

Poincaré–Birkhoff–Witt deformations of Calabi–Yau algebras

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Abstract. Recently, Bocklandt proved a conjecture by Van den Bergh in its graded version, stating that a graded quiver algebra A (with relations) which is Calabi–Yau of dimension 3 is defined from a homogeneous potential W . In this paper, we prove that if we add to W any potential of smaller degree, we get a Poincaré–Birkhoff–Witt deformation of A . Such PBW deformations are Calabi–Yau and are characterised among all the PBW deformations of A . Various examples are presented.

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1. Introduction

The notion of Calabi–Yau algebra has its origin in algebraic geometry: a smooth projective variety is Calabi–Yau if its canonical bundle is trivial, i.e. if it admits a global volume form (see e.g. [20]). By Serre duality, it is equivalent to say that the bounded derived category of coherent sheaves is endowed with a Serre functor (see [8]) given by a power of the shift functor. This condition was used by Kontsevich to define the notion of a Calabi–Yau triangulated category. Given a (noncommutative) algebra A over a field \mathbb{k} , Calabi–Yau conditions for A can be defined by considering various triangulated categories associated with A , for example the bounded derived category (or the stable category) of finite dimensional A -modules (that is, of A -modules that are finite dimensional as \mathbb{k} -vector spaces). Such categories, and the noncommutative algebras giving rise to them, are currently playing an important role in representation theory and in algebraic geometry (resolutions of singularities) in connection with cluster algebras or McKay correspondence [19], [18], [16].

String theory physicists have a construction of noncommutative Calabi–Yau algebras, which are quiver algebras with relations [24]. For a fixed quiver Q , this construction only depends on a noncommutative polynomial (or series) in several variables called *potential*. Van den Bergh conjectured that any 3-dimensional Calabi–Yau algebra derives from a potential [24]. This was actually proved in the graded

situation by Bocklandt [7]. Our paper deals with deformations of the graded algebras $A(Q, W_{N+1})$ obtained by Bocklandt. Here W_{N+1} denotes the potential (which can be chosen homogeneous since A is graded) and $N + 1$ is its degree.

Our first observation is that the graded algebras $A(Q, W_{N+1})$ are N -Koszul, so we can consider Poincaré–Birkhoff–Witt (PBW) deformations of $A(Q, W_{N+1})$ in the sense of [5] (see also [15] for the case in which Q has only one vertex). The main result of this paper is the following.

Theorem 1.1. *Let $A = A(Q, W_{N+1})$ be a graded Calabi–Yau algebra of dimension 3. Let W' be a not necessarily homogeneous potential of degree at most N . Then:*

- (i) $A' := A(Q, W_{N+1} + W')$ is a PBW deformation of A , and A' is Calabi–Yau.
- (ii) Assume that the characteristic of the ground field does not divide $N!$. If A' is any PBW deformation of A , then A' derives from a potential if and only if a certain condition (PBW2') (involving only the degree $N - 1$ part of the relations of A') holds.

This theorem is the combination of Theorems 3.1, 3.2 and 3.6 in the text below. Part (ii) can be viewed as an answer to an analogue of Van den Bergh's conjecture in a filtered situation.

We illustrate this theorem in Section 5 by several examples arising from various sources: Yang–Mills algebras (from theoretical physics), cubic Artin–Schelter regular (AS-regular) algebras (from noncommutative algebraic geometry), antisymmetriser algebras (from representation theory), and some quiver algebras with several vertices (introduced in [7]).

The definition of Calabi–Yau algebras used in this text is due to Ginzburg [16]; we relate it to that used by Bocklandt in Section 2 (see also [14], Section 3.2, where a similar class of algebras is introduced). Ginzburg's definition is a noncategorical definition involving Hochschild cohomology, and inspired by Van den Bergh's duality theorem. This definition is well adapted to the determination of Calabi–Yau algebras within the class of N -Koszul and Artin–Schelter Gorenstein (AS-Gorenstein) algebras (Proposition 4.5 below). In order to explain this, we need some general considerations on graded Calabi–Yau algebras presented in Subsection 4.1.

We would like to discuss here the following point that appears to us to be important and is a consequence of our results: the class \mathcal{C}_1 of graded Calabi–Yau algebras of dimension 3 is strictly contained in the class \mathcal{C}_2 of graded N -Koszul and AS-Gorenstein algebras of global dimension 3, and Proposition 4.5 specialised to global dimension 3 enables us to characterise \mathcal{C}_1 inside \mathcal{C}_2 . For instance, an AS-regular algebra of global dimension 3 is in \mathcal{C}_2 , and moreover it is in \mathcal{C}_1 if and only if it is of type A (Proposition 5.4). In [12], [13], Dubois-Violette describes the algebras inside \mathcal{C}_2 using multilinear forms called *3-regular*, and he conjectures that any 3-regular multilinear form provides an algebra in \mathcal{C}_2 . Therefore he offers, in a more general situation (but for quivers with only one vertex), a notion equivalent to that

of a “good” potential (that is, a potential that defines a Calabi–Yau algebra) with an explicit description. Moreover, he suggests an analogous description in dimension greater than 3.

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2. Preliminaries

Let \mathbb{k} be a field, and let A be an associative \mathbb{k} -algebra with unit which has a finite projective A -bimodule resolution by bimodules of finite type. The symbol \otimes with no subscript denotes the tensor product over the base field \mathbb{k} .

The space $A \otimes A$ is endowed with two A -bimodule structures: the *outer* structure defined by $a \cdot (x \otimes y) \cdot b = ax \otimes yb$, and the *inner* structure defined by $a \cdot (x \otimes y) \cdot b = xb \otimes ay$. Consequently, the Hom spaces $\text{Hom}_{A-A}(M, A \otimes A)$ of A -bimodule morphisms from M to $A \otimes A$ endowed with the outer structure are again A -bimodules using the inner structure of $A \otimes A$, and the same is true of the Hochschild cohomology spaces $H^k(A, A \otimes A)$.

We use Ginzburg’s definition [16] of a Calabi–Yau algebra:

Definition 2.1. We say that an algebra A as above is a *Calabi–Yau algebra* of dimension $n \geq 1$ if there are A -bimodule isomorphisms $H^k(A, A \otimes A) \cong \begin{cases} A & \text{if } k = n, \\ 0 & \text{otherwise.} \end{cases}$

Proposition 2.2. *If A is a Calabi–Yau algebra of dimension n , then the Hochschild dimension of A (that is, the projective dimension of A as an A -bimodule) is n .*

Proof. Since A has a finite projective A -bimodule resolution by assumption, the Hochschild dimension of A is finite. This Hochschild dimension is at least n , since $\text{Ext}_{A-A}^n(A, A \otimes A) \neq 0$ (see [9], VI.2.1).

By assumption, there exists a finite projective A -bimodule resolution of A by bimodules of finite type. Let m be the shortest length of such a resolution, and let $0 \rightarrow P_m \xrightarrow{\delta_m} P_{m-1} \xrightarrow{\delta_{m-1}} \dots \rightarrow P_0 \rightarrow A \rightarrow 0$ be a projective A -bimodule resolution of A by bimodules of finite type of length m . Clearly, m is greater than the Hochschild dimension of A , so we need only prove that $m = n$. Assume for a contradiction that $m > n$. Then $H^m(A, A \otimes A) = 0$, and therefore $H^m(A, P) = 0$ for any projective A -bimodule of finite type P . In particular, $H^m(A, P_m) = 0$, so the map $\text{Hom}_{A-A}(P_{m-1}, P_m) \xrightarrow{-\circ\delta_m} \text{Hom}_{A-A}(P_m, P_m)$ is onto. Therefore there exists $\sigma \in \text{Hom}_{A-A}(P_{m-1}, P_m)$ such that $\sigma \circ \delta_m = \text{id}_{P_m}$, i.e., δ_m splits, so P_m is a direct

summand in P_{m-1} . Therefore we can write $P_{m-1} = P_m \oplus Q$ with Q a projective A -bimodule of finite type. But then the sequence $0 \rightarrow Q \xrightarrow{\delta_{m-1|Q}} P_{m-2} \rightarrow \dots \rightarrow P_0 \rightarrow A \rightarrow 0$ is a projective A -bimodule resolution of A by bimodules of finite type of length strictly less than m , a contradiction. Therefore $m = n$ and the Hochschild dimension of A is n . \square

We shall use the results and notations of Bocklandt [7]. We recall some of them here.

Let Q be a connected quiver, fixed throughout. Let Q_j denote the set of paths in Q of length j . We assume throughout that Q_0 and Q_1 are finite. The maps $s, t : Q_1 \rightarrow Q_0$ send an arrow to its *source* and *target*.

A *potential* is an element in the vector space $\text{Pot}(Q) := \mathbb{k}Q / [\mathbb{k}Q, \mathbb{k}Q]$. It can be viewed as an element in $\mathbb{k}Q$ via the map $c : \text{Pot}(Q) \rightarrow \mathbb{k}Q$ that sends a cycle $a_n \dots a_1$ (read from right to left) to $\sum_i a_{i-1} \dots a_1 a_n \dots a_i$ (this map is denoted by \cup in [7]). If $p = a_n \dots a_2 a_1$ is a path and b is an arrow, then pb^{-1} denotes the path $a_n \dots a_2$ if $b = a_1$, and is 0 otherwise. Define similarly the path $b^{-1}p$. For each $a \in Q_1$, we shall consider the map $\partial_a : \text{Pot}(Q) \rightarrow \mathbb{k}Q$ that sends an element p in $\text{Pot}(Q)$ to $c(p)a^{-1} = a^{-1}c(p)$. Finally, if W is a potential, we shall denote by $A(Q, W)$ the algebra $\mathbb{k}Q / \mathcal{I}(\{\partial_a W; a \in Q_1\})$ (called a *vacualgebra* in [7]) where $\mathcal{I}(S)$ is the two-sided ideal of $\mathbb{k}Q$ generated by the set S . Bocklandt proved in [7], Theorem 3.1, that if a quiver algebra $A = \mathbb{k}Q / I$ is a graded Calabi–Yau algebra of dimension 3, then there exists a (homogeneous) potential W such that $A = A(Q, W)$. Moreover, if W is a potential, the algebra $A(Q, W)$ is Calabi–Yau of dimension 3 if and only if a certain complex C_W is exact [7], Section 4.2.

The results in [7] are still true with the definition of a Calabi–Yau algebra given here. Indeed, Bocklandt uses only [7], Property 2.2, applied to graded algebras, which is still true with Ginzburg’s definition of a Calabi–Yau algebra: the fact that the global dimension of graded Calabi–Yau algebras of dimension n is n follows from Remark 2.8, and we have the result that follows.

Proposition 2.3. *Let A be a Calabi–Yau algebra of dimension n over a field, and let X and Y be two finite-dimensional left A -modules. Let $(-)^*$ denote the ordinary \mathbb{k} -dual. Then $\text{Ext}_A^\bullet(X, Y)^* \cong \text{Ext}_A^{n-\bullet}(Y, X)$.*

The proof of this proposition relies on the lemmas below.

Lemma 2.4. *Let $P_\bullet \rightarrow A \rightarrow 0$ be a projective A -bimodule resolution of A . Then for any left A -module X , the complex $P_\bullet \otimes_A X \rightarrow X \rightarrow 0$ is a projective left A -module resolution of X .*

Proof. Each module $P_k \otimes_A X$ is a projective left A -module, since by adjunction there is an isomorphism $\text{Hom}_A(P_k \otimes_A X, \cdot) \cong \text{Hom}_{A-A}(P_k, \text{Hom}_{\mathbb{k}}(X, \cdot))$, and the functor on the right-hand side is exact since P_k is projective as an A -bimodule.

We must now prove that $P_\bullet \otimes_A X \rightarrow X \rightarrow 0$ is exact: since \mathbb{k} is a field, the algebra A is flat over \mathbb{k} , and hence the enveloping algebra $A^e = A \otimes A^{\text{op}}$ is flat as a left A -module and as a right A -module. Consequently, the projective A^e -modules P_k are flat as left A -modules and as right A -modules. Flatness is a property preserved by taking kernels of epimorphisms, so that all the syzygies $\Omega_{A^e}^k(A)$ are flat as left A -modules and right A -modules. It then follows that for all $k \in \mathbb{N}$, the sequence $0 \rightarrow \Omega_{A^e}^{k+1}(A) \otimes_A X \rightarrow P_k \otimes_A X \rightarrow \Omega_{A^e}^k(A) \otimes_A X \rightarrow 0$ is exact, and hence the sequence $P_\bullet \otimes_A X$ is exact. \square

We define the following structures: if X is a left A -module, then X^* is a right A -module with $\alpha^a : x \mapsto \alpha(ax)$ (for $\alpha \in X^*$, $a \in A$ and $x \in X$); if X is an A -bimodule, then X^* is an A -bimodule with ${}^a\alpha^b : x \mapsto \alpha(bxa)$ (for $\alpha \in X^*$, $a, b \in A$ and $x \in X$); if X is a left A -module and X' is a right A -module, then $X \otimes X'$ is an A -bimodule with $a \cdot (x \otimes x') \cdot b = ax \otimes x'b$ (for $a, b \in A$, $x \in X$ and $x' \in X'$); and if X and Y are left A -modules, $\text{Hom}_{\mathbb{k}}(X, Y)$ is an A -bimodule with ${}^a f^b : x \mapsto af(bx)$ (for $a, b \in A$, $f \in \text{Hom}_{\mathbb{k}}(X, Y)$ and $x \in X$).

