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Isometry groups of proper CAT(0)-spaces of rank one

Ursula Hamenstädt

Abstract. Let X be a proper CAT(0)-space and let G be a closed subgroup of the isometry group $\mathrm{Iso}(X)$ of X. We show that if G is non-elementary and contains a rank-one element then its second continuous bounded cohomology group with coefficients in the regular representation is non-trivial. As a consequence, up to passing to an open subgroup of finite index, either G is a compact extension of a totally disconnected group or G is a compact extension of a simple Lie group of rank one.

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Contents

1. Introduction

A geodesic metric space (X, d) is called *proper* if closed balls in X of finite radius are compact. A proper CAT(0)-metric space X can be compactified by adding the *visual boundary* ∂X. The isometry group $\mathrm{Iso}(X)$ of X, equipped with the compact open topology, is a locally compact σ-compact topological group which acts as a group of homeomorphisms on the compact space $\bar{X} = X \cup \partial X$. The *limit set* Λ of a subgroup G of $\mathrm{Iso}(X)$ is the set of accumulation points in ∂X of an orbit of the action of G on X. The group G is called *elementary* if either its limit set consists of at most two points or if G fixes a point in ∂X.

The *displacement function* of an isometry $g \in \mathrm{Iso}(X)$ is the function $x \to d(x, gx)$ on X. The isometry g is called *semisimple* if its displacement function assumes a minimum on X. If this minimum is zero then g has a fixed point in X and is called *elliptic*, and otherwise g is called *axial*. If g is axial then the closed convex subset of X of all minima of the displacement function of g is isometric to a product space $C \times \mathbb{R}$ where g acts on each of the geodesics $\{x\} \times \mathbb{R}$ as a translation. Such a geodesic is called an *axis* for g. We refer to the books [3], [4], [7] for basic properties of CAT(0)-spaces and for references.

Call an axial isometry g of X *rank-one* if there is an axis γ for g which does not bound a flat half-plane. Here by a flat half-plane we mean a totally geodesic embedded isometric copy of an euclidean half-plane in X.

A *compact extension* of a topological group H is a topological group G which contains a compact normal subgroup K such that $H = G/K$ as topological groups. Extending earlier results for isometry groups of proper hyperbolic geodesic metric spaces [15], [18], [20] we show

Theorem 1. *Let X be a proper* CAT(0)*-space and let $G < \mathrm{Iso}(X)$ be a closed subgroup. Assume that G is non-elementary and contains a rank-one element. Then one of the following two possibilities holds.*

(1) *Up to passing to an open subgroup of finite index, G is a compact extension of a simple Lie group of rank one.*
(2) *G is a compact extension of a totally disconnected group.*

Caprace and Monod showed the following version of Theorem 1 (Corollary 1.7 of [12]): A CAT(0)-space X is called *irreducible* if it is not a non-trivial metric product. Let $X \neq \mathbb{R}$ be an irreducible proper CAT(0)-space with finite dimensional Tits boundary. Assume that the isometry group $\mathrm{Iso}(X)$ of X does not have a global fixed point in ∂X and that its action on X does not preserve a non-trivial closed convex subset of X. Then $\mathrm{Iso}(X)$ is either totally disconnected or an almost connected simple Lie group with trivial center.

We also note the following consequence (see Corollary 1.24 of [12]). For its formulation, an isometry of a CAT(0)-space is called *parabolic* if it is not semisimple.

Corollary 1. *Let M be a closed Riemannian manifold of non-positive sectional curvature. If the universal covering $\tilde{M}$ of M is irreducible and if the isometry group of $\tilde{M}$ contains a parabolic element, then M is locally symmetric.*

Our proof of Theorem 1 is different from the approach of Caprace and Monod and uses second bounded cohomology for locally compact topological groups G with coefficients in a *separable Banach module* for G. Such a separable Banach module is a separable Banach space E together with a continuous homomorphism of G into the group of linear isometries of E. For every separable Banach module E for G and every $i \geq 1$, the group G naturally acts on the vector space $C_b(G^i, E)$ of continuous bounded maps $G^i \to E$. If we denote by $C_b(G^i, E)^G \subset C_b(G^i, E)$ the linear subspace of all G-invariant such maps, then the *second continuous bounded cohomology group* $H^2_{cb}(G, E)$ of G with coefficients E is defined as the second cohomology group of the complex

$$0 \to C_b(G, E)^G \xrightarrow{d} C_b(G^2, E)^G \xrightarrow{d} \cdots$$

with the usual homogeneous coboundary operator d (see [19]).

A closed subgroup G of Iso(X) is a locally compact and σ-compact topological group and hence it admits a left invariant locally finite Haar measure μ. In particular, for every $p \in (1, \infty)$ the separable Banach space $L^p(G, \mu)$ of functions on G which are p-integrable with respect to μ is a separable Banach module for G with respect to the isometric action of G by left translation. Extending an earlier result for isometry groups of proper hyperbolic spaces [15] (see also the work of Monod–Shalom [20], of Mineyev–Monod–Shalom [18], of Bestvina–Fujiwara [6] and of Caprace–Fujiwara [11] for closely related results) we obtain the following non-vanishing result for second bounded cohomology.

Theorem 2. *Let G be a closed non-elementary subgroup of the isometry group of a proper* CAT(0)*-space X.*

If G contains a rank-one element, then $H^2_{cb}(G, L^p(G, \mu)) \neq \{0\}$ for every $p \in (1, \infty)$.

It follows from the work of Burger and Monod [8] and Monod and Shalom [21] that the conclusion in Theorem 2 does not hold for simple Lie groups of non-compact type and higher rank. Such a group is the isometry group of a symmetric space of non-compact type which is a finite dimensional complete Riemannian manifold of non-positive curvature. Thus the assumption on the existence of a rank-one element in G in Theorem 2 can not be removed. More precisely, we obtain the following super-rigidity theorem as an application of Theorem 2.

Corollary 2. *Let G be a connected semi-simple Lie group with finite center, no compact factors and of rank at least* 2*. Let Γ be an irreducible lattice in G, let X be a proper* CAT(0)*-space and let $\rho \colon \Gamma \to \mathrm{Iso}(X)$ be a homomorphism. Let $H < \mathrm{Iso}(X)$ be the closure of $\rho(\Gamma)$. If H is non-elementary and contains a rank-one element, then H is a compact extension of a simple Lie group L of rank one, and up to passing to an open subgroup of finite index, ρ extends to a continuous homomorphism $G \to L$.*

Remark. As in [15], our proof of Theorem 2 also shows the following result of Bestvina and Fujiwara [6]: Let $G < \mathrm{Iso}(X)$ be a closed non-elementary subgroup with limit set Λ which contains a rank-one element. If G does not act transitively on the complement of the diagonal in $\Lambda \times \Lambda$ then the second continuous bounded cohomology group $H^2_{cb}(G,\mathbb{R})$ is infinite-dimensional. Moreover, the arguments in [15] together with the geometric discussion in Sections 2–5 of this paper show that if G acts transitively on the complement of the diagonal in $\Lambda\times\Lambda$ then $H^2_{cb}(G,\mathbb{R})=0$. Under the additional assumption that G acts on X cocompactly, this is due to Caprace and Fujiwara [11].

The organization of this note is as follows. In Section 2 we collect some geometric properties of a proper CAT(0)-space X needed in the sequel. In particular, we discuss contracting geodesics as introduced by Bestvina and Fujiwara [6].

In Section 3 we investigate for a fixed number $B>0$ the space of all B-contracting geodesics in X. We construct a family of finite distance functions on the space of pairs of endpoints of such geodesics which are parametrized by the points in X. These distance functions are equivariant under the natural action of the isometry group of X. This construction is the main novelty of this work.

Let $G<\mathrm{Iso}(X)$ be a closed non-elementary subgroup with limit set $\Lambda\subset\partial X$ which contains a rank-one element. In Section 4 we use the distance functions on the space of endpoints of B-contracting geodesics to construct continuous bounded cocycles for G with values in $L^p(G\times G,\mu\times\mu)$ on a G-invariant closed subspace of the space of triples of pairwise distinct points in Λ

If the action of the group G on the complement of the diagonal in $\Lambda\times\Lambda$ is transitive, then this space of triples equals the entire space of triples of pairwise distinct points in Λ. In this case standard arguments are used in Section 5 to show Theorem 2. The case that G does not act transitively on the complement of the diagonal in $\Lambda\times\Lambda$ is technically more difficult and is established in Section 6. The proof of Theorem 1 and of the corollaries is contained in Section 7.

Acknowledgement. I thank the referee for making me aware that the construction in Section 6 can be used to construct second continuous bounded cohomology classes.

2. Metric contraction in CAT(0)-spaces

In this section we collect some geometric properties of CAT(0)-spaces needed in the later sections. We use the books [3], [4], [7] as the main references.

2.1. Shortest distance projections. A proper CAT(0)-space has strong convexity properties which we summarize in this subsection.

In a complete CAT(0)-space X, any two points can be connected by a unique geodesic which varies continuously with the endpoints. The distance function is convex: If $\gamma,\zeta\colon J\to X$ are two geodesics in X parametrized on the same interval $J\subset\mathbb{R}$ then the function $t\to d(\gamma(t),\zeta(t))$ is convex. More generally, we call a function $f\colon X\to\mathbb{R}$ *convex* if for every geodesic $\gamma\colon J\to\mathbb{R}$ the function $t\to f(\gamma(t))$ is convex [3].

For a fixed point $x \in X$, let ∂X be the space of all geodesic rays issuing from $x \in X$ equipped with the topology of uniform convergence on compact sets. The topological space ∂X does not depend on the choice of x and is called the *visual boundary* of X. We denote the point in ∂X defined by a geodesic ray $\gamma\colon [0,\infty) \to X$ by $\gamma(\infty)$. We also say that γ *connects* x to $\gamma(\infty)$.

There is another description of the visual boundary of X as follows. For points $x, y, z \in X$ define

$$b_x(y,z) = d(x,z) - d(x,y).$$

Then we have

$$b_x(y,z) = -b_x(z,y) \quad \text{for all } y, z \in X$$

and

$$|b_x(y,z) - b_x(y,z')| \le d(z,z') \quad \text{for all } z, z' \in X$$

and hence the function $b_x(y,\,\cdot\,)\colon z \to b_x(y,z)$ is one-Lipschitz and vanishes at y. Moreover, the function $b_x(y,\,\cdot\,)$ is convex, and for $\tilde{y} \in X$ we have

$$b_x(\tilde{y},\,\cdot\,) = b_x(y,\,\cdot\,) + b_x(\tilde{y}, y). \tag{1}$$

Let $C(X)$ be the space of all continuous functions on X endowed with the topology of uniform convergence on bounded sets. For fixed $y \in X$, the assignment $x \to b_x(y,\,\cdot\,)$ is an embedding of X into $C(X)$. A sequence $\{x_n\} \subset X$ *converges at infinity* if $d(x_n, y) \to \infty$ and if the functions $b_{x_n}(y,\cdot)$ converge in $C(X)$. The visual boundary ∂X of X can also be defined as the subset of $C(X)$ of all functions which are obtained as limits of functions $b_{x_n}(y,\,\cdot\,)$ for sequences $\{x_n\} \subset X$ which converge at infinity. In particular, the union $X \cup \partial X$ is naturally homeomorphic to a closed subset of $C(X)$ (Chapter II.1 and II.2 of [3]). In this identification, each $\xi \in \partial X$ corresponds to a *Busemann function* $b_\xi(y,\,\cdot\,)$ at ξ normalized at y. If $\gamma\colon [0,\infty) \to X$ is the geodesic ray which connects y to ξ then the Busemann function $b_\xi(y,\,\cdot\,)$ satisfies $b_\xi(y,\gamma(t)) = -t$ for all $t \ge 0$.

From now on let X be a *proper* (i.e., complete and locally compact) CAT(0)-space. Then $X \cup \partial X$ is compact. A subset $C \subset X$ is *convex* if for all $x, y \in C$ the geodesic connecting x to y is contained in C. For every closed convex set $C \subset X$ and every $x \in X$ there is a unique point $\pi_C(x) \in C$ of smallest distance to x (Proposition II.2.4 of [7]).

Let $J \subset \mathbb{R}$ be a closed connected set and let $\gamma\colon J \to X$ be a geodesic arc. Then $\gamma(J) \subset X$ is closed and convex and hence there is a shortest distance projection

$$\pi_{\gamma(J)}\colon X \to \gamma(J).$$

The projection point $\pi_{\gamma(J)}(x)$ of $x \in X$ is the unique minimum for the restriction of the function $b_x(y,\,\cdot\,)$ to $\gamma(J)$. By equality (1), this does not depend on the choice of the basepoint $y \in X$. The projection $\pi_{\gamma(J)}\colon X \to \gamma(J)$ is distance non-increasing.

For $\xi \in \partial X$ the function $t \to b_\xi(y, \gamma(t))$ is convex. Let $\overline{\gamma(J)}$ be the closure of $\gamma(J)$ in $X \cup \partial X$. If $b_\xi(y, \cdot)|\gamma(J)$ assumes a minimum then we can define $\pi_{\gamma(J)}(\xi) \subset \overline{\gamma(J)}$ to be the closure in $\overline{\gamma(J)}$ of the connected subset of $\gamma(J)$ of all such minima. If $b_\xi(y, \cdot)|\gamma(J)$ does not assume a minimum then by continuity the set J is unbounded, and by convexity either $\lim_{t\to\infty} b_\xi(y, \gamma(t)) = \inf\{b_\xi(y, \gamma(s)) \mid s \in J\}$ or $\lim_{t\to-\infty} b_\xi(y, \gamma(t)) = \inf\{b_\xi(y, \gamma(s)) \mid s \in J\}$. In the first case we define $\pi_{\gamma(J)}(\xi) = \gamma(\infty) \in \partial X$, and in the second case we define $\pi_{\gamma(J)}(\xi) = \gamma(-\infty)$. Then for every $\xi \in \partial X$ the set $\pi_{\gamma(J)}(\xi)$ is a closed connected subset of $\overline{\gamma(J)}$ (which may contain points in both X and ∂X).

2.2. Contracting geodesics. A proper CAT(0)-space X can contain many totally geodesic embedded flat subspaces, and it can also contain subsets with hyperbolic behavior. To give a precise description of such hyperbolic behavior, Bestvina and Fujiwara introduced a geometric property for geodesics in a CAT(0)-space (Definition 3.1 of [6]) which we repeat in the following definition. For the remainder of this note, geodesics are always defined on closed connected subsets of $\mathbb{R}$.

Definition 2.1. A geodesic arc $\gamma: J \to X$ is *B-contracting* for some $B > 0$ if for every closed metric ball $K \subset X$ which is disjoint from $\gamma(J)$ the diameter of the projection $\pi_{\gamma(J)}(K)$ does not exceed B.

We call a geodesic *contracting* if it is B-contracting for some $B > 0$. Lemma 3.3 of [16] relates B-contraction for a geodesic line γ to the diameter of the projections $\pi_{\gamma(\mathbb{R})}(\xi)$ where $\xi \in \partial X$.

Lemma 2.2. *Let $\gamma: \mathbb{R} \to X$ be a B-contracting geodesic. Then for every $\xi \in \partial X - \{\gamma(-\infty), \gamma(\infty)\}$ the projection $\pi_{\gamma(\mathbb{R})}(\xi)$ is a compact subset of $\gamma(\mathbb{R})$ of diameter at most $6B + 4$.*

Lemma 3.2 and 3.5 of [6] show that a connected subarc of a contracting geodesic is contracting and that a triangle containing a B-contracting geodesic as one of its sides is uniformly thin.

Lemma 2.3. *Let $\gamma: J \to X$ be a B-contracting geodesic.*

(1) *For every closed connected subset $I \subset J$, the subarc $\gamma(I)$ of γ is $B + 3$-contracting.*

(2) *For $x \in X$ and for every $t \in J$ the geodesic connecting x to $\gamma(t)$ passes through the $3B + 1$-neighborhood of $\pi_{\gamma(J)}(x)$.*

Note that by convexity of the distance function, if $\zeta_i: [a_i, b_i] \to X$ $(i = 1, 2)$ are two geodesic segments such that $d(\zeta_1(a_1), \zeta_2(a_2)) \le R$, $d(\zeta_1(b_1), \zeta_2(b_2)) \le R$, then the *Hausdorff distance* between the subsets $\zeta_1[a_1, b_1]$, $\zeta_2[a_2, b_2]$ of X is at most R. Here the Hausdorff distance between closed (not necessarily compact) subsets

A, B of X is the infimum of all numbers $R > 0$ such that A is contained in the R-neighborhood of B and B is contained in the R-neighborhood of A. (This number can be infinite).

A *visibility point* is a point $\xi \in \partial X$ with the property that any $\eta \in \partial X - \{\xi\}$ can be connected to ξ by a geodesic line. By Lemma 3.5 of [16], the endpoint of a contracting geodesic ray is a visibility point. Geodesic rays which abut at an endpoint of a contracting geodesic ray are themselves contracting.

