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Algebraic aspects of connections: From torsion, curvature, and post-Lie algebras to Gavrilov's double exponential and special polynomials

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Abstract. Understanding the algebraic structure underlying a manifold with a general affine connection is a natural problem. In this context, A. V. Gavrilov introduced the notion of framed Lie algebra, consisting of a Lie bracket (the usual Jacobi bracket of vector fields) and a magmatic product without any compatibility relations between them. In this work, we will show that an affine connection with curvature and torsion always gives rise to a post-Lie algebra as well as a D-algebra. The notions of torsion and curvature together with Gavrilov's special polynomials and double exponential are revisited in this post-Lie algebraic framework. We unfold the relations among the post-Lie Magnus expansion, the Grossman–Larson product, and the K-map, α-map, and β-map, three particular functions introduced by Gavrilov with the aim of understanding the geometric and algebraic properties of the double-exponential, which can be understood as a geometric variant of the Baker–Campbell–Hausdorff formula. We propose a partial answer to a conjecture by Gavrilov, by showing that a particular class of geometrically special polynomials is generated by torsion and curvature. This approach unlocks many possibilities for further research such as numerical integrators and rough paths on Riemannian manifolds.

1. Introduction

Let $\mathcal{M}$ be a smooth manifold, and let $\mathcal{X}\mathcal{M} = \operatorname{Der} C^\infty(\mathcal{M})$ be the Lie algebra of vector fields on $\mathcal{M}$. An affine connection on $\mathcal{M}$ gives rise to a covariant derivative operator ∇ on $\mathcal{X}\mathcal{M}$. It is well-known that the induced binary product $\triangleright$ defined by

$$X \triangleright Y := \nabla_X Y$$

is left pre-Lie when the connection is flat and torsion-free. This fact can be traced back to A. Cayley's famous 1857 article on vector fields and rooted trees [6]. See for example [5, 24, 25]. An affine connection with constant torsion and vanishing curvature gives rise to a post-Lie algebra [28], a structure which appeared in a paper by B. Vallette on partition posets [33]. The closely related notion of D-algebra appeared independently in joint work by one of the present authors together with W. Wright [30] in the context of

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numerical schemes for differential equations on Lie groups and homogeneous spaces. Pre-Lie algebras, also known as chronological algebras [1], are rather well-studied in algebra, combinatorics, and geometry. On the other hand, research on post-Lie and D-algebras is more recent. See for instance [3, 7, 9, 12, 22, 26, 28].

Understanding the algebraic structure underlying a manifold with a more general affine connection is a natural problem which attracted some attention [14, 16, 18, 20]. An obvious structure on the space $\mathcal{XM}$ is that of a so-called framed Lie algebra, consisting of a Lie bracket (the usual Jacobi bracket of vector fields) and a binary product (the right triangle $\rhd$) without any compatibility relations between them. This setting was studied in depth by A. V. Gavrilov in the 2006 work [14]. Our contribution aims at exhibiting a natural post-Lie algebra $\mathfrak{g}$ associated with these geometric data and reinterpreting the main geometric notions in this framework, that is, torsion and curvature as well as Gavrilov's special polynomials and double exponential[1]. To sum up, we advocate in the present paper the relevance of the post-Lie framework for a deeper and more refined understanding of Gavrilov's major findings on framed Lie algebras which are not necessarily post-Lie, including vector fields on a manifold endowed with an affine connection with curvature and torsion.

The paper is organized as follows. Section 2 is devoted to post-Lie and D-algebras. Several of the lengthier algebraic computations in the proofs of the algebraic statements in this section have been collected in Appendix A. After the necessary background is recalled in Section 2.1, the free D-algebra and the free post-Lie algebra generated by a magmatic algebra M are defined, in terms of the free unital associative algebra (the tensor algebra) $TM = \bigoplus_{k\geq 0} M^{\cdot k}$ and the free Lie algebra $\mathrm{Lie}(M)$, respectively. The particular case of a free post-Lie and free D-algebra generated by a set A is detailed in Section 2.2.2 using planar rooted trees and forests decorated by elements from A. In Appendix B, we present briefly the notion of planar multi-grafting. Of particular interest is equation (2.10) relating right Butcher product and left grafting on rooted trees. Reminders on the enveloping algebra of a post-Lie algebra (Section 2.3) and the post-Lie Magnus expansion (Section 2.4) are followed by a longer subsection on Gavrilov's K-map, $K : TM \to TM$, seen from a post-Lie algebra point of view (Section 2.5). This map is reinterpreted as a linear automorphism of TM mapping the Grossman–Larson product $*$ onto the defining product $\cdot$ of TM (Theorem 3). An explicit non-recursive expression of the inverse map, K^{-1}, is given in terms of set partitions (Proposition 6). Section 2.5.3 concludes this section by stating equation (2.31) relating the K-map, the post-Lie Magnus expansion, χ, and the logarithm $Z : y \mapsto \log^{\cdot} K(\exp^{\cdot}(y))$.

Section 3 is devoted to Gavrilov's β-map [14], obtained by pre-composing the aforementioned logarithm Z with the canonical projection p from the tensor algebra onto the

[1] The latter is a formal binary product on vector fields which can be understood as a geometric variant of the Baker–Campbell–Hausdorff formula. It is not associative in general, unless the connection is flat. Concrete examples can be found in applied differential geometry, for instance, in the context of numerical schemes on manifolds [19].

enveloping algebra of a completed graded framed Lie algebra $(\widehat{\mathcal{L}}, \rhd, [\![.,.]\!])$. This map is a keystone in the expression of Gavrilov's double exponential [14] considered in Section 6. A simple formula is obtained in terms of the K-map, the post-Lie Magnus expansion, and the projection p (equation (3.1)).

In Section 4, we apply the mentioned algebraic results to the concrete setting of the framed Lie algebra $\mathcal{XM}$ of vector fields on a smooth manifold $\mathcal{M}$ endowed with an affine connection. Higher-order covariant derivatives are recast in the post-Lie framework in terms of the K-map. We show in Proposition 7 that the free Lie algebra $\mathfrak{g}$ (resp., the tensor algebra $\mathcal{A}$ of $\mathcal{XM}$) over the ring $\mathcal{R} := C^\infty(\mathcal{M})$ is a post-Lie algebra (resp., a D-algebra). As a consequence (Remark 12), we obtain in equation (4.5) an alternative expression of Gavrilov's β-map. The particular case of a flat connection with constant torsion, in which the framed Lie algebra $\mathcal{XM}$ itself is post-Lie, is detailed in Section 4.3.

Section 5 is devoted to Gavrilov's special polynomials [14]. We provide a partial answer to a conjecture put forward in [14], by showing that a natural (and rather broad) family of special polynomials can be expressed in terms of torsion, curvature, and their higher-order covariant derivatives (Theorem 5). An important intermediate result (Proposition 8) expresses the kernel $\mathcal{J}$ of the action ρ of $\mathfrak{g}$ by derivations on $C^\infty(\mathcal{M})$ (which is an ideal for the Grossman–Larson Lie bracket) in terms of the so-called curvature elements, denoted by $s(a \cdot b)$ and introduced in Definition 5. We also show that the kernel $\mathcal{K}$ of the action $\rhd$ of $\mathfrak{g}$ on $\mathcal{XM}$ is a Grossman–Larson ideal included in the kernel $\mathcal{J}$, and we exhibit a nonzero element of $\mathcal{K}$ by means of the first Bianchi identity. The inclusion $\mathcal{J} \subset \mathcal{K}$ is in general strict, manifesting the presence of curvature.

Finally, Section 6 discusses Gavrilov's double exponential, which we first describe heuristically by comparison of consecutive parallel transports. Then, we express it in precise terms using the post-Lie Magnus expansion (Theorem 6).

We close the paper with a short synthesis of the results followed by a brief outlook.

2. Post-Lie and D-algebras

2.1. Reminders on post-Lie and D-algebras

Let $\mathbf{k}$ be a field of characteristic zero, which will be the real numbers $\mathbb{R}$ whenever differential geometry comes into play.

Definition 1 ([33]). A post-Lie algebra on $\mathbf{k}$ is a Lie algebra $(\mathfrak{g}, [\cdot,\cdot])$ together with a bilinear mapping $\rhd\colon \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ compatible with the Lie bracket, in the following sense:

$$x \rhd [y, z] = [x \rhd y, z] + [y, x \rhd z], \tag{2.1}$$

$$[x, y] \rhd z = \mathrm{a}_\rhd(x, y, z) - \mathrm{a}_\rhd(y, x, z), \tag{2.2}$$

for any elements $x, y, z \in \mathfrak{g}$. Here, $\mathrm{a}_\rhd(x, y, z)$ is the associator with respect to the product $\rhd$ defined by

$$\mathrm{a}_\rhd(x, y, z) := x \rhd (y \rhd z) - (x \rhd y) \rhd z.$$

Any Lie algebra can be seen as a post-Lie algebra by setting the second product $\rhd$ to zero. Another possibility is to take for the second product $\rhd$ the opposite of the Lie bracket. A (left) pre-Lie algebra is just an Abelian post-Lie algebra, i.e., a post-Lie algebra with trivial Lie bracket, implying that (2.2) reduces to the (left) pre-Lie identity

$$0 = \mathrm{a}_{\rhd}(x, y, z) - \mathrm{a}_{\rhd}(y, x, z).$$

In other words, for a (left) pre-Lie algebra, the associator is symmetric in the first two entries. On any post-Lie algebra, particular combinations of the Lie bracket and the $\rhd$ product yield two other operations, as follows:

$$\begin{aligned} [\![x, y]\!] &:= x \rhd y - y \rhd x + [x, y], \\ x \blacktriangleright y &:= x \rhd y + [x, y], \end{aligned} \tag{2.3}$$

for all $x, y \in \mathfrak{g}$. From (2.1) and (2.2) above, one can easily deduce that $(\mathfrak{g}, [\![\cdot,\cdot]\!])$ forms a Lie algebra, and the triple $(\mathfrak{g}, -[\cdot,\cdot], \blacktriangleright)$ is another post-Lie algebra [7, 28] sharing the same double Lie bracket:

$$\begin{aligned} [\![x, y]\!] &= x \rhd y - y \rhd x + [x, y] \\ &= x \blacktriangleright y - y \blacktriangleright x - [x, y] \\ &= x \rhd y - y \blacktriangleright x. \end{aligned}$$

Example 1. Let $\mathcal{XM}$ be the space of vector fields on a smooth manifold $\mathcal{M}$, which is equipped with an affine connection. For vector fields $X, Y \in \mathcal{XM}$, the covariant derivative of Y in the direction of X is denoted $\nabla_X Y =: X \rhd Y$. This defines an $\mathbb{R}$-linear, non-associative binary product on $\mathcal{XM}$. The torsion t is defined by

$$t(X, Y) := X \rhd Y - Y \rhd X - \mathbf{[}X, Y\mathbf{]},$$

where the bracket $\mathbf{[}.\,,.\mathbf{]}$ on the right is the usual Jacobi bracket of vector fields. It admits a covariant differential ∇t. The curvature tensor r is given by

$$r(X, Y)Z := X \rhd (Y \rhd Z) - Y \rhd (X \rhd Z) - \mathbf{[}X, Y\mathbf{]} \rhd Z.$$

It is known that for a flat connection with constant torsion, $r = 0 = \nabla t$, we have that $(\mathcal{XM}, -t(\cdot,\cdot), \rhd)$ defines a post-Lie algebra. The first Bianchi identity (see (4.2) below) shows that $-t(\cdot,\cdot)$ obeys the Jacobi identity; skew-symmetry of t implies anti-symmetry. Flatness is equivalent to identity (2.2), whereas property (2.1) follows from the definition of the covariant differential of t:

$$0 = (\nabla_X t)(Y, Z) = X \rhd t(Y, Z) - t(Y, X \rhd Z) - t(X \rhd Y, Z).$$

Definition 2 ([30]). Let $(D, \cdot, \rhd)$ be an associative algebra with product $m_D(u \otimes v) = u \cdot v$ and unit $\mathbf{1}$, carrying another product $\rhd : D \otimes D \to D$ such that $\mathbf{1} \rhd v = v$ for all $v \in D$. Let

$$\mathfrak{d}(D) := \big\{u \in D \mid u \rhd (v \cdot w) = (u \rhd v) \cdot w + v \cdot (u \rhd w),\ \forall v, w \in D\big\}.$$

The triple $(D, \cdot, \rhd)$ is called a D-algebra if the algebra product $\cdot$ generates D from $\{\mathbf{1}, \mathfrak{d}(D)\}$ and furthermore for any $x \in \mathfrak{d}(D)$ and $v, w \in D$,

$$v \rhd x \in \mathfrak{d}(D),$$
$$(x \cdot v) \rhd w = \mathrm{a}_{\rhd}(x, v, w). \tag{2.4}$$

Lemma 1. *Let $(D, \cdot, \rhd)$ be as in Definition 2. Then, $\mathfrak{d}(D)$ together with $\rhd$ and Lie bracket $[u, v] := u \cdot v - v \cdot u$ is a post-Lie algebra.*

Proof. Remark that $\mathfrak{d}(D)$ is the set of $x \in D$ such that $L^{\rhd}_x := x \rhd -$ is a derivation for the associative product. One only needs to verify that $\mathfrak{d}(D)$ is stable under the Lie bracket $(x, y) \mapsto x \cdot y - y \cdot x$, which amounts to prove that $L^{\rhd}_{x \cdot y - y \cdot x}$ is a derivation. From

$$L^{\rhd}_{x \cdot y} = L^{\rhd}_x \circ L^{\rhd}_y - L^{\rhd}_{x \rhd y},$$

which is a reformulation of (2.4), we get

$$L^{\rhd}_{x \cdot y - y \cdot x} = [L^{\rhd}_x, L^{\rhd}_y] - L^{\rhd}_{x \rhd y - y \rhd x},$$

which proves the claim. Identity (2.1) results from the fact that any derivation for the associative product is also a derivation for the Lie bracket, and (2.2) is checked immediately. ∎

Example 2. The space $\mathcal{DM}$ of differential operators on the manifold $\mathcal{M}$ endowed with an affine connection ∇ with vanishing curvature and constant torsion is a D-algebra such that $\mathfrak{d}(\mathcal{DM}) = \mathcal{XM}$.

This non-trivial example is treated in detail in [28]. We shall revisit it in Section 4.3 below.

2.2. Free post-Lie and free D-algebras

2.2.1. The free D-algebra generated by a magmatic algebra. Recall that a magmatic algebra consists of a set equipped with a binary operation and no further relations.

Theorem 1. *Let $(M, \rhd)$ be any magmatic algebra. Let $T(M)$ be the tensor algebra over M with concatenation as product. Extending the magma product $\rhd$ to $T(M)$ as*

$$x \rhd (VW) = (x \rhd V)W + V(x \rhd W), \tag{2.5}$$
$$(xV) \rhd W = \mathrm{a}_{\rhd}(x, V, W), \tag{2.6}$$

for any element $x \in M$ and $V, W \in T(M)$, defines a D-algebra structure on $T(M)$.

Proof. It is clear that (2.5) and (2.6) uniquely define the extended magma product $\rhd$, by induction on the lengths of the elements involved. The tensor algebra is a Hopf algebra with the usual unshuffle coproduct, $\Delta_{\sqcup\!\sqcup}(A) := A_{(1)} \otimes A_{(2)}$ (we use Sweedler's notation).

