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Any generating set of an arbitrary property T von Neumann algebra has free entropy dimension ≤ 1

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Abstract. Suppose that N is a diffuse, property T von Neumann algebra and X is an arbitrary finite generating set of selfadjoint elements for N. By using rigidity/deformation arguments applied to representations of N in ultraproducts of full matrix algebras, we deduce that the microstate spaces of X are asymptotically discrete up to unitary conjugacy. We use this description to show that the free entropy dimension of X, $\delta_0(X)$ is less than or equal to 1. It follows that when N embeds into the ultraproduct of the hyperfinite II_1 factor, then $\delta_0(X) = 1$ and otherwise, $\delta_0(X) = -\infty$. This generalizes the earlier results of Voiculescu, and Ge, Shen pertaining to $\mathrm{SL}_n(\mathbb{Z})$ as well as the results of Connes, Shlyakhtenko pertaining to group generators of arbitrary property T algebras.

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Introduction

In [24] and [25], Voiculescu introduced the notion of free entropy dimension. For X a finite set of self-adjoint elements of a tracial von Neumann algebra, $\delta_0(X)$ is a kind of asymptotic Minkowski dimension of the set of matricial microstates for X. These notions led to the solution of several old operator algebra problems (see [27] for an overview). Closely tied to this is the invariance question for δ_0 which asks the following. If X and Y are two finite sets of selfadjoint elements generating the same tracial von Neumann algebra, then is it true that $\delta_0(X) = \delta_0(Y)$?

For certain X one can compute $\delta_0(X)$ and answer the invariance question in the affirmative. Suppose that $N = W^*(X)$ is diffuse and embeds into the ultraproduct of the hyperfinite II_1 factor. Then $\delta_0(X) = 1$ when N has property Γ, or has a Cartan subalgebra, or is nonprime, or can be decomposed as an amalgamated free product of these algebras over a common diffuse subalgebra (see [11], [14], [16], [25]).

Another class of algebras to investigate in regard to possible values of $\delta_0(X)$ and the invariance question are those with Kazhdan's property T ([7], [17], [20]). These

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first appeared in the von Neumann algebra context in Connes' seminal work [6]. In recent years, Popa introduced the technique of playing the rigidity properties of such algebras against deformation results; this has led to a number of significant advances in the theory of von Neumann algebras ([20], [21], [13]).

Voiculescu made the first computations of δ_0 for property T factors by showing that if $x_1, \dots, x_n$ are diffuse, selfadjoint elements in a tracial von Neumann algebra such that for each $1 \le i \le n-1$, $x_i x_{i+1} = x_{i+1} x_i$, then $\delta_0(x_1, \dots x_n) \le 1$ (see [26]). For $n \ge 3$, there exists a finite set of generators X_n for the group algebra $\mathbb{C}\,\mathrm{SL}_n(\mathbb{Z})$ with this property (this was first used in the context of measurable equivalence relations by Gaboriau [9] to prove that their cost is at most 1). Hence $L(\mathrm{SL}_n(\mathbb{Z}))$ has a set of generators X for which $\delta_0(X) \le 1$. This was generalized in [11] (see also [10] and references therein) where Ge and Shen weakened the conditions on the generators x_i and in particular obtained the stronger statement that $\delta_0(Y) \le 1$ for any other set Y of self-adjoint generators of the von Neumann algebra. However, all of these results rely on the special algebraic properties of certain generators (e.g. in $\mathrm{SL}_n(\mathbb{Z})$) and thus do not apply to the more general property T groups or von Neumann algebras.

In [8] a notion of L^2-cohomology for von Neumann algebras was introduced, and the values of the resulting L^2-Betti numbers were connected with free probability and the value of δ_0. Indeed, using cohomological ideas, it was proved in [8] that if $X \subset \mathbb{C}\Gamma$ is an arbitrary set of generators, then

$$\delta_0(X) \le \beta_1^{(2)}(\Gamma) - \beta_0^{(2)}(\Gamma) + 1.$$

Here $\beta_j^{(2)}(\Gamma)$ are the Atiyah–Cheeger–Gromov ℓ^2-Betti numbers of Γ (see e.g. [18]). This inequality is quite complicated to prove; indeed, one first proves the same inequality with δ_0 replaced by its "non-microstates" analog δ^*, and then uses a highly nontrivial result of Biane, Capitaine, Guionnet [2] that implies $\delta_0 \le \delta^*$.

