0 to all other branches in the chosen set. It follows that $V(T)$ is the non-negative orthant in the vector space $W(T)$ of all real assignments of weights to the branches of T (that satisfy the switch conditions) with basis $\{e_1, \dots, e_9\}$.

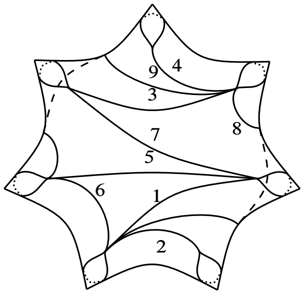

Figure 3. The bigon track T with a numbering of some of its branches.

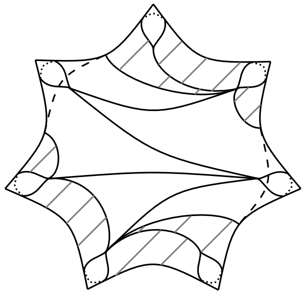

Figure 4. The bigon in the complement of T.

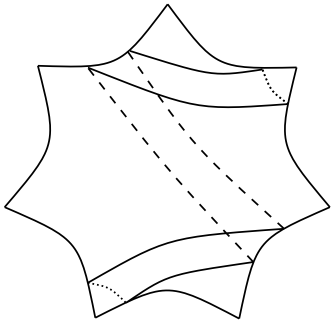

Figure 5. A curve on S that can be represented as a vector in $V(T)$ in two different ways.

The dimension of the space of measured laminations on S is equal to 8, and the natural map from $V(T)$ to $\mathcal{ML}(S)$ is not injective because T has a bigon. Namely, we can show that the following assertion is true.

Claim 4.1. *The space of measured laminations carried by T is naturally identified with the linear quotient cone $V'(T) = V(T)/\sim$, where for $\mu_1, \mu_2 \in V(T)$ we let $\mu_1 \sim \mu_2$ when $\mu_1 - \mu_2 \in \operatorname{span}(2e_2 - 2e_4 + e_6 - e_8 + e_9) \subset W(T)$.*

Proof. According to [38, Proposition 3.4.1] and since $\dim V(T) - \dim \mathcal{ML}(S) = 1$, it is sufficient to find two distinct vectors $v_1, v_2 \in V(T)$ that correspond to the same measured lamination. Indeed, it then follows that vectors $\mu_1, \mu_2 \in V(T)$ correspond to the same measured lamination if and only if $\operatorname{span}(\mu_1 - \mu_2) = \operatorname{span}(v_1 - v_2) \subset W(T)$. Consider $v_1 = 4e_2 + 2e_6 + 2e_9$ and $v_2 = 4e_4 + 2e_8$. We leave it for the reader to verify that both of them correspond to the curve in Figure 5. ∎

Proposition 4.2. *The bigon track T is Φ_i-invariant.*

Proof. It is enough to check that T is invariant under the mapping classes τ and $\tau \circ \rho$. We refer to Figures 6 and 7 for the verification. ∎

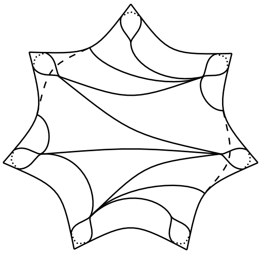

Figure 6. The action of τ on T.

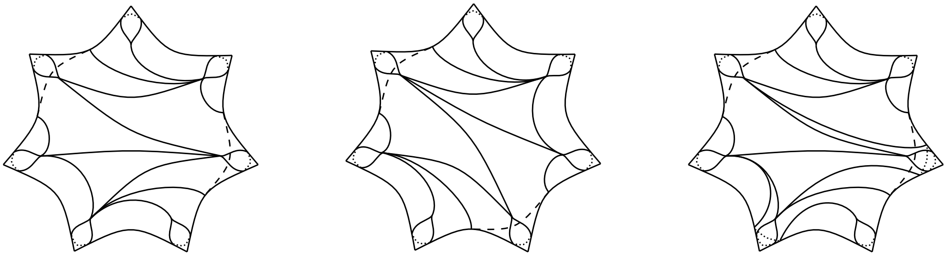

Figure 7. The action of ρ followed by τ on T.

Denote by A the matrix of the induced action of τ on the cone $V(T)$ in the basis $\{e_1, \dots, e_n\}$. Similarly, denote by B the matrix of the induced action of $\tau \circ \rho$ on the cone $V(T)$ in the same basis. We have the following proposition.

Proposition 4.3. *The matrices A and B are as follows:*

$$\begin{pmatrix} 2 & 1 & 0 & 0 & 1 & 0 & 1 & 2 & 0 \\ 1 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}.$$

Further, the vector $v = (\phi, 1, 0, 0, 0, 0, 0, 0, 0)^{\mathsf{T}}$ is an eigenvector of A with the eigenvalue ϕ^2, where $\phi = \frac{1+\sqrt{5}}{2}$.

Proof. Let $w_i = 2e_i$. The matrices A and B do not change if expressed in the basis $\{w_1, \dots, w_9\}$. It is sufficient to find the images of the vectors w_i, $i = 1, \dots, 9$. We refer to Figures 8–16 and leave the verification for the reader. Finally, the vector $v = (\phi, 1, 0, 0, 0, 0, 0, 0, 0)^{\mathsf{T}}$ corresponds to the unstable lamination of τ on S. ∎

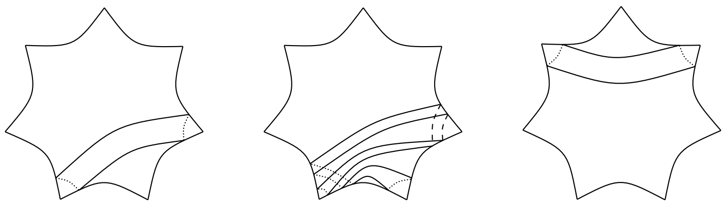

Figure 8. The curve corresponding to the vector w_1 and its images under τ and $\tau \circ \rho$, respectively.

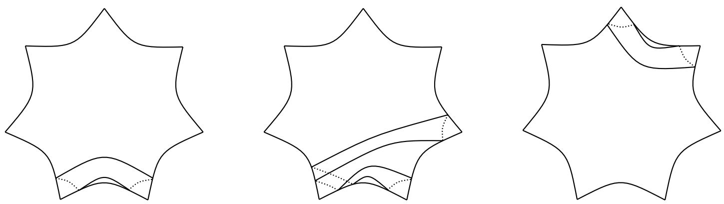

Figure 9. The curve corresponding to the vector w_2 and its images under τ and $\tau \circ \rho$, respectively.

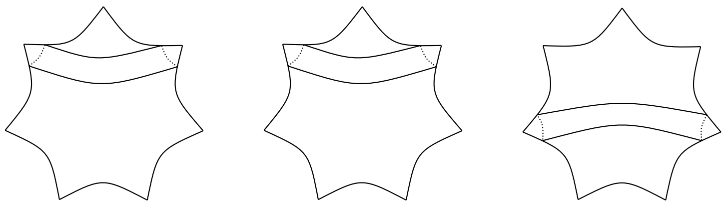

Figure 10. The curve corresponding to the vector w_3 and its images under τ and $\tau \circ \rho$, respectively.

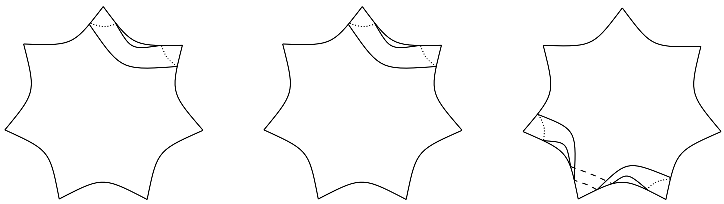

Figure 11. The curve corresponding to the vector w_4 and its images under τ and $\tau \circ \rho$, respectively.

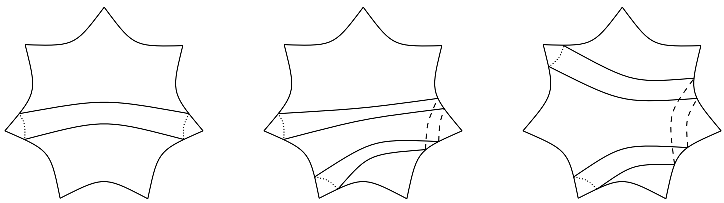

Figure 12. The curve corresponding to the vector w_5 and its images under τ and $\tau \circ \rho$, respectively.

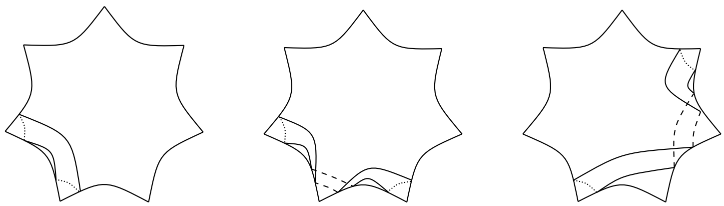

Figure 13. The curve corresponding to the vector w_6 and its images under τ and $\tau \circ \rho$, respectively.

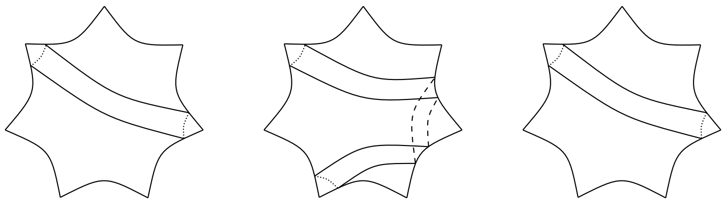

Figure 14. The curve corresponding to the vector w_7 and its images under τ and $\tau \circ \rho$, respectively.

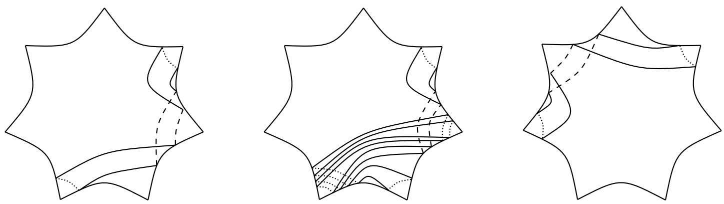

Figure 15. The curve corresponding to the vector w_8 and its images under τ and $\tau \circ \rho$, respectively.

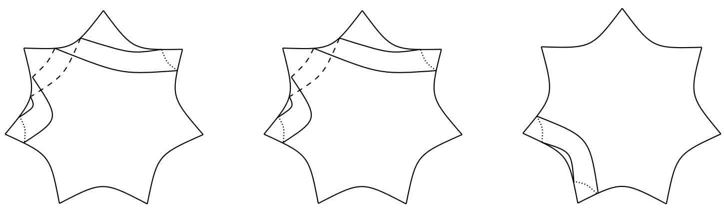

Figure 16. The curve corresponding to the vector w_9 and its images under τ and $\tau \circ \rho$, respectively.

5. Estimating the intersection numbers

Let γ_0 be the multicurve on S that corresponds to the vector $w_1 + w_3 \in V(T)$ as in Figure 17. Define the multicurves $\gamma_i = \Phi_i(\gamma_0)$. In this subsection, we will coarsely estimate the intersection numbers between pairs of multicurves in the sequence $\{\gamma_i\}$. To state the result, we introduce some notations.

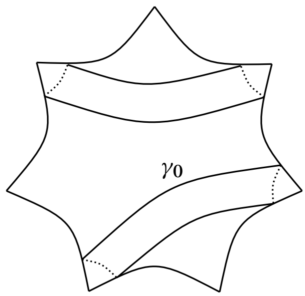

Figure 17. The multicurve γ_0 on S.

Let $f_0 = 0$, $f_1 = 1$, $f_n = f_{n-1} + f_{n-2}$ for $n \geqslant 2$ be the Fibonacci sequence. Define the numbers $c_i = 2f_{2r_i-2}$ for $i \geqslant 1$. We assume that the sequence $\{r_n\}$ is such that

$$c_i \neq 0, \quad c_{i+1} \geqslant c_i, \quad i \in \mathbb{N}, \quad \sum_{i=1}^{\infty} \frac{c_i}{c_{i+1}} < \infty. \tag{5.1}$$

We prove the following assertion.

Proposition 5.1. *There is a constant $i_0 \in \mathbb{N}$ such that for $i_0 \leqslant i < j$ with odd $j - i$, the following holds:*

$$i(\gamma_{i-1}, \gamma_j) \overset{*}{\asymp} i(\gamma_i, \gamma_j) \overset{*}{\asymp} c_{i+1}c_{i+3}\cdots c_j.$$

The multiplicative constants are independent of i and j.

To prove this proposition, we will study the asymptotic behavior of the matrix products involving matrices A and B from Proposition 4.3. We start with elementary observations about the Fibonacci sequence.

Claim 5.2. *For $m \geqslant 0$, the following holds:*

$$2f_{m+1} + f_m = f_{m+3}, \quad 2f_{m+1} + f_m + f_{m+2} = f_{m+4}.$$

Proof. We have

$$\begin{aligned} 2f_{m+1} + f_m &= f_{m+1} + f_{m+1} + f_m = f_{m+1} + f_{m+2} = f_{m+3}, \\ 2f_{m+1} + f_m + f_{m+2} &= f_{m+3} + f_{m+2} = f_{m+4}. \end{aligned}$$

∎

Let $\phi = \frac{1+\sqrt{5}}{2}$ be the golden ratio.

Claim 5.3. *For $m \geqslant 1$, the following holds:*

$$\phi^{-2} f_{2m} - \phi^{-2m} = f_{2m-2}, \quad \phi^{-1} f_{2m} - \phi^{-2m} = f_{2m-1},$$
$$\phi f_{2m} + \phi^{-2m} = f_{2m+1}, \quad \phi^2 f_{2m} + \phi^{-2m} = f_{2m+2}.$$

Proof. By Binet's formula, we have

$$f_{2m} = \frac{\phi^{2m} - \psi^{2m}}{\sqrt{5}}, \quad \text{where} \quad \psi = \frac{1-\sqrt{5}}{2}.$$

Since $\psi^2 = \phi^{-2}$, we have

$$f_{2m} = \frac{\phi^{2m} - \phi^{-2m}}{\sqrt{5}}.$$

Since $\phi^2 - \phi^{-2} = \phi - \phi^{-1} = \sqrt{5}$, we have

$$\begin{aligned}
\phi^{-2} f_{2m} - \phi^{-2m} &= \frac{\phi^{2m-2} - \phi^{-2m-2} - \phi^{-2m}\sqrt{5}}{\sqrt{5}} = \frac{\phi^{2m-2} - \phi^{-2m}(\phi^{-2} + \sqrt{5})}{\sqrt{5}} \\
&= \frac{\phi^{2m-2} - \phi^{-2m+2}}{\sqrt{5}} = f_{2m-2}, \\
\phi^{-1} f_{2m} - \phi^{-2m} &= \frac{\phi^{2m-1} - \phi^{-2m-1} - \phi^{-2m}\sqrt{5}}{\sqrt{5}} = \frac{\phi^{2m-1} - \phi^{-2m}(\phi^{-1} + \sqrt{5})}{\sqrt{5}} \\
&= \frac{\phi^{2m-1} - \phi^{-2m+1}}{\sqrt{5}} = f_{2m-1}, \\
\phi f_{2m} + \phi^{-2m} &= \frac{\phi^{2m+1} - \phi^{-2m+1} + \phi^{-2m}\sqrt{5}}{\sqrt{5}} = \frac{\phi^{2m+1} - \phi^{-2m}(\phi - \sqrt{5})}{\sqrt{5}} \\
&= \frac{\phi^{2m+1} - \phi^{-2m-1}}{\sqrt{5}} = f_{2m+1}, \\
\phi^2 f_{2m} + \phi^{-2m} &= \frac{\phi^{2m+2} - \phi^{-2m+2} + \phi^{-2m}\sqrt{5}}{\sqrt{5}} = \frac{\phi^{2m+2} - \phi^{-2m}(\phi^2 - \sqrt{5})}{\sqrt{5}} \\
&= \frac{\phi^{2m+2} - \phi^{-2m-2}}{\sqrt{5}} = f_{2m+2}.
\end{aligned}$$

∎

Next, we prove the following claim.

