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Pseudo-multiplicative unitaries on C*-modules and Hopf C*-families I

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Abstract. Pseudo-multiplicative unitaries on C*-modules generalize the multiplicative unitaries of Baaj and Skandalis [1], and are analogues of the pseudo-multiplicative unitaries on Hilbert spaces studied by Enock, Lesieur, Vallin [5], [10], [21]. We introduce Hopf C*-families on C*-bimodules and associate to special classes of pseudo-multiplicative unitaries two Hopf C*-families. Furthermore, we discuss dual pairings and counits on these Hopf C*-families, étalé and proper pseudo-multiplicative unitaries, and two classes of examples. In a later article, we will study regularity conditions on pseudo-multiplicative unitaries, coactions of Hopf C*-families on C*-algebras, and duality.

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1. Introduction

Multiplicative unitaries, introduced by Baaj and Skandalis [1], play a central rôle in operator-algebraic approaches to quantum groups and to generalizations of Pontrjagin duality: To each locally compact quantum group – that is, a Hopf C*-algebra equipped with a Haar weight – one can associate a manageable multiplicative unitary [7], [8], [11], and to every manageable multiplicative unitary, one can associate a pair of Hopf C*-algebras called the legs of the unitary [23]. One of these legs coincides with the initial quantum group, and the other is its Pontrjagin dual. A remarkable feature of the theory of quantum groups is the close interplay between the C*-algebraic (i.e., topological) and the von Neumann algebraic (i.e., measurable) level.

In the setting of von Neumann algebras, the theory of quantum groups was extended to a theory of measured quantum groupoids by Lesieur [10], building on work of Vallin and Enock [5], [6], [21]. Central concepts in this theory are Hopf–von Neumann bimodules and pseudo-multiplicative unitaries on Hilbert spaces, which generalize Hopf C*-algebras and multiplicative unitaries, respectively. Each measurable quantum groupoid gives rise to a manageable pseudo-multiplicative unitary, and each such unitary gives rise to a pair of Hopf–von Neumann bimodules called the legs of the unitary.

In the setting of C*-algebras, a theory of quantum groupoids is still elusive. The proper analogue of a (pseudo-)multiplicative unitary on Hilbert spaces – a pseudo-multiplicative unitary on C*-modules – is defined in this article; special examples were already discussed by O'uchi [13], [14]. The proper analogue of the notion of a Hopf C*-algebra and of a Hopf–von Neumann bimodule, however, is not known. The problem is to define the target of the comultiplication, which should be some fiber product of C*-algebras. In particular, it is not clear how to define the legs of a general pseudo-multiplicative unitary on C*-modules [15]. In this article, we propose a solution for this problem in a special case. We introduce C*-families which generalize C*-algebras, and define an internal tensor product of C*-families that leads to the notion of a Hopf C*-family. Given these notions, we can define the legs of suitable pseudo-multiplicative unitaries in the form of Hopf C*-families.

This work was supported by the SFB 478 "Geometrische Strukturen in der Mathematik". The article is an extract from my PhD thesis, which was supervised by Joachim Cuntz. In subsequent articles, we will discuss regularity conditions for pseudo-multiplicative unitaries, coactions on C*-algebras, and a duality theorem for such coactions.

Organization of the article. This article is organized as follows. First, we define pseudo-multiplicative unitaries on C*-modules and present two examples related to groupoids and to center-valued conditional expectations (Section 2). We explain the problems that obstruct the definition of the legs of a pseudo-multiplicative unitary, and outline our plan for a partial solution.

In Section 3, we introduce a general calculus of homogeneous operators on C*-bimodules. These operators twist the left and right module multiplication by some partial automorphisms of the underlying C*-algebras and have "twisted" adjoints. Moreover, we define C*-families of such operators and study homogeneous elements of C*-bimodules.

Using these concepts, we associate to each pseudo-multiplicative unitary two families of homogeneous operators (Section 4). Under certain assumptions, these families represent the legs of the unitary. We determine the legs of the unitaries considered in Section 1, and show that they are C*-families.

Next, we introduce internal tensor products and morphisms of C*-families, which enter the definition of a Hopf C*-family (Section 5). As a tool, we construct a functorial embedding of C*-families into C*-algebras.

In Section 6, we return to pseudo-multiplicative unitaries on C*-modules and introduce comultiplications on their legs. We study the examples introduced in Section 1 and show that these examples yield Hopf C*-families.

Finally, we discuss further properties of the legs like dual pairings, counits, fixed and cofixed elements (Section 7), and study those concepts for the examples mentioned before.

Conventions and preliminaries. Given a subset Y of a normed space X, we denote by $[Y] \subseteq X$ the closed linear span of Y.

Recall that a *partial automorphism* of a C*-algebra B is a $*$-isomorphism $\sigma\colon \mathrm{Dom}(\sigma) \to \mathrm{Im}(\sigma)$, where $\mathrm{Dom}(\sigma)$ and $\mathrm{Im}(\sigma)$ are closed ideals of B. Since the composition and the inverse of partial automorphisms are partial automorphisms again, the set $\mathrm{PAut}(B)$ of all partial automorphisms of B forms an inverse semigroup [16]. We denote the inverse of a partial automorphism σ by σ^*. Let $\sigma, \sigma' \in \mathrm{PAut}(B)$. We say that σ' extends σ and write $\sigma' \geq \sigma$ iff $\mathrm{Dom}(\sigma) \subseteq \mathrm{Dom}(\sigma')$ and $\sigma'|_{\mathrm{Dom}(\sigma)} = \sigma$. We put $\sigma \wedge \sigma' := \max\{\sigma'' \in \mathrm{PAut}(B) \mid \sigma'' \leq \sigma, \sigma'' \leq \sigma'\}$; thus, $\sigma \wedge \sigma' = \sigma|_I = \sigma'|_I$, where $I \subseteq \mathrm{Dom}(\sigma) \cap \mathrm{Dom}(\sigma')$ is the largest ideal on which σ and σ' coincide.

We consider (right) C*-modules, also known as Hilbert C*-modules or Hilbert modules, see, e.g., [9].

All sesquilinear maps (as, e.g., the inner product of a Hilbert space or a C*-module) are assumed to be conjugate-linear in the first component and linear in the second one.

Let A, B be C*-algebras. Given C*-modules E, F over B, we denote the set of all adjointable operators $E \to F$ by $\mathcal{L}_B(E, F)$, and the subset of all compact operators by $\mathcal{K}_B(E, F)$.

A *right C*-A-B-bimodule* is a C*-module E over B with a fixed non-degenerate $*$-homomorphism $\pi\colon A \to \mathcal{L}_B(E)$. If the representation π is understood, we loosely call E a right C*-bimodule and write $b\xi$ for $\pi(b)\xi$, where $b \in B$, $\xi \in E$; otherwise, we denote the right C*-bimodule by ${}_\pi E$. Given right C*-A-B-bimodules E, F, we put $\mathcal{L}^A_B(E, F) := \{T \in \mathcal{L}_B(E, F) \mid aT\xi = Ta\xi \text{ for all } a \in A,\ \xi \in E\}$.

Given a C*-A-module E and right C*-A-B-bimodule F, one can form an internal tensor product $E \otimes_A F$, which is a C*-module over B [9], Chapter 4. It is densely spanned by elements $\eta \otimes_A \xi$, where $\eta \in E$, $\xi \in F$, such that $\langle \eta' \otimes_A \xi' | \eta \otimes_A \xi\rangle = \langle \xi' | \langle \eta' | \eta\rangle \xi\rangle$ and $(\eta \otimes_A \xi)b = \eta \otimes_A \xi b$. We denote the internal tensor product by "$\ominus$"; thus, for example, $E \ominus F = E \otimes_A F$.

Given E and F as above, one can also form a *flipped internal tensor product* $F \oslash E$: we equip the algebraic tensor product $F \odot E$ with the structure maps $\langle \xi' \odot \eta' | \xi \odot \eta\rangle := \langle \xi' | \langle \eta' | \eta\rangle \xi\rangle$, $(\xi' \odot \eta')b := \xi' b \odot \eta'$, and by factoring out the null-space of the semi-norm $\zeta \mapsto \|\langle \zeta | \zeta\rangle\|^{1/2}$ and taking completion, we obtain a C*-B-module $F \oslash E$. It is densely spanned by elements $\xi \oslash \eta$, where $\eta \in E$, $\xi \in F$, such that $\langle \xi' \oslash \eta' | \xi \oslash \eta\rangle = \langle \xi' | \langle \eta' | \eta\rangle \xi\rangle$ and $(\xi \oslash \eta)b = \xi b \oslash \eta$.

The usual and the flipped internal tensor product are related by a unitary map $\Sigma\colon F \ominus E \xrightarrow{\cong} E \oslash F$, $\eta \ominus \xi \mapsto \xi \oslash \eta$.

If we want to emphasize that the factor F of a (flipped) internal tensor product $E \ominus F$ (or $F \oslash E$) is considered as a right C*-bimodule via a fixed representation π, we denote the product by $E \ominus {}_\pi F$ (or $F_\pi \oslash E$, respectively).

We shall frequently use the following result [3], Proposition 1.34:

Proposition 1.1. *Let E_1, E_2 be C*-A-modules, let F_1, F_2 be C*-A-B-bimodules, and let $S \in \mathcal{L}_A(E_1, E_2)$, $T \in \mathcal{L}^A_B(F_1, F_2)$. Then there exists a unique operator*

$S \ogreaterthan T \in \mathcal{L}_B(E_1 \ogreaterthan F_1, E_2 \ogreaterthan F_2)$ such that $(S \ogreaterthan T)(\eta \ogreaterthan \xi) = S\eta \ogreaterthan T\xi$ for all $\eta \in E_1$, $\xi \in F_1$. Moreover, $(S \ogreaterthan T)^* = S^* \ogreaterthan T^*$. □

The (flipped) internal tensor product of C*-bimodules is a C*-bimodule in a natural way, and the (flipped) internal tensor product is associative in a natural sense. More generally, (flipped) internal tensor products can be iterated in a natural way, and the C*-module obtained does not essentially depend on the order in which the tensor products are formed.

2. Pseudo-multiplicative unitaries

Recall that a multiplicative unitary [1], Définition 1.1, on a Hilbert space H is a unitary $V: H \otimes H \to H \otimes H$ that satisfies the so-called pentagon equation $V_{12}V_{13}V_{23} = V_{23}V_{12}$. Here, V_{12}, V_{13}, V_{23} are operators on $H \otimes H \otimes H$, defined by $V_{12} = V \otimes \mathrm{id}$, $V_{23} = \mathrm{id} \otimes V$, $V_{13} = (\Sigma \otimes \mathrm{id})V_{23}(\Sigma \otimes \mathrm{id}) = (\mathrm{id} \otimes \Sigma)V_{12}(\mathrm{id} \otimes \Sigma)$, where $\Sigma \in \mathcal{B}(H \otimes H)$ denotes the flip $\eta \otimes \xi \mapsto \xi \otimes \eta$. We extend this concept, replacing H by a C*-module E with representations $\hat{\beta}, \beta$.

Throughout this section, let B be a C*-algebra.

Definition 2.1. A *C*-trimodule* $(E, \hat{\beta}, \beta)$ over B is a full C*-B-module E with two commuting non-degenerate faithful representations $\hat{\beta}, \beta$ of B on E.

Let $(E, \hat{\beta}, \beta)$ be a C*-trimodule over B. Using Proposition 1.1, we define representations β_1, $\hat{\beta}_2$, β_2 of B on $E_{\hat{\beta}} \olessthan E$ by $\beta_1(b) := \beta(b) \olessthan 1$, $\hat{\beta}_2(b) := 1 \olessthan \hat{\beta}(b)$, $\beta_2(b) := 1 \olessthan \beta(b)$ for all $b \in B$, and similarly representations $\beta_1, \hat{\beta}_1, \hat{\beta}_2$ on $E \ogreaterthan_\beta E$. From Proposition 1.1 we deduce:

Lemma 2.2. *Let $W \in \mathcal{L}_B(E_{\hat{\beta}} \olessthan E, E \ogreaterthan_\beta E)$, and assume that for all $b \in B$,*

$$W\beta_2(b) = \hat{\beta}_1(b)W, \quad W\beta_1(b) = \beta_1(b)W, \quad W\hat{\beta}_2(b) = \hat{\beta}_2(b)W. \tag{1}$$

Then all operators in the following diagram are well defined:

$$\begin{array}{ccccc} & & E \ogreaterthan_\beta E_{\hat{\beta}} \olessthan E & & \\ & \overset{W \olessthan 1}{\nearrow} & & \overset{1 \ogreaterthan W}{\searrow} & \\ E_{\hat{\beta}} \olessthan E_{\hat{\beta}} \olessthan E & & & & E \ogreaterthan_\beta E \ogreaterthan_\beta E; \\ \downarrow{\scriptstyle 1 \olessthan W} & & & & \uparrow{\scriptstyle W \ogreaterthan 1} \\ E_{\hat{\beta}} \olessthan (E \ogreaterthan_\beta E) & & & & (E_{\hat{\beta}} \olessthan E) \ogreaterthan_\beta E \\ \downarrow{\scriptstyle 1 \olessthan \Sigma} & & & & \uparrow{\scriptstyle \Sigma_{23}} \\ E_{\hat{\beta}} \olessthan E_\beta \olessthan E & & \xrightarrow[W \olessthan 1]{} & & (E \ogreaterthan_\beta E)_{\hat{\beta}_1} \olessthan E \end{array} \tag{2}$$

where Σ_{23} *is given by* $(\xi_1 \otimes \xi_2) \otimes \xi_3 \mapsto (\xi_1 \otimes \xi_3) \otimes \xi_2$ *for all* $\xi_1, \xi_2, \xi_3 \in E$. □

Extending the leg notation to the operators in diagram (2), we write W_{12} for $W \otimes 1$ and $W \otimes 1$, W_{23} for $1 \otimes W$ and $1 \otimes W$, and W_{13} for $\Sigma_{23}(W \otimes 1)(1 \otimes \Sigma)$. Then diagram (2) commutes iff $W_{12}W_{13}W_{23} = W_{23}W_{12}$.

Definition 2.3. Suppose that $(E, \hat{\beta}, \beta)$ is a C*-trimodule over B. We call a unitary $W \in \mathcal{L}_B(E_{\hat{\beta}} \otimes E, E \otimes {}_{\beta}E)$ *pseudo-multiplicative* iff it satisfies the intertwining conditions (1) and diagram (2) commutes.

For commutative B, Definition 2.3 subsumes the following special cases:

(i) If $B = \mathbb{C}$, then $\beta(\lambda)\xi = \lambda\xi = \hat{\beta}(\lambda)\xi$ for all $\lambda \in \mathbb{C}$, $\xi \in E$, and W is a multiplicative unitary in the sense of Baaj and Skandalis [1].

(ii) If $\beta(b)\xi = \xi b = \hat{\beta}(b)\xi$ for all $\xi \in E$, $b \in B$, then W is a continuous field of multiplicative unitaries as defined by Blanchard [2].

(iii) If $\hat{\beta}(b)\xi = \xi b$ for all $\xi \in E$, $b \in B$, then W is a pseudo-multiplicative unitary in the sense of O'uchi [13].

Clearly Definition 2.3 is a C*-algebraic analogue of the definition of pseudo-multiplicative unitaries on Hilbert spaces given by Vallin [21].

Remark 2.4. Let $(E, \hat{\beta}, \beta)$ and $W: E_{\hat{\beta}} \otimes E \to E \otimes_{\beta} E$ be as in Definition 2.3. Then $(E, \beta, \hat{\beta})$ is a C*-trimodule over B, and the unitary $W^{\mathrm{op}} := \Sigma W^* \Sigma: E_{\beta} \otimes E \to E \otimes {}_{\hat{\beta}}E$, called the *opposite* of W, is pseudo-multiplicative.

Let us turn to the fundamental example discussed already in [13], the pseudo-multiplicative unitary associated to a locally compact groupoid. For background on groupoids and Haar systems see, e.g., [17] or [16].

Example 2.5. Let G be a locally compact, Hausdorff, second countable groupoid with left Haar system λ. We denote its unit space by G^0, its range map by r_G, its source map by s_G, and put $G^u := r_G^{-1}(\{u\})$ for $u \in G^0$.

Let $B := C_0(G^0)$. Denote by $L^2(G, \lambda)$ the C*-module over B associated to G and λ; this is the completion of the pre-C*-module $C_c(G)$, where $\langle \xi' | \xi \rangle(u) = \int_{G^u} \overline{\xi'(x)} \xi(x) \, d\lambda^u(x)$ and $(\xi f)(x) = \xi(x) f(r_G(x))$ for all $u \in G^0$, $x \in G$, $\xi, \xi' \in C_c(G)$, $f \in B$. Define representations $r, s: B \to \mathcal{L}_B(L^2(G, \lambda))$ by $(r(f)\xi)(x) := f(r_G(x))\xi(x)$ and $(s(f)\xi)(x) := f(s_G(x))\xi(x)$ for $x \in G$, $\xi \in C_c(G)$, $f \in B$. Then $(E, \hat{\beta}, \beta) := (L^2(G, \lambda), s, r)$ is a C*-trimodule over B.

For $k = r, s$, put $G^2_{k,r} := \{(x, y) \in G \times G \mid k_G(x) = r_G(y)\}$. Consider $C_c(G^2_{s,r})$ and $C_c(G^2_{r,r})$ as pre-C*-modules over B via the structure maps

$$\langle \zeta' | \zeta \rangle(u) := \int_{G^u} \int_{G^{s_G(x)}} \overline{\zeta'(x, y)} \zeta(x, y)\, d\lambda^{s_G(x)}(y) d\lambda^u(x) \quad \text{for } C_c(G^2_{s,r}),$$

$$\langle \zeta' | \zeta \rangle(u) := \int_{G^u} \int_{G^u} \overline{\zeta'(x, y)} \zeta(x, y)\, d\lambda^u(y) d\lambda^u(x) \quad \text{for } C_c(G^2_{r,r}),$$

$$(\zeta f)(x, y) := \zeta(x, y) f(r_G(x)) \quad \text{for both,}$$

and denote by $L^2(G^2_{s,r})$ and $L^2(G^2_{r,r})$ the respective completions. Then it is easy to see that $E_{\hat{\beta}} \otimes E \cong L^2(G^2_{s,r})$ and $E \otimes_{\beta} E \cong L^2(G^2_{r,r})$.

The map $W_0 \colon C_c(G^2_{s,r}) \to C_c(G^2_{r,r})$, $(W_0\zeta)(x, y) := \zeta(x, x^{-1}y)$, extends to a pseudo-multiplicative unitary $W_G \colon E_{\hat{\beta}} \otimes E \to E \otimes_{\beta} E$ [13]. Indeed, W_0 is a linear bijection because it is the transpose of a homeomorphism $G^2_{r,r} \to G^2_{s,r}$, it extends to a unitary W_G because λ is left-invariant, and a routine calculation shows that W_G satisfies the pentagon equation.

The pseudo-multiplicative unitary W_G is closely related to the pseudo-multiplicative unitary on Hilbert spaces associated to G in [21]; see [13].

The following example is a C*-algebraic analogue of a pseudo-multiplicative unitary on Hilbert spaces considered by Lesieur [10], Section 7.6.

Example 2.6. Let B be a unital C*-algebra, $C \subseteq Z(B)$ a C*-subalgebra containing 1_B, and $\tau \colon B \to C$ a faithful conditional expectation, that is, a faithful positive C-linear map such that $\tau|_C = \mathrm{id}_C$. We associate to τ a pseudo-multiplicative unitary W_τ as follows.

First, consider B as a pre-C*-module over C via the inner product $\langle a' | a \rangle := \tau(a'^* a)$ and via right multiplication, and denote by B_τ the completion. Next consider B as a right C*-B-B-bimodule in the natural way, and denote by $E := B_\tau \otimes B$ the internal tensor product over C. Thus E is generated by elements $a \otimes b$, where $a, b \in B$, and $\langle a' \otimes b' | a \otimes b \rangle = b'^* \tau(a'^* a) b$, $(a \otimes b)b' = a \otimes bb'$ for all $a, b, a', b' \in B$.

Routine arguments show that there exist representations $\hat{\beta}, \beta \colon B \to \mathcal{L}_B(E)$ such that $\hat{\beta}(b')(a \otimes b) := b'a \otimes b$ and $\beta(b')(a \otimes b) := a \otimes b'b$ for all $a, b, b' \in B$; here we use $\tau(B) \subseteq Z(B)$. Evidently $(E, \hat{\beta}, \beta)$ is a C*-trimodule.

We claim that there exist unitaries

$$X \colon E_{\hat{\beta}} \otimes E \to B_\tau \otimes B_\tau \otimes B, \quad (a \otimes b) \otimes (c \otimes d) \mapsto da \otimes c \otimes b,$$

$$Y \colon E \otimes_{\beta} E \to B_\tau \otimes B_\tau \otimes B, \quad (a \otimes b) \otimes (c \otimes d) \mapsto a \otimes c \otimes bd.$$

Indeed, for $x := (a \ogreaterthan b) \olessthan (c \ogreaterthan d)$ and $y := (a \ogreaterthan b) \ogreaterthan (c \ogreaterthan d)$ as above,

$$\begin{aligned} \|Xx\|^2 &= \|b^*\tau(c^*\tau(a^*d^*da)c)b\| = \|b^*\tau(a^*d^*\tau(c^*c)da)b\| = \|x\|^2, \\ \|Yy\|^2 &= \|d^*b^*\tau(c^*\tau(a^*a)c)bd\| = \|d^*\tau(c^*c)b^*\tau(a^*a)bd\| = \|y\|^2; \end{aligned}$$

here we use $\tau(B) \subseteq Z(B)$ and $\tau(e\tau(f)) = \tau(e)\tau(f)$ for $e, f \in B$. Now consider the unitary $W_\tau := Y^*X \colon E_{\hat\beta} \olessthan E \to E \ogreaterthan {}_\beta E$. Explicitly,

$$W_\tau((a \ogreaterthan b) \olessthan (c \ogreaterthan d)) = (da \ogreaterthan b) \ogreaterthan (c \ogreaterthan 1) \quad \text{for all } a, b, c, d \in B, \qquad (3)$$

since $Y((da \ogreaterthan b) \ogreaterthan (c \ogreaterthan 1)) = da \ogreaterthan c \ogreaterthan b = X((a \ogreaterthan b) \olessthan (c \ogreaterthan d))$. The following calculations show that W_τ is pseudo-multiplicative: for $a, b, c, d, e, f, g \in B$,

$$\begin{array}{ccc} (a \ogreaterthan b) \olessthan (c \ogreaterthan d) & \xmapsto{\beta_1(e)\beta_2(f)\hat\beta_2(g)} & (a \ogreaterthan eb) \olessthan (gc \ogreaterthan fd) \\ \downarrow{\scriptstyle W_\tau} & & \downarrow{\scriptstyle W_\tau} \\ (da \ogreaterthan b) \ogreaterthan (c \ogreaterthan 1) & \xmapsto[\beta_1(e)\hat\beta_1(f)\hat\beta_2(g)]{} & (fda \ogreaterthan eb) \ogreaterthan (gc \ogreaterthan 1), \end{array}$$

$$\begin{array}{ccc} & (da \ogreaterthan b) \ogreaterthan (c \ogreaterthan 1) \olessthan (e \ogreaterthan f) & \\ {\scriptstyle (W_\tau)_{12}} \nearrow & & \searrow {\scriptstyle (W_\tau)_{23}} \\ (a \ogreaterthan b) \olessthan (c \ogreaterthan d) \olessthan (e \ogreaterthan f) & & (da \ogreaterthan b) \ogreaterthan (fc \ogreaterthan 1) \ogreaterthan (e \ogreaterthan 1) \\ \downarrow{\scriptstyle (W_\tau)_{23}} & & \uparrow{\scriptstyle (W_\tau)_{12}} \\ (a \ogreaterthan b) \olessthan ((fc \ogreaterthan d) \ogreaterthan (e \ogreaterthan 1)) & \xmapsto[(W_\tau)_{13}]{} & ((a \ogreaterthan b) \olessthan (fc \ogreaterthan d)) \ogreaterthan (e \ogreaterthan 1). \end{array}$$

As indicated in the introduction, multiplicative unitaries are closely related to Hopf C*-algebras. Recall that a *Hopf C*-algebra* (more precisely, *bisimplifiable C*-bialgebra*, see also [1]) is a C*-algebra A with a $*$-homomorphism $\Delta \colon A \to M(A \otimes A)$ such that $[\Delta(A)(A \otimes 1)] = A \otimes A = [\Delta(A)(1 \otimes A)]$ and $(\mathrm{id} \otimes \Delta) \circ \Delta = (\Delta \otimes \mathrm{id}) \circ \Delta$ as maps $A \to M(A \otimes A \otimes A)$; note that $\mathrm{id} \otimes \Delta$ and $\Delta \otimes \mathrm{id}$ extend to $M(A \otimes A)$ by the first assumption. Here $A \otimes A$ denotes the minimal tensor product. Now each well behaved (e.g., regular [1] or manageable [23]) multiplicative unitary V on a Hilbert space H yields two Hopf C*-algebras $(\hat A(V), \hat\Delta)$ and $(A(V), \Delta)$, called the *legs* of V, as follows [1]. Denote by $\mathcal{L}(H)_*$ the predual of $\mathcal{L}(H)$. Each $\omega \in \mathcal{L}(H)_*$ yields slice maps $\mathrm{id} \mathbin{\bar\otimes} \omega, \omega \mathbin{\bar\otimes} \mathrm{id} \colon \mathcal{L}(H \otimes H) \to \mathcal{L}(H)$. Then

$$\begin{aligned} \hat A(V) &= \overline{\mathrm{span}}\{(\mathrm{id} \mathbin{\bar\otimes} \omega)(V) \mid \omega \in \mathcal{L}(H)_*\} \subseteq \mathcal{L}(H), & \hat\Delta(\hat a) &= V^*(1 \otimes \hat a)V, \\ A(V) &= \overline{\mathrm{span}}\{(\omega \mathbin{\bar\otimes} \mathrm{id})(V) \mid \omega \in \mathcal{L}(H)_*\} \subseteq \mathcal{L}(H), & \Delta(a) &= V(a \otimes 1)V^*. \end{aligned}$$

Naturally, the following question arises: Given a pseudo-multiplicative unitary $W \colon E_{\hat\beta} \olessthan E \to E \ogreaterthan {}_\beta E$, can we similarly associate to W two "legs" $(\hat A(W), \hat\Delta)$ and $(A(W), \Delta)$ in the form of generalized Hopf C*-algebras?

Let us first focus on the left leg $(\hat{A}(V), \hat{\Delta})$ and reformulate its definition. Note that functionals of the form $\omega_{\xi',\xi}\colon T \mapsto \langle \xi' | T\xi \rangle$, where $\xi', \xi \in H$, are linearly dense in $\mathcal{L}(H)_*$ [18], II Theorem 2.6, and that $(\mathrm{id} \bar{\otimes} \omega_{\xi',\xi})(V) = |\xi'\rangle_2^* V |\xi\rangle_2$, where $|\xi''\rangle_2\colon H \to H \otimes H, \zeta \mapsto \zeta \otimes \xi''$, for $\xi'' = \xi, \xi'$, and $|\xi'\rangle_2^*(\zeta \otimes \zeta') = \zeta\langle \xi' | \zeta' \rangle$. So, $\hat{A}(V)$ is the closed linear span of all operators $|\xi'\rangle_2^* V |\xi\rangle_2$, where $\xi', \xi \in H$.