In the case that Γ has property T, the first ℓ^2-Betti number vanishes (see [12], [1], [4]). So for Γ an infinite group, one has $\delta_0(X) \le 1$ for any finite generating set $X \subset \mathbb{C}\Gamma$. However, even in this case, an "elementary" proof of this bound was not available and, moreover, it was not known whether $\delta_0(X) \le 1$ for any finite generating set $X \subset L(\Gamma)$.

Our result settles the question of the value of $\delta_0(X)$ for an arbitrary set of self-adjoint generators of a property T factor in full generality:

Theorem. *Suppose that N is a diffuse, property T von Neumann algebra with a finite set of selfadjoint generators X, and let R^ω be an ultrapower of the hyperfinite II_1 factor. Then $\delta_0(X) \le 1$. Moreover, if N has an embedding into R^ω, then $\delta_0(X) = 1$, and if N has no embedding into R^ω, then $\delta_0(X) = -\infty$.*

Note that this result shows that the value of the free entropy dimension δ_0 is independent of the choice of generators of N. In particular, one gets as a corollary that if Γ is any infinite discrete group with property T, and X is any set of self-adjoint generators of the group von Neumann algebra $L(\Gamma)$ (we do not make the assumption that $X \subset \mathbb{C}\Gamma$ here), then $\delta_0(X) = 1$ or $-\infty$, depending on whether Γ embeds into the unitary group of R^ω.

The proof of the main theorem relies on a deformation/rigidity argument in the style of Popa, which is used to prove that the set of unitary conjugacy classes of embeddings of a property T von Neumann algebra N into the ultrapower of the hyperfinite II_1 factor is discrete. This fact can then be employed to show that if $X \subset N$ is a set of self-adjoint generators, then any $k \times k$ matricial microstate for X essentially lies in the unitary orbit of a certain discrete set S, all of whose elements are at least a certain fixed distance apart. One then turns this into an estimate for the packing dimension of the microstate space for X. We prove, effectively, that the packing dimension of the microstate set is essentially the same as that of a small number of disjoint copies of the k-dimensional unitary group.

1. Property T, embeddings, and unitary orbits

Throughout this section and the next we fix a property T finite von Neumann algebra N and a finite p-tuple of selfadjoint generators $X \subset N$. $\|\cdot\|_2$ denotes the L^2-norm induced by a specified trace on a von Neumann algebra. $M_k^{\mathrm{sa}}(\mathbb{C})$ denotes the set of selfadjoint $k \times k$ matrices, $M_k(\mathbb{C})$ denotes the set of $k \times k$ matrices, and tr_k is the trace on $M_k(\mathbb{C})$. If $\xi = \{y_1, \ldots, y_p\}$ and $\eta = \{z_1, \ldots, z_p\}$ are p-tuples in a von Neumann algebra and u, w are elements in a tracial von Neumann algebra, then $\xi - \eta = \{y_1 - z_1, \ldots, y_p - z_p\}$, $u\xi w = \{uy_1w, \ldots, uy_pw\}$, and $\|\xi\|_2 = \big(\sum_{i=1}^p \|y_j\|_2^2\big)^{\frac{1}{2}}$. $R > 0$ will be a fixed constant greater than any of the operator norms of the elements in X. $\Gamma_R(X; m, k, \gamma)$ will denote the standard microstate spaces introduced in [24].

The following theorem, stated for the reader's convenience, is by now among the standard results in the theory of rigid factors. Such deformation-conjugacy arguments have played a fundamental role in the recent startling results of Popa and others ([13], [19], [21], [22]).

Theorem 1.1. *Let X and N be as above. Then for any $t > 0$ there exists a corresponding $r_t > 0$ so that if (M, τ) is a tracial von Neumann algebra and $\pi, \sigma\colon N \to M$ are normal faithful trace-preserving $*$-homomorphisms such that for all $x \in X$, $\|\pi(x) - \sigma(x)\|_2 < r_t$, then there exist projections $e \in \pi(N)' \cap M$, $f \in \sigma(N)' \cap M$, a partial isometry $v \in M$ such that $v^*v = e$, $vv^* = f$, $\tau(e) > 1 - t$, and for all $x \in N$, $ve\pi(x)ev^* = f\sigma(x)f$.*

Proof. Recall (see [7] for the factor case or [21], Proposition 4.1.3°, for the general case) that there exist $K, \varepsilon_0 > 0$, and a finite set $F \subset N$ such that if $0 < \delta \le \varepsilon_0$ and H is a correspondence of N with a vector $\xi \in H$ satisfying $\|z\xi - \xi z\|_2 < \delta$,

$\|\langle \cdot \xi, \xi\rangle - \tau_N\| < \delta$, $\|\langle \xi \cdot, \xi\rangle - \tau_N\| < \delta$, $z \in F$, then there exists a vector $\eta \in H$ which is central for N and $\|\eta - \xi\|_2 < K\delta$.