Claim 5.4. *For $n \geqslant 1$, the matrix A^n is*

$$\begin{pmatrix}
f_{2n+1} & f_{2n} & 0 & 0 & f_{2n} & f_{2n-1}-1 & f_{2n} & f_{2n+2}-1 & 0 \\
f_{2n} & f_{2n-1} & 0 & 0 & f_{2n-1}-1 & f_{2n-2}+1 & f_{2n-1}-1 & f_{2n+1}-1 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix}.$$

Proof. The proof is by induction.

Base: $n = 1$. It holds since $f_2 = 1$, $f_3 = 2$, $f_4 = 3$.

Step. Using Claim 5.2 and introducing the shorthand notation $s_k := f_k + f_{k+1}$, we calculate the matrix $A^{n+1} = A^n \cdot A$:

$$\begin{pmatrix} s_{2n}+f_{2n+1} & s_{2n} & 0 & 0 & s_{2n} & s_{2n-1}-1 & s_{2n} & s_{2n}+s_{2n+1}-1 & 0 \\ s_{2n-1}+f_{2n} & s_{2n-1} & 0 & 0 & s_{2n-1}-1 & s_{2n-2}+1 & s_{2n-1}-1 & s_{2n-1}+s_{2n}-1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} f_{2n+3} & f_{2n+2} & 0 & 0 & f_{2n+2} & f_{2n+1}-1 & f_{2n}+2 & f_{2n+4}-1 & 0 \\ f_{2n+2} & f_{2n+1} & 0 & 0 & f_{2n+1}-1 & f_{2n2}+1 & f_{2n+1}-1 & f_{2n+3}-1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

This completes the proof. ■

Corollary 5.5. *For $n \geqslant 1$, the matrix $A^n B$ is*

$$\begin{pmatrix} 0 & 0 & f_{2n} & f_{2n+1}-1 & f_{2n+2} & f_{2n+2}-1 & f_{2n} & 0 & f_{2n-1}-1 \\ 0 & 0 & f_{2n-1}-1 & f_{2n}+1 & f_{2n+1}-1 & f_{2n+1}-1 & f_{2n-1}-1 & 0 & f_{2n-2}+1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}.$$

Proof. Direct check. ■

Claim 5.6. *For $n \geqslant 1$, the matrix $A^n B$ can be expressed as*

$$2f_{2n}N + M + \phi^{-2n}L, \tag{5.2}$$

where matrices N, M and L are, respectively,

$$\begin{pmatrix} 0 & 0 & 1/2 & \phi/2 & \phi^2/2 & \phi^2/2 & 1/2 & 0 & 1/2\phi \\ 0 & 0 & 1/2\phi & 1/2 & \phi/2 & \phi/2 & 1/2\phi & 0 & 1/2\phi^2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$\begin{pmatrix} 0 & 0 & 0 & -1 & 0 & -1 & 0 & 0 & -1 \\ 0 & 0 & -1 & 1 & -1 & -1 & -1 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 & 1 & 1 & -1 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Further, the following holds:

(1) $N^2 = 0$ *and* $\mathrm{rk}(N) = 1$.

(2) $(MN)^2 = MN$, *MN is non-negative and*

$$MN(v) = v \quad \text{for } v = (0, 0, \phi, 1, 0, 0, 0, 0, 0)^{\mathsf{T}}.$$

(3) $(NM)^2 = NM$ *and NM is non-negative.*

(4) $NL = LN = 0$.

(5) $L^2 = 0$.

Proof. Equation (5.2) holds by Corollary 5.5 together with Claim 5.3. The rest is a direct check. ∎

Let $\|\cdot\|$ denote the operator norm induced by the standard norm on $V(T)$ with basis $\{e_1, \ldots, e_9\}$.

Claim 5.7. *There is a constant $C > 0$ such that for $m, n \geqslant 1$, the following holds:*

$$\left\| \frac{A^n B}{2 f_{2n}} - N \right\| \leqslant C \cdot \frac{1}{f_{2n}}, \qquad \left\| \frac{A^m B A^n B}{2 f_{2n}} - MN \right\| \leqslant C \cdot \frac{f_{2m}}{f_{2n}}.$$

Proof. By Claim 5.6, we have

$$\frac{A^n B}{2 f_{2n}} - N = \frac{1}{2 f_{2n}} (M + \phi^{-2n} L).$$

Hence

$$\left\| \frac{A^n B}{2f_{2n}} - N \right\| \leqslant \frac{1}{2f_{2n}} \|M + \phi^{-2n} L\| \leqslant \frac{1}{2f_{2n}} (\|M\| + \|L\|).$$

By Claim 5.6, we have

$$\begin{aligned} A^m B A^n B &= (2f_{2m} N + M + \phi^{-2m} L)(2f_{2n} N + M + \phi^{-2n} L) \\ &= 2f_{2n} MN + 2f_{2m} NM + M^2 + \phi^{-2m} LM + \phi^{-2n} ML. \end{aligned} \tag{5.3}$$

Hence

$$\frac{A^m B A^n B}{2f_{2n}} - MN = \frac{f_{2m}}{f_{2n}} \Big(NM + \frac{1}{2f_{2m}} M^2 + \frac{\phi^{-2m}}{2f_{2m}} LM + \frac{\phi^{-2n}}{2f_{2m}} ML \Big).$$

Therefore,

$$\begin{aligned} \left\| \frac{A^m B A^n B}{2f_{2n}} - MN \right\| &\leqslant \frac{f_{2m}}{f_{2n}} \Big(\|NM\| + \frac{1}{2f_{2m}} \|M\|^2 + \frac{\phi^{-2m}}{2f_{2m}} \|LM\| + \frac{\phi^{-2n}}{2f_{2m}} \|ML\| \Big) \\ &\leqslant \frac{f_{2m}}{f_{2n}} (\|NM\| + \|M\|^2 + \|LM\| + \|ML\|). \end{aligned}$$

Letting $C = \max\{\frac{\|M\|+\|L\|}{2}, \|NM\| + \|M\|^2 + \|LM\| + \|ML\|\}$, we complete the proof. ∎

Observe that the matrix $A^n B$ is the induced matrix of the homeomorphism φ_{n+1} since $\varphi_{n+1} = \tau^{n+1} \circ \rho = \tau^n \circ (\tau \circ \rho)$.

Then the matrix P_i defined as

$$P_i = A^{r_i - 1} B A^{r_{i+1} - 1} B \quad \text{for } i \geqslant 1$$

corresponds to $\varphi_{r_i} \varphi_{r_{i+1}}$. We prove the following claim.

Claim 5.8. *There are constants $C' > 0$ and $j_0 \in \mathbb{N}$ such that for $j_0 \leqslant j < k$ with odd $k - j$, the following holds:*

$$\left\| \frac{P_j}{c_{j+1}} \cdot \frac{P_{j+2}}{c_{j+3}} \cdots \frac{P_{k-1}}{c_k} - MN \right\| \leqslant C' \cdot \sum_{i=j}^{\infty} \frac{c_i}{c_{i+1}},$$

$$\left\| \frac{A^{r_{j+1}-1} B}{c_{j+1}} \cdot \frac{P_{j+2}}{c_{j+3}} \cdot \frac{P_{j+4}}{c_{j+5}} \cdots \frac{P_{k-1}}{c_k} - NMN \right\| \leqslant C' \cdot \sum_{i=j}^{\infty} \frac{c_i}{c_{i+1}}.$$

Proof. By the definition of P_i and c_i, we have

$$\frac{P_i}{c_{i+1}} = \frac{A^{r_i - 1} B A^{r_{i+1} - 1} B}{2f_{2r_{i+1} - 2}},$$

therefore by Claim 5.7, we get

$$\left\| \frac{P_i}{c_{i+1}} - MN \right\| \leqslant C \cdot \frac{c_i}{c_{i+1}}.$$

It follows from equation (5.1) that

$$\sum_{i=1}^{\infty}\left\|\frac{P_i}{c_{i+1}}-MN\right\|<\infty.$$

Since the matrix MN is idempotent by Claim 5.6, we can invoke Lemma A.1 (see equation (A.3)) to conclude that there is a constant $j_0\in\mathbb{N}$ such that for $j_0\leqslant j<k$,

$$\left\|\frac{P_j}{c_{j+1}}\cdot\frac{P_{j+2}}{c_{j+3}}\cdots\frac{P_{k-1}}{c_k}-MN\right\|\leqslant 2\cdot\left(C\cdot\sum_{i=j}^{\infty}\frac{c_i}{c_{i+1}}\right)\cdot\|MN\|^2.$$

Together with the triangle inequality and the first inequality in Claim 5.7, we obtain

$$\begin{aligned}&\left\|\frac{A^{r_{j+1}-1}B}{c_{j+1}}\cdot\frac{P_{j+2}}{c_{j+3}}\cdots\frac{P_{k-1}}{c_k}-NMN\right\|\\&\quad\leqslant\left\|\frac{A^{r_{j+1}-1}B}{c_{j+1}}\cdot\frac{P_{j+2}}{c_{j+3}}\cdots\frac{P_{k-1}}{c_k}-\frac{A^{r_{j+1}-1}B}{c_{j+1}}MN\right\|+\left\|\frac{A^{r_{j+1}-1}B}{c_{j+1}}MN-NMN\right\|\\&\quad\leqslant\left(\|N\|+\frac{2C}{c_{j+1}}\right)\cdot 2\cdot\left(C\cdot\sum_{i=j}^{\infty}\frac{c_i}{c_{i+1}}\right)\cdot\|MN\|^2+\frac{2C}{c_{j+1}}\cdot\|MN\|.\end{aligned}$$

Letting $C'=(\|N\|+2C)\cdot 2C\,\|MN\|^2+2C\,\|MN\|$ concludes the proof. ■

We prove the main result of the section.

Proof of Proposition 5.1. Using equation (3.1), we can write

$$i(\gamma_{i-1},\gamma_j)=i(\Phi_{i-1}(\gamma_0),\Phi_{i-1}\varphi_{r_i}\dots\varphi_{r_j}(\gamma_0))=i(\gamma_0,\varphi_{r_i}\dots\varphi_{r_j}(\gamma_0)).$$

We can express the multicurve $\varphi_{r_i}\dots\varphi_{r_j}(\gamma_0)$ as the vector $P_iP_{i+2}\dots P_{j-1}(w_1+w_3)$ in $V(T)$. Recall that by Proposition 4.3, the unstable lamination of the homeomorphism τ can be represented (up to a positive scalar) as the vector $\phi\cdot w_1+w_2$. Notice that the measured lamination that corresponds to the vector

$$MN(w_1+w_3)=\frac{1}{2}w_3+\frac{1}{2\phi}w_4$$

is the unstable lamination of the homeomorphism $\rho\tau\rho^{-1}$, which has a positive intersection number with the curve that corresponds to w_3, hence also with the multicurve γ_0. Since the natural map $V(T)\to\mathcal{ML}(S)$ and the intersection number $i(\cdot,\cdot)$ are continuous, by Claim 5.8 we can choose $i_0\in\mathbb{N}$ so that for $i_0\leqslant i<j$, the intersection number of the measured lamination $\frac{P_i}{c_{i+1}}\cdot\frac{P_{i+2}}{c_{i+3}}\cdots\frac{P_{j-1}}{c_j}(w_1+w_3)$ and γ_0 is bounded above and below from zero, where the bound is independent of i and j. Hence for $i_0\leqslant i<j$, the intersection number $i(\gamma_{i-1},\gamma_j)$ is equal to $c_{i+1}c_{i+3}\cdots c_j$ up to a fixed multiplicative constant.

Similarly, we can write

$$i(\gamma_i, \gamma_j) = i(\Phi_i(\gamma_0), \Phi_i\varphi_{r_{i+1}} \dots \varphi_{r_j}(\gamma_0)) = i(\gamma_0, \varphi_{r_{i+1}} \dots \varphi_{r_j}(\gamma_0)).$$

We can express $\varphi_{r_{i+1}} \dots \varphi_{r_j}(\gamma_0)$ as the vector $A^{r_{i+1}-1}BP_{i+2}P_{i+4} \dots P_{j-1}(w_1 + w_3)$ in $V(T)$. Notice that the measured lamination that corresponds to the vector

$$NMN(w_1 + w_3) = \frac{1}{2}w_1 + \frac{1}{2\phi}w_2$$

is the unstable lamination of the homeomorphism τ, which has a positive intersection number with the curve that corresponds to w_1, hence also with the multicurve γ_0. By Claim 5.8, for $i_0 \leqslant i < j$ the intersection number of the measured lamination $\frac{A^{r_{i+1}-1}B}{c_{i+1}} \cdot \frac{P_{i+2}}{c_{i+3}} \cdot \frac{P_{i+4}}{c_{i+5}} \cdots \frac{P_{j-1}}{c_j}(w_1 + w_3)$ and γ_0 is bounded above and below from zero, where the bound is independent of i and j. Hence for $i_0 \leqslant i < j$, the intersection number $i(\gamma_i, \gamma_j)$ is equal to $c_{i+1}c_{i+3} \cdots c_j$ up to a fixed multiplicative constant. ∎

6. Non-unique ergodicity

In this section, we show that the ending lamination λ constructed in Section 3 is not uniquely ergodic. Namely, we prove that the appropriately scaled subsequences of multicurves $\{\gamma_i\}$ with even and odd indices converge to non-zero measured laminations that are not multiples of each other. Further, we show that the limiting measured laminations are ergodic and are the only ergodic transverse measures on λ.