The elements in the magmatic algebra M are primitive, i.e., $\Delta_{\sqcup\!\sqcup}(x) = x \otimes \mathbf{1} + \mathbf{1} \otimes x$ for all $x \in M$. A more explicit formula for the unshuffle coproduct is given by

$$\Delta_{\sqcup\!\sqcup}(x_1 \cdots x_n) = \sum_{I \sqcup J = \{1,\ldots,n\}} x_I \otimes x_J,$$

with the obvious word notation $x_I := x_{i_1} \cdots x_{i_p}$ for index set $I = \{i_1, \ldots, i_p\}$ with $i_1 < \cdots < i_p$. The set Prim $T(M)$ of primitive elements is actually the free Lie algebra Lie(M) generated by M. This is the vector subspace of $T(M)$ generated by iterated Lie brackets of elements of M [31]. An iteration of (2.5) yields

$$U \rhd (VW) = (U_{(1)} \rhd V)(U_{(2)} \rhd W), \tag{2.7}$$

which is easily checked on monomials $U = x_1 \cdots x_n$ (with $x_1, \ldots, x_n \in M$) by induction on the length n. We deduce from (2.7) that the set of $U \in T(M)$ such that $L_U^{\rhd}$ is a derivation is precisely Lie(M). It remains to show that (2.6) is valid for any element $x \in$ Lie(M). It suffices to check that the set of elements of $T(M)$ verifying (2.5) and (2.6), which contains M, is a Lie subalgebra. Let us choose two elements X and Y in $T(M)$ verifying (2.5) and (2.6). The claim follows from a straightforward computation detailed in Appendix A. ∎

Remark 1. It is easily checked that $T(M)$ is the free D-algebra generated by the magmatic algebra $(M, \rhd)$. In fact, for any D-algebra $(D, \diamond, \cdot)$, any magmatic morphism $\psi : M \to (D, \diamond)$ can be uniquely extended to an associative algebra morphism $\Psi : T(M) \to D$, which happens to be a D-algebra morphism. Similarly, Lie(M) is the free post-Lie algebra generated by the magmatic algebra M. This has been first remarked by L. Foissy [13, Theorem 1 and Proposition 2].

It is easily seen that $L_x^{\rhd}(y) := x \rhd y$ is a coderivation with respect to the coproduct $\Delta_{\sqcup\!\sqcup}$ for any $x \in M$. More generally, the coproduct is compatible with the extended magmatic product.

Proposition 1. *For any $U, V \in T(M)$, we have*

$$\Delta_{\sqcup\!\sqcup}(U \rhd V) = \Delta_{\sqcup\!\sqcup}(U) \rhd \Delta_{\sqcup\!\sqcup}(V) = U_{(1)} \rhd V_{(1)} \otimes U_{(2)} \rhd V_{(2)}.$$

Proof. We can suppose that U is a monomial, and we proceed by induction on its length ℓ. The details are given in Appendix A. ∎

We mention the following result for later use.

Proposition 2. *For any $U, V, W \in T(M)$, the following holds:*

$$U \rhd (V \rhd W) = \big(U_{(1)}(U_{(2)} \rhd V)\big) \rhd W. \tag{2.8}$$

Proof. We can suppose without loss of generality that U is a monomial. We proceed by induction on the length ℓ of U. The details are given in Appendix A. ∎

Definition 3. The Grossman–Larson product on $T(M)$ is defined by

$$U * V := U_{(1)}(U_{(2)} \rhd V). \tag{2.9}$$

It is easily seen to be associative:

$$\begin{aligned}(U * V) * W &= U_{(1)}(U_{(2)} \rhd V_{(1)})\big((U_{(3)}(U_{(4)} \rhd V_{(2)})) \rhd W\big)\\ &= U * (V * W).\end{aligned}$$

This is a straightforward computation using the cocommutativity of the unshuffle coproduct, left to the reader (see the proof of Proposition 3.3 in [9]).

Proposition 3. *The Grossman–Larson product* (2.9) *is compatible with the unshuffle coproduct: for any* $U, V \in T(M)$, *we have*

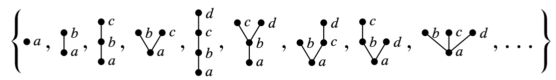
$$\Delta_{\sqcup\!\sqcup}(U * V) = \Delta_{\sqcup\!\sqcup}(U) * \Delta_{\sqcup\!\sqcup}(V) = U_{(1)} * V_{(1)} \otimes U_{(2)} * V_{(2)}.$$

Proof. This is a straightforward check using (2.9) and Proposition 1. ∎

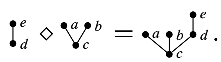
2.2.2. The free D-algebra generated by a set: Planar rooted trees and grafting. The following is immediate in view of Remark 1.

Proposition 4. *Let A be a finite alphabet and* $(\mathrm{Mag}(A), \rhd)$ *the free magmatic algebra over A. Then,* $\mathrm{Dalg}(A) = T(\mathrm{Mag}(A))$, *with the product $\rhd$ extended as in Theorem 1, is the free D-algebra generated by A. Similarly,* $\mathrm{PostLie}(A) = \mathrm{Lie}(\mathrm{Mag}(A))$ *is the free post-Lie algebra generated by A.*

It is known that the free magma generated by A can be represented in terms of planar rooted trees

$T_A^{\mathrm{pl}} =$ 

with nodes decorated by elements of A. Here, the magmatic product $\diamond : T_A^{\mathrm{pl}} \times T_A^{\mathrm{pl}} \to T_A^{\mathrm{pl}}$ is the right Butcher product defined as follows: $\sigma \diamond \tau$ is the A-decorated planar rooted tree obtained by grafting σ on the root of τ on the right; for example,



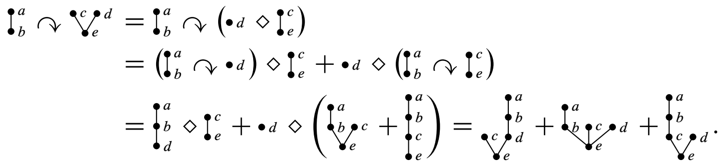
Then, $(T_A^{\mathrm{pl}}, \diamond)$ is the free magma generated by A via the inclusion $A \hookrightarrow T_A^{\mathrm{pl}}$ given by $a \mapsto \bullet_a$. Now, let $\mathcal{T}_A^{\mathrm{pl}}$ denote the vector space freely spanned by A-decorated planar rooted trees. The left-grafting $\curvearrowright : \mathcal{T}_A^{\mathrm{pl}} \times \mathcal{T}_A^{\mathrm{pl}} \to \mathcal{T}_A^{\mathrm{pl}}$ is defined as the $\mathbf{k}$-linear product given by

$$\tau \curvearrowright \bullet_a := \tau \diamond \bullet_a$$

and

$$\tau \curvearrowright (\tau_1 \diamond \tau_2) = (\tau \curvearrowright \tau_1) \diamond \tau_2 + \tau_1 \diamond (\tau \curvearrowright \tau_2) \tag{2.10}$$

for $\tau, \tau_1, \tau_2 \in T_A^{\mathrm{pl}}$. For example, $\mathsf{I}^{a}_{b} = \bullet_a \curvearrowright \bullet_b = \bullet_a \diamond \bullet_b$ and

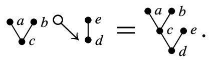


Lemma 2. *Let $\varphi : \mathrm{Mag}(A) \to \mathcal{T}_A^{\mathrm{pl}}$ be the unique linear map defined by $\varphi(a) := \bullet_a$ for all $a \in A$ and $\varphi(\sigma \rhd \tau) := \varphi(\sigma) \curvearrowright \varphi(\tau)$ for all $\sigma, \tau \in \mathrm{Mag}(A)$. The map φ is an isomorphism; i.e., the magmatic algebras $(\mathcal{T}_A^{\mathrm{pl}}, \curvearrowright)$ and $(\mathrm{Mag}(A), \rhd)$ are isomorphic.*

Proof. We have just seen that the correspondence κ, defined by $\kappa(a) = \bullet_a$ for all $a \in A$ and

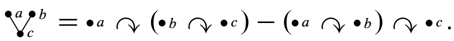
$$\kappa(\tau_1 \rhd \tau_2) = \kappa(\tau_1) \diamond \kappa(\tau_2),$$

is a magma isomorphism. This is obviously still true for the correspondence $\overline{\kappa}$ analogously defined with the right Butcher product $\diamond$ replaced by its left version $\searrow$, where $\tau_1 \searrow \tau_2$ is the A-decorated planar rooted tree obtained by grafting τ_1 on the root of τ_2 on the left; for example,

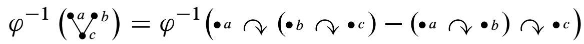


Now, both magmatic algebras $(\mathcal{T}_A^{\mathrm{pl}}, \searrow)$ and $(\mathcal{T}_A^{\mathrm{pl}}, \curvearrowright)$ are isomorphic. Indeed, the unique morphism $\Psi : (\mathcal{T}_A^{\mathrm{pl}}, \searrow) \to (\mathcal{T}_A^{\mathrm{pl}}, \curvearrowright)$ extending the identity on one-vertex trees can be put in upper-triangular matrix form with 1's on the diagonal, hence being a linear isomorphism. Full details are given in [2]. The isomorphism φ is therefore given by $\varphi = \Psi \circ \overline{\kappa}$. ∎

As an example, we consider the tree



Then, we have



$$= a \rhd (b \rhd c) - (a \rhd b) \rhd c$$

$$= \mathrm{a}_{\rhd}(a, b, c).$$

Remark 2. In view of Lemma 2, the free magmatic algebra $(\mathrm{Mag}(A), \rhd)$ can be represented by the linear span of planar A-decorated rooted trees, endowed by either the right Butcher product $\diamond$ or the left grafting $\curvearrowright$.

2.3. The enveloping algebra of a post-Lie algebra

Proposition 5. *If $(\mathfrak{g}, [\cdot,\cdot], \rhd)$ is a post-Lie algebra, then $(\mathcal{U}(\mathfrak{g}), \cdot, \rhd)$ is a D-algebra, and the set of primitive elements $\mathfrak{g} = \mathrm{Prim}\, \mathcal{U}(\mathfrak{g})$ is the post-Lie algebra $\mathfrak{d}(\mathcal{U}(\mathfrak{g}))$ of the D-algebra $\mathcal{U}(\mathfrak{g})$.*

Proof. Let us consider the D-algebra structure on $T(\mathfrak{g})$ given in Section 2.2.1. The ideal J generated by $\{j_{x,y} := x \cdot y - y \cdot x - [x, y],\ x, y \in \mathfrak{g}\}$ is also a two-sided ideal for the product $\rhd$. To see this, choose any $x, y \in \mathfrak{g}$ and any U, A, B in $T(\mathfrak{g})$. By iterating (2.5), we have

$$U \rhd (A \cdot j_{x,y} \cdot B) = (U_{(1)} \rhd A) \cdot j_{U_{(2)} \rhd x, U_{(3)} \rhd y} \cdot (U_{(4)} \rhd B) \in J,$$

using Sweedler's notation for the iterated unshuffle coproduct. Starting from $j_{x,y} \rhd U = 0$ which is a simple consequence of (2.6), we also have $(j_{x,y} \cdot B) \rhd U = 0$ by equation (2.8), taking primitiveness of $j_{x,y}$ into account. Equation (2.8) also proves $(A \cdot j_{x,y} \cdot B) \rhd U \in J$ by induction on the length of A. Hence, the D-algebra structure on $T(\mathfrak{g})$ naturally gives rise to a D-algebra structure on the quotient $\mathcal{U}(\mathfrak{g}) = T(\mathfrak{g})/J$. ■

Remark 3. From this follows the existence of a pair of adjoint functors between the categories of D-algebras and post-Lie algebras

$$\mathcal{U}(-) : \text{postLie} \leftrightarrows D\text{-algebra} : \mathfrak{g}(-).$$

In other words, there is a natural isomorphism

$$\mathrm{Hom}_{\text{postLie}}\big(\mathfrak{g}(A), B\big) \to \mathrm{Hom}_{D\text{-algebra}}\big(A, \mathcal{U}(B)\big).$$

Remark 4. The Grossman–Larson product also makes sense on $\mathcal{U}(\mathfrak{g})$ and is given by (2.9). From Proposition 3, it is compatible with the coproduct, making $(\mathcal{U}(\mathfrak{g}), *, \Delta)$ a Hopf algebra isomorphic to the enveloping algebra of $(\mathfrak{g}, [\![\cdot,\cdot]\!])$. This has been first established in [9]; see Proposition 3.3 therein. From Proposition 2 and (2.9), we have

$$U \rhd (V \rhd W) = (U * V) \rhd W \tag{2.11}$$

for any $U, V, W \in \mathcal{U}(\mathfrak{g})$.

Note however that both Hopf algebras have different antipodes. As an example, we compare the two antipodes for product $x \cdot y$ and $x * y$, $x, y \in \mathfrak{g}$:

$$\begin{aligned} S_*(x \cdot y) &= -x \cdot y - S_*(x) * y - S_*(y) * x \\ &= y * x + x \rhd y \\ &= y \cdot x + x \rhd y + y \rhd x \\ &= S(x \cdot y) + x \rhd y + y \rhd x \end{aligned}$$

and

$$\begin{aligned} S(x * y) &= -x * y - S(x) \cdot y - S(y) \cdot x \\ &= y \cdot x - x \rhd y \\ &= y * x - x \rhd y - y \rhd x \\ &= S_*(x * y) - x \rhd y - y \rhd x. \end{aligned}$$

The following theorem is key to many upcoming computations.

Theorem 2. *In the Hopf algebra* $(\mathcal{U}(\mathfrak{g}), \cdot, \Delta_{\sqcup\!\sqcup}, \varepsilon, S)$*, the product can be expressed in terms of the Grossman–Larson product* (2.9) *as follows:*

$$A \cdot B = A_{(1)} * \big(S_*(A_{(2)}) \rhd B\big), \quad A, B \in U(\mathfrak{g}). \tag{2.12}$$

Proof. We use (2.9) on the right-hand side of (2.12), which gives

$$\begin{aligned} A_{(1)} * \big(S_*(A_{(2)}) \rhd B\big) &= A_{(1)(1)} \cdot \big(A_{(1)(2)} \rhd \big(S_*(A_{(2)}) \rhd B\big)\big) \\ &\overset{(2.11)}{=} A_{(1)(1)} \cdot \big(\big(A_{(1)(2)} * S_*(A_{(2)})\big) \rhd B\big) \\ &= A_{(1)} \cdot \big(\big(A_{(2)(1)} * S_*(A_{(2)(2)})\big) \rhd B\big) \\ &= A_{(1)} \cdot \big(m_*(\mathrm{id} \otimes S_*)\Delta_{\sqcup\!\sqcup}(A_{(2)}) \rhd B\big) \\ &= A \cdot B. \end{aligned}$$

In the third equality, we used coassociativity. ∎

Remark 5. Using (2.7) and $x \in \mathfrak{g} \hookrightarrow \mathcal{U}(\mathfrak{g})$ being primitive, i.e., $\Delta_{\sqcup\!\sqcup}(x) = x \otimes 1 + 1 \otimes x$, implying $S_*(x) = -x$, we find from (2.12) the recursion

$$\begin{aligned} x_1 \cdots x_n &= x_1 * (x_2 \cdots x_n) - x_1 \rhd (x_2 \cdots x_n) \\ &\overset{(2.9)}{=} x_1 * (x_2 \cdots x_n) - \sum_{i=2}^{n} x_2 \cdots (x_1 \rhd x_i) \cdots x_n. \end{aligned} \tag{2.13}$$

We used that for $x \in \mathfrak{g} \hookrightarrow \mathcal{U}(\mathfrak{g})$, the left-multiplication operator $L^{\rhd}_x$ acts as a derivation on elements in $\mathcal{U}(\mathfrak{g})$. Further below, we will revisit these identities in the context of Gavrilov's K-map and special polynomials.