Choose r_t so small so that if $\rho_1, \rho_2 \colon N \to M$ are any two faithful, normal trace preserving $*$-homomorphisms such that $\|\rho_1(x) - \rho_2(x)\|_2 < r_t$ for all $x \in X$, then $\|\rho_1(z) - \rho_2(z)\|_2 < \min\{t, \varepsilon_0\} \cdot (4K)^{-1}$ for all $z \in F$. This can be done because X generates N.

Suppose that $\pi, \sigma \colon N \to M$ are two normal, faithful trace-preserving $*$-homomorphisms such that $\|\pi(x) - \sigma(x)\|_2 < r_t$ for all $x \in X$. Consider $L^2(M)$ as an $N - N$ bimodule where for any $\xi \in L^2(M)$, $x, y \in N$, $x\xi y = \pi(x)J\sigma(y)^*J\xi$. Denote by 1_M the vector associated to the unit of M. The hypothesis on π and σ guarantee that for all $x \in F$, $\|x1_M - 1_M x\|_2 = \|\pi(x) - \sigma(x)\|_2 < \min\{t, \varepsilon_0\} \cdot (4K)^{-1}$ and moreover that $\langle x1_M, 1_M\rangle = \langle 1_M x, 1_M\rangle = \tau_N(x)$, which in turn implies the existence of a central vector $\eta_0 \in L^2(M)$ for N such that $\|\eta_0 - 1_M\|_2 < t/4$. Regard η_0 as an unbounded operator on $L^2(M)$ by its left action. If $\eta_0 = u|\eta_0|$ is the polar decomposition of η_0, then $u \in M$ and $\|\eta_0 - 1_M\|_2 < t/4$ implies that $\|u - 1_M\|_2 < t/2$ and so $\|u^*u - 1_M\|_2 < t$. On the other hand, since for any $x \in N$, $x\eta_0 = \eta_0 x$, one concludes in the usual way that $xu = ux$. Consequently, $uu^* \in \pi(N)'$ and $u^*u \in \sigma(N)'$. Set $e = uu^* \in \pi(N)' \cap M$ and $f = u^*u \in \sigma(N)' \cap M$. It follows that $u^*e\pi(x)eu = f\sigma(x)f$ for all $x \in N$. Finally, $\tau(e) = \tau(f) > 1 - t$. □

For each $t > 0$, we now choose a critical $r = r_t > 0$ dependent on t as in Theorem 1.1.

We now need some notation.

Notation 1.2. (a) If $\eta \in (M_k^{\mathrm{sa}}(\mathbb{C}))^p$ and $r > 0$, then

$$\Theta_r(\eta) = \{\xi \in (M_k^{\mathrm{sa}}(\mathbb{C}))^p : \text{for some } u \in U_k, \|\xi - u^*\eta u\|_2 < r\}.$$

(b) If $\eta \in (M_k^{\mathrm{sa}}(\mathbb{C}))^p$ and $\kappa, s > 0$, then $\mathcal{G}_{\kappa,s}(\eta)$ consist of all p-tuples ξ such that there exists projections $e, f \in M_k^{\mathrm{sa}}(\mathbb{C})$ and $w \in M_k(\mathbb{C})$ with $w^*w = e$, $ww^* = f$, $\mathrm{tr}_k(e) = \mathrm{tr}_k(f) > s$ and $\|we\xi ew^* - f\eta f\|_2 < \kappa$.

Lemma 1.3. *For any $\kappa, t > 0$ there exists $m \in \mathbb{N}$ such that if $\xi, \eta \in \Gamma_R(X; m, k, m^{-1})$ and $\xi \in \Theta_{r_t}(\eta)$, then $\xi \in \mathcal{G}_{\kappa,1-t}(\eta)$.*