Claim 6.1. *There are $\lambda_e, \lambda_o \in \mathcal{ML}(S)$ such that the following holds as $n \to \infty$:*

$$\frac{\gamma_{2n}}{c_2c_4 \cdots c_{2n}} \to \lambda_e, \quad \frac{\gamma_{2n+1}}{c_1c_3 \cdots c_{2n+1}} \to \lambda_o.$$

Proof. Notice that the vectors $(\prod_{i=1}^{n} \frac{P_{2i-1}}{c_{2i}})(w_1+w_3)$ and $\frac{A^{r_1-1}B}{c_1}(\prod_{i=1}^{n} \frac{P_{2i}}{c_{2i+1}})(w_1+w_3)$ correspond to $\frac{\gamma_{2n}}{c_2c_4\cdots c_{2n}}$ and $\frac{\gamma_{2n+1}}{c_1c_3\cdots c_{2n+1}}$, respectively. In accordance with equation (5.1), Claim 5.7 and Lemma A.1, the infinite products $\prod_{i=1}^{\infty} \frac{P_{2i-1}}{c_{2i}}$ and $\prod_{i=1}^{\infty} \frac{P_{2i}}{c_{2i+1}}$ converge. So $(\prod_{i=1}^{n} \frac{P_{2i-1}}{c_{2i}})(w_1+w_3)$ and $\frac{A^{r_1-1}B}{c_1}(\prod_{i=1}^{n} \frac{P_{2i}}{c_{2i+1}})(w_1+w_3)$ converge as $n \to \infty$, and the result follows. ∎

Claim 6.2. *As $n \to \infty$,*

$$\frac{i(\gamma_{2n}, \lambda_e)}{i(\gamma_{2n}, \lambda_o)} \to 0, \quad \frac{i(\gamma_{2n+1}, \lambda_e)}{i(\gamma_{2n+1}, \lambda_0)} \to \infty.$$

Proof. Suppose that $2n \geqslant i_0$, where $i_0 \in \mathbb{N}$ is the constant from Proposition 5.1. Then by Proposition 5.1, for $m > n$, we have

$$i\left(\gamma_{2n}, \frac{\gamma_{2m}}{c_2c_4 \cdots c_{2m}}\right) = \frac{i(\gamma_{2n}, \gamma_{2m})}{c_2c_4 \cdots c_{2m}} \overset{*}{\asymp} \frac{c_{2n+2}c_{2n+4} \cdots c_{2m}}{c_2c_4 \cdots c_{2m}} = \frac{1}{c_2c_4 \cdots c_{2n}}.$$

Since it holds for every $m > n$, by passing to the limit as $m \to \infty$, we have $i(\gamma_{2n}, \lambda_e) \overset{*}{\asymp} \frac{1}{c_2 c_4 \cdots c_{2n}}$. In particular, $i(\gamma_{2n}, \lambda_e) \neq 0$.

Similarly, for $m > n$ we have

$$i\left(\gamma_{2n}, \frac{\gamma_{2m+1}}{c_1 c_3 \cdots c_{2m+1}}\right) = \frac{i(\gamma_{2n}, \gamma_{2m+1})}{c_1 c_3 \cdots c_{2m+1}} \overset{*}{\asymp} \frac{c_{2n+1} c_{2n+3} \cdots c_{2m+1}}{c_1 c_3 \cdots c_{2m+1}} = \frac{1}{c_1 c_3 \cdots c_{2n-1}}.$$

Since it holds for every $m > n$, by passing to the limit as $m \to \infty$, we have $i(\gamma_{2n}, \lambda_o) \overset{*}{\asymp} \frac{1}{c_1 c_3 \cdots c_{2n-1}}$. In particular, $i(\gamma_{2n}, \lambda_o) \neq 0$.

Putting this together, we obtain

$$\frac{i(\gamma_{2n}, \lambda_e)}{i(\gamma_{2n}, \lambda_o)} \overset{*}{\asymp} \frac{c_1 c_3 \cdots c_{2n-1}}{c_2 c_4 \cdots c_{2n}}.$$

It follows from equation (5.1) that $\frac{c_{2n-1}}{c_{2n}} \to 0$ as $n \to \infty$. Hence $\frac{c_1 c_3 \cdots c_{2n-1}}{c_2 c_4 \cdots c_{2n}} \to 0$, and therefore $\frac{i(\gamma_{2n}, \lambda_e)}{i(\gamma_{2n}, \lambda_o)} \to 0$ as $n \to \infty$.

Similarly, for $m > n$ we have

$$\begin{aligned} \frac{i(\gamma_{2n+1}, \frac{\gamma_{2m}}{c_2 c_4 \cdots c_{2m}})}{i(\gamma_{2n+1}, \frac{\gamma_{2m+1}}{c_1 c_3 \cdots c_{2m+1}})} &= \frac{c_1 c_3 \cdots c_{2m+1}}{c_2 c_4 \cdots c_{2m}} \cdot \frac{i(\gamma_{2n+1}, \gamma_{2m})}{i(\gamma_{2n+1}, \gamma_{2m+1})} \\ &\overset{*}{\asymp} \frac{c_1 c_3 \cdots c_{2m+1}}{c_2 c_4 \cdots c_{2m}} \cdot \frac{c_{2n+2} c_{2n+4} \cdots c_{2m}}{c_{2n+3} c_{2n+5} \cdots c_{2m+1}} = \frac{c_1 c_3 \cdots c_{2n+1}}{c_2 c_4 \cdots c_{2n}}. \end{aligned}$$

Since it holds for every $m > n$, by passing to the limit as $m \to \infty$, we have $\frac{i(\gamma_{2n+1}, \lambda_e)}{i(\gamma_{2n+1}, \lambda_o)} \overset{*}{\asymp} \frac{c_1 c_3 \cdots c_{2n+1}}{c_2 c_4 \cdots c_{2n}}$. It follows from equation (5.1) that $\frac{c_{2n+1}}{c_{2n}} \to \infty$, hence $\frac{c_1 c_3 \cdots c_{2n+1}}{c_2 c_4 \cdots c_{2n}} \to \infty$ and therefore $\frac{i(\gamma_{2n+1}, \lambda_e)}{i(\gamma_{2n+1}, \lambda_o)} \to \infty$ as $n \to \infty$. ∎

Corollary 6.3. *The measured laminations λ_e, λ_o are non-zero and are not multiples of each other.*

Proof. It was shown in Claim 6.2 that $i(\gamma_{2n}, \lambda_e) \neq 0$ and $i(\gamma_{2n}, \lambda_o) \neq 0$ for $2n \geqslant i_0$, hence $\lambda_e \neq 0$ and $\lambda_o \neq 0$. If λ_e, λ_o are multiples of each other, then the sequence $\frac{i(\gamma_{2n}, \lambda_e)}{i(\gamma_{2n}, \lambda_o)}$ is constant, which contradicts Claim 6.2. ∎

Proposition 6.4. *The ending lamination λ is not uniquely ergodic.*

Proof. The measured lamination λ_e can be expressed as $\lambda_e = \lambda'_e + \lambda''_e$, where λ'_e is the measured lamination that corresponds to the vector $(\prod_{i=1}^{\infty} \frac{P_{2i-1}}{c_{2i}})(w_1)$, λ''_e is the measured lamination that corresponds to the vector $(\prod_{i=1}^{\infty} \frac{P_{2i-1}}{c_{2i}})(w_3)$. The simple closed curve that corresponds to the vector $\prod_{i=1}^{n} P_{2i-1}(w_1)$ is at distance 2 from the curve α_{2n} in the curve graph for each $n \geqslant 1$, hence the sequence of curves $\prod_{i=1}^{n} P_{2i-1}(w_1), n \geqslant 1$ converges to λ in the Gromov boundary as $n \to \infty$. Then by Theorem 2.2, the measured lamination λ'_e is supported either on λ or at zero. Repeating the same argument for λ''_e and since $\lambda_e \neq 0$ by Corollary 6.3, we obtain that λ_e is supported on λ. By a similar argument, the measured lamination λ_o is supported on λ. By Corollary 6.3, λ is not uniquely ergodic. ∎

Let $C(\lambda)$ denote the convex cone of transverse measures supported on λ. Since the measured lamination λ_e is carried by T, the ending lamination λ, being the support of λ_e, is carried by T. Hence every measured lamination in $C(\lambda)$ is carried by T. In fact, we can show more:

Claim 6.5. *For every $n \geqslant 1$, the image of the convex cone $P_1 P_3 \ldots P_{2n-1}(V(T))$ under the natural map to $\mathcal{ML}(S)$ contains $C(\lambda)$.*

Proof. Notice that $P_1 P_3 \ldots P_{2n-1}(V(T))$ is isomorphic to the convex cone of the non-negative real assignments of weights to the branches of the train track $\varphi_{r_1} \ldots \varphi_{r_{2n}}(T)$ that satisfy the switch conditions. It is then sufficient to show that the measured lamination λ_e is carried by the train track $\varphi_{r_1} \ldots \varphi_{r_{2n}}(T)$. Indeed, in this case every measured lamination in $C(\lambda)$ is carried by $\varphi_{r_1} \ldots \varphi_{r_{2n}}(T)$. Since the measured lamination corresponding to the vector $(\prod_{i=n+1}^{\infty} \frac{P_{2i-1}}{c_{2i}})(w_1 + w_3)$ is carried by T, then the measured lamination corresponding to the vector $\frac{P_1}{c_2} \frac{P_3}{c_4} \cdots \frac{P_{2n-1}}{c_{2n}} \cdot (\prod_{i=n+1}^{\infty} \frac{P_{2i-1}}{c_{2i}})(w_1 + w_3)$ is carried by $\varphi_{r_1} \ldots \varphi_{r_{2n}}(T)$. Since the latter measured lamination is λ_e, the result follows. ■

To find all ergodic transverse measures on λ, we study the shapes of the convex cones $P_1 P_3 \ldots P_{2n-1}(V(T))$ as $n \to \infty$. Roughly speaking, we will show that for each $n \geqslant 1$, the set of the generators of the cone $P_1 P_3 \ldots P_{2n-1}(V(T))$ can be divided into two subsets such that the angles between pairs of generators within each of the subsets converge to zero as $n \to \infty$ (Lemma 6.10). From this the upper bound on the number of ergodic transverse measures will follow.

Endow $V(T)$ with the standard inner product with respect to the basis $\{e_1, \ldots, e_9\}$. We start with the following helpful observation.

Claim 6.6. *For $i \geqslant 1$,*

$$\langle P_i(e_1), e_1 \rangle = \frac{c_i}{2}, \quad \langle P_i(e_3), e_3 \rangle = \frac{c_{i+1}}{2}.$$

Proof. By equation (5.3), we have

$$P_i = c_{i+1} MN + c_i NM + M^2 + \phi^{-2(r_i - 1)} LM + \phi^{-2(r_{i+1} - 1)} ML.$$

Notice that $MN(e_1) = 0$ since $e_1 \in \ker N$. We have $NM(e_1) = \frac{1}{2} e_1 + \frac{1}{2\phi} e_2$. It is also a direct check that

$$\langle M^2(e_1), e_1 \rangle = \langle LM(e_1), e_1 \rangle = \langle ML(e_1), e_1 \rangle = 0,$$

hence $\langle P_i(e_1), e_1 \rangle = \frac{c_i}{2}$.

Similarly, we have $MN(e_3) = \frac{1}{2} e_3 + \frac{1}{2\phi} e_4$ and

$$\langle NM(e_3), e_3 \rangle = \langle M^2(e_3), e_3 \rangle = \langle LM(e_3), e_3 \rangle = \langle ML(e_3), e_3 \rangle = 0.$$

Hence $\langle P_i(e_3), e_3 \rangle = \frac{c_{i+1}}{2}$. ■

Next, we prove the following assertion.

Claim 6.7. *For every $n \geqslant 1$, the following holds. If $e_i \notin \ker N$, then*

$$\|P_1 P_3 \dots P_{2n-1} MN(e_i)\| \geqslant \frac{c_2 c_4 \cdots c_{2n}}{2^{n+1}\phi}.$$

If $e_i \in \ker N$, then

$$\|P_1 P_3 \dots P_{2n-1} NM(e_i)\| \geqslant \frac{c_1 c_3 \cdots c_{2n-1}}{2^{n+1}\phi}.$$

Proof. Notice that if $e_i \notin \ker N$, then $\langle MN(e_i), e_3\rangle \geqslant \frac{1}{2\phi}$. Since the matrices P_j are non-negative for $j \geqslant 1$, we have

$$\langle P_j(v), e_3\rangle \geqslant \langle P_j(\langle v, e_3\rangle \cdot e_3), e_3\rangle$$

for all $v \in V(T)$. Since the matrix MN is non-negative by Claim 5.6, applying Claim 6.6 n times, it follows that $\langle P_1 P_3 \dots P_{2n-1} MN(e_i), e_3\rangle \geqslant \frac{c_2 c_4 \cdots c_{2n}}{2^{n+1}\phi}$, therefore

$$\|P_1 P_3 \dots P_{2n-1} MN(e_i)\| \geqslant \frac{c_2 c_4 \cdots c_{2n}}{2^{n+1}\phi}.$$

Similarly, if $e_i \in \ker N$, then $\langle NM(e_i), e_1\rangle \geqslant \frac{1}{2\phi}$. Since the matrices P_j are non-negative for $j \geqslant 1$ and NM is non-negative by Claim 5.6, together with Claim 6.6 it follows that $\langle P_1 P_3 \dots P_{2n-1} NM(e_i), e_1\rangle \geqslant \frac{c_1 c_3 \cdots c_{2n-1}}{2^{n+1}\phi}$, hence $\|P_1 P_3 \dots P_{2n-1} NM(e_i)\| \geqslant \frac{c_1 c_1 \cdots c_{2n-1}}{2^{n+1}\phi}$. ∎

Let K_i be the matrix defined as $K_i = M^2 + \phi^{-2(r_i - 1)} LM + \phi^{-2(r_{i+1}-1)} ML$ for $i \geqslant 1$. Then

$$P_i = c_{i+1} MN + c_i NM + K_i. \tag{6.1}$$

Notice that $\|K_i\| \leqslant \|M\|^2 + \|LM\| + \|ML\|$ for all $i \geqslant 1$.