For example, let $x, y, z \in \mathfrak{g}$. Then, as these elements are primitive with respect to the unshuffle coproduct $\Delta_{\sqcup\!\sqcup}$, we find

$$\begin{aligned} x \cdot y &= x * y - x \rhd y, \\ x \cdot y \cdot z &= x * (y \cdot z) - x \rhd (y \cdot z) \\ &= x * y * z - x * (y \rhd z) - x \rhd (y * z) + x \rhd (y \rhd z). \end{aligned}$$

Remark 6. From (2.11), we immediately get

$$(x_1 * \cdots * x_n) \rhd B = L^{\rhd}_{x_1} \cdots L^{\rhd}_{x_n} B. \tag{2.14}$$

Hence, Grossman–Larson products of elements in $\mathfrak{g} \hookrightarrow \mathcal{U}(\mathfrak{g})$ transfer to compositions of left-multiplication maps. Therefore, any operator of the form $A \rhd -$, where $A \in \mathcal{U}(\mathfrak{g})$, translates into a $L^{\rhd}$-polynomial. For example, for any $x, y \in \mathfrak{g}$,

$$(x \cdot y) \rhd B = (x * y - x \rhd y) \rhd B = (L^{\rhd}_x L^{\rhd}_y - L^{\rhd}_{x \rhd y}) \rhd B.$$

In light of (2.14) and the recursion (2.13), we can extend the definition of the $L^{\triangleright}$-operator to words, $x_1 \cdots x_n \in \mathcal{U}(\mathfrak{g})$, by defining inductively

$$\hat{L}^{\triangleright}_{x_1\cdots x_n} := L^{\triangleright}_{x_1}\hat{L}^{\triangleright}_{x_2\cdots x_n} - \sum_{i=2}^{n} \hat{L}^{\triangleright}_{x_2\cdots x_1 \triangleright x_i \cdots x_n}.$$

One checks that $\hat{L}^{\triangleright}$ is an algebra morphism from $(\mathcal{U}(\mathfrak{g}), *)$ into $\mathrm{End}(\mathcal{U}(\mathfrak{g}))$; that is,

$$\hat{L}^{\triangleright}_{A*B} = \hat{L}^{\triangleright}_{A}\hat{L}^{\triangleright}_{B},$$

for $A, B \in \mathcal{U}(\mathfrak{g})$. For example,

$$\begin{aligned}\hat{L}^{\triangleright}_{x_1 * x_2} &= \hat{L}^{\triangleright}_{x_1x_2 + x_1\triangleright x_2} = \hat{L}^{\triangleright}_{x_1x_2} + L^{\triangleright}_{x_1\triangleright x_2}\\ &= L^{\triangleright}_{x_1}L^{\triangleright}_{x_2} - L^{\triangleright}_{x_1\triangleright x_2} + L^{\triangleright}_{x_1\triangleright x_2} = L^{\triangleright}_{x_1}L^{\triangleright}_{x_2}.\end{aligned}$$

Further below, we will see that these polynomials are closely related to Gavrilov's K-map.

2.4. Reminder on post-Lie Magnus expansion

We consider now the free D-algebra generated by a set A, i.e., the universal enveloping algebra $\mathcal{F}^{\mathrm{pl}}_A := \mathcal{U}(\mathcal{L}(\mathcal{T}^{\mathrm{pl}}_A))$ of the free post-Lie algebra $(\mathcal{L}(\mathcal{T}^{\mathrm{pl}}_A), [\cdot,\cdot], \triangleright)$, graded by the number of vertices of the forests[2]. Denote by $\widehat{\mathcal{U}(\mathcal{L}(\mathcal{T}^{\mathrm{pl}}_A))}$ its completion with respect to the grading. The unshuffle coproduct, $\Delta_{\sqcup\!\sqcup}$, is naturally extended to the completion. The set $\mathrm{Prim}(\mathcal{F}^{\mathrm{pl}}_A)$ consists in primitive elements, whereas $G(\mathcal{F}^{\mathrm{pl}}_A)$ denotes the set of group-like elements:

$$\begin{aligned}\mathrm{Prim}(\mathcal{F}^{\mathrm{pl}}_A) &:= \big\{\alpha \in \widehat{\mathcal{U}\big(\mathcal{L}(\mathcal{T}^{\mathrm{pl}}_A)\big)} \mid \Delta_{\sqcup\!\sqcup}(\alpha) = \mathbf{1}\otimes\alpha + \alpha\otimes\mathbf{1}\big\} = \widehat{\mathcal{L}(\mathcal{T}^{\mathrm{pl}}_A)},\\ G(\mathcal{F}^{\mathrm{pl}}_A) &:= \big\{\alpha \in \widehat{\mathcal{U}\big(\mathcal{L}(\mathcal{T}^{\mathrm{pl}}_A)\big)} \mid \Delta_{\sqcup\!\sqcup}(\alpha) = \alpha\otimes\alpha\big\}.\end{aligned}$$

Both products on $\mathcal{U}(\mathcal{L}(\mathcal{T}^{\mathrm{pl}}_A))$ – the concatenation and the Grossman–Larson product – can also be extended to products on the completion $\widehat{\mathcal{U}(\mathcal{L}(\mathcal{T}^{\mathrm{pl}}_A))}$. As a result, two different exponential functions can be defined on $\widehat{\mathcal{U}(\mathcal{L}(\mathcal{T}^{\mathrm{pl}}_A))}$; namely,

$$\begin{aligned}\exp^*(y) &= \sum_{n=0}^{\infty}\frac{y^{*n}}{n!} = \mathbf{1} + y + \frac{1}{2}y*y + \frac{1}{6}y*y*y + \cdots,\\ \exp^{\cdot}(y) &= \sum_{n=0}^{\infty}\frac{y^{\cdot n}}{n!} = \mathbf{1} + y + \frac{1}{2}y\cdot y + \frac{1}{6}y\cdot y\cdot y + \cdots.\end{aligned}$$

On a manifold, the exponential $\exp^*(y)$ will be seen to represent the exact flow of a vector field, while $\exp^{\cdot}(y)$ represents the flow along geodesics.

[2]Here, the magmatic product is not precised: in view of Remark 2, it can be either the right Butcher product $\diamond$ or the left grafting $\curvearrowright$.

Both these exponential functions map $\mathrm{Prim}(\mathcal{F}_A^{\mathrm{pl}})$ bijectively onto $G(\mathcal{F}_A^{\mathrm{pl}})$. See [9] for details. The post-Lie Magnus expansion χ is the bijective map from $\widehat{\mathcal{L}(\mathcal{T}_A^{\mathrm{pl}})}$ onto itself defined by the following relation between exponentials:

$$\exp^*\big(\chi(y)\big) = \exp^{\cdot}(y); \tag{2.15}$$

namely,

$$\chi(y) = \log^*\big(\exp^{\cdot}(y)\big).$$

The post-Lie Magnus expansion can be characterized by taking the derivation with respect to t of $\exp^*(\chi(ty)) = \exp^{\cdot}(ty)$. Recall the dexp-formulas [4] for derivations of the exp-map in a noncommutative setting

$$\begin{aligned}\frac{d}{dt}\exp\big(\Omega(t)\big) &= \mathrm{dexp}_{\Omega(t)}\big(\dot{\Omega}(t)\big)\exp\big(\Omega(t)\big)\\ &= \exp\big(\Omega(t)\big)\mathrm{dexp}_{-\Omega(t)}\big(\dot{\Omega}(t)\big).\end{aligned}$$

Using the group-likeness of the two exponentials, $\exp^*(\chi(ty))$ and $\exp^{\cdot}(ty)$, yields

$$\begin{aligned}\frac{d}{dt}\exp^*\big(\chi(ty)\big) &= \exp^{\cdot}(ty)\cdot y\\ &\overset{(2.12)}{=} \exp^{\cdot}(ty) * \Big(S_*\big(\exp^{\cdot}(ty)\big) \rhd y\Big)\\ &\overset{(2.15)}{=} \exp^*\big(\chi(ty)\big) * \Big(S_*\big(\exp^*\big(\chi(ty)\big)\big) \rhd y\Big)\\ &= \exp^*\big(\chi(ty)\big) * \Big(\exp^*\big(-\chi(ty)\big) \rhd y\Big),\end{aligned}$$

from which we deduce that

$$\exp^*\big(-\chi(ty)\big) * \frac{d}{dt}\exp^*\big(\chi(ty)\big) = \mathrm{dexp}^*_{-\chi(ty)}\big(\dot{\chi}(ty)\big) = \exp^*\big(-\chi(ty)\big) \rhd y.$$

Therefore, $\chi(ty)$ solves the Magnus-type differential equation

$$\dot{\chi}(ty) = \mathrm{dexp}^{*-1}_{-\chi(ty)}\Big(\exp^*\big(-\chi(ty)\big) \rhd y\Big), \quad \chi(0) = 0.$$

The classical formula $\frac{d}{dt}\exp(-A(t)) = -\exp(-A(t))\frac{d}{dt}(\exp(A(t)))\exp(-A(t))$ implies for the identity $\mathrm{dexp}^*_{-\chi(ty)}(\dot{\chi}(ty)) = \exp^*(-\chi(ty)) \rhd y$ that the function

$$\alpha(y,t) := \exp^*\big(-\chi(ty)\big) \rhd y$$

satisfies the differential equation

$$\begin{aligned}\frac{d}{dt}\alpha(y,t) &= \frac{d}{dt}\exp^*\big(-\chi(ty)\big) \rhd y\\ &= \left(\frac{d}{dt}\exp^*\big(-\chi(ty)\big)\right) \rhd y\\ &= \left(-\exp^*\big(-\chi(ty)\big) * \left(\frac{d}{dt}\exp^*\big(\chi(ty)\big)\right) * \exp^*\big(-\chi(ty)\big)\right) \rhd y\end{aligned}$$

$$= \big(-\operatorname{dexp}^*_{-\chi(ty)}\big(\dot\chi(ty)\big) * \exp^*\big(-\chi(ty)\big)\big) \rhd y$$
$$= -\operatorname{dexp}^*_{-\chi(ty)}\big(\dot\chi(ty)\big) \rhd \big(\exp^*\big(-\chi(ty)\big) \rhd y\big)$$
$$= -\alpha(y,t) \rhd \alpha(y,t).$$

Hence, $\alpha(y,t)$ satisfies the *post-Lie flow equation*

$$\begin{cases} \alpha(y,0) = y, \\ \dfrac{d}{dt}\alpha(y,t) = -\alpha(y,t) \rhd \alpha(y,t). \end{cases} \tag{2.16}$$

We can therefore describe the Grossman–Larson exponential $\exp^*(\chi(ty))$ as a solution of a linear non-autonomous initial value problem

$$\frac{d}{dt}\exp^*\big(\chi(ty)\big) = \exp^*\big(\chi(ty)\big) * \alpha(y,t). \tag{2.17}$$

The inclusion of any element y into the completed post-Lie algebra $\widehat{\mathcal{L}(\mathcal{T}_A^{\mathrm{pl}})}$ yields a unique injective morphism from the completed free magmatic algebra $\widehat{M_y}$ into it. We define now the map

$$\delta_y : \widehat{M_y} \to \widehat{M_y}$$

to be the unique derivation, with respect to the magmatic product $\rhd$, such that $\delta_y y = y \rhd y$. For instance,

$$\delta_y(\delta_y y) = \delta_y(y \rhd y) = (y \rhd y) \rhd y + y \rhd (y \rhd y).$$

It is then clear that $e^{t\delta_y}$ is a one-parameter group of automorphisms for the product $\rhd$. This yields

$$\frac{d}{dt}e^{-t\delta_y} y = -e^{-t\delta_y}\delta_y(y) = -e^{-t\delta_y}(y \rhd y) = -(e^{-t\delta_y} y \rhd e^{-t\delta_y} y).$$

This means that the function $t \mapsto e^{-t\delta_y} y$ solves as well the post-Lie flow equation (2.16) from which we deduce the intriguing identity

$$\exp^*\big(-\chi(ty)\big) \rhd y = e^{-t\delta_y} y. \tag{2.18}$$

The right-hand side of (2.18) is therefore the purely magmatic expression of the map α introduced by A. V. Gavrilov in [14], and the left-hand side is its post-Lie reformulation.

The inverse of $\chi(ty)$, which we denote by $\theta(ty)$, is obviously characterized by

$$\exp^*(ty) = \exp^{\cdot}\big(\theta(ty)\big). \tag{2.19}$$

From this we deduce in an analogous manner the differential equation

$$\dot\theta(ty) = \operatorname{dexp}^{\cdot -1}_{-\theta(ty)}\big(\exp^{\cdot}\big(\theta(ty)\big) \rhd y\big), \quad \theta(0) = 0.$$

The first terms of the post-Lie Magnus expansion χ and its inverse θ are given by [3, Appendix A]

$$\begin{aligned}\chi(y) &= y - \frac{1}{2} y \rhd y + \frac{1}{12} y \rhd (y \rhd y) + \frac{1}{4}(y \rhd y) \rhd y + \frac{1}{12}[y \rhd y, y] + \cdots \\ &= y - \frac{1}{2} y \rhd y + \frac{1}{6} y \rhd (y \rhd y) + \frac{1}{6}(y \rhd y) \rhd y + \frac{1}{12}[\![y \rhd y, y]\!] + \cdots, \qquad (2.20)\\ \theta(y) &= y + \frac{1}{2} y \rhd y + \frac{1}{6} y \rhd (y \rhd y) + \frac{1}{12}[y, y \rhd y] + \cdots \\ &= y + \frac{1}{2} y \rhd y + \frac{1}{12} y \rhd (y \rhd y) + \frac{1}{12}(y \rhd y) \rhd y + \frac{1}{12}[\![y, y \rhd y]\!] + \cdots. \qquad (2.21)\end{aligned}$$

Note that (2.20) and (2.21) follow from identity (2.3), which relates the Lie brackets

$$(y \rhd y) \rhd y - y \rhd (y \rhd y) + [y \rhd y, y] = [\![y \rhd y, y]\!].$$

We remark that the post-Lie Magnus expansion χ and its inverse θ make sense in any complete graded post-Lie algebra.

Remark 7. Returning to the linear initial value problem (2.17), we see that $\exp^*(\chi(ty)) = \exp^*(\Omega(\alpha(y,t)))$, and therefore

$$\chi(ty) = \Omega\big(\alpha(y,t)\big),$$

where the Magnus expansion

$$\Omega\big(\alpha(y,t)\big) = \int_0^t ds \sum_{n \geq 0} (-1)^n \frac{B_n}{n!} \mathrm{ad}_{\Omega}^{*(n)} \alpha(y,s).$$

Here, B_n are the Bernoulli numbers, $B_0 = 1$, $B_1 = 1/2$, $B_2 = 1/6$, $B_3 = 0, \ldots$, and ad^* refers to the fact that the Grossman–Larson Lie bracket is used. Note that the $(-1)^n$ factor on the right-hand side affects only the first Bernoulli number B_1. Computing up to fourth order in t, we find

$$\begin{aligned}\Omega\big(\alpha(y,t)\big) = ty &- \frac{t^2}{2} y \rhd y + \frac{t^3}{6}\big((y \rhd y) \rhd y + y \rhd (y \rhd y)\big) + \frac{t^3}{12}[\![y \rhd y, y]\!] \\ &- \frac{t^4}{24}\Big(\big((y \rhd y) \rhd y\big) \rhd y + \big(y \rhd (y \rhd y)\big) \rhd y + 2(y \rhd y) \rhd (y \rhd y) \\ &\qquad + y \rhd \big((y \rhd y) \rhd y\big) + y \rhd \big(y \rhd (y \rhd y)\big)\Big) \\ &+ \frac{t^4}{24}[\![y, (y \rhd y) \rhd y + y \rhd (y \rhd y)]\!] + \cdots.\end{aligned}$$

This should be compared with the terms in (2.20).