Similarly, $\hat{A}(W)$ should be spanned by operators $|\xi']_2^* W |\xi\rangle_2$, where

$$|\xi\rangle_2\colon E \to E_{\hat{\beta}} \otimes E,\ \zeta \mapsto \zeta \otimes \xi, \quad \text{and} \quad |\xi']_2\colon E \to E \otimes_{\beta} E,\ \zeta \mapsto \zeta \otimes \xi',$$

and $\xi, \xi' \in E$ are suitably chosen. But $|\xi']_2$ has no adjoint unless $\beta(b)\xi' = \xi' b$ for all $b \in B$, as we can see from the relations $|\xi']_2(\zeta b) = \zeta b \otimes \xi' = \zeta \otimes \beta(b)\xi'$ and $(|\xi']_2\zeta)b = \zeta \otimes \xi' b$, which are valid for all $\zeta \in E$, $b \in B$.

However, if there exists a partial automorphism θ' of B such that ξ' is *θ'-homogeneous* in the sense that $\xi' \in E\,\mathrm{Dom}(\theta')$ and $\beta(\theta'(b))\xi' = \xi' b$ for all $b \in \mathrm{Dom}(\theta')$, then $|\xi']_2$ is adjointable up to a twist by θ'. If also ξ is θ-homogeneous for some $\theta \in \mathrm{PAut}(B)$, then $|\xi']_2^* W |\xi\rangle_2$ is homogeneous in the sense that it is adjointable and commutes with $\hat{\beta}(B)$ up to a twist by θ' or θ, respectively. To put these ideas into the right perspective, we give a systematic account of homogeneous elements and homogeneous operators in Section 3 before we return to pseudo-multiplicative unitaries in Section 6.

3. C*-families of homogeneous operators

In this section we introduce a general calculus of homogeneous operators on C*-bimodules and of homogeneous elements of C*-bimodules. Furthermore, we define C*-families which can be thought of as generalized C*-algebras of homogeneous operators on C*-bimodules.

Throughout this section, let A and B be C*-algebras.

Homogeneous operators on C*-bimodules. We consider maps of right C*-bimodules which almost preserve the bimodule structure:

Definition 3.1. Let E, F be right C*-A-B-bimodules and let $\rho \in \mathrm{PAut}(A)$, $\sigma \in \mathrm{PAut}(B)$. We call a map $T\colon E \to F$ a *(ρ, σ)-homogeneous operator* iff

(i) $\mathrm{Im}(T) \subseteq [\mathrm{Im}(\rho)F]$, and $Ta\xi = \rho(a)T\xi$ for all $a \in \mathrm{Dom}(\rho)$, $\xi \in E$, and

(ii) there exists a map $S\colon F \to E$ such that $\langle SF | E \rangle \subseteq \mathrm{Dom}(\sigma)$ and $\langle \eta | T\xi \rangle = \sigma(\langle S\eta | \xi \rangle)$ for all $\xi \in E$, $\eta \in F$.

Let us collect some first properties of homogeneous operators.

Proposition 3.2. *Let T, S be as in the definition above. Then:*

(i) *T and S are linear and bounded, and $\|T\| = \|S\|$.*

(ii) $T(\xi b) = (T\xi)\sigma(b)$ *for all* $b \in \mathrm{Dom}(\sigma)$ *and* $\xi \in E$.

(iii) *There exists* $\sigma_0 \in \mathrm{PAut}(B)$ *such that whenever* T *is* (ρ', σ')*-homogeneous for* $\rho' \in \mathrm{PAut}(A)$*,* $\sigma' \in \mathrm{PAut}(B)$*, then* $\sigma_0 \leq \sigma'$.

(iv) S *is uniquely determined by* T *and condition* ii) *in Definition* 3.1.

(v) *If* $(u_\nu)_\nu$ *and* $(v_\mu)_\mu$ *are approximate units of* $\mathrm{Dom}(\rho)$ *and* $\mathrm{Dom}(\sigma)$*, respectively, then* $\lim_\nu T(u_\nu\xi) = T\xi = \lim_\mu T(\xi v_\mu)$ *for all* $\xi \in E$.

Proof. (i) This is similar to the case of ordinary adjointable operators.

(ii) This relation follows from the fact that for all $\eta, \xi \in E$ and $b \in \mathrm{Dom}(\sigma)$, $\langle\eta|T(\xi b)\rangle = \sigma(\langle S\eta|\xi b\rangle) = \sigma(\langle S\eta|\xi\rangle b) = \sigma(\langle S\eta|\xi\rangle)\sigma(b) = \langle\eta|(T\xi)\sigma(b)\rangle$.

(iii) Put $J := [\langle F|TE\rangle]$. Then J is contained in $\mathrm{Im}(\sigma)$ and is an ideal in B because $BJ \subseteq [\langle FB|TE\rangle]$ and $J\,\mathrm{Im}(\sigma) \subseteq [\langle F|TEB\rangle]$ by (ii). Denote by σ_0 the restriction of σ to $\sigma^*(J)$. Assume that T is also (ρ', σ')-homogeneous for $\rho' \in \mathrm{PAut}(A)$, $\sigma' \in \mathrm{PAut}(B)$, and that S' satisfies condition (ii) of Definition 3.1 for T and σ'. Then $\sigma(\langle S\eta|\xi\rangle b) = \langle\eta|T(\xi b)\rangle = \sigma'(\langle S'\eta|\xi\rangle b)$ for all $\eta, \xi \in E$, $b \in B$, and hence $\sigma(\sigma^*(a)b) = \sigma'(\sigma'^*(a)b)$ for all $a \in J$, $b \in B$. Let $(u_\nu)_\nu$ be an approximate unit for J and let $d \in J$. The last relation and the inclusion $J \subseteq \mathrm{Im}(\sigma')$ imply that $d = \lim_\nu \sigma'(\sigma'^*(d)\sigma'^*(u_\nu)) = \lim_\nu \sigma(\sigma^*(d)\sigma'^*(u_\nu))$, and hence $\sigma^*(d) = \lim_\nu \sigma^*(d)\sigma'^*(u_\nu)$ is in $\sigma'^*(J)$. Now $\sigma_0 \leq \sigma'$ because $d = \lim_\nu \sigma(\sigma^*(u_\nu)\sigma^*(d)) = \lim_\nu \sigma'(\sigma'^*(u_\nu)\sigma^*(d)) = \sigma'(\sigma^*(d))$.

(iv) As in the case of ordinary adjointable operators, one finds that S is uniquely determined by T and σ. But by (ii), S is independent of σ.

(v) This follows from standard arguments. □

Definition 3.3. If T and S are as in Definition 3.1, we call S the *adjoint* of T and denote it by T^*.

For later use, we note the following simple example.

Example 3.4. Consider $\mathrm{Dom}(\rho), \mathrm{Im}(\rho) \subseteq B$ as right sub-C*-bimodules of B. Then $\rho \in \mathcal{L}^{\rho}_{\rho}(\mathrm{Dom}(\rho), \mathrm{Im}(\rho))$. Indeed, condition (i) in Definition 3.1 is easily checked, and for condition (ii) note that $\langle c|\rho(b)\rangle = c^*\rho(b) = \rho(\rho^*(c^*)b) = \rho(\langle\rho^*(c)|b\rangle)$ for all $b \in \mathrm{Dom}(\rho)$, $c \in \mathrm{Im}(\rho)$.

Remark 3.5. Suppose that E, F are right C*-A-B-bimodules, and let $\rho \in \mathrm{PAut}(A)$, $\sigma \in \mathrm{PAut}(B)$. Consider $E_{(\rho,\sigma)} := [\mathrm{Dom}(\rho)E\,\mathrm{Dom}(\sigma)] \subseteq E$ and $F^{(\rho,\sigma)} := [\mathrm{Im}(\rho)F\,\mathrm{Im}(\sigma)] \subseteq F$ as right C*-$\mathrm{Dom}(\rho)$-$\mathrm{Dom}(\sigma)$-bimodules, where the structure maps of $E_{(\rho,\sigma)}$ are inherited from E and the structure maps of $F^{(\rho,\sigma)}$ are twisted by ρ and σ in a straightforward way. Then every (ρ, σ)-homogeneous operator $T : E \to F$ restricts to an operator $T_{(\rho,\sigma)} \in \mathcal{L}^{\mathrm{Dom}(\rho)}_{\mathrm{Dom}(\sigma)}(E_{(\rho,\sigma)}, F^{(\rho,\sigma)})$, whose adjoint is a restriction of T^*.

The preceding remark shows that homogeneous operators generalize ordinary operators on right C*-bimodules only slightly. The point is that we shall consider entire families of homogeneous operators.

Definition 3.6. Let E, F be right C*-A-B-bimodules and $\rho \in \mathrm{PAut}(A)$, $\sigma \in \mathrm{PAut}(B)$. We denote the set of all (ρ,σ)-homogeneous operators from E to F by $\mathcal{L}^{\rho}_{\sigma}(E,F)$ and put $\mathcal{L}^{\rho}_{\sigma}(E) := \mathcal{L}^{\rho}_{\sigma}(E,E)$. The *strict topology* on $\mathcal{L}^{\rho}_{\sigma}(E,F)$ is the topology given by the family of seminorms $T \mapsto \|T\xi\|$, $\xi \in E$, and $T \mapsto \|T^*\eta\|$, $\eta \in F$.

The family of all homogeneous operators has the following properties:

Proposition 3.7. *Let E, F, G be right C*-A-B-bimodules and let $\rho,\rho' \in \mathrm{PAut}(A)$, $\sigma,\sigma' \in \mathrm{PAut}(B)$. Then:*

(i) *$\mathcal{L}^{\rho}_{\sigma}(E,F)$ is a closed subspace of the space of all bounded linear maps from E to F and complete with respect to the strict topology.*

(ii) *$\mathcal{L}^{\rho'}_{\sigma'}(F,G)\mathcal{L}^{\rho}_{\sigma}(E,F) \subseteq \mathcal{L}^{\rho'\rho}_{\sigma'\sigma}(E,G)$.*

(iii) *$\mathcal{L}^{\rho}_{\sigma}(E,F)^* = \mathcal{L}^{\rho^*}_{\sigma^*}(F,E)$, and $(\lambda T)^* = \overline{\lambda}T^*$, $\|T^*\| = \|T\| = \|T^*T\|^{1/2}$, $(ST)^* = T^*S^*$ for all $\lambda \in \mathbb{C}$, $T \in \mathcal{L}^{\rho}_{\sigma}(E,F)$, $S \in \mathcal{L}^{\rho'}_{\sigma'}(F,G)$.*

(iv) *$\mathcal{L}^{\mathrm{id}}_{\mathrm{id}}(E,F) = \mathcal{L}^{A}_{B}(E,F)$, and for each pair of partial identities $\epsilon' \in \mathrm{PAut}(A)$, $\epsilon \in \mathrm{PAut}(B)$ the space $\mathcal{L}^{\epsilon'}_{\epsilon}(E)$ is a C*-subalgebra of $\mathcal{L}^{A}_{B}(E)$.*

(v) *$\mathcal{L}^{\rho}_{\sigma}(E,F)$ is a right C*- $\mathcal{L}^{\rho\rho^*}_{\sigma\sigma^*}(F)$-$\mathcal{L}^{\rho^*\rho}_{\sigma^*\sigma}(E)$-bimodule.*

(vi) *$\mathcal{L}^{\rho}_{\sigma}(E,F) \subseteq \mathcal{L}^{\rho'}_{\sigma'}(E,F)$ if $\rho \le \rho'$ and $\sigma \le \sigma'$.*

Proof. Most of these assertions generalize facts about ordinary operators on right C*-bimodules and can be proved in a similar way by the help of Proposition 3.2. Therefore we only prove (ii). Let $T \in \mathcal{L}^{\rho}_{\sigma}(E,F)$, $T' \in \mathcal{L}^{\rho'}_{\sigma'}(F,G)$. By Definition 3.1 (i) and Proposition 3.2 (v),

$$[T'TE] \subseteq [T'\,\mathrm{Im}(\rho)F] \subseteq [\rho'(\mathrm{Dom}(\rho') \cap \mathrm{Im}(\rho))G] = [\mathrm{Im}(\rho'\rho)G]$$

and $T'Tb\xi = \rho'(\rho(b))T'T\xi$ for all $b \in \mathrm{Dom}(\rho'\rho)$, $\xi \in E$. Moreover, by Definition 3.1 (ii) and Proposition 3.2 (v), $\langle T'^*G|TE\rangle \subseteq \mathrm{Dom}(\sigma') \cap \mathrm{Im}(\sigma)$ and

$$\langle T^*T'^*G|E\rangle = \sigma^*(\langle T'^*G|TE\rangle) \subseteq \sigma^*(\mathrm{Dom}(\sigma') \cap \mathrm{Im}(\sigma)) = \mathrm{Dom}(\sigma'\sigma).$$

Finally, $\langle\eta|T'T\xi\rangle = \sigma'(\langle T'^*\eta|T\xi\rangle) = (\sigma'\sigma)(\langle T^*T'^*\eta|\xi\rangle)$ for all $\xi \in E$, $\eta \in G$. Therefore, $T'T \in \mathcal{L}^{\rho'\rho}_{\sigma'\sigma}(E,G)$ and $(T'T)^* = T^*T'^*$. □

C*-families of homogeneous operators. We adopt the following notation. Let E, F be right C*-A-B-bimodules and let $\mathcal{C} = (\mathcal{C}_\sigma^\rho)_{\rho,\sigma}$ be a family of closed subspaces $\mathcal{C}_\sigma^\rho \subseteq \mathcal{L}_\sigma^\rho(E, F)$, where $\rho \in \mathrm{PAut}(A)$, $\sigma \in \mathrm{PAut}(B)$.

- Given a family $\mathcal{D} = (\mathcal{D}_\sigma^\rho)_{\rho,\sigma}$ of closed subspaces $\mathcal{D}_\sigma^\rho \subseteq \mathcal{L}_\sigma^\rho(E, F)$, we write $\mathcal{D} \subseteq \mathcal{C}$ iff $\mathcal{D}_\sigma^\rho \subseteq \mathcal{C}_\sigma^\rho$ for all $\rho \in \mathrm{PAut}(A)$, $\sigma \in \mathrm{PAut}(B)$.
- We define a family $\mathcal{C}^* \subseteq \mathcal{L}(F, E)$ by $(\mathcal{C}^*)_\sigma^\rho := (\mathcal{C}_{\sigma^*}^{\rho^*})^*$ for all ρ, σ.
- We put $[\mathcal{C}E] := \overline{\mathrm{span}}\{T\xi \mid T \in \mathcal{C}_\sigma^\rho,\ \rho \in \mathrm{PAut}(A),\ \sigma \in \mathrm{PAut}(B), \xi \in E\}$.
- Let G be a right C*-A-B-bimodule and $\mathcal{D} \subseteq \mathcal{L}(F, G)$ a family of closed subspaces. The product $[\mathcal{D}\mathcal{C}] \subseteq \mathcal{L}(E, G)$ is the family given by

  $$[\mathcal{D}\mathcal{C}]_{\sigma''}^{\rho''} := \overline{\mathrm{span}}\{T'T \mid T' \in \mathcal{D}_{\sigma'}^{\rho'},\ T \in \mathcal{C}_\sigma^\rho,\ \rho'\rho \le \rho'',\ \sigma'\sigma \le \sigma''\}$$

  for all $\rho'' \in \mathrm{PAut}(A)$, $\sigma'' \in \mathrm{PAut}(B)$. Clearly the product $(\mathcal{D}, \mathcal{C}) \mapsto [\mathcal{D}\mathcal{C}]$ is associative.
  Similarly, we define families $[\mathcal{D}T], [T'\mathcal{C}] \subseteq \mathcal{L}(E, G)$ for operators $T \in \mathcal{L}_\sigma^\rho(E, F)$, $T' \in \mathcal{L}_{\sigma'}^{\rho'}(F, G)$, where $\rho, \rho' \in \mathrm{PAut}(A)$, $\sigma, \sigma' \in \mathrm{PAut}(B)$.
- By a slight abuse of notation, we define $\mathcal{L}^{\mathrm{id}}(E, F) \subseteq \mathcal{L}(E, F)$ by $(\mathcal{L}^{\mathrm{id}}(E, F))_\sigma^{\mathrm{id}} := \mathcal{L}_\sigma^{\mathrm{id}}(E, F)$, $(\mathcal{L}^{\mathrm{id}}(E, F))_\sigma^\rho := 0$ for $\rho \neq \mathrm{id}$. Similarly, we define $\mathcal{L}_{\mathrm{id}}(E, F) \subseteq \mathcal{L}(E, F)$.

The following generalization of C*-algebras will play an important rôle.

Definition 3.8. We call a family $\mathcal{C} \subseteq \mathcal{L}(E)$ of closed subspaces a *C*-family on E* iff $[\mathcal{C}\mathcal{C}] \subseteq \mathcal{C}$, $\mathcal{C}^* \subseteq \mathcal{C}$ and $\mathcal{C}_{\sigma_1}^{\rho_1} \subseteq \mathcal{C}_{\sigma_2}^{\rho_2}$ whenever $\rho_1 \le \rho_2$ and $\sigma_1 \le \sigma_2$. We call a C*-family $\mathcal{C}$ *non-degenerate* iff $[\mathcal{C}E] = E$.

Remarks 3.9. Let $\mathcal{C} \subseteq \mathcal{L}(E)$ be a C*-family.

(i) For each pair of partial identities $\epsilon' \in \mathrm{PAut}(A)$, $\epsilon \in \mathrm{PAut}(B)$, the space $\mathcal{C}_\epsilon^{\epsilon'} \subseteq \mathcal{L}_{\mathrm{id}}^{\mathrm{id}}(E) = \mathcal{L}_B^A(E)$ is a C*-algebra because $(\mathcal{C}_\epsilon^{\epsilon'})^* = \mathcal{C}_{\epsilon^*}^{\epsilon'^*} = \mathcal{C}_\epsilon^{\epsilon'}$ and $\mathcal{C}_\epsilon^{\epsilon'}\mathcal{C}_\epsilon^{\epsilon'} \subseteq \mathcal{C}_{\epsilon\epsilon}^{\epsilon'\epsilon'} = \mathcal{C}_\epsilon^{\epsilon'}$.

(ii) For each $\rho \in \mathrm{PAut}(A)$ and $\sigma \in \mathrm{PAut}(B)$, the space $\mathcal{C}_\sigma^\rho$ is a C*-module over the C*-algebra $\mathcal{C}_{\sigma^*\sigma}^{\rho^*\rho}$ because $(\mathcal{C}_\sigma^\rho)^*\mathcal{C}_\sigma^\rho = \mathcal{C}_{\sigma^*}^{\rho^*}\mathcal{C}_\sigma^\rho \subseteq \mathcal{C}_{\sigma^*\sigma}^{\rho^*\rho}$ and $\mathcal{C}_\sigma^\rho\mathcal{C}_{\sigma^*\sigma}^{\rho^*\rho} \subseteq \mathcal{C}_{\sigma\sigma^*\sigma}^{\rho\rho^*\rho} = \mathcal{C}_\sigma^\rho$. Likewise, $\mathcal{C}_\sigma^\rho$ is a left C*-module over the C*-algebra $\mathcal{C}_{\sigma\sigma^*}^{\rho\rho^*}$ and, in fact, a C*-bimodule over $\mathcal{C}_{\sigma\sigma^*}^{\rho\rho^*}$ and $\mathcal{C}_{\sigma^*\sigma}^{\rho^*\rho}$.

(iii) $[\mathcal{C}_{\mathrm{id}}^{\mathrm{id}}\mathcal{C}_\sigma^\rho] = \mathcal{C}_\sigma^\rho = [\mathcal{C}_\sigma^\rho\mathcal{C}_{\mathrm{id}}^{\mathrm{id}}]$ for each $\rho \in \mathrm{PAut}(A)$, $\sigma \in \mathrm{PAut}(B)$; this follows from (ii) and a standard result on C*-modules [9], p. 5.

(iv) The C*-family $\mathcal{C}$ is non-degenerate iff the C*-algebra $\mathcal{C}_{\mathrm{id}}^{\mathrm{id}} \subseteq \mathcal{L}_B^A(E)$ is non-degenerate. This follows easily from (iii).

To every C*-family, one can associate a multiplier C*-family:

Definition 3.10. The *multiplier family* of a C*-family $\mathcal{C} \subseteq \mathcal{L}(E)$ is the family $\mathcal{M}(\mathcal{C}) \subseteq \mathcal{L}(E)$ given by

$$\mathcal{M}(\mathcal{C})^\rho_\sigma := \{T \in \mathcal{L}^\rho_\sigma(E) \mid [T\mathcal{C}], [\mathcal{C}T] \subseteq \mathcal{C}\}, \quad \rho \in \mathrm{PAut}(A),\ \sigma \in \mathrm{PAut}(B).$$

Evidently, $\mathcal{M}(\mathcal{C})$ is a C*-family and by Remark 3.9 (iii), $\mathcal{M}(\mathcal{C})^\rho_\sigma = \{T \in \mathcal{L}^\rho_\sigma(E) \mid T\mathcal{C}^{\mathrm{id}}_{\mathrm{id}} \subseteq \mathcal{C}^\rho_\sigma,\ \mathcal{C}^{\mathrm{id}}_{\mathrm{id}} T \subseteq \mathcal{C}^\rho_\sigma\}$ for each $\rho \in \mathrm{PAut}(A)$, $\sigma \in \mathrm{PAut}(B)$.

Homogeneous elements of right C*-bimodules. We consider elements of right C*-bimodules that almost intertwine left and right multiplication:

Definition 3.11. Let E be a right C*-B-B-bimodule and $\theta \in \mathrm{PAut}(B)$. An element $\xi \in E$ is *θ-homogeneous* iff $\xi \in [E\,\mathrm{Dom}(\theta)]$ and $\xi b = \theta(b)\xi$ for all $b \in \mathrm{Dom}(\theta)$. We denote the set of all θ-homogeneous elements of E by $\mathcal{H}_\theta(E)$. Moreover, we call E *decomposable* iff the family $\mathcal{H}(E) := (\mathcal{H}_\theta(E))_\theta$ is linearly dense in E.

Note that B can be regarded as a C*-module over B in a natural way, and left multiplication turns B into a right C*-B-B-bimodule. Thus we can speak of homogeneous elements of B; these will be studied later.

Let E be a right C*-B-B-bimodule. For each $\xi \in E$, we define maps

$$|\xi\rangle \colon B \to E,\ b \mapsto \xi b, \quad |\xi] \colon B \to E,\ b \mapsto b\xi.$$

Then $|\xi\rangle$ has an adjoint $\langle\xi| = |\xi\rangle^* \colon \eta \mapsto \langle\xi|\eta\rangle$ and $\||\xi\rangle\| = \|\xi\|$ ([9], p. 12–13).

Proposition 3.12. *Let $\xi \in E$ and $\theta \in \mathrm{PAut}(B)$. Then the following conditions are equivalent:*

(i) $\xi \in \mathcal{H}_\theta(E)$;

(ii) $|\xi\rangle \in \mathcal{L}^\theta_{\mathrm{id}}(B, E)$;

(iii) $|\xi] \in \mathcal{L}^{\mathrm{id}}_{\theta^*}(B, E)$.

If (i)–(iii) *hold, then* $\||\xi]\| = \|\xi\|$ *and* $[\xi| := |\xi]^*$ *is given by* $\eta \mapsto \theta(\langle\xi|\eta\rangle)$.

Proof. (i) ⇒ (ii), (iii): Assume that (i) holds. To prove (ii), we only need to show that $|\xi\rangle$ satisfies condition (i) of Definition 3.1. But by assumption, $\mathrm{Im}\,|\xi\rangle \subseteq [\mathrm{Im}(\theta)E]$ and $|\xi\rangle(bb') = \xi bb' = \theta(b)|\xi\rangle b'$ for all $b \in \mathrm{Dom}(\theta)$, $b' \in B$. Let us prove (iii). Evidently, $|\xi]$ commutes with left multiplication. By assumption, $\langle\xi|\eta\rangle \in \mathrm{Dom}(\theta)$ for all $\eta \in E$ so that the map $[\xi| \colon \eta \mapsto \theta(\langle\xi|\eta\rangle)$ is well defined. Let $(u_\nu)_\nu$ be an approximate unit of $\mathrm{Im}(\theta)$. Then

$$\langle\eta||\xi]b\rangle = \lim_\nu \langle\eta|u_\nu b\xi\rangle = \lim_\nu \langle\eta|\xi\rangle\theta^*(u_\nu b) = \theta^*(\theta(\langle\xi|\eta\rangle)^* b) = \theta^*(\langle[\xi|\eta|b\rangle)$$

for all $\eta \in E$, $b \in B$. Hence (iii) holds. Moreover, we may assume $\|u_\nu\| \leq 1$ for all ν, and then $\|\xi\| = \lim_\nu \||\xi]u_\nu\| \leq \||\xi]\|$. The reverse inequality is evident.

(ii) $\Rightarrow$ (i): If (ii) holds, then $\xi \in [\xi B] = [\operatorname{Im}|\xi\rangle] \subseteq [\operatorname{Im}(\theta)E]$, and $\xi c = \theta(c)\xi$ for each $c \in \operatorname{Dom}(\theta)$ because $\xi cb = |\xi\rangle cb = \theta(c)(|\xi\rangle b) = \theta(c)\xi b$ for each $b \in B$.

(iii) $\Rightarrow$ (i): This follows from a similar argument as (ii) $\Rightarrow$ (i). □

Let E be a C*-module over A and F a right C*-A-B-bimodule. For each $\eta \in E$ and $\xi \in F$, we define maps

$$|\eta\rangle_1 \colon F \to E \otimes F,\ \zeta \mapsto \eta \otimes \zeta, \quad |\xi]_2 \colon E \to E \otimes F,\ \zeta \mapsto \zeta \otimes \xi.$$

Then $|\eta\rangle_1$ has an adjoint $\langle\eta|_1 = |\eta\rangle_1^* \colon \zeta \otimes \zeta' \mapsto \langle\eta|\zeta\rangle\zeta'$, and $\||\xi\rangle\| = \|\xi\|$ if the representation $A \to \mathcal{L}_B(F)$ is injective ([9], Lemma 4.6).

Proposition 3.13. *Let E, F be right C*-B-B-bimodules and $\theta \in \operatorname{PAut}(B)$.*

(i) *If $\eta \in \mathcal{H}_\theta(E)$, then $|\eta\rangle_1 \in \mathcal{L}^{\theta}_{\mathrm{id}}(F, E \otimes F)$.*

(ii) *Let $\xi \in \mathcal{H}_\theta(F)$. Then $|\xi]_2 \in \mathcal{L}^{\mathrm{id}}_{\theta^*}(E, E \otimes F)$ and $[\xi|_2 := |\xi]_2{}^*$ is given by $\zeta \otimes \zeta' \mapsto \zeta\theta(\langle\xi|\zeta'\rangle)$. If E is full, then $\||\xi]\| = \|\xi\|$.*

Proof. The proof is similar to that of Proposition 3.12; we only sketch the main steps for (ii). Let $\xi \in \mathcal{H}_\theta(F)$. For all $\zeta, \zeta' \in E$ and $\xi' \in F$,

$$\langle\zeta' \otimes \xi'|\zeta \otimes \xi\rangle = \langle\xi'|\langle\zeta'|\zeta\rangle\xi\rangle = \theta^*(\theta(\langle\xi'|\xi\rangle)\langle\zeta'|\zeta\rangle) = \theta^*(\langle\zeta'\theta(\langle\xi|\xi'\rangle)|\zeta\rangle).$$

For $\zeta' = \zeta$, $\xi' = \xi$ this equation shows that $\||\xi]_2\zeta\|^2 \leq \|\theta(\langle\xi|\xi\rangle)\|\|\zeta\|^2$, and hence $\||\xi]_2\| \leq \|\xi\|$. If E is full, this inequality is an equality. Finally, the equation above shows that the formula for $[\xi|_2$ defines a bounded map $E \otimes F \to E$, and that $\langle\zeta' \otimes \xi'||\xi]_2\zeta\rangle = \theta^*(\langle[\xi|_2(\zeta' \otimes \xi')|\zeta\rangle)$ for all $\zeta, \zeta' \in E$ and $\xi' \in F$. □

Next we collect several useful formulas concerning homogeneous elements. Let E and F be right C*-B-B-bimodules, and for $\theta, \theta' \in \operatorname{PAut}(B)$ put $\mathcal{H}_\theta(E) \otimes \mathcal{H}_{\theta'}(F) := \overline{\operatorname{span}}\{\eta \otimes \xi \mid \eta \in \mathcal{H}_\theta(E),\ \xi \in \mathcal{H}_{\theta'}(F)\} \subseteq E \otimes F$.