Lemma 2.5. *Let X and Y be finite-dimensional left A -modules. Then there is an isomorphism $\text{Ext}_A^k(X, Y) \cong H^k(A, \text{Hom}_{\mathbb{k}}(X, Y))$.*

Proof. The result is in [9, IX.4.4], and can also be proved using Lemma 2.4 above. \square

Lemma 2.6. *For any A -bimodule M , we have $(H_k(A, M))^* \cong H^k(A, M^*)$.*

Proof. Let P_\bullet be a projective A -bimodule resolution. Then $H_\bullet(A, M)$ is the homology of $P_\bullet \otimes_{A^e} M \rightarrow 0$. Dualising this gives $0 \rightarrow (P_\bullet \otimes_{A^e} M)^*$. We have

$$(P_\bullet \otimes_{A^e} M)^* = \text{Hom}_{\mathbb{k}}(P_\bullet \otimes_{A^e} M, \mathbb{k}) \cong \text{Hom}_{A^e}(P_\bullet, \text{Hom}_{\mathbb{k}}(M, \mathbb{k})) = \text{Hom}_{A^e}(P_\bullet, M^*)$$

with adjunction. The differentials correspond via this isomorphism, therefore the cohomology of the dual complex is $H^\bullet(A, M^*)$. \square

Lemma 2.7. *Let X and Y be any finite-dimensional left A -modules. Then there is an isomorphism of A -bimodules $\text{Hom}_{\mathbb{k}}(X, Y)^* \cong \text{Hom}_{\mathbb{k}}(Y, X)$.*

Proof. We combine the following well-known \mathbb{k} -isomorphisms, which are also A -bimodule homomorphisms:

$$\begin{aligned} Y \otimes X^* &\rightarrow \text{Hom}_{\mathbb{k}}(X, Y), \\ y \otimes \alpha &\mapsto [x \mapsto \alpha(x)y], \\ X \otimes Y^* &\rightarrow (Y \otimes X^*)^*, \\ x \otimes \beta &\mapsto [y \otimes \alpha \mapsto \alpha(x)\beta(y)]. \end{aligned} \quad \square$$

We can now prove Proposition 2.3.

Proof. We first apply Van den Bergh’s duality theorem [23]: A has a finite projective resolution by A -bimodules of finite type; $H^k(A, A \otimes A) = 0$ for all $k \neq n$; and $U := H^n(A, A \otimes A) \cong A$ is an invertible A -bimodule (with $U^{-1} = A$). Therefore Van den Bergh’s duality theorem applies, and we have $H^k(A, M) \cong H_{n-k}(A, M)$.

Applying the lemmas above, we get:

$$\begin{aligned} \text{Ext}_A^k(X, Y)^* &\cong H^k(A, \text{Hom}_{\mathbb{k}}(X, Y))^* \cong H_{n-k}(A, \text{Hom}_{\mathbb{k}}(X, Y))^* \\ &\cong H^{n-k}(A, \text{Hom}_{\mathbb{k}}(X, Y)^*) \cong H^{n-k}(A, \text{Hom}_{\mathbb{k}}(Y, X)) \\ &\cong \text{Ext}_A^{n-k}(Y, X). \end{aligned} \quad \square$$

Remark 2.8. Let A be a Calabi–Yau algebra of dimension n such that there exists a nonzero finite-dimensional A -module X . Then the global dimension of A is n : we know from [9], IX.7.6, that the global dimension of A is at most the Hochschild dimension of A , that is, n . Moreover, by Proposition 2.3, we have $\text{Ext}_A^n(X, X) = \text{Ext}_A^0(X, X)^* = \text{Hom}_A(X, X)^* \neq 0$, so that the global dimension of A , which is larger than the projective dimension of the A -module X , is at least n . Therefore it is equal to n .

3. PBW deformations of graded Calabi–Yau algebras of dimension 3

3.1. Algebras associated with nonhomogeneous potentials

Theorem 3.1. *Suppose that $A = A(Q, W_{N+1})$ is a graded Calabi–Yau algebra of dimension 3, where W_{N+1} is a homogeneous potential of degree $N + 1$, and let $W = W_{N+1} + W' = W_{N+1} + W_N + \dots + W_0$ be a potential with $\deg W_j = j$ for each $0 \leq j \leq N + 1$. Then $A' := A(Q, W)$ is a PBW deformation of A .*

Proof. Denote by R the $\mathbb{k}Q_0$ -bimodule spanned by $\{\partial_a W_{N+1}; a \in Q_1\}$ (which can be identified as a set with the \mathbb{k} -vector space generated by $\{\partial_a W_{N+1}; a \in Q_1\}$) and by P the $\mathbb{k}Q_0$ -bimodule spanned by $\{\partial_a W; a \in Q_1\}$ (similarly, this can be identified with the \mathbb{k} -vector space generated by $\{\partial_a W; a \in Q_1\}$). Define $\varphi: R \rightarrow \mathbb{k}Q$ by setting $\varphi(\partial_a W_{N+1}) = -\partial_a W'$. Define also $\varphi_j: R \rightarrow (\mathbb{k}Q_1)^{\otimes_{\mathbb{k}Q_0} j} = \mathbb{k}Q_j$ by composing φ with the natural projection from $\mathbb{k}Q$ to $\mathbb{k}Q_j$ (so that $\varphi_j(\partial_a W_{N+1}) = -\partial_a W_{j+1}$). Set $\varphi_j^{1,N} = \varphi_j \otimes_{\mathbb{k}Q_0} \text{id}: R \otimes_{\mathbb{k}Q_0} \mathbb{k}Q_1 \rightarrow \mathbb{k}Q_{j+1}$ and $\varphi_j^{2,N+1} = \text{id} \otimes_{\mathbb{k}Q_0} \varphi_j: \mathbb{k}Q_1 \otimes_{\mathbb{k}Q_0} R \rightarrow \mathbb{k}Q_{j+1}$. Finally, set $F^j = \bigoplus_{k=0}^j \mathbb{k}Q_k$.

The \mathbb{k} -algebra A can be viewed as a $\mathbb{k}Q_0$ -algebra, with $\mathbb{k}Q_0$ semisimple. Now since A is Calabi–Yau, the complex $C_{W_{N+1}}$ in [7] is a projective A -bimodule resolution of A , from which we can see that A is N -Koszul (as a $\mathbb{k}Q_0$ -algebra). Therefore

the results in [5] apply, so that A' is a PBW deformation of A if and only if the following conditions are satisfied:

(PBW1) $P \cap F^{N-1} = \{0\}$.

(PBW2) $(\varphi_{N-1}^{1,N} - \varphi_{N-1}^{2,N+1})(\mathbb{k}Q_1 \otimes_{\mathbb{k}Q_0} R \cap R \otimes_{\mathbb{k}Q_0} \mathbb{k}Q_1) \subseteq R$.

(PBW3) For all $j, 1 \leq j \leq N - 1$,

$$(\varphi_j(\varphi_{N-1}^{1,N} - \varphi_{N-1}^{2,N+1}) + \varphi_{j-1}^{1,N} - \varphi_{j-1}^{2,N+1})(\mathbb{k}Q_1 \otimes_{\mathbb{k}Q_0} R \cap R \otimes_{\mathbb{k}Q_0} \mathbb{k}Q_1) = \{0\}.$$

(PBW4) $\varphi_0(\varphi_{N-1}^{1,N} - \varphi_{N-1}^{2,N+1})(\mathbb{k}Q_1 \otimes_{\mathbb{k}Q_0} R \cap R \otimes_{\mathbb{k}Q_0} \mathbb{k}Q_1) = \{0\}$.

Let us first give an alternative description of $R \otimes_{\mathbb{k}Q_0} \mathbb{k}Q_1 \cap \mathbb{k}Q_1 \otimes_{\mathbb{k}Q_0} R$: consider the \mathbb{k} -linear map $\theta: \mathbb{k}Q_0 \rightarrow \mathbb{k}Q_1 \otimes_{\mathbb{k}Q_0} R$ that sends a vertex e to $\theta(e) = \sum_{a \in eQ_1} a \partial_a W_{N+1}$. For any cycle p , we have

$$\begin{aligned} \sum_{a \in eQ_1} a \partial_a p &= \sum_{a \in eQ_1; a \in p} aa^{-1}c(p) \\ &= ec(p) = c(p)e \\ &= \sum_{b \in Q_1e; b \in p} c(p)b^{-1}b = \sum_{b \in Q_1e} \partial_b p b. \end{aligned}$$

Therefore $\theta(e)$ is contained in $R \otimes_{\mathbb{k}Q_0} \mathbb{k}Q_1$, and so we have a $\mathbb{k}Q_0$ -linear map $\theta: \mathbb{k}Q_0 \rightarrow \mathbb{k}Q_1 \otimes_{\mathbb{k}Q_0} R \cap R \otimes_{\mathbb{k}Q_0} \mathbb{k}Q_1$, and $\theta(e) = \sum_{a \in eQ_1} a \partial_a W_{N+1} = \sum_{b \in Q_1e} \partial_b W_{N+1} b = ec(W_{N+1})$.

The map θ is one-to-one: if $\theta(\sum_{s \in Q_0} \lambda_s s) = 0$, then for a fixed $e \in Q_0$ we have

$$0 = e\theta\left(\sum_{s \in Q_0} \lambda_s s\right) = \sum_{s \in Q_0} \lambda_s se c(W_{N+1}) = \lambda_e e c(W_{N+1}) = \lambda_e \theta(e),$$

so we only need to check that $\theta(e) \neq 0$. We use [7], Theorem 3.1: A is Calabi–Yau, so we know that e is (for instance) the source of an arrow b , and moreover b is contained in $c(W_{N+1})$, therefore e is contained in $c(W_{N+1})$, so $\theta(e) = ec(W_{N+1}) \neq 0$.

To prove that θ is an isomorphism, we shall prove that $R \otimes_{\mathbb{k}Q_0} \mathbb{k}Q_1 \cap \mathbb{k}Q_1 \otimes_{\mathbb{k}Q_0} R$ and $\mathbb{k}Q_0$ have same dimension. We know from [5] that

$$R \otimes_{\mathbb{k}Q_0} \mathbb{k}Q_1 \cap \mathbb{k}Q_1 \otimes_{\mathbb{k}Q_0} R = (\text{Tor}_3^A(\mathbb{k}Q_0, \mathbb{k}Q_0))_{N+1} = \text{Tor}_3^A(\mathbb{k}Q_0, \mathbb{k}Q_0),$$

so $\dim_{\mathbb{k}} R \otimes_{\mathbb{k}Q_0} \mathbb{k}Q_1 \cap \mathbb{k}Q_1 \otimes_{\mathbb{k}Q_0} R = \dim_{\mathbb{k}} \text{Tor}_3^A(\mathbb{k}Q_0, \mathbb{k}Q_0)$. As before (cf. Lemma 2.4), $C_{W_{N+1}} \otimes_A \mathbb{k}Q_0$ is a projective left A -module resolution of $\mathbb{k}Q_0$. Applying the functor $\mathbb{k}Q_0 \otimes_A -$ gives $\text{Tor}_3^A(\mathbb{k}Q_0, \mathbb{k}Q_0) = \text{Ker}(\text{id} \otimes_A \delta_3 \otimes_A \text{id})$ using

the notations in [7]. Therefore

$$\begin{aligned} \#Q_0 &\leq \dim_{\mathbb{k}} R \otimes_{\mathbb{k}Q_0} \mathbb{k}Q_1 \cap \mathbb{k}Q_1 \otimes_{\mathbb{k}Q_0} R \\ &= \dim_{\mathbb{k}} \operatorname{Tor}_3^A(\mathbb{k}Q_0, \mathbb{k}Q_0) \\ &\leq \dim_{\mathbb{k}} \mathbb{k}Q_0 \otimes_A A \otimes_{\mathbb{k}Q_0} \operatorname{span}_{\mathbb{k}}\{ec(W_{N+1}); e \in Q_0\} \otimes_{\mathbb{k}Q_0} A \otimes_A \mathbb{k}Q_0 \\ &= \dim_{\mathbb{k}} \operatorname{span}_{\mathbb{k}}\{ec(W_{N+1}); e \in Q_0\} \leq \#Q_0 \end{aligned}$$

so finally $\dim_{\mathbb{k}} R \otimes_{\mathbb{k}Q_0} \mathbb{k}Q_1 \cap \mathbb{k}Q_1 \otimes_{\mathbb{k}Q_0} R = \dim_{\mathbb{k}} \mathbb{k}Q_0$ and θ is an isomorphism. In particular, $\{\theta(e); e \in Q_0\}$ is a basis for $R \otimes_{\mathbb{k}Q_0} \mathbb{k}Q_1 \cap \mathbb{k}Q_1 \otimes_{\mathbb{k}Q_0} R$.

Let us now check that conditions (PBW1) to (PBW4) hold:

(PBW1): Clearly

$$P \cap F^{N-1} = \left\{ \sum_{a \in Q_1} \lambda_a \partial_a W; \lambda_a \in \mathbb{k} \text{ and } \sum_{a \in Q_1} \lambda_a \partial_a W_{N+1} = 0 \right\}.$$

However, the $\partial_a W_{N+1}$ for $a \in Q_1$ are linearly independent: consider the resolution $C_{W_{N+1}}$ in [7]. By Lemma 2.4, $C_{W_{N+1}} \otimes_A \mathbb{k}Q_0$ is a projective resolution of $\mathbb{k}Q_0$ as a left A -module. The maps in the complex $\operatorname{Hom}_A(C_{W_{N+1}} \otimes_A \mathbb{k}Q_0, \mathbb{k}Q_0)$ vanish, so $\#Q_1 = \dim_{\mathbb{k}} \operatorname{Ext}_A^1(\mathbb{k}Q_0, \mathbb{k}Q_0) = \dim_{\mathbb{k}} \operatorname{Ext}_A^2(\mathbb{k}Q_0, \mathbb{k}Q_0) = \dim_{\mathbb{k}} \operatorname{span}_{\mathbb{k}}\{\partial_a W_{N+1}; a \in Q_1\}$ using the Calabi–Yau property. Therefore, $P \cap F^{N-1} = \{0\}$.

(PBW2): This condition is equivalent to: $(\varphi_{N-1}^{1,N} - \varphi_{N-1}^{2,N+1})(\theta(e)) \in R$ for all $e \in Q_0$. We have

$$\begin{aligned} (\varphi_{N-1}^{1,N} - \varphi_{N-1}^{2,N+1})(\theta(e)) &= \varphi_{N-1}^{1,N} \left(\sum_{b \in Q_1 e} \partial_b W_{N+1} b \right) - \varphi_{N-1}^{2,N+1} \left(\sum_{a \in e Q_1} a \partial_a W_{N+1} \right) \\ &= \sum_{b \in Q_1 e} \varphi_{N-1}(\partial_b W_{N+1}) b - \sum_{a \in e Q_1} a \varphi_{N-1}(\partial_a W_{N+1}) \\ &= - \sum_{b \in Q_1 e} \partial_b W_N b + \sum_{a \in e Q_1} a \partial_a W_N = 0 \in R \end{aligned}$$

as above.