2.5. Gavrilov's K-map

We recall here A. V. Gavrilov's K-map [14,18]. We give an explicit formula for its inverse in terms of set partitions, and we show the link with noncommutative Bell polynomials [8, 22, 32]. Finally, we recall the differential equation satisfied by the generating series [14, Lemma 2]

$$K\big(\exp^{\cdot}(ty)\big) = \sum_{n\geq 0} \frac{t^n}{n!} K(y^{\cdot n}).$$

As a consequence, its logarithm can be expressed in terms of the Magnus expansion (in its right-sided version) applied to Gavrilov's α-map [14, Section 5].

2.5.1. The K-map and the D-algebra structure. Let $(M, \triangleright)$ be any magmatic algebra, and let $T(M)$ be the tensor algebra over M, endowed with the D-algebra structure of Section 2.2.1. For any element $y \in M$, let $\tau_y^{\triangleright}$ be the linear endomorphism of $T(M)$ defined by $\tau_y^{\triangleright}(w) = y \triangleright w$ for any $w \in T(M)$. By (2.5), it is a derivation. The map

$$K : T(M) \to T(M)$$

is recursively defined by $K(1) = 1$, $K(y) = y$ for any $y \in M$, and

$$K(yU) = yK(U) - K \circ \tau_y^{\triangleright}(U) \tag{2.22}$$

for any $y \in M$ and $U \in T(M)$. In particular,

$$K(y_1 y_2) = y_1 y_2 - y_1 \triangleright y_2$$

and

$$\begin{aligned} K(y_1 y_2 y_3) &= y_1 K(y_2 y_3) - K \circ \tau_{y_1}^{\triangleright}(y_2 y_3) \\ &= y_1 y_2 y_3 - y_1(y_2 \triangleright y_3) - (y_1 \triangleright y_2) y_3 \\ &\quad - y_2(y_1 \triangleright y_3) + y_2 \triangleright (y_1 \triangleright y_3) + (y_1 \triangleright y_2) \triangleright y_3. \end{aligned}$$

The map K is clearly invertible, as $K(U) - U$ is a linear combination of terms of strictly smaller length than the length of $U \in T(M)$. The inverse K^{-1} is uniquely determined by $K^{-1}(1) = 1$, $K^{-1}(y) = y$ for any $y \in M$, and the recursive relation

$$(L_y + \tau_y^{\triangleright}) \circ K^{-1} = K^{-1} \circ L_y \tag{2.23}$$

for any $y \in M$, where $L_y : T(M) \to T(M)$ is defined by $L_y(U) := yU$. Recall the Grossman–Larson product (2.9) on $T(M)$ given in Definition 3.

Theorem 3. *The K-map is a unital algebra isomorphism from $T(M)$ equipped with the Grossman–Larson product, $(T(M), *)$, onto the tensor algebra $(T(M), \cdot)$.*

Proof. Recall that for $U, V \in T(M)$, we have $U * V := U_{(1)}(U_{(2)} \triangleright V)$. We prove

$$K(U * V) = K(U) \cdot K(V)$$

by induction on the length of the tensor $U \in T(M)$. The length zero case is trivial, and the length one case is given by (2.22). Now, let us compute, with $x \in M$, using the induction hypothesis:

$$\begin{aligned} K\big((xU) * V\big) &= K(x * U * V) - K\big((x \rhd U) * V\big) \\ &= K(x) \cdot K(U * V) - K(x \rhd U) \cdot K(V) \\ &= \big(xK(U) - K(x \rhd U)\big) \cdot K(V) \\ &= K(xU) \cdot K(V). \end{aligned}$$

∎

2.5.2. An explicit formula for K^{-1} in terms of set partitions. An explicit formula for $K^{-1}(y_1 \cdots y_n)$ is available in terms of set partitions of the strictly ordered set $[n] := \{1, \ldots, n\}$: for any such partition π, let us denote its blocks by $B_1, \ldots, B_{|\pi|}$, where we have ordered them according to their maximum:

$$\max B_1 < \cdots < \max B_{|\pi|}.$$

For any block B, say, of size ℓ, define the element $y_B \in M$ by

$$y_B := y_{b_1} \rhd \big(y_{b_2} \rhd \big(\cdots \rhd (y_{b_{\ell-1}} \rhd y_{b_\ell}) \cdots\big)\big),$$

where the elements $b_1 < \cdots < b_\ell$ of B are arranged in increasing order. For any partition π, let $(y_1 \cdots y_n)^\pi \in T(M)$ be the element given by

$$(y_1 \cdots y_n)^\pi := y_{B_1} \cdots y_{B_{|\pi|}}.$$

Proposition 6.

$$K^{-1}(y_1 \cdots y_n) = \sum_{\substack{\pi \text{ set partition} \\ \text{of } \{1,\ldots,n\}}} (y_1 \cdots y_n)^\pi.$$

Proof. We prove this result by induction on the length n. The cases $n = 0$ and $n = 1$ are trivial, and the cases $n = 2$ and $n = 3$ read

$$\begin{aligned} K^{-1}(y_1 y_2) &= y_1 y_2 + y_1 \rhd y_2, \\ K^{-1}(y_1 y_2 y_3) &= y_1 y_2 y_3 + y_1(y_2 \rhd y_3) + y_2(y_1 \rhd y_3) \\ &\quad + (y_1 \rhd y_2) y_3 + y_1 \rhd (y_2 \rhd y_3). \end{aligned}$$

In the case of $n - 2$, we have a sum of two terms which correspond to the two partitions of the set $\{1, 2\}$, one with two blocks and one with two single blocks, respectively. The case $n = 3$ includes all set partitions of order three. Supposing the result is true up to length n, we have, using (2.23),

$$\begin{aligned} K^{-1}(y_0 y_1 \cdots y_n) &= y_0 K^{-1}(y_1 \cdots y_n) + y_0 \rhd K^{-1}(y_1 \cdots y_n) \\ &= \sum_{\substack{\pi \text{ set partition} \\ \text{of } \{1,\ldots,n\}}} y_0 (y_1 \cdots y_n)^\pi + y_0 \rhd (y_1 \cdots y_n)^\pi \end{aligned}$$

$$= \sum_{\substack{\rho \text{ set partition} \\ \text{of } \{0,\ldots,n\},\, B_1=\{0\}}} (y_0 y_1 \cdots y_n)^\rho$$

$$+ \sum_{\substack{\pi \text{ set partition} \\ \text{of } \{1,\ldots,n\}}} \sum_{j=0}^{|\pi|} y_{B_1} \cdots y_{B_{j-1}} y_{B_j \sqcup \{0\}} y_{B_{j+1}} \cdots y_{B_{|\pi|}}$$

$$= \sum_{\substack{\rho \text{ set partition} \\ \text{of } \{0,\ldots,n\},\, B_1=\{0\}}} (y_0 y_1 \cdots y_n)^\rho + \sum_{\substack{\rho \text{ set partition} \\ \text{of } \{0,\ldots,n\},\, B_1\neq\{0\}}} (y_0 y_1 \cdots y_n)^\rho$$

$$= \sum_{\substack{\rho \text{ set partition} \\ \text{of } \{0,\ldots,n\}}} (y_0 y_1 \cdots y_n)^\rho. \qquad \blacksquare$$

Corollary 1. *Gavrilov's K-map is recursively given by*

$$K(y_1 \cdots y_n) \overset{(2.13)}{=} y_1 K(y_2 \cdots y_n) - \sum_{i=2}^{n} K\big(y_2 \cdots (y_1 \rhd y_i) \cdots y_n\big) \tag{2.24}$$

$$= y_1 \cdots y_n - \sum_{\substack{\pi \text{ set partition} \\ \text{of } \{1,\ldots,n\},\, \pi\neq\hat{0}}} K\big((y_1 \cdots y_n)^\pi\big), \tag{2.25}$$

where $\hat{0}$ stands for the unique partition of $\{1,\ldots,n\}$ with n blocks.

In other words, from Theorem 3 and Proposition 6, we have

$$y_1 * \cdots * y_n = y_1 \cdots y_n + \sum_{\substack{\pi \text{ set partition} \\ \text{of } \{1,\ldots,n\},\, \pi\neq\hat{0}}} (y_1 \cdots y_n)^\pi,$$

which is in line with Remark 5.

Extending the derivation $\tau^\rhd : M \to \mathrm{Der}(T(M))$ to an algebra homomorphism

$$\hat{\tau}^\rhd : T(M) \to \mathrm{End}_{\mathbf{k}}\big(T(M)\big),$$

and using (2.13) and (2.14) (we replace here $\hat{L}^\rhd$ with $\hat{\tau}^\rhd$), together with (2.24), we can deduce a particular formula for the GL-product which already appeared in Gavrilov [16, p. 1003]:

$$A * B = A_{(1)}(A_{(2)} \rhd B) = A_{(1)} \hat{\tau}^\rhd_{K(A_{(2)})} B.$$

Remark 8. Let us consider the terms $b_n := K^{-1}(y^n)$ for some $y \in M$. We can see these expressions as elements of the free magmatic algebra M_y generated by y. From (2.23), we have

$$b_n = (L_y + \delta_y) b_{n-1}.$$

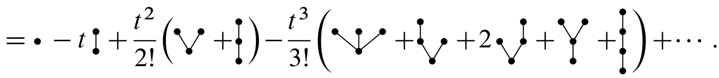
Hence, the b_n's are the noncommutative Bell polynomials ([32], see also [8, 22]).

2.5.3. Gavrilov's α and λ maps and the logarithm of $K(\exp^{\cdot}(ty))$. The recursive expression (2.25) does not deliver easily a closed formula. Gavrilov tackled the problem from another angle in [14], by solving a first-order linear differential equation verified by the generating function $t \mapsto K(\exp^{\cdot}(ty)) \in T(M_y)[\![t]\!]$. Namely, following reference [14, Lemma 2], we have

$$\frac{d}{dt} K\big(\exp^{\cdot}(ty)\big) = K\big(\exp^{\cdot}(ty)\big) \cdot \alpha(y,t). \tag{2.26}$$

Here, $\alpha = \alpha(y,t) \in M_y[\![t]\!]$ solves the initial value problem (2.16) [14, Lemma 1]. The first terms of the series $\alpha(y,t)$ are given by

$$\begin{aligned}
\alpha(y,t) &= e^{-t\delta_y} y \\
&= y - ty \rhd y + \frac{t^2}{2}\big((y \rhd y) \rhd y + y \rhd (y \rhd y)\big) \\
&\quad - \frac{t^3}{3!}\Big(\big((y \rhd y) \rhd y\big) \rhd y + \big(y \rhd (y \rhd y)\big) \rhd y + 2(y \rhd y) \rhd (y \rhd y) \\
&\qquad\qquad + y \rhd \big((y \rhd y) \rhd y\big) + y \rhd \big(y \rhd (y \rhd y)\big)\Big) + \cdots
\end{aligned}$$

 (2.27)

In the last equality, we have represented the monomials in the free magmatic algebra $(M_y, \rhd)$ as planar binary trees, the magmatic product $\rhd$ being the right Butcher product $\diamond$ here. In view of (2.10), the derivation δ_y is given by the left grafting $y \curvearrowright -$. We encounter a planar version of the so-called Connes–Moscovici coefficients in front of the trees, which can be interpreted as the number of possible levelings of the corresponding planar binary tree obtained by inverse (right) Knuth rotation [10]. The ordinary (non-planar) Connes–Moscovici coefficient of a rooted tree is obtained by summing up the coefficients of its planar representatives. Introducing one more generator z, the element

$$\lambda(ty,z) := e^{-t\delta_y} z \in T(M_{y,z})[\![t]\!] \tag{2.28}$$

satisfies the differential equation [14, Lemma 3]

$$\frac{d}{dt}\lambda(ty,z) = -\delta_y(e^{-t\delta_y} z) = -e^{-t\delta_y}(y \rhd z) = -\alpha(y,t) \rhd \lambda(ty,z).$$

Remark 9. Going back to (2.18), we deduce from (2.27) and (2.28), using Proposition 2,

$$\begin{aligned}
\alpha(y,t) &= e^{-t\delta_y} y = \exp^{*}\big(-\chi(ty)\big) \rhd y, \\
\lambda(ty,z) &= e^{-t\delta_y} z = \exp^{*}\big(-\chi(ty)\big) \rhd z.
\end{aligned} \tag{2.29}$$

The solution of (2.26) is given by the exponential of the Magnus expansion [23] (in its right-sided version):

$$K\big(\exp^{\cdot}(ty)\big) = \exp^{\cdot}\left(\int_0^t \dot{\Omega}\big[\alpha(y,-)\big](s)\,ds\right),$$

where $\dot{\Omega} = \dot{\Omega}[A]$ is implicitly defined for any series $A = A[t]$ in the indeterminate t by

$$\frac{d}{dt}\Omega[A](t) = \frac{\mathrm{ad}_{\Omega[A]}}{1 - e^{-\mathrm{ad}_{\Omega[A]}}} A(t) = \sum_{n=0}^{\infty} \frac{(-1)^n B_n}{n!} \mathrm{ad}^n_{\Omega[A]} A(t). \tag{2.30}$$

As before, B_n are the modified Bernoulli numbers. The integration of equation (2.30) leads to an infinite series for $\Omega[A]$, the first terms of which are

$$\Omega[A](t) = \int_0^t A(s_1)ds_1 + \frac{1}{2}\int_0^t \left[\int_0^{s_1} A(s_2)ds_2, A(s_1)\right]ds_1 + \cdots .$$

Indeed, we have

$$\dot{\Omega}[sA] = \sum_{r\geq 1} s^r \widetilde{A}_r,$$

where s is an indeterminate, with $\widetilde{A}_1 = A$, $\widetilde{A}_2 = \frac{1}{2} A \rhd A$ and the recursive procedure

$$\widetilde{A}_r = \sum_{m\geq 0} \frac{(-1)^m B_m}{m!} \sum_{r_1+\cdots+r_m=r-1} \widetilde{A}_{r_1} \rhd \big(\widetilde{A}_{r_2} \rhd \big(\cdots \rhd (\widetilde{A}_{r_m} \rhd A)\cdots\big)\big).$$

The binary operation $\rhd$ is the pre-Lie product defined by

$$(A \rhd B)(t) := \left[\int_0^t A(s)\,ds,\ B(t)\right].$$

For $A(t) := \alpha(y,t) = \sum_{\ell\geq 0} \alpha_\ell(y)t^\ell$, we therefore get

$$Z(ty) := \log^{\cdot} K\big(\exp^{\cdot}(ty)\big) = \sum_{n\geq 1} Z_n(y)t^n,$$

where the coefficients $Z_n(y)$ are recursively given by $Z_1(y) = y$ and

$$(n+1)Z_{n+1}(y) = \sum_{m\geq 0} \frac{(-1)^m B_m}{m!} \sum_{\substack{a_1+\cdots+a_m+\ell=n,\\ a_j\geq 1,\ell\geq 0}} \mathrm{ad}_{Z_{a_1}(y)} \circ \cdots \circ \mathrm{ad}_{Z_{a_m}(y)}\big(\alpha_\ell(y)\big).$$

It turns out that the series $Z = Z(ty)$ is closely related to the post-Lie Magnus expansion $\chi = \chi(ty)$. Indeed, the definition of χ, recalled in Section 2.4, and Theorem 3, which states that the K-map is a unital algebra isomorphism from $(T(M), *)$ to $(T(M), \cdot)$, together with identity (2.15) yield

$$\begin{aligned} Z(ty) &= \log^{\cdot} K\big(\exp^{\cdot}(ty)\big) \\ &= \log^{\cdot} K\big(\exp^{*}\big(\chi(ty)\big)\big) \\ &= \log^{\cdot} \exp^{\cdot}\big(K\big(\chi(ty)\big)\big) \\ &= K\big(\chi(ty)\big). \end{aligned} \tag{2.31}$$

Safely dropping the indeterminate t, the map $Z := K \circ \chi$ makes sense as a map from $\widehat{\mathrm{Lie}(M)}$ into itself, where M is any graded magmatic algebra and where the hat stands for completion.