Lemma 2.4. *For every $B > 0$ there is a number $C = C(B) > B$ with the following property. Let $\gamma: [0,\infty) \to X$ be a B-contracting ray and let $\xi \in \partial X - \gamma(\infty)$. Then every geodesic ζ connecting $\xi = \zeta(-\infty)$ to $\gamma(\infty) = \zeta(\infty)$ passes through the $9B + 6$-neighborhood of every point $x \in \pi_{\gamma[0,\infty)}(\xi)$. If $t \in \mathbb{R}$ is such that $d(\zeta(t), x) \le 9B + 6$ then the geodesic ray $\zeta[t,\infty)$ is C-contracting.*

Proof. Let $\gamma: [0,\infty) \to X$ be a B-contracting geodesic ray and let $\xi \in \partial X - \gamma(\infty)$. Let $s \ge 0$ be such that $\gamma(s) \in \pi_{\gamma[0,\infty)}(\xi)$. By Lemma 2.2, the projection $\pi_{\gamma[0,\infty)}(\xi)$ is contained in $\gamma[s - 6B - 4, s + 6B + 4]$. Let $\zeta: \mathbb{R} \to X$ be a geodesic connecting ξ to $\gamma(\infty)$. Lemma 2.2 of [16] shows that for sufficiently large t we have

$$\pi_{\gamma[0,\infty)}(\zeta(-t)) \in \gamma[s - 6B - 5, s + 6B + 5].$$

Thus by Lemma 2.3, the geodesic ray $\zeta[-t,\infty)$ connecting $\zeta(-t)$ to $\gamma(\infty)$ passes through the $9B + 6$-neighborhood of $\gamma(s)$. Since ζ was an arbitrary geodesic connecting ξ to $\gamma(\infty)$, this shows the first part of the lemma.

From this and Lemma 3.8 of [6], the second part of the lemma is immediate as well. Namely, let again ζ be a geodesic connecting ξ to $\gamma(\infty)$ and let $\gamma(s) \in \pi_{\gamma[0,\infty)}(\xi)$. Assume that ζ is parametrized in such a way that $d(\zeta(0), \gamma(s)) \le 9B+6$. The geodesic ray $\zeta[0,\infty)$ is a locally uniform limit as $t \to \infty$ of the geodesics ζ_t connecting $\zeta(0)$ to $\gamma(t)$. By Lemma 3.8 of [6], there is a number $C > 0$ only depending on B such that each of the geodesics ζ_t is C-contracting. Now Lemma 3.3 of [16] shows that a limit of a sequence of C-contracting geodesics is C-contracting. This completes the proof of the lemma. □

The next observation is an extension of Lemma 2.3. For its formulation, define an *ideal geodesic triangle* to consist of three biinfinite geodesics γ_1, γ_2, γ_3 with $\gamma_i(\infty) = \gamma_{i+1}(-\infty)$ (where indices are taken modulo three). The points $\gamma_i(\infty)$ $(i = 1,2,3)$ are called the vertices of the ideal geodesic triangle. If $a, b \in \partial X$ are visibility points then for every $\xi \in \partial X - \{a,b\}$ there is an ideal geodesic triangle with vertices a, b, ξ. Note that such a triangle need not be unique.

Lemma 2.5. *Let $B > 0$ and let $\gamma: \mathbb{R} \to X$ be a B-contracting geodesic. Then for every ideal geodesic triangle T with side γ there is a point $x \in X$ whose distance to each of the sides of T does not exceed $9B + 6$. The diameter of the set of all such points does not exceed $54B + 36$.*

Proof. Let $\gamma: \mathbb{R} \to X$ be a B-contracting geodesic and let T be an ideal geodesic triangle with side γ and vertex $\xi \in \partial X - \{\gamma(\infty), \gamma(-\infty)\}$ opposite to γ. Assume that γ is parametrized in such a way that $\gamma(0) \in \pi_{\gamma(\mathbb{R})}(\xi)$. Let $c: [0,\infty) \to X$ be the geodesic ray connecting $c(0) = \gamma(0)$ to ξ. By Lemma 2.4 and by CAT(0)-comparison, the side α of T connecting ξ to $\gamma(\infty)$ is contained in the $9B+6$-tubular neighborhood of $c[0,\infty) \cup \gamma[0,\infty)$, and the side β of T connecting ξ to $\gamma(-\infty)$ is contained in the $9B+6$-tubular neighborhood of $c[0,\infty) \cup \gamma(-\infty, 0]$. The distance between $\gamma(0) = c(0)$ and every side of T does not exceed $9B+6$.

Since the projection $\pi_{\gamma(\mathbb{R})}$ is distance non-increasing and since $\pi_{\gamma(\mathbb{R})}(c[0,\infty)) = \gamma(0)$ (see [7] and the proof of Lemma 3.5 in [16]), if as before α, β are the sides of T connecting ξ to $\gamma(\infty), \gamma(-\infty)$, respectively, then

$$\pi_{\gamma(\mathbb{R})}(\alpha) \subset \gamma[-9B-6, \infty) \quad \text{and} \quad \pi_{\gamma(\mathbb{R})}(\beta) \subset \gamma(-\infty, 9B+6].$$

Now if $x \in X$ is such that the distance between x and each side of T is at most $9B+6$ then using again that $\pi_{\gamma(\mathbb{R})}$ is distance non-increasing we conclude that $\pi_{\gamma(\mathbb{R})}(x) \in \gamma[-18B-12, 18B+12]$. But $d(x,\gamma(\mathbb{R})) \leq 9B+6$ and hence $d(x,\gamma(0)) \leq 27B+18$. This completes the proof of the lemma. □

2.3. Isometries. For an isometry g of a proper CAT(0)-space X define the displacement function d_g of g to be the function $x \to d_g(x) = d(x, gx)$. An isometry g of X is called *semisimple* if d_g assumes a minimum in X. If g is semisimple and $\min d_g = 0$ then g is called *elliptic*. Thus an isometry is elliptic if and only if it fixes at least one point in X. A semisimple isometry g with $\min d_g > 0$ is called *axial*. By Proposition 3.3 of [3], an isometry g of X is axial if and only if there is a geodesic $\gamma: \mathbb{R} \to X$ such that $g\gamma(t) = \gamma(t+\tau)$ for every $t \in \mathbb{R}$ where $\tau = \min d_g > 0$ is the *translation length* of g. Such a geodesic is called an *oriented axis* for g. Note that the geodesic $t \to \gamma(-t)$ is an oriented axis for g^{-1}. The endpoint $\gamma(\infty)$ of γ is a fixed point for the action of g on ∂X, which is called the *attracting fixed point*. The closed convex set $A \subset X$ of all points for which the displacement function d_g of g is minimal is isometric to a metric product $A_0 \times \mathbb{R}$. For each $x \in A_0$ the set $\{x\} \times \mathbb{R}$ is an axis of g. The endpoints of this axis do not depend on x.

Bestvina and Fujiwara introduced the following notion to identify isometries of a CAT(0)-space with geometric properties similar to the properties of isometries in a hyperbolic geodesic metric space (Definition 5.1 of [6]).

Definition 2.6. For a number $B > 0$, an isometry $g \in \mathrm{Iso}(X)$ is called *B-rank-one* if g is axial and admits a B-contracting axis.

We call an isometry $g \in \mathrm{Iso}(X)$ *rank-one* if g is B-rank-one for some $B > 0$. By Lemma 2.4, if g is a rank-one isometry then there is a number $C > 0$ such that every axis of g is C-contracting.

The following statement is Theorem 5.4 of [6].

Proposition 2.7. *An axial isometry of X with axis γ is rank-one if and only if γ does not bound a flat half-plane.*

Let $G < \mathrm{Iso}(X)$ be a subgroup of the isometry group of X. The *limit set* Λ of G is the set of accumulation points in ∂X of one (and hence every) orbit of the action of G on X. The limit set is a compact G-invariant subset of ∂X (which may be empty). Call G *non-elementary* if its limit set contains at least three points and if moreover G does not fix a point in ∂X.

A compact space is *perfect* if it does not have isolated points. The action of a group G on a topological space Λ is called *minimal* if every orbit is dense. A homeomorphism g of a space Λ is said to act with *north-south dynamics* if it admits two fixed points $a \neq b \in \Lambda$ with the following property. For every neighborhood U of a, V of b there is some $k > 0$ such that $g^k(\Lambda - V) \subset U$ and $g^{-k}(\Lambda - U) \subset V$. The point a is called the *attracting fixed point* of g, and b is the repelling fixed point. The following is shown in [16] (see also [3] for a similar discussion).

Lemma 2.8. *Let $G < \mathrm{Iso}(X)$ be a non-elementary group which contains a rank-one element. Then the limit set Λ of G is perfect, and it is the smallest closed G-invariant subset of ∂X. The action of G on Λ is minimal. An element $g \in G$ is rank-one if and only if g acts on ∂X with north-south dynamics.*

Since X is proper by assumption, the isometry group $\mathrm{Iso}(X)$ of X can be equipped with a natural locally compact σ-compact metrizable topology, the so-called *compact open topology*. With respect to this topology, a sequence $(g_i) \subset \mathrm{Iso}(X)$ converges to some isometry g if and only if $g_i \to g$ uniformly on compact subsets of X. In this topology, a closed subset $A \subset \mathrm{Iso}(X)$ is compact if and only if there is a compact set $K \subset X$ such that $gK \cap K \neq \emptyset$ for every $g \in A$. In particular, the action of $\mathrm{Iso}(X)$ on X is proper. In the sequel we always equip subgroups of $\mathrm{Iso}(X)$ with the compact open topology.

Denote by Δ the diagonal in $\partial X \times \partial X$. Lemma 6.1 of [15] shows the following.

Lemma 2.9. *Let $G < \mathrm{Iso}(X)$ be a closed subgroup with limit set Λ. Let $(a, b) \in \Lambda \times \Lambda - \Delta$ be the pair of fixed points of a rank-one element of G. Then the G-orbit of (a, b) is a closed subset of $\Lambda \times \Lambda - \Delta$.*

The following technical observation is useful in Section 6.

Lemma 2.10. *Let $G < \mathrm{Iso}(X)$ be a closed non-elementary group with limit set Λ. If G contains a rank-one element $g \in G$ with fixed points $a \neq b \in \Lambda$ and if G does not act transitively on the complement of the diagonal in $\Lambda \times \Lambda$, then there is some $h \in G$ such that $hb \neq b$ and the stabilizer in G of the pair of points $(b, hb) \in \Lambda \times \Lambda$ is compact.*

Proof. Let $g \in G$ be a rank-one element with attracting fixed point $a \in \Lambda$, repelling fixed point $b \in \Lambda$. Let $\gamma : \mathbb{R} \to X$ be an axis for g connecting b to a. Then γ is $B - 3$-contracting for some $B > 3$.

Let $h \in G$ be such that $hb \neq b$. By Lemma 2.3, the rays $\gamma(-\infty, 0]$ and $h(\gamma(-\infty, 0])$ are B-contracting, with endpoints $b, hb \in \Lambda$. Now a biinfinite geodesic $\xi : \mathbb{R} \to X$ with the property that there are numbers $-\infty < s < t < \infty$, $C > 0$ such that the rays $\xi(-\infty, s]$, $\xi[t, \infty)$ are C-contracting is C'-contracting for a number $C' > C$ only depending on C and on $[s, t]$. Therefore Lemma 2.4 implies that there is a number $B_0 > B$ such that each geodesic connecting b to hb is B_0-contracting. As a consequence, the set $A \subset X$ of all points which are contained in a geodesic connecting b to hb is closed and convex and isometric to $K_0 \times \mathbb{R}$ for a compact convex subset K_0 of X.

An isometry u of X which fixes the pair of points (b, hb) preserves the closed convex set $K_0 \times \mathbb{R}$. The restriction of u to $K_0 \times \mathbb{R}$ can be represented in the form (u_1, u_2) where u_1 is an isometry of K_0 and u_2 is an isometry of $\mathbb{R}$. Since K_0 is a compact convex subset of a CAT(0)-space, the map u_1 has a fixed point. As a consequence, u is semi-simple. Moreover, if u is not elliptic then u is rank-one. Since G is a closed subgroup of $\mathrm{Iso}(X)$, this implies that either the stabilizer of (b, hb) in G is compact or it contains a rank-one element.

Now assume that there is no $h \in G$ with $hb \neq b$ such that the stabilizer of (b, hb) in G is compact. Then each such stabilizer contains a rank-one element. Our goal is to show that G acts transitively on the complement of the diagonal in $\Lambda \times \Lambda$.

Let $h \in G$ with $hb \notin \{a, b\}$ and let again γ be an oriented axis for g connecting b to a. By Lemma 2.2, we can assume that $\gamma(0) \in \pi_{\gamma(\mathbb{R})}(hb)$. Since γ is B-contracting, by Lemma 2.4 and convexity the ray $\gamma(-\infty, 0]$ is contained in the $9B+6$-neighborhood of every geodesic connecting b to hb. Let G_b be the stabilizer of b in G. By assumption and the above discussion, there is a rank-one element $u \in G_b$ with attracting fixed point hb and repelling fixed point b. By Lemma 2.8, u acts with north-south dynamics on Λ and hence we have $u^i a \to hb$ $(i \to \infty)$. Since $ga = a$, for every sequence $(k(i)) \subset \mathbb{Z}$ we also have

$$u^i g^{-k(i)} a \to hb \quad (i \to \infty).$$

Let $\tau > 0$ be the translation length of g and let K be the closed $18B + 12 + 2\tau$-neighborhood of $\gamma(0)$. By the choice of the set K and the fact that u preserves a geodesic connecting b to hb which contains the ray $\gamma(-\infty, 0]$ in its $9B + 6$-neighborhood, for every $i > 0$ there is some $k(i) > 0$ such that $u^i g^{-k(i)} \gamma(0) \in K$. Since G is a closed subgroup of $\mathrm{Iso}(X)$ and $G_b < G$ is closed, up to passing to a subsequence the sequence $\{u^i g^{-k(i)}\} \subset G_b$ converges to an element $v \in G_b$ with $va = hb$.

As a consequence, for every $\xi \in Gb - \{b\}$ there is some $v \in G_b$ with $va = \xi$. This implies that the image of (a, b) under the action of the group G is dense in $\Lambda \times \Lambda - \Delta$. Namely, by Lemma 2.8, the G-orbit of b is dense in Λ. Thus it suffices to show that for every $u \in G$ and every $\xi \in Gb - \{b, ub\}$ there is some $v \in G$ with $v(a,b) = (\xi, ub)$. For this let $y = u^{-1}\xi \in Gb$. Then $y \neq b$ and hence there is some $w \in G_b$ with $w(a) = y$. Then the isometry $v = uw$ satisfies $v(b) = u(b)$, $v(a) = \xi$.

By Lemma 2.9, the G-orbit of (a,b) is closed in $\Lambda \times \Lambda - \Delta$. We showed in the previous paragraph that it is also dense and therefore G acts transitively on the complement of the diagonal in $\Lambda \times \Lambda$. The lemma follows. □

A free group with two generators is hyperbolic in the sense of Gromov [13]. In particular, it admits a Gromov boundary which can be viewed as a compactification of the group. The following result is contained in [6] (see also Proposition 5.8 of [16] and [11], [5]).

Lemma 2.11. *Let $G < \mathrm{Iso}(X)$ be a closed non-elementary group which contains a rank-one element. Let $\Lambda \subset \partial X$ be the limit set of G. If G does not act transitively on $\Lambda \times \Lambda - \Delta$, then G contains a free subgroup Γ with two generators and the following properties.*

(1) *Every element $e \neq g \in \Gamma$ is rank-one.*
(2) *There is a Γ-equivariant embedding of the Gromov boundary of Γ into Λ.*
(3) *There are infinitely many elements $u_i \in \Gamma$ $(i > 0)$ with fixed points a_i, b_i such that for all i the G-orbit of $(a_i, b_i) \in \Lambda \times \Lambda - \Delta$ is distinct from the orbit of (b_j, a_j) $(j > 0)$ or (a_j, b_j) $(j \neq i)$.*

3. The space of B-contracting geodesics

In the previous section we introduced for a number $B > 0$ a B-contracting geodesic in a proper CAT(0)-space X. In this section we investigate in more detail the space of all such geodesics in X.

The main idea is as follows: Even though the geometry of a CAT(0)-space X may be very different from the geometry of a hyperbolic geodesic metric space, if X admits B-contracting geodesics then by Lemma 2.5, these geodesics have the same global geometric properties as geodesics in a δ-hyperbolic geodesic metric space where $\delta > 0$ only depends on B. As a consequence, given a fixed point $x \in X$, we can describe the position of two such geodesics γ, ζ relative to each other as seen from x by introducing a metric quantity which can be thought of being equivalent to the (oriented) sum of the Gromov distances at x of their endpoints in the case that the space X is hyperbolic.