(PBW3): Using the fact that $(\varphi_{N-1}^{1,N} - \varphi_{N-1}^{2,N+1})(\theta(e)) = 0$ for all $e \in Q_0$, this condition becomes $(\varphi_{j-1}^{1,N} - \varphi_{j-1}^{2,N+1})(\theta(e)) = 0$ for all $e \in Q_0$. This is true by a calculation similar to that above.

(PBW4): This condition is trivially true using the fact that for all $e \in Q_0$ we have $(\varphi_{N-1}^{1,N} - \varphi_{N-1}^{2,N+1})(\theta(e)) = 0$.

Therefore A' is a PBW deformation of A . □

This leads to the following question: when is a PBW deformation of a Calabi–Yau algebra of dimension 3 obtained from a (nonhomogeneous) potential?

Let $A = A(Q, W_{N+1})$ be a Calabi–Yau algebra of dimension 3, and let A' be a PBW deformation of A . We recall here the notations of [5]: let P be a sub- $\mathbb{k}Q_0$ -bimodule of F^N such that $A' = \mathbb{k}Q/\mathcal{I}(P)$, let $\pi : F^N \rightarrow \mathbb{k}Q_N$ be the natural projection, and set $R = \pi(P)$, with $A = \mathbb{k}Q/\mathcal{I}(R)$. The condition $P \cap F^{N-1} = \{0\}$ holds since A' is a PBW deformation of A , so $\pi : P \rightarrow R$ is an isomorphism. For each $a \in Q_1$, there exists a unique element in P whose image by π is $\partial_a W_{N+1}$; this element can be written $\partial_a W_{N+1} - \varphi(\partial_a W_{N+1})$ with $\varphi(\partial_a W_{N+1}) \in F^{N-1}$. This defines a $\mathbb{k}Q_0$ -linear map $\varphi : R \rightarrow F^{N-1}$, and $P = \text{span}_{\mathbb{k}Q_0} \{ \partial_a W_{N+1} - \varphi(\partial_a W_{N+1}); a \in Q_1 \}$. Define maps $\varphi_j : R \rightarrow \mathbb{k}Q_j$ as above by composing φ with the projections $\mathbb{k}Q \rightarrow \mathbb{k}Q_j$ for $0 \leq j \leq N - 1$.

If A' can be obtained from a potential, that is, if there exists a potential $W = W_{N+1} + W'$ with $\deg W' \leq N$ such that $A' = A(Q, W)$, then we have $\varphi(\partial_a W_{N+1}) = -\partial_a W'$, and therefore

$$(PBW2') \quad (\varphi_{N-1}^{1,N} - \varphi_{N-1}^{2,N+1})(\theta(e)) = 0 \quad \text{for all } e \in Q_0,$$

since $(\varphi_{N-1}^{1,N} - \varphi_{N-1}^{2,N+1})(\theta(e)) = -\sum_{b \in Q_1 e} \partial_b W_N b + \sum_{a \in e Q_1} a \partial_a W_N = 0$. This condition (PBW2') is in fact sufficient:

Theorem 3.2. *Let $A = A(Q, W_{N+1})$ be a Calabi–Yau algebra of dimension 3, and let A' be a PBW deformation of A . Assume that $\text{char}(\mathbb{k})$ does not divide $N!$. Then A' can be obtained from a potential if and only if condition (PBW2') holds.*

Proof. Let A' be a PBW deformation of A satisfying (PBW2'). Since φ_{j-1} is $\mathbb{k}Q_0$ -linear, $\varphi_{j-1}(\partial_a W_{N+1})$ is a linear combination of paths all starting at $t(a)$ and all ending at $s(a)$. Therefore A' satisfies, for all $1 \leq j \leq N$,

$$(*_j) \quad \sum_{a \in e Q_1} a \varphi_{j-1}(\partial_a W_{N+1}) - \sum_{b \in Q_1 e} \varphi_{j-1}(\partial_b W_{N+1}) b = 0 \quad \text{for all } e \in Q_0.$$

The element $\varphi_{j-1}(\partial_a W_{N+1})$ is a linear combination of paths of length $j - 1$ starting at $t(a)$ and ending at $s(a)$, therefore we can write

$$\begin{aligned} \varphi_{j-1}(\partial_a W_{N+1}) &= \sum_{q \in s(a) Q_{j-1} t(a)} \lambda_{a,q} q = \sum_{\substack{q \in Q_{j-1}: \\ a q \in t(a) Q_j t(a)}} \lambda_{a,q} q \\ &= \sum_{\substack{q \in Q_{j-1}: \\ q a \in s(a) Q_j s(a)}} \lambda_{a,q} q \end{aligned}$$

for some scalars $\lambda_{a,q} \in \mathbb{k}$. Then $(*_j)$ becomes

$$\sum_{\substack{a \in e Q_1, q \in Q_{j-1}: \\ a q \in e Q_j e}} \lambda_{a,q} a q = \sum_{\substack{b \in Q_1 e, q \in Q_{j-1}: \\ q b \in e Q_j e}} \lambda_{b,q} q b \quad \text{for all } e \in Q_0.$$

If σ is a cycle of length $j \geq 2$, then we can write $\sigma = aqb$ with $a, b \in Q_1$ and $q \in Q_{j-2}$. This cycle occurs once on each side of the equality above with $e = t(a) = s(a)$, so $\lambda_{a,qb} = \lambda_{b,aq}$. Moreover, if $j = 1$, we also have $\lambda_{a,e} = \lambda_{a,e}$ if $a \in eQ_1e$. Now if we write $\sigma = a_j \dots a_1$, we have $\lambda_{a_j, a_{j-1} \dots a_1} = \lambda_{a_1, a_j \dots a_2} = \lambda_{a_2, a_1 a_j \dots a_3} = \dots$, so that the coefficient only depends on $c(\sigma)$. We denote this common coefficient by $\hat{\lambda}_\sigma$.

Let σ be a cycle in Q . Define a new element $c'(\sigma)$ in $\mathbb{k}Q$ as follows: if σ is not a power of a cycle, then we set $c'(\sigma) = c(\sigma)$; if $\sigma = \tau^m$ is a power of a cycle, with m maximal, then $c'(\sigma)$ is the sum of the m^{th} powers of all the terms in $c(\tau)$. We shall denote by $\bar{\sigma}$ the class of a cycle σ modulo cyclic permutations.

Now define $W_j := -\frac{1}{j} \sum_{e \in Q_0} \sum_{\sigma \in eQ_1e} \hat{\lambda}_\sigma c'(\sigma)$. Then, if $a \in Q_1$ we want to compute $\partial_a W_j$. For this, we shall need the following lemmas.

Lemma 3.3. *Let σ be a cycle of length j which can be written $\sigma = ap$ where a is an arrow and p is a path, and such that there exists a cyclic permutation of length at most $j - 1$ which when applied to σ again gives ap . Then σ is a power of a cycle.*

Proof. The hypothesis implies that a occurs in p , so we can write $p = \alpha\beta$ for some paths α and β , and these paths can be chosen such that $\sigma = \alpha\alpha\beta = \beta\beta\alpha$ (\otimes).

Let $\ell(q)$ denote the length of a path q . If $\ell(\beta) = \ell(\alpha)$, then necessarily $\alpha = \beta$ and therefore $\sigma = (\alpha\alpha)^2$ is a power of a cycle.

If $\ell(\beta) > \ell(\alpha)$, then the expressions of σ above show that $\beta = \alpha\alpha\beta'$ for some path β' , with $\ell(\beta') < \ell(\beta)$. Therefore, $\sigma = \alpha\alpha\alpha\alpha\beta' = \alpha\alpha\beta'\alpha\alpha$ and so $\alpha\alpha\beta' = \alpha\beta'\alpha\alpha$. We have obtained an identity similar to (\otimes), but with a path of smaller length. A descending induction shows that σ is a power of a cycle. \square

Lemma 3.4. *Let σ be a cycle of length j and let a be an arrow. Then $\partial_a c'(\sigma) = j \sum_{q \in Q_{j-1}; \bar{aq} = \bar{\sigma}} q$.*

Proof. First assume that σ is not a power of a cycle. Then $c'(\sigma) = c(\sigma)$. This is a sum of j terms, and all the terms are obtained by applying cyclic permutations to σ , and therefore they all give the same derivatives. So we need only look at the derivative with respect to a of σ , and multiply the result by j .

No cyclic permutation of length at most $j - 1$ of σ is equal to σ , and therefore if $\bar{\sigma} = \bar{ap} = \bar{aq}$ for two different choices of a within σ , then $p \neq q$ (Lemma 3.3). Therefore $\partial_a \sigma = \sum_{q \in Q_{j-1}; \bar{aq} = \bar{\sigma}} q$ and we get $\partial_a c'(\sigma) = j \sum_{q \in Q_{j-1}; \bar{aq} = \bar{\sigma}} q$.

Now assume that σ is a power of a cycle, and write $\sigma = \tau^m$ with m maximal (so that τ is not a power of a cycle). As before, there are $\frac{j}{m}$ terms in $c'(\sigma)$, all obtained by applying cyclic permutations to σ , so we need only consider $\partial_a \sigma$ and multiply the result by $\frac{j}{m}$.

To compute $\partial_a \sigma = \partial_a \tau^m$, we choose a factor τ , then for each a in this factor, we apply the appropriate cyclic permutation to σ so that the a occurs at the end of the

cycle, and then we remove the a . We then do the same for each of the other factors τ , and in each case we will obtain the same result. Therefore a factor m appears when we compute this derivative:

$$\partial_a \tau^m = m \sum_{\substack{(p_1, p_2): \\ \tau = p_2 a p_1}} p_1 \tau^{m-1} p_2$$

(in this sum, $p_1 p_2$ is necessarily a path of length $\frac{j}{m} - 1$). Now consider the map φ from $\{(p_1, p_2); p_2 a p_1 = \tau\}$ to $\{q; \overline{a q} = \bar{\sigma}\}$ given by $\varphi(p_1, p_2) = p_1 \tau^{m-1} p_2$. This map is one-to-one: if $\varphi(p_1, p_2) = \varphi(p'_1, p'_2)$ with $(p_1, p_2) \neq (p'_1, p'_2)$, assume that $\ell(p_1) < \ell(p'_1)$ so that $\ell(p'_2) < \ell(p_2)$. Since $p_1 \tau^{m-1} p_2 = p'_1 \tau^{m-1} p'_2$, we have $p'_1 = p_1 p''_1$ and $p_2 = p''_2 p'_2$ for some paths p''_1 and p''_2 , so we get $\tau^{m-1} p''_2 = p''_1 \tau^{m-1}$. By [17], Lemma 2.4, there exist a cycle γ and an integer $t \geq 2$ such that $\tau = \gamma^t$, a contradiction. Therefore φ is one-to-one. The map φ is also onto: if $\bar{\sigma} = \overline{a q}$, the a in this equality is chosen within σ and therefore within one of the factors τ , so we can write $\tau = p_2 a p_1$ and $\sigma = \tau^u p_2 a p_1 \tau^v$ for some integers u and v such that $u + v = m - 1$, and therefore $q = p_1 \tau^{m-1} p_2 = \varphi(p_1, p_2)$.

Finally, we have $\partial_a \sigma = m \sum_{q \in Q_{j-1}; \overline{a q} = \bar{\sigma}} q$, and this gives the result. □

Now we have

$$\begin{aligned} \partial_a W_j &= -\frac{1}{j} \sum_{e \in Q_0} \sum_{\sigma \in e Q_j e} \hat{\lambda}_\sigma \partial_a c'(\sigma) = -\frac{1}{j} \sum_{\substack{e \in Q_0 \\ \sigma \in e Q_j e}} \sum_{\substack{q \in Q_{j-1}: \\ \overline{a q} = \bar{\sigma}}} j \hat{\lambda}_\sigma q \\ &= - \sum_{\substack{q \in Q_{j-1}: \\ a q \in t(a) Q_j t(a)}} \lambda_{a, q} q = -\varphi_{j-1}(\partial_a W_{N+1}). \end{aligned}$$

Therefore $A' = A(Q, W)$ with $W = W_{N+1} + W_N + \dots + W_1$. □

Remark 3.5. In the result above, the potential $W = W_{N+1} + W_N + \dots + W_1$ defining A' is unique. Indeed, it is easy to see that $\bigcap_{a \in Q_1} \text{Ker}(\partial_a) = \mathbb{k} Q_0$, and since the degree of each W_j is nonzero, it follows that two such potentials must be equal.

3.2. The algebras $A(Q, W)$ are Calabi–Yau . We now prove the following result.

Theorem 3.6. *Let $A = A(Q, W_{N+1})$ be a graded Calabi–Yau algebra of dimension 3. Let $A' = A(Q, W)$ be a PBW deformation of A defined by a potential $W = W_{N+1} + W'$ with $\text{deg } W' \leq N$. Then A' is Calabi–Yau of dimension 3.*

The proof will use the following characterisation of Calabi–Yau algebras (the ungraded analogue of [7], Theorem 4.2).