From (2.20), we therefore get

$$Z(ty) = ty - \frac{t^2}{2} y \rhd y + \frac{t^3}{6} y \rhd (y \rhd y) + \frac{t^3}{6} (y \rhd y) \rhd y + \frac{t^3}{12} [y \rhd y, y] + \cdots .$$

Here, we use $K(\llbracket y \rhd y, y \rrbracket) = [y \rhd y, y]$. The series Z in turn gives rise to Gavrilov's β-map [14, Lemma 6]. We will return to this point in Section 3 below.

3. Framed Lie algebras and Gavrilov's β map

We now recall Gavrilov's notion of framed Lie algebra [14].

Definition 4 ([14]). A framed Lie algebra is a triple $(\mathfrak{l}, \rhd, [\![. , .]\!])$ where $\rhd$ is any bilinear product on the vector space $\mathfrak{l}$, and where $[\![. , .]\!]$ is a Lie bracket on $\mathfrak{l}$.

No compatibility relations of any sort are requested between the magmatic product $\rhd$ and the Lie bracket, which we denote in bold, $[\![. , .]\!]$, to stress the distinction with the two Lie brackets, $[\cdot, \cdot]$ and $\llbracket \cdot, \cdot \rrbracket$, of a post-Lie algebra. An obvious example is given by the Lie algebra of vector fields, $\mathfrak{l} = \mathcal{XM}$, on a manifold $\mathcal{M}$ endowed with an affine connection ∇. In this case, the magmatic product is given by $X \rhd Y := \nabla_X Y$, and the natural Lie bracket is the usual Jacobi bracket.

The free framed Lie algebra generated by a single element y is denoted by $\mathcal{L}_y$, and its completion with respect to the natural grading is denoted by $\widehat{\mathcal{L}}_y$. One also denotes by $\mathcal{U}(\mathcal{L}_y)$ (resp., $\widehat{\mathcal{U}}(\mathcal{L}_y)$) the enveloping algebra of $(\mathcal{L}_y, [\![. , .]\!])$ (resp., its completion). The canonical projection

$$p : T(\mathcal{L}_y) \to \mathcal{U}(\mathcal{L}_y)$$

readily extends to the completion. Its restriction to the free Lie algebra $\mathrm{Lie}(\mathcal{L}_y)$ generated by $\mathcal{L}_y$ is a Lie algebra morphism from $(\mathrm{Lie}(\mathcal{L}_y), [\cdot, \cdot])$ onto $(\mathcal{L}_y, [\![. , .]\!])$. Gavrilov showed the following lemma.

Lemma 3 ([14, Lemma 6]). *There exists a unique series $\beta = \beta(y)$ in the completion $\widehat{\mathcal{L}}_y$ such that*

$$p \circ K\big(\exp^{\cdot}(y)\big) = \exp\big(\beta(y)\big) \in \widehat{\mathcal{U}}(\mathcal{L}_y).$$

The series $\beta(ty)$ verifies $\beta(0) = 0$ and

$$\frac{d}{dt}\beta(ty) = \frac{\mathrm{ad}_{\beta(ty)}}{1 - e^{-\mathrm{ad}_{\beta(ty)}}}\alpha(y, t).$$

Proof. The series $\beta(ty) = p(Z(ty))$, with Z defined in Section 2.5.3, is a solution to the problem. Uniqueness, as a solution to an initial value problem, follows. ∎

From (2.31), the series $\beta(ty)$ is explicitly given by

$$\beta(ty) = p \circ K \circ \chi(ty). \tag{3.1}$$

Safely dropping the indeterminate t, the map

$$\beta := p \circ K \circ \chi_{|\widehat{\mathcal{L}}_y} \tag{3.2}$$

is a linear endomorphism of the completed framed Lie algebra $\widehat{\mathcal{L}}_y$. We therefore have the following from (2.20):

$$\beta(y) = y - \frac{1}{2} y \rhd y + \frac{1}{6} y \rhd (y \rhd y) + \frac{1}{6}(y \rhd y) \rhd y + \frac{1}{12}[\![y \rhd y, y]\!] + \cdots .$$

Remark 10. Note that (3.2) defines Gavrilov's β-map in the completion of any graded framed Lie algebra. It is obviously bijective due to the fact that β is equal to the identity modulo higher degree terms.

4. Affine connections on manifolds and covariant derivation

4.1. Reminders on connections, torsion, curvature, and Bianchi identities

Let E be a vector bundle on a smooth manifold $\mathcal{M}$, and let $\Gamma(E)$ be the $C^\infty(\mathcal{M})$-module of its smooth sections. An affine connection on E is a bilinear map

$$\begin{aligned} \nabla : \mathcal{X}\mathcal{M} \times \Gamma(E) &\to \Gamma(E) \\ (X, s) &\mapsto \nabla_X s, \end{aligned}$$

subject to the relations

$$\begin{aligned} \nabla_{fX} s &= f \nabla_X s, \\ \nabla_X (fs) &= (X \cdot f)s + f \nabla_X s \end{aligned} \tag{4.1}$$

for any $f \in C^\infty(\mathcal{M})$ and any $s \in \Gamma(E)$. For clarity, we write ∇^E if the vector bundle has to be made precise, and we use the convenient notation

$$X \rhd s := \nabla_X s.$$

This includes the case of the trivial line bundle L, where $\Gamma(L)$ then coincides with $C^\infty(\mathcal{M})$, and the natural connection is given by

$$X \rhd f := X \cdot f = \langle Df, X \rangle.$$

The Leibniz rule (4.1) can therefore be rewritten as follows:

$$X \rhd (fs) = (X \rhd f)s + f(X \rhd s).$$

Given a connection on two vector bundles, E and F, connections on $E \otimes F$ and $\mathrm{Lin}(E, F) = E^* \otimes F$ are naturally given respectively by the Leibniz rules

$$X \rhd (s \otimes s') = (X \rhd s) \otimes s' + s \otimes (X \rhd s'),$$
$$(X \rhd \varphi)(s) = X \rhd \varphi(s) - \varphi(X \rhd s).$$

The curvature of an affine connection is given by

$$r(X, Y)s := X \rhd (Y \rhd s) - Y \rhd (X \rhd s) - [\![X, Y]\!] \rhd s$$

and is also denoted by $R(X, Y, s)$. It is skew symmetric in (X, Y) and C^∞-linear with respect to each of the three arguments. In the case $E = T\mathcal{M}$, the torsion is given by

$$t(X, Y) := X \rhd Y - Y \rhd X - [\![X, Y]\!].$$

This is skew-symmetric, and C^∞-linear with respect to both arguments. The two Bianchi identities are given by

$$\oint_{XYZ} R(X, Y, Z) = \oint_{XYZ} (X \rhd t)(Y, Z) - \oint_{XYZ} t\big(X, t(Y, Z)\big), \tag{4.2}$$

$$\oint_{XYZ} (X \rhd R)(Y, Z, W) = \oint_{XYZ} R\big(X, t(Y, Z), W\big) \tag{4.3}$$

for any $X, Y, Z, W \in \mathcal{X}\mathcal{M}$. Here, the symbol $\oint$ stands for summing over circular permutations of the three arguments. For a detailed account, see e.g. [20].

4.2. Higher-order covariant derivatives

We keep the notations from the previous paragraph and consider the post-Lie algebra associated with a manifold with connection. Let $\mathcal{R} := C^\infty(\mathcal{M})$. The $\mathcal{R}$-module $\mathcal{X}\mathcal{M} = \mathrm{Der}(\mathcal{R})$ is denoted by $\mathcal{V}$. We adopt the notations $X \rhd Y := \nabla_X Y$, for $X, Y \in \mathcal{V}$, and $X \rhd f := X \cdot f$, for $f \in \mathcal{R}$. Let $\mathcal{D}\mathcal{M}$ be the algebra of differential operators on $\mathcal{M}$, which is the subalgebra of linear operators on $C^\infty(\mathcal{M})$ generated by the vector fields and the multiplication operators $\mu_f : g \mapsto fg$.

Let $A(T_m\mathcal{M})$ and $\mathrm{Lie}(T_m\mathcal{M})$ be the free unital associative algebra (i.e. the tensor algebra) and the free Lie algebra both defined over the tangent space at any point $m \in \mathcal{M}$. Each of these free algebras put together form respectively the free unital algebra bundle $A_\mathcal{M}$ and the free Lie algebra bundle $\mathrm{Lie}_\mathcal{M}$. The free $\mathcal{R}$-associative unital algebra $\mathcal{A} = T_\mathcal{R}(\mathcal{V})$ on $\mathcal{V}$ is the $\mathcal{R}$-module of sections of $A_\mathcal{M}$. We clearly have

$$\mathcal{A} = T_\mathbb{R}(\mathcal{V})/\mathcal{C},$$

where $\mathcal{C}$ is the two-sided ideal generated by the elements $fX \cdot Y - X \cdot fY$ with $f \in C^\infty(\mathcal{M})$ and $X, Y \in \mathcal{V}$. We denote by π the natural projection from $T_\mathbb{R}(\mathcal{V})$ onto $\mathcal{A}$ (let us recall that $T_\mathbb{R}(\mathcal{V})$ is the free D-algebra generated by the magmatic algebra $(\mathcal{V}, \rhd)$).

Similarly, the $\mathcal{R}$-module $\mathfrak{g}$ of sections of $\mathrm{Lie}_{\mathcal{M}}$ is the free $\mathcal{R}$-Lie algebra $\mathrm{Lie}_{\mathcal{R}}(\mathcal{V})$ on the vector fields, and we have

$$\mathfrak{g} = \tilde{\mathfrak{g}}/(\mathcal{C} \cap \tilde{\mathfrak{g}}),$$

where $\tilde{\mathfrak{g}} = \mathrm{Lie}_{\mathbb{R}}(\mathcal{V})$ is the free post-Lie algebra generated by $(\mathcal{V}, \rhd)$. The tautological action of $\mathcal{V}$ by derivations on $\mathcal{R}$ is extended to $\tilde{\mathfrak{g}}$ as follows:

$$[X,Y] \rhd f := X \rhd (Y \rhd f) - (X \rhd Y) \rhd f - Y \rhd (X \rhd f) - (Y \rhd X) \rhd f,$$

and similarly, for $X \in \mathcal{V}$ and $U \in T_{\mathbb{R}}(\mathcal{V})$,

$$(X \cdot U) \rhd f := X \rhd (U \rhd f) - (X \rhd U) \rhd f.$$

These rules, which define the action recursively with respect to the degree, are adapted from the second post-Lie axiom and the second D-algebra axiom, respectively.

Theorem 4. *The map $\rhd : T_{\mathbb{R}}(\mathcal{V}) \times \mathcal{R} \to \mathcal{R}$ defined above factorizes into a map*

$$\rhd : \mathcal{A} \times \mathcal{R} \to \mathcal{R}.$$

In other words, $\mathrm{Id}_{\mathcal{V}}$ *extends to a surjective $\mathcal{R}$-linear morphism*

$$\rho : \mathcal{A} \to \mathcal{DM}.$$

It restricts to

$$\rho : \mathfrak{g} \to \mathcal{V}.$$

Proof. It suffices to prove that

$$\rho(x_1 \cdots x_n) = (x_1 \cdots x_n) \rhd y$$

is $C^\infty(\mathcal{M})$-linear in each argument $x_i \in \mathcal{XM}$. This is obvious for $n = 1$ and proven by induction for $n \geq 2$ using (2.5) and (2.6):

$$\begin{aligned}
&(f_1 x_1 \cdots f_n x_n) \rhd y \\
&\quad = f_1 x_1 \rhd \big((f_2 x_2 \cdots f_n x_n) \rhd y\big) - \big(f_1 x_1 \rhd (f_2 x_2 \cdots f_n x_n)\big) \rhd y \\
&\quad = f_1 x_1 \rhd \big(f_2 \cdots f_n (x_2 \cdots x_n) \rhd y\big) - \big(f_1 x_1 \rhd (f_2 x_2 \cdots f_n x_n)\big) \rhd y \\
&\quad = f_1 \cdots f_n x_1 \rhd (x_2 \cdots x_n \rhd y) + \big(f_1 x_1 \rhd (f_2 \cdots f_n)\big)(x_2 \cdots x_n \rhd y) \\
&\qquad - f_1 \sum_{i=2}^{n} \big(f_2 x_2 \cdots \big((x_1 \rhd f_i) x_i + f_i (x_1 \rhd x_i)\big) \cdots x_n\big) \rhd y \\
&\quad = f_1 \cdots f_n \big(x_1 \rhd \big((x_2 \cdots x_n) \rhd y\big) - \big(x_1 \rhd (x_2 \cdots x_n)\big) \rhd y\big).
\end{aligned}$$

■

Remark 11. The previous formalism can be generalized to any vector bundle E endowed with a connection ∇^E, replacing $\mathcal{XM}$ by the $C^\infty(\mathcal{M})$-module $\Gamma(E)$ of smooth sections of E. The higher-order covariant derivatives are also recursively defined by (2.6), and Theorem 4 holds for the action $\rhd : T_{\mathbb{R}}(V) \times \Gamma(E) \to \Gamma(E)$.

We note that several notations are used in the literature for higher covariant derivatives; namely [16],

$$\begin{aligned}\rho(x_1\cdots x_n)y &= (x_1\cdots x_n)\rhd y\\ &= \nabla^n_{x_1\cdots x_n}y\\ &= (x_1\cdots x_n)\nabla^n y\\ &= \nabla^n y(x_n;\cdots;x_1).\end{aligned}$$

Proposition 7. $\mathfrak{g}$ *(resp.,* $\mathcal{A}$*) is a post-Lie algebra (resp., a* D*-algebra), and* $\mathfrak{g}=\mathfrak{d}(\mathcal{A})$.