We continue to use the assumptions and notations from Section 2. In the remainder of this section, a geodesic in X is always defined on a closed connected subset J of $\mathbb{R}$. For some $B > 0$ denote by

$$\mathcal{A}(B) \subset \partial X \times \partial X - \Delta$$

the set of all ordered pairs of points in ∂X which are connected by a B-contracting geodesic. We have

Lemma 3.1. *$\mathcal{A}(B)$ is a closed subset of $\partial X \times \partial X - \Delta$.*

Proof. Let $\{(\xi_i, \eta_i)\} \subset \mathcal{A}(B)$ be a sequence which converges in $\partial X \times \partial X - \Delta$ to a point (ξ, η). For each i let γ_i be a B-contracting geodesic connecting ξ_i to η_i. We first claim that the geodesics γ_i pass through a fixed compact subset of X.

Namely, choose a point $x \in X$ and let $x_i = \pi_{\gamma_i(\mathbb{R})}(x)$. If the geodesics γ_i do not pass through a fixed compact subset of X then we have $d(x_i, x) \to \infty$. Since $X \cup \partial X$ is compact, after passing to a subsequence we may assume that $x_i \to \alpha \in \partial X$ as $i \to \infty$. On the other hand, the geodesic γ_i is B-contracting and therefore by Lemma 2.3 the geodesics connecting x to $\xi_i = \gamma_i(-\infty), \eta_i = \gamma_i(\infty)$ both pass through the $3B + 1$-neighborhood of x_i. By CAT(0)-comparison, this implies that $\xi_i \to \alpha, \eta_i \to \alpha$, which contradicts the assumption that $\xi_i \to \xi, \eta_i \to \eta \neq \xi$.

Thus the geodesics γ_i pass through a fixed compact subset of X and therefore after passing to a subsequence we may assume that $\gamma_i \to \gamma$ locally uniformly where γ is a geodesic connecting ξ to η. The limit geodesic is B-contracting by Lemma 3.6 of [16]. □

For a positive number B, a point $x \in X$ and an ordered pair $(\zeta_1 \colon J_1 \to X, \zeta_2 \colon J_2 \to X)$ of oriented geodesics in X which share at most one endpoint in ∂X define a number $\tau_B(x, \zeta_1, \zeta_2) \geq 0$ as follows.

By convexity of the distance function, there are (perhaps empty) closed connected subsets $[a_1, b_1] \subset J_1, [a_2, b_2] \subset J_2$ such that

$$[a_i, b_i] = \{t \mid d(\zeta_i(t), \zeta_{i+1}(J_{i+1})) \leq 6B + 2\}.$$

(Here $i = 1, 2$ and indices are taken modulo two. If ζ_1, ζ_2 have a common endpoint in ∂X then one of the numbers a_1, b_1 may be infinite, and this is the case if and only if $[a_i, b_i] \neq \emptyset$ and if one of the numbers a_2, b_2 is infinite.)

If $[a_i, b_i] \neq \emptyset$ then let $s_i, t_i \in J_i \cup \{\pm\infty\}$ be such that

$$\pi_{\zeta_i(J_i)}(\zeta_{i+1}(a_{i+1})) = \zeta_i(s_i) \quad \text{and} \quad \pi_{\zeta_i(J_i)}(\zeta_{i+1}(b_{i+1})) = \zeta_i(t_i)$$

($i = 1, 2$ and indices are taken modulo two).

Let $x_i = \pi_{\zeta_i(J_i)}(x)$ $(i = 1, 2)$. If $s_i < t_i$ and if $x_i \in \zeta_i[a_i, b_i]$ for $i = 1, 2$, then define

$$\tau_B(x, \zeta_1, \zeta_2) = \min\{d(x_i, \zeta_i(a_i)), d(x_i, \zeta_i(b_i)) \mid i = 1, 2\}.$$

In all other cases define $\tau_B(x, \zeta_1, \zeta_2) = 0$. Note that $\tau_B(x, \zeta_1, \zeta_2)$ depends on the orientation of ζ_1, ζ_2 but not on the parametrization of ζ_1, ζ_2 defining a fixed orientation.

We collect some first easy properties of the function τ_B.

Lemma 3.2. *For any two geodesics ζ_1, ζ_2 in X and any $x \in X$ the following holds true.*

(1) $\tau_B(x, \zeta_1, \zeta_2) = \tau_B(x, \zeta_2, \zeta_1)$.
(2) *If* $\hat{\zeta}_i$ *equals the geodesic obtained from* ζ_i *by reversing the orientation, then* $\tau_B(x, \hat{\zeta}_1, \hat{\zeta}_2) = \tau_B(x, \zeta_1, \zeta_2)$.
(3) $\tau_B(x, \zeta_1, \zeta_2) \leq \tau_B(y, \zeta_1, \zeta_2) + d(x, y)$ *for all* $x, y \in X$.

Proof. The first and the second property in the lemma is obvious from the definition. To show the third property simply note that for a geodesic $\zeta\colon J \to X$ the projection $\pi_{\zeta(J)}$ is distance non-increasing. □

Moreover we observe

Lemma 3.3. *Let* $\zeta_i\colon J_i \to X$ *be* B*-contracting geodesics* $(i = 1, 2)$ *such that* $\tau_B(x, \zeta_1, \zeta_2) > 0$. *Then we have*

$$d(\pi_{\zeta_1(J_1)}(x), \pi_{\zeta_2(J_2)}(x)) \leq 24B + 8.$$

Proof. If $\tau_B(x, \zeta_1, \zeta_2) > 0$ and if (after reparametrization) we have $\pi_{\zeta_i(J_i)}(x) = \zeta_i(0)$ $(i = 1, 2)$ then by definition of the function τ_B there is some $t \in J_2$ such that $d(\zeta_1(0), \zeta_2(t)) \leq 6B + 2$. This shows that

$$d(x, \zeta_2(J_2)) \leq d(x, \zeta_1(J_1)) + 6B + 2.$$

By symmetry we conclude that

$$|d(x, \zeta_1(J_1)) - d(x, \zeta_2(J_2))| \leq 6B + 2.$$

Since $d(\zeta_1(0), \zeta_2(t)) \leq 6B + 2$ we have

$$d(x, \zeta_2(t)) \leq d(x, \zeta_1(J_1)) + 6B + 2 \leq d(x, \zeta_2(J_2)) + 12B + 4. \tag{2}$$

On the other hand, by Lemma 2.3, the geodesic connecting x to $\zeta_2(t)$ passes through the $3B + 1$-neighborhood of $\zeta_2(0)$ and hence

$$d(x, \zeta_2(t)) \geq d(x, \zeta_2(J_2)) + |t| - 6B - 2. \tag{3}$$

The two inequalities (2) and (3) together show that $|t| \leq 18B + 6$ and therefore $d(\pi_{\zeta_1(J_1)}(x), \pi_{\zeta_2(J_2)}(x)) \leq 24B + 8$, as claimed. □

We also have

Lemma 3.4. *Let* $\zeta_i\colon [0, \infty) \to X$ $(i = 1, 2)$ *be two geodesic rays with the same endpoint* $\zeta_1(\infty) = \zeta_2(\infty)$. *Let* $s \in [1, \infty)$ *be such that*

$$p = \tau_B(\zeta_1(s), \zeta_1, \zeta_2) \geq 1.$$

Then $\tau_B(\zeta_1(s + t), \zeta_1, \zeta_2) \geq p + t - 12B - 4$ *for all* $t \geq 0$.

Proof. Let $\zeta_i : [0,\infty) \to X$ be geodesic rays in X $(i = 1,2)$ with $\zeta_1(\infty) = \zeta_2(\infty)$. Let $s \in [0,\infty)$ be such that $\tau_B(\zeta_1(s), \zeta_1, \zeta_2) \geq 1$. Then the geodesic ray $\zeta_2[0,\infty)$ passes through the $6B + 2$-neighborhood of $\zeta_1(s)$. If $s' \in [0,\infty)$ is such that $d(\zeta_1(s), \zeta_2(s')) \leq 6B + 2$, then by convexity of the distance function we have

$$d(\zeta_1(s+t), \zeta_2(s'+t)) \leq 6B + 2 \quad \text{for all } t \geq 0.$$

Now let $t \geq 0$ and let $\sigma \in \mathbb{R}$ be such that $\pi_{\zeta_2[0,\infty)}(\zeta_1(s+t)) = \zeta_2(\sigma)$. Then $d(\zeta_1(s+t), \zeta_2(\sigma)) \leq 6B + 2$ and hence the triangle inequality shows that $\sigma \in [s' + t - 12B - 4, s' + t + 12B + 4]$. From this and the definition of the function τ_B the lemma follows. □

The next observation is the analog of the familiar ultrametric inequality for Gromov products in hyperbolic spaces.

Lemma 3.5. *There is a number $L > 0$ such that for every $B > 0$ and for all B-contracting geodesics $\zeta_i : J_i \to X$ $(i = 1,2,3)$ we have*

$$\tau_B(x, \zeta_1, \zeta_3) \geq \min\{\tau_B(x, \zeta_1, \zeta_2), \tau_B(x, \zeta_2, \zeta_3)\} - LB.$$

Proof. Let $\zeta_i : J_i \to X$ be B-contracting geodesics $(i = 1,2,3)$ and let $x \in X$. Taking indices modulo 3, assume without loss of generality that

$$\tau_B(x, \zeta_1, \zeta_3) = \min\{\tau_B(x, \zeta_i, \zeta_{i+1}) \mid i = 1,2,3\}$$

and that $r_1 = \tau_B(x, \zeta_1, \zeta_2) \leq r_2 = \tau_B(x, \zeta_2, \zeta_3)$. If $r_1 = 0$ then there is nothing to show. So assume that $r_1 > 0$. By Lemma 3.3 we then have

$$d(\pi_{\zeta_2(J_2)}(x), \pi_{\zeta_j(J_j)}(x)) \leq 24B + 8 \, (j = 1,3) \tag{4}$$

and hence $d(\pi_{\zeta_1(J_1)}(x), \pi_{\zeta_3(J_3)}(x)) \leq 48B + 16$. Since the lemma is only significant if r_1 is large, without explicit mentioning we successively increase a lower bound for r_1 by a controlled amount in the course of the proof so that all the geometric estimates are meaningful.

For simplicity parametrize the geodesics ζ_i in such a way that $\pi_{\zeta_i(J_i)}(x) = \zeta_i(0)$ $(i = 1,2,3)$. By definition of the function τ_B there is a number $t_2 \geq 0$ such that $d(\zeta_1(r_1), \zeta_2(t_2)) \leq 6B + 2$. By the distance estimate (4), we have

$$t_2 = d(\zeta_2(t_2), \pi_{\zeta_2(J_2)}(x)) \in [r_1 - 30B - 10, r_1 + 30B + 10]$$

and hence $d(\zeta_1(r_1), \zeta_2(r_1)) \leq 36B + 12$.

Now $r_2 \geq r_1$ by assumption and therefore using once more the definition of the function τ_B we have $d(\zeta_2(r_1), \zeta_3(J_3)) \leq 6B + 2$. Thus if we write $R_0 = 42B + 18$ then we have $d(\zeta_1(r_1), \zeta_3(J_3)) \leq R_0$ and similarly $d(\zeta_1(-r_1), \zeta_3(J_3)) \leq R_0$. Since $d(\zeta_1(0), \zeta_3(0)) \leq 48B + 16$ by the estimate (4) above, we conclude that for $R_1 = 48B + 16 + R_0$ there are numbers $s_3, t_3 \geq r_1 - R_1$ such that $[-s_3, t_3] \subset J_3$ and

$$d(\zeta_1(-r_1), \zeta_3(-s_3)) \leq R_0 \quad \text{and} \quad d(\zeta_1(r_1), \zeta_3(t_3)) \leq R_0. \tag{5}$$

By assumption, ζ_1 and ζ_3 are B-contracting. Let $\rho\colon [0,b] \to X$ be the geodesic connecting $\zeta_1(-r_1) = \rho(0)$ to $\zeta_3(t_3) = \rho(b)$. Let

$$z = \pi_{\zeta_1(J_1)}(\zeta_3(t_3)).$$

Then $d(z, \zeta_3(t_3)) \le d(\zeta_1(r_1), \zeta_3(t_3))$ and hence by the estimate (5), the distance between z and $\zeta_3(t_3)$ is at most R_0. It follows from (5) and the triangle inequality that the distance between z and $\zeta_1(r_1)$ is bounded from above by $2R_0$.

Since ζ_1 is B-contracting, by Lemma 2.3 and the remark thereafter, there is a number $T \le b$ such that the Hausdorff distance between the subarc of ζ_1 connecting $\zeta_1(-r_1)$ to z and the arc $\rho[0,T]$ is at most $3B+1$. Moreover, by (5) above we can choose T in such a way that $T \ge b - R_0$.

Similarly, if

$$w = \pi_{\zeta_3(J_3)}(\zeta_1(-r_1))$$

then the distance between w and $\zeta_1(-r_1)$ is at most R_0. By (5) above, there is a number $S \le R_0$ such that the Hausdorff distance between $\rho[S,b]$ and the subarc of ζ_3 connecting w to $\zeta_3(t_3)$ is at most $3B+1$.

As a consequence, there are two subarcs ζ_1' of $\zeta_1(J_1)$, ζ_3' of $\zeta_3(J_3)$ whose Hausdorff distance to the geodesic arc $\rho[S,T]$ is at most $3B+1$. Hence the Hausdorff distance between ζ_1' and ζ_3' is at most $6B+2$.

To summarize the above discussion, if r_1 is sufficiently large depending on B then there is a number $L > 0$ and there is a subarc ζ_1', ζ_3' of $\zeta_1(J_1), \zeta_3(J_3)$ with the following property. The arc ζ_1', ζ_3' contains x_1, x_3 as an interior point, and the distance of x_1, x_3 to the endpoints of ζ_1', ζ_2' is at least $r_1 - LB$. Moreover, the Hausdorff distance in X between ζ_1', ζ_3' is smaller than $6B+2$. This shows that

$$\tau_B(x, \zeta_1, \zeta_3) \ge r_1 - LB \ge \tau_B(x, \zeta_1, \zeta_2) - LB,$$

which completes the proof of the lemma. □

For distinct pairs of points $(\xi_1, \eta_1), (\xi_2, \eta_2) \in \mathcal{A}(B)$ define

$$\tau_B(x, (\xi_1, \eta_1), (\xi_2, \eta_2)) \ge 0$$

to be the infimum of the numbers $\tau_B(x, \zeta_1, \zeta_2)$ over all B-contracting geodesics ζ_i connecting ξ_i to η_i $(i = 1, 2)$. Clearly we have

$$\tau_B(x, \alpha_1, \alpha_2) = \tau_B(x, \alpha_2, \alpha_1) \quad \text{for all } x \in X,\ \alpha_1, \alpha_2 \in \mathcal{A}(B).$$

Moreover, by Lemma 3.5, there is a number $L > 0$ such that for all $\alpha_1, \alpha_2, \alpha_3 \in \mathcal{A}(B)$ we have

$$\tau_B(x, \alpha_1, \alpha_3) \ge \min\{\tau_B(x, \alpha_1, \alpha_2), \tau_B(x, \alpha_2, \alpha_3)\} - LB.$$

Now we follow Section 7.3 of [13]. Namely, let $\chi > 0$ be sufficiently small that $\chi' = e^{\chi LB} - 1 < \sqrt{2} - 1$. Note that χ only depends on B. For this number χ and for $x \in X, \alpha_1, \alpha_2 \in \mathcal{A}(B) \times \mathcal{A}(B)$ define

$$\tilde{\delta}_x(\alpha_1, \alpha_2) = e^{-\chi\tau_B(x,\alpha_1,\alpha_2)}. \tag{6}$$

From Lemma 3.5 and Proposition 7.3.10 of [13] we obtain the following.

Corollary 3.6. *There is a family* $\{\delta_x\}$ $(x \in X)$ *of distances on* $\mathcal{A}(B)$ *with the following properties.*

(1) *The topology on* $\mathcal{A}(B)$ *defined by the distances* δ_x *is the restriction of the product topology on* $\partial X \times \partial X - \Delta$. *In particular,* $(\mathcal{A}(B), \delta_x)$ *is locally compact.*
(2) *The distances* δ_x *are invariant under the involution* ι *of* $\mathcal{A}(B)$ *which exchanges the two points* $\xi \neq \eta \in \partial X$ *in a pair* $(\xi, \eta) \in \mathcal{A}(B)$.
(3) $(1 - 2\chi')\tilde{\delta}_x \le \delta_x \le \tilde{\delta}_x$ *for all* $x \in X$.
(4) $e^{-\chi d(x,y)}\delta_x \le \delta_y \le e^{\chi d(x,y)}\delta_x$ *for all* $x, y \in X$.
(5) *The family* $\{\delta_x\}$ *is invariant under the action of* $\mathrm{Iso}(X)$ *on* $\mathcal{A}(B) \times X$.