Lemma 3.7. *Let B be an associative \mathbb{k} -algebra with unit. If B has a self-dual (with respect to the functor $(\cdot)^\vee = \text{Hom}_{B-B}(\cdot, B \otimes B)$) projective B -bimodule resolution of finite length n by B -bimodules of finite type, then B is Calabi–Yau of dimension n .*

Proof. By assumption, we have a diagram

$$\begin{array}{ccccccc}
 P_n & \xrightarrow{d_n} & P_{n-1} & \xrightarrow{d_{n-1}} & \cdots & \xrightarrow{d_2} & P_1 & \xrightarrow{d_1} & P_0 \\
 \downarrow \alpha_n & & \downarrow \alpha_{n-1} & & & & \downarrow \alpha_1 & & \downarrow \alpha_0 \\
 P_0^\vee & \xrightarrow{-d_1^\vee} & P_1^\vee & \xrightarrow{-d_2^\vee} & \cdots & \xrightarrow{-d_{n-1}^\vee} & P_{n-1}^\vee & \xrightarrow{-d_n^\vee} & P_n^\vee
 \end{array}$$

in which all the squares commute and where all the maps α_i are isomorphisms. Then:

- B has a finite projective B -bimodule resolution by bimodules of finite type.
- For $0 < k < n$, we have

$$H^k(B, B \otimes B) = \frac{\text{Ker}(P_k^\vee \xrightarrow{d_{k+1}^\vee} P_{k+1}^\vee)}{\text{Im}(P_{k-1}^\vee \xrightarrow{d_k^\vee} P_k^\vee)} \cong \frac{\text{Ker}(P_{n-k} \xrightarrow{-d_{n-k}} P_{n-k-1})}{\text{Im}(P_{n-k+1} \xrightarrow{-d_{n-k+1}} P_{n-k})} = 0.$$

- For $k > n$, $H^k(B, B \otimes B) = 0$.
- $H^n(B, B \otimes B) = \frac{\text{Ker}(P_n^\vee \rightarrow 0)}{\text{Im}(P_{n-1}^\vee \xrightarrow{d_n^\vee} P_n^\vee)} \cong \frac{P_0}{\text{Im}(P_1 \xrightarrow{-d_1} P_0)} \cong B$.

Therefore B is Calabi–Yau of dimension n . □

We now prove Theorem 3.6.

Proof. We shall construct a self-dual projective resolution of A' .

Let R denote the $\mathbb{k}Q_0$ -bimodule spanned by $\{\partial_a W_{N+1}; a \in Q_1\}$ and P the $\mathbb{k}Q_0$ -bimodule spanned by $\{\partial_a W; a \in Q_1\}$. Let $\pi: P \rightarrow R$ be the projection defined by $\partial_a W \mapsto \partial_a W_{N+1}$. Consider the $\mathbb{k}Q_0$ -bimodule map $\tilde{\theta}: \mathbb{k}Q_0 \rightarrow P \otimes_{\mathbb{k}Q_0} \mathbb{k}Q_1$ that sends a vertex e to $\sum_{a \in Q_1 e} \partial_a W a$. For every $e \in Q_0$, we have $\tilde{\theta}(e) \in \mathbb{k}Q_1 \otimes_{\mathbb{k}Q_0} P$ since

$$\begin{aligned}
 \tilde{\theta}(e) &= \theta(e) - \sum_{a \in Q_1 e} \varphi(\partial_a W_{N+1})a = \theta(e) + \sum_{a \in Q_1 e} \partial_a W' a \\
 &= \theta(e) + \sum_{b \in e Q_1} b \partial_b W' = \theta(e) - \sum_{b \in e Q_1} b \varphi(\partial_b W_{N+1}) = \sum_{b \in e Q_1} b \partial_b W.
 \end{aligned}$$

Therefore $\tilde{\theta}$ takes values in $P \otimes_{\mathbb{k}Q_0} \mathbb{k}Q_1 \cap \mathbb{k}Q_1 \otimes_{\mathbb{k}Q_0} P$. We wish to compose $\tilde{\theta}$ with $\pi \otimes_{\mathbb{k}Q_0} \text{id}$ and with $\text{id} \otimes_{\mathbb{k}Q_0} \pi$. To this end we take an element $x =$

$\sum_{a,b \in Q_1} a \partial_b W = \sum_{c,d \in Q_1} \partial_c W d$ in $P \otimes_{\mathbb{k}Q_0} \mathbb{k}Q_1 \cap \mathbb{k}Q_1 \otimes_{\mathbb{k}Q_0} P$; then looking at the terms of degree $N + 1$ in x gives $\sum_{a,b \in Q_1} a \partial_b W_{N+1} = \sum_{c,d \in Q_1} \partial_c W_{N+1} d$, that is, $(\text{id} \otimes_{\mathbb{k}Q_0} \pi)(x) = (\pi \otimes_{\mathbb{k}Q_0} \text{id})(x)$. Therefore $(\pi \otimes_{\mathbb{k}Q_0} \text{id})\tilde{\theta} = (\text{id} \otimes_{\mathbb{k}Q_0} \pi)\tilde{\theta}$ takes values in $R \otimes_{\mathbb{k}Q_0} \mathbb{k}Q_1 \cap \mathbb{k}Q_1 \otimes_{\mathbb{k}Q_0} R$ and is equal to θ , which is an isomorphism. We deduce that $\tilde{\theta}: \mathbb{k}Q_0 \rightarrow \text{Im } \tilde{\theta}$ is an isomorphism, and that $(\pi \otimes_{\mathbb{k}Q_0} \text{id})|_{\text{Im } \tilde{\theta}}$ is an isomorphism from $\text{Im } \tilde{\theta}$ to $\text{Im } \theta$.

The sequence \mathcal{C} below is a complex of projective A' -bimodules of finite type:

$$\begin{aligned} \mathcal{C}: 0 \rightarrow A' \otimes_{\mathbb{k}Q_0} \text{Im } \tilde{\theta} \otimes_{\mathbb{k}Q_0} A' \xrightarrow{\delta_3} A' \otimes_{\mathbb{k}Q_0} P \otimes_{\mathbb{k}Q_0} A' \\ \xrightarrow{\delta_2} A' \otimes_{\mathbb{k}Q_0} \mathbb{k}Q_1 \otimes_{\mathbb{k}Q_0} A' \xrightarrow{\delta_1} A' \otimes_{\mathbb{k}Q_0} \mathbb{k}Q_0 \otimes_{\mathbb{k}Q_0} A' \xrightarrow{\mu} A' \rightarrow 0 \end{aligned}$$

where μ is the multiplication map and $\delta_1, \delta_2, \delta_3$ are morphisms of A' -bimodules defined by

$$\begin{aligned} \delta_1(1 \otimes_{\mathbb{k}Q_0} a \otimes_{\mathbb{k}Q_0} 1) &= a \otimes_{\mathbb{k}Q_0} s(a) \otimes_{\mathbb{k}Q_0} 1 - 1 \otimes_{\mathbb{k}Q_0} t(a) \otimes_{\mathbb{k}Q_0} a, \\ \delta_2(1 \otimes_{\mathbb{k}Q_0} \partial_a W \otimes_{\mathbb{k}Q_0} 1) &= \Delta(\partial_a W), \\ \delta_3(1 \otimes_{\mathbb{k}Q_0} \tilde{\theta}(e) \otimes_{\mathbb{k}Q_0} 1) &= \sum_{a \in e Q_1} a \otimes_{\mathbb{k}Q_0} \partial_a W \otimes_{\mathbb{k}Q_0} 1 \\ &\quad - \sum_{b \in Q_1 e} 1 \otimes_{\mathbb{k}Q_0} \partial_b W \otimes_{\mathbb{k}Q_0} b. \end{aligned}$$

For δ_2 , the map Δ is the linear map defined on the monomials by $\Delta(a_k \dots a_1) = \sum_{i=1}^k a_k \dots a_{i+1} \otimes_{\mathbb{k}Q_0} a_i \otimes_{\mathbb{k}Q_0} a_{i-1} \dots a_1$, and $\Delta(1_{\mathbb{k}}) = 0$ (the bar denotes the class of an element in $\mathbb{k}Q$ modulo P).

Clearly $\mu \delta_1 = 0$. We have $\delta_1 \circ \delta_2(1 \otimes_{\mathbb{k}Q_0} \partial_a W \otimes_{\mathbb{k}Q_0} 1) = \partial_a W \otimes_{\mathbb{k}Q_0} t(a) \otimes_{\mathbb{k}Q_0} 1 +$ terms that simplify $- 1 \otimes_{\mathbb{k}Q_0} s(a) \otimes_{\mathbb{k}Q_0} \partial_a W = 0$. Finally,

$$\begin{aligned} \delta_2 \circ \delta_3(1 \otimes_{\mathbb{k}Q_0} \tilde{\theta}(e) \otimes_{\mathbb{k}Q_0} 1) &= \sum_{a \in e Q_1} a \Delta(\partial_a W) - \sum_{b \in Q_1 e} \Delta(\partial_b W) b \\ &= \left[\Delta(\tilde{\theta}(e)) - \sum_{a \in e Q_1} 1 \otimes_{\mathbb{k}Q_0} a \otimes_{\mathbb{k}Q_0} \partial_a W \right] \\ &\quad - \left[\Delta(\tilde{\theta}(e)) - \sum_{b \in Q_1 e} \partial_b W \otimes_{\mathbb{k}Q_0} b \otimes_{\mathbb{k}Q_0} 1 \right] = 0. \end{aligned}$$

We will now prove that \mathcal{C} is a resolution of A' : the complex \mathcal{C} is isomorphic to the complex \mathcal{C}' defined as follows:

$$\begin{aligned} \mathcal{C}': 0 \rightarrow A' \otimes_{\mathbb{k}Q_0} \text{Im } \theta \otimes_{\mathbb{k}Q_0} A' \xrightarrow{\delta'_3} A' \otimes_{\mathbb{k}Q_0} R \otimes_{\mathbb{k}Q_0} A' \\ \xrightarrow{\delta'_2} A' \otimes_{\mathbb{k}Q_0} \mathbb{k}Q_1 \otimes_{\mathbb{k}Q_0} A' \xrightarrow{\delta_1} A' \otimes_{\mathbb{k}Q_0} \mathbb{k}Q_0 \otimes_{\mathbb{k}Q_0} A' \xrightarrow{\mu} A' \rightarrow 0 \end{aligned}$$

with $\delta'_2(1 \otimes_{\mathbb{k}Q_0} \partial_a W_{N+1} \otimes_{\mathbb{k}Q_0} 1) = \Delta(\partial_a W)$ and $\delta'_3(1 \otimes_{\mathbb{k}Q_0} \theta(e) \otimes_{\mathbb{k}Q_0} 1) = \sum_{a \in eQ_1} a \otimes_{\mathbb{k}Q_0} \partial_a W_{N+1} \otimes_{\mathbb{k}Q_0} 1 - \sum_{b \in Q_1 e} 1 \otimes_{\mathbb{k}Q_0} \partial_b W_{N+1} \otimes_{\mathbb{k}Q_0} b$. The isomorphism between the two complexes is given by $\text{id}_{A'} \otimes_{\mathbb{k}Q_0} (\pi \otimes_{\mathbb{k}Q_0} \text{id}) \otimes_{\mathbb{k}Q_0} \text{id}_{A'}$, $\text{id}_{A'} \otimes_{\mathbb{k}Q_0} \pi \otimes_{\mathbb{k}Q_0} \text{id}_{A'}$ for the first two and by id for the other three.

The complex \mathcal{C}' is filtered, and the graded complex associated to \mathcal{C}' is the complex $C_{W_{N+1}}$ in [7], which is exact (since A is Calabi–Yau). Therefore \mathcal{C}' is exact (since the natural functor which associates a graded object to a filtered object is faithful), hence \mathcal{C} is exact.

Using Lemma 3.7, we need only check that \mathcal{C} is self-dual to prove that A' is Calabi–Yau of dimension 3. We must first define the isomorphisms α_i .

We use Bocklandt's notation [7]: if T is a finite-dimensional $\mathbb{k}Q_0$ -bimodule, F_T is the A' -bimodule $A' \otimes_{\mathbb{k}Q_0} T \otimes_{\mathbb{k}Q_0} A'$. If T^* is the \mathbb{k} -dual of T , we have an isomorphism of A' -bimodules $F_{T^*} \xrightarrow{\sim} F_T^\vee$ given by $1 \otimes_{\mathbb{k}Q_0} \alpha \otimes_{\mathbb{k}Q_0} 1 \mapsto [1 \otimes_{\mathbb{k}Q_0} t \otimes_{\mathbb{k}Q_0} 1 \mapsto \sum_{i,j \in Q_0} \alpha(itj)i \otimes j]$ for $t \in T$ and $\alpha \in T^*$.

For $\alpha_0: F_{\mathbb{k}Q_0} \rightarrow F_{\text{Im } \tilde{\theta}}^\vee$, consider now the composition of $\theta: \mathbb{k}Q_0 \rightarrow \text{Im } \tilde{\theta}$ with the \mathbb{k} -linear isomorphism $\text{Im } \tilde{\theta} \rightarrow (\text{Im } \tilde{\theta})^*$ defined on the basis $\{\tilde{\theta}(e); e \in Q_0\}$ by sending the basis elements to the corresponding elements in the dual basis. This composition is an isomorphism of $\mathbb{k}Q_0$ -bimodules. Now since $\mathbb{k}Q_0$ is semisimple, A' is flat over $\mathbb{k}Q_0$, so the functor $\text{id}_{A'} \otimes_{\mathbb{k}Q_0} \cdot \otimes_{\mathbb{k}Q_0} \text{id}_{A'}$ is an isomorphism of A' -bimodules from $F_{\mathbb{k}Q_0}$ to $F_{(\text{Im } \tilde{\theta})^*}$. Composing with the isomorphism $F_{(\text{Im } \tilde{\theta})^*} \cong F_{\text{Im } \tilde{\theta}}^\vee$ above gives an A' -bimodule isomorphism $\alpha_0: F_{\mathbb{k}Q_0} \rightarrow F_{\text{Im } \tilde{\theta}}^\vee$. This isomorphism can be expressed explicitly:

$$\alpha_0(1 \otimes_{\mathbb{k}Q_0} e \otimes_{\mathbb{k}Q_0} 1)(1 \otimes_{\mathbb{k}Q_0} \tilde{\theta}(s) \otimes_{\mathbb{k}Q_0} 1) = es \otimes es.$$

For $\alpha_1: F_{\mathbb{k}Q_1} \rightarrow F_P^\vee$, we proceed similarly, using the composition of the \mathbb{k} -linear isomorphism $\mathbb{k}Q_1 \rightarrow P$ sending a to $\partial_a W$ with the \mathbb{k} -linear isomorphism $P \rightarrow P^*$ sending the basis $\{\partial_a W; a \in Q_1\}$ to its dual basis (this composition is an isomorphism of $\mathbb{k}Q_0$ -bimodules). We get

$$\alpha_1(1 \otimes_{\mathbb{k}Q_0} a \otimes_{\mathbb{k}Q_0} 1)(1 \otimes_{\mathbb{k}Q_0} \partial_b W \otimes_{\mathbb{k}Q_0} 1) = \begin{cases} s(a) \otimes t(a) & \text{if } a = b, \\ 0 & \text{otherwise.} \end{cases}$$