Proposition 3.14. *Let $\theta, \theta', \sigma, \rho \in \operatorname{PAut}(B)$. Then:*

(i) *$\mathcal{H}_\theta(E) = [\mathcal{H}_{\theta'}(E)\operatorname{Dom}(\theta)] \subseteq \mathcal{H}_{\theta'}(E)$ if $\theta \leq \theta'$.*

(ii) *$\langle\mathcal{H}_\theta(E)|\mathcal{H}_{\theta'}(E)\rangle \subseteq \mathcal{H}_{\theta^*\theta'}(B)$.*

(iii) *For each $\xi \in E$, the set $\{\theta' \in \operatorname{PAut}(B) \mid \xi \in \mathcal{H}_{\theta'}(E)\}$ either is empty or has a minimal element.*

(iv) *$\mathcal{L}^{\rho}_{\sigma}(E, F)\mathcal{H}_\theta(E) \subseteq \mathcal{H}_{\rho\theta\sigma^*}(F)$; $\mathcal{H}_\rho(A)\mathcal{H}_\theta(E)\mathcal{H}_\sigma(B) \subseteq \mathcal{H}_{\rho\theta\sigma}(E)$.*

(v) *The space $I_\theta := [\langle\mathcal{H}_\theta(E)|\mathcal{H}_\theta(E)\rangle]$ is an ideal in $Z(B)$ and $\mathcal{H}_\theta(E)$ is a right C*-$Z(B)$-I_θ-bimodule. In particular, $\mathcal{H}_\theta(E)I_\theta = \mathcal{H}_\theta(E)$.*

(vi) *If E is full and decomposable, then B is decomposable and the ideal of $Z(B)$ spanned by all $I_{\theta'}$, where $\theta' \in \mathrm{PAut}(B)$, is non-degenerate in B.*

(vii) $\mathcal{H}_{\theta}(E) \cap \mathcal{H}_{\theta'}(E) = \mathcal{H}_{(\theta \wedge \theta')}(E)$.

(viii) $\mathcal{H}_{\theta}(E) \otimes \mathcal{H}_{\theta'}(F) \subseteq \mathcal{H}_{\theta\theta'}(E \otimes F)$.

Proof. We only prove assertions (iii), (iv), (vi), (vii); the others follow from straightforward calculations or can be deduced from Propositions 3.7, 3.12.

(iii) Given $\xi \in E$, apply Propositions 3.2 (iii) and 3.12 to $|\xi]$.

(iv) Let $T \in \mathcal{L}^{\rho}_{\sigma}(E, F)$, $\xi \in \mathcal{H}_{\theta}(E)$. Choose approximate units $(u_{\kappa})_{\kappa}$, $(v_{\mu})_{\mu}$, $(w_{\nu})_{\nu}$ of $\mathrm{Dom}(\rho)$, $\mathrm{Im}(\theta)$, $\mathrm{Dom}(\sigma)$, respectively. By Proposition 3.2,

$$T\xi = \lim_{\kappa,\mu,\nu} T(u_{\kappa} v_{\mu} \xi w_{\nu}) = \lim_{\kappa,\mu,\nu} T(\xi \theta^*(u_{\kappa} v_{\mu}) w_{\nu}) = \lim_{\kappa,\mu,\nu} (T\xi)\sigma(\theta^*(u_{\kappa} v_{\mu}) w_{\nu}).$$

Since $(\sigma(\theta^*(u_{\kappa} v_{\mu}) w_{\nu}))_{\kappa,\mu,\nu}$ is an approximate unit for $\mathrm{Dom}(\rho\theta\sigma^*)$, the equation above implies that $T\xi \in [F\,\mathrm{Dom}(\rho\theta\sigma^*)]$. Finally, for all $b \in \mathrm{Dom}(\rho\theta\sigma^*)$ we have $(T\xi)b = T(\xi\sigma^*(b)) = T\theta(\sigma^*(b))\xi = ((\rho\theta\sigma^*)(b))T\xi$. This proves the first inclusion in (iv) and the second one follows similarly.

(vi) The assumptions imply that B is contained in the closure of

$$\sum_{\theta,\theta'} \langle \mathcal{H}_{\theta'}(E) | \mathcal{H}_{\theta}(E) \rangle = \sum_{\theta,\theta'} I_{\theta'} \langle \mathcal{H}_{\theta'}(E) | \mathcal{H}_{\theta}(E) \rangle \subseteq \sum_{\theta,\theta'} I_{\theta'} \mathcal{H}_{\theta'^*\theta}(B);$$

here we used (ii) and (v). The claims follow.

(vii) By (i) we have that $\mathcal{H}_{(\theta \wedge \theta')}(E) \subseteq \mathcal{H}_{\theta}(E) \cap \mathcal{H}_{\theta'}(E)$. Conversely, if $\xi \in \mathcal{H}_{\theta}(E) \cap \mathcal{H}_{\theta'}(E)$ and $\theta'' \in \mathrm{PAut}(B)$ is minimal with $\xi \in \mathcal{H}_{\theta''}(E)$ (see (iii)), then $\theta'' \leq \theta$ and $\theta'' \leq \theta'$, whence $\theta'' \leq \theta \wedge \theta'$ and $\xi \in \mathcal{H}_{(\theta \wedge \theta')}(E)$. □

The preceding proposition suggests the following notation. Let E be a right C*-B-B-bimodule and let $\mathcal{E} = (\mathcal{E}_{\theta})_{\theta}$ and $\mathcal{E}' = (\mathcal{E}'_{\theta})_{\theta}$ be families of closed subspaces $\mathcal{E}_{\theta} \subseteq \mathcal{H}_{\theta}(E)$, $\mathcal{E}'_{\theta} \subseteq \mathcal{H}_{\theta}(E)$, where $\theta \in \mathrm{PAut}(B)$.

- We write $\mathcal{E}' \subseteq \mathcal{E}$ iff $\mathcal{E}'_{\theta} \subseteq \mathcal{E}_{\theta}$ for all $\theta \in \mathrm{PAut}(B)$.
- We define a family $[\langle \mathcal{E}' | \mathcal{E} \rangle] \subseteq \mathcal{H}(B)$ by $[\langle \mathcal{E}' | \mathcal{E} \rangle]_{\theta''} = \overline{\mathrm{span}}\{\langle \xi' | \xi \rangle \mid \xi \in \mathcal{E}_{\theta}, \xi' \in \mathcal{E}'_{\theta'}, \theta, \theta' \in \mathrm{PAut}(B), \theta'^*\theta \leq \theta''\}$.
- Given a family $\mathcal{C} \subseteq \mathcal{L}(E, F)$, where F is a right C*-B-B-bimodule, we define a family $[\mathcal{C}\mathcal{E}] \subseteq \mathcal{H}(F)$ by $[\mathcal{C}\mathcal{E}]_{\theta} = \overline{\mathrm{span}}\{S\xi \mid S \in \mathcal{C}^{\rho}_{\sigma}, \xi \in \mathcal{E}_{\theta'}, \rho, \theta', \sigma \in \mathrm{PAut}(B), \rho\theta'\sigma^* \leq \theta\}$. Similarly, we define a family $[S\mathcal{E}] \subseteq \mathcal{H}(F)$ for each homogeneous operator $S \colon E \to F$.
- Given a right C*-B-B-bimodule F and a family $\mathcal{F} \subseteq \mathcal{H}(F)$, we define a family $[\mathcal{E} \otimes \mathcal{F}] \subseteq \mathcal{H}(E \otimes F)$ by $[\mathcal{E} \otimes \mathcal{F}]_{\theta''} := \overline{\mathrm{span}}\{\eta \otimes \xi \mid \eta \in \mathcal{E}_{\theta}, \xi \in \mathcal{F}_{\theta'}, \theta, \theta' \in \mathrm{PAut}(B), \theta\theta' \leq \theta''\}$.

Sometimes it is easy to determine a dense subspace $E^0 \subseteq E$ spanned by homogeneous elements and desirable to know whether $E^0 \cap \mathcal{H}_\theta(E)$ is dense in $\mathcal{H}_\theta(E)$ for each $\theta \in \mathrm{PAut}(B)$.

Proposition 3.15. *Let E be a decomposable right C*-B-B-bimodule and $E^0_\theta \subseteq \mathcal{H}_\theta(E)$ a subspace for each $\theta \in \mathrm{PAut}(B)$ such that $\sum_\theta E^0_\theta \subseteq E$ is dense and $E^0_\theta \mathcal{H}_\sigma(B) \subseteq E^0_{\theta\sigma}$ for all $\theta, \sigma \in \mathrm{PAut}(B)$. Then E^0_θ is dense in $\mathcal{H}_\theta(E)$ for each $\theta \in \mathrm{PAut}(B)$.*

Proof. Put $K := \overline{\mathrm{span}}\{|\eta\rangle\langle\eta'| \mid \eta \in E^0_\theta, \eta' \in E^0_{\theta'}, \theta\theta'^* \leq \mathrm{id}\} \subseteq \mathcal{K}_B(E)$. Proposition 3.14 (ii), (iv) implies that K is a C*-algebra. Moreover, considering $\overline{E^0_\theta}$ as a C*-module over $[\langle E^0_\theta | E^0_\theta\rangle] \subseteq B$ for each $\theta \in \mathrm{PAut}(B)$, we find that $E = \left[\sum_\theta E^0_\theta\right] = \left[\sum_\theta E^0_\theta \langle E^0_\theta | E^0_\theta\rangle\right] \subseteq [KE]$. Hence K is non-degenerate.

Now let $\xi \in \mathcal{H}_\theta(E)$, $\theta \in \mathrm{PAut}(B)$. We prove that $\xi \in \overline{E^0_\theta}$. Choose an approximate unit $(\kappa_\nu)_\nu$ of K of the form $\kappa_\nu = \sum_i |\eta_{\nu,i}\rangle\langle\eta'_{\nu,i}|$, where $\eta_{\nu,i} \in E^0_{\theta_{\nu,i}}$, $\eta'_{\nu,i} \in E^0_{\theta'_{\nu,i}}$, $\theta_{\nu,i}\theta'_{\nu,i}{}^* \leq \mathrm{id}$. Since K is non-degenerate, $\xi = \lim_\nu \kappa_\nu \xi = \lim_\nu \sum_i \xi_{\nu,i}$, where $\xi_{\nu,i} = \eta_{\nu,i}\langle\eta'_{\nu,i}|\xi\rangle$. By Proposition 3.14 (ii), (iv) and assumption on $(E^0_\theta)_\theta$, we have $\xi_{\nu,i} \in E^0_{\theta_{\nu,i}} \cdot \mathcal{H}_{(\theta'_{\nu,i}{}^*\theta)}(B) \subseteq E^0_\theta$. Thus, $\xi \in \overline{E^0_\theta}$. □

Before collecting corollaries we prove another useful result by a similar technique. Let E, F be right C*-B-B-bimodules. For $\theta'' \in \mathrm{PAut}(B)$, put $\mathcal{K}^{\theta''}_{\mathrm{id}}(E, F) := \overline{\mathrm{span}}\{|\xi\rangle\langle\xi'| \mid \xi \in \mathcal{H}_\theta(F), \xi' \in \mathcal{H}_{\theta'}(E), \theta\theta'^* \leq \theta''\}$. Then by Proposition 3.14 (v),

$$E = \Big[\sum_\theta \mathcal{H}_\theta(E)\Big] = \Big[\sum_\theta \mathcal{H}_\theta(E)\langle\mathcal{H}_\theta(E)|\mathcal{H}_\theta(E)\rangle\Big] \subseteq [\mathcal{K}^{\mathrm{id}}_{\mathrm{id}}(E)E]. \tag{4}$$

Proposition 3.16. *If E or F is decomposable, then for each $\theta \in \mathrm{PAut}(B)$ we have $\mathcal{K}^\theta_{\mathrm{id}}(E, F) = \mathcal{K}_B(E, F) \cap \mathcal{L}^\theta_{\mathrm{id}}(E, F)$.*

Proof. Let $\theta \in \mathrm{PAut}(B)$. By Proposition 3.12, $\mathcal{K}^\theta_{\mathrm{id}}(E, F) \subseteq \mathcal{K}_B(E, F) \cap \mathcal{L}^\theta_{\mathrm{id}}(E, F)$. We prove the reverse inclusion for the case that F is decomposable; the case that E is decomposable is similar. Choose a bounded approximate unit $(\kappa_\nu)_\nu$ of $\mathcal{K}^{\mathrm{id}}_{\mathrm{id}}(E)$ of the form $\kappa_\nu = \sum_i |\eta_{\nu,i}\rangle\langle\eta'_{\nu,i}|$, where $\eta_{\nu,i} \in \mathcal{H}_{\theta_{\nu,i}}(E)$, $\eta'_{\nu,i} \in \mathcal{H}_{\theta'_{\nu,i}}(E)$, $\theta_{\nu,i}\theta'_{\nu,i}{}^* \leq \mathrm{id}$. Let $T \in \mathcal{K}_B(E, F) \cap \mathcal{L}^\theta_{\mathrm{id}}(E, F)$. Then (4) implies $T = \lim_\nu T\kappa_\nu = \lim_\nu \sum_i |T\eta_{\nu,i}\rangle\langle\eta'_{\nu,i}|$. Using Proposition 3.14 (iv) and the relation $\theta\theta_{\nu,i}\theta'_{\nu,i}{}^* \leq \theta$, we find $|T\eta_{\nu,i}\rangle\langle\eta'_{\nu,i}| \in \mathcal{K}^\theta_{\mathrm{id}}(E, F)$. Therefore, $T \in \mathcal{K}^\theta_{\mathrm{id}}(E, F)$. □

Proposition 3.17. *Let E, F be decomposable right C*-B-B-bimodules. Then $E \otimes F$ is decomposable, and $\mathcal{H}(E \otimes F) = [\mathcal{H}(E) \otimes \mathcal{H}(F)]$.*

Proof. By Proposition 3.14 (viii), $[\mathcal{H}(E) \otimes \mathcal{H}(F)] \subseteq \mathcal{H}(E \otimes F)$. For the reverse inclusion apply Proposition 3.15 to $H([\mathcal{H}(E) \otimes \mathcal{H}(F)]_\theta)_\theta$. □

Let E be a C*-A-module, F a right C*-B-B-bimodule and $\pi\colon A \to \mathcal{L}_B^B(F)$ a $*$-homomorphism. Then $E \otimes_\pi F$ is a right C*-B-B-bimodule via the representation $B \to \mathcal{L}_B(E \otimes_\pi F)$, $b \mapsto \mathrm{id} \otimes b$ (use Proposition 1.1). Given a family $\mathcal{F} \subseteq \mathcal{H}(F)$, we define a family $[E \otimes_\pi \mathcal{F}] \subseteq \mathcal{H}(E \otimes_\pi F)$ by $[E \otimes_\pi \mathcal{F}]_\theta := \overline{\mathrm{span}}\{\eta \otimes \xi \mid \eta \in E,\ \xi \in \mathcal{F}_\theta\}$.

Proposition 3.18. *If F is decomposable, then $E \otimes_\pi F$ is decomposable and $\mathcal{H}(E \otimes_\pi F) = [E \otimes_\pi \mathcal{H}(F)]$.*

Proof. A short calculation shows that $[E \otimes_\pi \mathcal{H}(F)] \subseteq \mathcal{H}(E \otimes_\pi F)$. For the reverse inclusion apply Proposition 3.15 to $([E \otimes_\pi \mathcal{H}_\theta(F)]_\theta)_\theta$. □

Homogeneous elements of C*-algebras

Proposition 3.19. *Let $b \in \mathcal{H}_\theta(B)$, $\theta \in \mathrm{PAut}(B)$ and denote by $I_b \subseteq B$ the ideal generated by b^*b. Then:*

(i) *b is normal and b^*b is central.*

(ii) *There exists a unitary $u \in M(I_b)$ such that $b = u(b^*b)^{1/2}$.*

(iii) *With u as in* (ii)*, the map $\mathrm{Ad}_u\colon I_b \to I_b$ is the minimal partial automorphism of B with respect to which b is homogeneous.*

(iv) *$\theta(b) = b$; in particular, $b \in \mathrm{Dom}(\theta^*)$ and $\theta^*(b) = b$.*

Proof. (i) The positive elements b^*b and bb^* are central by Proposition 3.14 (ii), whence $bb^* \cdot bb^* = b^*bbb^* = b^*b \cdot b^*b$. Consequently, $bb^* = b^*b$.

(ii) Put $D := \mathrm{spec}(b) \setminus \{0\}$. For $n \geq 1$, define $f_n \in C_0(D)$ by $f_n(z) := z/|z|$ if $|z| \geq 1/n$, and $f_n(z) := nz$ if $|z| \leq 1/n$. Then $(f_n)_n$ converges in $M(D)$ strictly to a unitary and functional calculus shows that the sequence $(f_n(b))_n$ converges in $M(I_b)$ strictly to some unitary u. Denote by $\mathrm{id}_D \in C_0(D)$ the identity map. Then $\lim_n f_n|\mathrm{id}_D| = \mathrm{id}_D$ in $C_0(D)$, and hence $u(b^*b)^{1/2} = \lim_n f_n(b)|\mathrm{id}_D(b)| = \mathrm{id}_D(b) = b$.

(iii) Evidently, $b \in I_b$ and $bd = u(b^*b)^{1/2}d = udu^*u(b^*b)^{1/2} = \mathrm{Ad}_u(d)b$ for all $d \in I_b$, so $b \in \mathcal{H}_{\mathrm{Ad}_u}(B)$. If $b \in \mathcal{H}_{\theta'}(B)$ for some $\theta' \in \mathrm{PAut}(B)$, then $I_b \subseteq \mathrm{Dom}(\theta')$ because $b \in \mathrm{Dom}(\theta')$, and $\mathrm{Ad}_u \leq \theta'$ by Proposition 3.14 (iii).

(iv) $\theta(b) = \mathrm{Ad}_u(b) = u(u(b^*b)^{1/2})u^* = u(b^*b)^{1/2} = b$ by (iii) and because $(b^*b)^{1/2}$ is central. The relations $b \in \mathrm{Dom}(\theta^*)$ and $b = \theta^*(b)$ follow. □

Proposition 3.20. *Let $\theta, \theta', \rho \in \mathrm{PAut}(B)$. Then:*

(i) *$bc = \theta(cb)$ and $cb = \theta^*(bc)$ for all $b \in \mathcal{H}_\theta(B)$, $c \in B$.*

(ii) *$\mathcal{H}_\theta(B) = \mathcal{H}_\theta(B) \cap \mathrm{Dom}(\theta \wedge \mathrm{id})$.*

(iii) *$\rho(\mathcal{H}_\theta(B) \cap \mathrm{Dom}(\rho)) \subseteq \mathcal{H}_{\rho\theta\rho^*}(B)$.*

(iv) *$\mathcal{H}_{\theta'}(B)\mathcal{H}_{\theta}(B) \subseteq \mathcal{H}_{\theta'\theta}(B)$ and $\mathcal{H}_{\theta}(B)^* = \mathcal{H}_{\theta^*}(B)$.*

(v) *B is decomposable iff the inclusion $Z(B) \subseteq B$ is non-degenerate. In particular, every unital C*-algebra is decomposable.*

Proof. (i) Let $b \in \mathcal{H}_{\theta}(B)$, $c \in B$, and let $(u_\nu)_\nu$ be an approximate unit of $\mathrm{Dom}(\theta)$. Then $bc = \lim_\nu bu_\nu c = \lim_\nu \theta(u_\nu c)b = \theta(c\theta^*(b)) = \theta(cb)$ by Proposition 3.19 (iv), and similarly $cb = \theta^*(bc)$.

(ii) This follows from Proposition 3.19 (iv).

(iii) Combine Example 3.4 with Proposition 3.14 (iv).

(iv) Straightforward.

(v) If B is decomposable, then $[BZ(B)] = B$ by Proposition 3.14 (vi). Conversely, assume that $[Z(B)B] = B$. For each unitary $u \in M(B)$ and each $b \in Z(B)$, the product bu is contained in $\mathcal{H}_{\mathrm{Ad}_u}(B)$ since $buc = (ucu^*)bu$ for all $c \in B$. By [12], Remark 2.2.2, each element of B can be written as a sum of four unitaries in $M(B)$. Therefore B is decomposable. □

To every C*-bimodule E we associate a C*-family $\mathcal{O}(E)$ as follows:

Proposition 3.21. *Let A, B be C*-algebras and E a right C*-A-B-bimodule.*

(i) *Let $a \in \mathcal{H}_\rho(A)$, let $\rho \in \mathrm{PAut}(A)$ and let $b \in \mathcal{H}_\sigma(B)$, $\sigma \in \mathrm{PAut}(B)$. Then $o_{a,b}\colon E \to E$, $\xi \mapsto a\xi b$, is (ρ,σ^*)-homogeneous and $(o_{a,b})^* = o_{a^*,b^*}$.*

(ii) *Put $\mathcal{O}^\rho_\sigma(E) := \overline{\mathrm{span}}\{o_{a,b} \mid a \in \mathcal{H}_\rho(A),\ b \in \mathcal{H}_{\sigma^*}(B)\}$ for all $\rho \in \mathrm{PAut}(A)$, $\sigma \in \mathrm{PAut}(B)$. Then $\mathcal{O}(E) \subseteq \mathcal{L}(E)$ is a C*-family.*

Proof. (i) Let a, b as above. Then $o_{a,b}$ satisfies condition (i) of Definition 3.1 because $\mathrm{Im}(o_{a,b}) \subseteq aE \subseteq \mathrm{Im}(\rho)E$ and $o_{a,b}a'\xi = aa'\xi b = \rho(a')a\xi b = \rho(a')o_{a,b}\xi$ for all $a' \in \mathrm{Dom}(\rho)$, $\xi \in E$. Moreover, by Proposition 3.20 (i), (iv) and 3.19 (iv), $a^* \in \mathcal{H}_{\rho^*}(A)$, $\sigma(b^*) = b^* \in \mathcal{H}_{\sigma^*}(B)$, $\langle o_{a^*,b^*}E|E\rangle \subseteq b\langle E|E\rangle \subseteq \mathrm{Dom}(\sigma^*)$, and $\langle \eta|a\xi b\rangle = \langle a^*\eta|\xi\rangle b = \sigma^*(b\langle a^*\eta|\xi\rangle) = \sigma^*(\langle a^*\eta b^*|\xi\rangle)$ for all $\eta, \xi \in E$. The claim follows.

(ii) Obvious from (i) and Proposition 3.20 (iv). □

Definition 3.22. Let E be a right C*-A-B-bimodule, where A and B are decomposable. A family $\mathcal{C} \subseteq \mathcal{L}(E)$ is called an *$\mathcal{O}(E)$-module* iff $[\mathcal{O}(E)\mathcal{C}] \subseteq \mathcal{C}$, and is called a *non-degenerate $\mathcal{O}(E)$-module* iff additionally $\mathcal{C}^\rho_\sigma = [\mathcal{O}^{\rho\rho^*}_{\sigma\sigma^*}(E)\mathcal{C}^\rho_\sigma]$ for all $\rho \in \mathrm{PAut}(A)$, $\sigma \in \mathrm{PAut}(B)$.

Remark 3.23. The C*-family $\mathcal{O}(E)$ defined above is interesting primarily if A and B are decomposable. However, we can consider E as a right C*-$M(A)$-$M(B)$-bimodule via the identification $E \cong A \otimes_A E \otimes_B M(B)$, and $M(A)$ and $M(B)$ are decomposable by Proposition 3.20 (v).

4. The legs of a decomposable pseudo-multiplicative unitary and C*-families

We return to the study of a pseudo-multiplicative unitary $W\colon E_{\hat{\beta}} \otimes E \to E \otimes {}_{\beta}E$, where $(E, \hat{\beta}, \beta)$ is a C*-trimodule over a C*-algebra B, and define the legs of W in the form of families of homogeneous operators. Our definition of the left and of the right leg will be useful only if the right C*-bimodule ${}_{\beta}E$ or ${}_{\hat{\beta}}E$, respectively, is decomposable. From Proposition 3.13, equation (1) and Proposition 3.7 (ii) we deduce:

Lemma 4.1. *Let* $\rho, \sigma \in \mathrm{PAut}(B)$.

(i) *Let* $\xi \in \mathcal{H}_{\rho}({}_{\beta}E)$, $\xi' \in \mathcal{H}_{\sigma}({}_{\beta}E)$. *Then we have homogeneous operators*

$$ {}_{\hat{\beta}}E \xrightarrow[(\rho,\mathrm{id})\text{-}hmg.]{|\xi\rangle_2} {}_{\beta_2}(E_{\hat{\beta}} \otimes E) \xrightarrow[(\mathrm{id},\mathrm{id})\text{-}hmg.]{W} {}_{\hat{\beta}_1}(E \otimes {}_{\beta}E) \xrightarrow[(\mathrm{id},\sigma)\text{-}hmg.]{|\xi']_2^*} {}_{\hat{\beta}}E, $$

where $|\xi\rangle_2\zeta = \zeta \otimes \xi$ *and* $|\xi']_2\zeta = \zeta \otimes \xi'$ *for all* $\zeta \in E$. *Put* $[\xi'|_2 := |\xi']_2^*$. *The composition* $\hat{a}_{(\xi',\xi)} := [\xi'|_2 W|\xi\rangle_2$ *belongs to* $\mathcal{L}^{\rho}_{\sigma}({}_{\hat{\beta}}E)$ *and satisfies* $\langle\zeta'|\hat{a}_{(\xi',\xi)}\zeta\rangle = \sigma(\langle\zeta' \otimes \xi'|W(\zeta \otimes \xi)\rangle)$ *for all* $\zeta, \zeta' \in E$.