Proof. The existence of a family $\{\delta_x\}$ $(x \in X)$ of distance functions on $\mathcal{A}(B)$ with the property stated in the third part of the corollary is immediate from Lemma 3.5 and Proposition 3.7.10 of [13]. The fourth part follows from the construction of the distance δ_x from the functions $\tilde{\delta}_x$ and from the third part of Lemma 3.2. Invariance under the action of the isometry group and under the involution ι is an immediate consequence of invariance of the function τ_B.

We are left with showing that for a given $x \in X$ the distance δ_x induces the restriction of the product topology on $\partial X \times \partial X - \Delta$. By the definition of the distances δ_x, if $(\xi_i, \eta_i) \to (\xi, \eta)$ in $(\mathcal{A}(B), \delta_x)$ then there are B-contracting geodesics γ_i connecting ξ_i to η_i which have longer and longer subsegments contained in a tubular neighborhood of radius $6B + 2$ about some geodesic γ connecting ξ to η. Moreover, these segments all pass through a fixed compact subset of X. By the definition of the topology on ∂X, this implies that $(\xi_i, \eta_i) \to (\xi, \eta)$ in $\partial X \times \partial X - \Delta$. As a consequence, the inclusion $(\mathcal{A}(B), \delta_x) \to \partial X \times \partial X - \Delta$ is continuous.

Continuity of the identity $\mathcal{A}(B) \subset \partial X \times \partial X - \Delta \to (\mathcal{A}(B), \delta_x)$ follows in the same way. Namely, by Lemma 3.1 and its proof, if $(\xi_i, \eta_i) \subset \mathcal{A}(B)$, if $(\xi_i, \eta_i) \to (\xi, \eta) \in \partial X \times \partial X - \Delta$ with respect to the product topology and if γ_i is a B-contracting geodesic connecting ξ_i to η_i then up to passing to a subsequence, we may assume that the geodesics γ_i converge uniformly on compact sets to a B-contracting geodesic γ connecting ξ to η. By convexity, by Lemma 2.3 and by the definition of the function τ_B, this implies that $(\xi_i, \eta_i) \to (\xi, \eta)$ in $(\mathcal{A}(B), \delta_x)$ for every $x \in X$. □

Using Corollary 3.6, we obtain the following analog of Lemma 2.1 of [15] (with identical proof).

Lemma 3.7. *$\mathcal{A}(B) \times X$ admits a natural* Iso(X)*-invariant ι-invariant distance function $\tilde{d}$ inducing the product topology. There is a number $c > 0$ such that for every $x \in X$, the restriction of $\tilde{d}$ to $\mathcal{A}(B) \times \{x\}$ satisfies*

$$c\delta_x(\alpha, \beta) \le \tilde{d}((\alpha, x), (\beta, x)) \le \delta_x(\alpha, \beta) \quad \textit{for all } \alpha, \beta \in \mathcal{A}(B).$$

Remark. The results of this section are valid for general proper geodesic metric spaces which admit a family $\mathcal{A}(B)$ of B-contracting geodesics. However, due to the lack of convexity of geodesics in this more general setting, the bounds in the estimates can change. This is in the spirit of [6].

Even more generally, for a number $C > 0$ define a *coarse geodesic* γ in a metric space (X, d) to be a map $\gamma\colon J \to X$ such that $|d(\gamma(s), \gamma(t)) - |s - t|| \le C$ for all $s, t \in J$. The construction above also applies to a family of B-contracting C-coarse geodesics in X.

4. Continuous bounded cocycles

In this section we consider again a proper CAT(0)-space X. Let G be a *closed* non-elementary subgroup of the isometry group of X with limit set Λ. Then G is a locally compact σ-compact topological group. Assume that G contains a rank-one element.

As in Section 3, for a number $B > 0$ denote by $\mathcal{A}(B) \subset \partial X \times \partial X - \Delta$ the set of pairs of distinct points in ∂X which can be connected by a B-contracting geodesic. Define $T \subset \Lambda^3$ to be the space of triples of pairwise distinct points in Λ. By Lemma 2.8, T is a locally compact uncountable topological G-space without isolated points. Let moreover

$$T(B) \subset T$$

be the set of triples $(a_1, a_2, a_3) \in T$ with the additional property that $(a_i, a_{i+1}) \in \mathcal{A}(B)$ ($1 \le i \le 3$ and where indices are taken modulo three). By Lemma 3.1, $T(B)$ is closed subset of T which is invariant under the diagonal action of G.

The goal of this section is to construct continuous bounded cocycles for the action of G on $T(B)$ (see Definition 4.2 below). We begin with constructing G-equivariant continuous bounded functions on $\mathcal{A}(B)$ with values in the topological vector space $C_b(G \times G)$ of continuous bounded functions on $G \times G$, equipped with the compact open topology (which is strictly weaker than the Banach space topology). The group G acts continuously on $C_b(G \times G)$ by left translation via $(gf)(h, u) = f(g^{-1}h, g^{-1}u)$.

Proposition 4.1. *Let X be a proper* CAT(0)*-space and let $G <$* Iso(X) *be a closed non-elementary subgroup which contains a B-rank-one element for some $B > 0$. Then for every triple $(a, b, \nu) \in T(B)$ such that $(a, b) \in \mathcal{A}(B) \cap \Lambda \times \Lambda$ is the pair of fixed points of a rank-one element of G, there is a continuous map $\alpha\colon \mathcal{A}(B) \to C_b(G \times G)$ with the following properties.*

(1) $g^{-1}\alpha(g\xi, g\eta) = \alpha(\xi, \eta) = -\alpha(\eta, \xi)$ *for all* $(\xi, \eta) \in \mathcal{A}(B)$ *and all* $g \in G$.
(2) *For every point* $x_0 \in X$, *all* $(\xi, \eta) \in \mathcal{A}(B)$ *and all neighborhoods* A_1 *of* ξ, A_2 *of* η *in* $X \cup \partial X$ *the intersection of the support of* $\alpha(\xi, \eta)$ *with the set* $\{(g, h) \in G \times G \mid gx_0 \in X - (A_1 \cup A_2)\}$ *is compact.*
(3) *There are elements* g, h *in* G *such that* $\alpha(a, b)(g, h) \neq 0$ *and* $\alpha(a, \nu)(g, h) = \alpha(b, \nu)(g, h) = 0$.

Proof. Let $G < \mathrm{Iso}(X)$ be a closed non-elementary subgroup which contains a B-rank-one element for some $B > 0$. We divide the proof of the proposition into three steps.

Step 1: Let $x_0 \in X$ be an arbitrary point and denote by G_{x_0} the stabilizer of x_0 in G. Then G_{x_0} is a compact subgroup of G, and the quotient space G/G_{x_0} is G-equivariantly homeomorphic to the orbit $Gx_0 \subset X$ of x_0. Note that Gx_0 is a closed subset of X and hence it is locally compact. The group G acts on the locally compact space $\mathcal{A}(B) \times Gx_0$ as a group of homeomorphisms.

The $\mathrm{Iso}(X)$-invariant metric $\tilde{d}$ on $\mathcal{A}(B) \times X$ constructed in Lemma 3.7 induces a G-invariant metric on $\mathcal{A}(B) \times G/G_{x_0}$ which defines the product topology. Hence we obtain a G-invariant symmetrized product metric $\hat{d}$ on

$$V = \mathcal{A}(B) \times G/G_{x_0} \times G/G_{x_0}$$

by defining

$$\hat{d}((\xi, x, y), (\xi', x', y')) = \tfrac{1}{2}(\tilde{d}((\xi, x), (\xi', x')) + \tilde{d}((\xi, y), (\xi', y'))). \tag{7}$$

The topology defined on V by this metric is the product topology, in particular it is locally compact.

Since V is a locally compact G-space, the quotient space $W = G\backslash V$ admits a natural metric d_0 as follows. Let

$$P : V \to W$$

be the canonical projection and define

$$d_0(x, y) = \inf\{\hat{d}(\tilde{x}, \tilde{y}) \mid P\tilde{x} = x, P\tilde{y} = y\}. \tag{8}$$

The topology induced by this metric is the quotient topology for the projection P. In particular, W is a locally compact metric space. A set $U \subset W$ is open if and only if $P^{-1}(U) \subset V$ is open. In other words, open subsets of W correspond precisely to G-invariant open subsets of V. The projection P is open and distance non-increasing.

The distance $\tilde{d}$ on $\mathcal{A}(B) \times X$ is invariant under the involution $\iota\colon (\xi, \eta, x) \to (\eta, \xi, x)$ exchanging the two components of a point in $\mathcal{A}(B)$ and hence the distance $\hat{d}$ on V is invariant under the natural extension of ι (again denoted by ι). Since the action of G commutes with the isometric involution ι, the map ι descends to an isometric involution of the metric space (W, d_0), which we denote again by ι.

For some $R_1, R_2 > 0$, an open subset U of W is said to have *property* (R_1, R_2) if for every $((\xi,\eta), gx_0, hx_0) \in P^{-1}(U) \subset V$ the distance in X between gx_0, hx_0 and any geodesic in X connecting ξ to η is at most R_1 and if moreover $d(gx_0, hx_0) \le R_2$.

We claim that for every $w \in W$ there are numbers $R_1, R_2 > 0$ and there is a neighborhood of w in W which has property (R_1, R_2). Namely, let $v = ((\xi,\eta), gx_0, hx_0) \in P^{-1}(w)$. Then ξ can be connected to η by a B-contracting geodesic γ and therefore by Lemma 2.3, any geodesic connecting ξ to η is contained in the $3B+1$-tubular neighborhood of γ. By the discussion in Section 3 (see the proof of Lemma 3.1), there is a neighborhood A of (ξ,η) in $\mathcal{A}(B)$ such that for all $(\xi',\eta') \in A$, any geodesic connecting ξ' to η' passes through a fixed compact neighborhood of gx_0. Thus by continuity, there are numbers $R_1 > 0$, $R_2 > 0$ and there is an open neighborhood U' of v in V such that for every $((\xi',\eta'), g'x_0, h'x_0) \in U'$ the distance between $g'x_0, h'x_0$ and any geodesic connecting ξ' to η' is at most R_1 and that moreover $d(g'x_0, h'x_0) \le R_2$. However, distances and geodesics are preserved under isometries and hence every point in $\widetilde{U} = \bigcup_{g\in G} gU'$ has this property. Since $\widetilde{U}$ is open, G-invariant and contains v, the set $\widetilde{U}$ projects to an open neighborhood of w in W. This neighborhood has property (R_1, R_2).

Step 2: In equation (8) in step 1 above, we defined a distance d_0 on the space $W = G\backslash V$. With respect to this distance, the involution ι acts non-trivially and isometrically. Choose a small closed metric ball D in W which is disjoint from its image under ι. In step 3 below we will construct explicitly such balls D; however for the moment, we simply assume that such a ball exists. By step 1 above, we may assume that D has property (R_1, R_2) for some $R_1, R_2 > 0$.

Let $\mathcal{H}$ be the vector space of all Hölder continuous functions $f: W \to \mathbb{R}$ supported in D. An example of such a function can be obtained as follows.

Let z be an interior point of D and let $r > 0$ be sufficiently small that the closed metric ball $B(z,r)$ of radius r about z is contained in D. Choose a smooth function $\chi: \mathbb{R} \to [0,1]$ such that $\chi(t) = 1$ for $t \in (-\infty, r/2]$ and $\chi(t) = 0$ for $t \in [r, -\infty)$ and define $f(y) = \chi(d_0(z,y))$. Since the function $y \to d_0(z,y)$ on W is one-Lipschitz and χ is smooth, the function f on W is Lipschitz, does not vanish at z and is supported in D.

Since D is disjoint from $\iota(D)$ by assumption and since ι is an isometry, every function $f \in \mathcal{H}$ admits a natural extension to a Hölder continuous function f_0 on W supported in $D \cup \iota(D)$ whose restriction to D coincides with the restriction of f and which satisfies $f_0(\iota z) = -f_0(z)$ for all $z \in W$. The function $\hat{f} = f_0 \circ P: V \to \mathbb{R}$ is invariant under the action of G, and it is *anti-invariant* under the involution ι of V, i.e., it satisfies $\hat{f}(\iota(v)) = -\hat{f}(v)$ for all $v \in V$ (here as before, $P: V \to W$ denotes the canonical projection).

Equip $\widetilde{V} = \mathcal{A}(B) \times G \times G$ with the product topology. The group G acts on $G \times G$ by left translation, and it acts diagonally on $\widetilde{V}$. There is a natural continuous projection $\Pi: \widetilde{V} \to V$ which is equivariant with respect to the action of G and with respect to the action of the involution ι on $\widetilde{V}$ and V. The function $\hat{f}$ on V lifts to a

G-invariant ι-anti-invariant continuous function $\tilde{f} = \hat{f} \circ \Pi$ on $\widetilde{V}$.

For $(\xi, \eta) \in \mathcal{A}(B)$ write

$$F(\xi, \eta) = \{(\xi, \eta, z) \mid z \in G \times G\}.$$

The sets $F(\xi, \eta)$ define a G-invariant foliation $\mathcal{F}$ of $\widetilde{V}$. The leaf $F(\xi, \eta)$ of $\mathcal{F}$ can naturally be identified with $G \times G$. For all $(\xi, \eta) \in \mathcal{A}(B)$ and every function $f \in \mathcal{H}$ we denote by $f_{\xi,\eta}$ the restriction of the function $\tilde{f}$ to $F(\xi, \eta)$, viewed as a continuous bounded function on $G \times G$. For every $f \in \mathcal{H}$, all $(\xi, \eta) \in \mathcal{A}(B)$ and all $g \in G$ we then have $f_{g\xi,g\eta} \circ g = f_{\xi,\eta} = -f_{\eta,\xi}$. Since the functions $f_{\xi,\eta}$ are restrictions to the leaves of the foliation $\mathcal{F}$ of a globally continuous bounded function on $\widetilde{V}$, the assignment

$$(\xi, \eta) \in \mathcal{A}(B) \to \alpha(\xi, \eta) = f_{\xi,\eta} \in C_{\mathrm{b}}(G \times G)$$

is a continuous map of $\mathcal{A}(B)$ into $C_{\mathrm{b}}(G \times G)$. By construction, it satisfies

$$g^{-1}\alpha(g\xi, g\eta) = \alpha(\xi, \eta) = -\alpha(\eta, \xi) \quad \text{for all } (\xi, \eta) \in \mathcal{A}(B) \text{ and all } g \in G.$$

Thus the map α fulfills the first requirement in the statement of the proposition. The second requirement is also satisfied since the set D is assumed to have property (R_1, R_2) and since moreover for every geodesic $\zeta \colon \mathbb{R} \to X$, for every open neighborhood A of $\zeta(\infty) \cup \zeta(-\infty)$ in $X \cup \partial X$ and for every $R > 0$ the intersection of the closed R-neighborhood of ζ with $X - A$ is compact.

Step 3: To show that we can construct the function α in such a way that is satisfies the third property in the proposition, let $g \in G$ be a B-rank-one isometry, let $a \neq b \in \partial X$ be the attracting and repelling fixed point for the action of g on ∂X, respectively, and let $\nu \in \Lambda - \{a, b\}$ be such that $(a, b, \nu) \in T(B)$. We have to show that we can choose α in such a way that $\alpha(a, b)(g, h) \neq 0$ and $\alpha(a, \nu)(g, h) = \alpha(b, \nu)(g, h) = 0$ for some $g, h \in G$.

For this let γ be a B-contracting oriented axis for g and let $x_0 \in \pi_{\gamma(\mathbb{R})}(\nu)$ be the basepoint for the above construction. The orbit of x_0 under the infinite cyclic subgroup of G generated by g is contained in the geodesic γ. Write $\kappa_0 = 9B + 6$. Since $g^j x_0 \to a, g^{-j} x_0 \to b$ $(j \to \infty)$, there are numbers $k < \ell$, $R_1 > 2\kappa_0$ such that the $R_1 + \kappa_0$-neighborhood of a geodesic connecting a to ν and the $R_1 + \kappa_0$-neighborhood of a geodesic connecting b to ν contains at most one of the points $g^k x_0$, $g^\ell x_0$ and that moreover the distance between $g^k x_0$, $g^\ell x_0$ is at least $4\kappa_0$ (compare Lemma 2.5 and its proof). Choose $R_2 > 2d(g^k x_0, g^\ell x_0)$.

We claim that there is no $h \in G$ with $hg^k x_0 = g^k x_0$, $hg^\ell x_0 = g^\ell x_0$ and $h(a) = b$, $h(b) = a$. Namely, any isometry h which exchanges a and b and fixes a point on the axis γ of g, say the point $\gamma(t)$, maps the geodesic ray $\gamma[t, \infty)$ to the geodesic ray $\gamma(-\infty, t]$. Thus a fixed point of h on γ is unique which shows the claim. As a consequence, the projection of $(a, b, g^k x_0, g^\ell x_0) \in V$ into the space W is not fixed by the involution ι.