For $\alpha_2: F_P \rightarrow F_{\mathbb{k}Q_1}^\vee$, we use the same construction starting with the $\mathbb{k}Q_0$ -bilinear map sending $\partial_a W$ to the opposite of the corresponding element in the dual basis of $\{a; a \in Q_1\}$ of $\mathbb{k}Q_1$. This gives

$$\alpha_2(1 \otimes_{\mathbb{k}Q_0} \partial_a W \otimes_{\mathbb{k}Q_0} 1)(1 \otimes_{\mathbb{k}Q_0} b \otimes_{\mathbb{k}Q_0} 1) = \begin{cases} -t(a) \otimes s(a) & \text{if } a = b, \\ 0 & \text{otherwise.} \end{cases}$$

Finally, the same procedure gives $\alpha_3: F_{\text{Im } \tilde{\theta}} \rightarrow F_{\mathbb{k}Q_0}^\vee$, with

$$\alpha_3(1 \otimes_{\mathbb{k}Q_0} \tilde{\theta}(e) \otimes_{\mathbb{k}Q_0} 1)(1 \otimes_{\mathbb{k}Q_0} s \otimes_{\mathbb{k}Q_0} 1) = -es \otimes es.$$

So we have a diagram:

$$\begin{array}{ccccccc}
 F_{\text{Im } \tilde{\theta}} & \xrightarrow{\delta_3} & F_P & \xrightarrow{\delta_2} & F_{\mathbb{k}Q_1} & \xrightarrow{\delta_1} & F_{\mathbb{k}Q_0} \\
 \downarrow \alpha_3 & & \downarrow \alpha_2 & & \downarrow \alpha_1 & & \downarrow \alpha_0 \\
 (F_{\mathbb{k}Q_0})^\vee & \xrightarrow{-\delta_1^\vee} & (F_{\mathbb{k}Q_1})^\vee & \xrightarrow{-\delta_2^\vee} & (F_P)^\vee & \xrightarrow{-\delta_3^\vee} & (F_{\text{Im } \tilde{\theta}})^\vee
 \end{array}$$

We must prove that it commutes.

Recall that M^\vee is the set of A' -bimodule morphisms from M to $A' \otimes A'$ when $A' \otimes A'$ is endowed with the *outer* bimodule structure. The result M^\vee is then an A' -bimodule when we endow $A' \otimes A'$ with the *inner* bimodule structure.

Let $\tau: A' \otimes A' \rightarrow A' \otimes A'$ be the map that sends $x \otimes y$ to $y \otimes x$.

On the one hand,

$$\begin{aligned}
 & \alpha_0 \circ \delta_1(1 \otimes_{\mathbb{k}Q_0} a \otimes_{\mathbb{k}Q_0} 1)(1 \otimes_{\mathbb{k}Q_0} \tilde{\theta}(e) \otimes_{\mathbb{k}Q_0} 1) \\
 &= \alpha_0(a \otimes_{\mathbb{k}Q_0} s(a) \otimes_{\mathbb{k}Q_0} 1 - 1 \otimes_{\mathbb{k}Q_0} t(a) \otimes_{\mathbb{k}Q_0} a)(1 \otimes_{\mathbb{k}Q_0} \tilde{\theta}(e) \otimes_{\mathbb{k}Q_0} 1) \\
 &= s(a)e \otimes as(a)e - t(a)ea \otimes t(a)e \\
 &= \begin{cases} e \otimes a & \text{if } s(a) = e, \\ -a \otimes e & \text{if } t(a) = e, \\ 0 & \text{otherwise} \end{cases}
 \end{aligned}$$

(here it is the inner action on $A' \otimes A'$ that is involved).

On the other hand,

$$\begin{aligned}
 & \delta_3^\vee \circ \alpha_1(1 \otimes_{\mathbb{k}Q_0} a \otimes_{\mathbb{k}Q_0} 1)(1 \otimes_{\mathbb{k}Q_0} \tilde{\theta}(e) \otimes_{\mathbb{k}Q_0} 1) \\
 &= \alpha_1(1 \otimes_{\mathbb{k}Q_0} a \otimes_{\mathbb{k}Q_0} 1) \\
 & \quad \left(\sum_{b \in eQ_1} b \otimes_{\mathbb{k}Q_0} \partial_b W \otimes_{\mathbb{k}Q_0} 1 - \sum_{c \in Q_1 e} 1 \otimes_{\mathbb{k}Q_0} \partial_c W \otimes_{\mathbb{k}Q_0} c \right) \\
 &= \begin{cases} \alpha_1(1 \otimes_{\mathbb{k}Q_0} a \otimes_{\mathbb{k}Q_0} 1)(-1 \otimes_{\mathbb{k}Q_0} \partial_a W \otimes_{\mathbb{k}Q_0} a) = -e \otimes a & \text{if } s(a) = e, \\ \alpha_1(1 \otimes_{\mathbb{k}Q_0} a \otimes_{\mathbb{k}Q_0} 1)(a \otimes_{\mathbb{k}Q_0} \partial_a W \otimes_{\mathbb{k}Q_0} 1) = a \otimes e & \text{if } t(a) = e, \\ 0 & \text{otherwise} \end{cases}
 \end{aligned}$$

(here it is the outer action on $A' \otimes A'$ that is involved).

Therefore $\alpha_0 \circ \delta_1 = -\delta_3^\vee \circ \alpha_1$.

Similarly, we can prove that $\alpha_2 \circ \delta_3 = -\delta_1^\vee \circ \alpha_3$.

Finally, we must prove that $\alpha_1 \circ \delta_2 = -\delta_2^\vee \circ \alpha_2$.

Following Van den Bergh in [24] (for the one-vertex case), we define the partial derivative of a path p with respect to an arrow a by $\frac{\partial}{\partial a} p = \sum_{p=uv} u \otimes v$. This partial derivative extends naturally to a $\mathbb{k}Q_0$ -linear map $\frac{\partial}{\partial a} : A' \rightarrow A' \otimes A'$.

Now if p is a cycle and if a and b are arrows, we have the following relation:

$$\frac{\partial}{\partial a}(\partial_b p) = \tau \left[\frac{\partial}{\partial b}(\partial_a p) \right].$$

Indeed,

$$\begin{aligned} \frac{\partial}{\partial a}(\partial_b p) &= \sum_{p=ubv} \frac{\partial}{\partial a}(vu) \\ &= \sum_{p=ubv_1av_2} v_1 \otimes v_2u + \sum_{p=u_1au_2bv} vu_1 \otimes u_2 \\ &= \tau \left[\sum_{p=ubv_1av_2} v_2u \otimes v_1 + \sum_{p=u_1au_2bv} u_2 \otimes vu_1 \right] \\ &= \tau \left[\sum_{p=xay} \left(\sum_{y=u_2bv} u_2 \otimes vx + \sum_{x=ubv_1} yu \otimes v_1 \right) \right] \\ &= \tau \left[\sum_{p=xay} \frac{\partial}{\partial b}(yx) \right] = \tau \left[\frac{\partial}{\partial b}(\partial_a p) \right]. \end{aligned}$$

This relation extends to any element in $\text{Pot}(Q)$.

Now consider $\delta_2^\vee \circ \alpha_2(1 \otimes_{\mathbb{k}Q_0} \partial_a W \otimes_{\mathbb{k}Q_0} 1)(1 \otimes_{\mathbb{k}Q_0} \partial_b W \otimes_{\mathbb{k}Q_0} 1) = \alpha_2(1 \otimes_{\mathbb{k}Q_0} \partial_a W \otimes_{\mathbb{k}Q_0} 1)(\Delta(\partial_b W))$. The element $\alpha_2(1 \otimes_{\mathbb{k}Q_0} \partial_a W \otimes_{\mathbb{k}Q_0} 1)(\lambda \otimes_{\mathbb{k}Q_0} c \otimes_{\mathbb{k}Q_0} \mu)$ is nonzero only if $c = a$, and $\alpha_2(1 \otimes_{\mathbb{k}Q_0} \partial_a W \otimes_{\mathbb{k}Q_0} 1)(\lambda \otimes_{\mathbb{k}Q_0} a \otimes_{\mathbb{k}Q_0} \mu) = -\lambda \otimes \mu$, so applying $-\alpha_2(1 \otimes_{\mathbb{k}Q_0} \partial_a W \otimes_{\mathbb{k}Q_0} 1)$ to $\Delta(\partial_b W)$ is the same as replacing (one at a time) the copies of a inside $\partial_b W$ by the symbol \otimes , that is, applying $\frac{\partial}{\partial a}$ to $\partial_b W$. Therefore $\delta_2^\vee \circ \alpha_2(1 \otimes_{\mathbb{k}Q_0} \partial_a W \otimes_{\mathbb{k}Q_0} 1)(1 \otimes_{\mathbb{k}Q_0} \partial_b W \otimes_{\mathbb{k}Q_0} 1) = -\frac{\partial}{\partial a}(\partial_b W)$.

On the other hand, $\alpha_1 \circ \delta_2(1 \otimes_{\mathbb{k}Q_0} \partial_a W \otimes_{\mathbb{k}Q_0} 1)(1 \otimes_{\mathbb{k}Q_0} \partial_b W \otimes_{\mathbb{k}Q_0} 1) = \alpha_1(\Delta \partial_a W)(1 \otimes_{\mathbb{k}Q_0} \partial_b W \otimes_{\mathbb{k}Q_0} 1)$. Moreover, the element $\alpha_1(\lambda \otimes_{\mathbb{k}Q_0} c \otimes_{\mathbb{k}Q_0} \mu)(1 \otimes_{\mathbb{k}Q_0} \partial_b W \otimes_{\mathbb{k}Q_0} 1)$ is nonzero only if $c = b$ and $\alpha_1(\lambda \otimes_{\mathbb{k}Q_0} b \otimes_{\mathbb{k}Q_0} \mu)(1 \otimes_{\mathbb{k}Q_0} \partial_b W \otimes_{\mathbb{k}Q_0} 1) = \mu \otimes \lambda = \tau(\lambda \otimes \mu)$ (we must use the inner action here). So, as above, $\alpha_1 \circ \delta_2(1 \otimes_{\mathbb{k}Q_0} \partial_a W \otimes_{\mathbb{k}Q_0} 1)(1 \otimes_{\mathbb{k}Q_0} \partial_b W \otimes_{\mathbb{k}Q_0} 1) = \tau \left[\frac{\partial}{\partial b}(\partial_a W) \right] = \frac{\partial}{\partial a}(\partial_b W)$. □

4. Connected graded Calabi–Yau algebras

4.1. Calabi–Yau algebras, graded Calabi–Yau algebras, and AS-Gorenstein algebras. Let A be an associative \mathbb{k} -algebra with unit. Denote by $A \overset{\text{out}}{\otimes} A$ (resp. $A \overset{\text{inn}}{\otimes} A$)

the vector space $A \otimes A$ endowed with its outer (resp. inner) bimodule structure. Consider the \mathbb{k} -algebra $A^e = A \otimes A^{\text{op}}$, so that the categories $A\text{-Mod-}A$, $A^e\text{-Mod}$ and $\text{Mod-}A^e$ are naturally isomorphic. Clearly, via these isomorphisms, $A \otimes^{\text{out}} A$ (resp. $A \otimes^{\text{inn}} A$) is identified with the left (resp. right) A^e -module A^e with the left (resp. right) multiplication as action.

The bifunctor Hom in the categories $A\text{-Mod-}A$, $A^e\text{-Mod}$ and $\text{Mod-}A^e$ are denoted by $\text{Hom}_{A-A}(\cdot, \cdot)$, $\text{Hom}_{A^e}(\cdot, \cdot)$, $\text{Hom}_{-A^e}(\cdot, \cdot)$ respectively, and similarly for Ext . The right A^e -module $\text{Ext}_{A^e}^i(A, A^e)$ and the A - A -bimodule $\text{Ext}_{A-A}^i(A, A \otimes^{\text{out}} A)$ are naturally identified, as well as $\text{Ext}_{-A^e}^i(A, A^e)$ and $\text{Ext}_{A-A}^i(A, A \otimes^{\text{inn}} A)$. If A is an associative \mathbb{k} -algebra which has a finite projective A -bimodule resolution by bimodules of finite type, then the left version in the following definition coincides with Ginzburg’s definition.

Definition 4.1. We say that A is a left Calabi–Yau algebra of dimension $n \geq 1$ if

$$\text{Ext}_{A^e}^i(A, A^e) \cong \begin{cases} A & \text{if } i = n, \\ 0 & \text{otherwise} \end{cases}$$

in $\text{Mod-}A^e$. Replacing A^e by $-A^e$ and $\text{Mod-}A^e$ by $A^e\text{-Mod}$, we say that A is a right Calabi–Yau algebra of dimension n .

Since $\text{Ext}_{A^{\text{op}}-A^{\text{op}}}^i(A^{\text{op}}, A^{\text{op}} \otimes^{\text{out}} A^{\text{op}}) \cong \text{Ext}_{A-A}^i(A, A \otimes^{\text{inn}} A)$, A^{op} is left Calabi–Yau if and only if A is right Calabi–Yau. We are now interested in a graded version of Definition 4.1.

Definition 4.2. Let A be a connected \mathbb{N} -graded \mathbb{k} -algebra. We say that A is a left graded Calabi–Yau algebra of dimension $n \geq 1$ if

$$\underline{\text{Ext}}_{A^e}^i(A, A^e) \cong \begin{cases} A(\ell) & \text{if } i = n, \\ 0 & \text{otherwise} \end{cases}$$

in $\text{grMod-}A^e$ for some $\ell \in \mathbb{Z}$ (called the parameter of A). Replacing A^e by $-A^e$ and $\text{grMod-}A^e$ by $A^e\text{-grMod}$, we say that A is a right graded Calabi–Yau algebra of dimension n .