(ii) *Let* $\eta \in \mathcal{H}_{\rho^*}({}_{\hat{\beta}}E)$, $\eta' \in \mathcal{H}_{\sigma^*}({}_{\hat{\beta}}E)$. *Then we have homogeneous operators*

$$ {}_{\beta}E \xrightarrow[(\mathrm{id},\sigma)\text{-}hmg.]{|\eta]_1} {}_{\beta_2}(E_{\hat{\beta}} \otimes E) \xrightarrow[(\mathrm{id},\mathrm{id})\text{-}hmg.]{W} {}_{\hat{\beta}_1}(E \otimes {}_{\beta}E) \xrightarrow[(\rho,\mathrm{id})\text{-}hmg.]{|\eta'\rangle_1^*} {}_{\beta}E, $$

where $|\eta]_1\zeta = \eta \otimes \zeta$ *and* $|\eta'\rangle_1\zeta = \eta' \otimes \zeta$ *for all* $\zeta \in E$. *Put* $\langle\eta'|_1 := |\eta'\rangle_1^*$. *The composition* $a_{(\eta',\eta)} := \langle\eta'|_1 W|\eta]_1$ *belongs to* $\mathcal{L}^{\rho}_{\sigma}({}_{\beta}E)$ *and satisfies* $\langle\zeta'|a_{(\eta',\eta)}\zeta\rangle = \langle\eta' \otimes \zeta'|W(\eta \otimes \zeta)\rangle$ *for all* $\zeta, \zeta' \in E$. □

We define families $\hat{\mathcal{A}}(W) \subseteq \mathcal{L}({}_{\hat{\beta}}E)$ and $\mathcal{A}(W) \subseteq \mathcal{L}({}_{\beta}E)$ as follows: for each $\rho, \sigma \in \mathrm{PAut}(B)$, we let $\hat{\mathcal{A}}(W)^{\rho}_{\sigma}$ and $\mathcal{A}(W)^{\rho}_{\sigma}$ be the closure of

$$ \hat{\mathcal{A}}_a(W)^{\rho}_{\sigma} := \mathrm{span}\{\hat{a}_{(\xi',\xi)} \mid \xi \in \mathcal{H}_{\rho}({}_{\beta}E),\ \xi' \in \mathcal{H}_{\sigma}({}_{\beta}E)\} \subseteq \mathcal{L}^{\rho}_{\sigma}({}_{\hat{\beta}}E) $$

and

$$ \mathcal{A}_a(W)^{\rho}_{\sigma} := \mathrm{span}\{a_{(\eta',\eta)} \mid \eta \in \mathcal{H}_{\sigma^*}({}_{\hat{\beta}}E),\ \eta' \in \mathcal{H}_{\rho^*}({}_{\hat{\beta}}E)\} \subseteq \mathcal{L}^{\rho}_{\sigma}({}_{\beta}E), $$

respectively. Applying the ket-bra notation to families of homogeneous elements, we can rewrite the definition of $\hat{\mathcal{A}}(W)$ and $\mathcal{A}(W)$ as follows. Define $|{}_{\beta}\mathcal{E}\rangle \subseteq \mathcal{L}_{\mathrm{id}}(B, {}_{\beta}E)$ and $|{}_{\beta}\mathcal{E}] \subseteq \mathcal{L}^{\mathrm{id}}(B, {}_{\beta}E)$ by (see Proposition 3.12)

$$ |{}_{\beta}\mathcal{E}\rangle^{\rho}_{\mathrm{id}} := \{|\xi\rangle \mid \xi \in \mathcal{H}_{\rho}({}_{\beta}E)\}, \quad |{}_{\beta}\mathcal{E}]^{\mathrm{id}}_{\sigma} := \{|\xi'] \mid \xi' \in \mathcal{H}_{\sigma^*}({}_{\beta}E)\}. $$

Put $\langle{}_{\beta}\mathcal{E}| := |{}_{\beta}\mathcal{E}\rangle^*$ and $[{}_{\beta}\mathcal{E}| := |{}_{\beta}\mathcal{E}]^*$. Replacing ${}_{\beta}E$ by ${}_{\hat{\beta}}E$ we similarly define $|{}_{\hat{\beta}}\mathcal{E}\rangle$, $\langle{}_{\hat{\beta}}\mathcal{E}|$, $|{}_{\hat{\beta}}\mathcal{E}]$, $[{}_{\hat{\beta}}\mathcal{E}|$. To all of these families we apply the leg notation just like to individual ket-bra operators. Then

$$ \hat{\mathcal{A}}(W) = [[{}_{\beta}\mathcal{E}|_2 W|{}_{\beta}\mathcal{E}\rangle_2] \quad \text{and} \quad \mathcal{A}(W) = [\langle{}_{\hat{\beta}}\mathcal{E}|_1 W|{}_{\hat{\beta}}\mathcal{E}]_1]. $$

If we pass from W to W^{op}, the legs of the unitary get switched as follows:

Proposition 4.2. $\hat{\mathcal{A}}(W) = \mathcal{A}(W^{\mathrm{op}})^*$ *and* $\mathcal{A}(W) = \hat{\mathcal{A}}(W^{\mathrm{op}})^*$.

Proof. For all homogeneous $\xi, \xi' \in {}_{\beta}E$, $\eta, \eta' \in {}_{\hat{\beta}}E$, we have $[\xi'|_2 W|\xi\rangle_2 = (\langle\xi|_2 W^*|\xi']_2)^* = (\langle\xi|_1 W^{\mathrm{op}}|\xi']_1)^*$ and $\langle\eta'|_1 W|\eta]_1 = ([\eta|_2 W^{\mathrm{op}}|\eta'\rangle_2)^*$. □

For each $\theta \in \mathrm{PAut}(B)$, $b \in \mathcal{H}_\theta(B)$ we have an (id, θ^*)-homogeneous operator (see the proof of Proposition 3.21)

$$\alpha(b)\colon E \to E, \quad \xi \mapsto \xi b.$$

Lemma 4.3. *Let* $b \in B$, $\xi, \xi' \in {}_{\hat{\beta}}E$, $\eta, \eta' \in {}_{\beta}E$ *be homogeneous. Then*

$$\hat{a}_{(\xi',\xi)}\hat{\beta}(b) = \hat{a}_{(\xi',\xi b)}, \qquad \hat{a}_{(\xi',\xi)}\alpha(b) = \hat{a}_{(\xi' b^*,\xi)}, \quad \hat{a}_{(\xi',\xi)}\beta(b) = \beta(b)\hat{a}_{(\xi',\xi)},$$

$$\hat{\beta}(b)a_{(\eta',\eta)} = a_{(\eta',\eta)}\hat{\beta}(b), \quad \alpha(b)a_{(\eta',\eta)} = a_{(\eta',\eta b)}, \quad \beta(b)a_{(\eta',\eta)} = a_{(\eta' b^*,\eta)}.$$

Proof. We only prove the equations concerning $\hat{a}_{(\xi',\xi)} = [\xi'|_2 W|\xi\rangle_2$.

First, $[\xi'|_2 W|\xi\rangle_2\hat{\beta}(b) = [\xi'|_2 W|\xi b\rangle_2 = \hat{a}_{(\xi',\xi b)}$ because $|\xi\rangle_2\hat{\beta}(b)\zeta = \hat{\beta}(b)\zeta \otimes \xi = \zeta \otimes \xi b = |\xi b\rangle_2\zeta$ for all $\zeta \in E$.

Next $[\xi'|_2 W|\xi\rangle_2\alpha(b) = [\xi'|_2\alpha(b)W|\xi\rangle_2 = [\xi' b^*|_2 W|\xi\rangle_2 = \hat{a}_{(\xi' b^*,\xi)}$ because $([\xi'|_2\alpha(b))^*\zeta = \alpha(b)^*(\zeta \otimes \xi') = \zeta \otimes \xi' b^* = ([\xi' b^*|_2)^*\zeta$ for all $\zeta \in E$.

Finally, $[\xi'|_2 W|\xi\rangle_2$ commutes with $\beta(b)$ because $|\xi\rangle_2\beta(b) = \beta_1(b)|\xi\rangle_2$, $W\beta_1(b) = \beta_1(b)W$ and $[\xi'|_2\beta_1(b) = \beta(b)[\xi'|_2$. □

For brevity we denote the family $\mathcal{H}(B)$ by $\mathcal{B}$. Define $\hat{\beta}(\mathcal{B}) \subseteq \mathcal{L}_{\mathrm{id}}({}_{\hat{\beta}}E)$ and $\alpha(\mathcal{B}) \subseteq \mathcal{L}^{\mathrm{id}}({}_{\hat{\beta}}E)$ by

$$\hat{\beta}(\mathcal{B})^{\rho}_{\mathrm{id}} := \{\hat{\beta}(b) \mid b \in \mathcal{H}_\rho(B)\}, \quad \alpha(\mathcal{B})^{\mathrm{id}}_{\sigma} := \{\alpha(b) \mid b \in \mathcal{H}_{\sigma^*}(B)\},$$

and similarly $\beta(\mathcal{B}) \subseteq \mathcal{L}_{\mathrm{id}}({}_{\beta}E)$, $\alpha(\mathcal{B}) \subseteq \mathcal{L}^{\mathrm{id}}({}_{\beta}E)$. Given a right C*-bimodule F and a family $\mathcal{C} \subseteq \mathcal{L}(F)$, we denote by $\mathcal{C}' \subseteq \mathcal{L}(F)$ the family of all homogeneous operators that commute with all operators of $\mathcal{C}$.

Proposition 4.4. (i) $[\hat{\mathcal{A}}(W)\alpha(\mathcal{B})] = [\hat{\mathcal{A}}(W)\hat{\beta}(\mathcal{B})] = \hat{\mathcal{A}}(W) \subseteq \beta(\mathcal{B})'$. *If* $\hat{\mathcal{A}}(W)$ *is a C*-family, then it is a non-degenerate* $\mathcal{O}({}_{\hat{\beta}}E)$*-module.*

(ii) $[\alpha(\mathcal{B})\mathcal{A}(W)] = [\beta(\mathcal{B})\mathcal{A}(W)] = \mathcal{A}(W) \subseteq \hat{\beta}(\mathcal{B})'$. *If* $\mathcal{A}(W)$ *is a C*-family, then it is a non-degenerate* $\mathcal{O}({}_{\beta}E)$*-module.*

Proof. We will only prove assertion (i). By Lemma 4.3, it is sufficient to show that $\hat{\mathcal{A}}(W)^{\rho}_{\sigma} \subseteq [\hat{\mathcal{A}}(W)^{\rho}_{\sigma} \cdot \hat{\beta}(\mathrm{Dom}(\rho^*\rho))]$ and $\hat{\mathcal{A}}(W)^{\rho}_{\sigma} \subseteq [\hat{\mathcal{A}}(W)^{\rho}_{\sigma} \cdot \alpha(\mathrm{Dom}(\sigma^*\sigma))]$

for each $\rho, \sigma \in \mathrm{PAut}(B)$. But if $(u_\nu)_\nu$ and $(v_\mu)_\mu$ are bounded approximate units of $\mathrm{Dom}(\rho^*\rho)$ and $\mathrm{Dom}(\sigma^*\sigma)$, respectively, and if $\hat{a}_{(\xi',\xi)}$ is as in Lemma 4.1 (i), then $\hat{a}_{(\xi',\xi)} = \lim_\nu \hat{a}_{(\xi',\xi u_\nu)} = \lim_\nu \hat{a}_{(\xi',\xi)}\hat{\beta}(u_\nu)$ and $\hat{a}_{(\xi',\xi)} = \lim_\mu \hat{a}_{(\xi' v_\mu^*,\xi)} = \lim_\mu \hat{a}_{(\xi',\xi)}\alpha(v_\mu)$ by Lemma 4.3. □

The families $\hat{\mathcal{A}}(W)$ and $\mathcal{A}(W)$ are non-degenerate in the following sense.

Proposition 4.5. (i) $[\hat{\mathcal{A}}(W)^* E] = E$ *if* ${}_\beta E$ *is decomposable.*

(ii)$[\mathcal{A}(W)E] = E$ *if* ${}_{\hat{\beta}}E$ *is decomposable.*

(iii) *If* ${}_\beta E$ *and* ${}_{\hat{\beta}}E$ *are decomposable, then* $[\hat{\mathcal{A}}(W)^*\mathcal{H}({}_{\hat{\beta}}E)] = \mathcal{H}({}_{\hat{\beta}}E)$ *and* $[\mathcal{A}(W)\mathcal{H}({}_\beta E)] = \mathcal{H}({}_\beta E)$.

Proof. We prove the first part of (iii); the other assertions follow similarly. By Proposition 3.14 (iv), $[\hat{\mathcal{A}}(W)^*\mathcal{H}({}_{\hat{\beta}}E)] \subseteq \mathcal{H}({}_{\hat{\beta}}E)$. Let us now prove the reverse inclusion. We have $[\hat{\mathcal{A}}(W)^*\mathcal{H}({}_{\hat{\beta}}E)] = [\langle {}_\beta\mathcal{E}|_2 W^* |{}_\beta\mathcal{E}]_2 \mathcal{H}({}_{\hat{\beta}}E)]$ by definition. Next $[W^*|{}_\beta\mathcal{E}]_2\mathcal{H}({}_{\hat{\beta}}E)] = [W^*\mathcal{H}({}_{\hat{\beta}_1}(E \ominus {}_\beta E))] = \mathcal{H}({}_{\beta_2}(E_{\hat{\beta}} \oslash E)) = [\mathcal{H}({}_{\hat{\beta}}E) \oslash \mathcal{H}({}_\beta E)]$ by Propositions 3.17, 3.14 and equation (1). Therefore, $[\hat{\mathcal{A}}(W)^*\mathcal{H}({}_{\hat{\beta}}E)] = [\langle {}_\beta\mathcal{E}|_2(\mathcal{H}({}_{\hat{\beta}}E) \oslash \mathcal{H}({}_\beta E))]$. For all homogeneous $\eta \in {}_{\hat{\beta}}E$ and $\xi, \xi' \in {}_\beta E$, we have $\langle \xi'|_2(\eta_{\hat{\beta}}\oslash\xi) = \hat{\beta}(\langle \xi'|\xi\rangle)\eta$. Therefore, $[\hat{\beta}(I)\mathcal{H}({}_{\hat{\beta}}E)] \subseteq [\hat{\mathcal{A}}(W)^*\mathcal{H}({}_{\hat{\beta}}E)]$ with $I = [\langle \mathcal{H}({}_\beta E)|\mathcal{H}({}_\beta E)\rangle]_{\mathrm{id}}$. By Proposition 3.14 (ii), $I \subseteq Z(B)$ and by 3.14 (vi), IB is linearly dense in B. Hence $[\hat{\beta}(I)\mathcal{H}({}_{\hat{\beta}}E)] = \mathcal{H}({}_{\hat{\beta}}E)$, and the claim follows. □

Next we show that $\hat{\mathcal{A}}(W)$ and $\mathcal{A}(W)$ are closed under multiplication. The proof involves the following observation. If ${}_\beta E$ is decomposable, then

$$\begin{aligned} [W[\mathcal{H}({}_\beta E) \oslash E]] &= [W\mathcal{H}({}_{\beta_1}(E_{\hat{\beta}} \oslash E))] && \text{(by Proposition 3.18)} \\ &= \mathcal{H}({}_{\beta_1}(E \ominus {}_\beta E)) && \text{(by equation (1))} \\ &= [\mathcal{H}({}_\beta E) \ominus \mathcal{H}({}_\beta E)] && \text{(by Proposition 3.17).} \end{aligned} \tag{5}$$

Proposition 4.6. (i) $[\hat{\mathcal{A}}(W)\hat{\mathcal{A}}(W)] = \hat{\mathcal{A}}(W)$ *if* ${}_\beta E$ *is decomposable.*

(ii) $[\mathcal{A}(W)\mathcal{A}(W)] = \mathcal{A}(W)$ *if* ${}_{\hat{\beta}}E$ *is decomposable.*

Proof. We only prove (i). By definition, $[\hat{\mathcal{A}}(W)\hat{\mathcal{A}}(W)] \subseteq \mathcal{L}({}_{\hat{\beta}}E)$ is the family of closed subspaces spanned by all compositions of the form

$$\hat{a}_{(\xi',\xi)}\hat{a}_{(\zeta',\zeta)} \colon E \xrightarrow{|\zeta\rangle_2} E_{\hat{\beta}}\oslash E \xrightarrow{W} E \ominus_\beta E \xrightarrow{[\zeta'|_2} E \xrightarrow{|\xi\rangle_2} E_{\hat{\beta}}\oslash E \xrightarrow{W} E\ominus_\beta E \xrightarrow{[\xi'|_2} E,$$

where $\xi, \xi', \zeta, \zeta' \in {}_\beta E$ are homogeneous. Moving $[\zeta'|_2$ to the left and $|\xi\rangle_2$ to the right, we can write $\hat{a}_{(\xi',\xi)}\hat{a}_{(\zeta',\zeta)}$ in the form

$$E \xrightarrow{|\xi \ominus \zeta\rangle_2} E_{\hat{\beta}} \otimes (E \ominus {}_\beta E) \xrightarrow{W_{13}} (E_{\hat{\beta}} \otimes E) \ominus {}_\beta E \xrightarrow{W_{12}} E \ominus {}_\beta E \ominus {}_\beta E \xrightarrow{[\xi' \ominus \zeta'|_2} E.$$

Using the pentagon equation (2) and Proposition 3.17, we find that the product $[\hat{\mathcal{A}}(W)\hat{\mathcal{A}}(W)]$ is equal to the family spanned by all compositions

$$E \xrightarrow{|\omega\rangle_2} E_{\hat{\beta}} \otimes (E \ominus {}_\beta E) \xrightarrow{W_{23}^*} E_{\hat{\beta}} \otimes E_{\hat{\beta}} \otimes E \xrightarrow{W_{12}} E \ominus_\beta E_{\hat{\beta}} \otimes E$$
$$\xrightarrow{W_{23}} E \ominus {}_\beta E \ominus {}_\beta E \xrightarrow{[\omega'|_2} E,$$

where $\omega, \omega' \in {}_{\beta_1}(E \ominus {}_\beta E)$ are homogeneous. Now equation (5) implies that $[\hat{\mathcal{A}}(W)\hat{\mathcal{A}}(W)]$ is equal to the family spanned by all compositions

$$E \xrightarrow{|\vartheta\rangle_2} E_{\hat{\beta}} \otimes E \xrightarrow{\mathrm{id}_E \ominus |\eta\rangle_2} E_{\hat{\beta}} \otimes E_{\hat{\beta}} \otimes E \xrightarrow{W_{12}} E \ominus_\beta E_{\hat{\beta}} \otimes E$$
$$\xrightarrow{\mathrm{id}_E \ominus \langle\eta'|_2} E \ominus {}_\beta E \xrightarrow{[\vartheta'|_2} E,$$

where $\vartheta, \vartheta' \in {}_\beta E$ are homogeneous and $\eta, \eta' \in E$ are arbitrary. Because $(\mathrm{id} \ominus \langle\eta'|_2)W_{12} = W(\mathrm{id} \ominus \langle\eta'|_2)$ and $(\mathrm{id} \ominus \langle\eta'|_2)(\mathrm{id} \ominus |\eta\rangle_2) = \mathrm{id} \ominus \hat{\beta}(\langle\eta'|\eta\rangle)$ the composition above is equal to

$$E \xrightarrow{|\vartheta\rangle_2} E_{\hat{\beta}} \otimes E \xrightarrow{\hat{\beta}_2(\langle\eta'|\eta\rangle)W} E \ominus {}_\beta E \xrightarrow{[\vartheta'|_2} E,$$

that is, equal to $\hat{a}_{(\vartheta',\vartheta'')}$ where $\vartheta'' = \hat{\beta}(\langle\eta'|\eta\rangle)\vartheta$. Note that $\vartheta'' \in {}_\beta E$ is homogeneous because $\hat{\beta}$ commutes with β. Using the fact that E is full and that $\hat{\beta}$ is non-degenerate, we find that $[\hat{\mathcal{A}}(W)\hat{\mathcal{A}}(W)]$ is equal to the family spanned by all operators $\hat{a}_{(\vartheta',\vartheta'')}$, where $\vartheta', \vartheta'' \in {}_\beta E$ are homogeneous. This is $\hat{\mathcal{A}}(W)$. □

Example: the pseudo-multiplicative unitary W_G. Let us determine the legs of the pseudo-multiplicative unitary W_G associated to a groupoid G (see Example 2.5). We use the same notation as in Example 2.5 and write ${}_rL^2(G,\lambda)$ or ${}_sL^2(G,\lambda)$ to indicate whether we consider $L^2(G,\lambda)$ as a right C*-bimodule via the representation r or s. Given $f, g \in C_c(G)$, we denote by fg, $\bar{f}$, f^*, $f \star g \in C_c(G)$ the functions given by $(fg)(x) := f(x)g(x)$, $\bar{f}(x) := \overline{f(x)}$, $f^*(x) := \overline{f(x^{-1})}$, $(f \star g)(x) := \int_{G^{r_G(x)}} f(y)g(y^{-1}x)\, d\lambda^{r_G(x)}(y)$ for all $x \in G$ ($f \star g \in C_c(G)$ by [17], Proposition 1.1).

The right C*-bimodule ${}_rL^2(G,\lambda)$ is always decomposable, and using Proposition 3.14 (i) we find:

Lemma 4.7. $\mathcal{H}_{\mathrm{id}}({}_rL^2(G,\lambda)) = L^2(G,\lambda)$ *and for each* $\theta \in \mathrm{PAut}(C_0(G^0))$ *we have* $\mathcal{H}_\theta({}_rL^2(G,\lambda)) = L^2(G,\lambda)\,\mathrm{Dom}(\theta \wedge \mathrm{id})$. □

The essential information of $\hat{\mathcal{A}}(W_G)$ is contained in the space $\hat{\mathcal{A}}(W_G)^{\mathrm{id}}_{\mathrm{id}}$, which can be defined without the concepts introduced Section 3. However, for completeness we shall determine the whole family $\hat{\mathcal{A}}(W_G)$.

It is easy to see that for each $f \in C_0(G)$, there exists a multiplication operator $m(f) \in \mathcal{L}_{C_0(G^0)}(L^2(G,\lambda))$, $m(f)\xi = f\xi$ for all $\xi \in C_c(G)$, and that the map $m\colon C_0(G) \to \mathcal{L}_{C_0(G^0)}(L^2(G,\lambda))$ is an injective $*$-homomorphism.

Proposition 4.8. (i) *If* $\xi,\xi' \in C_c(G)$*, then* $\hat{a}_{(\xi',\xi)} = m(\overline{\xi'} \star \overline{\xi^*})$.

(ii) $\hat{\mathcal{A}}(W_G)^{\mathrm{id}}_{\mathrm{id}} = m(C_0(G))$ *and for all* $\rho,\sigma \in \mathrm{PAut}(C_0(G^0))$ *we have* $\hat{\mathcal{A}}(W_G)^{\rho}_{\sigma} = r(\mathrm{Dom}(\sigma \wedge \mathrm{id}))m(C_0(G))s(\mathrm{Dom}(\rho \wedge \mathrm{id})) = m(C_0(G^V_U))$*, where the open subsets* $U,V \subseteq G^0$ *are determined by* $\mathrm{Dom}(\rho \wedge \mathrm{id}) = C_0(U)$*,* $\mathrm{Dom}(\sigma \wedge \mathrm{id}) = C_0(V)$ *and* $G^V_U = r_G^{-1}(V) \cap s_{G^{-1}}(U)$.

(iii) $\hat{\mathcal{A}}(W_G)$ *is a C*-family.*

Proof. (i) Let $\zeta \in C_c(G)$ and $x \in G$. By definition we have $(W_G|\xi\rangle_2\zeta)(x,y) = (W_G(\zeta \otimes \xi))(x,y) = \zeta(x)\xi(x^{-1}y)$ for each $y \in G^{r_G(x)}$, and hence

$$\begin{aligned}(\hat{a}_{(\xi',\xi)}\zeta)(x) &= \int_{G^{r_G(x)}} \overline{\xi'(y)}\zeta(x)\xi(x^{-1}y)\,d\lambda^{r_G(x)}(y)\\ &= \zeta(x)\int_{G^{r_G(x)}} \overline{\xi'}(y)\overline{\xi^*(y^{-1}x)}\,d\lambda^{r_G(x)}(y) = \zeta(x)(\overline{\xi'} \star \overline{\xi^*})(x).\end{aligned}$$

(ii) Let $\rho,\sigma \in \mathrm{PAut}(C_0(G^0))$. For each element $\xi \in C_c(G)$ we have that $\xi \in L^2(G,\lambda)\,\mathrm{Dom}(\rho\wedge\mathrm{id})$ iff $\xi^* \in s(\mathrm{Dom}(\rho\wedge\mathrm{id}))L^2(G,\lambda)$. Hence, by Lemma 4.7 and (i), $\hat{\mathcal{A}}(W_G)^{\rho}_{\sigma}$ is the closed linear span of all operators of the form $m(\xi' \star \xi'')$, where $\xi'' \in s(\mathrm{Dom}(\rho \wedge \mathrm{id}))C_c(G)$, $\xi' \in r(\mathrm{Dom}(\sigma \wedge \mathrm{id}))C_c(G)$. But from [17], Proposition 1.9, it follows that $C_c(G) \star C_c(G) \subseteq C_c(G)$ is dense with respect to the supremum norm, which implies the claim.

(iii) This follows from (ii) and from the relations $\mathrm{Dom}(\theta^*\wedge\mathrm{id}) = \mathrm{Dom}(\theta\wedge\mathrm{id})$ and $\mathrm{Dom}(\theta\wedge\mathrm{id})\,\mathrm{Dom}(\theta'\wedge\mathrm{id}) \subseteq \mathrm{Dom}(\theta\theta'\wedge\mathrm{id})$, which hold for all $\theta,\theta' \in \mathrm{PAut}(C_0(G^0))$. □

Let us turn to $\mathcal{A}(W_G)$. The right C*-bimodule ${}_sL^2(G,\lambda)$ is decomposable if the groupoid G itself is decomposable in the following sense.

Definition 4.9. We call an open subset $U \subseteq G$ *homogeneous* iff $r_G(x) = r_G(y) \Leftrightarrow s_G(x) = s_G(y)$ for all $x,y \in U$. We call G *decomposable* iff it is the union of open homogeneous subsets.

Remarks 4.10. (i) Recall that an open subset $U \subseteq G$ is called a G-*set* iff the restrictions $r|_U : U \to r(U)$ and $s|_U : U \to s(U)$ are homeomorphisms and $r(U)$, $s(U) \subseteq G^0$ are open. Moreover, recall that G is r-discrete iff it is the union of open G-sets [17], Proposition 2.8. Evidently, every G-set is homogeneous and if G is r-discrete, then it is decomposable.

(ii) If $U, V \subseteq G$ are homogeneous subsets, then also U^{-1} and $UV = \{xy \mid (x, y) \in G^2_{s,r} \cap (U \times V)\}$ are homogeneous.

Denote by $\mathrm{PHom}(G^0)$ the set of all partial homeomorphisms of G^0, that is, of all homeomorphisms between open subsets of G^0. Every open homogeneous subset $U \subseteq G$ defines a partial homeomorphism $q_U : s_G(U) \to r_G(U)$ of G^0 by $s_G(x) \mapsto r_G(x)$, and partial automorphisms $q_{U*} : C_0(s_G(U)) \to C_0(r_G(U))$, $q^*_U : C_0(r_G(U)) \to C_0(s_G(U))$ of $C_0(G^0)$. For each $q \in \mathrm{PHom}(G^0)$ denote by $\mathcal{H}_q(G) \subseteq G$ the union of all open homogeneous subsets $U \subseteq G$ that satisfy $q_U \leq q$. Note that $\mathcal{H}_q(G)$ is open and homogeneous again.

Proposition 4.11. *Assume that G is decomposable. Then* ${}_sL^2(G, \lambda)$ *is decomposable and* $\mathcal{H}_{q^*}({}_sL^2(G, \lambda)) = \overline{C_c(\mathcal{H}_q(G))}$ *for each* $q \in \mathrm{PHom}(G^0)$.