Let $\widetilde{D} \subset V$ be a neighborhood of $(a, b, g^k x_0, g^\ell x_0)$ in V which is small enough that for all $(a', b', x, y) \in \widetilde{D}$ a geodesic connecting a' to b' passes through the $2\kappa_0$-neighborhood of $g^k x_0, g^\ell x_0$, and $d(x, g^k x_0) \le \kappa_0$, $d(y, g^\ell x_0) \le \kappa_0$. The projection of $\widetilde{D}$ into W contains a ball $D \subset W$ about the projection of $(a, b, g^k x_0, g^\ell x_0)$ with property (R_1, R_2) which is disjoint from its image under ι. Let $f \in \mathcal{H}$ be a Hölder continuous function supported in D which does not vanish at the projection of $(a, b, g^k x_0, g^\ell x_0)$. Then the lift $\tilde{f}$ of f to $\widetilde{V}$ does not vanish at (a, b, g^k, g^ℓ). By the choice of D, we have $f_{a,b}(g^k, g^\ell) \neq 0$ and $f_{b,v}(g^k, g^\ell) = f_{v,a}(g^k, g^\ell) = 0$. In other words, the function α constructed as above from f has property (3) stated in the proposition. □

Proposition 4.1 is used for the construction of continuous bounded cocycles for the action of G on $T(B)$.

Definition 4.2. Let E be a separable Banach-module for G. An *E-valued continuous bounded two-cocycle* for the action of G on $T(B)$ is a continuous *bounded* G-equivariant map $\omega\colon T(B) \to E$ which satisfies the following two properties.

(1) For every permutation σ of the three variables, the *anti-symmetry condition* $\omega \circ \sigma = \mathrm{sgn}(\sigma)\omega$ holds.

(2) For every quadruple (a_1, a_2, a_3, a_4) of distinct points in Λ such that $(a_i, a_j) \in \mathcal{A}(B)$ for $i \neq j$ the *cocycle equality*

$$\omega(a_2, a_3, a_4) - \omega(a_1, a_3, a_4) + \omega(a_1, a_2, a_4) - \omega(a_1, a_2, a_3) = 0 \qquad (9)$$

is satisfied.

The separable Banach modules for G we are interested in are as follows. Every locally compact σ-compact topological group G admits a left invariant locally finite Haar measure μ. For $p \in (1, \infty)$ denote by $L^p(G \times G, \mu \times \mu)$ the Banach space of all functions on $G \times G$ which are p-integrable with respect to the product measure $\mu \times \mu$. The group G acts continuously and isometrically on $L^p(G \times G, \mu \times \mu)$ by left translation via $(gf)(h, u) = f(g^{-1}h, g^{-1}u)$.

The following theorem is the main result in this section.

Theorem 4.3. *Let $\alpha\colon \mathcal{A}(B) \to C_b(G \times G)$ be a continuous map as in Proposition 4.1. Then for every $p \in (1, \infty)$ the assignment*

$$\omega\colon (\sigma, \eta, \beta) \in T(B) \to \omega(\sigma, \eta, \beta) = \alpha(\sigma, \eta) + \alpha(\eta, \beta) + \alpha(\beta, \sigma)$$

is an $L^p(G \times G, \mu \times \mu)$-valued continuous bounded two-cocycle for the action of G on $T(B)$.

Proof. As in step 1 of the proof of Proposition 4.1, let $x_0 \in X$ be a fixed point and let $V = \mathcal{A}(B) \times G/G_{x_0} \times G/G_{x_0}$. Choose a compact ball $D \subset W = G\backslash V$ which is disjoint from its image under the involution ι and which has property (R_1, R_2) for some $R_1, R_2 > 0$. Let $\mathcal{H}$ be the vector space of all Hölder continuous functions $f: W \to \mathbb{R}$ supported in D. Every $f \in \mathcal{H}$ lifts to a Hölder continuous G-invariant ι-anti-invariant function

$$\hat{f}: V \to \mathbb{R}$$

and to a continuous function $\tilde{f}$ on $\mathcal{A}(B) \times G \times G$. As in the proof of Proposition 4.1 we denote by $f_{\xi,\eta}$ the restriction of $\tilde{f}$ to $(\xi, \eta) \times G \times G$, viewed as a function on $G \times G$.

For $f \in \mathcal{H}$ and for an ordered triple $(\xi, \eta, \beta) \in T(B)$ define

$$\omega(\xi, \eta, \beta) = f_{\xi,\eta} + f_{\eta,\beta} + f_{\beta,\xi} \in C_b(G \times G). \tag{10}$$

Since $f_{\xi,\eta} = -f_{\eta,\xi}$ for all $(\xi, \eta) \in \mathcal{A}(B)$, we have

$$\omega \circ \sigma = (\text{sgn}(\sigma))\omega$$

for every permutation σ of the three variables. As a consequence, the cocycle condition for ω is also satisfied. The assignment $(\xi, \eta, \beta) \in T(B) \to \omega(\xi, \eta, \beta) \in C_b(G \times G)$ is continuous with respect to the compact open topology on $C_b(G \times G)$. Moreover, it is equivariant with respect to the natural action of G on the space $T(B)$ and on $C_b(G \times G)$. This means that ω is a continuous bounded cocycle for the action of G on $T(B)$ with values in the topological vector space $C_b(G \times G)$.

For the proof of the theorem, we have to show the following.

(1) $\omega(\xi, \eta, \beta) \in L^p(G \times G, \mu \times \mu)$ for every $p \in (1, \infty)$, with L^p-norm bounded from above by a constant which does not depend on $(\xi, \eta, \beta) \in T(B)$.

(2) The assignment $(\xi, \eta, \beta) \to \omega(\xi, \eta, \beta) \in L^p(G \times G, \mu \times \mu)$ is continuous.

For this let $(\xi, \eta, \beta) \in T(B)$ and let γ be a B-contracting geodesic connecting ξ to η. By Lemma 2.3 and Lemma 2.5 there is a point $y_0 \in X$ which is contained in the $\kappa_0 = \kappa_0(B) = 9B + 6$-neighborhood of every side of any geodesic triangle Q with vertices ξ, η, β and side γ.

Assume that γ is parametrized in such a way that $d(\gamma(0), y_0) \leq \kappa_0$. Also, let $\rho: \mathbb{R} \to X$ be the side of Q connecting β to η which is parametrized in such a way that $d(y_0, \rho(0)) \leq \kappa_0$. We may assume that ρ is B-contracting. Then $\gamma[0, \infty), \rho[0, \infty)$ are two sides of a geodesic triangle in X with vertices $\gamma(0), \rho(0), \eta$. Since $d(\gamma(0), \rho(0)) \leq 2\kappa_0$, the convexity of the distance function implies that $d(\gamma(t), \rho(t)) \leq 2\kappa_0$ for all $t \geq 0$. In particular, the R_1-neighborhood of $\rho[0, \infty)$ is contained in the $R_1 + 2\kappa_0$-neighborhood of $\gamma[0, \infty)$.

Let $R_1 > 0, R_2 > 0$ be as in the beginning of this proof. For a subset C of X write

$$C_{G,R_2} = \{(u, h) \in G \times G \mid ux_0 \in C, d(ux_0, hx_0) \leq R_2\}.$$

Let $\nu = \mu \times \mu$ be the left invariant product measure on $G \times G$. We claim that there is a number $m > 0$ such that for every subset C of X of diameter at most $2R_1 + 4\kappa_0 + 1$

the ν-mass of the set C_{G,R_2} is at most m. Namely, the set

$$K = \{(u,h) \in G \times G \mid d(ux_0, x_0) \le 2R_1 + 4\kappa_0 + 1, d(ux_0, hx_0) \le R_2\}$$

of $G \times G$ is compact and hence its ν-mass is finite, say this mass equals $m > 0$. On the other hand, if $C \subset X$ is a set of diameter at most $2R_1 + 4\kappa_0 + 1$ and if there is some $g \in G$ such that $gx_0 \in C$ then any pair $(u,h) \in C_{G,R_2}$ is contained in gK. Our claim now follows from the fact that ν is invariant under left translation.

By construction, if $N(C,r)$ denotes the r-neighborhood of a set $C \subset X$, then the support of the function $f_{\xi,\eta}$ is contained in $N(\gamma(\mathbb{R}), R_1)_{G,R_2}$ and similarly for the functions $f_{\eta,\beta}$, $f_{\beta,\xi}$. As a consequence, the support of the function ω defined in (10) above is the union of the three sets

$$N(\gamma[0,\infty), R_1 + 2\kappa_0)_{G,R_2}, \quad N(\gamma(-\infty,0], R_1 + 2\kappa_0)_{G,R_2}, \\ N(\rho(-\infty,0], R_1 + 2\kappa_0)_{G,R_2}.$$

Moreover, there is a number $\tau > 0$ only depending on R_1, R_2 and κ_0 such that the restriction of ω to $N(\gamma[\tau,\infty), R_1 + 2\kappa_0)_{G,R_2}$ coincides with the restriction of the function $f_{\xi,\eta} + f_{\eta,\beta}$ and similarly for the other two sets in the above decomposition of the support of ω.

Since ω is uniformly bounded, to show that ω is contained in $L^p(G \times G, \nu)$ it is now enough to show that there is constant $c_p > 0$ only depending on p and the Hölder norm of f such that

$$\int_{N(\gamma[\tau,\infty), R_1+2\kappa_0)_{G,R_2}} |f_{\xi,\eta} + f_{\eta,\beta}|^p d\nu < c_p.$$

However, we observed above that for every integer $k \ge 0$ the ν-mass of the set $N(\gamma[\tau+k, \tau+k+1], R_1+2\kappa_0)_{G,R_2}$ is bounded from above by a universal constant $m > 0$. Thus it suffices to show that there are numbers $r > 0, \sigma > 0$ such that the value of the function $|f_{\xi,\eta} + f_{\beta,\eta}|$ on this set does not exceed $re^{-\sigma(\tau+k)}$. Then the inequality holds true with $c_p = mr^p \sum_{k=0}^{\infty} e^{-p\sigma(\sigma+k)}$.

For an estimate of $|f_{\xi,\eta} + f_{\eta,\beta}|$, apply Lemma 2.3 to the geodesic ray $\gamma[0,\infty)$ and a geodesic ζ connecting β to η. We conclude that there is a subray of ζ whose Hausdorff distance to $\gamma[2\kappa_0, \infty)$ is bounded from above by $3B + 1$. Then by the definition of the distances δ_x on $\mathcal{A}(B)$ and by Lemma 3.4 and Corollary 3.6, there is a number $r_0 > 0$ depending on κ_0, R_1, R_2 such that if $t \ge 0$ and if $y \in X$ satisfies $d(\gamma(t), y) < \kappa_0 + R_1 + R_2$ then

$$\delta_y((\xi,\eta),(\beta,\eta)) \le r_0 e^{-\chi t},$$

where $\chi > 0$ is as in Corollary 3.6. Moreover, by the definition (7) of the distance function $\hat{d}$ on V and by the estimate in Lemma 3.7 for the distance function $\tilde{d}$ on $\mathcal{A}(B) \times X$, we have

$$\begin{aligned} \hat{d}((\xi,\eta,ux_0,hx_0),(\beta,\eta,ux_0,hx_0)) &\le \tfrac{1}{2}(\delta_{ux_0}((\xi,\eta),(\beta,\eta)) + \delta_{hx_0}((\xi,\eta),(\beta,\eta))) \\ &\le r_0 e^{-\chi t} \end{aligned} \tag{11}$$

whenever $d(ux_0, \gamma(t)) \leq \kappa_0 + R_1$ and $((\xi, \eta), ux_0, hx_0)$ is contained in the support of $f_{\xi,\eta}$ or of $f_{\eta,\beta}$.

The function $\hat{f}: V \to \mathbb{R}$ is Hölder continuous and ι-anti-invariant. Therefore by the estimate (11) there are numbers $\theta > 0, r_1 > r_0$ only depending on the Hölder norm for f with the following property. Let $0 \leq t$ and let $u, h \in G$ be such that $d(ux_0, \gamma(t)) < \kappa_0 + R_1, d(hx_0, \gamma(t)) < \kappa_0 + R_1 + R_2$; then

$$|\hat{f}(\xi, \eta, ux_0, hx_0) + \hat{f}(\eta, \beta, ux_0, hx_0)| \leq r_1 e^{-\theta\chi t}. \tag{12}$$

The function f is bounded in absolute value by a universal constant. Hence from the definition of the functions $f_{\xi,\eta}$ and $f_{\eta,\beta}$ and from the estimate (12) we obtain the existence of a constant $r > r_1$ (depending on the Hölder norm of f) such that

$$|(f_{\xi,\eta} + f_{\eta,\beta})(u, h)| \leq r e^{-\theta\chi t}$$

whenever $d(ux_0, \gamma(t)) \leq \kappa_0 + R_1$. This is the estimate we were looking for.

To show continuity of the assignment $(\xi, \eta, \beta) \in T(B) \to \omega(\xi, \eta, \beta) \in L^p(G \times G, \nu)$, we use again the above estimate. Namely, let $(\xi_i, \zeta_i, \eta_i) \subset T(B)$ be a sequence of triples of pairwise distinct points converging to a triple $(\xi, \eta, \beta) \in T(B)$. By the above consideration, for every $\varepsilon > 0$ there is a compact subset A of $G \times G$ such that $\int_{G\times G-A} |\omega(\xi_i, \eta_i, \beta_i)|^p d\nu \leq \varepsilon$ for all sufficiently large $i > 0$ and that the same holds true for $\omega(\xi, \eta, \zeta)$. Let χ_A be the characteristic function of A. By continuity of the function $\hat{f}$ on $\tilde{V}$ and compactness, the functions $\chi_A\omega(\xi_i, \eta_i, \beta_i)$ converge as $i \to \infty$ in $L^p(G \times G, \nu)$ to $\chi_A\omega(\xi, \eta, \zeta)$. Since $\varepsilon > 0$ was arbitrary, the required continuity follows.

By construction, the assignment $(\xi, \eta, \beta) \to \omega(\xi, \eta, \beta)$ is equivariant under the action of G on the space $T(B)$ and on $L^p(G\times G, \nu)$ and satisfies the cocycle equality (9). In other words, ω defines a continuous $L^p(G \times G, \nu)$-valued bounded cocycle for the action of G on $T(B)$ as required. This completes the proof of the theorem. □

5. Second continuous bounded cohomology I

Let X be a proper CAT(0)-space with isometry group $\mathrm{Iso}(X)$. Let $G < \mathrm{Iso}(X)$ be a closed non-elementary subgroup with limit set Λ which contains a rank-one element. Under the additional assumption that G acts transitively on the complement of the diagonal in $\Lambda \times \Lambda$ we use Theorem 4.3 to construct nontrivial second bounded cohomology classes for G.

We use the arguments from Section 3 of [15] (see also [18], [20] for earlier results along the same line). Namely, let $\mathcal{P}(\Lambda)$ be the space of all probability measures on Λ, equipped with the weak*-topology. Denote moreover by $\mathcal{P}_{\geq 3}(\Lambda) \subset \mathcal{P}(\Lambda)$ the set of all probability measures which are not concentrated on at most two points. We first show

Lemma 5.1. *Let $G < \mathrm{Iso}(X)$ be a non-elementary closed subgroup with limit set Λ. If G contains a rank-one element and acts transitively on the complement of the diagonal in $\Lambda \times \Lambda$, then the action of G on $\mathcal{P}_{\geq 3}(\Lambda)$ is tame with compact point stabilizers.*

Proof. Let $G < \mathrm{Iso}(X)$ be a closed non-elementary group with limit set Λ which acts transitively on the complement of the diagonal Δ in $\Lambda \times \Lambda$ and contains a rank-one element $g \in G$. Then there is a number $B > 0$ and for every pair $(\xi, \eta) \in \Lambda \times \Lambda - \Delta$ there is a B-contracting geodesic $\gamma\colon \mathbb{R} \to X$. This geodesic is the image of an axis of g under an element of G.

Let $T \subset \Lambda^3$ be the space of triples of pairwise distinct points in Λ. By Lemma 2.3, if $\gamma\colon \mathbb{R} \to X$ is a B-contracting geodesic then every other geodesic connecting $\gamma(-\infty)$ to $\gamma(\infty)$ is contained in the $3B + 1$-tubular neighborhood of γ. Thus by Lemma 2.5, for every triple $(a, b, c) \in T$ there is a point $x_0 \in X$ whose distance to any of the sides of some geodesic triangle in X with vertices a, b, c is at most $12B + 7$. The set $K(a, b, c)$ of all points with this property is clearly closed. Lemma 2.5 shows that its diameter is bounded from above by a constant not depending on $(a, b, c) \in T(B)$. In other words, $K(a, b, c)$ is compact and hence it has a unique center $\Phi(a, b, c) \in X$ where a center of a compact set $K \subset X$ is a point $x \in X$ such that the radius of the smallest closed ball about x containing K is minimal (see p. 10 in [4]).