Claim 6.8. *There is a constant $D > 0$ such that for every $n \geqslant 1$, the following holds:*

$$\begin{aligned} \|P_1 P_3 \dots P_{2n-1} NM\| &\leqslant D^{n+1} \cdot c_1 c_3 \cdots c_{2n-1}, \\ \|P_1 P_3 \dots P_{2n-1} MN\| &\leqslant D^{n+1} \cdot c_2 c_4 \cdots c_{2n}, \\ \|P_1 P_3 \dots P_{2n-1} K_{2n+1}\| &\leqslant D^{n+1} \cdot c_2 c_4 \cdots c_{2n}. \end{aligned}$$

Proof. Consider the first inequality. Expressing each matrix P_i in $P_1 P_3 \dots P_{2n-1} NM$ as in equation (6.1) and expanding the brackets, we obtain a sum of 3^n terms, where each term is an $(n+1)$-factor product of matrices MN, NM and K_i for $i = 1, 3, \dots, 2n-1$ multiplied by a coefficient which is certain product of numbers c_j for $j = 1, 2, \dots, 2n$. It follows from the identity $N^2 = 0$ (Claim 5.6) that some of the matrices in the sum are zero. Eliminating all terms in the sum whose matrices have an N^2 in their expression, we get from equation (5.1) that $c_1c_3\cdots c_{2n-1}$ is the largest of the remaining coefficients. Since the operator norm is sub-multiplicative, the norm of each matrix in the sum is at most $(\max\{\|MN\|, \|NM\|, \|M\|^2 + \|LM\| + \|ML\|\})^{n+1}$, hence by the triangle inequality we have

$$\begin{aligned}&\|P_1P_3\dots P_{2n-1}NM\|\\&\qquad\leqslant 3^n\cdot c_1c_3\cdots c_{2n-1}\cdot(\max\{\|MN\|, \|NM\|, \|M\|^2 + \|LM\| + \|ML\|\})^{n+1}.\end{aligned}$$

Letting $D = 3\cdot\max\{\|MN\|, \|NM\|, \|M\|^2 + \|LM\| + \|ML\|\}$ concludes the first inequality.

For the second and the third inequalities, notice that by equations (6.1) and (5.1) for $i \geqslant 1$ we have

$$\begin{aligned}&\|P_i\| \leqslant c_{i+1}\|MN\| + c_i\|NM\| + \|K_i\|\\&\qquad\leqslant 3\cdot c_{i+1}\cdot\max\{\|MN\|, \|NM\|, \|M\|^2 + \|LM\| + \|ML\|\} = D\cdot c_{i+1},\end{aligned}$$

from which the upper bound follows. ∎

Remark 6.9. It is possible to obtain better upper bounds for $\|P_1P_3\dots P_{2n-1}MN\|$ and $\|P_1P_3\dots P_{2n-1}K_{2n+1}\|$ using Claim 5.8, but weaker bounds will suffice for our purposes.

Lemma 6.10. *There is a constant $D' > 0$ such that for every $n \in \mathbb{N}$ the following holds. If $e_i, e_j \notin \ker N$, then*

$$1 - \cos\angle(P_1P_3\dots P_{2n+1}(e_i), P_1P_3\dots P_{2n+1}(e_j)) \leqslant (D')^{n+1}\cdot\frac{c_1c_3\cdots c_{2n+1}}{c_2c_4\cdots c_{2n+2}}.$$

If $e_i, e_j \in \ker N$, then

$$1 - \cos\angle(P_1P_3\dots P_{2n+1}(e_i), P_1P_3\dots P_{2n+1}(e_j)) \leqslant (D')^{n+1}\cdot\frac{c_2c_4\cdots c_{2n}}{c_1c_3\cdots c_{2n+1}}.$$

Further, in either case $\angle(P_1P_3\dots P_{2n+1}(e_i), P_1P_3\dots P_{2n+1}(e_j)) \to 0$ as $n \to \infty$.

Proof. By equation (6.1), we can write

$$P_1P_3\dots P_{2n+1} = P_1P_3\dots P_{2n-1}(c_{2n+2}MN + c_{2n+1}NM + K_{2n+1}).$$

Consider the first inequality. Let

$$\begin{aligned}v_1 &= c_{2n+2}P_1P_3\dots P_{2n-1}MN(e_i),\\v_2 &= c_{2n+2}P_1P_3\dots P_{2n-1}MN(e_j),\\w_1 &= P_1P_3\dots P_{2n-1}(c_{2n+1}NM + K_{2n+1})(e_i),\\w_2 &= P_1P_3\dots P_{2n-1}(c_{2n+1}NM + K_{2n+1})(e_j).\end{aligned}$$

Notice that $v_1 + w_1 = P_1 P_3 \dots P_{2n+1}(e_i)$ and $v_2 + w_2 = P_1 P_3 \dots P_{2n+1}(e_j)$. We also have $\angle(v_1, v_2) = 0$ since the vectors $MN(e_i)$ and $MN(e_j)$ are collinear. By Claims 6.7 and 6.8, we have

$$\frac{c_2 c_4 \cdots c_{2n+2}}{2^{n+1}\phi} \leqslant \|v_1\|, \|v_2\| \leqslant D^{n+1} \cdot c_2 c_4 \cdots c_{2n+2},$$
$$\|w_1\|, \|w_2\| \leqslant D^{n+1} \cdot (c_1 c_3 \cdots c_{2n+1} + c_2 c_4 \cdots c_{2n}) \leqslant 2 \cdot D^{n+1} \cdot c_1 c_3 \cdots c_{2n+1}.$$

Then by Lemma A.2, we have

$$\begin{aligned}
&1 - \cos \angle(P_1 P_3 \dots P_{2n+1}(e_i), P_1 P_3 \dots P_{2n+1}(e_j)) \\
&\quad \leqslant \frac{2(4 \cdot D^{2n+2} \cdot c_2 c_4 \cdots c_{2n+2} \cdot c_1 c_3 \cdots c_{2n+1} + 4 \cdot D^{2n+2} \cdot (c_1 c_3 \cdots c_{2n+1})^2)}{\frac{c_2 c_4 \cdots c_{2n+2}}{2^{n+1}\phi} \cdot \frac{c_2 c_4 \cdots c_{2n+2}}{2^{n+1}\phi}} \\
&\quad \leqslant 2^{2n+2}\phi^2 \cdot 8 \cdot D^{2n+2} \cdot \frac{2 c_2 c_4 \cdots c_{2n+2} \cdot c_1 c_3 \cdots c_{2n+1}}{(c_2 c_4 \cdots c_{2n+2})^2} \\
&\quad \leqslant 2^{2n+6}\phi^2 \cdot D^{2n+2} \cdot \frac{c_1 c_3 \cdots c_{2n+1}}{c_2 c_4 \cdots c_{2n+2}}.
\end{aligned}$$

For the second inequality, notice that

$$c_{2n+2} P_1 P_3 \dots P_{2n-1} MN(e_i) = c_{2n+2} P_1 P_3 \dots P_{2n-1} MN(e_j) = 0$$

since $N(e_i) = N(e_j) = 0$. We let

$$\begin{aligned}
v_1' &= c_{2n+1} P_1 P_3 \dots P_{2n-1} NM(e_i), \\
v_2' &= c_{2n+1} P_1 P_3 \dots P_{2n-1} NM(e_j), \\
w_1' &= P_1 P_3 \dots P_{2n-1} K_{2n+1}(e_i), \\
w_2' &= P_1 P_3 \dots P_{2n-1} K_{2n+1}(e_j).
\end{aligned}$$

Notice that $v_1' + w_1' = P_1 P_3 \dots P_{2n+1}(e_i)$ and $v_2' + w_2' = P_1 P_3 \dots P_{2n+1}(e_j)$. We also have $\angle(v_1', v_2') = 0$ since the vectors $NM(e_i)$ and $NM(e_j)$ are collinear. Then by Lemma A.2 and Claims 6.7 and 6.8, we have

$$\begin{aligned}
&1 - \cos \angle(P_1 P_3 \dots P_{2n+1}(e_i), P_1 P_3 \dots P_{2n+1}(e_j)) \\
&\quad \leqslant \frac{2(2 \cdot D^{2n+2} \cdot c_1 c_3 \cdots c_{2n+1} \cdot c_2 c_4 \cdots c_{2n} + D^{2n+2} \cdot (c_2 c_4 \cdots c_{2n})^2)}{\frac{c_1 c_3 \cdots c_{2n+1}}{2^{n+1}\phi} \cdot \frac{c_1 c_3 \cdots c_{2n+1}}{2^{n+1}\phi}} \\
&\quad \leqslant 2^{2n+2}\phi^2 \cdot 2 \cdot D^{2n+2} \cdot \frac{3 c_1 c_3 \cdots c_{2n+1} \cdot c_2 c_4 \cdots c_{2n}}{(c_1 c_3 \cdots c_{2n+1})^2} \\
&\quad \leqslant 2^{2n+3}\phi^2 \cdot 3 \cdot D^{2n+2} \cdot \frac{c_2 c_4 \cdots c_{2n}}{c_1 c_3 \cdots c_{2n+1}}.
\end{aligned}$$

Letting $D' = 2^4\phi \cdot 3 \cdot D^2$ concludes the desired inequalities. By equation (5.1), we have $\frac{c_n}{c_{n+1}} \to 0$ as $n \to \infty$. Hence for sufficiently large n, the upper bounds for $1 - \cos \angle(P_1 P_3 \dots P_{2n+1}(e_i), P_1 P_3 \dots P_{2n+1}(e_j))$ decrease at least exponentially with n, therefore

$$\angle(P_1 P_3 \dots P_{2n+1}(e_i), P_1 P_3 \dots P_{2n+1}(e_j)) \to 0 \quad \text{as } n \to \infty. \qquad \blacksquare$$

Proposition 6.11. *The measured laminations λ_e, λ_o are ergodic. Further, any transverse measure on λ is a linear combination of λ_e and λ_o.*

Proof. Let $\Delta = \{x_1e_1 + \cdots + x_9e_9 \mid \sum_{i=1}^{9} x_i = 1\} \subset V(T)$ be the standard unit simplex. Notice that the set $P_1P_3 \ldots P_{2n-1}(V(T)) \cap \Delta$ is the convex hull of the points $\operatorname{span}(P_1P_3 \ldots P_{2n-1}(e_i)) \cap \Delta$ for $i = 1, \ldots, 9$. Since $\prod_{i=1}^{\infty} \frac{p_{2i-1}}{c_{2i}}$ converges (see the proof of Claim 6.1), the sequence of compact sets $P_1P_3 \ldots P_{2n-1}(V(T)) \cap \Delta$ converges in the Hausdorff metric on $V(T)$ as $n \to \infty$. It follows from Lemma 6.10 that the limiting set is either an interval or a point. If λ admits at least three ergodic transverse measures up to scalar, then the set of points in Δ that correspond to measured laminations in $C(\lambda)$ contains a convex triangle. Let $R > 0$ be the radius of the circumscribed circle of such triangle. For sufficiently large n such that the Hausdorff distance from $P_1P_3 \ldots P_{2n-1}(V(T)) \cap \Delta$ to the limiting set is less than $\frac{R}{8}$, the set $P_1P_3 \ldots P_{2n-1}(V(T)) \cap \Delta$ cannot contain a 2-dimensional disk of radius r: it is immediate if the limiting set is a point and if the limiting set is an interval it follows from Lemma A.3 since $R > 2\sqrt{2(8-1)}\frac{R}{8}$. Since the projective class of every non-zero measure in $C(\lambda)$ is represented in $P_1P_3 \ldots P_{2n-1}(V(T)) \cap \Delta$ by Claim 6.5, we arrive at a contradiction. Hence λ admits at most two ergodic transverse measures up to a scalar. Together with Proposition 6.4, we obtain that λ admits exactly two ergodic transverse measured up to scalar.

Together with Claim 6.2, it now follows that λ_e and λ_o are ergodic by a well-known argument, see, for example, [9, Lemma 6.3]. ∎

7. Relative twisting bounds

In this section, we prove that the lamination λ constructed in Section 3 has uniformly bounded annular projection distances. To show this, we return to the sequence of curves $\{\alpha_i\}$, defined in Section 3.1. First we prove the following lemma.

Lemma 7.1. *For every $i \geqslant 2$,*

$$d_{\alpha_i}(\alpha_{i-2}, \alpha_{i+2}) \leqslant 7.$$

Proof. By equation (3.2), the triangle inequality and Lemma 2.3, we have

$$\begin{aligned}
d_{\alpha_i}(\alpha_{i-2}, \alpha_{i+2}) &= d_{\alpha_2}(\alpha_0, \varphi_{r_{i-1}}\alpha_3) = d_{\alpha_2}(\alpha_0, \tau^{r_{i-1}}(\rho\alpha_3)) \\
&\leqslant d_{\alpha_2}(\alpha_0, \rho\alpha_3) + d_{\alpha_2}(\rho\alpha_3, \tau^{r_{i-1}}(\rho\alpha_3)) \\
&\leqslant (i(\alpha_0, \rho\alpha_3) + 1) + d_{\alpha_2}(\rho\alpha_3, \tau^{r_{i-1}}(\rho\alpha_3)) \\
&= 3 + d_{\alpha_2}(\rho\alpha_3, \tau^{r_{i-1}}(\rho\alpha_3)).
\end{aligned}$$

We will prove that $d_{\alpha_2}(\rho\alpha_3, \tau^{r_{i-1}}(\rho\alpha_3)) \leqslant 4$. We remark that this is not trivial since by equation (5.1) the sequence $\{r_n\}$ converges to infinity, and a partial pseudo-Anosov homeomorphism can act as a root of the Dehn twist on a boundary curve.