Proof. We proceed by contradiction. Assume that there exists some $\kappa_0, t_0 > 0$ such that for each $m \in \mathbb{N}$ there are $k_m \in \mathbb{N}$ and $\xi_m, \eta_m \in \Gamma_R(X; m, k_m, m^{-1})$ with

$$\xi_m \in \Theta_r(\eta_m) \quad \text{and} \quad \xi_m \notin \mathcal{G}_{\kappa_0, 1-t_0}(\eta_m),$$

where $r = r_{t_0}$. Fix a free ultrafilter ω, and consider the ultraproduct

$$R^\omega = \prod^\omega M_{k_m}(\mathbb{C}) = \frac{\prod_{m=1}^\infty M_{k_m}(\mathbb{C})}{\{\langle x_m\rangle_{m=1}^\infty : \lim_\omega \mathrm{tr}_{m_k}(x_m^* x_m) = 0\}}.$$

Denote by $Q\colon \prod M_{k_m} \to R^\omega$ the quotient map. Set $\xi = \langle \xi_m \rangle_{m=1}^\infty$ and $\eta = \langle \eta_m \rangle_{m=1}^\infty$.

For each m we can find a $k_m \times k_m$ unitary u_m such that $\|u_m^* \xi_m u_m - \eta\|_2 < r$. Set $u = \langle u_m \rangle_{m=1}^\infty$. It follows that there exist two normal faithful trace-preserving $*$-homomorphisms $\pi, \sigma\colon N \to R^\omega$ such that $\pi(X) = Q(U)^* Q(\xi) Q(U)$ and $\sigma(X) = Q(\eta)$. Clearly $\|\pi(X) - \sigma(X)\|_2 < r$. By Theorem 1.1 there exist projections $e \in \pi(N)' \cap R^\omega$, $f \in \sigma(N)' \cap R^\omega$ and a partial isometry $v \in R^\omega$ with initial domain e and final range f such that for all $x \in N$, $ve\pi(x)ev^* = f\sigma(x)f$ and $\tau(e) = \tau(f) > 1 - t_0$. v is a partial isometry and $\tau(v^*v) = \tau(e) > 1 - t_0$. There exist sequences of projections $\langle e_m \rangle_{m=1}$ and $\langle f_m \rangle_{m=1}^\infty$ such that for each m, $e_m, f_m \in M_{k_m}(\mathbb{C})$ and $Q(\langle e_m \rangle_{m=1}^\infty) = e$, $Q(\langle f_m \rangle_{m=1}^\infty) = f$. Similarly there exists a sequence of partial isometries $\langle v_m \rangle_{m=1}^\infty$ such that for each m, $v_m \in M_{k_m}(\mathbb{C})$ and $Q(\langle v_m \rangle_{m=1}^\infty) = v$. We can also arrange that $v_m v_m^* = f_m$ and $v_m^* v_m = e_m$ for each m. Now, the equation $ve\pi(x)ev^* = f\sigma(x)f$, $x \in M$, implies in particular that $\|v_{m_{\lambda_0}} e_{m_{\lambda_0}} \xi_{m_{\lambda_0}} e_{m_{\lambda_0}} v_{m_{\lambda_0}}^* - f_{m_{\lambda_0}} \eta_{m_\lambda} f_{m_{\lambda_0}}\|_2 < \kappa_0$ for some $\lambda_0 \in \omega$, and that the normalized trace of both $f_{m_{\lambda_0}}$ and $e_{m_{\lambda_0}}$ is strictly greater than $1 - t_0$. But this means that $\xi_{m_{\lambda_0}} \in \mathcal{G}_{\kappa_0, 1-t_0}(\eta)$, which contradicts our initial assumption. □

Remark 1.4. Observe that in Lemma 1.3 the quantity r_t is independent of κ.

2. The main estimate

In this section we maintain the notation for $\mathbb{K}_\varepsilon$ introduced in [15] taken now with respect to the microstate spaces with the operator norm cutoffs. Set $K = \|X\|_2$. We first state a technical lemma on the covering numbers for the spaces $\mathcal{G}_{\kappa,s}(\eta)$.

Lemma 2.1. *If $\eta \in (M_k^{\mathrm{sa}}(\mathbb{C}))^p$ and $\varepsilon, \kappa, s > 0$ with $\varepsilon > \kappa$, then there exists a $5K\varepsilon$-net for $\mathcal{G}_{\kappa,s}(\eta)$ with cardinality no greater than*

$$\left(\frac{2\pi}{\varepsilon}\right)^{2k^2 - s^2k^2} \cdot \left(\frac{K+1}{\varepsilon}\right)^{4(1-s)^2k^2}.$$