Proof. It is sufficient to show that the Lie ideal $\mathcal{C}' := \mathcal{C}\cap\tilde{\mathfrak{g}}$, generated by the elements $[fA,B]-[A,fB]$, $f\in\mathcal{R}$, $A,B\in\tilde{\mathfrak{g}}$, is also an ideal for the product $\rhd$ extended to $\tilde{\mathfrak{g}}$. Let $U,A,B\in\mathfrak{g}$ and $f\in\mathcal{R}$. From the straightforward computation

$$\begin{aligned}U\rhd\big([fA,B]-[A,fB]\big) = \big[(U\rhd f)A,B\big]+\big[f(U\rhd A),B\big]+[fA,U\rhd B]\\ -[U\rhd A,fB]-\big[A,(U\rhd f)B\big]-\big[A,f(U\rhd B)\big],\end{aligned}$$

we get $U\rhd\mathcal{C}'\subseteq\mathcal{C}'$ for any $U\in\mathfrak{g}$. From

$$\begin{aligned}&\big([fA,B]-[A,fB]\big)\rhd U\\ &\quad= fA\rhd(B\rhd U)-(fA\rhd B)\rhd U-B\rhd(fA\rhd U)\\ &\qquad+(B\rhd fA)\rhd U-A\rhd(fB\rhd U)+(A\rhd fB)\rhd U\\ &\qquad+fB\rhd(A\rhd U)-(fB\rhd A)\rhd U\\ &\quad= -(A\rhd f)B\rhd U+(A\rhd f)B\rhd U-(B\rhd f)A\rhd U+(B\rhd f)A\rhd U\\ &\quad= 0,\end{aligned}$$

we get $\mathcal{C}'\rhd U\subseteq\mathcal{C}'$ for any $U\in\mathfrak{g}$. The proof of the fact that $\mathcal{A}$ is a D-algebra is similar, and the last assertion is clear. ∎

Let us recall two more Lie brackets at hand:

- The usual Jacobi bracket $[\![.,.]\!]$ on $\mathcal{V}$, defined by

  $$[\![X,Y]\!]\rhd f = X\rhd(Y\rhd f)-Y\rhd(X\rhd f)$$

  for any $X,Y\in\mathfrak{g}$ and any $f\in\mathcal{R}$.
- The Grossman–Larson bracket $[\![\cdot,\cdot]\!]$ on $\mathfrak{g}$, which satisfies for any $Z\in\mathfrak{g}$

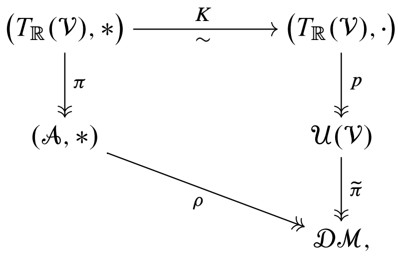
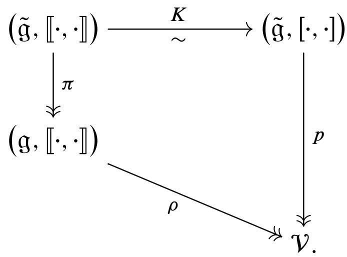
  $$[\![X,Y]\!]\rhd Z = X\rhd(Y\rhd Z)-Y\rhd(X\rhd Z).$$

From the post-Lie algebra identity $[\![X,Y]\!]=[X,Y]+X\rhd Y-Y\rhd X$, for $X,Y\in\mathfrak{g}$, we get

$$\rho[\![X,Y]\!]=\rho(X)\circ\rho(Y)-\rho(Y)\circ\rho(X)=[\![\rho(X),\rho(Y)]\!]. \tag{4.4}$$

Hence, $(\mathrm{Lie}_{\mathcal{M}}, [\![\cdot,\cdot]\!])$ is a Lie algebroid on $\mathcal{M}$ with anchor map ρ. We refer the reader to [29] which discusses in detail the fact that post-Lie algebroids are action algebroids.

The canonical projection $p : T_{\mathbb{R}}(\mathcal{V}) \to \mathcal{U}(\mathcal{V})$ restricts to $p : \tilde{\mathfrak{g}} \to \mathcal{V}$. The two following diagrams commute (see also [16, Lemma 2]):

Here, the map $\tilde{\pi}$ stands for the natural projector from the universal enveloping algebra $\mathcal{U}(\mathcal{V}, [\![.,.]\!])$ onto $\mathcal{DM}$, and, in view of Theorem 3, all arrows are algebra (resp., Lie algebra) morphisms.

Remark 12. Gavrilov's β-map is a bijection from $\overline{\mathcal{V}}$ into itself, where $\overline{\mathcal{V}}$ is the complete filtered framed Lie algebra obtained from $\mathcal{V} = \mathcal{XM}$ by extending the scalars from real numbers $\mathbb{R}$ to power series without constant terms, in one or several indeterminates; e.g., $\overline{\mathcal{V}} = t\mathcal{V}[\![t]\!]$ or $\overline{\mathcal{V}} = t\mathcal{V}[\![t,s]\!] + s\mathcal{V}[\![t,s]\!]$. On the other hand, both graded post-Lie algebras, $\tilde{\mathfrak{g}}$ and $\mathfrak{g}$, come with their own post-Lie Magnus expansions, $\tilde{\chi}$ and χ. Using the natural extensions of $\tilde{\chi}$, χ and the projection π to the associated completed versions of $\tilde{\mathfrak{g}}$ and $\mathfrak{g}$, we have

$$\pi \circ \tilde{\chi} = \chi \circ \pi.$$

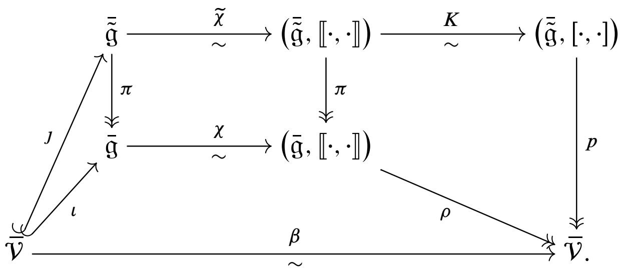
In view of (3.1), we therefore have

$$\beta = p \circ K \circ \tilde{\chi} \circ \jmath = \rho \circ \pi \circ \tilde{\chi} \circ \jmath = \rho \circ \chi \circ \iota$$

where $\jmath$ (resp., ι) is the natural injection of $\overline{\mathcal{V}}$ into $\bar{\tilde{\mathfrak{g}}}$ (resp., $\bar{\mathfrak{g}}$). Hence,

$$\beta = \rho \circ \chi_{|\overline{\mathcal{V}}}. \tag{4.5}$$

The situation can be summarized by the following commutative diagram:



4.3. Differential operators

It is easily seen that we can identify the algebra $\mathcal{DM}$ of differential operators with $\mathcal{U}(\mathcal{V})/\mathcal{I}$, where $\mathcal{I}$ is the two-sided ideal generated by the elements $X(fv) - (fX)v - (X \rhd f)v$ for $f \in C^\infty(\mathcal{M})$, $X \in \mathcal{V}$ and $v \in \mathcal{U}(\mathcal{V})$.

Now, we develop Example 2 outlined above. Supposing that the connection is flat with constant torsion, the D-algebra structure on $\mathcal{DM}$ will immediately arise in view of the following result.

Lemma 4. *When the affine connection ∇ on $\mathcal{M}$ is flat with constant torsion, $(\mathcal{V}, \rhd, [\![.,.]\!])$ is a post-Lie algebra, and both projections $T(\mathcal{XM}) \overset{p}{\twoheadrightarrow} \mathcal{U}(\mathcal{XM}) \overset{\tilde{\pi}}{\twoheadrightarrow} \mathcal{DM}$ are D-algebra morphisms.*

Proof. It suffices to show that the ideal $\mathcal{I}$ is also a two-sided ideal with respect to the magmatic product $\rhd$. The proof, left to the reader, uses the cocommutativity of the coproduct on $\mathcal{U}(\mathcal{XM})$ and proceeds similarly to the one of Proposition 5. ■

5. Special polynomials

We now study Gavrilov's special polynomials [14] from the post-Lie viewpoint.

5.1. Torsion and curvature revisited in the post-Lie framework

We use the notations introduced at the beginning of Section 4.

Definition 5. The torsion of two elements $a, b \in \mathcal{V}$ is defined by

$$t(a \cdot b) := a \rhd b - b \rhd a - [\![a, b]\!].$$

The curvature of three elements $a, b, c \in \mathcal{V}$ is defined by

$$r(a \cdot b)(c) := a \rhd (b \rhd c) - b \rhd (a \rhd c) - [\![a, b]\!] \rhd c.$$

This is sometimes denoted by $R(a \cdot b \cdot c)$. One can show that $t \in \mathrm{Lin}_{\mathcal{R}}(\mathcal{V} \otimes_{\mathcal{R}} \mathcal{V}, \mathcal{V})$ (a tensor of type $(2,1)$) and that $R \in \mathrm{Lin}_{\mathcal{R}}(\mathcal{V} \otimes_{\mathcal{R}} \mathcal{V} \otimes_{\mathcal{R}} \mathcal{V}, \mathcal{V})$ (a tensor of type $(3,1)$). We can rewrite the curvature in our post-Lie framework as follows:

$$r(a \cdot b)(c) = s(a \cdot b) \rhd c. \tag{5.1}$$

On the right-hand side of (5.1), we identify a new element.

Definition 6. The *curvature element* $s(a \cdot b) \in \mathfrak{g}$ is defined by

$$s(a \cdot b) := [\![a, b]\!] - [\![a, b]\!] = [a, b] + t(a \cdot b). \tag{5.2}$$

In turn, $s(a \cdot b)$ permits to express the torsion in terms of the three Lie brackets involved:

$$t(a \cdot b) = [\![a, b]\!] - [a, b] - [\![a, b]\!].$$

5.2. The ideals $\mathcal{J}$ and $\mathcal{K}$

Proposition 8. *Let* $\mathcal{J} := \operatorname{Ker}\rho \subset \mathfrak{g}$*, and let* $\tilde{\mathcal{J}}$ *be the ideal of* $\mathfrak{g}$*, for the Grossman–Larson bracket (GL-bracket), generated by the curvature elements* $s(a \cdot b)$*,* $a, b \in \mathcal{V}$*. Then,*

$$\tilde{\mathcal{J}} = \mathcal{J}.$$

We also have the decomposition

$$\mathfrak{g} = \mathcal{V} \oplus \mathcal{J}. \tag{5.3}$$

Proof. The direct sum decomposition (5.3) is immediate in view of $\mathcal{V} = \rho(\mathfrak{g})$ and $\mathcal{J} = (I - \rho)(\mathfrak{g})$. From (4.4) and (5.2), we immediately get the inclusion $\tilde{\mathcal{J}} \subseteq \mathcal{J}$, as well as the fact that $\mathcal{J}$ is an ideal for the GL-bracket.

Conversely, we use the grading $\mathfrak{g} = \oplus_{n \geq 0} \mathfrak{g}_n$ by the length of the iterated brackets. Looking at the definition of the GL-bracket, it is easy to show that $\mathfrak{g}^{(m)} := \oplus_{i=1}^{m} \mathfrak{g}_i$ is the $\mathcal{R}$-linear span of iterated GL-brackets of length smaller than or equal to m. By using Jacobi identity as many times as necessary, any such iterated bracket can be rewritten as a sum

$$\sum_i [\![a_i, v_i]\!]$$

with $a_i \in \mathcal{V}$ and $v_i \in \mathfrak{g}^{(n-1)}$. Suppose now that any element of $\mathfrak{g}^{(n-1)} \cap \mathcal{J}$ is in $\mathfrak{g}^{(n-1)} \cap \tilde{\mathcal{J}}$. This is trivial for $n - 1 = 1$ and clear for $n - 1 = 2$ in view of (5.2). Considering any element $u = \sum_i [\![a_i, v_i]\!] \in \mathfrak{g}^{(n)}$, we have

$$\begin{aligned}
u - \rho(u) &= \sum_i \big([\![a_i, v_i]\!] - [\![a_i, \rho(v_i)]\!]\big) \\
&= \sum_i [\![a_i, v_i - \rho(v_i)]\!] + \sum_i \big([\![a_i, \rho(v_i)]\!] - \mathbf{[}a_i, \rho(v_i)\mathbf{]}\big) \\
&= \sum_i [\![a_i, v_i - \rho(v_i)]\!] + \sum_i s\big(a_i \cdot \rho(v_i)\big),
\end{aligned}$$

which proves $\mathcal{J} \subseteq \tilde{\mathcal{J}}$ by induction on n. ∎

Proposition 9. *For any* $u \in \mathfrak{g}$ *and* $b, c, d \in \mathcal{V}$*, we have*

$$\begin{aligned}
\big(u \triangleright r(b \cdot c)\big)(d) &= [\![u, s(b \cdot c)]\!] \triangleright d, \\
\big((u \triangleright r)(b \cdot c)\big)(d) &= \Big([\![u, s(b \cdot c)]\!] - s\big((u \triangleright b) \cdot c\big) - s\big(b \cdot (u \triangleright c)\big)\Big) \triangleright d.
\end{aligned}$$

Proof. From the Leibniz rule, we have

$$\begin{aligned}
u \triangleright \big(s(b \cdot c) \triangleright d\big) &= u \triangleright \big(r(b \cdot c)(d)\big) \\
&= \big(u \triangleright r(b \cdot c)\big)(d) + r(b \cdot c)(u \triangleright d) \\
&= \big(u \triangleright r(b \cdot c)\big)(d) + s(b \cdot c) \triangleright (u \triangleright d).
\end{aligned} \tag{5.4}$$

Now, we have

$$u \rhd \big(s(b\cdot c) \rhd d\big) - s(b\cdot c) \rhd (u \rhd d) = [\![u, s(b\cdot c)]\!] \rhd d. \tag{5.5}$$

From (5.4) and (5.5), we get

$$\big(u \rhd r(b\cdot c)\big)(d) = [\![u, s(b\cdot c)]\!] \rhd d,$$

which proves the first assertion. The second one comes from the first together with the Leibniz rule

$$\begin{aligned}(u \rhd r)(b\cdot c)(d) &= \big(u \rhd r(b\cdot c)\big)(d) - r\big((u \rhd b)\cdot c\big)(d) - r\big(b\cdot (u \rhd c)\big)(d)\\ &= \big(u \rhd r(b\cdot c)\big)(d) - s\big((u \rhd b)\cdot c\big) \rhd d - s\big(b\cdot (u \rhd c)\big) \rhd d. \qquad \blacksquare\end{aligned}$$

Now, let us consider the map

$$\begin{aligned}\Phi : \mathfrak{g} &\to \mathrm{End}_{\mathbb{R}}(\mathcal{V})\\ u &\mapsto u \rhd -,\end{aligned}$$

and introduce $\mathcal{K} := \mathrm{Ker}\,\Phi$. Using earlier notation, $\Phi(u) = L^{\rhd}_u$.

Proposition 10. *The restriction of* Φ *to* $\mathcal{J}$ *takes its values into* $\mathrm{End}_{\mathcal{R}}(\mathcal{V})$. *Moreover,* $\mathcal{K}$ *is an ideal for the Grossman–Larson bracket, and we have the strict inclusions*

$$\{0\} \subsetneq \mathcal{K} \subsetneq \mathcal{J}.$$

Proof. The first assertion is immediate from the Leibniz rule. From

$$[\![u, v]\!] \rhd a = u \rhd (v \rhd a) - v \rhd (u \rhd a)$$

for any $u, v \in \mathfrak{g}$ and $a \in \mathcal{V}$, we get $\Phi([\![u, v]\!]) = [\Phi(u), \Phi(v)]$ (bracket of operators on $\mathcal{V}$); hence, $\mathcal{K}$ is an ideal for the Grossman–Larson bracket.