The following proposition is the Calabi–Yau analogue of a result by D. Stephenson and J. Zhang concerning graded (i.e. Artin–Schelter) Gorenstein algebras (Proposition 3.1 in [21]).

Proposition 4.3. Assume that A is a left graded Calabi–Yau algebra of dimension n and parameter ℓ . Assume that A has a finite global dimension D . Then:

- (i) We have $n = D$.
- (ii) The bimodule A has a graded free resolution of finite type in $A\text{-grMod-}A$.
- (iii) For any minimal projective resolution \mathcal{P} of A in $A^e\text{-grMod}$, $\underline{\text{Hom}}_{A^e-}(\mathcal{P}, A^e)$ is a minimal projective resolution of $A(\ell)$ in $\text{grMod-}A^e$.
- (iv) A is AS-Gorenstein of dimension n and parameter ℓ , and ℓ is nonnegative.
- (v) A is right graded Calabi–Yau of dimension n and parameter ℓ .
- (vi) A is left and right Calabi–Yau of dimension n .

Proof. Let $\mathcal{P} = (P_i)_{i \geq 0}$ be a minimal projective resolution of A in $A^e\text{-grMod}$. We know that the length of \mathcal{P} equals D [3], Théorème 3.3. In particular, $P_i = 0$ if and only if $i > D$. For $0 \leq i \leq D$, set $P_i = A^e \otimes E_i$, where E_i is in $\mathbb{k}\text{-grMod}$ such that for $i > 0$, E_i is a graded vector subspace of P_{i-1} and the differential $d_i : P_i \rightarrow P_{i-1}$ is the natural A^e -linear extension of the inclusion $E_i \hookrightarrow P_{i-1}$.

Set $\underline{\text{Hom}}_{A^e-}(P_i, A^e) = P_i^\vee$ and $\underline{\text{Hom}}_{A^e-}(d_i, A^e) = d_i^\vee$. We identify P_i^\vee with $\underline{\text{Hom}}_{\mathbb{k}}(E_i, A^e)$, so that the following direct sum holds in $\mathbb{k}\text{-grMod}$:

$$P_i^\vee = \underline{\text{Hom}}_{\mathbb{k}}(E_i, \mathbb{k}) \oplus \underline{\text{Hom}}_{\mathbb{k}}(E_i, A_{>0}^e), \tag{4.1}$$

where $A_{>0}^e = \bigoplus_{u+v>0} A_u \otimes A_v$. The projection p_i of P_i^\vee onto the first component assigns to each $f \in P_i^\vee$ the linear form $\epsilon \circ f$, where $\epsilon : A^e \rightarrow \mathbb{k}$ is the natural projection. Since d_{i-1} is injective on E_{i-1} , E_i is contained in $A_{>0}^e \otimes E_{i-1}$. By A^e -linearity of $g \in P_{i-1}^\vee$, $d_i^\vee(g)(v) = g(d_i(v))$ belongs to $A_{>0}^e$ for any $v \in E_i$. Therefore, $p_i(d_i^\vee(g)) = 0$ for any $g \in P_{i-1}^\vee$. Thus we have obtained

$$\text{Im}(d_i^\vee) \subseteq \underline{\text{Hom}}_{\mathbb{k}}(E_i, A_{>0}^e). \tag{4.2}$$

We have $\text{Ext}_{A^e-}^D(A, A^e) \cong P_D^\vee / \text{Im}(d_D^\vee)$, and (4.1), (4.2) show that this quotient maps onto $\underline{\text{Hom}}_{\mathbb{k}}(E_D, \mathbb{k})$. The latter is nonzero, thus $D = n$.

Note that $\underline{\text{Hom}}_{\mathbb{k}}(E_i, A_{>0}^e)$ is a sub- A^e -module of P_i^\vee , and the structure of A^e -module inherited by $\underline{\text{Hom}}_{\mathbb{k}}(E_i, \mathbb{k})$ is trivial, i.e. the action of $x \in A^e$ on $f : E_i \rightarrow \mathbb{k}$ is given by $f.x = f\epsilon(x)$. But now the surjective arrow $A \rightarrow \underline{\text{Hom}}_{\mathbb{k}}(E_n, \mathbb{k})$ of $\text{grMod-}A^e$ implies that $\underline{\text{Hom}}_{\mathbb{k}}(E_n, \mathbb{k})$ is generated as a vector space by the image of 1_A , therefore it is 1-dimensional, thus E_n is 1-dimensional as well.

Let us prove that E_i is finite dimensional by using a finite decreasing induction. Fix $0 \leq i \leq D - 1$ such that E_{i+1} is finite dimensional and E_i infinite dimensional. There exist graded vector subspaces F and G in E_i such that $E_i = F \oplus G$, with F finite dimensional and $E_{i+1} \subseteq A^e \otimes F$, so that $\text{Im}(d_{i+1}) \subseteq A^e \otimes F$. Thus $(A^e \otimes F)^\perp \subseteq \text{Ker}(d_{i+1}^\vee)$. But $\text{Ker}(d_{i+1}^\vee) = \text{Im}(d_i^\vee)$ since $\text{Ext}_{A^e-}^i(A, A^e) = 0$ by assumption. As $G \neq 0$, there exists $f \in \underline{\text{Hom}}_{\mathbb{k}}(E_i, \mathbb{k})$ vanishing on F and nonvanishing on G . Therefore $f \in (A^e \otimes F)^\perp \subseteq \text{Im}(d_i^\vee)$, which contradicts (4.1) and (4.2). Consequently, (i) and (ii) are proved.

A consequence of (ii) is that $P_i^\vee \cong E_i^* \otimes A^e$ in $\text{grMod-}A^e$, where E_i^* stands for the graded \mathbb{k} -dual. Thus $\underline{\text{Hom}}_{A^e-}(\mathcal{P}, A^e)$ is a projective resolution of $A(\ell)$ in $\text{grMod-}A^e$ (using the fact that A is left graded Calabi–Yau). Another consequence of (ii) is that any module P_i^\vee is bounded below. In order to prove that the resolution is minimal, it is sufficient to prove that the differential of the complex $\underline{\text{Hom}}_{A^e-}(\mathcal{P}, A^e) \otimes_{A^e} \mathbb{k}$ vanishes [3]. But this is straightforward from the inclusion (4.2) which can be written now as $\text{Im}(d_i^\vee) \subseteq E_i^* \otimes A^e_{>0}$. Hence (iii).

Following [3], $\mathcal{Q} = \mathcal{P} \otimes_A \mathbb{k}$ is a minimal projective resolution of $\mathbb{k}(\ell)$ in $A\text{-grMod}$. For the same reason, $\mathbb{k} \otimes_A \underline{\text{Hom}}_{A^e-}(\mathcal{P}, A^e)$ is a minimal projective resolution of $\mathbb{k}(\ell)$ in $\text{grMod-}A$. But

$$\mathbb{k} \otimes_A \underline{\text{Hom}}_{A^e-}(\mathcal{P}, A^e) \cong \underline{\text{Hom}}_{A-}(\mathcal{Q}, A)$$

in $\text{grMod-}A$. In particular, $\underline{\text{Hom}}_{A-}(\mathcal{Q}, A)$ is a resolution of $\mathbb{k}(\ell)$ so that

$$\underline{\text{Ext}}_{A-}^i(\mathbb{k}, A) \cong \begin{cases} \mathbb{k}(\ell) & \text{if } i = n, \\ 0 & \text{otherwise} \end{cases}$$

in $\text{grMod-}A$. Thus A is AS-Gorenstein of dimension n , and the Gorenstein parameter is ℓ . It is known that $\ell \geq 0$ [21]. Hence (iv).

To prove (v), we apply the functor $\underline{\text{Hom}}_{A^e-}(\cdot, A^e)$ to the projective resolution $\underline{\text{Hom}}_{A^e-}(\mathcal{P}, A^e)$ of $A(\ell)$. Since the E_i 's are finite dimensional, we recover the resolution \mathcal{P} of A . Thus

$$\underline{\text{Ext}}_{A^e-}^i(A(\ell), A^e) \cong \begin{cases} A & \text{if } i = n, \\ 0 & \text{otherwise.} \end{cases}$$

Shifting by ℓ , we get (v).

A consequence of (ii) is that

$$\text{Ext}_{A^e-}^i(A, A^e) \cong \underline{\text{Ext}}_{A^e-}^i(A, A^e)$$

in $\text{Mod-}A^e$, and the same is true for right A^e -modules. Hence (vi). □

Remark 4.4. Let A be a connected \mathbb{N} -graded \mathbb{k} -algebra. Assume that (ii) holds. If A is left Calabi–Yau of dimension n , then A is left graded Calabi–Yau of dimension n .

4.2. A criterion for certain graded algebras to be Calabi–Yau. This criterion (Proposition 4.5 below) will be useful to us for Examples 5.1, 5.3 and 5.5. Let A be a connected \mathbb{N} -graded \mathbb{k} -algebra. Let us assume that A is N -Koszul for a certain $N \geq 2$, and that A is AS-Gorenstein of global dimension n . It is known [6], Section 6, that $\text{Ext}_{A^e-}^i(A, A^e) = 0$ if $i \neq n$ and

$$\text{Ext}_{A^e-}^n(A, A^e) \cong_{\varepsilon^{n+1}\phi} A(\zeta_N(n))$$

in the category $A\text{-grMod-}A$ of graded bimodules, where ζ_N stands for the jump function of the N -Koszul algebra A . Here ε and ϕ are certain automorphisms of the graded algebra A . We refer to [6] for more details.

Proposition 4.5. *With the notations and assumptions above, the algebra A is Calabi–Yau if and only if $\varepsilon^{n+1}\phi = \text{id}_A$.*

Proof. It is clear from Ginzburg’s definition that if $\varepsilon^{n+1}\phi = \text{id}_A$, A is Calabi–Yau of dimension n . Conversely, assume that A is Calabi–Yau (necessarily of dimension n). Remark 4.4 and Proposition 4.3 show that A is graded Calabi–Yau of dimension n , and that the Calabi–Yau parameter coincides with the Gorenstein parameter $\zeta_N(n)$. Thus there is an isomorphism $f: A \rightarrow_{\varepsilon^{n+1}\phi} A$ in $A\text{-grMod-}A$, so that $f(1)$ is a nonvanishing element in \mathbb{k} . For any a, b in A , we have $f(ab) = \varepsilon^{n+1}\phi(a)f(1)b$, implying that $f(a) = \varepsilon^{n+1}\phi(a)f(1) = f(1)a$, and $\varepsilon^{n+1}\phi(a) = a$ follows. \square

5. Examples

Throughout Examples 5.1, 5.3 and 5.5 the characteristic of \mathbb{k} is assumed to be zero.

Example 5.1 (Yang–Mills algebra). The Yang–Mills algebra A with $s + 1$ generators is graded, 3-Koszul, AS-Gorenstein of global dimension 3 [10]. Following [11], the identity $\varepsilon^{n+1}\phi = \text{id}_A$ holds, thus A is Calabi–Yau by Proposition 4.5 (equivalently, it would be possible to extract from [11] a self-dual resolution of A by bimodules, and then A is Calabi–Yau by Theorem 4.2 in [7]).

Let us show directly that A derives from a potential (see also [16]). We use the material contained in [4]. The generators of the Yang–Mills algebra A are denoted by $\nabla_0, \nabla_1, \dots, \nabla_s$, where $s \geq 1$. Let $(g^{\alpha\beta})_{0 \leq \alpha, \beta \leq s}$ be a nondegenerate symmetric matrix with entries in \mathbb{k} . The relations of A are denoted by W^0, W^1, \dots, W^s , where $W^\rho = \sum_{\lambda, \mu, \nu} W^{\rho\lambda\mu\nu} \nabla_\lambda \nabla_\mu \nabla_\nu$, and

$$W^{\rho\lambda\mu\nu} = g^{\rho\lambda} g^{\mu\nu} + g^{\rho\nu} g^{\lambda\mu} - 2g^{\rho\mu} g^{\lambda\nu}.$$

Using the concepts introduced in [12], [13], A is defined from the following “volume form”

$$W_4 = \sum_{\rho, \lambda, \mu, \nu} W^{\rho\lambda\mu\nu} \nabla_\rho \nabla_\lambda \nabla_\mu \nabla_\nu.$$

Actually, this volume form W_4 is a potential because of the relation $W^{\rho\lambda\mu\nu} = W^{\lambda\mu\nu\rho}$. Moreover we have $\partial_\rho W_4 = 4W^\rho$ for any ρ . In other words, $A = A(Q, W_4)$, where Q is the quiver having one vertex and $s + 1$ loops.

Following [4], a PBW deformation U of A satisfying (PBW2') has the following relations

$$W^\rho + \sum_{\mu, \nu} c^{\mu\nu\rho} \nabla_\mu \nabla_\nu + \sum_{\lambda} s^{\lambda\rho} \nabla_\lambda + s^\rho = 0, \quad 0 \leq \rho \leq s,$$

where the coefficients are elements of \mathbb{k} such that $c^{\mu\nu\rho} = c^{\rho\mu\nu}$, $s^{\lambda\rho} = s^{\rho\lambda}$. Then it is easy to verify that U derives from a potential, as stated in Theorem 3.2. In fact, setting

$$W_3 = \frac{1}{3} \sum_{\mu, \nu, \rho} c^{\mu\nu\rho} \nabla_\mu \nabla_\nu \nabla_\rho, \quad W_2 = \frac{1}{2} \sum_{\lambda, \rho} s^{\lambda\rho} \nabla_\lambda \nabla_\rho, \quad W_1 = \sum_{\rho} s^\rho \nabla_\rho,$$

we have $U = A(Q, W_4 + W_3 + W_2 + W_1)$.

All the PBW deformations of A are determined in [4]. From these computations, it is immediate to see that there exist PBW deformations of A for which (PBW2') does not hold.