Proof. Find the smallest $m \in \mathbb{N}$ such that $sk \le m \le k$. Denote by V the set of partial isometries in $M_k(\mathbb{C})$ whose range has dimension m. Denote by P_m the set of projections of trace mk^{-1}. It follows from [23] that there exists an ε-net for P_m (with respect to the operator norm) with cardinality no greater than $(\frac{2\pi}{\varepsilon})^{k^2 - m^2 - (k-m)^2}$. There exists again by [23] an ε-net for the unitary group of $M_m(\mathbb{C})$ (with respect to the operator norm) with cardinality no greater than $(\frac{2\pi}{\varepsilon})^{m^2}$. These two facts imply that there exists an ε-net $\langle v_{jk} \rangle_{j \in J_k}$ for V with respect to the operator norm such that

$$\#J_k < \left(\frac{2\pi}{\varepsilon}\right)^{4km - 3m^2}.$$

Now fix $j \in J_k$. Denote by $G(\eta, j)$ the set of all $\xi \in (M_k^{sa}(\mathbb{C}))^p$ such that $\|\xi\|_2 \leq K$ and $\|v_{jk}(e_{jk}\xi e_{jk})v_{jk}^* - f_{jk}\eta f_{jk}\|_2 < 5K\varepsilon$ where $e_{jk} = v_{jk}^* v_{jk}$ and $f_{jk} = v_{jk}v_{jk}^*$. There exists a 2ε-cover $\langle \xi_{ijk}\rangle_{i\in\theta(j)}$ for $G(\eta, j)$ such that $\#\theta(j) < \left(\frac{K+1}{\varepsilon}\right)^{4(1-s)^2k^2}$.

Consider the set $\langle \xi_{ijk}\rangle_{i\in\theta(j), j\in J_k}$. It is clear that this set has cardinality no greater than

$$\left(\frac{2\pi}{\varepsilon}\right)^{4km-3m^2} \cdot \left(\frac{K+1}{\varepsilon}\right)^{4(1-s)^2k^2}.$$

It remains to show that this set is a $5K\varepsilon$-cover for $\mathcal{G}_{\kappa,s}(\eta)$. Towards this end suppose that $\xi \in \mathcal{G}_{\kappa,s}(\eta)$. Then there exists a partial isometry $v \in M_k(\mathbb{C})$ such that $v^*v = e$, $vv^* = f$, $\|ve\xi ev^* - f\eta f\|_2 < \kappa$, and $\mathrm{tr}_k(e) = \mathrm{tr}_k(f) > s$. By cutting the domain and range of the projection, we can assume that e and f are projections onto subspaces of dimension exactly m and that the inequality with tolerance κ is preserved. Obviously $v \in V$, whence there exists $j_0 \in J_k$ such that $\|v_{j_0k} - v\| < \varepsilon$. This condition immediately implies that $\|v_{j_0k}e_{j_0k} - ve\|, \|f_{j_0k} - f\| < 2\varepsilon$ and thus

$$\|v_{j_0k}e_{j_0k}\xi e_{j_0k}v_{j_0k}^* - f_{j_0k}\eta f_{j_0k}\|_2 \leq 4\varepsilon K + \|ve\xi ev^* - f\eta f\|_2 < 5K\varepsilon.$$

By definition, $\xi \in G(\eta, j_0)$. Thus, there exists some i_0 such that $i_0 \in \theta(j_0)$ and $\|\xi_{i_0j_0k} - \xi\|_2 < 5K\varepsilon$. □

We can now prove the main result of the paper:

Theorem 2.2. *Let N be a diffuse, property T von Neumann algebra with a finite set of selfadjoint generators X, and let R^ω be an ultrapower of the hyperfinite II_1 factor.*
(a) *If N has an embedding into R^ω, then $\delta_0(X) = 1$.*
(b) *If N has no embedding into R^ω, then $\delta_0(X) = -\infty$.*

Proof. Fix $1 > a > 0$. For any $\varepsilon > 0$, setting $\kappa = \varepsilon$ and $t = 1 - a$ in Lemma1.3 shows that there exists $m \in \mathbb{N}$, $m > p^2$, such that if $\xi, \eta \in \Gamma_R(X; m, k, m^{-1})$ and $\xi \in \Theta_{r_a}(\eta)$, then $\xi \in \mathcal{G}_{\varepsilon,1-a}(\eta)$. Consider the ball B_k of $(M_k^{sa}(\mathbb{C}))^p$ of $\|\cdot\|_2$-radius $K + 1$. For each k find an r_a-net $\langle \eta_{jk}\rangle_{j\in J_k}$ of $\Gamma_R(X; m, k, m^{-1})$ with minimal cardinality such that each element of the net lies in $\Gamma(X; m, k, m^{-1})$. The standard volume comparison test of this set with B_k (remember that $\Gamma_R(X; m, k, m^{-1}) \subset (M_k^{sa}(\mathbb{C}))_K^p$) implies that