This construction defines a map $\Phi\colon T \to X$ which is equivariant with respect to the action of G. Moreover, it is continuous. Namely, if $(a_i^1, a_i^2, a_i^3) \to (a^1, a^2, a^3)$ in T then by the discussion in the proof of Lemma 3.1 there is a compact neighborhood A of $K(a^1, a^2, a^3)$ such that for all sufficiently large i, every geodesic connecting a pair of points $(a_i^j, a_i^{j+1}) \in \Lambda \times \Lambda - \Delta$ passes through A. Since the diameters of the sets $K(a_i^1, a_i^2, a_i^3)$ are uniformly bounded, there is a compact subset Q of X which contains the sets $K(a_i^1, a_i^2, a_i^3)$ for all sufficiently large i. Therefore up to passing to a subsequence, we may assume that the compact sets $K(a_i^1, a_i^2, a_i^3)$ converge in the Hausdorff topology for compact subsets of X to a compact set K. The set K is contained in the set of all limits of sequences (b_{i_ℓ}), where i_ℓ is a subsequence of the set of natural numbers and $b_{i_\ell} \in K(a_{i_\ell}^1, a_{i_\ell}^2, a_{i_\ell}^3)$.

On the other hand, up to passing to a subsequence and reparametrization, a sequence of geodesics γ_i^j connecting a_i^j to a_i^{j+1} converges as $i \to \infty$ locally uniformly to a geodesic γ^j connecting a^j to a^{j+1}. By continuity of the distance function and the definitions, this implies that $K \subset K(a^1, a^2, a^3)$.

Let $G_{(a^j, a^{j+1})} < G$ be the stabilizer of (a^j, a^{j+1}) in G. Since G acts transitively on $\Lambda \times \Lambda - \Delta$, the topological space $\Lambda \times \Lambda - \Delta$ is G-equivariantly homeomorphic to $G/G_{(a^j, a^{j+1})}$. In particular, there is a sequence $g_i \subset G$ converging to the identity such that $(a_i^j, a_i^{j+1}) = g_i(a^j, a^{j+1})$. However, this implies that every geodesic connecting a^j to a^{j+1} is a limit as $i \to \infty$ of a sequence of geodesics connecting a_i^j to a_i^{j+1} and therefore $K = K(a, b, c)$. Since the map which associates to a compact subset of X its center is continuous with respect to the Hausdorff topology on compact subsets of X, we conclude that the map Φ is in fact continuous. Since the action of G on X is proper, Lemma 3.4 of [1] then shows that the action of G on T is proper as well.

Let $\mathcal{P}(T)$ be the space of probability measures on the locally compact space T. Since the action of G on T is proper, the action of G on $\mathcal{P}(T)$ is tame, with compact point stabilizers. For $\mu \in \mathcal{P}_{\geq 3}(\Lambda)$ let $a_\mu > 0$ be the total mass of the open set $T \subset \Lambda^3$ with respect to the measure $\mu \times \mu \times \mu$. The map $\mathcal{P}_{\geq 3}(\Lambda) \to \mathcal{P}(T)$ which associates to a measure $\mu \in \mathcal{P}_{\geq 3}(\Lambda)$ the normalized product $\mu \times \mu \times \mu / a_\mu$ is Borel and G-equivariant (see Theorem 5.2 of [1]). Since the action of G on $\mathcal{P}(T)$ is tame, with compact point stabilizers, Lemma 3.4 of [1] shows that the action of G on $\mathcal{P}_{\geq 3}(\Lambda)$ is tame with compact point stabilizers. This completes the proof of the lemma. □

The next easy consequence of a result of Adams and Ballmann [2] will be important for the proof of Theorem 1. For later reference, recall that the closure of a normal subgroup of a topological group G is normal, and the closure of an amenable subgroup of G is amenable (Lemma 4.1.13 of [22]).

Lemma 5.2. *Let $G < \mathrm{Iso}(X)$ be a closed non-elementary subgroup which contains a rank-one element. Then a closed normal amenable subgroup N of G is compact, and N fixes the limit set of G pointwise.*

Proof. Let $G < \mathrm{Iso}(X)$ be a closed non-elementary group which contains a rank-one element and let $N \lhd G$ be a closed normal amenable subgroup. Since N is amenable, either N fixes a point $\xi \in \partial X$ or N fixes a *flat* $F \subset X$, i.e., a closed convex subspace of X which is isometric to a finite dimensional euclidean space [2].

Assume first that N fixes a point $\xi \in \partial X$. Since N is normal in G, for every $g \in G$ the point $g\xi$ is a fixed point for $gNg^{-1} = N$. On the other hand, by Lemma 2.8, the closure in ∂X of every orbit for the action of G contains the limit set Λ of G, and the action of G on Λ is minimal. Therefore by continuity, N fixes Λ pointwise. Then Lemma 5.1 shows that N is compact.

If N fixes a flat $F \subset X$ of dimension at least two then we argue in the same way. Namely, the image of F under an isometry of X is a flat. Let $a \neq b \in \Lambda$ be the attracting and repelling fixed points, respectively, of a rank-one element g of G. Then a, b are visibility points in ∂X and hence there is no flat in X of dimension at least two whose boundary in ∂X contains one of the points a, b. Thus the boundary $\partial F \subset \partial X$ of F is contained in $\partial X - \{b\}$ and consequently $g^k \partial F \to \{a\}$ $(k \to \infty)$. But by the argument in the previous paragraph, N fixes $g^k \partial F = \partial(g^k F)$ and therefore N fixes a by continuity. In other words, N fixes a point in ∂X. The first part of this proof then shows that indeed N is compact.

The same reasoning is also valid if N fixes a flat F of dimension one, i.e., a geodesic. Namely, in this case Lemma 2.8 shows that there is a rank-one element $g \in G$ such that the repelling fixed point of g is not an endpoint of F. Then the above argument applies and shows that N is indeed compact. This completes the proof of the lemma. □

Remark. Caprace and Monod (Theorem 1.6 of [12], see also [11]) found geometric conditions which guarantee that an amenable normal subgroup of a non-elementary group G of isometries of X is trivial. This however need not be true under the above more general assumptions. A simple example is a space of the form $X = \mathbf{H}^2 \times X_2$ where $\mathbf{H}^2$ is the hyperbolic plane and where X_2 is a compact CAT(0)-space whose isometry group H is non-trivial. Then the compact group H is a normal subgroup of the isometry group of X. On the other hand, any axial isometry of $\mathbf{H}^2$ acts as a rank-one isometry on X.

As in [15] we use Lemma 5.1 and Lemma 5.2 to show

Proposition 5.3. *Let $G < \mathrm{Iso}(X)$ be a non-elementary closed subgroup with limit set Λ. If G contains a rank-one element and acts transitively on the complement of the diagonal in $\Lambda \times \Lambda$, then $H^2_{\mathrm{cb}}(G, L^p(G, \mu)) \neq 0$ for every $p \in (1, \infty)$.*

Proof. A *strong boundary* for a locally compact topological group G is a standard Borel space (B, ν) with a probability measure ν and a measure class preserving amenable action of G which is *doubly ergodic* (we refer to [19] for a detailed explanation of the significance of a strong boundary). A strong boundary exists for every locally compact topological group G [17].

Let $G < \mathrm{Iso}(X)$ be a non-elementary closed subgroup with limit set Λ. Assume that G acts transitively on the complement of the diagonal in $\Lambda \times \Lambda$. Since the action of G on its strong boundary (B, ν) is amenable, there is a G-equivariant measurable Furstenberg map $\varphi\colon (B, \nu) \to \mathcal{P}(\Lambda)$ [22]. By ergodicity of the action of G on (B, ν), either the set of all $x \in B$ with $\varphi(x) \in \mathcal{P}_{\geq 3}(\Lambda)$ has full mass or vanishing mass.

Assume that this set has full mass. By Lemma 5.1, the action of G on $\mathcal{P}_{\geq 3}(\Lambda)$ is tame, with compact point stabilizers. Thus φ induces a G-invariant measurable map $(B, \nu) \to \mathcal{P}_{\geq 3}(\Lambda)/G$ which is almost everywhere constant by ergodicity. Therefore by changing the map φ on a set of measure zero, we can assume that φ is a G-equivariant map $(B, \nu) \to G/G_\mu$ where G_μ is the stabilizer of a point in $\mathcal{P}_{\geq 3}(\Lambda)$ and hence it is compact.

Following the reasoning in Section 3 of [20], since the action of G on $(B, \nu) \times (B, \nu)$ is ergodic as well, the cocycle defined by the identity homomorphism $G \to G$ is equivalent to a cocycle ranging in a compact subgroup of G (see the proof of Lemma 3.4 of [20]). By Lemma 3.2 and Lemma 3.1 of [20], this implies that G is elementary which is a contradiction (the proofs of these lemmas are valid in the situation at hand without any change).

As a consequence, the image under φ of ν-almost every $x \in B$ is a measure supported on at most two points. By Lemma 5.1 and its proof, the action of G on the space of triples of pairwise distinct points in Λ is proper and hence the assumptions in Lemma 23 of [18] are satisfied. We can then use Lemma 23 of [18] as in the proof of Lemma 3.5 of [20] to conclude that the image under φ of almost every $x \in B$ is supported in a single point. In other words, φ is a G-equivariant Borel map of (B, ν) into Λ. Note that since the action of G on Λ is minimal, by equivariance the support of the measure class $\varphi_*(\nu)$ is all of Λ.

Let μ be a Haar measure on G. By invariance under the action of G, there is some $B > 0$ such that $(a, b) \in \mathcal{A}(B)$ for all $(a, b) \in \Lambda \times \Lambda - \Delta$. Thus by Theorem 4.3, for every $p \in (1, \infty)$ there is a nontrivial bounded continuous $L^p(G \times G, \mu \times \mu)$-valued cocycle ω on the space of triples of pairwise distinct points in Λ. Then the $L^p(G \times G, \mu \times \mu)$-valued $\nu \times \nu \times \nu$-measurable bounded cocycle $\omega \circ \varphi^3$ on $B \times B \times B$ is non-trivial on a set of positive measure. Since B is a strong boundary for G, this cocycle then defines a *non-trivial* class in $H^2_{cb}(G, L^p(G \times G, \mu \times \mu))$ (see [19]). On the other hand, the isometric G-representation space $L^p(G \times G, \mu \times \mu)$ is a direct integral of copies of the isometric G-representation space $L^p(G, \mu)$ and therefore by Corollary 2.7 of [20] and Corollary 3.4 of [21], if $H^2_{cb}(G, L^p(G, \mu)) = \{0\}$ then also $H^2_{cb}(G, L^p(G \times G, \mu \times \mu)) = \{0\}$. This shows the proposition. □

6. Second continuous bounded cohomology II

In this section we investigate non-elementary closed subgroups of $\mathrm{Iso}(X)$ with limit set Λ which contain a rank-one element and which do not act transitively on the complement of the diagonal in $\Lambda \times \Lambda$. Such a group G is a locally compact σ-compact group which admits a Haar measure μ. Our goal is to show that for every $p \in (1, \infty)$ the second continuous bounded cohomology group $H^2_{cb}(G, L^p(G, \mu))$ is infinite-dimensional.

Unlike in Section 5, for this we can not use Theorem 4.3 directly since the cocycle constructed in this theorem may not be defined on the entire space of triples of pairwise distinct points in Λ. Instead we use the strategy from the proof of Theorem 4.3 to construct explicitly for every $p \in (1, \infty)$ bounded cocycles for G with values in $L^p(G, \mu)$ which define an infinite-dimensional subspace of $H^2_{cb}(G, L^p(G, \mu))$.

For the construction of these classes we use a relative version of the construction in Section 3. The following lemma is analogous to Corollary 3.6.

Lemma 6.1. *Let $G < \mathrm{Iso}(X)$ be a closed non-elementary group with limit set Λ. Assume that G does not act transitively on the complement of the diagonal in $\Lambda \times \Lambda$ and that it contains a rank-one element with pair of fixed points $(a, b) \in \Lambda \times \Lambda - \Delta$. Let $\mathcal{A}(b)$ be the union of the set of ordered pairs of distinct points in Gb with the G-translates of (a, b), (b, a). Then for some $\chi > 0$ there is a family of distance functions δ^{rel}_x $(x \in X)$ on $\mathcal{A}(b)$ with the following properties.*

(1) *The distances δ^{rel}_x are invariant under the involution ι of $\mathcal{A}(b)$ which exchanges the two points $\xi \neq \eta \in \partial X$ in a pair $(\xi, \eta) \in \mathcal{A}(b)$.*

(2) $e^{-\chi d(x,y)}\delta_x^{\rm rel} \leq \delta_y^{\rm rel} \leq e^{\chi d(x,y)}\delta_x^{\rm rel}$ *for all* $x, y \in X$.

(3) *The family* $\{\delta_x^{\rm rel}\}$ *is invariant under the action of* G *on* $\mathcal{A}(b) \times X$.

(4) *The function* $\mathcal{A}(b) \times \mathcal{A}(b) \times X \to [0,\infty)$ *defined by* $(\zeta, \eta, x) \to \delta_x^{\rm rel}(\zeta, \eta)$ *is Borel for the restriction of the product topology on* $(\partial X)^4 \times X$.

(5) *The point* $(b, a) \in \mathcal{A}(b)$ *is not isolated for the distances* $\delta_x^{\rm rel}$.

Proof. Let $G < \mathrm{Iso}(X)$ be a closed non-elementary group with limit set $\Lambda \subset \partial X$. Assume that G does not act transitively on the complement of the diagonal in $\Lambda \times \Lambda$ and that it contains a rank-one element g with fixed points $a \neq b \in \Lambda$. Let $B_0 > 0$ be such that every geodesic in X connecting b to a is B_0-contracting. Such a number exists by Lemma 2.4. By Lemma 2.10, we can find some $h \in G$ such that $hb \neq b$ and that the stabilizer $\mathrm{Stab}(b, hb)$ in G of the ordered pair of points (b, hb) is compact.

By the consideration in the proof of Lemma 2.10, there is a number $B > B_0$ depending on B_0 and h such that every geodesic connecting b to hb is B-contracting. Thus by invariance under isometries, for every $k \in \mathbb{Z}$, every geodesic connecting b to $g^k h^{-1} b$ is B-contracting. Moreover, the set of all points in X which are contained in a geodesic connecting b to hb is isometric to $K \times \mathbb{R}$ where K is a compact CAT(0)-space. The group $\mathrm{Stab}(b, hb)$ acts on $K \times \mathbb{R}$ as a group of isometries preserving the orientation of the lines $\{x\} \times \mathbb{R}$.

Every isometry φ of $K \times \mathbb{R}$ can be represented as a product $\varphi = (\varphi_1, \varphi_2)$ where φ_1 is an isometry of K and φ_2 is an isometry of $\mathbb{R}$. A group of isometries acting on a compact CAT(0)-space has a fixed point. Since $\mathrm{Stab}(b, hb)$ is compact, this implies that there is a geodesic γ connecting b to hb which is fixed pointwise by $\mathrm{Stab}(b, hb)$. Let γ^{-1} be the geodesic obtained by reversing the orientation of γ (note that for all of our constructions, only the orientation of a geodesic but not an explicit parametrization plays any role). For $v \in G$, the geodesic $v\gamma^{-1}$ connects vhb to vb, and it is B-contracting. Moreover, it depends continuously on v with respect to the topology of uniform convergence on compact sets.

We use the translates of γ under G (which are all B-contracting) to construct uniformly contracting rays with endpoints in Gb. Similar to the approach in Section 3, these geodesics are used to define the distance functions $\delta_x^{\rm rel}$ on $\mathcal{A}(b)$. There are additional technical difficulties we have to overcome, and the remainder of this proof is devoted to address these technical points.

For this let $x_0 \in \gamma$ be a fixed point and let $\mathcal{G}_0$ be the space of all geodesic lines $\xi\colon \mathbb{R} \to X$ which are parametrized in such a way that $\xi(0) = \pi_\xi(x_0)$. The space $\mathcal{G}_0$ is equipped with the topology of uniform convergence on compact sets. The group G acts on $\mathcal{G}_0$ as a group of transformations. For the number $B > 0$ as above let $C = C(B) > 0$ be as in Lemma 2.4. The G-orbit of b consists of visibility points. Thus for $v \in G$ and $z \in \partial X - vb$ there is an oriented geodesic $\xi \in \mathcal{G}_0$ connecting z to vb. By Lemma 2.4, the geodesic ξ passes through the $9B + 6$-neighborhood of every point in $\pi_{v\gamma^{-1}(\mathbb{R})}(z)$.