Choose a marked complete hyperbolic metric X on S of finite volume. In what follows, we refer to Figure 18. Let $\widetilde{\alpha_2}$ and $\widetilde{\rho\alpha_3}$ be geodesic lifts of α_2 and $\rho\alpha_3$ in the universal cover $\widetilde{X} \cong \mathbb{H}^2$ intersecting at the point $O \in \mathbb{H}^2$ (geodesics intersecting perpendicularly is not meant literally, but for the convenience of the picture). Let δ_0, δ_1 be the curves on S shown in Figure 1. Let $\widetilde{\delta_0}$ and $\widetilde{\delta_0'}$ be the geodesic lifts of δ_0 in $\mathbb{H}^2$ that intersect $\widetilde{\rho\alpha_3}$ at the points $A, A' \in \mathbb{H}^2$, respectively, such that the geodesic segment $[AA'] \subset \mathbb{H}^2$ contains the point O and does not contain any other intersection points of lifts of δ_0 with $\widetilde{\rho\alpha_3}$ (such lifts exist since $i(\rho\alpha_3, \delta_0) > 0$). Let $q, r \in \partial\mathbb{H}^2$ be the endpoints of $\widetilde{\delta_0}$ and let $q', r' \in \partial\mathbb{H}^2$ be the endpoints of $\widetilde{\delta_0'}$. Let $(qr) \subset \partial\mathbb{H}^2$ be an open interval such that $q', r' \notin (qr)$ and let $(q'r') \subset \partial\mathbb{H}^2$ be an open interval such that $q, r \notin (q'r')$. We will show that there is a lift $\widetilde{\tau}\colon \mathbb{H}^2 \to \mathbb{H}^2$ of τ such that the endpoints of $\widetilde{\tau}^n(\widetilde{\rho\alpha_3})$ are in $(qr) \cup (q'r')$ for all $n \in \mathbb{N}$. Assuming this, we can conclude as follows. Consider the orbit of $\widetilde{\rho\alpha_3}$ under the hyperbolic isometry $\langle\alpha_2\rangle$ corresponding to the curve α_2 that fixes $\widetilde{\alpha_2}$. Notice that the endpoints of the elements in the orbit are contained in the corresponding elements in the orbits of the intervals (qr), $(q'r')$. Since the curve δ_0 is simple, the lift $\widetilde{\rho\alpha_3}$ is the only element in the orbit with endpoints in $(qr) \cup (q'r')$. It follows that for every $n \in \mathbb{N}$, the geodesic representative of $\widetilde{\tau}^n(\widetilde{\rho\alpha_3})$ intersects at most one such element. Projecting to the annulus $\mathbb{H}^2/\langle\alpha_2\rangle$, we obtain lifts of $\rho\alpha_3$ and $\tau^n(\rho\alpha_3)$ connecting two boundary components that intersect at most once. Then from the definition of the distance in the curve graph of an annulus, it follows that

$$d_{\alpha_2}(\rho\alpha_3, \tau^n(\rho\alpha_3)) \leqslant 2 + 1 + 1 = 4 \quad \text{for every } n \in \mathbb{N}.$$

Now we show that there is a lift $\widetilde{\tau}\colon \mathbb{H}^2 \to \mathbb{H}^2$ of τ such that an endpoint of $\widetilde{\tau}^n(\widetilde{\rho\alpha_3})$ is in (qr) for all $n \in \mathbb{N}$, the argument for $(q'r')$ is similar. Let $\widetilde{H_{\delta_0}}\colon \mathbb{H}^2 \to \mathbb{H}^2$ be a lift of the half-twist H_{δ_0} such that $\widetilde{H_{\delta_0}}(O) = O$. Similarly, let $\widetilde{H_{\delta_1}^{-1}}\colon \mathbb{H}^2 \to \mathbb{H}^2$ be a lift of the half-twist $H_{\delta_1}^{-1}$ such that $\widetilde{H_{\delta_1}^{-1}}(O) = O$. Then the map $\widetilde{\tau}\colon \mathbb{H}^2 \to \mathbb{H}^2$ defined as $\widetilde{\tau} = \widetilde{H_{\delta_1}^{-1}} \circ \widetilde{H_{\delta_0}}$ is a lift of τ. Let $p \in (qr)$ be an endpoint of $\widetilde{\rho\alpha_3}$. Let $\widetilde{\delta_1}$ be the geodesic lift of δ_1 in $\mathbb{H}^2$ that intersects $\widetilde{\delta_0}$ at the point $B \in \mathbb{H}^2$ such that $B \in [A, q)$ and such that the geodesic segment $[AB] \subset \mathbb{H}^2$ does not contain any other intersection points of lifts of δ_1 with $\widetilde{\delta_0}$ (such

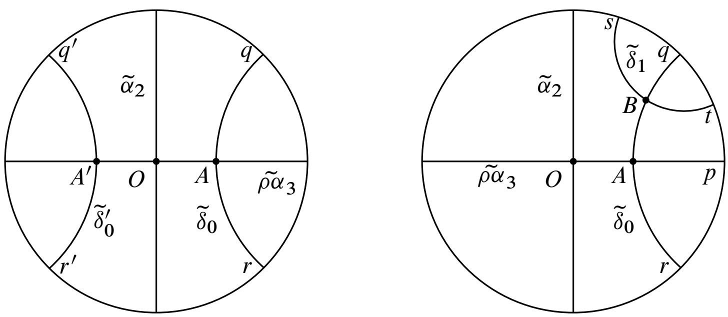

Figure 18. Left: lifts of the curves α_2, $\rho\alpha_3$, δ_0 in the universal cover. Right: lifts of the curves α_2, $\rho\alpha_3$, δ_0, δ_1 in the universal cover.

a lift exists since $i(\delta_0,\delta_1)>0$). Let $s,t\in\partial\mathbb{H}^2$ be the endpoints of $\widetilde{\delta_1}$ and let $(st)\subset\partial\mathbb{H}^2$ be an open interval such that $q\in(st)$. Since δ_1 does not intersect $\rho\alpha_3$ essentially, we have $p\notin(st)$. It follows from [41, Proposition 2.1] that the boundary extension of $\widetilde{H_{\delta_0}}$ fixes $q,r\in\partial\mathbb{H}^2$ and moves all points in (qr) counterclockwise (Proposition 2.1 is about Dehn twists, but the argument applies to half-twists as well). Similarly, the boundary extension of $\widetilde{H_{\delta_1}^{-1}}$ fixes $s,t\in\partial\mathbb{H}^2$ and moves all points in (st) clockwise. Then since $H_{\delta_0}(\rho\alpha_3)$ intersects δ_1 essentially, it follows that $\widetilde{H_{\delta_0}}(p)\in(qt)$. Then $\widetilde{\tau}(p)\in(qt)$ and hence $\widetilde{\tau}^n(p)\in(qr)$ for all $n\geqslant 1$ since $\widetilde{\tau}^n((q,t))\subset(q,t)\subset(q,r)$. ■

Let $\mu=\{\alpha_0,\alpha_5\}$ be a collection of curves on S. By Lemma 3.5, μ is a marking on S. We prove the following assertion.

Proposition 7.2. *There is a constant $E>0$ such that the following holds. For every curve γ on S, there is $j_\gamma\in\mathbb{N}$ such that for all $j\geqslant j_\gamma$, the curve α_j intersects γ essentially and*

$$d_\gamma(\mu,\alpha_j)\leqslant E.$$

Proof. If the curve γ intersects every curve α_j essentially for $j\in\mathbb{N}$, then by Corollary 2.5 we have

$$d_\gamma(\mu,\alpha_j)=\max\{d_\gamma(\alpha_0,\alpha_j),d_\gamma(\alpha_5,\alpha_j)\}\leqslant A$$

for every $j\in\mathbb{N}$. Otherwise, the curve γ is disjoint from some curves in the sequence $\{\alpha_i\}$. If γ is disjoint from α_0, then by Lemma 3.5, γ intersects every α_j essentially for $j\geqslant 5$. Then by Corollary 2.5,

$$d_\gamma(\mu,\alpha_j)=d_\gamma(\alpha_5,\alpha_j)\leqslant A$$

for every $j\geqslant 5$. If γ intersects α_0 essentially, let $\ell\in\mathbb{N}$ be the smallest index so that γ is disjoint from α_ℓ and $r\geqslant\ell$ be the largest index so that γ is disjoint from α_r. By Lemma 3.5, we have $r-\ell\leqslant 4$. Let $j_\gamma=r+1$. Then for $j\geqslant j_\gamma=r+1$, by the triangle inequality we have

$$d_\gamma(\mu,\alpha_j)\leqslant d_\gamma(\mu,\alpha_{\ell-1})+d_\gamma(\alpha_{\ell-1},\alpha_{r+1})+d_\gamma(\alpha_{r+1},\alpha_j). \tag{7.1}$$

By Corollary 2.5, $d_\gamma(\alpha_{r+1},\alpha_j)\leqslant A$. From now on, we assume that $A\geqslant 4$. Next, we show that $d_\gamma(\mu,\alpha_{\ell-1})\leqslant A+d_\gamma(\alpha_{\ell-1},\alpha_{r+1})$, thus it will remain to find an upper bound for $d_\gamma(\alpha_{\ell-1},\alpha_{r+1})$. If γ intersects α_0 essentially and is disjoint from α_5, then by Corollary 2.5,

$$d_\gamma(\mu,\alpha_{\ell-1})=d_\gamma(\alpha_0,\alpha_{\ell-1})\leqslant A.$$

Suppose that γ intersects both α_0 and α_5 essentially. If $\ell-1\geqslant 5$, then by Corollary 2.5,

$$d_\gamma(\mu,\alpha_{\ell-1})=\max\{d_\gamma(\alpha_0,\alpha_{\ell-1}),d_\gamma(\alpha_5,\alpha_{\ell-1})\}\leqslant A.$$

If $\ell - 1 < 5$ and $r + 1 \leqslant 5$, then by the triangle inequality and Corollary 2.5,

$$d_\gamma(\alpha_5, \alpha_{\ell-1}) \leqslant d_\gamma(\alpha_5, \alpha_{r+1}) + d_\gamma(\alpha_{r+1}, \alpha_{\ell-1}) \leqslant A + d_\gamma(\alpha_{\ell-1}, \alpha_{r+1}).$$

Therefore, we have

$$\begin{aligned} d_\gamma(\mu, \alpha_{\ell-1}) &= \max\{d_\gamma(\alpha_0, \alpha_{\ell-1}), d_\gamma(\alpha_5, \alpha_{\ell-1})\} \leqslant \max\{A, A + d_\gamma(\alpha_{\ell-1}, \alpha_{r+1})\} \\ &= A + d_\gamma(\alpha_{\ell-1}, \alpha_{r+1}). \end{aligned}$$

If $\ell - 1 < 5$ and $r + 1 > 5$, then since γ intersects α_5 essentially, we have $\ell < 5$ and $r > 5$. Then the curves in $\{\alpha_i\}$ that are disjoint from γ are not consecutive. It follows from Corollary 3.11 that either $r - 1 = 5$ or $r - 2 = 5$ and that γ intersects α_{r-1} essentially. In the first case, by Lemma 2.3 and equation (3.2) we have

$$d_\gamma(\alpha_5, \alpha_{r+1}) = d_\gamma(\alpha_{r-1}, \alpha_{r+1}) \leqslant i(\alpha_{r-1}, \alpha_{r+1}) + 1 = i(\alpha_0, \alpha_2) + 1 = 3.$$

In the second case, by the triangle inequality

$$d_\gamma(\alpha_5, \alpha_{r+1}) = d_\gamma(\alpha_{r-2}, \alpha_{r+1}) \leqslant d_\gamma(\alpha_{r-2}, \alpha_{r-1}) + d_\gamma(\alpha_{r-1}, \alpha_{r+1}).$$

Notice that $d_\gamma(\alpha_{r-2}, \alpha_{r-1}) = 1$ since α_{r-2} and α_{r-1} are disjoint. As $d_\gamma(\alpha_{r-1}, \alpha_{r+1}) \leqslant 3$, we have $d_\gamma(\alpha_5, \alpha_{r+1}) \leqslant 4$. We obtain that if $\ell - 1 < 5$ and $r + 1 > 5$, then

$$\begin{aligned} d_\gamma(\mu, \alpha_{\ell-1}) &= \max\{d_\gamma(\alpha_0, \alpha_{\ell-1}), d_\gamma(\alpha_5, \alpha_{\ell-1})\} \\ &\leqslant \max\{A, d_\gamma(\alpha_5, \alpha_{r+1}) + d_\gamma(\alpha_{r+1}, \alpha_{\ell-1})\} \\ &\leqslant \max\{A, 4 + d_\gamma(\alpha_{r+1}, \alpha_{\ell-1})\} \\ &\leqslant A + d_\gamma(\alpha_{\ell-1}, \alpha_{r+1}). \end{aligned} \tag{7.2}$$

Now we find an upper bound for $d_\gamma(\alpha_{\ell-1}, \alpha_{r+1})$. Depending on the value of $r - \ell$, we consider the following cases.

Case $r - \ell = 4$. By Claim 3.7, we have $\gamma = \beta_\ell$. By equation (3.2), we have

$$d_\gamma(\alpha_{\ell-1}, \alpha_{r+1}) = d_{\beta_\ell}(\alpha_{\ell-1}, \alpha_{\ell+5}) = d_{\beta_0}(\varphi_{r_\ell}^{-1}(\alpha_0), \varphi_{r_{\ell+1}}\varphi_{r_{\ell+2}}(\alpha_3)).$$

Recall that $\varphi_n = \tau^n \rho$, where τ is partial pseudo-Anosov supported on Y_2. Let ν^s and ν^u be the stable and unstable laminations of τ contained in Y_2. Let $\delta_0, \delta_1 \subset Y_2$ be the curves as in Figure 1. Curves in Y_2 converge to ν^u under positive powers of τ and to ν^s under negative powers of τ in the Hausdorff topology, possibly together with finitely many extra leaves (for the convergence argument, see, for example, [12, pp. 24–26]). Thus, by Lemma 2.7 we can find $N \in \mathbb{N}$ sufficiently large so that for all $n \geqslant N$, $d_{\delta_0}(\nu^u, \tau^n(\delta_1)) \leqslant 9$ and $d_{\delta_1}(\nu^s, \tau^{-n}(\delta_0)) \leqslant 9$.

Since Y_2 and $\rho(Y_2) = Y_3$ are disjoint, for $m, n \geqslant 1$ we have

$$\varphi_m \varphi_n(\rho^{-1}\delta_1) = \rho\tau^n(\delta_1).$$

Note that δ_0 is disjoint from α_0 and that $\beta_0 = \rho^{-1}(\delta_1)$. Likewise, $\rho^{-1}(\delta_1)$ is disjoint from α_3 and $\beta_0 = \rho(\delta_0)$. Also note that $\beta_0 \subset Y_1 = \rho^{-1}(Y_2)$, so $\rho^{-1}(\nu^s)$ intersects β_0 essentially. Likewise, $\beta_0 \subset Y_3 = \rho(Y_2)$, so $\rho(\nu^u)$ also intersects β_0 essentially. Combining these facts, and using the triangle inequality, we get that for all $k, m, n \geqslant N$,

$$\begin{aligned}
&d_{\beta_0}(\varphi_k^{-1}(\alpha_0), \varphi_m\varphi_n(\alpha_3)) \\
&\quad \leqslant 4 + d_{\beta_0}(\varphi_k^{-1}(\delta_0), \varphi_m\varphi_n(\rho^{-1}\delta_1)) = 4 + d_{\beta_0}(\rho^{-1}\tau^{-k}(\delta_0), \rho\tau^n(\delta_1)) \\
&\quad \leqslant 4 + d_{\delta_1}(\tau^{-k}(\delta_0), \nu^s) + d_{\beta_0}(\rho^{-1}(\nu^s), \rho(\nu^u)) + d_{\delta_0}(\nu^u, \tau^n(\delta_1)) \\
&\quad \leqslant 22 + d_{\beta_0}(\rho^{-1}(\nu^s), \rho(\nu^u)).
\end{aligned}$$

Recall that by equation (5.1), the sequence $\{r_n\}$ goes to infinity. Then there exists $I \in \mathbb{N}$ such that $r_i \geqslant N$ for all $i \geqslant I$. Let

$$K = \max_{1 \leqslant i \leqslant I}\{d_{\beta_0}(\varphi_{r_i}^{-1}(\alpha_0), \varphi_{r_{i+1}}\varphi_{r_{i+2}}(\alpha_3))\}.$$

Then we have

$$d_\gamma(\alpha_{\ell-1}, \alpha_{r+1}) \leqslant 22 + d_{\beta_0}(\rho^{-1}(\nu^s), \rho(\nu^u)) + K. \tag{7.3}$$

Case $r - \ell = 3$. By Claim 3.8 and the triangle inequality, we can write

$$\begin{aligned}
d_\gamma(\alpha_{\ell-1}, \alpha_{r+1}) &= d_\gamma(\alpha_{\ell-1}, \alpha_{\ell+4}) \\
&\leqslant d_\gamma(\alpha_{\ell-1}, \alpha_{\ell+1}) + d_\gamma(\alpha_{\ell+1}, \alpha_{\ell+2}) + d_\gamma(\alpha_{\ell+2}, \alpha_{\ell+4}).
\end{aligned}$$

Notice that $d_\gamma(\alpha_{\ell+1}, \alpha_{\ell+2}) = 1$ since $\alpha_{\ell+1}$ and $\alpha_{\ell+2}$ are disjoint. By Lemma 2.3, we also have

$$d_\gamma(\alpha_{\ell-1}, \alpha_{\ell+1}) \leqslant i(\alpha_{\ell-1}, \alpha_{\ell+1}) + 1 = i(\alpha_0, \alpha_2) + 1 = 3.$$

Similarly, $d_\gamma(\alpha_{\ell+2}, \alpha_{\ell+4}) \leqslant 3$. Hence $d_\gamma(\alpha_{\ell-1}, \alpha_{r+1}) \leqslant 7$.