Now, for any $f \in \mathcal{R}$, $a \in \mathcal{V}$, and $u \in \mathcal{K}$, we have $u \rhd a = u \rhd fa = 0$, and therefore, $u \rhd f = 0$ by the Leibniz rule; hence, $\mathcal{K} \subset \mathcal{J}$. Moreover, for any $a, b \in \mathcal{V}$, the curvature element $s(a\cdot b)$ belongs to $\mathcal{J}$ but has no reason to belong to $\mathcal{K}$ unless the connection is flat. Finally, from the second identity of Proposition 9 and from the differential Bianchi identity (4.3), the expression

$$\begin{aligned}[\![a, s(b\cdot c)]\!] + [\![b, s(c\cdot a)]\!] + [\![c, s(a\cdot b)]\!] - s\big(a\cdot t(b\cdot c)\big) - s\big(b\cdot t(c\cdot a)\big) - s\big(c\cdot t(a\cdot b)\big)\\ + s\big(a\cdot (b \rhd c - c \rhd b)\big) + s\big(b\cdot (c \rhd a - a \rhd c)\big) + s\big(c\cdot (a \rhd b - b \rhd a)\big)\end{aligned}$$

defines a nontrivial element of $\mathcal{K}$, which can also be rewritten as

$$\begin{aligned}[\![a, s(b\cdot c)]\!] + [\![b, s(c\cdot a)]\!] + [\![c, s(a\cdot b)]\!] + s\big(a\cdot [b, c]\big)\\ + s\big(b\cdot [c, a]\big) + s\big(c\cdot [a, b]\big) \in \mathcal{K}.\end{aligned}$$

$\blacksquare$

5.3. Lie monomials

We denote by $T^r_s(\mathcal{V})$ the space of tensors of type (r,s); namely,

$$T^r_s(\mathcal{V}) := \underbrace{\mathcal{V} \otimes_{\mathcal{R}} \cdots \otimes_{\mathcal{R}} \mathcal{V}}_{r} \otimes_{\mathcal{R}} \underbrace{\mathcal{V}^* \otimes_{\mathcal{R}} \cdots \otimes_{\mathcal{R}} \mathcal{V}^*}_{s},$$

such that $T^1_0(\mathcal{V}) := \mathcal{V}$ and $T^0_0(\mathcal{V}) := \mathcal{R}$.

Definition 7. A Lie monomial of degree n is an $\mathcal{R}$-linear map $\alpha : T^n_0(\mathcal{V}) \to \mathfrak{g}_n$ defined by an iteration of Lie brackets. In particular, it is a tensor of type (n,n).

As an example, consider

$$\alpha(a \cdot b \cdot c \cdot d \cdot e) := \big[[a,b],\big[c,[d,e]\big]\big],$$

which defines a Lie monomial of degree 5. The following statement is straightforward and left to the reader.

Proposition 11. *A degree n Lie monomial $w \mapsto \alpha(w) \in \mathfrak{g}_n$ defines three tensors*

$$\begin{aligned} \alpha &\in T^n_n(\mathcal{V}), \\ -t_\alpha &:= \rho \circ \alpha \in T^1_n(\mathcal{V}), \\ R_\alpha &:= x_1 \cdots x_{n+1} \mapsto \big(\big((I-\rho)\circ\alpha\big)(x_1 \cdots x_n)\big) \triangleright x_{n+1} \in T^1_{n+1}(\mathcal{V}). \end{aligned} \tag{5.6}$$

Definition 8. The tensors t_α and R_α are respectively the generalized torsion and the generalized curvature associated with the Lie monomial α. Note the minus sign in (5.6), so that the definition matches torsion and curvature for $\alpha(a \cdot b) := [a,b]$. For later use, we also define the generalized curvature element

$$s_\alpha := (I-\rho)\circ\alpha,$$

so that

$$R_\alpha(x_1 \cdots x_{n+1}) = s_\alpha(x_1 \cdots x_n) \triangleright x_{n+1}.$$

Let us give the $\mathcal{V} \oplus \mathcal{J}$ decomposition of Lie monomials of low degrees:

- Degree one: $\alpha = \mathrm{Id}_{\mathcal{V}}$ and the $\mathcal{J}$-part is equal to zero.
- Degree two:

$$\alpha(a \cdot b) = [a,b] = \underbrace{s(a \cdot b)}_{\in \mathcal{J}} - \underbrace{t(a \cdot b)}_{\in \mathcal{V}}.$$

  The torsion $t = \rho \circ \alpha$ belongs to $T^1_2(\mathcal{V})$; the curvature $R_\alpha = R$ belongs to $T^1_3(\mathcal{V})$.

- Degree three:

$$\begin{aligned} \alpha(a \cdot b \cdot c) &= \big[[a,b],c\big] \\ &= \underbrace{[\![s(a \cdot b), c]\!] + s\big((c \triangleright a)\cdot b\big) + s\big(a \cdot (c \triangleright b)\big) - s\big(t(a \cdot b)\cdot c\big)}_{\in \mathcal{J}} \\ &\quad \times \underbrace{-s(a \cdot b) \triangleright c + (c \triangleright t)(a \cdot b) + t\big(t(a \cdot b)\cdot c\big)}_{\in \mathcal{V}}. \end{aligned} \tag{5.7}$$

5.4. Special polynomials

We use the language of operads here. Let us recall that a $\mathbb{S}$-module $\mathcal{P}$ is a collection $(\mathcal{P}_n)_{n\geq 0}$ of modules over some base commutative unital ring, together with a right action of the symmetric groupoid $\mathbb{S} = \coprod_{n\geq 0} S_n$, i.e., a right action of the symmetric group S_n on $\mathcal{P}_n$ for each $n \geq 0$. An operad is a $\mathbb{S}$-module $\mathcal{P}$ together with global compositions

$$\gamma : \mathcal{P}_n \otimes \mathcal{P}_{k_1} \otimes \cdots \otimes \mathcal{P}_{k_n} \to \mathcal{P}_{k_1+\cdots+k_n}$$

functorial with respect to symmetric group actions, and subject to associativity and unitality axioms [21, 27]. We denote by $\mathcal{P}(\mathcal{V})$ the operad of $\mathbb{R}$-multilinear maps[3] on $\mathcal{V}$; namely, $\mathcal{P}_n(\mathcal{V}) = \mathrm{Hom}_{\mathbb{R}}(\mathcal{V}^{\otimes n}, \mathcal{V})$. The symmetric groups act on the right by permuting the variables, and the compositions γ are obviously defined. The unit for the composition is $\mathrm{Id}_{\mathcal{V}} \in \mathcal{P}_1(\mathcal{V})$.

A monomial of degree (also named arity) $n \geq 0$ is a nonzero element of $\mathcal{P}_n(\mathcal{V})$. A polynomial is a finite sum of monomials, possibly of different arities.

Definition 9. Let us introduce three particular families of polynomials, defined as follows.

- A geometrically special polynomial [17] is a polynomial ω for which there exists an $\mathcal{R}$-linear map $\widetilde{\omega} : \mathcal{A} \to \mathcal{V}$ such that $\omega = \widetilde{\omega} \circ \pi$, where π is the natural projection from $T_{\mathbb{R}}(\mathcal{V})$ onto $\mathcal{A}$.
- A special polynomial [14] is a polynomial made, by means of iterated compositions, of derivatives of torsion and curvature, possibly permuted.
- A polynomial of Lie type is a polynomial made, by means of iterated compositions, of derivatives of t_α's and R_α's (where α is a Lie monomial), possibly permuted.

The corresponding sets are respectively denoted by $\mathcal{P}_{\mathcal{R}}(\mathcal{V})$, $\mathcal{S}(\mathcal{V})$, and $\mathcal{P}_{\mathrm{Lie}}(\mathcal{V})$. It is obvious from Definition 9 above that those are three suboperads of $\mathcal{P}(\mathcal{V})$. More precisely, $\mathcal{S}(\mathcal{V})$ is the suboperad generated by $\{\nabla^n t, \nabla^n R,\ n \geq 0\}$, and $\mathcal{P}_{\mathrm{Lie}}(\mathcal{V})$ is the suboperad generated by $\{\nabla^n t_\alpha, \nabla^n R_\alpha,\ n \geq 0,\ \alpha \text{ Lie monomial}\}$. For later use, we define an extended version.

Definition 10. An extended geometrically special polynomial [17] is a finite sum of monomials $\omega : \mathcal{V}^{\otimes n} \to \mathcal{DM}$ for which there exists an $\mathcal{R}$-linear map $\widetilde{\omega} : \mathcal{V}_{\mathcal{R}}^{\otimes n} \to \mathcal{DM}$ such that $\omega = \widetilde{\omega} \circ \pi$.

The higher-order covariant derivative $\rho : \mathcal{V}^{\otimes n} \to \mathcal{DM}$ is an example of extended geometrically special monomial.

The inclusion $\mathcal{S}(\mathcal{V}) \subset \mathcal{P}_{\mathcal{R}}(\mathcal{V})$ holds; i.e., any special polynomial is geometrically special, and the reciprocal is conjectured [14, Section 7]. The inclusion $\mathcal{P}_{\mathrm{Lie}}(\mathcal{V}) \subset \mathcal{P}_{\mathcal{R}}(\mathcal{V})$ is obvious from Proposition 11. We partly answer Gavrilov's conjecture as follows.

Theorem 5. $\mathcal{P}_{\mathrm{Lie}}(\mathcal{V}) = \mathcal{S}(\mathcal{V})$.

---

[3] The standard notation in the literature is End $\mathcal{V}$ or Endop $\mathcal{V}$.

Proof. The inclusion $\mathcal{S}(\mathcal{V}) \subset \mathcal{P}_{\mathrm{Lie}}(\mathcal{V})$ is obvious. In order to show the reverse inclusion, it suffices to prove that the generalized curvature and torsion t_α and R_α are special polynomials for any Lie monomial α. We proceed by induction on the degree: the claim is obvious in degrees one and two, and for $\alpha(a \cdot b \cdot c) = [[a, b], c]$, we have, from (5.7),

$$t_\alpha(a \cdot b \cdot c) = R(a \cdot b \cdot c) - (c \rhd t)(a \cdot b) - t\big(t(a \cdot b) \cdot c\big)$$

and

$$R_\alpha(a \cdot b \cdot c \cdot d) = -(c \rhd R)(a \cdot b \cdot d) - R\big(t(a \cdot b) \cdot c \cdot d\big),$$

which proves the claim. By an iterated use of the Jacobi identity, any Lie monomial of degree $n+1$ can be written as a linear combination of Lie monomials of the form

$$\alpha(x_1 \cdots x_{n+1}) = \big[x_j, \beta(x_1 \cdots \widehat{x_j} \cdots x_{n+1})\big],$$

where β is a Lie monomial of degree n. For example, consider the equality

$$\big[[a, b], [c, d]\big] = -\big[d, \big[[a, b], c\big]\big] + \big[c, \big[[a, b], d\big]\big].$$

We therefore compute, with $X := x_1 \cdots \widehat{x_j} \cdots x_{n+1}$:

$$\begin{aligned}
\alpha(x_1 \cdots x_{n+1}) &= \big[x_j, \beta(x_1 \cdots \widehat{x_j} \cdots x_{n+1})\big] \\
&= [\![x_j, \beta(X)]\!] - x_j \rhd \beta(X) + \beta(X) \rhd x_j \\
&\overset{(2.5)}{=} [\![x_j, \beta(X)]\!] - \beta(x_j \rhd X) + \beta(X) \rhd x_j \\
&= -[\![x_j, t_\beta(X)]\!] + t_\beta(x_j \rhd X) - t_\beta(X) \rhd x_j \\
&\quad + [\![x_j, s_\beta(X)]\!] - s_\beta(x_j \rhd X) + s_\beta(X) \rhd x_j \\
&= -[\![x_j, t_\beta(X)]\!] + \mathbf{[}x_j, t_\beta(X)\mathbf{]} + t_\beta(x_j \rhd X) - t_\beta(X) \rhd x_j \\
&\quad - \mathbf{[}x_j, t_\beta(X)\mathbf{]} + [\![x_j, s_\beta(X)]\!] - s_\beta(x_j \rhd X) + s_\beta(X) \rhd x_j \\
&= -s\big(x_j \cdot t_\beta(X)\big) - t\big(x_j \cdot t_\beta(X)\big) - (x_j \rhd t_\beta)(X) \\
&\quad + [\![x_j, s_\beta(X)]\!] - s_\beta(x_j \rhd X) + s_\beta(X) \rhd x_j
\end{aligned}$$

from which we get

$$t_\alpha(x_1 \cdots x_{n+1}) = t\big(x_j \cdot t_\beta(X)\big) + (x_j \rhd t_\beta)(X) - R_\beta(X \cdot x_j)$$

and

$$\begin{aligned}
R_\alpha(x_1 \cdots x_{n+2}) &= \big([\![x_j, s_\beta(X)]\!] - s\big(x_j \cdot t_\beta(X)\big) - s_\beta(x_j \rhd X)\big) \rhd x_{n+2} \\
&= x_j \rhd \big(s_\beta(X) \rhd x_{n+2}\big) - s_\beta(X) \rhd (x_j \rhd x_{n+2}) \\
&\quad - R(x_j \cdot t_\beta(X) \cdot x_{n+2}) - R_\beta\big((x_j \rhd X \cdot x_{n+2})\big) \\
&= x_j \rhd R_\beta(X \cdot x_{n+2}) - R_\beta\big(X \cdot (x_j \rhd x_{n+2})\big) \\
&\quad - R(x_j \cdot t_\beta(X) \cdot x_{n+2}) - R_\beta\big((x_j \rhd X) \cdot x_{n+2}\big) \\
&= (x_j \rhd R_\beta)(X \cdot x_{n+2}) - R_\beta\big((x_j \rhd X) \cdot x_{n+2}\big),
\end{aligned}$$

which ends up the induction step and therefore proves the result. ∎

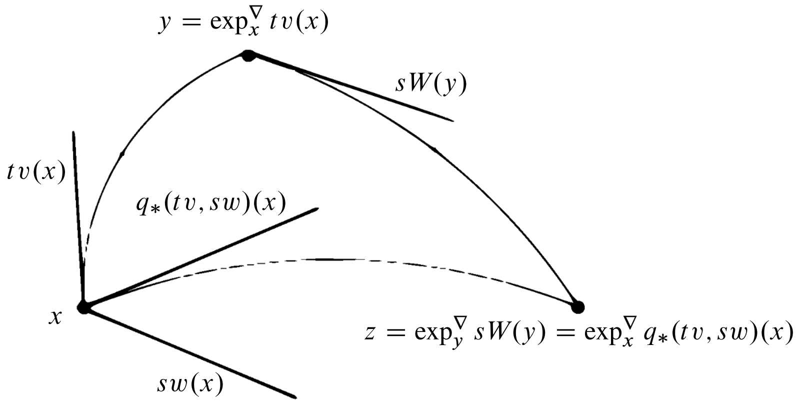


Figure 1. Gavrilov's double exponential.