Let $\beta(v,\xi) \in \mathbb{R} \cup \{-\infty\}$ be the infimum of all numbers $t \in \mathbb{R}$ such that $\xi(t)$ is contained in the closed $9B + 6$-neighborhood of a point in $v\gamma^{-1}(\mathbb{R})$. Note that we retain the information on $v \in G$ since the stabilizer of b in G is unbounded and the geodesic $v\gamma^{-1}$ is not determined by vb. By convexity, the geodesic ray $\xi(\beta(v,\xi),\infty)$ is contained in the closed $9B+6$-neighborhood of $v\gamma^{-1}(\mathbb{R})$, and it is C-contracting. Since γ is fixed pointwise by $\mathrm{Stab}(b, hb)$, the ray $\xi(\beta(v,\xi),\infty)$ only depends on the geodesic line ξ and on the coset $[v]$ of v in $G/\mathrm{Stab}(b,hb)$. If $v = g^k h^{-1}$ and if ξ is a geodesic connecting b to vb then we have $b(v,\xi) = -\infty$.

Let $\mathcal{B}$ be the space of all pairs (v,ξ) where $v \in G$ and where $\xi \in \mathcal{G}_0$ satisfies $\xi(\infty) = vb$. Equip $\mathcal{B}$ with the topology induced from the product topology of $G\times\mathcal{G}_0$ (where G carries the compact open topology). The function

$$\beta\colon \mathcal{B} \to \mathbb{R} \cup \{-\infty\}, \quad (v,\xi) \mapsto \beta(v,\xi),$$

is lower semi-continuous and hence Borel. Namely, if $v_i \to v$ in G then the geodesics $v_i\gamma^{-1}$ converge to $v\gamma^{-1}$ pointwise. Now if for each i there is a geodesic $\xi_i \in \mathcal{G}_0$ with $\xi_i(\infty) = v_i b$ and if $\xi_i \to \xi$ in $\mathcal{G}_0$ (i.e., locally uniformly as parametrized geodesics) then $\xi(0) = \pi_\xi(x_0)$. Moreover, up to passing to a subsequence, the points $\xi_i(\beta(v_i,\xi_i))$ converge as $i \to \infty$ to a point $y \in \xi$ whose distance to $v\gamma^{-1}$ does not exceed $9B+6$. But this just means that this limit is contained in $\xi[\beta(v,\xi),\infty)$ and hence $\beta(v,\xi) \le \liminf_{i\to\infty} \beta(v_i,\xi_i)$. This shows lower semi-continuity of the function β as claimed.

If both endpoints of $\xi \in \mathcal{G}_0$ are contained in Gb, i.e., if the endpoints of ξ are ub, vb for some $u, v \in G$, then we can define similarly a number $\alpha(u,\xi) \in \mathbb{R} \cup \{\infty\}$ using the above procedure for the inverse of the geodesic ξ and the geodesic $u\gamma^{-1}$. The resulting geodesic ray $\xi(-\infty, a(u,\xi))$ only depends on the geodesic ξ and on $[u] \in G/\,\mathrm{Stab}(b,hb)$. The function α is upper semi-continuous.

Let $\mathcal{A}(b)$ be the union of the ordered pairs of distinct points in Gb with the G-translates of $(a,b),(b,a)$. Denote by $\mathcal{G}_0(b)$ the subspace of $\mathcal{G}_0$ of all oriented geodesics whose ordered pair of endpoints is contained in $\mathcal{A}(b)$. The group G acts on $\mathcal{G}_0$ as a group of transformations.

For a geodesic line $\xi \in \mathcal{G}_0$ with an ordered pair of endpoints (ub, vb) $(u, v \in G)$ define

$$\beta(\xi) = \inf\{\beta(\tilde v,\xi) \mid \tilde v \in G, \tilde v(b) = v(b)\}$$

and

$$\alpha(\xi) = \sup\{\alpha(\tilde u,\xi) \mid \tilde u \in G, \tilde u(b) = u(b)\}.$$

If the endpoints of ξ are contained in a G-translate of (a,b) then define $a(\xi) = \infty, b(\xi) = -\infty$. The rays $\xi(-\infty, a(\xi)), \xi(b(\xi),\infty)$ are C-contracting. Moreover, since for every $v \in G$ the set $\{\tilde v \in G \mid \tilde v(b) = v(b)\} \subset G$ is closed, the above discussion shows that their dependence on ξ is Borel. Note that we may have $\alpha(\xi) = \infty$ or $\beta(\xi) = -\infty$.

Let $\mathcal{G}(C)$ be the set of all triples consisting of an (oriented) geodesic $\eta\colon \mathbb{R} \to X$ and two closed C-contracting subrays $\eta(-\infty, \alpha(\eta)], \eta[\beta(\eta), \infty)$ (which are not necessarily proper). For $\eta \in \mathcal{G}(C)$, the subrays $\eta(-\infty, \alpha(\eta)], \eta[\beta(\eta), \infty)$ are part of the structure of η. Thus the same geodesic with distinct distinguished subrays defines two distinct points in $\mathcal{G}(C)$. The group G naturally acts on $\mathcal{G}(C)$ from the left.

The above construction associates to any ordered pair of points $(\sigma, \eta) \in \mathcal{A}(b)$ and every geodesic ξ connecting σ to η two subrays $\xi(-\infty, \alpha(\xi)], \xi[\beta(\xi), \infty)$ of ξ in such a way that $(\xi, \xi(-\infty, \alpha(\xi)], \xi[\beta(\xi), \infty)) \in \mathcal{G}(C)$. The assignment

$$\Pi\colon \mathcal{G}_0(b) \to \mathcal{G}(C), \quad \xi \mapsto \Pi(\xi) = (\xi, \xi(-\infty, \alpha(\xi)], \xi[\beta(\xi), \infty)),$$

satisfies the following properties.

(1) Invariance under change of orientation: If $\hat{\xi}$ is the geodesic obtained from ξ by reversal of orientation then $\hat{\xi}[\beta(\hat{\xi}), \infty) = \xi(-\infty, \alpha(\xi)]$ and $\hat{\xi}(-\infty, \alpha(\hat{\xi})] = \xi[\beta(\xi), \infty)$ (as subsets of X).

(2) Invariance under the action of G: For $u \in G$ we have $\Pi(u\xi) = u\Pi(\xi)$.

(3) Borel dependence on ξ: The assignments $\xi(\mathbb{R}) \mapsto \xi(-\infty, \alpha(\xi)]$ and $\xi(\mathbb{R}) \mapsto \xi[\beta(\xi), \infty)$ are Borel for the topology on $\mathcal{G}_0$ and the Hausdorff topology for compact subsets of $X \cup \partial X$.

Recall from Section 3 the definition of the function τ_C which associates to a point $x \in X$ and two (finite or infinite) geodesic arcs ζ_1, ζ_2 with at most one common endpoint in ∂X a number $\tau_C(x, \zeta_1, \zeta_2) \geq 0$. For $x \in X$ and for two geodesics $\xi_1, \xi_2 \in \mathcal{G}(C)$ with at most one common endpoint define

$$\begin{aligned}\tau_{C\,\mathrm{rel}}(x, \xi_1, \xi_2) = \max\{&\tau_C(x, \xi_1[\beta(\xi_1), \infty), \xi_2[\beta(\xi_2), \infty)),\\ &\tau_C(x, \xi_1(-\infty, \alpha(\xi_1)], \xi_2(-\infty, \alpha(\xi_2)])\}.\end{aligned}$$

Notice that if $\alpha(\xi_i) = \infty$ or $\beta(\xi_i) = -\infty$ $(i = 1, 2)$, then $\tau_{C\,\mathrm{rel}}(x, \xi_1, \xi_2) = \tau_C(x, \xi_1, \xi_2)$. Note also that $\tau_{C\,\mathrm{rel}}(x, \xi_1, \xi_2)$ depends on an orientation of ξ_1, ξ_2 but not on a specific parametrization.

For $(\sigma_1, \eta_1), (\sigma_2, \eta_2) \in \mathcal{A}(b)$ and $x \in X$ define

$$\tau_{C\,\mathrm{rel}}(x, (\sigma_1, \eta_1), (\sigma_2, \eta_2)) = \inf \tau_{C\,\mathrm{rel}}(x, \gamma_1, \gamma_2),$$

where the infimum is taken over all elements $\Pi(\gamma_1), \Pi(\gamma_2) \in \mathcal{G}(C)$ defined by all geodesics γ_1, γ_2 connecting σ_1 to η_1 and σ_2 to η_2. By construction, we have

$$\tau_{C\,\mathrm{rel}}(x, (\sigma_1, \eta_1), (\sigma_2, \eta_2)) = \tau_{C\,\mathrm{rel}}(x, (\sigma_2, \eta_2), (\sigma_1, \eta_1))$$

for all $(\sigma_1, \eta_1), (\sigma_2, \eta_2) \in \mathcal{A}(b)$.

Lemma 3.5 shows that the function $\tau_{C\,\mathrm{rel}}$ on $\mathcal{A}(b)$ satisfies the ultrametric inequality. Thus as in Section 3, for each $x \in X$ we can use the function $\tau_{C\,\mathrm{rel}}(x, \cdot, \cdot)$ to define a distance δ_x^{rel} on $\mathcal{A}(b)$. The family $\{\delta_x^{\mathrm{rel}}\}$ is invariant under the natural action of G on $\mathcal{A}(b) \times X$, and it is invariant under the natural involution ι defined by $\iota(\sigma, \eta) = (\eta, \sigma)$. Thus these distances have properties 1)-4) stated in the lemma.

Since for each geodesic ξ in X connecting a to b or connecting b to $g^k h^{-1} b$ we have $a(\xi) = \infty$ and $b(\xi) = -\infty$, for each $x \in X$ the points $(b, g^k h^{-1} b) \in \mathcal{A}(b)$ converge as $k \to \infty$ in $(\mathcal{A}(b), \delta_x^{C\,\mathrm{rel}})$ to (b, a). In particular, the point $(b, a) \in \mathcal{A}(b)$ is not isolated for δ_x^{rel} and hence property 5) holds as well. □

A *twisted* $L^p(G, \mu)$*-valued quasi-morphism* for a closed subgroup G of $\mathrm{Iso}(X)$ is a map $\psi : G \to L^p(G, \mu)$ such that

$$\sup_{g,h} \|\psi(g) + g\psi(h) - \psi(gh)\|_p < \infty$$

where $\| \, \|_p$ is the L^p-norm for functions on G.

Every unbounded twisted $L^p(G, \mu)$-valued quasi-morphism for G defines a second bounded cohomology class in $H_b^2(G, L^p(G, \mu))$ which vanishes if and only if there is a cocycle $\rho : G \to L^p(G, \mu)$ (i.e., ρ satisfies the cocycle equation $\rho(g) + g\rho(h) - \rho(gh) = 0$) such that $\psi - \rho$ is bounded (compare the discussion in [14]). We use twisted quasimorphisms to complete the proof of Theorem 2 from the introduction.

Proposition 6.2. *Let $G < \mathrm{Iso}(X)$ be a closed non-elementary subgroup with limit set Λ which contains a rank-one element. If G does not act transitively on the complement of the diagonal in $\Lambda \times \Lambda$, then for every $p \in (1, \infty)$ the second continuous bounded cohomology group $H_{cb}^2(G, L^p(G, \mu))$ is infinite-dimensional.*

Proof. Let $G < \mathrm{Iso}(X)$ be a closed subgroup with limit set $\Lambda \subset \partial X$ which contains a rank-one element and which does not act transitively on the complement of the diagonal Δ in $\Lambda \times \Lambda$.

By Lemma 2.11 there are infinitely many rank-one elements $g \in G$ with attracting and repelling fixed points $a, b \in \Lambda$ and the additional property that there is no $u \in G$ with $u(a, b) = (b, a)$. Thus let $g \in G$ be such a rank-one element with fixed points $a \neq b \in \Lambda$. Let $\mathcal{A}(b)$ be the union of the set of all ordered pairs of distinct points in Gb with the G-translates of (a, b), (b, a). For $x \in X$ denote by δ_x^{rel} the distance function on $\mathcal{A}(b)$ constructed in Lemma 6.1. In the sequel we always equip $\mathcal{A}(b)$ with the topology induced by one (and hence each) of these distance functions. As in Lemma 3.7, we use the distances δ_x^{rel} to construct a G-invariant distance ρ on $\mathcal{A}(b) \times X$ with the properties stated in Lemma 3.7. Then ρ induces the product topology on $\mathcal{A}(b) \times X$.

We now use the strategy from the proof of Theorem 4.3. Namely, let $x_0 \in X$ be a point on an axis for the rank-one element $g \in G$. Let G_{x_0} be the stabilizer of x_0 in G and let

$$V(b) = \mathcal{A}(b) \times G/G_{x_0} = \mathcal{A}(b) \times G x_0.$$

The group G acts on $V(b)$ as a group of isometries with respect to the restriction of the distance ρ. Define $W = G \backslash V(b)$ and let

$$P : V(b) \to W$$

be the canonical projection. The distance ρ on $V(b)$ induces a distance $\hat{\rho}$ on W by defining $\hat{\rho}(x,y) = \inf\{\rho(\tilde{x},\tilde{y}) \mid P\tilde{x} = x, P\tilde{y} = y\}$. Note that we have $\hat{\rho}(x,y) > 0$ for $x \neq y$ by the definition of the distance ρ and the fact that the distances $\{\delta_x^{\mathrm{rel}}\}$ depend uniformly Lipschitz continuously on $x \in X$.

The isometric involution ι of $(\mathcal{A}(b)\times X, \rho)$ which exchanges the two components of the point in $\mathcal{A}(b)$ descends to an isometric involution on W again denoted by ι. Since there is no $u \in G$ with $u(a,b) = (b,a)$, we can find an open neighborhood D of

$$w = P((b,a), x_0) \in W$$

which is disjoint from its image under ι. We choose D to be contained in the image under the projection P of the set $\mathcal{A}(b) \times K$ where K is the closed ball of radius 1 about x_0 in $Gx_0 \subset X$.

Let $C_{\mathrm{b}}(Gx_0)$ be the vector space of continuous bounded functions on $Gx_0 \subset X$, equipped with the topology of uniform convergence on compact sets. Call a set $A \subset C_{\mathrm{b}}(Gx_0)$ *bounded* if the norm of every element in A is bounded from above by a fixed constant. As in the proof of Theorem 4.3, we use the induced distance on W to construct from a Hölder continuous function f supported in D with $f(w) > 0$ a G-invariant ι-anti-invariant bounded continuous map $\tilde{\sigma}\colon \mathcal{A}(b) \to C_{\mathrm{b}}(Gx_0)$ (i.e., a map with bounded range) which lifts to a bounded continuous map $\sigma\colon \mathcal{A}(b) \to C_{\mathrm{b}}(G)$ with the equivariance properties as stated in this theorem. Since the function $(\xi,\eta,x) \in \mathcal{A}(b)\times\mathcal{A}(b)\times X \to \delta_x^{\mathrm{rel}}(\xi,\eta)$ is Borel for the restriction of the product topology on $\Lambda^4 \times X$, the map $\{(u,v) \in G\times G \mid ub \neq vb\} \to \sigma(ub,vb) \in C_{\mathrm{b}}(G)$ is Borel.

Now we use the constructions and notations in the proof of Lemma 6.1. Namely, by invariance and the definition of the distances δ_x^{rel}, if $(\zeta,\eta) \in \mathcal{A}(b)$ and if $z \in G$ is such that zx_0 is contained in the support of the function $\tilde{\sigma}(\zeta,\eta)$, then there is a geodesic $\xi \in \mathcal{G}(C)$ connecting ζ to η so that zx_0 is contained in a tubular neighborhood of $\xi(-\infty, a(\xi)) \cup \xi(b(\xi),\infty)$ of uniformly bounded radius.

Let $h \in G$ be as in the construction of the distances δ_x^{rel}, i.e., such that $hb \neq b$ and that the stabilizer of (b, hb) in G is compact. Let A be a small compact neighborhood of b in $X \cup \partial X$ which does not contain the attracting fixed point a of g and is disjoint from $h^{-1}A$. In particular, we have $b \notin h^{-1}A$. For $u \in G$ with $ub \neq b$ define a function $\Psi_\sigma(u)\colon G \to \mathbb{R}$ by

$$\Psi_\sigma(u)(w) = \sigma(b, ub)(w)$$

if $wx_0 \in X - (A \cup uA)$ and let $\Psi_\sigma(u)(w) = 0$ otherwise. If $ub = b$ then define $\Psi_\sigma(u) \equiv 0$. By the construction of the function σ, the function $(u,w) \in G\times G \to \Psi_\sigma(u)(w)$ is Borel and pointwise uniformly bounded.

For fixed $u \in G$, the support of $\Psi_\sigma(u)$ is compact. More precisely, for every compact subset K_0 of G there is a compact subset C of G containing the support of each of the functions $\Psi_\sigma(u)$ $(u \in K_0)$. In particular, we have $\Psi_\sigma(u) \in L^p(G,\mu)$

for every $p > 1$, and for every compact subset K_0 of G the set $\{\Psi_\sigma(u) \mid u \in K_0\} \subset L^p(G, \mu)$ is bounded.