$$\#J_k \leq \left(\frac{K+2}{r_a}\right)^{pk^2}.$$

For each such $j \in J_k$ find a $5K\varepsilon$-net $\langle \xi_{ij}\rangle_{i\in\theta(j)}$ for $\mathcal{G}_{\varepsilon,1-a}(\eta_{jk})$ where $\theta(j)$ is an indexing set satisfying

$$\#\theta(j) \leq \left(\frac{2\pi}{\varepsilon}\right)^{2k^2-(1-a)^2k^2} \cdot \left(\frac{K+2}{\varepsilon}\right)^{4a^2k^2}.$$

Consider now the set $\langle \xi_{ij} \rangle_{i\in\theta(j), j\in J_k}$. It is clear that this set has cardinality no greater than

$$\left(\frac{K+2}{r_a}\right)^{pk^2} \left(\frac{2\pi}{\varepsilon}\right)^{(1+2a-a^2)k^2} \cdot \left(\frac{K+2}{\varepsilon}\right)^{4a^2k^2}.$$

Moreover, if $\xi \in \Gamma_R(X; m, k, m^{-1})$, then there exists some $j_0 \in J_k$ such that $\|\xi - \eta_{j_0 k}\|_2 < r_a$. Clearly $\xi \in \Theta_{r_a}(\eta_{j_0 k})$, which implies that $\xi \in \mathscr{G}_{\varepsilon,1-a}(\eta_{j_0 k})$. Consequently there exists some $i_0 \in \theta(j_0)$ such that $\|\xi - \xi_{i_0 j_0}\|_2 < 5K\varepsilon$. Therefore, $\langle \xi_{ij} \rangle_{i\in\theta(j), j\in J_k}$ is a $5K\varepsilon$-net for $\Gamma_R(X; m, k, m^{-1})$.

The preceding paragraph implies that for $\varepsilon > 0$,

$$\begin{aligned}
\mathbb{K}_{5K\varepsilon}(X) &\le \limsup_{k\to\infty} k^{-2} \cdot \log \left[\left(\frac{K+2}{r_a}\right)^{pk^2} \left(\frac{2\pi}{\varepsilon}\right)^{(1+2a-a^2)k^2} \cdot \left(\frac{K+2}{\varepsilon}\right)^{4a^2k^2} \right] \\
&= p|\log r_a| + (1+2a-a^2)|\log \varepsilon| + \log[(2\pi)^2 (K+2)^{p+4}].
\end{aligned}$$

Keeping in mind that a and ε are independent it now follows from [15]

$$\begin{aligned}
\delta_0(X) &= \limsup_{\varepsilon\to 0} \frac{\mathbb{K}_\varepsilon(X)}{|\log \varepsilon|} \\
&= \limsup_{\varepsilon\to 0} \frac{\mathbb{K}_{5K\varepsilon}(X)}{|\log \varepsilon|} \\
&\le \limsup_{\varepsilon\to 0} p \cdot \frac{|\log r_a|}{|\log \varepsilon|} + 1 + 2a - a^2 + \frac{\log((2\pi)^2 (K+2)^{p+4})}{|\log \varepsilon|} \\
&= 1 + 2a - a^2.
\end{aligned}$$

As $1 > a > 0$ was arbitrary, $\delta_0(X) \le 1$. The rest of the assertions follow from [14]. □

Remark 2.3. For $\varepsilon > 0$ consider the set $X + \varepsilon S = \{x_1 + \varepsilon s_1, \ldots, x_n + \varepsilon s_n\}$ where $\{s_1, \ldots, s_n\}$ is a semicircular family free with respect to X. In [3] it is shown that for sufficiently small $\varepsilon > 0$ the von Neumann algebras M^ε generated by $X + \varepsilon S$ are not isomorphic to the free group factors and yet, if X'' embeds into the ultraproduct of the hyperfinite II_1 factor, then $\chi(X + \varepsilon S) > -\infty$. Theorem 2.2 implies that if X'' embeds into the ultraproduct of the hyperfinite II_1 factor, then M^ε cannot have property T. Also observe that the usual rigidity/deformation argument shows that for sufficiently small $\varepsilon > 0$, there exists a II_1 property T subfactor N^ε of M^ε.

Remark 2.4. Unfortunately, we were not able to settle the question of whether N must be strongly 1-bounded in the sense of [16].

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