Proof. Let $q \in \mathrm{PHom}(G^0)$. Then $C_c(\mathcal{H}_q(G)) \subseteq \mathcal{H}_{q^*}({}_sL^2(G, \lambda))$ because each $\xi \in C_c(\mathcal{H}_q(G))$ belongs to $L^2(G, \lambda)C_0(r_G(\mathcal{H}_q(G))) \subseteq L^2(G, \lambda)\,\mathrm{Dom}(q^*)$ and satisfies $(\xi f)(x) = \xi(x) f(r_G(x)) = \xi(x) f(q(s_G(x))) = (s(q^*(f))\xi)(x)$ for all $x \in \mathcal{H}_q(G)$, $f \in \mathrm{Dom}(q^*)$. A partition of unity argument shows that the sum of all $C_c(\mathcal{H}_{q'}(G))$, where $q' \in \mathrm{PHom}(G^0)$, is equal to $C_c(G)$. In particular, ${}_sL^2(G, \lambda)$ is decomposable. Proposition 3.15, applied to $E = {}_sL^2(G, \lambda)$ and $E_0 = C_c(G)$, shows that $\mathcal{H}_{q^*}({}_sL^2(G, \lambda)) = \overline{C_c(\mathcal{H}_q(G))}$. □

If G is r-discrete and λ is a Haar-system on G, then for each $u \in G^0$, the set G^u is discrete and the measure λ^u is the counting measure multiplied by $\lambda^u(\{u\})$ [16], Proposition 2.2.5. To simplify the discussion, we assume:

Assumption 4.12. If G is r-discrete, then $\lambda^{r_G(x)}(\{x\}) = 1$ for all $x \in G$.

Lemma 4.13. (i) *For every* f *in* $C_c(G, \lambda)$ *there exists an operator* $L(f)$ *in* $\mathcal{L}(L^2(G, \lambda))$ *such that* $L(f)\xi = f * \xi$ *for all* $\xi \in C_c(G)$. *Moreover,* $L(f)L(g) = L(f \star g)$ *for all* $f, g \in C_c(G)$.

(ii) *Let G be r-discrete,* $U \subseteq G$ *open and homogeneous,* $f \in C_c(U)$ *and put* $q := q_U$. *Then* $L(f) \in \mathcal{L}^{q*}_{q*}({}_rL^2(G, \lambda))$ *and* $L(f)^* = L(f^*)$.

Proof. (i) The boundedness of $L(f)$ can be seen by a similar proof as in [17], Proposition 1.8, or [16], Proposition 3.1.1. The last relation follows from associativity of the convolution [16], Theorem 2.2.1.

(ii) It is easy to see that $\operatorname{Im} L(f) \subseteq r(\operatorname{Im}(q_*))L^2(G,\lambda)$ and $L(f)r(b) = r(q_*(b))L(f)$ for all $b \in \operatorname{Dom}(q_*)$. Let $\xi, \eta \in C_c(G)$. Then $\langle \eta | L(f)\xi \rangle$ and $\langle L(f^*)\eta | \xi \rangle$, considered as functions on G^0, vanish outside $r_G(U)$ and $s_G(U)$, respectively, and for $u \in s_G(U)$, we find that $\langle \eta | L(f)\xi \rangle (q(u))$ is equal to

$$\sum_{x,y \in G^{q(u)}} \overline{\eta(x)} f(y) \xi(y^{-1}x) = \sum_{x,y \in G^{q(u)}} \overline{f^*(y^{-1})\eta(x)} \xi(y^{-1}x)$$
$$= \sum_{x',y' \in G^u} \overline{f^*(y')\eta(y'^{-1}x')} \xi(x') = \langle L(f^*)\eta | \xi \rangle (u).$$

Therefore $q_*(\langle \eta | L(f)\xi \rangle) = \langle L(f^*)\eta | \xi \rangle$, and the claims follow. □

Proposition 4.14. (i) *Let $\eta \in C_c(U)$, $\eta' \in C_c(U')$, where $U, U' \subseteq G$ are open and homogeneous. Then $a_{(\eta',\eta)} = L(\overline{\eta'}\eta)$.*

(ii) *$\mathcal{A}(W_G)^\rho_\sigma$ is the closure of $\{L(g) \mid g \in C_c(\mathcal{H}_\rho(G) \cap \mathcal{H}_\sigma(G))\}$ for all ρ, σ.*

(iii) *If G is r-discrete, then $\mathcal{A}(W_G)$ is a C*-family.*

Proof. (i) If $\zeta \in C_c(G)$, $(x,y) \in G^2_{r,r}$, then $(W_G|\eta]_1\zeta)(x,y) = \eta(x)\zeta(x^{-1}y)$ and $(a_{(\eta',\eta)}\zeta)(x) = \int_{G^{r_G(x)}} \overline{\eta'(x)}\eta(x)\zeta(x^{-1}y)\, d\lambda^{r_G(x)}(y) = ((\overline{\eta'}\eta) \star \zeta)(x)$.

(ii), (iii) Combine (i) with Proposition 4.11 and Lemma 4.13. □

In a subsequent article we will show that $\mathcal{A}(W_G)$ is a C*-family whenever G is decomposable; here the difficulty is to prove that $\mathcal{A}(W_G)^* = \mathcal{A}(W_G)$.

Example: the pseudo-multiplicative unitary W_τ. Let us consider the pseudo-multiplicative unitary W_τ associated to a center-valued conditional expectation $\tau \colon B \to C \subseteq Z(B)$; see Example 2.6. Recall that the underlying C*-module $E = B_\tau \circledcirc B$ of W_τ is generated by elements $a \circledcirc b$, where $a, b \in B$, such that $(a \circledcirc b)b' = a \circledcirc bb'$, $\hat{\beta}(b')(a \circledcirc b) = b'a \circledcirc b$, $\beta(b')(a \circledcirc b) = a \circledcirc b'b$ and $\langle a' \circledcirc b' | a \circledcirc b \rangle = b'^*\tau(a'^*a)b$ for all $a, a', b, b' \in B$.

Recall that C (hence also B) was assumed to be unital. In particular, B is decomposable (Proposition 3.20 (v)). From Proposition 3.18 we deduce:

Lemma 4.15. *${}_\beta E$ is decomposable and $\mathcal{H}({}_\beta E) = B_\tau \circledcirc \mathcal{H}(B)$.* □

Lemma 4.16. *Let $d \in \mathcal{H}_\rho(B)$, $d' \in \mathcal{H}_\sigma(B)$, $\rho, \sigma \in \operatorname{PAut}(B)$ and $c, c' \in B$. Moreover, put $\xi := c \circledcirc d$ and $\xi' := c' \circledcirc d'$. Then $\xi \in \mathcal{H}_\rho({}_\beta E)$, $\xi' \in \mathcal{H}_\sigma({}_\beta E)$ and $\hat{a}_{(\xi',\xi)} = o_{d,d''} \in \mathcal{O}({}_{\hat{\beta}}E)^\rho_\sigma$, where $d'' = d'^*\tau(c'^*c) \in \mathcal{H}_{\sigma^*}(B)$.*

Proof. By Proposition 3.20, $d'' \in \mathcal{H}_\sigma(B)^* Z(B) \subseteq \mathcal{H}_{\sigma^*}(B)$. Let $a, b \in B$. Then $W_\tau|\xi\rangle_2(a \circledcirc b) = W_\tau((a \circledcirc b) \otimes (c \circledcirc d)) = (da \circledcirc b) \otimes (c \circledcirc 1)$ and hence $\hat{a}_{(\xi',\xi)}(a \circledcirc b) = da \circledcirc b\sigma(\langle c' \circledcirc d' | c \circledcirc 1 \rangle) = \hat{\beta}(d)(a \circledcirc b)\sigma(d'') = o_{d,d''}(a \circledcirc b)$; note that $\sigma(d'') = d''$ by Proposition 3.19 (iv). □

Proposition 4.17. *$\hat{\mathcal{A}}(W_\tau) = \mathcal{O}({}_{\hat{\beta}}E)$; in particular, $\hat{\mathcal{A}}(W_\tau)$ is a C*-family.*

Proof. By Lemma 4.15 and 4.16, $\hat{\mathcal{A}}(W_\tau) \subseteq \mathcal{O}({}_{\hat{\beta}}E)$. Conversely, if $d'' \in \mathcal{H}_{\sigma^*}(B)$, $d \in \mathcal{H}_\rho(B)$, $\rho, \sigma \in \mathrm{PAut}(B)$, then $\xi := 1 \otimes d \in \mathcal{H}_\rho({}_\beta E)$, $\xi' := 1 \otimes d''^* \in \mathcal{H}_\sigma({}_\beta E)$ and $o_{d,d''} = \hat{a}_{(\xi',\xi)} \in \hat{\mathcal{A}}(W_\tau)^\rho_\sigma$. □

In general the C*-module ${}_{\hat{\beta}}E$ will not be decomposable.

5. Hopf C*-families

In this section, we introduce the internal tensor product of C*-families, and the notion of a morphism of C*-families. These concepts are needed for the definition of a Hopf C*-family, which is given afterwards. Throughout this section, let A, B, C be C*-algebras.

The internal tensor product. Let E be a right C*-A-B-bimodule and F a right C*-B-C-bimodule. We define an internal tensor product of operators as a map $\mathcal{L}^\rho_\sigma(E) \times \mathcal{L}^{\rho'}_{\sigma'}(F) \to \mathcal{L}^\rho_{\sigma'}(E \otimes F)$ for all $\rho, \sigma, \rho', \sigma'$, where σ and ρ' are compatible in the following sense:

Definition 5.1. Two partial automorphisms $\rho, \sigma \in \mathrm{PAut}(B)$ are called *compatible*, denoted by $\rho \curlyvee \sigma$, iff $\rho\sigma^* \leq \mathrm{id}$ and $\rho^*\sigma \leq \mathrm{id}$.

Lemma 5.2. *Let $\rho, \sigma \in \mathrm{PAut}(B)$ such that $\rho \curlyvee \sigma$. Then:*

(i) $\rho^* \curlyvee \sigma^*$;

(ii) $\rho(a) = \sigma(a)$ *for all* $a, b \in \mathrm{Dom}(\rho) \cap \mathrm{Dom}(\sigma)$;

(iii) $\rho(\mathrm{Dom}(\rho) \cap \mathrm{Dom}(\sigma)) = \mathrm{Im}(\rho) \cap \mathrm{Im}(\sigma) = \sigma(\mathrm{Dom}(\rho) \cap \mathrm{Dom}(\sigma))$;

(iv) $\rho(ab) = \rho(a)\sigma(b) = \sigma(ab)$ *for all* $a \in \mathrm{Dom}(\rho)$, $b \in \mathrm{Dom}(\sigma)$;

(v) *if* $\rho' \curlyvee \sigma'$ *for* $\rho', \sigma' \in \mathrm{PAut}(B)$, *then* $\rho\rho' \curlyvee \sigma\sigma'$.

Proof. Assertions (i) and (ii) follow immediately from the definition.

(iii) By (ii), $\rho(\mathrm{Dom}(\rho) \cap \mathrm{Dom}(\sigma)) = \sigma(\mathrm{Dom}(\rho) \cap \mathrm{Dom}(\sigma))$ is contained in $\mathrm{Im}(\rho) \cap \mathrm{Im}(\sigma)$. To obtain the reverse inclusion, replace ρ, σ by ρ^*, σ^*.

(iv) Let $a \in \mathrm{Dom}(\rho)$ and $b \in \mathrm{Dom}(\sigma)$. By (ii), $\rho(ab)\sigma(c) = \rho(ab)\rho(c) = \rho(a)\rho(bc) = \rho(a)\sigma(bc) = \rho(a)\sigma(b)\sigma(c)$ for each $c \in \mathrm{Dom}(\rho) \cap \mathrm{Dom}(\sigma)$. If $(u_\nu)_\nu$ is an approximate unit for $\mathrm{Dom}(\rho) \cap \mathrm{Dom}(\sigma)$, then by (iii), $(\sigma(u_\nu))_\nu$ is an approximate unit for $\mathrm{Im}(\rho) \cap \mathrm{Im}(\sigma)$. Therefore, $\rho(ab) = \lim_\nu \rho(a)\rho(bu_\nu) = \lim_\nu \rho(a)\sigma(bu_\nu) = \lim_\nu \rho(a)\sigma(b)\sigma(u_\nu) = \rho(a)\sigma(b)$. Symmetrically, $\rho(a)\sigma(b) = \sigma(ab)$ for all $a \in \mathrm{Dom}(\rho)$, $b \in \mathrm{Dom}(\sigma)$.

(v) $(\rho\rho')(\sigma\sigma')^* = \rho(\rho'\sigma'^*)\sigma^* \leq \rho\sigma^* \leq \mathrm{id}$; similarly, $(\rho\rho')^*(\sigma\sigma') \leq \mathrm{id}$. □

In general compatibility is not transitive: the automorphism of the ideal $\{0\}$ is compatible with every other partial automorphism of B.

Proposition 5.3. *Let E_1, E_2 be right C*-A-B-bimodules, F_1, F_2 right C*-B-C-bimodules, and let $S \in \mathcal{L}^{\rho_S}_{\sigma_S}(E_1, E_2)$, $T \in \mathcal{L}^{\rho_T}_{\sigma_T}(F_1, F_2)$, where $\rho_S \in \mathrm{PAut}(A)$, $\sigma_S, \rho_T \in \mathrm{PAut}(B)$, $\sigma_T \in \mathrm{PAut}(C)$. If $\sigma_S \curlyvee \rho_T$, then there exists an operator $S \otimes T \in \mathcal{L}^{\rho_S}_{\sigma_T}(E_1 \otimes F_1, E_2 \otimes F_2)$ such that $(S \otimes T)(\eta \otimes \xi) = S\eta \otimes T\xi$ for all $\eta \in E_1$, $\xi \in F_1$, and $\|S \otimes T\| \leq \|S\|\|T\|$, $(S \otimes T)^* = S^* \otimes T^*$.*

Proof. To simplify notation, we put $E := E_1 \oplus E_2$, $F := F_1 \oplus F_2$ and consider S and T as elements of $\mathcal{L}^{\rho_S}_{\sigma_S}(E)$ and $\mathcal{L}^{\rho_T}_{\sigma_T}(F)$, respectively, in the natural way. Let $\eta, \eta' \in E$ and $\xi, \xi' \in F$. Then

$$\langle \eta' \otimes \xi' | S\eta \otimes T\xi \rangle = \langle \xi' | \langle \eta' | S\eta \rangle T\xi \rangle = \langle \xi' | \sigma_S(\langle S^*\eta' | \eta \rangle) T\xi \rangle.$$

Suppose that $(u_\nu)_\nu$ is an approximate unit for $\mathrm{Dom}(\rho_T)$. Then Proposition 3.2 (v) and Lemma 5.2 (iv) imply that $\sigma_S(\langle S^*\eta'|\eta\rangle)T\xi = \lim_\nu \rho_T(u_\nu)\sigma_S(\langle S^*\eta'|\eta\rangle)T\xi = \lim_\nu Tu_\nu\langle S^*\eta'|\eta\rangle\xi = T\langle S^*\eta'|\eta\rangle\xi$. Thus we have

$$\begin{aligned}\langle \eta' \otimes \xi' | S\eta \otimes T\xi \rangle &= \langle \xi' | T\langle S^*\eta'|\eta\rangle\xi \rangle = \sigma_T(\langle T^*\xi' | \langle S^*\eta'|\eta\rangle\xi\rangle) \\ &= \sigma_T(\langle S^*\eta' \otimes T^*\xi' | \eta \otimes \xi \rangle).\end{aligned} \tag{6}$$

Let us show that the map $\eta \otimes \xi \mapsto S\eta \otimes T\xi$ is well defined and bounded. By equation (6), $\| \sum_i S\eta_i \otimes T\xi_i \|^2 = \| \sum_{i,j} \langle S^*S\eta_i \otimes T^*T\xi_i | \eta_j \otimes \xi_j \rangle \|$ for all $\eta_i \in E$, $\xi_i \in F$. Now $T^*T \in \mathcal{L}^B_C(F)$ and by Proposition 1.1 the operators $S^*S \otimes 1$, $1 \otimes T^*T$, $S^*S \otimes T^*T$ in $\mathcal{L}_C(E \otimes F)$ are well defined. Since $S^*S \otimes T^*T = (S^*S \otimes 1)(1 \otimes T^*T) = (1 \otimes T^*T)(S^*S \otimes 1)$, we obtain that $\|S \otimes T\|^2 \leq \|S^*S \otimes T^*T\| \leq \|S^*S \otimes 1\|\|1 \otimes T^*T\| \leq \|S\|^2\|T\|^2$.

Obviously the image of $S \otimes T$ is contained in $\mathrm{Im}(\rho_S)(E \otimes F)$ and $(S \otimes T)a(\eta \otimes \xi) = Sa\eta \otimes T\xi = \rho_S(a)S\eta \otimes T\xi = \rho_S(a)(S \otimes T)(\eta \otimes \xi)$ for all $\eta \in E$, $\xi \in F$, $a \in \mathrm{Dom}(\rho_S)$. Replacing S and T by their adjoints, we obtain a bounded map $S^* \otimes T^* \colon E \otimes F \to E \otimes F$, and equation (6) shows that $S \otimes T$ is (ρ_S, σ_T)-homogeneous with adjoint $(S \otimes T)^* = S^* \otimes T^*$. □

Next we introduce the internal tensor product of C*-families.

Definition 5.4. Suppose that E_1, E_2 are right C*-A-B-bimodules and F_1, F_2 right C*-B-C-bimodules. The *internal tensor product* of families of closed subspaces $\mathcal{C} \subseteq \mathcal{L}(E_1, E_2)$ and $\mathcal{D} \subseteq \mathcal{L}(F_1, F_2)$ is the family $\mathcal{C} \otimes \mathcal{D} \subseteq \mathcal{L}(E_1 \otimes F_1, E_2 \otimes F_2)$ given by $(\mathcal{C} \otimes \mathcal{D})^\rho_\sigma := \overline{\mathrm{span}}\{S \otimes T \mid S \in \mathcal{C}^\rho_{\sigma_S}, T \in \mathcal{D}^{\rho_T}_\sigma, \sigma_S, \rho_T \in \mathrm{PAut}(B), \sigma_S \curlyvee \rho_T\}$.

Remark 5.5. Let E be a right C*-A-B-bimodule and F a right C*-B-C-bimodule, and let $\mathcal{A}, \mathcal{C} \subseteq \mathcal{L}(E)$ and $\mathcal{B}, \mathcal{D} \subseteq \mathcal{L}(F)$ be families of closed subspaces. Then $[(\mathcal{A} \otimes \mathcal{B})(\mathcal{C} \otimes \mathcal{D})] \subseteq [\mathcal{A}\mathcal{C}] \otimes [\mathcal{B}\mathcal{D}]$. *This inclusion may be strict and fail to be an equality.* As a simple example assume that all spaces comprising the families $\mathcal{C}$ and $\mathcal{D}$ are 0 except for $\mathcal{C}_{\sigma_1}^{\rho_1}$ and $\mathcal{D}_{\sigma_2}^{\rho_2}$, where σ_1 and ρ_2 are not compatible. Then $\mathcal{C}^* \otimes \mathcal{D}^* = 0 = \mathcal{C} \otimes \mathcal{D}$, but $\mathcal{C}^*\mathcal{C} \otimes \mathcal{D}^*\mathcal{D}$ need not be 0.

Lemma 5.2 and routine arguments show:

Proposition 5.6. *Let E be a right C*-A-B-bimodule, F a right C*-B-C-bimodule, and let $\mathcal{C} \subseteq \mathcal{L}(E)$ and $\mathcal{D} \subseteq \mathcal{L}(F)$ be C*-families. Then:*

(i) *If $\mathcal{C}$ and $\mathcal{D}$ are (non-degenerate) C*-families, then so is $\mathcal{C} \otimes \mathcal{D}$.*

(ii) *If $\mathcal{C}$ is a (non-degenerate) $\mathcal{O}(E)$-module and $\mathcal{D}$ is a (non-degenerate) $\mathcal{O}(F)$-module, then $\mathcal{C} \otimes \mathcal{D}$ is a (non-degenerate) $\mathcal{O}(E \otimes F)$-module.*

(iii) $\mathcal{M}(\mathcal{C}) \otimes \mathcal{M}(\mathcal{D}) \subseteq \mathcal{M}(\mathcal{C} \otimes \mathcal{D})$. □

It is easy to see that the internal tensor product is associative:

Proposition 5.7. *Let A, B, C, D be C*-algebras, let E be a right C*-A-B-bimodule, F a right C*-B-C-bimodule and G a right C*-C-D-bimodule. Furthermore, let $\mathcal{B} \subseteq \mathcal{L}(E)$, $\mathcal{C} \subseteq \mathcal{L}(F)$, $\mathcal{D} \subseteq \mathcal{L}(G)$ be C*-families. Then the natural isomorphism $(E \otimes F) \otimes G \cong E \otimes (F \otimes G)$ induces an isomorphism of C*-families $(\mathcal{B} \otimes \mathcal{C}) \otimes \mathcal{D} \cong \mathcal{B} \otimes (\mathcal{C} \otimes \mathcal{D})$.* □

The constructions introduced above can easily be adapted to the flipped internal tensor product of right C*-bimodules and give rise to a flipped internal tensor product of homogeneous operators and of C*-families.

Embedding C*-families into C*-algebras. We construct an embedding of C*-families into C*-algebras that will be used in the next section. This construction involves two right C*-bimodules $\mathfrak{I}A$, $\mathfrak{I}B$. Let us first define $\mathfrak{I}A$. Consider A as a C*-A-module. Then for each $\theta \in \mathrm{PAut}(A)$, the ideal $\mathrm{Dom}(\theta) \subseteq A$ is a C*-submodule and routine calculations show:

Lemma 5.8. *There exists a representation $\pi_\theta \colon A \to \mathcal{L}_A(\mathrm{Dom}(\theta))$ such that $\pi_\theta(a)x = \theta^*(a\theta(x))$ for all $a \in A$, $x \in \mathrm{Dom}(\theta)$.* □

Consider the direct sum of C*-modules $\mathfrak{I}A := \bigoplus_{\theta \in \mathrm{PAut}(A)} \mathrm{Dom}(\theta)$ as a right C*-A-A-bimodule via the representations π_θ defined above. For each $\theta \in \mathrm{PAut}(A)$, denote by $v_\theta \colon \mathrm{Dom}(\theta) \to \mathfrak{I}A$, $x \mapsto v_\theta x$, the canonical map. Then sums of the form $\sum_\theta v_\theta x_\theta$, where $x_\theta \in \mathrm{Dom}(\theta)$ is zero for all but finitely many θ, form a dense subspace $\mathfrak{I}_0 A \subseteq \mathfrak{I} A$ and $(v_\theta x)a = v_\theta(xa)$, $\langle v_{\theta'} x' | v_\theta x\rangle = \delta_{\theta,\theta'} x'^* x$, $a(v_\theta x) = v_\theta \theta^*(a\theta(x))$ for all $x \in \mathrm{Dom}(\theta)$, $x' \in \mathrm{Dom}(\theta')$, $\theta, \theta' \in \mathrm{PAut}(A)$. Replacing A by B, we obtain a right C*-B-B-bimodule $\mathfrak{I} B$.

Lemma 5.9. *For all $\sigma \in \mathrm{PAut}(A)$, $\rho \in \mathrm{PAut}(B)$, the maps $V_\sigma \colon \mathfrak{I}_0 A \to \mathfrak{I}_0 A$ and $W_\rho \colon \mathfrak{I}_0 B \to \mathfrak{I}_0 B$ given by*

$$V_\sigma \colon \sum_\theta v_\theta x_\theta \mapsto \sum_{\theta = \theta\sigma^*\sigma} v_{(\theta\sigma^*)} \sigma(x_\theta), \quad W_\rho \colon \sum_\theta v_\theta x_\theta \mapsto \sum_{\theta = \rho^*\rho\theta} v_{(\rho\theta)} x_\theta$$

extend to operators $V_\sigma \in \mathcal{L}^{\mathrm{id}}_\sigma(\mathfrak{I} A)$ and $W_\rho \in \mathcal{L}^\rho_{\mathrm{id}}(\mathfrak{I} B)$. For all $\sigma, \sigma' \in \mathrm{PAut}(A)$, $\rho, \rho' \in \mathrm{PAut}(B)$, we have $(V_\sigma)^* = V_{\sigma^*}$, $(W_\rho)^* = W_{\rho^*}$, $V_\sigma V_{\sigma'} = V_{\sigma\sigma'}$, $W_\rho W_{\rho'} = W_{\rho\rho'}$, and $\|V_\sigma\| = 1$ if $\sigma \neq \mathrm{id}_{\{0\}}$, $\|W_\rho\| = 1$ if $\rho \neq \mathrm{id}_{\{0\}}$.

Proof. Given a logical expression e, put $[\![e]\!] := 0$ if e is false, and $[\![e]\!] := 1$ if e is true. Fix $\sigma \in \mathrm{PAut}(A)$, $\sigma \neq \mathrm{id}_{\{0\}}$.

The map V_σ extends to a bounded linear map on $\mathfrak{I} A$ of norm 1 because $V_\sigma v_\theta \,\mathrm{Dom}(\theta)$ is orthogonal to $V_\sigma v_{\theta'} \,\mathrm{Dom}(\theta')$ whenever $\theta \neq \theta'$. Indeed, if $\theta\sigma^*\sigma \neq \theta$ or $\theta'\sigma^*\sigma \neq \theta'$, one of these spaces is zero; if $\theta\sigma^*\sigma = \theta$, $\theta'\sigma^*\sigma = \theta'$ and $\theta \neq \theta'$, then $\theta\sigma^* \neq \theta'\sigma^*$, and again the spaces above are orthogonal.