Case $r - \ell = 2$. If γ is disjoint from α_ℓ and $\alpha_{\ell+2}$, then one of the following holds: $\gamma = \beta_{\ell-1}$, $\gamma = \alpha_{\ell+1}$, $\gamma \subset Y_{\ell+1}$. Indeed, by applying the homeomorphism $\Phi_{\ell-1}^{-1}$ and by equation (3.2) it is enough to consider the case $\ell = 1$, which follows from Figure 1. If $\gamma = \beta_{\ell-1}$, then by Claim 3.7 γ is disjoint from $\alpha_{\ell+3}$, which is impossible. If $\gamma = \alpha_{\ell+1}$, then by Lemma 7.1, we have

$$d_\gamma(\alpha_{\ell-1}, \alpha_{r+1}) = d_{\alpha_{\ell+1}}(\alpha_{\ell-1}, \alpha_{\ell+3}) \leqslant 7.$$

If $\gamma \subset Y_{\ell+1}$, we have

$$d_\gamma(\alpha_{\ell-1}, \alpha_{r+1}) = d_\gamma(\alpha_{\ell-1}, \alpha_{\ell+3}) = d_{\Phi_{\ell-1}^{-1}\gamma}(\alpha_0, \tau^{r_\ell}(\rho\alpha_3)),$$

where $\Phi_{\ell-1}^{-1}\gamma$ is a curve in Y_2. Denote the curve $\Phi_{\ell-1}^{-1}\gamma$ by γ'.

Notice that $\pi_{Y_2}(\alpha_0) = \delta_0$ and that α_0 is disjoint from δ_0. Likewise, $\pi_{Y_2}(\rho\alpha_3) = \delta_1$ and $\rho\alpha_3$ is disjoint from δ_1. Note that if a curve on S intersects Y_2 essentially, then it intersects γ' essentially if and only if its projection to Y_2 intersects γ' essentially. This is because every essential arc of intersection with Y_2 projects to a disjoint curve, and any two distinct curves in Y_2 intersect each other. It follows that δ_0 intersects γ' essentially. Further, if $\tau^k(\delta_1)$ intersects γ' essentially for some $k \in \mathbb{N}$, then $\tau^k(\rho\alpha_3)$ does as well. Then if $n \in \mathbb{N}$ is sufficiently large so that $d_{\gamma'}(\nu^u, \tau^n(\delta_1)) \leqslant 9$, we have

$$\begin{aligned} d_{\gamma'}(\alpha_0, \tau^n(\rho\alpha_3)) &\leqslant 4 + d_{\gamma'}(\delta_0, \tau^n(\delta_1)) \\ &\leqslant 4 + d_{\gamma'}(\delta_0, \nu^u) + d_{\gamma'}(\nu^u, \tau^n(\delta_1)) \\ &\leqslant 13 + d_{\gamma'}(\delta_0, \nu^u). \end{aligned}$$

By Proposition 2.8, we have $d_{\gamma'}(\delta_0, \nu^u) \leqslant C_{\tau,\delta_0}$. Then using the triangle inequality, we can write

$$\begin{aligned} d_{\gamma'}(\alpha_0, \tau^{r_\ell}(\rho\alpha_3)) &\leqslant d_{\gamma'}(\alpha_0, \tau^n(\rho\alpha_3)) + d_{\gamma'}(\tau^n(\rho\alpha_3), \tau^{r_\ell}(\rho\alpha_3)) \\ &\leqslant (13 + C_{\tau,\delta_0}) + d_{\tau^{-r_\ell}(\gamma')}(\tau^{n-r_\ell}(\rho\alpha_3), \rho\alpha_3). \end{aligned}$$

Denote the curve $\tau^{-r_\ell}(\gamma')$ by γ''. Notice that

$$d_{\gamma''}(\tau^{n-r_\ell}(\delta_1), \nu^u) = d_{\gamma'}(\tau^n(\delta_1), \tau^{r_\ell}(\nu^u)) = d_{\gamma'}(\tau^n(\delta_1), \nu^u) \leqslant 9.$$

Then we similarly have

$$\begin{aligned} d_{\gamma''}(\tau^{n-r_\ell}(\rho\alpha_3), \rho\alpha_3) &\leqslant 4 + d_{\gamma''}(\tau^{n-r_\ell}(\delta_1), \delta_1) \\ &\leqslant 4 + d_{\gamma''}(\tau^{n-r_\ell}(\delta_1), \nu^u) + d_{\gamma''}(\nu^u, \delta_1) \\ &\leqslant 13 + C_{\tau,\delta_1}. \end{aligned}$$

Therefore,

$$d_\gamma(\alpha_{\ell-1}, \alpha_{r+1}) \leqslant C_{\tau,\delta_0} + C_{\tau,\delta_1} + 26. \tag{7.4}$$

Case $r - \ell = 1$. This case is impossible by Claim 3.10.

Case $r - \ell = 0$. By Lemma 2.3, we have

$$d_\gamma(\alpha_{\ell-1}, \alpha_{r+1}) = d_\gamma(\alpha_{\ell-1}, \alpha_{\ell+1}) \leqslant i(\alpha_{\ell-1}, \alpha_{\ell+1}) + 1 = i(\alpha_0, \alpha_2) + 1 = 3.$$

Finally, according to equations (7.1) and (7.2), if we set $E = 2A + 2F$, where $F > 0$ is the maximum of the expressions obtained in equations (7.3) and (7.4), then

$$d_\gamma(\mu, \alpha_j) \leqslant E$$

for every $j \geqslant j_\gamma = r + 1$, which concludes the proof. ■

In the following corollary, λ is the non-uniquely ergodic ending lamination on S constructed in Section 3.

Corollary 7.3. *There is a constant $E' > 0$ such that $d_\gamma(\mu, \lambda) \leqslant E'$ for all curves γ on S.*

Proof. By Corollary 3.4, there is a subsequence of $\{\alpha_i\}$ that converges in the Hausdorff topology on $\mathcal{GL}(S)$ to a geodesic lamination λ' that contains λ. Taking an index $i \in \mathbb{N}$ in the subsequence sufficiently large so that Lemma 2.7 applies to the annular subsurface of a curve γ on S and so that α_i intersects γ essentially, we obtain

$$d_\gamma(\mu, \alpha_i) \overset{+}{\asymp}_8 d_\gamma(\mu, \lambda').$$

Since $\lambda \subset \lambda'$, we have $d_\gamma(\mu, \lambda) \leqslant d_\gamma(\mu, \lambda')$. Taking $i \in \mathbb{N}$ sufficiently large so that Proposition 7.2 applies as well, we have

$$d_\gamma(\mu, \lambda) \leqslant d_\gamma(\mu, \lambda') \leqslant d_\gamma(\mu, \alpha_i) + 8 \leqslant E + 8.$$

Letting $E' = E + 8$ concludes the proof. ∎

We remark that not all projection distances for λ are uniformly bounded. We prove the following assertion.

Claim 7.4. *Let ν be a minimal, filling geodesic lamination on S such that $d_Y(\mu, \nu) \leqslant G$ for some constant $G > 0$ and all subsurfaces $Y \subset S$. Then*

$$d_{Y_i}(\nu, \lambda) \geqslant c r_{i-1} - G - 18$$

for all $i \geqslant 2$.

Proof. By Lemma 3.2, we have

$$d_{Y_i}(\mu, \alpha_j) \geqslant c r_{i-1} - 9$$

for all $i \geqslant 2$ and $j \geqslant i + 2$. By an argument, similar to the one in Corollary 7.3, we have

$$d_{Y_i}(\mu, \lambda) \geqslant (c r_{i-1} - 9) - 9$$

for all $i \geqslant 2$. Then by the triangle inequality, we have

$$d_{Y_i}(\nu, \lambda) \geqslant d_{Y_i}(\mu, \lambda) - d_{Y_i}(\mu, \nu) \geqslant c r_{i-1} - 18 - G$$

for all $i \geqslant 2$. ∎

We obtain the following corollary which in contrast with Theorem 1.1.

Corollary 7.5. *Suppose X_t is a Teichmüller geodesic such that the support of the lamination that corresponds to its vertical foliation contains the support of λ constructed in Section 3, and such that the support of the lamination that corresponds to its horizontal foliation contains the support of ν as in Claim 7.4. Then for all sufficiently large $i \in \mathbb{N}$, the minimal length ℓ_{α_i} of the curve α_i along X_t satisfies*

$$\ell_{\alpha_i} \overset{*}{\prec} \frac{1}{r_{i-1}}.$$

Proof. Since by equation (5.1), the sequence $\{r_n\}$ converges to infinity, we can choose $i_G \geqslant 2$ such that $cr_{i-1} - G - 18 \geqslant \frac{c}{2} r_{i-1}$ for all $i \geqslant i_G$. Then the statement follows from Claim 7.4 and [39, Theorem 6.1]. In particular, X_t does not stay in the thick part of the Teichmüller space. Moreover, it follows from Theorem 1.2 that X_t diverges in the moduli space as $t \to \infty$. ∎

8. Geodesics in the thick part

In this section, we prove Theorem 1.1. First, we prove some technical lemmas.

Lemma 8.1. *Let $X_n \in \mathcal{T}(S)$ be a sequence in Teichmüller space converging to $[\xi]$ in the Thurston boundary, and let η_n be a curve on S such that $\ell_{\eta_n}(X_n) \leqslant C$ for some $C > 0$. If $[\eta]$ is a limit point of the sequence $[\eta_n]$ in the Thurston boundary, then $i(\xi, \eta) = 0$.*

Proof. By definition, there is a sequence $\{a_n\}$ of positive numbers, such that $a_n X_n \to \xi$ as geodesic currents. We have (see [4, Proposition 15])

$$i(a_n X_n, a_n X_n) = a_n^2\, i(X_n, X_n) = a_n^2 \pi^2 |\chi(S)|.$$

By the continuity of the intersection number, $i(a_n X_n, a_n X_n) \to i(\xi, \xi) = 0$, since $\xi \in \mathcal{ML}(S)$. Hence $a_n^2 \to 0$, and in particular, $a_n \to 0$. By definition, there is a sequence $\{b_n\}$ of non-negative numbers, such that $b_n \eta_n \to \eta$ as geodesic currents. Let γ be a filling collection of curves on S, then $i(\gamma, b_n \eta_n) = b_n i(\gamma, \eta_n) \geqslant b_n$. We also have $i(\gamma, b_n \eta_n) \to i(\gamma, \eta) < \infty$. Hence the sequence $\{b_n\}$ is bounded from above, so suppose $b_n \leqslant B$ for some $B > 0$. Then

$$i(a_n X_n, b_n \eta_n) = a_n b_n \ell_{\eta_n}(X_n) \leqslant a_n BC.$$

Since $i(a_n X_n, b_n \eta_n) \to i(\xi, \eta)$, we obtain $i(\xi, \eta) = 0$. ∎

Let $B(S)$ be a Bers constant of S. We prove the following lemma.

Lemma 8.2. *Let $X_n, Y_n \in \mathcal{T}(S)$ be sequences in Teichmüller space converging to $[\xi]$ and $[\zeta]$ in the Thurston boundary, respectively. Suppose that the supports of ξ and ζ are minimal and filling. If α is a curve on S such that $\ell_\alpha(X_n), \ell_\alpha(Y_n) > B(S)$ for all $n \in \mathbb{N}$, then*

$$d_\alpha(X_n, Y_n) \stackrel{+}{\asymp} d_\alpha(\xi, \zeta)$$

for infinitely many $n \in \mathbb{N}$.

Proof. It follows from the definition of a Bers constant that for every $n \in \mathbb{N}$, there are curves η_n and ν_n on S that intersect α essentially such that $\ell_{\eta_n}(X_n), \ell_{\nu_n}(X_n) \leqslant B(S)$. By the triangle inequality, we have

$$|d_\alpha(X_n, Y_n) - d_\alpha(\eta_n, \nu_n)| \leqslant d_\alpha(X_n, \eta_n) + d_\alpha(Y_n, \nu_n).$$

Hence it is sufficient to show that $d_\alpha(X_n, \eta_n), d_\alpha(Y_n, \nu_n) \stackrel{+}{\asymp} 0$ and that $d_\alpha(\eta_n, \nu_n) \stackrel{+}{\asymp} d_\alpha(\xi, \zeta)$ for infinitely many $n \in \mathbb{N}$.