Remark 5.2. Let us show how we can deduce the formula $\varepsilon^{n+1}\phi = \text{id}_A$ from the definitions of ε and ϕ contained in [6]. Since $n = 3$, $\varepsilon^{n+1} = \text{id}_A$ is clear. The automorphism ϕ is uniquely determined by its component ϕ_1 , and ϕ_1 is defined as the transposed linear map of ν_1 where ν is the Nakayama automorphism of the Yoneda algebra $E(A)$ of A . Our aim is now to prove that $\nu_1 = \text{id}_{E(A)_1}$. We have to compute the Frobenius pairing (\cdot, \cdot) of $E(A)$. Actually, it suffices to compute (x, y) and (y, x) when $x \in E(A)_1 = V^*$ and $y \in E(A)_2 = R^*$. As usual, V denotes the generator space of A and R denotes the relation space. We have (formula (6.3) in [6])

$$(x, y) = \langle x \otimes g, W_4 \rangle, \quad (y, x) = \langle g \otimes x, W_4 \rangle,$$

where g is any linear form on $V^{\otimes 3}$ extending $y: R \rightarrow \mathbb{k}$, and consequently $x \otimes g$ and $g \otimes x$ are linear forms on $V^{\otimes 4}$. Then ν_1 is defined by $(x, y) = (y, \nu_1(x))$.

Since $W_4 = \sum_{\alpha} \nabla_{\alpha} W^{\alpha}$, we have $(\nabla_{\lambda}^*, W^{\alpha*}) = \delta_{\lambda\alpha}$ (Kronecker symbol). Similarly, $W_4 = \sum_{\alpha} W^{\alpha} \nabla_{\alpha}$ implies $(W^{\alpha*}, \nabla_{\lambda}^*) = \delta_{\lambda\alpha}$. Thus $\nu_1 = \text{id}_{E(A)_1}$.

Example 5.3 (Artin–Schelter regular algebras of global dimension 3). In this example, \mathbb{k} is the field of complex numbers. Applying Proposition 4.5 in the quadratic case [22], Corollary 9.3, and in the cubic case [6], Proposition 6.5, we get the following.

Proposition 5.4. *Let A be an AS-regular algebra (with polynomial growth) of global dimension 3. Then A is Calabi–Yau if and only if A is of type A in the classification of Artin and Schelter [1].*

It is easy to show directly that A of type A derives from a potential. Let us limit ourselves to the cubic case and let us examine the PBW deformations in this case. If

A is cubic of type A, then $A = A(Q, W_4)$ where the quiver Q has one vertex and two loops x and y , and the potential is

$$W_4 = y^4 + a(x^2y^2 + xy^2x + y^2x^2 + yx^2y) + b(xyxy + yxyx) + x^4.$$

We have $\partial_x W_4 = 4f$ and $\partial_y W_4 = 4g$, where

$$f = ay^2x + byxy + axy^2 + x^3, \quad g = y^3 + ayx^2 + bxyx + ax^2y$$

are the relations of A . Here a and b are the usual generic complex parameters.

In [15], Fløystad and Vatne have determined all the PBW deformations of A (actually for any type, not necessarily type A). From their computations, it results that any PBW deformation U of A (of type A) satisfies condition (PBW2'). Moreover U has the following relations

$$\begin{aligned} f + a_{11}x^2 + b_{11}xy + b_{11}yx + a_{14}y^2 + a_{21}x + a_{22}y + a_3 &= 0, \\ g + b_{11}x^2 + a_{14}xy + a_{14}yx + b_{14}y^2 + a_{22}x + b_{22}y + b_3 &= 0, \end{aligned}$$

where the coefficients are arbitrary complex numbers. Then it is easy to verify that U derives from a potential, as stated in Theorem 3.2. In fact, setting

$$\begin{aligned} W_3 &= \frac{1}{3} (a_{11}x^3 + b_{11}(x^2y + xyx + yx^2) + a_{14}(y^2x + yxy + xy^2) + b_{14}y^3), \\ W_2 &= \frac{1}{2} (a_{21}x^2 + a_{22}(xy + yx) + b_{22}y^2), \\ W_1 &= a_3x + b_3y, \end{aligned}$$

we have $U = A(Q, W_4 + W_3 + W_2 + W_1)$.

Example 5.5 (Antisymmetriser algebras). Let us introduce some general notations. If p is a cycle of a quiver Q , denote by \bar{p} the class of p up to cyclic permutation of the arrows. Denote by \mathcal{C} the set of cycles of Q and by $\bar{\mathcal{C}}$ the set of classes of cycles in \mathcal{C} modulo cyclic permutations. The natural map from \mathcal{C} into the set $\text{Pot}(Q)$ of potentials of Q induces a linear isomorphism from $\mathbb{k}\bar{\mathcal{C}}$ to $\text{Pot}(Q)$. We shall often identify $\mathbb{k}\bar{\mathcal{C}}$ and $\text{Pot}(Q)$ via this isomorphism, and $\bar{\mathcal{C}}$ will be considered as a basis of the vector space $\text{Pot}(Q)$.

Now assume that \mathbb{k} has characteristic zero, Q has one vertex and $n \geq 2$ loops x_1, \dots, x_n . Any path is a cycle. Introduce $W_n = \text{Ant}(x_1, \dots, x_n)$ where Ant stands for the antisymmetriser of the variables on which this operator is acting, i.e.

$$W_n = \sum_{\sigma \in S_n} \text{sgn}(\sigma) x_{\sigma(1)} \dots x_{\sigma(n)}.$$

Here S_n denotes the group of permutations of $1, \dots, n$.

Lemma 5.6. *If n is even, $\bar{W}_n = 0$. If n is odd, then for any $i = 1, \dots, n$ we have the formulas*

$$\begin{aligned} \bar{W}_n &= n \sum_{\sigma(1)=i} \operatorname{sgn}(\sigma) \overline{x_{\sigma(1)} \dots x_{\sigma(n)}}, \\ \partial_{x_i} \bar{W}_n &= n(-1)^{i+1} \operatorname{Ant}(x_1, \dots, \hat{x}_i, \dots, x_n), \end{aligned}$$

in which \hat{x}_i means that the variable x_i is removed.

Proof. Fix i and write

$$\begin{aligned} W_n &= \sum_{\sigma(1)=i} \operatorname{sgn}(\sigma) (x_{\sigma(1)} \dots x_{\sigma(n)} + (-1)^{n+1} x_{\sigma(2)} \dots x_{\sigma(n)} x_{\sigma(1)} \\ &\quad + (-1)^{2(n+1)} x_{\sigma(3)} \dots x_{\sigma(n)} x_{\sigma(1)} x_{\sigma(2)} + \dots \\ &\quad + (-1)^{(n-1)(n+1)} x_{\sigma(n)} x_{\sigma(1)} \dots x_{\sigma(n-1)}). \end{aligned}$$

Hence

$$\bar{W}_n = \sum_{\sigma(1)=i} \operatorname{sgn}(\sigma) (1 + (-1)^{n+1} + (-1)^{2(n+1)} + \dots + (-1)^{(n-1)(n+1)}) \overline{x_{\sigma(1)} \dots x_{\sigma(n)}}.$$

Clearly, $\bar{W}_n = 0$ if n is even, and we have the first formula if n is odd. From this first formula, we deduce

$$\partial_{x_i} \bar{W}_n = n \sum_{\sigma(1)=i} \operatorname{sgn}(\sigma) x_{\sigma(2)} \dots x_{\sigma(n)},$$

and the second formula follows, since the sign of the permutation $1 \mapsto i, 2 \mapsto 1, \dots, i \mapsto i - 1$ is $(-1)^{i+1}$. □

For the remainder of this example, we assume $n \geq 3$. Denote by A the $(N = n - 1)$ -homogeneous algebra whose generators are x_1, \dots, x_n and whose relations are $\operatorname{Ant}(x_1, \dots, \hat{x}_i, \dots, x_n)$ for $i = 1, \dots, n$, and call it the *antisymmetriser algebra* (it was introduced in [2] for any $2 \leq N \leq n$). Lemma 5.6 shows that $A = A(Q, W_n)$ if n is odd. When n is even, we can improve the statement $\bar{W}_n = 0$.

Lemma 5.7. *If n is even, there exists no potential W such that $A = A(Q, W)$.*

Proof. Assume that W exists. Write $W = \sum_{\bar{p} \in \bar{\mathcal{C}}_n} \lambda_{\bar{p}} \bar{p}$, where $\bar{\mathcal{C}}_n$ denotes the set of cycles of length n . Let W' be the part of W for which the summation is performed only over the cycles p containing (at least) twice the same loop. The fact that $\partial_{x_i} W$ belongs to $\mathbb{k} \cdot \operatorname{Ant}(x_1, \dots, \hat{x}_i, \dots, x_n)$ implies $\partial_{x_i} W' = 0$, and this holds for $i = 1, \dots, n$.

Now it is easy to see in general that

$$\bigcap_{a \in Q_1} \text{Ker}(\partial_a) = \mathbb{k} Q_0.$$

Here $W' \in \bigcap_{i=0}^n \text{Ker} \partial_{x_i}$ and W' is homogeneous of length $n \geq 3$, so $W' = 0$. Therefore we can write

$$W = \sum_{\bar{\sigma} \in \bar{\mathcal{S}}_n} \lambda_{\bar{\sigma}} \overline{x_{\sigma(1)} \dots x_{\sigma(n)}},$$

with obvious notations for $\bar{\sigma}$ and $\bar{\mathcal{S}}_n$. Fixing i , we have a unique decomposition

$$W = \sum_{\sigma(1)=i} \lambda_{\sigma}^{(i)} \overline{x_{\sigma(1)} \dots x_{\sigma(n)}},$$

since in each class $\bar{\sigma}$ there is a unique permutation mapping 1 to i . The superscript (i) in $\lambda_{\sigma}^{(i)}$ is there to remind us that σ depends on i (since $\sigma(1) = i$). Therefore, we obtain

$$\partial_{x_i} W = \sum_{\sigma(1)=i} \lambda_{\sigma}^{(i)} x_{\sigma(2)} \dots x_{\sigma(n)}.$$

The fact that $\partial_{x_i} W$ belongs to $\mathbb{k} \cdot \text{Ant}(x_1, \dots, \hat{x}_i, \dots, x_n)$ implies that

$$\partial_{x_i} W = \lambda_{\tau_i}^{(i)} \text{Ant}(x_{i+1}, \dots, x_n, x_1, \dots, x_{i-1}),$$

where τ_i is the permutation

$$1 \mapsto i, 2 \mapsto i + 1, \dots, n - i + 1 \mapsto n, n - i + 2 \mapsto 1, \dots, n \mapsto i - 1.$$

Decompose $\sigma = \sigma' \tau_i$, σ' leaving i fixed. The comparison of the two previous decompositions of $\partial_{x_i} W$ provides $\lambda_{\sigma}^{(i)} = \lambda_{\tau_i}^{(i)} \text{sgn}(\sigma')$, and we get

$$W = \lambda_{\tau_i}^{(i)} \sum_{\sigma'(i)=i} \text{sgn}(\sigma') \overline{x_{\sigma'(i)} \dots x_{\sigma'(n)} x_{\sigma'(1)} \dots x_{\sigma'(i-1)}}. \tag{5.1}$$

For $\sigma' = \text{id}$ in (5.1), we see that $\lambda_{\tau_i}^{(i)}$ coincides with the coefficient λ of $\overline{x_1 \dots x_n}$ in the linear decomposition of W in the basis $\bar{\mathcal{C}}_n$. The case $i = 1$ in (5.1) gives

$$W = \lambda[\overline{x_1 \dots x_n} - \overline{x_1 x_3 x_2 x_4 \dots x_n} + \dots],$$

while the case $i = 2$ gives

$$\begin{aligned} W &= \lambda[\overline{x_2 x_3 \dots x_n x_1} + (-1)^{n-2} \overline{x_2 x_4 \dots x_n x_1 x_3} + \dots] \\ &= \lambda[\overline{x_1 \dots x_n} + \overline{x_1 x_3 x_2 x_4 \dots x_n} + \dots]. \end{aligned}$$

The comparison of the two so obtained linear decompositions in the basis $\bar{\mathcal{C}}_n$ implies that $\lambda = 0$, hence $W = 0$, and we have a contradiction. □

For any $n \geq 3$, it is known that the antisymmetriser algebra A is $(n - 1)$ -Koszul of global dimension 3 [2], Theorem 3.13, AS-Gorenstein [6], Corollary 5.10. Moreover, the automorphism $\varepsilon^{n+1}\phi$ is id_A or $-\text{id}_A$ according to whether n is odd or even; see [6], end of the paper. Thus Proposition 4.5 shows that A is Calabi–Yau if and only if n is odd.

Fløystad and Vatne have determined the PBW deformations of any antisymmetriser algebra A such that $N \leq n - 2$ (Theorem 4.1 and Theorem 4.2 in [15]). The case $N = n - 1$ of interest for us is not treated in [15]. However Theorems 3.1 and 3.2 provide the following partial answer to the determination of PBW deformations of A when $n \geq 3$ is odd and $N = n - 1$:

- (a) For any $W_j \in \text{Pot}(Q)_j$, $1 \leq j \leq n - 1$, the algebra $A(Q, \sum_{1 \leq j \leq n} W_j)$ is a PBW deformation of $A = A(Q, W_n)$.
- (b) Any PBW deformation in (a) satisfies (PBW2’).
- (c) Any PBW deformation of A satisfying (PBW2’) is as in (a).

This leads to the following questions:

Question 1. Determine $\dim(\text{Pot}(Q)_j)$ for $1 \leq j \leq n - 1$.

Question 2. Does (PBW2’) hold for any PBW deformation of A ?

We examine now Questions 1 and 2 in the simplest case $n = 3$. In this case, $A = k[x, y, z]$. The answer to Question 1 is easy since $\bar{\mathcal{C}}_1 = \{x, y, z\}$ and $\bar{\mathcal{C}}_2 = \{x^2, y^2, z^2, xy, yz, zx\}$. Let us examine the *linear* PBW deformations of A , i.e. such that $\varphi_0 = 0$. It is well known that the linear PBW deformations of the polynomial algebra A are exactly the Lie algebras having $V = kx \oplus ky \oplus kz$ as underlying vector space (see e.g. Example 3.7 in [5]). More precisely, let $\varphi_1 : R \rightarrow V$ be linear, and denote by $[\cdot, \cdot] : V \times V \rightarrow V$ the antisymmetric bilinear map extending naturally φ_1 (R is spanned by $yz - zy, zx - xz, xy - yx$). Then φ_1 is a PBW deformation of A if and only if the relation

$$\varphi_1(\varphi_1 \otimes \text{id}_V - \text{id}_V \otimes \varphi_1)(W_3) = 0$$

holds, which in turn is equivalent to the Jacobi identity of the bracket $[\cdot, \cdot]$.