We claim that $V_\sigma a = a V_\sigma$ for each $a \in A$. Indeed, if $\theta \in \mathrm{PAut}(A)$, $\theta\sigma^*\sigma = \theta$, and $\theta' := \theta\sigma^*$, then for all $x \in \mathrm{Dom}(\theta)$,

$$a V_\sigma v_\theta x = a v_{\theta'} \sigma(x) = v_{\theta'} \theta'^*(a(\theta'\sigma(x))) = v_{\theta'} \sigma(\theta^*(a\theta(x))) = V_\sigma a v_\theta x.$$

Moreover, for all $\sigma, \sigma', \theta, \theta' \in \mathrm{PAut}(A)$ and $x \in \mathrm{Dom}(\theta)$, $x' \in \mathrm{Dom}(\theta')$,

$$\begin{aligned} \langle v_{\theta'} x' | V_\sigma v_\theta x\rangle &= x'^* \sigma(x) \cdot [\![\theta\sigma^*\sigma = \theta \wedge \theta' = \theta\sigma^*]\!] \\ &= \sigma(\sigma^*(x')^* x) \cdot [\![\theta'\sigma\sigma^* = \theta' \wedge \theta'\sigma = \theta]\!] = \sigma(\langle V_{\sigma^*} v_{\theta'} x' | v_\theta x\rangle), \\ V_\sigma V_{\sigma'} v_\theta x &= v_{(\theta\sigma'^*\sigma^*)} \sigma(\sigma'(x)) \cdot [\![\theta\sigma'^*\sigma' = \theta \wedge \theta\sigma'^*\sigma^*\sigma = \theta\sigma'^*]\!] \\ &= v_{(\theta(\sigma\sigma')^*)} \sigma(\sigma'(x)) \cdot [\![\theta(\sigma\sigma')^*(\sigma\sigma') = \theta]\!] = V_{\sigma\sigma'} v_\theta x. \end{aligned}$$

The claims concerning V_σ follow and the claims concerning W_ρ are proved similarly. □

Theorem 5.10. *Let E be a right C*-A-B-bimodule. For $\rho \in \mathrm{PAut}(A)$, $\sigma \in \mathrm{PAut}(B)$, define $\iota^\rho_\sigma \colon \mathcal{L}^\rho_\sigma(E) \to \mathcal{L}^A_B(\mathfrak{I} A \otimes E \otimes \mathfrak{I} B)$ by $T \mapsto V_\rho \otimes T \otimes W_\sigma$. Then $\|\iota^\rho_\sigma(T)\| = \|T\|$, $\iota^\rho_\sigma(T)^* = \iota^{\rho^*}_{\sigma^*}(T^*)$ and $\iota^\rho_\sigma(T)\iota^{\rho'}_{\sigma'}(T') = \iota^{\rho\rho'}_{\sigma\sigma'}(TT')$ for all $T \in \mathcal{L}^\rho_\sigma(E)$, $T' \in \mathcal{L}^{\rho'}_{\sigma'}(E)$, $\rho, \rho' \in \mathrm{PAut}(A)$, $\sigma, \sigma' \in \mathrm{PAut}(B)$.*

Proof. Let T, T', ρ, ρ', σ, σ' be as above. By Lemma 5.9 and Proposition 5.3, $\iota^\rho_\sigma(T)^* = \iota^{\rho^*}_{\sigma^*}(T^*)$, $\iota^\rho_\sigma(T)\iota^{\rho'}_{\sigma'}(T') = \iota^{\rho\rho'}_{\sigma\sigma'}(TT')$ and $\|\iota^\rho_\sigma(T)\| \le \|T\|$. Let us prove that $\|\iota^{\rho}_{\sigma}(T)\| \geq \|T\|$. Fix $\xi \in E$. Note that for all $\theta \in \mathrm{PAut}(A)$, $x \in \mathrm{Dom}(\theta)$ and $\theta' \in \mathrm{PAut}(B)$, $x' \in \mathrm{Dom}(\theta')$,

$$\|v_{\theta}x \otimes \xi \otimes v_{\theta'}x'\|^2 = \|x'^*\theta'^*(\langle\xi|x^*x\xi\rangle\theta'(x'))\| = \|\langle x\xi\theta'(x')|x\xi\theta'(x')\rangle\|$$

and hence $\|v_{\theta}x \otimes \xi \otimes v_{\theta'}x'\| = \|x\xi\theta'(x')\|$. Choose approximate units $(u_{\nu})_{\nu}$ and $(u'_{\nu'})_{\nu'}$, bounded in norm by 1, for the ideals $\mathrm{Dom}(\rho)$ and $\mathrm{Im}(\sigma)$, respectively, and put $\xi_{\nu,\nu'} := v_{\rho}u_{\nu} \otimes \xi \otimes v_{\sigma^*}u'_{\nu'}$ for all ν, ν'. Then $\|\xi_{\nu,\nu'}\| = \|u_{\nu}\xi\sigma^*(u'_{\nu'})\| \leq \|\xi\|$ and $\|\iota^{\rho}_{\sigma}(T)\xi_{\nu,\nu'}\| = \|v_{(\rho\rho^*)}\rho(u_{\nu}) \otimes T\xi \otimes v_{(\sigma\sigma^*)}u'_{\nu'}\| = \|\rho(u_{\nu})(T\xi)u'_{\nu'}\|$ for all ν, ν'. By Proposition 3.2, $\lim_{\nu,\nu'} \|\iota^{\rho}_{\sigma}(T)\xi_{\nu,\nu'}\| = \|T\xi\|$, and hence, $\|\iota^{\rho}_{\sigma}(T)\| \geq \|T\|$. □

By Theorem 5.10 we can embed every C*-family into some C*-algebra. Nevertheless, we continue to work with C*-families, because it is not clear how to define the internal tensor product, which is crucial for the concept of a Hopf C*-family, intrinsically on the level of the ambient C*-algebras.

Morphisms of C*-families. It seems difficult to find a notion of a morphism between C*-families that makes the internal tensor product bifunctorial (with respect to these morphisms). We adopt a pragmatic approach:

Definition 5.11. Let $\mathcal{C}$ and $\mathcal{D}$ be C*-families on right C*-A-B-bimodules. By a *family of linear maps* $\phi\colon \mathcal{C} \to \mathcal{D}$ we mean a family $\phi = (\phi^{\rho}_{\sigma})_{\rho,\sigma}$ of linear maps $\phi^{\rho}_{\sigma}\colon \mathcal{C}^{\rho}_{\sigma} \to \mathcal{D}^{\rho}_{\sigma}$ defined for all $\rho \in \mathrm{PAut}(A)$, $\sigma \in \mathrm{PAut}(B)$. We call a family of linear maps $\phi\colon \mathcal{C} \to \mathcal{D}$

- *A'-B'-extendible*, where A' and B' are C*-algebras, iff for each right C*-A'-A-bimodule X and each right C*-B-B'-bimodule Y, there exists a linear map $\phi^X_Y\colon (\mathcal{L}(X) \otimes \mathcal{C} \otimes \mathcal{L}(Y))^{\mathrm{id}}_{\mathrm{id}} \to (\mathcal{L}(X) \otimes \mathcal{D} \otimes \mathcal{L}(Y))^{\mathrm{id}}_{\mathrm{id}}$ such that $\phi^X_Y(R \otimes S \otimes T) = R \otimes \phi^{\rho}_{\sigma}(S) \otimes T$ for all $R \in \mathcal{L}^{\mathrm{id}}_{\sigma'}(X)$, $S \in \mathcal{C}^{\rho}_{\sigma}$, $T \in \mathcal{L}^{\rho'}_{\mathrm{id}}(Y)$, where $\sigma', \rho \in \mathrm{PAut}(A)$, $\sigma, \rho' \in \mathrm{PAut}(B)$, $\sigma' \curlyvee \rho$, $\sigma \curlyvee \rho'$;
- *extendible* iff ϕ is A'-B'-extendible for every C*-algebra A' and B';
- *injective* iff each component ϕ^{ρ}_{σ} is injective;
- a *morphism* iff ϕ is extendible and ϕ^X_Y always is a $*$-homomorphism.

We call a morphism $\phi\colon \mathcal{C} \to \mathcal{M}(\mathcal{D})$ *non-degenerate* iff $[\phi(\mathcal{C})\mathcal{D}] = \mathcal{D}$.

Let $\mathcal{B}, \mathcal{C}, \mathcal{D}$ be C*-families on right C*-A-B-bimodules. The *composition* of two families of linear maps $\phi\colon \mathcal{B} \to \mathcal{C}$ and $\psi\colon \mathcal{C} \to \mathcal{D}$ is the family $\psi \circ \phi\colon \mathcal{B} \to \mathcal{D}$ given by $(\psi \circ \phi)^{\rho}_{\sigma} := \psi^{\rho}_{\sigma} \circ \phi^{\rho}_{\sigma}$ for all ρ, σ.

Remark 5.12. (i) $(\mathcal{L}(X) \otimes \mathcal{C} \otimes \mathcal{L}(Y))^{\mathrm{id}}_{\mathrm{id}}$ and $(\mathcal{L}(X) \otimes \mathcal{D} \otimes \mathcal{L}(Y))^{\mathrm{id}}_{\mathrm{id}}$ are C*-subalgebras of $\mathcal{L}^{A'}_{B'}(X \otimes E \otimes Y)$ and $\mathcal{L}^{A'}_{B'}(X \otimes F \otimes Y)$, respectively.

(ii) Clearly, the composition of (extendible) families of linear maps/of morphisms is a (extendible) family of linear maps/a morphism again, and the collection of all C*-families on right C*-A-B-bimodules and all (extendible) families of linear maps/all morphisms forms a category.

Proposition 5.13. *Let $\phi\colon \mathcal{C} \to \mathcal{D}$ be a morphism of C*-families, and let $c \in \mathcal{C}_{\sigma}^{\rho}$, $c' \in \mathcal{C}_{\sigma'}^{\rho'}$, $\rho, \rho' \in \mathrm{PAut}(A)$, $\sigma, \sigma' \in \mathrm{PAut}(B)$. Then $\phi_{\sigma'}^{\rho'}(c')\phi_{\sigma}^{\rho}(c) = \phi_{\sigma'\sigma}^{\rho'\rho}(c'c)$, $\phi_{\sigma}^{\rho}(c)^* = \phi_{\sigma^*}^{\rho^*}(c^*)$, $\|\phi_{\sigma}^{\rho}(c)\| \leq \|c\|$, and $\phi_{\sigma}^{\rho}(c) = \phi_{\sigma'}^{\rho'}(c)$ if $(\rho, \sigma) \leq (\rho', \sigma')$. In particular, $\phi_{\mathrm{id}}^{\mathrm{id}}\colon \mathcal{C}_{\mathrm{id}}^{\mathrm{id}} \to \mathcal{D}_{\mathrm{id}}^{\mathrm{id}}$ is a $*$-homomorphism (of C*-algebras).*

Proof. This follows from the existence of a $*$-homomorphism $\phi_{\mathfrak{J}B}^{\mathfrak{J}A}$ which makes the diagram below commute for all $\rho \in \mathrm{PAut}(A)$ and $\sigma \in \mathrm{PAut}(B)$:

$$\begin{array}{ccc}
\mathcal{C}_{\sigma}^{\rho} & \xrightarrow{\iota_{\sigma}^{\rho}} & [\iota(\mathcal{C})] \subseteq (\mathcal{L}(\mathfrak{J}A) \otimes \mathcal{C} \otimes \mathcal{L}(\mathfrak{J}B))_{\mathrm{id}}^{\mathrm{id}} \\
\downarrow{\scriptstyle \phi_{\sigma}^{\rho}} & & \downarrow{\scriptstyle \phi_{\mathfrak{J}B}^{\mathfrak{J}A}} \\
\mathcal{D}_{\sigma}^{\rho} & \xrightarrow{\iota_{\sigma}^{\rho}} & [\iota(\mathcal{D})] \subseteq (\mathcal{L}(\mathfrak{J}A) \otimes \mathcal{D} \otimes \mathcal{L}(\mathfrak{J}B))_{\mathrm{id}}^{\mathrm{id}}.
\end{array}$$ □

Remarks 5.14. (i) A morphism $\phi\colon \mathcal{C} \to \mathcal{D}$ of C*-families is injective iff the component $\phi_{\mathrm{id}}^{\mathrm{id}}$ is injective because $\|\phi_{\sigma}^{\rho}(c)\|^2 = \|\phi_{\sigma}^{\rho}(c)^*\phi_{\sigma}^{\rho}(c)\| = \|\phi_{\sigma^*\sigma}^{\rho^*\rho}(c^*c)\| = \|\phi_{\mathrm{id}}^{\mathrm{id}}(c^*c)\|$ for all $c \in \mathcal{C}_{\sigma}^{\rho}$ and all ρ, σ.

(ii) A morphism $\phi\colon \mathcal{C} \to \mathcal{M}(\mathcal{D})$ of C*-families is non-degenerate iff the natural map $\phi_{\mathrm{id}}^{\mathrm{id}}\colon \mathcal{C}_{\mathrm{id}}^{\mathrm{id}} \to \mathcal{M}(\mathcal{D})_{\mathrm{id}}^{\mathrm{id}} \to M(\mathcal{D}_{\mathrm{id}}^{\mathrm{id}})$ is a non-degenerate $*$-homomorphism of C*-algebras. This follows from Remark 3.9 (iii).

Proposition 5.15. *Let $\phi\colon \mathcal{C} \to \mathcal{D}$ be a family of linear maps between C*-families that is $\mathbb{C}$-$\mathbb{C}$-extendible. Then ϕ is extendible.*

Proof. Given C*-algebras A', B', we show that ϕ is A'-B'-extendible. Let X' be a right C*-A'-A-bimodule and Y' a right C*-B-B'-bimodule. Denote by X the C*-module X' considered as a right C*-$\mathbb{C}$-A-bimodule via multiplication by scalars. Choose a faithful representation of B' on a Hilbert space H and put $Y := Y' \otimes_{B'} H$. For $G = E, F$, the embedding $\mathcal{L}_{B'}^{A'}(X' \otimes_A G \otimes_B Y') \hookrightarrow \mathcal{L}_{\mathbb{C}}^{\mathbb{C}}(X \otimes_A G \otimes_B Y' \otimes_{B'} H)$, $T \mapsto T \otimes_{B'} \mathrm{id}_H$, maps $(\mathcal{L}(X') \otimes \mathcal{B} \otimes \mathcal{L}(Y'))_{\mathrm{id}}^{\mathrm{id}}$ to $(\mathcal{L}(X) \otimes \mathcal{B} \otimes \mathcal{L}(Y))_{\mathrm{id}}^{\mathrm{id}}$, where $\mathcal{B} = \mathcal{C}, \mathcal{D}$, respectively. Restricting the map ϕ_Y^X (which exists by assumption), we obtain the desired map $\phi_{Y'}^{X'}$. □

The internal tensor product of C*-families is bifunctorial:

Proposition 5.16. *Let $\phi\colon \mathcal{A} \to \mathcal{C}$ and $\psi\colon \mathcal{B} \to \mathcal{D}$ be extendible families of linear maps/(non-degenerate) morphisms of C*-families on right C*-A-B-bimodules and right C*-B-C-bimodules, respectively. Then there exists an extendible family of linear maps/(non-degenerate) morphism $\phi \otimes \psi\colon \mathcal{A} \otimes \mathcal{B} \to \mathcal{C} \otimes \mathcal{D}$ such that*

$(\phi \ogreaterthan \psi)^{\rho}_{\sigma'}(a \ogreaterthan b) = \phi^{\rho}_{\sigma}(a) \ogreaterthan \psi^{\rho'}_{\sigma'}(b)$ *for all* $a \in \mathcal{A}^{\rho}_{\sigma}$, $b \in \mathcal{B}^{\rho'}_{\sigma'}$, *where* $\rho \in \mathrm{PAut}(A)$, $\sigma, \rho' \in \mathrm{PAut}(B)$, $\sigma' \in \mathrm{PAut}(C)$ *and* $\sigma \curlyvee \rho'$.

Proof. If we can prove the assertion for the case that $\mathcal{B} = \mathcal{D}$, $\psi = \mathrm{id}_{\mathcal{B}}$ and for the case that $\mathcal{A} = \mathcal{C}$, $\phi = \mathrm{id}_{\mathcal{A}}$, then we can simply put $\phi \ogreaterthan \psi := (\phi \ogreaterthan \mathrm{id}) \circ (\mathrm{id} \ogreaterthan \psi)$. We treat the first case, the second one is similar.

Let $\rho \in \mathrm{PAut}(A)$, $\sigma' \in \mathrm{PAut}(C)$. Denote by F the right C*-bimodule on which $\mathcal{B}$ acts. If $\sigma, \rho' \in \mathrm{PAut}(B)$, $\sigma \curlyvee \rho'$, then the diagram

$$\begin{array}{ccc}
\mathcal{A}^{\rho}_{\sigma} \ogreaterthan \mathcal{B}^{\rho'}_{\sigma'} & \xhookrightarrow{\iota^{\rho}_{\sigma'}} & \iota^{\rho}_{\sigma'}((\mathcal{A} \ogreaterthan \mathcal{B})^{\rho}_{\sigma'}) \subseteq (\mathcal{L}(\mathfrak{I}A) \ogreaterthan \mathcal{A} \ogreaterthan \mathcal{L}(F \ogreaterthan \mathfrak{I}C))^{\mathrm{id}}_{\mathrm{id}} \\
{\scriptstyle \phi^{\rho}_{\sigma} \ogreaterthan \mathrm{id}} \downarrow & & \downarrow {\scriptstyle \phi^{\mathfrak{I}A}_{F \ogreaterthan \mathfrak{I}C}} \\
\mathcal{C}^{\rho}_{\sigma} \ogreaterthan \mathcal{B}^{\rho'}_{\sigma'} & \xhookrightarrow{\iota^{\rho}_{\sigma'}} & \iota^{\rho}_{\sigma'}((\mathcal{C} \ogreaterthan \mathcal{B})^{\rho}_{\sigma'}) \subseteq (\mathcal{L}(\mathfrak{I}A) \ogreaterthan \mathcal{C} \ogreaterthan \mathcal{L}(F \ogreaterthan \mathfrak{I}C))^{\mathrm{id}}_{\mathrm{id}}
\end{array}$$

commutes. So we can insert a unique linear map $(\phi \ogreaterthan \mathrm{id})^{\rho}_{\sigma'} \colon (\mathcal{A} \ogreaterthan \mathcal{B})^{\rho}_{\sigma'} \to (\mathcal{C} \ogreaterthan \mathcal{B})^{\rho}_{\sigma'}$ that does not depend on σ, ρ' such that the diagram still commutes.

The family $((\phi \ogreaterthan \mathrm{id})^{\rho}_{\sigma'})_{\rho,\sigma'}$ is extendible. For let X be a right C*-$\mathbb{C}$-A-bimodule and Y a right C*-C-$\mathbb{C}$-bimodule. Then $F \ogreaterthan Y$ is a right C*-B-$\mathbb{C}$-bimodule, so the linear map $\phi^{X}_{F \ogreaterthan Y} : (\mathcal{L}(X) \ogreaterthan \mathcal{A} \ogreaterthan \mathcal{L}(F \ogreaterthan Y))^{\mathrm{id}}_{\mathrm{id}} \to (\mathcal{L}(X) \ogreaterthan \mathcal{C} \ogreaterthan \mathcal{L}(F \ogreaterthan Y))^{\mathrm{id}}_{\mathrm{id}}$ restricts to a linear map $(\phi \ogreaterthan \mathrm{id})^{X}_{Y} : (\mathcal{L}(X) \ogreaterthan \mathcal{A} \ogreaterthan \mathcal{B} \ogreaterthan \mathcal{L}(Y))^{\mathrm{id}}_{\mathrm{id}} \to (\mathcal{L}(X) \ogreaterthan \mathcal{C} \ogreaterthan \mathcal{B} \ogreaterthan \mathcal{L}(Y))^{\mathrm{id}}_{\mathrm{id}}$ that has the desired properties. If ϕ is a morphism, then $\phi^{X}_{F \ogreaterthan Y}$ and hence also $(\phi \ogreaterthan \mathrm{id})^{X}_{Y}$ are always $*$-homomorphisms, so $\phi \ogreaterthan \mathrm{id}$ is a morphism. □

Remark 5.17. Let $\mathcal{A}$, $\mathcal{C}$ be C*-families on right C*-A-B-bimodules and let $\mathcal{B}$, $\mathcal{D}$ be C*-families on right C*-B-C-bimodules. If $\phi \colon \mathcal{A} \to \mathcal{M}(\mathcal{C})$ and $\psi \colon \mathcal{B} \to \mathcal{M}(\mathcal{D})$ are non-degenerate morphisms, then the morphism $\phi \ogreaterthan \psi \colon \mathcal{A} \ogreaterthan \mathcal{B} \to \mathcal{M}(\mathcal{C}) \ogreaterthan \mathcal{M}(\mathcal{D}) \to \mathcal{M}(\mathcal{C} \ogreaterthan \mathcal{D})$ evidently is non-degenerate.

Non-degenerate morphisms of C*-families can be extended to multipliers:

Proposition 5.18. *Let* $\phi \colon \mathcal{C} \to \mathcal{M}(\mathcal{D})$ *be a non-degenerate morphism of C*-families. If the C*-family* $\mathcal{D}$ *is non-degenerate, then* ϕ *extends uniquely to a morphism* $\mathcal{M}(\mathcal{C}) \to \mathcal{M}(\mathcal{D})$.

Proof. Uniqueness follows once existence is proved by a standard argument. Denote by F the underlying right C*-bimodule of $\mathcal{D}$. Choose an approximate unit $(u_{\nu})_{\nu}$ for the C*-algebra $\mathcal{C}^{\mathrm{id}}_{\mathrm{id}}$ such that $0 \leq u_{\nu} \leq 1$ for all ν.

We construct an extension $\overline{\phi}^{\rho}_{\sigma} \colon \mathcal{M}(\mathcal{C})^{\rho}_{\sigma} \to \mathcal{M}(\mathcal{D})^{\rho}_{\sigma}$ of ϕ^{ρ}_{σ} for each $\sigma \in \mathrm{PAut}(A)$ and $\rho \in \mathrm{PAut}(B)$ as follows. Let $c \in \mathcal{M}(\mathcal{C})^{\rho}_{\sigma}$. Since ϕ and $\mathcal{D}$ are non-degenerate, the net $(\phi^{\rho}_{\sigma}(cu_{\nu}))_{\nu}$ converges strictly to some $\overline{\phi}^{\rho}_{\sigma}(c) \in \mathcal{L}^{\rho}_{\sigma}(F)$ (see Proposition 3.7 (i)).

Since $\overline{\phi}^{\rho}_{\sigma}(c)\mathcal{D}^{\mathrm{id}}_{\mathrm{id}} = \overline{\phi}^{\rho}_{\sigma}(c)[\phi^{\mathrm{id}}_{\mathrm{id}}(\mathcal{C}^{\mathrm{id}}_{\mathrm{id}})\mathcal{D}^{\mathrm{id}}_{\mathrm{id}}] \subseteq [\phi^{\rho}_{\sigma}(c\mathcal{C}^{\mathrm{id}}_{\mathrm{id}})\mathcal{D}^{\mathrm{id}}_{\mathrm{id}}] \subseteq \mathcal{D}^{\rho}_{\sigma}$ and likewise $\mathcal{D}^{\mathrm{id}}_{\mathrm{id}}\overline{\phi}^{\rho}_{\sigma}(c) \subseteq \mathcal{D}^{\rho}_{\sigma}$, it follows that $\overline{\phi}^{\rho}_{\sigma}(c) \in \mathcal{M}(\mathcal{D})^{\rho}_{\sigma}$.

We show that the family $\overline{\phi}\colon \mathcal{M}(\mathcal{C}) \to \mathcal{M}(\mathcal{D})$ is a morphism. Let X be a right C*-$\mathbb{C}$-A-bimodule and Y a right C*-B-$\mathbb{C}$-bimodule. By assumption on ϕ, the $*$-homomorphism ϕ^X_Y is non-degenerate and extends to a $*$-homomorphism $\overline{\phi^X_Y}\colon M((\mathcal{L}(X) \otimes \mathcal{C} \otimes \mathcal{L}(Y))^{\mathrm{id}}_{\mathrm{id}}) \to M((\mathcal{L}(X) \otimes \mathcal{M}(\mathcal{D}) \otimes \mathcal{L}(Y))^{\mathrm{id}}_{\mathrm{id}})$. For all $R \in \mathcal{L}^{\mathrm{id}}_{\sigma'}(X)$, $S \in \mathcal{M}(\mathcal{C})^{\rho}_{\sigma}$, $T \in \mathcal{L}^{\rho'}_{\mathrm{id}}(Y)$, where $\sigma', \rho \in \mathrm{PAut}(A)$, $\sigma, \rho' \in \mathrm{PAut}(B)$, and $\sigma' \curlyvee \rho, \sigma \curlyvee \rho'$, the operators $\overline{\phi^X_Y}(R \otimes S \otimes T)$ and $R \otimes \overline{\phi}^{\rho}_{\sigma}(S) \otimes T$ are equal because they coincide with the strict limit of the net $(R \otimes \phi^{\rho}_{\sigma}(Su_{\nu}) \otimes T)_{\nu}$. Hence $\overline{\phi^X_Y}$ restricts to a $*$-homomorphism $\overline{\phi}^X_Y\colon (\mathcal{L}(X) \otimes \mathcal{C} \otimes \mathcal{L}(Y))^{\mathrm{id}}_{\mathrm{id}} \to (\mathcal{L}(X) \otimes M(\mathcal{D}) \otimes \mathcal{L}(Y))^{\mathrm{id}}_{\mathrm{id}}$, as desired. □

We are primarily concerned with the following examples of morphisms.

Examples 5.19. (i) An inclusion of C*-families is a morphism.

Let $\mathcal{C}$ be a C*-family on a right C*-A-B-bimodule E.

(ii) Let F be a right C*-A-B-bimodule and $V \in \mathcal{L}^A_B(E, F)$ an isometry. Then $\mathrm{Ad}_V(\mathcal{C}) := [V\mathcal{C}V^*] \subseteq \mathcal{L}(F)$ is a C*-family and the formula $c \mapsto VcV^*$ defines an isomorphism $\mathrm{Ad}_V\colon \mathcal{C} \to \mathrm{Ad}_V(\mathcal{C})$. If $\mathcal{C}$ is a (non-degenerate) $\mathcal{O}(E)$-module, then $\mathrm{Ad}_V(\mathcal{C})$ is a (non-degenerate) $\mathcal{O}(F)$-module; if V is unitary and $\mathcal{C}$ non-degenerate, then $\mathrm{Ad}_V(\mathcal{C})$ is non-degenerate.

(iii) Let F be a C*-module over C and $\pi\colon C \to \mathcal{L}_B(E)$ a $*$-homomorphism such that $\pi(C)$ commutes with each operator in $\mathcal{C}$. Consider $F \otimes_{\pi} E$ as a right C*-A-B-bimodule via $a(\eta \otimes \xi) := \eta \otimes a\xi$ for all $a \in A$, $\eta \in F$, $\xi \in E$. By a slight abuse of notation, we denote by $1 \otimes \mathcal{C} \subseteq \mathcal{L}(F \otimes_{\pi} E)$ the internal tensor product of $\mathcal{C}$ with the C*-family generated by the identity operator on F. Then $1 \otimes \mathcal{C}$ is a C*-family, and the map $T \mapsto 1 \otimes T$ defines a non-degenerate morphism $\mathcal{C} \to 1 \otimes \mathcal{C}$. If $\pi(\langle F|F\rangle) \subseteq \mathcal{L}_B(E)$ is non-degenerate, then this morphism is injective. If the C*-family $\mathcal{C}$ is non-degenerate, then $1 \otimes \mathcal{C}$ is non-degenerate.

Now we have gathered all concepts needed to define Hopf C*-families.

Definition 5.20. A (*flipped*) *Hopf C*-family* over B is a non-degenerate C*-family $\mathcal{A}$ on a right C*-B-B-bimodule equipped with a non-degenerate morphism $\Delta\colon \mathcal{A} \to \mathcal{M}(\mathcal{A} \otimes \mathcal{A})$ (or $\Delta\colon \mathcal{A} \to \mathcal{M}(\mathcal{A} \varoslash \mathcal{A})$, respectively) such that

(i) $[\Delta(\mathcal{A})(1 \otimes \mathcal{A})] = \mathcal{A} \otimes \mathcal{A} = [\Delta(\mathcal{A})(\mathcal{A} \otimes 1)]$ (or $[\Delta(\mathcal{A})(1 \varoslash \mathcal{A})] = \mathcal{A} \varoslash \mathcal{A} = [\Delta(\mathcal{A})(\mathcal{A} \varoslash 1)]$, respectively), and

(ii) $(\mathrm{id} \otimes \Delta) \circ \Delta = (\Delta \otimes \mathrm{id}) \circ \Delta$ (or $(\mathrm{id} \varoslash \Delta) \circ \Delta = (\Delta \varoslash \mathrm{id}) \circ \Delta$, respectively).

Note that condition (i) implies that Δ is non-degenerate; therefore we can extend $\mathrm{id} \otimes \Delta$, $\Delta \otimes \mathrm{id}$ (or $\mathrm{id} \varoslash \Delta$, $\Delta \varoslash \mathrm{id}$, respectively) to multipliers.