6. Gavrilov's double exponential

Gavrilov's double exponential [14] is a formal series in two indeterminates, t and s, without a constant term, which can be explicitly written as follows:

$$q_*(tv, sw) = \beta^{-1}\big(\mathrm{BCH}\,\{\beta(tv), \beta\big(s\lambda(tv, w)\big)\}\big). \tag{6.1}$$

Here, v and w are two vector fields on a smooth manifold $\mathcal{M}$ endowed with an affine connection ∇. The notation BCH refers to the usual Baker–Campbell–Hausdorff series in the completed Lie algebra

$$\bar{\mathcal{V}} = \big(t\mathcal{XM}[\![s,t]\!] + s\mathcal{XM}[\![s,t]\!], [\![\cdot\,,\cdot]\!]\big),$$

and β stands for Gavrilov's β-map described earlier in Section 3. The map λ was introduced in Section 2.5.3. The double exponential (6.1) can be informally described in geometrical terms as follows: starting from a point $x \in \mathcal{M}$ in the direction given by the vector field v at x and following the geodesic $\exp_x^{\nabla} t'v(x)$ up to time $t' = t$, one reaches the point $y = \exp_x^{\nabla} tv(x) \in \mathcal{M}$. Let $W(y)$ be the vector field w at x parallel-transported to the point y. Following the geodesic $\exp_y^{\nabla} t'W(y)$ up to time $t' = s$, one reaches a third point $z = \exp_y^{\nabla} sW(y)$ on $\mathcal{M}$. Gavrilov's double exponential permits to express this point following a geodesic starting from $x \in \mathcal{M}$:

$$z = \exp_x^{\nabla} q_*(tv, sw)(x) \in \mathcal{M}.$$

6.1. Heuristic approach

Let us briefly outline how formula (6.1) can be heuristically obtained from this geometric description. Any tangent vector u at any point $x \in \mathcal{M}$ gives rise to a vector field on $\mathcal{M}$ (at least on a sufficiently small neighborhood of x) by parallel-transporting u at any point x' along the unique geodesic joining x to x'. We denote somewhat abusively by $\beta(u)$ this vector field, and we denote its flow by $\exp t\beta(u)$ or $\exp \beta(tu)$. Let δ be the unique tangent vector at x such that $x' = \exp_x^{\nabla}(\delta)$. We denote by $\lambda(\delta, u)$ the parallel transport of u at x' along the geodesic $t \mapsto \exp_m^{\nabla}(t\delta)$. Figure 1 can be read as the composition of two flows:

$$\exp \beta(tv) \exp \beta(sW) = \exp \beta\big(q_*(tv, sw)\big);$$

hence,

$$\exp \beta(tv) \exp \beta\big(\lambda(tv, sw)\big) = \exp \beta\big(q_*(tv, sw)\big), \tag{6.2}$$

which gives (6.1). The value of the series $q_*(tv, sw)$ at point x indeed depends only on the values at x of the two vector fields v and w [14, Proposition 4]. We will give in Section 6.2 our own proof of this crucial fact (Remark 13).

6.2. Another expression of the double exponential

As before, let $\mathcal{R} = C^\infty(\mathcal{M})$, let $\mathcal{V} = (\mathcal{X}\mathcal{M}, \rhd, [\![.\,,.]\!])$ be the framed Lie algebra of vector fields, and let $(\mathfrak{g}, \rhd, [\cdot,\cdot]) = \mathrm{Lie}_{\mathcal{R}}(\mathcal{V})$ be the post-Lie algebra defined in Section 4.2. Let $\widehat{\mathcal{U}(\mathfrak{g})}$ be the completion of the enveloping algebra of $\mathfrak{g}$, endowed with both associative products $\cdot$ and the GL-product, $*$.

Proposition 12. *Let $v, w \in \mathfrak{g}$, and let $\tilde{w} \in \hat{\mathfrak{g}}$ such that $(\exp^{\cdot} v) \rhd \tilde{w} = w$. Then, the following holds in $\widehat{\mathcal{U}(\mathfrak{g})}$:*

$$\exp^{\cdot} v * \exp^{\cdot} \tilde{w} = \exp^{\cdot} z^{\cdot}(v, w)$$

with

$$z^{\cdot}(v, w) = \mathrm{BCH}^{\cdot}(v, w) = v + w + \frac{1}{2}[v, w] + \frac{1}{12}\big[[v, w], w - v\big] + \cdots.$$

Proof. Using that $\exp^{\cdot} v$ is grouplike for the coproduct $\Delta_{\sqcup\!\sqcup}$ and that $L^{\rhd}_{\exp^{\cdot} v} = (\exp^{\cdot} v) \rhd -$ is an automorphism for the product $\cdot$, we get

$$\begin{aligned}
\exp^{\cdot} v * \exp^{\cdot} \tilde{w} &= \exp^{\cdot} v \cdot \big((\exp^{\cdot} v) \rhd \exp^{\cdot} \tilde{w}\big) \\
&= \exp^{\cdot} v \cdot \big(\exp^{\cdot} \big((\exp^{\cdot} v) \rhd \tilde{w}\big)\big) \\
&= \exp^{\cdot} v \cdot \exp^{\cdot} w \\
&= \exp^{\cdot} z^{\cdot}(v, w).
\end{aligned}$$

∎

From (4.5), $\beta = \rho \circ \chi|_{\overline{\mathcal{V}}}$, and (6.2), we get

$$\rho\big(\exp^* \chi(tv) * \exp^* \chi\big(\lambda(tv, sw)\big)\big) = \rho\big(\exp^* \chi\big(q_*(tv, sw)\big)\big);$$

hence,

$$\rho\big(\exp^{\cdot}(tv) * \exp^{\cdot}\big(\lambda(tv, sw)\big)\big) = \rho\big(\exp^{\cdot}\big(q_*(tv, sw)\big)\big).$$

From Proposition 12 together with (2.29), saying that $\lambda(tv, sw) = \exp^*(-\chi(tv)) \rhd sw$, we therefore get

$$\rho\big(\exp^{\cdot}\big(z^{\cdot}(tv, sw)\big)\big) = \rho\big(\exp^{\cdot}\big(q_*(tv, sw)\big)\big). \tag{6.3}$$

This in turn yields

$$\chi\big(z^{\cdot}(tv,sw)\big) = \chi\big(q_*(tv,sw)\big) \text{ modulo } \mathcal{J}.$$

Bearing in mind that $q_*(tv,sw)$ belongs to $\overline{\mathcal{V}}$ contrarily to $z^{\cdot}(tv,sw)$, and using (4.5) again, we finally get the following theorem.

Theorem 6. *Gavrilov's double exponential can be alternatively expressed as follows:*

$$q_*(tv,sw) = \beta^{-1}\Big(\rho \circ \chi\big(z^{\cdot}(tv,sw)\big)\Big).$$

Remark 13. As already proved by Gavrilov ([15, main Theorem] and [14, Section 4]), for any $x \in \mathcal{M}$, the (formal) tangent vector $q_*(tv,sw)(x)$ depends only on the two tangent vectors $v(x)$ and $w(x)$. Our interpretation of this fact in the post-Lie framework is the following: observe that the expression $\rho(\exp^{\cdot}(z^{\cdot}(tv,sw)))$ is geometrically special in the sense of Definition 10. In other words, for any function $f \in \mathcal{R}$, the expression $\rho(\exp^{\cdot}(z^{\cdot}(tv,sw)))f(x)$ depends on the vector fields v and w through $v(x)$ and $w(x)$ alone. Identity (6.3) then implies the same for $\rho(\exp^{\cdot}(q_*(tv,sw)))f(x)$. As $q_*(tv,sw)$ is a (formal) vector field, we have that

$$\rho\big(\exp^{\cdot}\big(q_*(tv,sw)\big)\big)f(x) = \rho\big(\exp^*(Q)\big)f(x) = \mathrm{Exp}(Q)f(x), \tag{6.4}$$

where $Q \in \overline{\mathcal{V}}$ is the unique geodesic formal vector field such that $Qf(x) = q_*(tv,sw)f(x)$ for any function f in $\mathcal{R}$, and where $\mathrm{Exp}(Q) \in \mathcal{DM}[\![s,t]\!]$ stands for its formal flow. The geodesic property of Q is expressed as

$$Q \rhd Q = 0, \tag{6.5}$$

from which we get

$$\exp^*(Q) = \exp^{\cdot}(Q). \tag{6.6}$$

Here, we used (2.19) together with the fact that (6.5) implies that the inverse post-Lie Magnus expansion reduces to the identity map. From (6.3), (6.4), and (6.6), the evaluation $\mathrm{Exp}(Q)f(x) = \rho(\exp^{\cdot}(Q))f(x)$ of the formal flow at x depends on v and w only through $v(x)$ and $w(x)$. The same is therefore true for

$$Qf(x) = \rho\big(\log^{\cdot}\exp^{\cdot}(Q)\big)f(x).$$

We finally deduce from $Qf(x) = q_*(tv,sw)f(x)$ that $q_*(tv,sw)f(x)$ depends on v and w through $v(x)$ and $w(x)$ only.

Conclusion

In this work, we have explored Gavrilov's results in [14, 16–18] from the post-Lie algebra perspective, thus showing the important role of this notion in differential geometry. This approach should be relevant in even broader contexts, such as Lie algebroids [29] and Lie–Rinehart algebras [11], their algebraic counterparts.

A. Proofs of Section 2

The final computation of the proof of Theorem 1 is as follows:

$$
\begin{aligned}
\big((XY-YX)U\big) \rhd V &= \big(X(YU)\big) \rhd V - \big(Y(XU)\big) \rhd V \\
&= X \rhd (YU \rhd V) - (X \rhd YU) \rhd V - (X \leftrightarrow Y) \\
&= X \rhd \big(Y \rhd (U \rhd V)\big) - X \rhd \big((Y \rhd U) \rhd V\big) \\
&\quad - \big((X \rhd Y)U\big) \rhd V - \big(Y(X \rhd U)\big) \rhd V - (X \leftrightarrow Y) \\
&= X \rhd \big(Y \rhd (U \rhd V)\big) - X \rhd \big((Y \rhd U) \rhd V\big) \\
&\quad - (X \rhd Y) \rhd (U \rhd V) + \big((X \rhd Y) \rhd U\big) \rhd V \\
&\quad - Y \rhd \big((X \rhd U) \rhd V\big) + \big(Y \rhd (X \rhd U)\big) \rhd V - (X \leftrightarrow Y) \\
&= XY \rhd (U \rhd V) - YX \rhd (U \rhd V) - \big((XY - YX) \rhd U\big) \rhd V \\
&= \mathrm{a}_{\rhd}(XY - YX, U, V),
\end{aligned}
$$

which proves Theorem 1.

The induction in the proof of Proposition 1 is given here. The length zero case is trivial and the length one case is the coderivation property mentioned above. Supposing $U = x \cdot U'$, we compute, using the induction hypothesis:

$$
\begin{aligned}
\Delta_{\sqcup\!\sqcup}(U \rhd V) &= \Delta_{\sqcup\!\sqcup}(xU' \rhd V) \\
&= \Delta_{\sqcup\!\sqcup}\big(x \rhd (U' \rhd V) - (x \rhd U') \rhd V\big) \\
&= (x \otimes 1 + 1 \otimes x) \rhd \Delta_{\sqcup\!\sqcup}(U \rhd V) - \Delta_{\sqcup\!\sqcup}(x \rhd U') \rhd \Delta_{\sqcup\!\sqcup}(V) \\
&= x \rhd (U'_{(1)} \rhd V_{(1)}) \otimes U'_{(2)} \rhd V_{(2)} + U'_{(1)} \rhd V_{(1)} \otimes x \rhd (U'_{(2)} \rhd V_{(2)}) \\
&\quad - (x \rhd U'_{(1)}) \rhd V_{(1)} \otimes U'_{(2)} \rhd V_{(2)} - U'_{(1)} \rhd V_{(1)} \otimes (x \rhd U'_{(2)}) \rhd V_{(2)} \\
&= xU'_{(1)} \rhd V_{(1)} \otimes U'_{(2)} \rhd V_{(2)} + U'_{(1)} \rhd V_{(1)} \otimes xU'_{(2)} \rhd V_{(2)} \\
&= U_{(1)} \rhd V_{(1)} \otimes U_{(2)} \rhd V_{(2)},
\end{aligned}
$$

which proves Proposition 1.

The induction in the proof of Proposition 2 is given here. The case $\ell = 1$ is just a reformulation of (2.6). For $\ell \geq 2$, we can suppose $U = xU'$ where $x \in M$ and where U' is a monomial of length $\ell - 1$. We compute

$$
\begin{aligned}
&U \rhd (V \rhd W) \\
&\quad = \ xU' \rhd (V \rhd W) \\
&\quad \overset{(2.6)}{=} x \rhd \big(U' \rhd (V \rhd W)\big) - (x \rhd U') \rhd (V \rhd W) \\
&\quad = \ x \rhd \big((U'_{(1)}(U'_{(2)} \rhd V)) \rhd W\big) - (x \rhd U') \rhd (V \rhd W) && \text{(from induction)} \\
&\quad \overset{(2.6)}{=} \big(xU'_{(1)}(U'_{(2)} \rhd V) + x \rhd \big(U'_{(1)}(U'_{(2)} \rhd V\big) - (x \rhd U')_{(1)}\big((x \rhd U')_{(2)} \rhd V\big)\big) \rhd W \\
&&& \text{(from induction again)}
\end{aligned}
$$

$$\overset{(2.5)}{=} \big(xU'_{(1)}(U'_{(2)} \rhd V) + U'_{(1)}\big(x \rhd (U'_{(2)} \rhd V)\big) - (x \rhd U'_{(2)}) \rhd V\big) \rhd W$$

(from the fact that L_x is a coderivation)

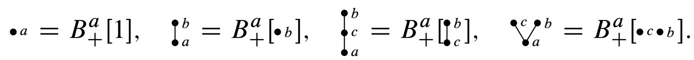
$$= \big(xU'_{(1)}(U'_{(2)} \rhd V) + U'_{(1)}(xU'_{(2)} \rhd V)\big) \rhd W$$

$$= \big(U_{(1)}(U_{(2)} \rhd V)\big) \rhd W,$$

which yields (2.8) and therefore proves Proposition 2.

B. Planar multi-grafting

We use the left grafting representation here (see Remark 2). We have $T((\mathrm{Mag}(A), \rhd)) = (\mathcal{F}_A^{\mathrm{pl}}, \rhd)$, where $\mathcal{F}_A^{\mathrm{pl}}$ is the linear span of ordered forests of planar rooted trees, and $\rhd$ is extended by means of (2.5) and (2.6). Recall that any planar rooted tree $\tau \in T_A^{\mathrm{pl}}$ with the root decorated by $a \in A$ can be written in terms of the so-called B_+^a-operator; that is, $\tau = B_+^a[\tau_1 \cdots \tau_n]$, for $\tau_1 \cdots \tau_n \in \mathcal{F}_A^{\mathrm{pl}}$. It adds a root decorated by $a \in A$ and connects it via an edge to every root in the forest $\tau_1 \cdots \tau_n$. For example, denoting the empty tree by 1, we have



We will now consider a multivariate extension of the grafting operation by defining the following brace operations:

$$\rhd^{n+1} : T_{n+1}\big(\mathrm{Mag}(A)\big) \times \mathrm{Mag}(A) \to \mathrm{Mag}(A). \tag{B.1}$$

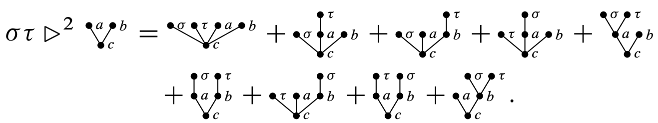
Here, $T_k(\mathrm{Mag}(A))$ denotes the k-th component in the tensor algebra. The multi-grafting in (B.1) is recursively defined for $\tau_1, \tau_2 \in T_A^{\mathrm{pl}}$ and a planar forest ω of length n by

$$(\tau_1\omega) \rhd^{n+1} \tau_2 := \tau_1 \rhd (\omega \rhd^n \tau_2) - (\tau_1 \rhd \omega) \rhd^n \tau_2. \tag{B.2}$$

In the case of $\tau_1, \ldots, \tau_n \in T_A^{\mathrm{pl}}$ and $\tau_2 = \bullet_a$, this simplifies to

$$(\tau_1 \cdots \tau_n) \rhd^n \bullet_a = (\tau_1 \cdots \tau_n) \rhd^n B_+^a[1] = B_+^a[\tau_1 \cdots \tau_n].$$

Rule (B.2) can be summarized combinatorially as follows: graft the trees $\tau_1, \ldots, \tau_n$ in all possible ways onto the tree τ, (i) excluding the grafting of any τ_i onto any τ_j and (ii) when grafting several trees, $\tau_{j_1}, \ldots, \tau_{j_k}$, $1 \le j_1 < \cdots < j_k \le n$, onto a vertex v of τ, then they must be grafted to the left of the leftmost edge going out from the vertex v of τ in such a way that the order among those trees is preserved. As an example, we consider



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