Let us give the matrix $\alpha = (\alpha_{ij})_{1 \leq i, j \leq 3}$ of coefficients of φ_1 , that is,

$$\begin{aligned} \varphi_1(yz - zy) &= \alpha_{11}x + \alpha_{12}y + \alpha_{13}z, \\ \varphi_1(zx - xz) &= \alpha_{21}x + \alpha_{22}y + \alpha_{23}z, \\ \varphi_1(xy - yx) &= \alpha_{31}x + \alpha_{32}y + \alpha_{33}z. \end{aligned}$$

Then we obtain

$$\begin{aligned} (\varphi_1 \otimes \text{id}_V - \text{id}_V \otimes \varphi_1)(W_3) &= (\alpha_{32} - \alpha_{23})(yz - zy) \\ &\quad + (\alpha_{13} - \alpha_{31})(zx - xz) \\ &\quad + (\alpha_{21} - \alpha_{12})(xy - yx), \end{aligned} \tag{5.2}$$

and the Jacobi identity is equivalent to the algebraic system formed by the equation

$$\alpha_{12}\alpha_{31} - \alpha_{13}\alpha_{21} + \alpha_{11}(\alpha_{23} - \alpha_{32}) = 0,$$

and the two other equations deduced by cyclic permutations of indices. The following is immediate from (5.2).

Lemma 5.8. *Condition (PBW2') is equivalent to saying that the matrix α is symmetric. In this case, the potential is given by $W_2 = -\frac{1}{2}\alpha_{11}x^2 - \frac{1}{2}\alpha_{22}y^2 - \frac{1}{2}\alpha_{33}z^2 - \alpha_{12}xy - \alpha_{23}yz - \alpha_{31}zx$.*

Thus the answer to Question 2 is no, since it is easy to find α non-symmetric and satisfying the Jacobi identity. Actually, it is possible to be more precise. Let us fix an arbitrary element $r = a(yz - zy) + b(zx - xz) + c(xy - yx)$ in R . Then there exists a unique *antisymmetric* α (denoted by α_0) satisfying the Jacobi identity and $(\varphi_1 \otimes \text{id}_V - \text{id}_V \otimes \varphi_1)(W_3) = r$. Furthermore, the set of these α 's (not necessarily antisymmetric) is equal to

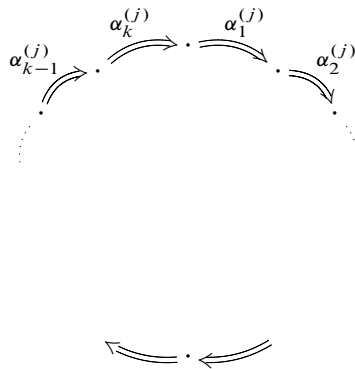
$$\left\{ \beta + \alpha_0; \beta \text{ symmetric and } \beta \begin{pmatrix} a \\ b \\ c \end{pmatrix} = 0 \right\},$$

which is a linear affine space of dimension 6 if $a = b = c = 0$, and of dimension 3 otherwise.

It would be interesting to have an analogue of the above discussion for higher n .

We now consider quiver algebras with several vertices, described in [7].

Example 5.9. Set $N = \ell k - 1$ with $\ell \geq 2$ and $k \geq 3$, and let $A = A(Q, W_{N+1})$ be the Calabi–Yau algebra of dimension 3 defined by the quiver Q ,



containing k vertices $1, \dots, k$ and n_i arrows $\alpha_i^{(1)}, \dots, \alpha_i^{(n_i)}$ from the vertex i to the

vertex $i + 1$ with $n_i \geq 2$ for all i with $1 \leq i \leq k$, and by the potential

$$W_{N+1} = \sum_{i \in Q_0} \alpha_i^{(1)} \alpha_{i-1}^{(1)} \alpha_{i-2}^{(2)} \cdots \alpha_{i-N}^{(2)} + \sum_{\substack{c \in Q_1 \\ c \notin \{\alpha_i^{(1)}, \alpha_i^{(2)}\}}} c \alpha_{t(c)-1}^{(1)} \alpha_{t(c)-2}^{(2)} \cdots \alpha_{t(c)-k+1}^{(2)} \cdot (c \alpha_{t(c)-1}^{(2)} \alpha_{t(c)-2}^{(2)} \cdots \alpha_{t(c)-k+1}^{(2)})^{\ell-1}.$$

We assume that the characteristic of \mathbb{k} does not divide $N!$.

Now let A' be a PBW deformation of A . Then A' satisfies the following condition:

$$\sum_{b \in Q_1 e} \varphi_{N-1}(\partial_b W_{N+1}) b - \sum_{a \in e Q_1} a \varphi_{N-1}(\partial_a W_{N+1}) \tag{*}$$

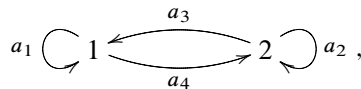
$$\in \text{span}_{\mathbb{k}} \{ \partial_c W_{N+1}; c \in Q_1 \} \quad \text{for all } e \in Q_0.$$

However, $\varphi_{N-1}(\partial_\alpha W_{N+1})$ has length $N - 1 = \ell k - 2$, starts at $t(\alpha)$ and ends at $s(\alpha)$ so that $\ell k - 2$ must be of the form $\lambda k - 1$, which is impossible since $k \geq 3$. Therefore $\varphi_{N-1}(\partial_\alpha W_{N+1}) = 0$ for all α . In particular, in the relation (*), the left-hand side is equal to 0, and therefore A' is defined by a potential (since (PBW2') holds).

Note that, in the same way as above, we can check that $\varphi_{j-1}(\partial_\alpha W_{N+1}) = 0$ whenever $j \not\equiv 0 \pmod{k}$.

Let us now present a more specific example. Assume that $n_i = 2$ for every $i \in Q_0$ (there are exactly two arrows between consecutive vertices) so that $W_{N+1} = \sum_{i \in Q_0} \alpha_i^{(1)} \alpha_{i-1}^{(1)} \alpha_{i-2}^{(2)} \cdots \alpha_{i-N}^{(2)}$. Moreover, let A' be defined by $\varphi_{\lambda k-1}(\partial_{\alpha_i^{(1)}} W_{N+1}) = \alpha_{i+\lambda k-1}^{(1)} \cdots \alpha_{i+2}^{(1)} \alpha_{i+1}^{(1)}$ and $\varphi_{\lambda k-1}(\partial_{\alpha_i^{(2)}} W_{N+1}) = \alpha_{i+\lambda k-1}^{(2)} \cdots \alpha_{i+2}^{(2)} \alpha_{i+1}^{(2)}$. Then the potential defining A' is $W = \sum_{p=1}^{N+1} W_p$ with $W_p = 0$ if $p \not\equiv 0 \pmod{k}$, and $W_{\lambda k} = \sum_{i \in Q_0} (\alpha_{i+\lambda k-1}^{(1)} \cdots \alpha_{i+1}^{(1)} \alpha_i^{(1)} + \alpha_{i+\lambda k-1}^{(2)} \cdots \alpha_{i+1}^{(2)} \alpha_i^{(2)})$.

Example 5.10. Let N be an integer with $N \geq 3$, and let $A = A(Q, W_{N+1})$ be the Calabi–Yau algebra defined by the quiver Q ,



and by the potential $W_{N+1} = a_3 a_4 a_1^{N-1} + a_4 a_3 a_2^{N-1}$.

Assume that the characteristic of \mathbb{k} does not divide $(N + 1)!$, and let A' be a PBW deformation of A . Then A' satisfies the following conditions:

$$\varphi_{N-1}(\partial_{a_1} W_{N+1}) a_1 + \varphi_{N-1}(\partial_{a_4} W_{N+1}) a_4 - a_1 \varphi_{N-1}(\partial_{a_1} W_{N+1}) - a_3 \varphi_{N-1}(\partial_{a_3} W_{N+1}) = \mu \partial_{a_1} W_{N+1} \tag{*1}$$

$$\begin{aligned} & \varphi_{N-1}(\partial_{a_2} W_{N+1})a_2 + \varphi_{N-1}(\partial_{a_3} W_{N+1})a_3 \\ & - a_2\varphi_{N-1}(\partial_{a_2} W_{N+1}) - a_4\varphi_{N-1}(\partial_{a_4} W_{N+1}) = \nu\partial_{a_2} W_{N+1} \end{aligned} \quad (*_2)$$

with $\mu, \nu \in \mathbb{k}$ (recall that $\varphi_{j-1}(\partial_a W_{N+1})$ is a linear combination of paths from $t(a)$ to $s(a)$).

The only terms occurring in the $\varphi_{N-1}(\partial_{a_i} W_{N+1})$ which give terms in $\partial_{a_1} W_{N+1} = \sum_{u+v=N-2} a_1^u a_3 a_4 a_1^v$ and in $\partial_{a_2} W_{N+1} = \sum_{u+v=N-2} a_2^u a_4 a_3 a_2^v$ are given as follows:

$$\varphi_{N-1}(\partial_{a_1} W_{N+1}) = \sum_{u+v=N-3} \lambda_{1,u} a_1^u a_3 a_4 a_1^v + X_1,$$

$$\varphi_{N-1}(\partial_{a_2} W_{N+1}) = \sum_{u+v=N-3} \lambda_{2,u} a_2^u a_4 a_3 a_2^v + X_2,$$

$$\varphi_{N-1}(\partial_{a_3} W_{N+1}) = \sigma_3 a_2^{N-2} a_4 + \lambda_3 a_4 a_1^{N-2} + X_3,$$

$$\varphi_{N-1}(\partial_{a_4} W_{N+1}) = \sigma_4 a_1^{N-2} a_3 + \lambda_4 a_3 a_2^{N-2} + X_4$$

with $\lambda_{1,u}, \lambda_{2,u}, \lambda_3, \lambda_4, \sigma_3$ and σ_4 in \mathbb{k} , and where X_1, X_2, X_3, X_4 are linear combinations of paths of length $N-1$ starting and ending at appropriate vertices.

The relation $(*_1)$ gives

$$\lambda_4 - \lambda_{1,0} = \lambda_{1,0} - \lambda_{1,1} = \cdots = \lambda_{1,p} - \lambda_{1,p+1} = \cdots = \lambda_{1,N-3} - \sigma_3 = \mu$$

and $\lambda_3 = \sigma_4$. The relation $(*_2)$ gives

$$\lambda_3 - \lambda_{2,0} = \lambda_{2,0} - \lambda_{2,1} = \cdots = \lambda_{2,p} - \lambda_{2,p+1} = \cdots = \lambda_{2,N-3} - \sigma_4 = \nu$$

and $\lambda_4 = \sigma_3$. They imply

$$(N+1)(\lambda_{1,0} - \lambda_{1,1}) = \left(\sum_{p=0}^{N-2} \lambda_{1,p} - \lambda_{1,p+1} \right) + \lambda_4 - \lambda_{1,0} + \lambda_{1,N-3} - \sigma_3 = \lambda_4 - \sigma_3 = 0$$

so that $\lambda_{1,0} = \lambda_{1,1}$ and $\mu = 0$. Similarly, $\nu = 0$. Therefore the relations $(*_1)$ and $(*_2)$ now show that (PBW2') is satisfied so that A' is an algebra defined by a potential.

Thus if the characteristic of \mathbb{k} does not divide $(N+1)!$, any PBW deformation of A is defined by a potential. This is not true however in arbitrary characteristic, as the following example shows. Let us fix $N = 3$ and $\text{char}(\mathbb{k}) = 2$. Choose a PBW deformation A' of A defined by $\varphi_0 = 0$, $\varphi_1 = 0$, and $\varphi_2(\partial_{a_1} W_4) = 0$, $\varphi_2(\partial_{a_2} W_4) = a_4 a_3$, $\varphi_2(\partial_{a_3} W_4) = a_4 a_1$, $\varphi_2(\partial_{a_4} W_4) = a_1 a_3$. It is easy to check that this is indeed a PBW deformation of A ((PBW1), (PBW3) and (PBW4) are obvious, and (PBW2) is equivalent to $(*_1)$ and $(*_2)$, which are simply the expressions of $\partial_{a_1} W_4$ and $\partial_{a_2} W_4$). However, (PBW2') does not hold, so that A' is not defined by a potential (condition (PBW2') is a necessary condition for A' to be defined by a potential, regardless of the characteristic of \mathbb{k}).

Note that if in this last example we change φ_2 to $\varphi_2(\partial_{a_1}W_4) = a_3a_4$, $\varphi_2(\partial_{a_2}W_4) = 0$, $\varphi_2(\partial_{a_3}W_4) = a_4a_1$, $\varphi_2(\partial_{a_4}W_4) = a_1a_3$, then we have a PBW deformation such that (PBW2') is satisfied and that is defined by a potential ($W_3 + W_2$ where $W_2 = a_4a_1a_3$). Note that Theorem 3.2 does not apply here.

Finally, if in the previous example we assume that $\text{char}(\mathbb{k}) = 3$ and we again change φ_2 to $\varphi_2(\partial_{a_1}W_4) = a_1^2$, $\varphi_2(\partial_{a_2}W_4) = 0$, $\varphi_2(\partial_{a_3}W_4) = 0$, $\varphi_2(\partial_{a_4}W_4) = 0$, then we have a PBW deformation such that (PBW2') is satisfied, but it is not defined from a potential (the only candidate being given by $W_2 = -a_1^3$, but $\partial_{a_1}(W_2) = -3a_1^2 = 0 \neq -\varphi_2(\partial_{a_1}W_4)$), thus showing that Theorem 3.2 is not applicable when the characteristic of \mathbb{k} divides $N!$.

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