6. Legs of a decomposable pseudo-multiplicative unitary and Hopf C*-families

We return to the study of a pseudo-multiplicative unitary $W\colon E_{\hat{\beta}} \otimes E \to E \otimes {}_{\beta}E$, where $(E, \hat{\beta}, \beta)$ is a C*-trimodule over a C*-algebra B, and construct comultiplications on the legs $\hat{\mathcal{A}}(W)$ and $\mathcal{A}(W)$ defined in Section 4. As before, our constructions are interesting only if the right C*-bimodule ${}_{\beta}E$ or ${}_{\hat{\beta}}E$, respectively, is decomposable.

Denote by $\hat{\mathcal{B}} \subseteq \mathcal{L}({}_{\hat{\beta}}E)$ and $\mathcal{B} \subseteq \mathcal{L}({}_{\beta}E)$ the C*-families generated by $\hat{\mathcal{A}}(W)$ and $\mathcal{A}(W)$, respectively. Since $\hat{\mathcal{B}}$ and $\mathcal{B}$ commute with $\beta(B)$ and $\hat{\beta}(B)$, respectively, see Lemma 4.1, we can define morphisms $\hat{\mathcal{B}} \to \mathcal{L}({}_{\hat{\beta}_2}(E \otimes {}_{\beta}E))$, $\hat{a} \mapsto 1 \otimes \hat{a}$, and $\mathcal{B} \to \mathcal{L}({}_{\beta_1}(E_{\hat{\beta}} \otimes E))$, $a \mapsto a \otimes 1$ (see Example 5.19 (iii)). Composing with conjugation by W^* or W, respectively, we obtain morphisms (see Example 5.19 (ii) and equation (1))

$$\hat{\Delta}\colon \hat{\mathcal{B}} \to \mathcal{L}({}_{\hat{\beta}_2}(E_{\hat{\beta}} \otimes E)), \quad \hat{a} \mapsto W^*(1 \otimes \hat{a})W,$$
$$\Delta\colon \mathcal{B} \to \mathcal{L}({}_{\beta_1}(E \otimes {}_{\beta}E)), \quad a \mapsto W(a \otimes 1)W^*.$$

On the operators $\hat{a}_{(\xi',\xi)}$ and $a_{(\eta',\eta)}$ of Lemma 4.1, $\hat{\Delta}$ and Δ act as follows:

Lemma 6.1. (i) *Let $\xi, \xi' \in {}_{\beta}E$ be homogeneous. Then $\hat{\Delta}(\hat{a}_{(\xi',\xi)})$ equals*

$$[\xi'|_3 W_{13}W_{23}|\xi\rangle_3\colon E_{\hat{\beta}} \otimes E \to E_{\hat{\beta}} \otimes E_{\hat{\beta}} \otimes E \to (E_{\hat{\beta}} \otimes E) \otimes {}_{\beta}E \to E_{\hat{\beta}} \otimes E,$$

where $|\xi\rangle_3(\eta \otimes \zeta) = \eta \otimes \zeta \otimes \xi$, $[\xi'|_3^*(\eta \otimes \zeta) = (\eta \otimes \zeta) \otimes \xi'$ for $\eta, \zeta \in E$.

(ii) *Let $\eta, \eta' \in {}_{\hat{\beta}}E$ be homogeneous. Then $\Delta(a_{(\eta',\eta)})$ is equal to the map*

$$\langle\eta'|_1 W_{12}W_{13}|\eta]_1\colon E \otimes {}_{\beta}E \to E_{\hat{\beta}} \otimes (E \otimes {}_{\beta}E) \to E \otimes {}_{\beta}E \otimes {}_{\beta}E \to E \otimes {}_{\beta}E,$$

where $|\eta]_1(\zeta \otimes \xi) = \eta \otimes (\zeta \otimes \xi)$, $\langle\eta'|_1^*(\zeta \otimes \xi) = \eta' \otimes \zeta \otimes \xi$ for $\zeta, \xi \in E$.

Proof. We only prove (i). By definition, $\hat{\Delta}(\hat{a}_{(\xi',\xi)})$ is equal to the composition

$$E_{\hat{\beta}} \otimes E \xrightarrow{W} E \otimes {}_{\beta}E \xrightarrow{\mathrm{id} \otimes |\xi\rangle_2} E \otimes {}_{\beta}E_{\hat{\beta}} \otimes E \xrightarrow{W_{23}} E \otimes {}_{\beta}E \otimes {}_{\beta}E$$
$$\xrightarrow{\mathrm{id} \otimes [\xi'|_2} E \otimes {}_{\beta}E \xrightarrow{W^*} E_{\hat{\beta}} \otimes E,$$

and this is equal to the map $[\xi'|_3 W_{12}^* W_{23} W_{12}|\xi\rangle_3\colon E_{\hat{\beta}} \otimes E \to E_{\hat{\beta}} \otimes E_{\hat{\beta}} \otimes E \to (E_{\hat{\beta}} \otimes E) \otimes {}_{\beta}E \to E_{\hat{\beta}} \otimes E$. But $W_{12}^* W_{23} W_{12} = W_{13}W_{23}$. □

Proposition 6.2. (i) *If ${}_{\beta}E$ is decomposable and $[\hat{\Delta}(\hat{\mathcal{B}})(1 \otimes \hat{\mathcal{B}})] = \hat{\mathcal{B}} \otimes \hat{\mathcal{B}} = [\hat{\Delta}(\hat{\mathcal{B}})(\hat{\mathcal{B}} \otimes 1)]$, then $(\hat{\mathcal{B}}, \hat{\Delta})$ is a flipped Hopf C*-family.*

(ii) *If ${}_{\hat{\beta}}E$ is decomposable and $[\Delta(\mathcal{B})(1 \otimes \mathcal{B})] = \mathcal{B} \otimes \mathcal{B} = [\Delta(\mathcal{B})(\mathcal{B} \otimes 1)]$, then $(\mathcal{B}, \Delta)$ is a Hopf C*-family.*

Proof. We only prove assertion (i); the proof of assertion (ii) is similar. Let us make the assumptions stated in (i). By Proposition 4.5, the C*-family $\hat{\mathcal{B}}$ is non-degenerate and by the second assumption, $\hat{\Delta}$ is a non-degenerate morphism $\hat{\mathcal{B}} \to \mathcal{M}(\hat{\mathcal{B}} \odot \hat{\mathcal{B}})$. It remains to show that $\hat{\Delta}$ is coassociative. Let $\hat{a} \in \hat{\mathcal{B}}^{\rho}_{\sigma}$, $\rho, \sigma \in \mathrm{PAut}(B)$. By definition, $\hat{\Delta}(\hat{a}) = W^*(1 \odot \hat{a})W$, and hence

$$(\hat{\Delta} \odot \mathrm{id})(\hat{\Delta}(\hat{a})) = W_{12}^* W_{23}^* (1 \odot 1 \odot \hat{a}) W_{23} W_{12},$$

where $W_{23}W_{12} \colon E_{\hat{\beta}} \odot E_{\hat{\beta}} \odot E \to E \odot_{\beta} E_{\hat{\beta}} \odot E \to E \odot_{\beta} E \odot_{\beta} E$. Now we can squeeze in conjugation by W_{12} and find

$$(\hat{\Delta} \odot \mathrm{id})(\hat{\Delta}(\hat{a})) = W_{12}^* W_{23}^* W_{12} ((1 \odot 1) \odot \hat{a}) W_{12}^* W_{23} W_{12},$$

where $W_{12}^*(W_{23}W_{12}) \colon E_{\hat{\beta}} \odot E_{\hat{\beta}} \odot E \to E \odot_{\beta} E \odot_{\beta} E \to (E_{\hat{\beta}} \odot E) \odot_{\beta} E$. From the pentagon equation (2) it follows that $W_{12}^* W_{23} W_{12}$ is equal to the composition $W_{13}W_{23} \colon E_{\hat{\beta}} \odot E_{\hat{\beta}} \odot E \to E_{\hat{\beta}} \odot (E \odot_{\beta} E) \to (E_{\hat{\beta}} \odot E) \odot_{\beta} E$. Therefore,

$$(\hat{\Delta} \odot \mathrm{id})(\hat{\Delta}(\hat{a})) = W_{23}^* W_{13}^* ((1 \odot 1) \odot \hat{a}) W_{13} W_{23} = (\mathrm{id} \odot \hat{\Delta})(\hat{\Delta}(\hat{a})). \qquad \square$$

Example: the pseudo-multiplicative unitary W_G. Let us consider the pseudo-multiplicative unitary W_G associated to a groupoid G and determine the comultiplications on its legs. We use the same notation as in Example 2.5 and Section 4.

Recall that the left leg $\hat{\mathcal{A}}(W_G) \subseteq \mathcal{L}({}_sL^2(G,\lambda))$ corresponds to (a filtration of) the C*-algebra $C_0(G)$, and that the internal tensor product $L^2(G,\lambda)_s \odot L^2(G,\lambda)$ can be identified with $L^2(G^2_{s,r})$.

Lemma 6.3. $(\hat{\Delta}^{\mathrm{id}}_{\mathrm{id}}(m(f))\zeta)(x,y) = f(xy)\zeta(x,y)$ *for all* $f \in C_0(G)$, $\zeta \in L^2(G^2_{s,r})$, $(x,y) \in G^2_{s,r}$.

Proof. If f, ζ, x, y are as above then $\hat{\Delta}(m(f)) = W_G^*(1 \odot m(f))W_G$ and

$$\begin{aligned}(W_G^*(1 \odot m(f))W_G\zeta)(x,y) &= ((1 \odot m(f))W_G\zeta)(x,xy) \\ &= f(xy)(W_G\zeta)(x,xy) = f(xy)\zeta(x,y). \qquad \square\end{aligned}$$

Define $\hat{\delta} \colon C_0(G) \to C_b(G^2_{s,r})$ by $(\hat{\delta}(f))(x,y) = f(xy)$ and denote by

$$m^2_{s,r} \colon C_b(G^2_{s,r}) \to L^2(G^2_{s,r})$$

the representation given by pointwise multiplication. Then the lemma above says that $\hat{\Delta}^{\mathrm{id}}_{\mathrm{id}} \circ m = m^2_{s,r} \circ \hat{\delta}$.

Theorem 6.4. $(\hat{\mathcal{A}}(W_G), \hat{\Delta})$ *is a Hopf C*-family.*

Proof. Put $\hat{\mathcal{A}} := \hat{\mathcal{A}}(W_G)$. By Proposition 4.8 (iii) and 6.2, it suffices to show that $[\hat{\Delta}(\hat{\mathcal{A}})(1 \otimes \hat{\mathcal{A}})] = [\hat{\mathcal{A}} \otimes \hat{\mathcal{A}}] = [\hat{\Delta}(\hat{\mathcal{A}})(\hat{\mathcal{A}} \otimes 1)]$. We prove the first equality, and the second one follows similarly. Denote by $p_2^*\colon C_0(G) \to C_b(G^2_{s,r})$ the map given by $(p_2^* f)(x,y) := f(y)$ for all $(x,y) \in G^2_{s,r}$, $f \in C_0(G)$. Routine arguments show that $[\hat{\delta}(C_0(G))p_2^*(C_0(G))] = C_0(G^2_{s,r})$. Let $\rho, \sigma \in \mathrm{PAut}(C_0(G^0))$ and put $D_\rho := \mathrm{Dom}(\rho \wedge \mathrm{id})$, $D_\sigma := \mathrm{Dom}(\sigma \wedge \mathrm{id})$. Then by Proposition 4.8 (ii), Lemma 6.3, and the preceding observation,

$$\begin{aligned}
[\hat{\mathcal{A}} \otimes \hat{\mathcal{A}}]^{\rho}_{\sigma} &= [r(D_\sigma)m(C_0(G)) \otimes s(D_\rho)m(C_0(G))] \\
&= [(r(D_\sigma) \otimes s(D_\rho))m^2_{s,r}(C_0(G^2_{s,r}))] \\
&= [(r(D_\sigma) \otimes s(D_\rho))m^2_{s,r}(\hat{\delta}(C_0(G))p_2^*(C_0(G)))] \\
&= [\hat{\Delta}(\hat{\mathcal{A}})(1 \otimes \hat{\mathcal{A}})]^{\rho}_{\sigma}. \qquad \square
\end{aligned}$$

Recall that the right leg $\mathcal{A}(W_G)$ corresponds to the left regular representation of G, and that $L^2(G,\lambda) \otimes {}_r L^2(G,\lambda)$ can be identified with $L^2(G^2_{r,r})$. As before, we impose Assumption 4.12.

Lemma 6.5. *Let $f \in C_c(U)$, where $U \subseteq G$ is open and homogeneous.*

(i) $(\Delta(L(f))\zeta)(x,y) = \int_{G^{r_G(x)}} f(z)\zeta(z^{-1}x, z^{-1}y)d\lambda^{r_G(x)}(z)$ *for all* $(x,y) \in G^2_{r,r}$, $\zeta \in L^2(G^2_{r,r})$.

(ii) *Assume that G is r-discrete, U a G-set, $g,h \in C_c(U)$ and $gh = f$. Then* $\Delta(L(f)) = L(g) \otimes L(h)$.

Proof. Let f, ζ, x, y be as above. Then $\Delta(L(f)) = W_G(L(f) \otimes 1)W_G^*$ and

$$\begin{aligned}
(W_G(L(f) \otimes 1)W_G^*\zeta)(x,y) &= ((L(f) \otimes 1)W_G^*\zeta)(x, x^{-1}y) \\
&= \int_{G^{r_G(x)}} f(z)(W_G^*\zeta)(z^{-1}x, x^{-1}y)\, d\lambda^{r_G(x)}(z) \\
&= \int_{G^{r_G(x)}} f(z)\zeta(z^{-1}x, z^{-1}xx^{-1}y)\, d\lambda^{r_G(x)}(z).
\end{aligned}$$

Assertion (i) follows. Let us prove (ii). If $r(x) \in r(U)$, there exists a unique element $z \in U$ such that $r(z) = r(x)$ and

$$\begin{aligned}
(\Delta(L(f))\zeta)(x,y) &= f(z)\zeta(z^{-1}x, z^{-1}y) \\
&= g(z)h(z)\zeta(z^{-1}x, z^{-1}y) = ((L(g) \otimes L(h))\zeta)(x,y). \qquad \square
\end{aligned}$$

Theorem 6.6. *If G is r-discrete, then $(\mathcal{A}(W_G), \Delta)$ is a Hopf C*-family.*

Proof. Put $\mathcal{A} = \mathcal{A}(W_G)$. By Proposition 4.14 (iii) and Lemma 6.5 (ii), it suffices to show that $[\mathcal{A} \otimes \mathcal{A}] \subseteq [\Delta(\mathcal{A})(1 \otimes \mathcal{A})]$ and $[\mathcal{A} \otimes \mathcal{A}] \subseteq [\Delta(\mathcal{A})(\mathcal{A} \otimes 1)]$. We prove the first inclusion, the second one follows similarly. Let $\rho, \sigma \in \mathrm{PAut}(B)$. By Proposition 4.14 (ii), $[\mathcal{A} \ogreaterthan \mathcal{A}]^{\rho}_{\sigma}$ is the closed linear span of all operators of the form $L(f) \ogreaterthan L(g)$, where $f \in C_c(U)$, $g \in C_c(V)$ for some open G-sets U, V and $\rho \geq q_{U*} \curlyvee q_{V*} \leq \sigma$. Fix such an operator, choose $\chi \in C_c(U)$ such that $\chi f = f$, and put $\omega := \chi \star \chi^*$. Then $\omega \in C_c(G^0)$, $\omega \star f = f$ and $L(\omega) = r(\omega) = L(\chi)L(\chi^*)$. Using Lemma 6.5 (ii), we find

$$\begin{aligned} L(f) \ogreaterthan L(g) &= r(\omega)L(f) \ogreaterthan L(g) = L(f) \ogreaterthan r(\omega)L(g) \\ &= L(f) \ogreaterthan L(\chi)L(\chi^* \star g) = \Delta(L(f))(1 \ogreaterthan L(\chi^* \star g)). \end{aligned}$$

Here $L(\chi^* \star g) = L(\chi)^* L(g) \in \mathcal{A}^{\mathrm{id}}_{\rho^*\sigma}$ by Proposition 4.14 and because $q_{U*} \curlyvee q_{V*}$. Therefore, $L(f) \ogreaterthan L(g) \in \Delta(\mathcal{A}^{\rho}_{\rho})(1 \ogreaterthan \mathcal{A}^{\mathrm{id}}_{\rho^*\sigma}) \subseteq [\Delta(\mathcal{A})(1 \ogreaterthan \mathcal{A})]^{\rho}_{\sigma}$. □

In a subsequent article we will show that $(\mathcal{A}(W_G), \Delta)$ is a Hopf C*-family whenever G is decomposable.

Example: the pseudo-multiplicative unitary W_τ. Let us consider the pseudo-multiplicative unitary W_τ associated to a center-valued conditional expectation $\tau \colon B \to C \subseteq Z(B)$, see Example 2.6 and Section 4, and determine the comultiplication on the leg $\hat{\mathcal{A}}(W_\tau) = \mathcal{O}({}_{\hat{\beta}}E)$.

Lemma 6.7. *$\hat{\Delta}(o_{e,f}) = o_{1,f} \ogreaterthan o_{e,1}$ for all $e \in \mathcal{H}_{\rho}(B)$, $f \in \mathcal{H}_{\sigma^*}(B)$, $\rho, \sigma \in \mathrm{PAut}(B)$.*

Proof. By Proposition 3.21, $o_{1,f} \in \mathcal{L}^{\mathrm{id}}_{\sigma}({}_{\hat{\beta}}E)$ and $o_{e,1} \in \mathcal{L}^{\rho}_{\mathrm{id}}({}_{\hat{\beta}}E)$, and by Proposition 5.3, $o_{1,f} \ogreaterthan o_{e,1} \in \mathcal{L}^{\rho}_{\sigma}({}_{\hat{\beta}_2}(E_{\hat{\beta}} \ogreaterthan E))$ is well defined. The following diagram shows that $\hat{\Delta}(o_{e,f}) = W^*_\tau(1 \ogreaterthan o_{e,f})W_\tau = o_{1,f} \ogreaterthan o_{e,1}$: for all $a, b, c, d \in B$,

$$\begin{array}{ccc} (a \ogreaterthan b) \ogreaterthan (c \ogreaterthan d) & \xmapsto{\;o_{1,f} \ogreaterthan o_{e,1}\;} & (a \ogreaterthan bf) \ogreaterthan (ec \ogreaterthan d) \\ \downarrow{\scriptstyle W_\tau} & & \downarrow{\scriptstyle W_\tau} \\ (da \ogreaterthan b) \ogreaterthan (c \ogreaterthan 1) & \xmapsto[1 \ogreaterthan o_{e,f}]{} & (da \ogreaterthan b) \ogreaterthan (ec \ogreaterthan f) = (da \ogreaterthan bf) \ogreaterthan (ec \ogreaterthan 1). \end{array}$$ □

In the next proposition we use the following equation: in $E_{\hat{\beta}} \ogreaterthan E$ we have

$$o_{a,b}\eta \ogreaterthan o_{c,de}\xi = o_{a,b}\eta \ogreaterthan (o_{c,d}\xi)e = \hat{\beta}(e)o_{a,b}\eta \ogreaterthan o_{c,d}\xi = o_{ea,b}\eta \ogreaterthan o_{c,d}\xi \quad (7)$$

for all $\eta, \xi \in E$ and all homogeneous $a, b, c, d, e \in B$.

Theorem 6.8. *$(\hat{\mathcal{A}}(W_\tau), \hat{\Delta})$ is a Hopf C*-family.*

Proof. Put $\mathcal{O} := \mathcal{O}({}_{\hat{\beta}}E) = \hat{\mathcal{A}}(W_\tau)$. By Proposition 6.2 and Lemma 6.7, we only need to prove $[\mathcal{O} \otimes \mathcal{O}] \subseteq [\hat{\Delta}(\mathcal{O})(1 \otimes \mathcal{O})]$ and $[\mathcal{O} \otimes \mathcal{O}] \subseteq [\hat{\Delta}(\mathcal{O})(\mathcal{O} \otimes 1)]$. For each $\rho, \sigma \in \mathrm{PAut}(B)$, the space $(\mathcal{O} \otimes \mathcal{O})^{\rho}_{\sigma}$ is the closed linear span of all elements of the form $o_{a,b} \otimes o_{c,d}$, where $a, b, c, d \in B$ are ρ'-/σ^*-/ρ-/σ'^*-homogeneous, and $\rho' \curlyvee \sigma'$. For such an element, $da \in \mathcal{H}_{\sigma'^*\rho'}(B) \subseteq \mathcal{H}_{\mathrm{id}}(B)$ by Proposition 3.20 (iv), whence $1 \otimes o_{1,da}$ and $o_{da,1} \otimes 1$ are well defined, and by Lemma 6.7 and equation (7), $o_{a,b} \otimes o_{c,d} = o_{da,b} \otimes o_{c,1} = \hat{\Delta}(o_{c,b})(o_{da,1} \otimes 1) \in [\hat{\Delta}(\mathcal{O})(\mathcal{O} \otimes 1)]^{\rho}_{\sigma}$ and $o_{a,b} \otimes o_{c,d} = o_{1,b} \otimes o_{c,da} = \hat{\Delta}(o_{c,b})(1 \otimes o_{1,da}) \in [\hat{\Delta}(\mathcal{O})(1 \otimes \mathcal{O})]^{\rho}_{\sigma}$. □

7. Additional structure on the legs

As before let B be a C*-algebra, let $(E, \hat{\beta}, \beta)$ be a C*-trimodule over B and let $W\colon E_{\hat{\beta}} \otimes E \to E \otimes {}_{\beta}E$ be a pseudo-multiplicative unitary.

The dual pairing of the legs. Similar to the case of multiplicative unitaries [1], Définition 1.3, there is a pairing between the spaces $\hat{A}_a(W) := \sum_{\rho,\sigma} \hat{\mathcal{A}}_a(W)^{\rho}_{\sigma} \subseteq \mathcal{L}(E)$ and $A_a(W) := \sum_{\rho',\sigma'} \mathcal{A}_a(W)^{\rho'}_{\sigma'} \subseteq \mathcal{L}(E)$. This pairing is interesting primarily if ${}_{\beta}E$ and ${}_{\hat{\beta}}E$ are decomposable.

Lemma 7.1. *For all homogeneous $\xi, \xi' \in {}_{\beta}E$ and $\eta, \eta' \in {}_{\hat{\beta}}E$, the compositions $[\xi'|a_{(\eta',\eta)}|\xi\rangle\colon B \to {}_{\beta}E \to {}_{\beta}E \to B$ and $\langle\eta'|\hat{a}_{(\xi',\xi)}|\eta]\colon B \to {}_{\hat{\beta}}E \to {}_{\hat{\beta}}E \to B$ are equal.*

Proof. $[\xi'|a_{(\eta',\eta)}|\xi\rangle = [\xi'|\langle\eta'|_1 W|\eta]_1|\xi\rangle = \langle\eta'|[\xi'|_2 W|\xi\rangle_2|\eta] = \langle\eta'|\hat{a}_{(\xi',\xi)}|\eta]$ because $|\eta]_1|\xi\rangle b = \eta \otimes \xi b = b\eta \otimes \xi = |\xi\rangle_2|\eta]b$ and $|\eta'\rangle_1|\xi']b = \eta' \otimes b\xi' = \eta' b \otimes \xi' = |\xi']_2|\eta'\rangle b$ for all $b \in B$. □

The next proposition involves the *weak topology* on $\mathcal{L}(E)$, which is the locally convex topology generated by all seminorms of the form $T \mapsto \|\langle\zeta'|T\zeta\rangle\|$ where $\zeta, \zeta' \in E$. Denote by $\overline{X}^w$ the weak closure of a subset $X \subseteq \mathcal{L}(E)$.

Proposition 7.2. *There exists a bilinear map $(\cdot|\cdot)\colon \hat{A}_a(W) \times A_a(W) \to \mathcal{L}(B)$ such that $[\xi'|a_{(\eta',\eta)}|\xi\rangle = (\hat{a}_{(\xi',\xi)}|a_{(\eta',\eta)}) = \langle\eta'|\hat{a}_{(\xi',\xi)}|\eta]$ for all homogeneous $\xi, \xi' \in {}_{\beta}E$ and $\eta, \eta' \in {}_{\hat{\beta}}E$. This map has the following properties:*

(i) *It extends to a bilinear map $(\cdot|\cdot)^w\colon \hat{A}_a(W) \times \overline{A_a(W)}^w \to \mathcal{L}(B)$ such that for each $\hat{a} \in \hat{A}_a(W)$ the map $a \mapsto (\hat{a}|a)^w$ is continuous with respect to the weak topology on $\overline{A_a(W)}^w$ and the norm topology on $\mathcal{L}(B)$, and it extends to a bilinear map ${}^w(\cdot|\cdot)\colon \overline{\hat{A}_a(W)}^w \times A_a(W) \to \mathcal{L}(B)$ such that for each $a \in A_a(W)$ the* *map* $\hat{a} \mapsto {}^{w}(\hat{a}|a)$ *is continuous with respect to the weak topology on* $\overline{\hat{A}_a(W)}^{w}$ *and the norm topology on* $\mathcal{L}(B)$.

(ii) $(\hat{\mathcal{A}}_a(W)^{\rho}_{\sigma}|\mathcal{A}_a(W)^{\rho'}_{\sigma'}) \subseteq \mathcal{L}^{\rho'\rho}_{\sigma\sigma'}(B)$ *for all* $\rho, \sigma, \rho', \sigma' \in \mathrm{PAut}(B)$.

(iii) *If* ${}_{\hat{\beta}}E$ *is decomposable, then* $(\hat{a}|A_a(W)) \neq 0$ *whenever* $\hat{a} \neq 0$*, and if* ${}_{\beta}E$ *is decomposable, then* $(\hat{A}_a(W)|a) \neq 0$ *whenever* $a \neq 0$.

(iv) $(\hat{a}|\alpha(b)a) = (\hat{a}\alpha(b)|a)$ *for all* $\hat{a} \in \hat{A}_a(W)$, $a \in A_a(W)$ *and all homogeneous* $b \in B$.

Proof. Existence follows from Lemma 7.1: If $\hat{a} = \sum_i \hat{a}_{(\xi'_i,\xi_i)}$, $a = \sum_j a_{(\eta'_j,\eta_j)}$, where $\xi_i, \xi'_i \in {}_{\beta}E$, $\eta_j, \eta'_j \in {}_{\hat{\beta}}E$ are homogeneous, then we can define $(\hat{a}|a)$ to be $\sum_i [\xi'_i|a|\xi_i\rangle = \sum_{i,j} [\xi'_i|a_{(\eta'_j,\eta_j)}|\xi_i\rangle = \sum_{i,j} \langle\eta'_j|\hat{a}_{(\xi'_i,\xi_i)}|\eta_j] = \sum_j \langle\eta'_j|\hat{a}|\eta_j]$.