We claim that Ψ_σ is unbounded. For this recall from Lemma 6.1 and its proof that as $k \to \infty$ we have $(b, g^k h^{-1} b) \to (b, a)$ in $\mathcal{A}(b)$. Since $h^{-1}A$ is compact and does not contain b, Lemma 2.8 shows that $g^k h^{-1} A \to \{a\}$ in ∂X. In particular, if ξ is the axis of the rank-one element g containing the point x_0 then $X - A - g^k h^{-1} A$ contains longer and longer subsegments of ξ which uniformly fellow-travel the geodesic $g^k \gamma$ connecting b to $g^k h^{-1} b$. Now the function $\sigma(b, a)$ is invariant under the action of the rank-one element g and its support contains the point x_0. This implies that $\sigma(b, a)$ is *not* integrable. But then for $p > 1$ the L^p-norm of the functions $\Psi_\sigma(g^k h^{-1})$ tends to infinity as $k \to \infty$.

Define a Borel function $\omega \colon G^3 \to L^p(G, \mu)$ by

$$\omega(u, uw, uv) = \omega(e, w, v) = \Psi_\sigma(w) + w\Psi_\sigma(v) - \Psi_\sigma(wv)$$

if ub, uwb, uvb are pairwise distinct, and let $\omega(u, uw, uv) = 0$ otherwise. Then ω is invariant under the diagonal action of G, and we have $\omega \circ \sigma = \mathrm{sgn}(\sigma)\omega$ for every permutation of the three variables. Moreover, ω satisfies the cocycle identity

$$\omega(v, w, z) - \omega(u, w, z) + \omega(u, v, z) - \omega(u, v, w) = 0.$$

This is immediate if the points ub, vb, wb, zb are pairwise distinct. If two of these points coincide, say if $ub = vb$, then $\omega(v, w, z) = \omega(u, w, z)$ and $\omega(u, v, z) = 0 = \omega(u, v, w)$ and hence in this case the cocycle equality holds as well. In other words, for every $p \in (1, \infty)$, ω is a Borel two-cocycle for G with values in $L^p(G, \mu)$.

We claim that the image of ω is uniformly bounded. For this we argue as in the proof of Theorem 4.3. Namely, by assumption, if $\Psi_\sigma(v)(w) \neq 0$ then the point wx_0 is contained in a uniformly bounded neighborhood of a geodesic $\tilde{v} h^{-1} \gamma$ for some $\tilde{v} \in G$ with $\tilde{v} b = vb$. Let ξ be a geodesic connecting b to vb. Let N be a bounded neighborhood of the ray $\xi(b(\xi), \infty)$. By the choice of the rays $\xi(-\infty, a(\xi)), \xi(b(\xi), \infty)$ and by the estimate in Lemma 3.4, for every $z \in G$ with $zb \neq vb$ the L^p-norm of the restriction of $\alpha(b, vb) + \alpha(vb, zb)$ to the set $\{y \in G \mid yx_0 \in N\}$ is uniformly bounded. By symmetry and the properties of the support of the functions $\alpha(b, vb)$, this implies as in the proof of Theorem 4.3 that ω is a bounded cocycle.

As a consequence, ω is a bounded $L^p(G, \mu)$-valued Borel two-cocycle for G. By construction, this cocycle is moreover nontrivial on a set of positive Haar measure on $G \times G \times G$. Since $L^p(G, \mu)$ is a coefficient G-module in the sense of [19] (recall that $p \in (1, \infty)$ by assumption), Proposition 7.5.1 of [19] shows that ω defines an element in $H^2_{\mathrm{cb}}(G, L^p(G, \mu))$.

We are now left with showing that the cocycles constructed in this way define an infinite-dimensional subspace of $H^2_{\mathrm{cb}}(G, L^p(G, \mu))$.

Since G does not act transitively on its limit set, Lemma 2.11 shows that G contains a free subgroup Γ with two generators consisting of rank-one elements which contains elements from infinitely many conjugacy classes of G. Using the above notations, we may assume that Γ is generated by g, h. If ω is any $L^p(G,\mu)$-valued bounded cocycle which defines a trivial cohomology class for Γ then there is a bounded function $\rho\colon \Gamma \to L^p(G,\mu)$ such that

$$\omega(e, v, w) = \rho(v) + v\rho(w) - \rho(vw).$$

By construction of the cocycle ω, in this case there is an unbounded function $\Psi_\sigma : \Gamma \to L^p(G,\mu)$ such that

$$\rho(v) + v\rho(w) - \rho(vw) = \Psi_\sigma(v) + v\Psi_\sigma(w) - \Psi_\sigma(vw)$$

whenever b, vb, wb are pairwise distinct. In other words, $\rho - \Psi_\sigma$ is the restriction to the set of all $v \in \Gamma$ with $vb \neq b$ of a $L^p(G,\mu)$-valued one-cocycle, i.e., a function $\beta\colon \Gamma \to L^p(G,\mu)$ which satisfies

$$\beta(v) + v\beta(w) - \beta(vw) \equiv 0.$$

Since Ψ_σ is unbounded and ρ is bounded, the map $\Psi_\sigma - \rho$ is unbounded and therefore the one-cocycle determined in this way is non-trivial.

Now a one-cocycle is determined by its values on a generating set. On the other hand, since the G-orbit of any pair of fixed points of rank-one elements in G is a closed subset of $\Lambda \times \Lambda - \Delta$, if $(a_i, b_i) \in \Lambda \times \Lambda$ are fixed points of elements $g_i \in \Gamma$ ($i = 1, \dots, k$) in distinct conjugacy classes in G then for each i we can choose the function f in the above construction in such a way that $\Psi_\sigma(g_j^k) = 0$ for $j \neq i$ and all $k \in \mathbb{Z}$ and such that $\Psi_\sigma(g_i^k)$ is unbounded as $k \to \pm\infty$. This implies that there are indeed infinitely many linearly independent distinct such classes which pairwise can not be obtained from each other by adding a bounded function. This shows the proposition. ☐

7. Structure of the isometry group

In this section we use the results from Section 5 and Section 6 to complete the proof of Theorem 1 from the introduction.

Proposition 7.1. *Let X be a proper* CAT(0)*-space and let $G < \mathrm{Iso}(X)$ be a closed subgroup which contains a rank-one element. Then one of the following three possibilities holds.*

(1) *G is elementary.*
(2) *G contains an open subgroup G' of finite index which is a compact extension of a simple Lie group of rank one.*
(3) *G is a compact extension of a totally disconnected group.*

Proof. Let G be a closed subgroup of the isometry group $\mathrm{Iso}(X)$ of a proper CAT(0)-space X. Then G is locally compact. Assume that G is non-elementary and contains a rank-one element. Then by Lemma 5.2, the maximal normal amenable subgroup N of G is compact, and the quotient $L = G/N$ is a locally compact σ-compact group. Moreover, N acts trivially on the limit set Λ of G.

By the solution to Hilbert's fifth problem (see Theorem 11.3.4 in [19]), after possibly replacing L by an open subgroup of finite index (which we denote again by L for simplicity), the group L splits as a direct product $L = H \times Q$ where H is a connected semisimple Lie group with finite center and without compact factors and Q is totally disconnected.

We show next that one of the groups H, Q is trivial. For this assume that H is nontrivial. Let $H_0 < G$ and $Q_0 < G$ be the preimage of H, Q under the projection $G \to L$. Then H_0 is not compact and the limit set $\Lambda_0 \subset \Lambda$ of $H_0 < G$ is nontrivial. Since Q commutes with H and the group N acts trivially on Λ, the group Q_0 acts trivially on Λ_0 (this is discussed in the proof of Proposition 4.3 of [15], and the proof given there is valid in our situation as well). In particular, if Λ_0 consists of a single point then G is elementary. Since G is non-elementary by assumption, Λ_0 contains at least two points.

We claim that $\Lambda_0 = \Lambda$. Since $\Lambda_0 \subset \Lambda$ is closed, by Lemma 2.8 it suffices to show that the fixed points of every rank-one element of G are contained in Λ_0. Thus let $g \in G$ be a rank-one element. By Lemma 2.8, g acts on ∂X with north-south dynamics, with attracting fixed point $a \in \Lambda$ and repelling fixed point $b \in \Lambda$. Since Λ_0 contains at least two points, if $a \notin \Lambda_0$ then there is a point $\xi \in \Lambda_0 - \{a, b\}$. Write $g = g_0 q$ with $g_0 \in H_0$, $q \in Q_0$. Since g_0 and q commute up to a compact normal subgroup which fixes $\Lambda \supset \Lambda_0$ pointwise, we have $g_0^k \xi = g_0^k q^k \xi = g^k \xi \to a$ $(k \to \infty)$. But $g_0^k \xi \in \Lambda_0$ for all $k > 0$ and therefore by compactness we have $a \in \Lambda_0$ (which contradicts the assumption that $a \notin \Lambda_0$). Now a was an arbitrary fixed point of a rank-one element in G and therefore $\Lambda_0 = \Lambda$ and Q_0 fixes the limit set of G pointwise. However, since G is non-elementary by assumption, in this case the argument in the proof of Lemma 5.2 shows that Q_0 is compact and hence Q is trivial.

To summarize, if G is non-elementary then up to passing to an open subgroup of finite index, either G is a compact extension of a totally disconnected group or G is a compact extension of a connected semisimple Lie group H with finite center and without compact factors.

We are left with showing that if the group G is a compact extension of a connected semisimple Lie group H with finite center and without compact factors then H is simple and of rank one. Now Propositions 5.3 and 6.2 show that $H^2_{\mathrm{cb}}(G, L^2(G, \mu)) \neq \{0\}$. By Corollary 8.5.2 of [19], this implies that $H^2_{\mathrm{cb}}(H, L^2(H, \mu)) \neq \{0\}$ as well. By the super-rigidity result for bounded cohomology of Burger and Monod [8], we conclude that H is simple of rank one (see [20], [15] for details of this argument). The proposition is proven. ∎

Now we are ready for the proof of the corollary from the introduction (which also follows from Corollary 1.24 of [12]). For this recall that a simply connected complete Riemannian manifold $\widetilde{M}$ of non-positive sectional curvature is called *irreducible* if $\widetilde{M}$ does not split as a non-trivial product. A *parabolic isometry* of $\widetilde{M}$ is an isometry which is not semisimple. We have

Corollary 7.2. *Let M be a closed Riemannian manifold of non-positive sectional curvature. If the universal covering $\widetilde{M}$ of M is irreducible and if* $\mathrm{Iso}(\widetilde{M})$ *contains a parabolic element, then M is locally symmetric.*

Proof. Let M be a closed Riemannian manifold of non-positive sectional curvature with irreducible universal covering $\widetilde{M}$. The fundamental group $\pi_1(M)$ of M acts cocompactly on the Hadamard space $\widetilde{M}$ as a group of isometries. By the celebrated rank-rigidity theorem (we refer to [3] for a discussion and for references), either $\pi_1(M)$ contains a rank-one element or M is locally symmetric of higher rank. These two possibilities are exclusive.

Assume that $\pi_1(M)$ contains a rank-one element. By Lemma 5.2, the amenable radical N of $\mathrm{Iso}(\widetilde{M})$ is compact and hence it fixes a point $x \in \widetilde{M}$ by convexity. Moreover, it fixes the limit set of $\pi_1(M)$ pointwise. Since the action of $\pi_1(M) < \mathrm{Iso}(\widetilde{M})$ on $\widetilde{M}$ is cocompact, the limit set Λ of $\pi_1(M)$ is the entire ideal boundary $\partial\widetilde{M}$ of $\widetilde{M}$. Then N fixes every geodesic ray issuing from x. This implies that N is trivial.

By Theorem 1, either the isometry group of $\widetilde{M}$ is an almost connected simple Lie group G of rank one or $\mathrm{Iso}(\widetilde{M})$ is totally disconnected. In the first case, $\pi_1(M)$ is necessarily a cocompact lattice in $G = \mathrm{Iso}(\widetilde{M})$ since the action of $\mathrm{Iso}(\widetilde{M})$ on $\widetilde{M}$ is proper and cocompact. The dimension of the symmetric space G/K associated to G coincides with the cohomological dimension of any of its uniform lattices and hence it coincides with the dimension of M. But then the action of G on $\widetilde{M}$ is open. Since this action is also closed, the action is transitive and hence $\widetilde{M}$ is a symmetric space of rank one.

We are left with showing that if $\mathrm{Iso}(\widetilde{M})$ contains a parabolic element then the isometry group of $\widetilde{M}$ is not totally disconnected. Assume to the contrary that $\mathrm{Iso}(\widetilde{M})$ is totally disconnected. Since the action of $\mathrm{Iso}(\widetilde{M})$ on $\widetilde{M}$ is cocompact, there is no non-trivial closed convex $\mathrm{Iso}(\widetilde{M})$-invariant subset of $\widetilde{M}$. Since $\mathrm{Iso}(\widetilde{M})$ is totally disconnected, by Theorem 5.1 of [12], point stabilizers of $\mathrm{Iso}(\widetilde{M})$ are open. Then Corollary 3.3 of [10] implies that every element of $\mathrm{Iso}(\widetilde{M})$ with vanishing translation length is elliptic. This is a contradiction to the assumption that $\mathrm{Iso}(\widetilde{M})$ contains a parabolic element. □

Finally we show Corollary 2 from the Introduction.

Corollary 7.3. *Let G be a semi-simple Lie group with finite center, no compact factors and rank at least* 2. *Let $\Gamma < G$ be an irreducible lattice, let X be a proper* CAT(0)-*space and let $\rho\colon \Gamma \to \mathrm{Iso}(X)$ be a homomorphism. If $\rho(\Gamma)$ is non-elementary and* *contains a rank-one element, then there is closed subgroup H of* $\mathrm{Iso}(X)$ *which is a compact extension of a simple Lie group L of rank one and there is a surjective homomorphism $\rho\colon G \to L$.*

Proof. Let $\Gamma < G$ be an irreducible lattice and let $\rho\colon \Gamma \to \mathrm{Iso}(X)$ be a homomorphism. Let $H < \mathrm{Iso}(X)$ be the closure of $\rho(\Gamma)$. Then H is a closed subgroup of $\mathrm{Iso}(X)$ which admits a Haar measure μ.

If $\rho(\Gamma)$ is non-elementary and contains a rank-one element then the same is true for H. By Proposition 5.3 and Proposition 6.2, in this case the second bounded cohomology group $H^2_{\mathrm{cb}}(H, L^2(H,\mu))$ is non-trivial. Via pullback by ρ, the second bounded cohomology group $H^2_{\mathrm{b}}(\Gamma, L^2(H,\mu))$ is non-trivial as well.

By Proposition 4.2 of [8], via inducing we deduce that the second continuous bounded cohomology group $H^2_{\mathrm{cb}}(G, L^{[2]}(G/\Gamma, L^2(H,\mu)))$ does not vanish. Here $L^{[2]}(G/\Gamma, L^2(H,\mu))$ denotes the Hilbert G-module of all measurable maps $G/\Gamma \to L^2(H,\mu)$ with the additional property that for each such map φ the function $x \to \|\varphi(x)\|$ is square integrable on G/Γ with respect to the projection of the Haar measure. The G-action is the twisted action determined by the homomorphism ρ.

We can now conclude as in the proof of Proposition 5.2 of [15]. Namely, let $\Omega \subset G$ be a Borel fundamental domain for the action of Γ on G. We obtain a measurable cocycle $\beta\colon G \times G/\Gamma \to H$ as follows. For $z \in \Omega$ and $g \in G$, let $\eta(g,z) \in \Gamma$ be the unique element such that $gz \in \eta(g,z)\Omega$ and define $\beta(g,z) = \rho(\eta(g,z))$. Since the rank of G is at least two by assumption, the results of Monod and Shalom [20] show that if G is simple then there is a β-equivariant map $G/\Gamma \to L^2(H,\mu)$. However, this implies that H is compact (see Section 3 of [20]) which is impossible since H contains a rank-one element.

If $G = G_1 \times G_2$ for semi-simple Lie groups G_1, G_2 with finite center and no compact factor then the results of Burger and Monod [8], [9] show that via possibly exchanging G_1 and G_2 we may assume that there is a map $G/\Gamma \to L^2(H,\mu)$ which is equivariant with respect to the restriction of β to G_1. Since Γ is irreducible by assumption, the action of G_1 on G/Γ is ergodic [22]. By Lemma 5.2, the amenable radical N of H is compact and we deduce as in [20] that there is a continuous homomorphism $\psi\colon G \to H/N$. Since G is connected, the image $\psi(G) = H/N$ is connected and hence by Proposition 7.1, H/L is a simple Lie group of rank one. □

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U. Hamenstädt, Mathematisches Institut, Rheinische Friedrich-Wilhelms-Universität Bonn, Endenicher Allee 60, 53115 Bonn, Germany

E-mail: ursula@math.uni-bonn.de