We show that the relative twisting coefficients $d_\alpha(X_n, \eta_n)$ are uniformly bounded, the case of $d_\alpha(Y_n, \nu_n)$ is identical. Let $\ell_n = \ell_\alpha(X_n)$. By the collar lemma [17, Section 13.5], the ω_n-neighborhood (collar) of the geodesic representative of α in X_n for $\omega_n = \operatorname{arcsinh}(\frac{1}{\sinh(\ell_n/2)})$ is embedded in X_n. Consider an arc $\widehat{\eta}_n$ of the geodesic representative of η_n inside the collar of α in X_n with one endpoint on α and the other endpoint on the boundary of the neighborhood. Since the collar is embedded, the length of $\widehat{\eta}_n$ is at most $B(S)$. From the trigonometry of right triangles, we find a lower bound on the angle δ_n that $\widehat{\eta}_n$ makes with α in X_n:

$$\sin \delta_n \geqslant \frac{\sinh \omega_n}{\sinh B(S)}.$$

Denote by L_n the length of the orthogonal projection of a lift of η_n on a lift of α in the universal cover of X_n that intersect at the angle δ_n. Then from the angle of parallelism formula, we have $\cosh \frac{L_n}{2} \sin \delta_n = 1$. Since $\sinh x \leqslant \frac{e^x}{2}$ and $\operatorname{arccosh} x \leqslant \ln 2x$ for $x > 0$, we find

$$\begin{aligned} L_n &\leqslant 2 \operatorname{arccosh} \frac{\sinh B(S)}{\sinh \omega_n} = 2 \operatorname{arccosh}\Big(\sinh B(S) \sinh \frac{\ell_n}{2}\Big) \\ &\leqslant 2 \ln(\sinh B(S) e^{\ell_n/2}) \leqslant \ell_n + 2B(S) - 2\ln 2 < 3\ell_n. \end{aligned}$$

We estimate the relative twisting coefficients (see [33, Section 3])

$$d_\alpha(X_n, \eta_n) \stackrel{+}{\asymp}_2 \frac{L_n}{\ell_n} \stackrel{+}{\asymp}_3 0.$$

We show that $d_\alpha(\eta_n, \nu_n) \stackrel{+}{\asymp} d_\alpha(\xi, \zeta)$ for infinitely many $n \in \mathbb{N}$. Let $[\eta] \in \mathbb{P}\mathcal{ML}(S)$ be the limit of a subsequence of $[\eta_n]$. By Lemma 8.1, $i(\xi, \eta) = 0$. Since ξ is minimal and filling, we have $\operatorname{supp}(\xi) = \operatorname{supp}(\eta)$, in particular, η intersects α essentially and $d_\alpha(\xi, \zeta) = d_\alpha(\eta, \zeta)$. Let η' be the limit of a further subsequence of $\{\eta_n\}$ in $\mathcal{GL}(S)$. Then $\operatorname{supp}(\eta) \subset \operatorname{supp}(\eta')$, hence $d_\alpha(\eta, \zeta) \stackrel{+}{\asymp}_1 d_\alpha(\eta', \zeta)$. By Lemma 2.7, $d_\alpha(\eta', \zeta) \stackrel{+}{\asymp} d_\alpha(\eta_n, \zeta)$ for infinitely many $n \in \mathbb{N}$. By a similar argument for $\{\nu_n\}$, we have $d_\alpha(\eta_n, \zeta) \stackrel{+}{\asymp} d_\alpha(\eta_n, \nu_n)$, hence $d_\alpha(\xi, \zeta) \stackrel{+}{\asymp} d_\alpha(\eta_n, \nu_n)$ for infinitely many $n \in \mathbb{N}$, which proves the lemma. ∎

Together with Theorem 2.10, we obtain the following corollary.

Corollary 8.3 (Bounded annular combinatorics implies cobounded). *Let $\mathcal{G}(t)$, $t \in \mathbb{R}$ be a stretch path in $\mathcal{T}(S)$ with the horocyclic foliation $[\xi]$ such that $\mathcal{G}(t) \to [\zeta] \in \mathbb{P}\mathcal{ML}(S)$ as $t \to -\infty$. Suppose that the supports of ξ and ζ are minimal and filling. If there exists a number $K \in \mathbb{N}$ such that $d_\alpha(\xi, \zeta) \leqslant K$ for all curves α on S, then there exists $\varepsilon(K) > 0$ such that $\mathcal{G}(t)$ lies in the thick part $\mathcal{T}_\varepsilon(S)$ for all $t \in \mathbb{R}$.*

Proof. Suppose that there is a curve α on S that gets shorter than ε_0 along the geodesic $\mathcal{G}(t)$, where $\varepsilon_0 > 0$ is the constant in the statement of Theorem 2.10, otherwise there is nothing to prove. Since $\mathcal{G}(t)$ is a stretch path, Theorem 2.10 is applicable. Let $[a, b]$ be the ε_0-active interval for α. Indeed, this interval is bounded: for example, if there is a sequence $t_i \to \infty$ such that $\ell_\alpha(\mathcal{G}(t_i)) \leqslant \varepsilon_0$, then by Lemma 8.1 we have $i(\alpha, \xi) = 0$, which is impossible since ξ is minimal and filling. By a similar argument, it can be shown that there are infinitely many numbers $m \in \mathbb{N}$ such that $\ell_\alpha(\mathcal{G}(-m)) > B(S)$. By choosing large enough n, so that the interval $[-n, n]$ contains the interval $[a, b]$ and Lemma 8.2 applies for $X_n = \mathcal{G}(-n)$, $Y_n = \mathcal{G}(n)$, we conclude by combining Theorems 2.9 and 2.10 with the condition $d_\alpha(\xi, \zeta) \leqslant K$ that there is a lower bound on the minimal length of α along $\mathcal{G}(t)$ that depends only on K. ■

Finally, we prove our main result.

Proof of Theorem 1.1. Let $[\lambda]$ be the projective class of some non-zero transverse measure on the non-uniquely ergodic ending lamination λ constructed in Section 3. Let ν be an unstable or stable lamination of a pseudo-Anosov map Ψ on S, and let $\widehat{\nu}$ be a maximal lamination on S obtained from ν by adding finitely many leaves. Consider the projective measured foliation on S that corresponds to $[\lambda]$ and that is standard near the cusps; we also denote it by $[\lambda]$. Since ν is minimal, filling and uniquely ergodic, the set of projective measured foliations transverse to $\widehat{\nu}$ contains $[\lambda]$. Thus there is a point $X \in \mathcal{T}(S)$ such that $[\mathcal{F}_{\widehat{\nu}}(X)] = [\lambda]$ (see Section 2.9). Since $\operatorname{stump}(\widehat{\nu}) = \nu$, by Theorem 2.9 the stretch path $\operatorname{stretch}(X, \widehat{\nu}, t)$ converges to $[\lambda]$ as $t \to \infty$ and to $[\nu]$ as $t \to -\infty$.

By Corollary 8.3, to prove that $\operatorname{stretch}(X, \widehat{\nu}, t)$ stays in the thick part, it is sufficient to show that the relative twisting coefficients $d_\alpha(\nu, \lambda)$ are uniformly bounded for all curves α on S. Let μ be the marking on S from Proposition 7.2. By the triangle inequality, we have

$$d_\alpha(\nu, \lambda) \leqslant d_\alpha(\nu, \mu) + d_\alpha(\mu, \lambda). \tag{8.1}$$

By Proposition 2.8 and Corollary 7.3, we have $d_\alpha(\nu, \lambda) \leqslant C_{\Psi,\mu} + E'$, which completes the proof. ■

A. Appendix

A.1. Convergence lemma

Let $\|\cdot\|$ denote the operator norm. Then $\|Y\| \geqslant 1$ for any non-trivial idempotent matrix Y. The following lemma is a slight improvement over Lemma 11.1 in [3].

Lemma A.1. *Let Y be an idempotent matrix and let $\{\Delta_i\}_{i=1}^\infty$ be a sequence of matrices such that $\sum_{i=1}^\infty \|\Delta_i\| < \infty$. Let $\varepsilon_j = \sum_{i=j}^\infty \|\Delta_i\|$ for $j \in \mathbb{N}$. Then there is $j_0 \in \mathbb{N}$ such that for every $j \geqslant j_0$, the infinite product*

$$\prod_{i=j}^\infty (Y + \Delta_i)$$

converges to a matrix X_j with $\|X_j - Y\| \leqslant 2\varepsilon_j \|Y\|^2$. Moreover, the kernel of Y is contained in the kernel of X_j.

Proof. Let $j_0 \in \mathbb{N}$ be such that $\varepsilon_{j_0} \leqslant \frac{1}{2\|Y\|}$. Now fix some $j \geqslant j_0$. For $k \geqslant j$, write

$$Y + \Sigma_k = \prod_{i=j}^{k} (Y + \Delta_i).$$

Then $(Y + \Sigma_k)(Y + \Delta_{k+1}) = Y + \Sigma_{k+1}$ and since $Y^2 = Y$, it follows that

$$\Sigma_{k+1} = \Sigma_k Y + Y\Delta_{k+1} + \Sigma_k \Delta_{k+1}. \tag{A.1}$$

Multiplying on the right by Y and using $Y^2 = Y$, we get

$$\Sigma_{k+1} Y = \Sigma_k Y + Y\Delta_{k+1} Y + \Sigma_k \Delta_{k+1} Y$$

and applying the norm to the both sides of the equation, we get

$$\|\Sigma_{k+1} Y\| \leqslant \|\Sigma_k Y\| + \|\Delta_{k+1}\| \cdot \|Y\|^2 + \|\Sigma_k\| \cdot \|\Delta_{k+1}\| \cdot \|Y\|.$$

For $m > j$, by applying these inequalities for $k = j, \dots, m-1$, we get

$$\begin{aligned}\|\Sigma_m Y\| \leqslant{}& \|\Sigma_j Y\| + (\|\Delta_{j+1}\| + \cdots + \|\Delta_m\|) \cdot \|Y\|^2 \\ &+ (\|\Sigma_j\| \cdot \|\Delta_{j+1}\| + \cdots + \|\Sigma_{m-1}\| \cdot \|\Delta_m\|) \cdot \|Y\|.\end{aligned}$$

Since $\|Y\| \leqslant \|Y\|^2$ and using $\Sigma_j = \Delta_j$, we can write

$$\begin{aligned}\|\Sigma_m Y\| \leqslant{}& (\|\Delta_j\| + \cdots + \|\Delta_m\|) \cdot \|Y\|^2 \\ &+ (\|\Sigma_j\| \cdot \|\Delta_{j+1}\| + \cdots + \|\Sigma_{m-1}\| \cdot \|\Delta_m\|) \cdot \|Y\|.\end{aligned}$$

Putting this together with equation (A.1) and using $\|Y\|^2 \geqslant \|Y\|$, $\|Y\| \geqslant 1$, we get

$$\begin{aligned}\|\Sigma_{k+1}\| &\leqslant \|\Sigma_k Y\| + \|\Delta_{k+1}\| \cdot \|Y\| + \|\Sigma_k\| \cdot \|\Delta_{k+1}\| \\ &\leqslant (\|\Delta_j\| + \cdots + \|\Delta_{k+1}\|) \cdot \|Y\|^2 \\ &\quad + (\|\Sigma_j\| \cdot \|\Delta_{j+1}\| + \cdots + \|\Sigma_k\| \cdot \|\Delta_{k+1}\|) \cdot \|Y\| \\ &\leqslant \varepsilon_j \|Y\|^2 + (\|\Sigma_j\| \cdot \|\Delta_{j+1}\| + \cdots + \|\Sigma_k\| \cdot \|\Delta_{k+1}\|) \cdot \|Y\|.\end{aligned} \tag{A.2}$$

Now we show by induction that $\|\Sigma_k\| \leqslant \frac{\varepsilon_j \|Y\|^2}{1-\varepsilon_{j+1}\|Y\|}$ for all $k \geqslant j$.

Base: $k = j$. Since $\Sigma_j = \Delta_j$, we have $\|\Sigma_j\| = \|\Delta_j\| = \varepsilon_j - \varepsilon_{j+1}$. Next, using $\|Y\|^2 \geqslant 1$ we trivially have

$$(\varepsilon_j - \varepsilon_{j+1})(1 - \varepsilon_{j+1}\|Y\|) \leqslant \varepsilon_j - \varepsilon_{j+1} \leqslant \varepsilon_j \leqslant \varepsilon_j \|Y\|^2.$$

By the choice of j_0, we have that $1 - \varepsilon_{j+1}\|Y\| > 0$, hence by dividing both sides by $(1 - \varepsilon_{j+1}\|Y\|)$, we obtain

$$\|\Sigma_j\| \leqslant \frac{\varepsilon_j \|Y\|^2}{1 - \varepsilon_{j+1}\|Y\|}$$

as desired.

Step. By equation (A.2), we have

$$\begin{aligned}\|\Sigma_{k+1}\| &\leqslant \varepsilon_j\|Y\|^2 + \frac{\varepsilon_j\|Y\|^2}{1-\varepsilon_{j+1}\|Y\|}(\|\Delta_{j+1}\| + \cdots + \|\Delta_{k+1}\|)\cdot\|Y\| \\ &\leqslant \varepsilon_j\|Y\|^2 + \frac{\varepsilon_j\|Y\|^2}{1-\varepsilon_{j+1}\|Y\|}\cdot\varepsilon_{j+1}\|Y\| = \frac{\varepsilon_j\|Y\|^2}{1-\varepsilon_{j+1}\|Y\|}.\end{aligned}$$

By the choice of j_0, we also have $\frac{\varepsilon_j\|Y\|^2}{1-\varepsilon_{j+1}\|Y\|} \leqslant 2\varepsilon_j\|Y\|^2$. This shows that

$$\|\Sigma_k\| \leqslant 2\varepsilon_j\|Y\|^2. \tag{A.3}$$

It also follows that $\|X_j - Y\| \leqslant 2\varepsilon_j\|Y\|^2$ if we assume the convergence.

To prove the convergence, we show that the partial products form a Cauchy sequence. For $j < k < m$,

$$\prod_{i=j}^{m}(Y+\Delta_i) - \prod_{i=j}^{k}(Y+\Delta_i) = \prod_{i=j}^{k-1}(Y+\Delta_i)\left(\prod_{i=k}^{m}(Y+\Delta_i) - (Y+\Delta_k)\right)$$

and applying the norm to the both sides of the equation, we get

$$\begin{aligned}\left\|\prod_{i=j}^{m}(Y+\Delta_i) - \prod_{i=j}^{k}(Y+\Delta_i)\right\| &\leqslant \left\|\prod_{i=j}^{k-1}(Y+\Delta_i)\right\|\left(\left\|\prod_{i=k}^{m}(Y+\Delta_i) - Y\right\| + \|\Delta_k\|\right) \\ &\leqslant (\|Y\| + 2\varepsilon_j\|Y\|^2)(2\varepsilon_k\|Y\|^2 + \|\Delta_k\|),\end{aligned}$$

which proves the sequence is Cauchy.

For the last statement, let v be a unit vector with $Yv = 0$. Then for $k \geqslant j$,

$$\|X_k v\| = \|(X_k - Y)v\| \leqslant \|X_k - Y\|\cdot\|v\| \leqslant 2\varepsilon_k\|Y\|^2.$$

Since $X_j = (Y + \Sigma_{k-1})X_k$, we have

$$\|X_j v\| \leqslant \|(Y+\Sigma_{k-1})\|\|X_k v\| \leqslant (\|Y\| + 2\varepsilon_j\|Y\|^2)(2\varepsilon_k\|Y\|^2).$$

Since this is true for all $k \geqslant j$, letting $k \to \infty$ yields $X_j v = 0$. ■

A.2. Angle estimate lemma

Let V be an inner product space.

Lemma A.2. *Let $v_1, v_2, w_1, w_2 \in V$ be such that $\angle(v_1, v_2) = 0$ and $v_1 + w_1 \neq 0$, $v_2 + w_2 \neq 0$. Then*

$$1 - \cos\angle(v_1 + w_1, v_2 + w_2) \leqslant \frac{2(\|v_1\|\cdot\|w_2\| + \|v_2\|\cdot\|w_1\| + \|w_1\|\cdot\|w_2\|)}{\|v_1\|\cdot\|v_2\|}.$$

Proof. Writing the definition of the cosine of the angle, using the triangle inequality, the fact that $\langle v_1, v_2\rangle = \|v_1\|\cdot\|v_2\|$ and that $\langle v, w\rangle \geqslant -\|v\|\cdot\|w\|$ for $v, w \in V$, we get

$$\begin{aligned}
&\cos\angle(v_1 + w_1, v_2 + w_2)\\
&\quad= \frac{\langle v_1 + w_1, v_2 + w_2\rangle}{\|v_1 + w_1\|\cdot\|v_2 + w_2\|} \geqslant \frac{\langle v_1 + w_1, v_2 + w_2\rangle}{(\|v_1\| + \|w_1\|)(\|v_2\| + \|w_2\|)}\\
&\quad= \frac{\langle v_1, v_2\rangle + \langle v_1, w_2\rangle + \langle v_2, w_1\rangle + \langle w_1, w_2\rangle}{(\|v_1\| + \|w_1\|)(\|v_2\| + \|w_2\|)}\\
&\quad\geqslant \frac{\|v_1\|\cdot\|v_2\| - \|v_1\|\cdot\|w_2\| - \|v_2\|\cdot\|w_1\| - \|w_1\|\cdot\|w_2\|}{(\|v_1\| + \|w_1\|)(\|v_1\| + \|w_1\|)}.
\end{aligned}\tag{A.4}$$

Then by equation (A.4) and since $\|w_1\|, \|w_2\| \geqslant 0$,

$$\begin{aligned}
1 - \cos\angle(v_1 + w_1, v_2 + w_2) &\leqslant \frac{2(\|v_1\|\cdot\|w_2\| + \|v_2\|\cdot\|w_1\| + \|w_1\|\cdot\|w_2\|)}{(\|v_1\| + \|w_1\|)(\|v_2\| + \|w_2\|)}\\
&\leqslant \frac{2(\|v_1\|\cdot\|w_2\| + \|v_2\|\cdot\|w_1\| + \|w_1\|\cdot\|w_2\|)}{\|v_1\|\cdot\|v_2\|}.
\end{aligned}$$

This completes the proof. ∎

A.3. Interval neighborhood lemma

Lemma A.3. *Let $I \subset \mathbb{R}^n$, $n \geqslant 2$ be a closed line segment. Let $I_r \subset \mathbb{R}^n$ be the r-neighborhood of I for $r > 0$. Then for every 2-dimensional closed disk $D_R \subset \mathbb{R}^n$ of radius $R > 2\sqrt{2(n-1)}r$, $D_R \not\subset I_r$.*

Proof. Without loss of generality, assume that $I = \{(x_1, 0, \dots, 0) \mid -t \leqslant x_1 \leqslant t\}$ for some $t > 0$. Let $B_r = \{(x_1, x_2, \dots, x_n) \mid -t - r \leqslant x_1 \leqslant t + r, -r \leqslant x_i \leqslant r, 2 \leqslant i \leqslant n\}$. Notice that $I_r \subset B_r$. We prove that $D_R \not\subset B_r$, hence $D_R \not\subset I_r$.

Assume on the contrary that $D_R \subset B_r$. Let $c \in D_R$ be the center of D_R and $a, b \in \partial D_R$ be such that the vector $a - c$ is perpendicular to the vector $b - c$. Thus there are points $a, b, c \in B_r$ such that $\|a - c\| = R$, $\|b - c\| = R$ and $\langle a - c, b - c\rangle = 0$. Let v_i denote the i-th coordinate of a vector $v \in \mathbb{R}^n$. Since $|(a - c)_i|, |(b - c)_i| \leqslant 2r$ for $2 \leqslant i \leqslant n$, we have

$$\begin{aligned}
R^2 &= \|a - c\|^2 \leqslant (a - c)_1^2 + (n - 1)\cdot 4r^2,\\
R^2 &= \|b - c\|^2 \leqslant (b - c)_1^2 + (n - 1)\cdot 4r^2.
\end{aligned}$$

Hence $(a - c)_1^2 \geqslant R^2 - 4(n - 1)r^2$, $(b - c)_1^2 \geqslant R^2 - 4(n - 1)r^2$. Then by the triangle inequality, we have

$$\begin{aligned}
|\langle a - c, b - c\rangle| &\geqslant |(a - c)_1|\cdot|(b - c)_1| - \sum_{i=2}^{n} |(a - c)_i|\cdot|(b - c)_i|\\
&\geqslant R^2 - 4(n - 1)r^2 - 4(n - 1)r^2 = R^2 - 8(n - 1)r^2.
\end{aligned}$$

Since $R > 2\sqrt{2(n-1)}r$, we have $|\langle a - c, b - c\rangle| > 0$, contradiction. ∎

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References

[1] T. Aougab, Uniform hyperbolicity of the graphs of curves. *Geom. Topol.* **17** (2013), no. 5, 2855–2875 Zbl 1273.05050 MR 3190300

[2] J. A. Behrstock, Asymptotic geometry of the mapping class group and Teichmüller space. *Geom. Topol.* **10** (2006), 1523–1578 Zbl 1145.57016 MR 2255505

[3] M. Bestvina, R. Gupta, and J. Tao, Limit sets of unfolding paths in Outer space. 2022, arXiv:2207.06992

[4] F. Bonahon, The geometry of Teichmüller space via geodesic currents. *Invent. Math.* **92** (1988), no. 1, 139–162 Zbl 0653.32022 MR 931208

[5] F. Bonahon, Shearing hyperbolic surfaces, bending pleated surfaces and Thurston's symplectic form. *Ann. Fac. Sci. Toulouse Math. (6)* **5** (1996), no. 2, 233–297 Zbl 0880.57005 MR 1413855

[6] B. H. Bowditch, Intersection numbers and the hyperbolicity of the curve complex. *J. Reine Angew. Math.* **598** (2006), 105–129 Zbl 1119.32006 MR 2270568

[7] B. H. Bowditch, Uniform hyperbolicity of the curve graphs. *Pacific J. Math.* **269** (2014), no. 2, 269–280 Zbl 1330.57023 MR 3238474

[8] M. R. Bridson and A. Haefliger, *Metric spaces of non-positive curvature*. Grundlehren Math. Wiss. 319, Springer, Berlin, 1999 Zbl 0988.53001 MR 1744486

[9] J. Brock, C. Leininger, B. Modami, and K. Rafi, Limit sets of Teichmüller geodesics with minimal nonuniquely ergodic vertical foliation, II. *J. Reine Angew. Math.* **758** (2020), 1–66 Zbl 1440.37056 MR 4048441

[10] J. Brock and B. Modami, Recurrent Weil–Petersson geodesic rays with non-uniquely ergodic ending laminations. *Geom. Topol.* **19** (2015), no. 6, 3565–3601 Zbl 1332.30067 MR 3447110

[11] A. Calderon and J. Farre, Shear-shape cocycles for measured laminations and ergodic theory of the earthquake flow. 2023, arXiv:2102.13124, to appear in *Geom. Topol.*

[12] D. Calegari, *Foliations and the geometry of* 3*-manifolds*. Oxford Math. Monogr., Oxford University Press, Oxford, 2007 Zbl 1118.57002 MR 2327361

[13] R. D. Canary, D. B. A. Epstein, and P. Green, Notes on notes of Thurston. In *Analytical and geometric aspects of hyperbolic space (Coventry/Durham, 1984)*, pp. 3–92, London Math. Soc. Lecture Note Ser. 111, Cambridge University Press, Cambridge, 1987 Zbl 0612.57009 MR 903850

[14] Y.-E. Choi and K. Rafi, Comparison between Teichmüller and Lipschitz metrics. *J. Lond. Math. Soc. (2)* **76** (2007), no. 3, 739–756 Zbl 1132.30024 MR 2377122

[15] M. Clay, K. Rafi, and S. Schleimer, Uniform hyperbolicity of the curve graph via surgery sequences. *Algebr. Geom. Topol.* **14** (2014), no. 6, 3325–3344 Zbl 1309.57015 MR 3302964

[16] D. Dumas, A. Lenzhen, K. Rafi, and J. Tao, Coarse and fine geometry of the Thurston metric. *Forum Math. Sigma* **8** (2020), article no. e28 Zbl 1475.30107 MR 4108919

[17] B. Farb and D. Margalit, *A primer on mapping class groups*. Princeton Math. Ser. 49, Princeton University Press, Princeton, 2011 Zbl 1245.57002 MR 2850125

[18] A. Fathi, F. Laudenbach, and V. Poénaru, *Thurston's work on surfaces*. Math. Notes 48, Princeton University Press, Princeton, 2012 Zbl 1244.57005 MR 3053012

[19] U. Hamenstädt, Train tracks and the Gromov boundary of the complex of curves. In *Spaces of Kleinian groups*, pp. 187–207, London Math. Soc. Lecture Note Ser. 329, Cambridge University Press, Cambridge, 2006 Zbl 1117.30036 MR 2258749

[20] W. J. Harvey, Boundary structure of the modular group. In *Riemann surfaces and related topics: Proceedings of the 1978 Stony Brook Conference (State Univ. New York, Stony Brook, N.Y., 1978)*, pp. 245–251, Ann. of Math. Stud. 97, Princeton University Press, Princeton, 1981 Zbl 0461.30036 MR 624817

[21] J. Hempel, 3-manifolds as viewed from the curve complex. *Topology* **40** (2001), no. 3, 631–657 Zbl 0985.57014 MR 1838999

[22] S. Hensel, P. Przytycki, and R. C. H. Webb, 1-slim triangles and uniform hyperbolicity for arc graphs and curve graphs. *J. Eur. Math. Soc. (JEMS)* **17** (2015), no. 4, 755–762 Zbl 1369.57018 MR 3336835

[23] E. Klarreich, The boundary at infinity of the curve complex and the relative Teichmüller space. *Groups Geom. Dyn.* **16** (2022), no. 2, 705–723 Zbl 1507.57021 MR 4502619

[24] C. Leininger, A. Lenzhen, and K. Rafi, Limit sets of Teichmüller geodesics with minimal non-uniquely ergodic vertical foliation. *J. Reine Angew. Math.* **737** (2018), 1–32 Zbl 1392.37032 MR 3781329

[25] A. Lenzhen, K. Rafi, and J. Tao, Bounded combinatorics and the Lipschitz metric on Teichmüller space. *Geom. Dedicata* **159** (2012), 353–371 Zbl 1254.30075 MR 2944537

[26] A. Lenzhen, K. Rafi, and J. Tao, The shadow of a Thurston geodesic to the curve graph. *J. Topol.* **8** (2015), no. 4, 1085–1118 Zbl 1367.30030 MR 3431669

[27] Z. Li, Length spectrums of Riemann surfaces and the Teichmüller metric. *Bull. Lond. Math. Soc.* **35** (2003), no. 2, 247–254 Zbl 1024.32011 MR 1952403

[28] J. Mangahas, A recipe for short-word pseudo-Anosovs. *Amer. J. Math.* **135** (2013), no. 4, 1087–1116 Zbl 1277.57019 MR 3086070

[29] B. Martelli, An introduction to geometric topology. 2022, arXiv:1610.02592

[30] H. Masur, Hausdorff dimension of the set of nonergodic foliations of a quadratic differential. *Duke Math. J.* **66** (1992), no. 3, 387–442 Zbl 0780.30032 MR 1167101

[31] H. A. Masur and Y. N. Minsky, Geometry of the complex of curves I: Hyperbolicity. *Invent. Math.* **138** (1999), no. 1, 103–149 Zbl 0941.32012 MR 1714338

[32] H. A. Masur and Y. N. Minsky, Geometry of the complex of curves II: Hierarchical structure. *Geom. Funct. Anal.* **10** (2000), no. 4, 902–974 Zbl 0972.32011 MR 1791145

[33] Y. N. Minsky, Extremal length estimates and product regions in Teichmüller space. *Duke Math. J.* **83** (1996), no. 2, 249–286 Zbl 0861.32015 MR 1390649

[34] Y. N. Minsky, Kleinian groups and the complex of curves. *Geom. Topol.* **4** (2000), 117–148 Zbl 0953.30027 MR 2182094

[35] M. Mirzakhani, Ergodic theory of the earthquake flow. *Int. Math. Res. Not. IMRN* **2008** (2008), no. 3, article no. 116 Zbl 1189.30087 MR 2416997

[36] A. Papadopoulos, On Thurston's boundary of Teichmüller space and the extension of earthquakes. *Topology Appl.* **41** (1991), no. 3, 147–177 Zbl 0761.57009 MR 1135095

[37] A. Papadopoulos and G. Théret, On Teichmüller's metric and Thurston's asymmetric metric on Teichmüller space. In *Handbook of Teichmüller theory. Vol. I*, pp. 111–204, IRMA Lect. Math. Theor. Phys. 11, European Mathematical Society, Zürich, 2007 Zbl 1129.30030 MR 2349669

[38] R. C. Penner and J. L. Harer, *Combinatorics of train tracks*. Ann. of Math. Stud. 125, Princeton University Press, Princeton, 1992 Zbl 0765.57001 MR 1144770

[39] K. Rafi, A characterization of short curves of a Teichmüller geodesic. *Geom. Topol.* **9** (2005), 179–202 Zbl 1082.30037 MR 2115672

[40] K. Rafi, Hyperbolicity in Teichmüller space. *Geom. Topol.* **18** (2014), no. 5, 3025–3053 Zbl 1314.30082 MR 3285228

[41] I. Smith, Geometric monodromy and the hyperbolic disc. *Q. J. Math.* **52** (2001), no. 2, 217–228 Zbl 0981.57013 MR 1838364

[42] G. Théret, On the negative convergence of Thurston's stretch lines towards the boundary of Teichmüller space. *Ann. Acad. Sci. Fenn. Math.* **32** (2007), no. 2, 381–408 Zbl 1125.30040 MR 2337484

[43] G. Théret, Convexity of length functions and Thurston's shear coordinates. 2014, arXiv:1408.5771

[44] W. P. Thurston, Minimal stretch maps between hyperbolic surfaces. In *Collected works of William P. Thurston with commentary. Vol. I. Foliations, surfaces and differential geometry*, pp. 533–585, American Mathematical Society, Providence, 1998 MR 4554454

[45] R. C. H. Webb, Uniform bounds for bounded geodesic image theorems. *J. Reine Angew. Math.* **709** (2015), 219–228 Zbl 1335.57029 MR 3430880

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Ivan Telpukhovskiy
Department of Mathematics, University of Toronto, Bahen Centre, Room 6290, 40 St. George St., Toronto, ON M5S 2E4, Canada; Saint Petersburg State University, 7/9 University Embankment, 199034 Saint Petersburg, Russia; ivantelp@math.toronto.edu