(i) We prove existence of $(\cdot|\cdot)^{w}$; for ${}^{w}(\cdot|\cdot)$ the proof is similar. Let $\hat{a}_{(\xi',\xi)} \in \hat{\mathcal{A}}_a(W)^{\rho}_{\sigma}$ be as in Lemma 4.1 (i) and let $(a_\mu)_\mu$ be a net in $A_a(W)$ with weak limit $a \in \mathcal{L}(E)$. Then the net $((\hat{a}_{(\xi',\xi)}|a_\mu))_\mu$ converges in norm to $[\xi'|a|\xi\rangle =: (\hat{a}_{(\xi',\xi)}|a)^{w}$. Indeed, $\|(\hat{a}_{(\xi',\xi)}|a_\mu) - [\xi'|a|\xi\rangle\| = \|[\xi'|(a_\mu - a)|\xi\rangle\| = \|\langle\xi'|(a_\mu - a)\xi\rangle\| \to 0$ because $[\xi'|(a_\mu - a)|\xi\rangle b = \sigma(\langle\xi'|(a_\mu - a)\xi\rangle b)$ for all $b \in B$. Using bilinearity of $(\cdot|\cdot)$, we can replace $\hat{a}_{(\xi',\xi)}$ by an arbitrary $\hat{a} \in \hat{A}_a(W)$.

(ii) Given $\hat{a}_{(\xi',\xi)} \in \hat{\mathcal{A}}_a(W)^{\rho}_{\sigma}$ as in Lemma 4.1 (i) and $a \in \mathcal{A}_a(W)^{\rho'}_{\sigma'}$, we have $(\hat{a}_{(\xi',\xi)}|a) = [\xi'|a|\xi\rangle \in \mathcal{L}^{\mathrm{id}}_{\sigma}({}_{\beta}E, B)\mathcal{L}^{\rho'}_{\sigma'}({}_{\beta}E)\mathcal{L}^{\rho}_{\mathrm{id}}(B, {}_{\beta}E) \subseteq \mathcal{L}^{\rho'\rho}_{\sigma\sigma'}(B)$ by Proposition 3.12. The claim follows.

(iii) If ${}_{\hat{\beta}}E$ is decomposable and $\langle\eta'|\hat{a}|\eta] = (\hat{a}|a_{(\eta',\eta)}) = 0$ for some $\hat{a} \in \hat{A}_a(W)$ and all homogeneous $\eta, \eta' \in {}_{\hat{\beta}}E$, then $\langle E|\hat{a}\hat{\beta}(B)E\rangle = 0$ and hence $\hat{a} = 0$. The second assertion follows similarly.

(iv) Let $\xi, \xi' \in {}_{\beta}E$, $\eta, \eta' \in {}_{\hat{\beta}}E$, $b \in B$ be homogeneous. Using the proof of Lemma 7.1 and the relation $\alpha(b)\langle\eta'|_1 W|\xi\rangle_2 = \langle\eta'|_1 W|\xi\rangle_2\alpha(b)$, we find that $(\hat{a}_{(\xi',\xi)}|\alpha(b)a_{(\eta',\eta)}) = [\xi'|\alpha(b)\langle\eta'|_1 W|\xi\rangle_2|\eta] = [\xi'|\langle\eta'|_1 W|\xi\rangle_2\alpha(b)|\eta] = (\hat{a}_{(\xi',\xi)}\alpha(b)|a_{(\eta',\eta)})$. The claim follows. □

The $\mathcal{L}(B)$-valued pairing $(\cdot|\cdot)$ yields a B-valued pairing $(\!(\cdot|\cdot)\!)$ as follows:

Corollary 7.3. *Assume that* B *is decomposable and let* $(u_\nu)_\nu$ *be an approximate unit of* $Z(B)$. *Then for all* $\hat{a} \in \hat{A}_a(W)$, $a \in A_a(W)$, *the limit* $(\!(\hat{a}|a)\!) := \lim_\nu (\hat{a}|a)u_\nu$ *exists and does not depend on the choice of* $(u_\nu)_\nu$. *The map* $(\!(\cdot|\cdot)\!)\colon \hat{A}_a(W) \times A_a(W) \to B$, $(\hat{a}, a) \mapsto (\!(\hat{a}|a)\!)$, *is bilinear and* $(\!(\hat{\mathcal{A}}_a(W)^{\rho}_{\sigma}|\mathcal{A}_a(W)^{\rho'}_{\sigma'})\!) \subseteq \mathcal{H}_{(\rho'\rho\sigma^*\sigma'^*)}(B)$ *for all* $\rho, \sigma, \rho', \sigma' \in \mathrm{PAut}(B)$.

Proof. Since $Z(B) \subseteq B$ is non-degenerate (Proposition 3.20 (v)), we have that $\lim_\nu (\hat{a}_{(\xi',\xi)}|a)u_\nu = \lim_\nu [\xi'|a|\xi\rangle u_\nu = \lim_\nu [\xi'|a\xi u_\nu = [\xi'|a\xi \in B$ for all homogeneous $\xi, \xi' \in {}_{\beta}E$ and $a \in A_a(W)$. The first claims follow. The last assertion follows from Proposition 7.2 (ii) and 3.14 (iv). □

Example 7.4. Consider the pseudo-multiplicative unitary W_G of a decomposable groupoid G (see Example 2.5 and Sections 4, 6). By Proposition 4.8 and 4.14 (and a partition of unity argument in the case of $A_a(W_G)$),

$$m(C_c(G)) \subseteq \hat{A}_a(W_G) \subseteq \overline{m(C_c(G))}^w, \quad L(C_c(G)) \subseteq A_a(W_G) \subseteq \overline{L(C_c(G))}^w.$$

Let $\eta, \eta' \in C_c(G)$ be homogeneous elements of ${}_sL^2(G,\lambda)$ and let $\xi, \xi' \in C_c(G)$. Then $\hat{a}_{(\xi',\xi)} = m(f)$ where $f = \overline{\xi'} \star \overline{\xi}^*$, and $a_{(\eta',\eta)} = L(g)$ where $g = \overline{\eta'}\eta$ (Proposition 4.8 and 4.14). We compute $T := (m(f)|L(g))$. By definition we have $T = (m(f)|a_{(\eta',\eta)}) = \langle \eta'|m(f)|\eta]$. Let $h \in C_0(G^0)$. Then $m(f)|\eta]h = m(f)s(h)\eta \in {}_sL^2(G,\lambda)$ is given by $x \mapsto f(x)h(s_G(x))\eta(x)$, and

$$(Th)(u) = \int_{G^u} \overline{\eta'}(x) f(x) h(s_G(x))\eta(x)\, d\lambda^u(x) = \int_{G^u} f(x)g(x)h(s_G(x))\, d\lambda^u(x)$$

for all $u \in G^0$. Thus we find: If $f, g \in C_c(G)$ and $T = (m(f)|L(g)) \in \mathcal{L}(C_0(G^0))$, then $(Th)(u) = \int_{G^u} f(x)g(x)h(s_G(x))\,d\lambda^u(x)$ for all $h \in C_0(G^0)$ and $u \in G^0$, and $(\!(m(f)|L(g))\!) \in C_0(G^0)$ is given by $u \mapsto \int_{G^u} f(x)g(x)\,d\lambda^u(x)$.

Fixed and cofixed multipliers. For (pseudo-) multiplicative unitaries on Hilbert spaces, fixed and cofixed elements were studied by Baaj and Skandalis [1], paragraphe 1, and later by Enock [4], Section 5. We carry over the definition and some of their results to the present situation. The discussion involves multipliers of C*-modules, which we briefly review.

Recall that E can be identified with $\mathcal{K}_B(B,E) \subseteq \mathcal{L}_B(B,E)$ via $\xi \leftrightarrow |\xi\rangle$, and that elements of $\mathcal{L}_B(B,E)$ are called *multipliers* of E. We extend the ket-bra notation to multipliers as follows. Let $S \in \mathcal{L}_B^B(B, {}_{\hat{\beta}}E)$. Consider the maps $S \otimes \mathrm{id}\colon B \otimes {}_\beta E \to E \otimes {}_\beta E$ and $S \otimes \mathrm{id}\colon B \otimes E \to E_{\hat{\beta}} \otimes E$ (see Proposition 1.1). Identifying $B \otimes {}_\beta E$ and $B \otimes E$ with E, we obtain maps $|S\rangle_1\colon E \to E \otimes {}_\beta E$ and $|S]_1\colon E \to E_{\hat{\beta}} \otimes E$. Similarly, we define for $T \in \mathcal{L}_B^B(B, {}_\beta E)$ maps $|T]_2\colon E \cong E \otimes B \to E \otimes {}_\beta E$ and $|T\rangle_2\colon E \cong E_{\hat{\beta}} \otimes B \to E_{\hat{\beta}} \otimes E$. Put $\langle S|_1 := |S\rangle_1^*$, $[S|_1 := |S]_1^*$, $[T|_2 := |T]_2^*$, $\langle T|_2 := |T\rangle_2^*$ and $S \otimes \xi := |S\rangle_1\xi$, $S \otimes \xi := |S]_1\xi$, $\eta \otimes T := |T]_2\eta$, $\eta \otimes T := |T\rangle_2\eta$ for all $\xi, \eta \in E$.

We extend $\beta, \hat{\beta}$ to the multiplier algebra $M(B)$ and denote the extensions by $\beta, \hat{\beta}$ again. Using the fact that $EB = E$ [9], Lemma 4.4, it is easy to see that for each $\zeta \in E$ and $T \in M(B)$, there exists a unique element $\zeta T \in E$ such that $(\zeta T)b = \zeta(Tb)$ for all $b \in B$.

Definition 7.5. Let us say that a multiplier $\eta_0 \in \mathcal{L}_B^B(B, {}_{\hat{\beta}}E)$ is *fixed by W* iff $W(\eta_0 \otimes \xi) = \eta_0 \otimes \xi$ for all $\xi \in E$, and that a multiplier $\xi_0 \in \mathcal{L}_B^B(B, {}_\beta E)$ is *cofixed by W* iff $W(\eta \otimes \xi_0) = \eta \otimes \xi_0$ for all $\eta \in E$. We denote the set of all fixed/cofixed multipliers by $\mathrm{Fix}(W)$/$\mathrm{Cofix}(W)$.

Remarks 7.6. (i) We speak of fixed and cofixed elements of E, identifying ξ with $|\xi\rangle$ for each $\xi \in E$. Note that by Proposition 3.12, $\mathcal{K}_B^B(B, {}_{\hat{\beta}}E) = |\mathcal{H}_{\mathrm{id}}({}_{\hat{\beta}}E)\rangle$ and $\mathcal{K}_B^B(B, {}_{\beta}E) = |\mathcal{H}_{\mathrm{id}}({}_{\beta}E)\rangle$.

(ii) If $\eta_0 \in E$ satisfies $W(\eta_0 \otimes \xi) = \eta_0 \otimes \xi$ for all $\xi \in E$, then automatically $\eta_0 \in \mathcal{H}_{\mathrm{id}}({}_{\hat{\beta}}E)$. For then $\eta_0 b \otimes \xi = (\eta_0 \otimes \xi)b = (W^*(\eta_0 \otimes \xi))b = W^*(\eta_0 \otimes \xi b) = \eta_0 \otimes \xi b = \hat{\beta}(b)\eta_0 \otimes \xi$ for all $\xi \in E$, $b \in B$, and since E is full, $\eta_0 \in \mathcal{H}_{\mathrm{id}}({}_{\hat{\beta}}E)$. Likewise, if $\xi_0 \in E$ satisfies $W(\eta \otimes \xi_0) = \eta \otimes \xi_0$ for all $\eta \in E$, then automatically $\xi_0 \in \mathcal{H}_{\mathrm{id}}({}_{\beta}E)$.

(iii) Clearly $\mathrm{Fix}(W) = \mathrm{Cofix}(W^{\mathrm{op}})$ and $\mathrm{Cofix}(W) = \mathrm{Fix}(W^{\mathrm{op}})$.

Lemma 7.7. (i) $\langle \eta_0'|_1 W |\eta_0]_1 \xi = \beta(\langle \eta_0'|\eta_0\rangle)\xi = \xi\langle \eta_0'|\eta_0\rangle$ *for all* $\eta_0, \eta_0' \in \mathrm{Fix}(W)$, $\xi \in E$.

(ii) $[\xi_0'|_2 W|\xi_0\rangle_2 \eta = \hat{\beta}(\langle \xi_0'|\xi_0\rangle)\eta = \eta\langle \xi_0'|\xi_0\rangle$ *for all* $\xi_0, \xi_0' \in \mathrm{Cofix}(W)$, $\eta \in E$.

Proof. We only prove assertion (i). Let $\eta_0, \eta_0' \in \mathrm{Fix}(W)$ and $\xi \in E$. Then we have $\langle \eta_0'|_1 W|\eta_0]_1\xi = \langle \eta_0'|_1 W(\eta_0 \otimes \xi) = \langle \eta_0'|_1(\eta_0 \otimes \xi) = \beta(\langle \eta_0'|\eta_0\rangle)\xi$ and $(\langle \eta_0'|_1 W|\eta_0]_1)^*\xi = [\eta_0|_1 W^*|\eta_0'\rangle_1\xi = [\eta_0|_1(\eta_0' \otimes \xi) = \xi\langle \eta_0|\eta_0'\rangle$. □

For $\gamma = \beta, \hat{\beta}$ put $Z({}_{\gamma}E) := \{T \in M(B) \mid \gamma(T)\xi = \xi T \text{ for all } \xi \in E\}$. Note that $Z({}_{\gamma}E) \subseteq Z(M(B))$ because $\langle \xi'|\xi\rangle TR = \langle \xi'|\xi T\rangle R = \langle \xi'|\gamma(T)\xi\rangle R = \langle \xi'|\gamma(T)\xi R\rangle = \langle \xi'|\xi R\rangle T = \langle \xi'|\xi\rangle RT$ for all $\xi', \xi \in E$, $R \in M(B)$, $T \in Z({}_{\gamma}E)$, and because E is full.

Proposition 7.8. (i) $\beta(M(B))\,\mathrm{Fix}(W) = \mathrm{Fix}(W)$*; furthermore, the space* $[\mathrm{Fix}(W)^*\,\mathrm{Fix}(W)] \subseteq M(B)$ *is a C*-subalgebra of* $Z({}_{\beta}E)$.

(ii) $\hat{\beta}(M(B))\,\mathrm{Cofix}(W) = \mathrm{Cofix}(W)$*, and* $[\mathrm{Cofix}(W)^*\,\mathrm{Cofix}(W)] \subseteq M(B)$ *is a C*-subalgebra of* $Z({}_{\hat{\beta}}E)$.

Proof. We only prove (i). For all $R \in M(B)$, $\eta_0 \in \mathrm{Fix}(W)$, $\xi \in E$ we have $\beta(R)\eta_0 \in \mathcal{L}_B^B(B, {}_{\hat{\beta}}E)$ and, by equation (1), $W(\beta(R)\eta_0 \otimes \xi) = W\beta_1(R)(\eta_0 \otimes \xi) = \beta_1(R)W(\eta_0 \otimes \xi) = \beta_1(R)(\eta_0 \otimes \xi) = \beta(R)\eta_0 \otimes \xi$. These relations show that $\beta(M(B))\,\mathrm{Fix}(W) \subseteq \mathrm{Fix}(W)$. By Lemma 7.7, $\mathrm{Fix}(W)^*\,\mathrm{Fix}(W) \subseteq Z({}_{\beta}E)$. Finally, $[\mathrm{Fix}(W)^*\,\mathrm{Fix}(W)]$ is a C*-algebra because $\mathrm{Fix}(W)\,\mathrm{Fix}(W)^*\,\mathrm{Fix}(W)$ is contained in $\mathrm{Fix}(W)Z({}_{\beta}E) \subseteq \beta(M(B))\,\mathrm{Fix}(W) = \mathrm{Fix}(W)$. □

Definition 7.9. We say that W is *étalé* iff $\langle \eta_0|\eta_0\rangle = \mathrm{id}_B$ for some $\eta_0 \in \mathrm{Fix}(W)$, *proper* iff $\langle \xi_0|\xi_0\rangle = \mathrm{id}_B$ for some $\xi_0 \in \mathrm{Cofix}(W)$, and *compact* iff it is proper and if B is unital.

Note that by Remark 7.6 (iii), W is proper/étalé iff W^{op} is étalé/proper.

Proposition 7.10. (i) *If* W *is proper, then* $\mathcal{O}({}_{\hat{\beta}}E) \subseteq \hat{\mathcal{A}}(W)$.

(ii) *If* W *is étalé, then* $\mathcal{O}({}_{\beta}E) \subseteq \mathcal{A}(W)$.

Proof. We only prove (i). Assume that $\xi_0 \in \mathrm{Cofix}(W)$ satisfies $\langle \xi_0|\xi_0\rangle = \mathrm{id}_B$; then $[\xi_0|_2 W|\xi_0\rangle_2 = \hat{\beta}(\langle\xi_0|\xi_0\rangle) = \mathrm{id}_E$ (Lemma 7.7). Let $b \in \mathcal{H}_\rho(B)$ and let $c \in \mathcal{H}_{\sigma^*}(B)$, $\rho, \sigma \in \mathrm{PAut}(B)$. Then $\xi_0 c^* \in \mathcal{H}_\sigma({}_{\hat{\beta}}E)$ and $\xi_0 b \in \mathcal{H}_\rho({}_{\hat{\beta}}E)$ by Proposition 3.14 (iv), and a similar calculation as in Lemma 4.3 shows that $o_{b,c} = \hat{\beta}(b)\alpha(c) = [\xi_0 c^*|_2 W|\xi_0 b\rangle_2 \in \hat{\mathcal{A}}(W)^\rho_\sigma$. □

If W is a multiplicative unitary, then the converse of the implications in Proposition 7.10 holds; see [1], Proposition 1.10.

Example: the pseudo-multiplicative unitary W_G. Let us consider the pseudo-multiplicative unitary W_G of a groupoid G (see Example 2.5 and Sections 4, 6) and determine the fixed and cofixed elements. We identify $M(L^2(G,\lambda))$ in the natural way with the completion of the space

$$\{f \in C(G) \mid r\colon \operatorname{supp} f \to G \text{ is proper, } \sup_{u\in G^0} \textstyle\int_{G^u} |f(x)|^2 d\lambda^u(x) \text{ is finite}\}$$

with respect to the norm $\|\cdot\|_{\infty,2}\colon f \mapsto \sup_{u\in G^0}(\int_{G^u}|f(x)|^2 d\lambda^u(x))^{1/2}$. Standard arguments and the relations $\eta(x)\xi(y) = (\eta \ominus \xi)(x,y)$ and $(W(\eta \otimes \xi))(x,y) = \eta(x)\xi(x^{-1}y)$, valid for all $(x,y) \in G^2_{r,r}$ and $\eta, \xi \in L^2(G,\lambda)$, show:

Lemma 7.11. (i) *A multiplier $\eta_0 \in M(L^2(G,\lambda))$ is fixed iff for each $u \in G^0$, $\eta_0|_{G^u\setminus\{u\}} = 0$ almost everywhere with respect to λ^u.*

(ii) *A multiplier $\xi_0 \in M(L^2(G,\lambda))$ is cofixed iff for each $u \in G^0$, $\xi_0|_{G^u} = \xi_0 \circ s_G|_{G^u}$ almost everywhere with respect to λ^u.*

Theorem 7.12. *W_G is étalé/proper/compact iff G is r-discrete/proper/compact.*

Proof. Assume that W_G is étalé, and that $\eta_0 \in \mathrm{Fix}(W_G)$ satisfies $\langle\eta_0|\eta_0\rangle = \mathrm{id}_B$. Define $f\colon G \to \mathbb{R}$ by $y \mapsto \int_{G^{r_G(y)}} \overline{\eta_0(x)}\eta_0(x^{-1}y)\, d\lambda^{r_G(y)}(x)$. Then f is continuous, $f|_{G^0} \equiv \eta^*\eta \equiv 1$ and $f|_{G\setminus G^0} = 0$ by Lemma 7.11. Therefore, $G^0 \subseteq G$ is open. Conversely, assume that G is r-discrete. Define $\eta_0\colon G \to [0,1]$ by $\eta_0|_{G^0} = 1$, $\eta_0|_{G\setminus G^0} = 0$. Then $\eta_0 \in M(L^2(G,\lambda))$ since η_0 is continuous, $\langle\eta_0|\eta_0\rangle = \mathrm{id}_B$, and $\eta_0 \in \mathrm{Fix}(W)$ by Lemma 7.11. Hence W_G is étalé.

Assume that W_G is proper and $\xi_0 \in \mathrm{Cofix}(W_G)$ satisfies $\langle\xi_0|\xi_0\rangle = \mathrm{id}_B$. Define $c\colon G^0 \to [0,\infty)$ by $u \mapsto \overline{\xi_0(u)}\xi_0(u)$. Then

$$\int_{G^u} c(s_G(x))d\lambda^u(x) = \int_{G^u} \overline{\xi_0(x)}\xi_0(x)\, d\lambda^u(x) = 1$$

for all $u \in G^0$ (see Lemma 7.11). By [19], Proposition 6.10, G is proper. Conversely, assume that G is proper. By [20], Proposition 6.11, there exists a continuous function $c\colon G^0 \to [0,\infty)$ such that the map $r\colon \operatorname{supp}(c \circ s) \to G^0$ is proper and $\int_{G^u} c(s_G(x))\, d\lambda^u(x) = 1$ for all $u \in G^0$. Define $\xi_0 \in M(L^2(G,\lambda))$ by

$x \mapsto c(s_G(x))^{1/2}$. By construction and by Lemma 7.11, $\xi_0 \in \mathrm{Cofix}(W_G)$ and $\langle \xi_0 | \xi_0 \rangle = \mathrm{id}_B$. Hence W_G is proper.

Finally, we conclude: G is compact $\Leftrightarrow$ G is proper and G^0 is compact $\Leftrightarrow$ W_G is proper and $C_0(G^0)$ is unital $\Leftrightarrow$ W_G is compact. □

Example: the pseudo-multiplicative unitary W_τ. Let us now consider the pseudo-multiplicative unitary W_τ associated to a center-valued conditional expectation $\tau \colon B \to C \subseteq Z(B)$, see Example 2.6 and Sections 4, 6.

Proposition 7.13. $\mathrm{Cofix}(W_\tau) = [B_\tau \ominus 1]$*, and* W_τ *is compact.*

Proof. Clearly $[B_\tau \ominus 1] \subseteq \mathrm{Cofix}(W_\tau)$, and W_τ is compact because $1 \ominus 1 \in \mathrm{Cofix}(W_\tau)$, $\langle 1 \ominus 1 | 1 \ominus 1 \rangle = 1$. Assume that $\xi_0 \in \mathrm{Cofix}(W_\tau)$. We consider the map $Y \colon E \ominus {}_\beta E \to B_\tau \ominus B_\tau \ominus B$ of Example 2.6. In $B_\tau \ominus B_\tau \ominus B$, we have that $1 \ominus \xi_0 = Y((1 \ominus 1) \ominus \xi_0) = YW_\tau((1 \ominus 1) \oslash \xi_0)$ and $YW_\tau((1 \ominus 1) \oslash (c \ominus d)) = Y((d \ominus 1) \ominus (c \ominus 1)) = (d \ominus c \ominus 1) \in B_\tau \ominus B_\tau \ominus 1$ for all $c, d \in B$. An application of the map $\langle 1| \ominus \mathrm{id} \ominus \mathrm{id} \colon B_\tau \ominus B_\tau \ominus B \to B_\tau \ominus B$ shows that $\xi_0 \in [B_\tau \ominus 1]$. □

Recall that a *quasi-basis* for τ is a finite set of elements $(u_i)_i$ of B satisfying $\sum_i \tau(bu_i)u_i^* = b$ for all $b \in B$, and that τ is said to be of *index-finite type* iff there exists a quasi-basis for τ. Moreover, if τ is of index-finite type with a quasi-basis $(u_i)_i$, then the element $\mathrm{Index}(\tau) := \sum u_i u_i^* \in B$ is central, invertible and independent of the choice of $(u_i)_i$. For details, see, e.g., [22].

Lemma 7.14. *If* $(u_i)_i$ *is a quasi-basis for* τ*, then* $\sum_i u_i \ominus u_i^* \in \mathcal{H}_{\mathrm{id}}({}_{\hat\beta}E)$.

Proof. $\sum_i u_i \ominus u_i^* \in \mathcal{H}_{\mathrm{id}}({}_{\hat\beta}E)$ since $\langle c \ominus d | \sum_i u_i \ominus u_i^* b\rangle = \sum_i d^* \tau(c^* u_i) u_i^* b = d^* c^* b = \sum_i d^* \tau(c^* b u_i) u_i^* = \langle c \ominus d | \sum_i b u_i \ominus u_i^* \rangle$ for all $b, c, d \in B$. □

Proposition 7.15. $\mathrm{Fix}(W_\tau) = \mathcal{H}_{\mathrm{id}}({}_{\hat\beta}E)$*, and if* τ *is of index-finite type, then* W_τ *is étalé.*

Proof. If $\eta_0 \in \mathcal{H}_{\mathrm{id}}({}_{\hat\beta}E)$, then $W_\tau(\eta_0 \oslash (c \ominus d)) = \hat\beta(d)\eta_0 \ominus (c \ominus 1) = \eta_0 d \ominus (c \ominus 1) = \eta_0 \ominus \beta(d)(c \ominus 1) = \eta_0 \ominus (c \ominus d)$ for all $c, d \in B$, whence $\eta_0 \in \mathrm{Fix}(W_\tau)$. If τ has a quasi-basis $(u_i)_i$, then $\eta_0 := \sum_i u_i \ominus u_i^* \,\mathrm{Index}(\tau)^{-1/2}$ satisfies $\eta_0 \in \mathcal{H}_{\mathrm{id}}({}_{\hat\beta}E) = \mathrm{Fix}(W)$ because $\mathrm{Index}(\tau)$ is central and by Lemma 7.14, and $\langle \eta_0 | \eta_0 \rangle = \sum_{i,j} u_i \tau(u_i^* u_j) u_j^* \,\mathrm{Index}(\tau)^{-1} = \sum_i u_i u_i^* \,\mathrm{Index}(\tau)^{-1} = 1$. □

The counits on the legs. Let us return to the legs of a pseudo-multiplicative unitary $W \colon E_{\hat\beta} \ominus E \to E \ominus_\beta E$. As before, we denote by $\hat{\mathcal{B}}$ and $\mathcal{B}$ the C*-families generated by $\hat{\mathcal{A}}(W)$ and $\mathcal{A}(W)$, respectively.

If id_E belongs to $\overline{A_a(W)}^w$, then we can define a linear map $\hat{\epsilon}_a \colon \hat{A}_a(W) \to \mathcal{L}(B)$, $\hat{a} \mapsto (\hat{a}|\mathrm{id}_E)^w$, which should be considered as the counit on the left leg of W. In this case, $\hat{\epsilon}_a(\hat{\mathcal{A}}_a(W)^\rho_\sigma) \subseteq \mathcal{L}^\rho_\sigma(B)$ and $\hat{\epsilon}_a(\hat{a}_{(\xi',\xi)}) = [\xi'||\xi\rangle$ for all $\rho, \sigma \in \mathrm{PAut}(B)$ and all homogeneous $\xi, \xi' \in {}_\beta E$; see Proposition 7.2. Similarly, if id_E belongs to $\overline{\hat{A}_a(W)}^w$, then we can define a "counit" $\epsilon_a \colon A_a(W) \to \mathcal{L}(B)$, $a \mapsto {}^w(\mathrm{id}_E|a)$, on the right leg of W.

Theorem 7.16. (i) *Assume that W is étalé. Then there exists a morphism $\hat{\epsilon} \colon \hat{\mathcal{B}} \to \mathcal{L}(B)$ such that $\hat{\epsilon}^\rho_\sigma(\hat{a}) = \hat{\epsilon}_a(\hat{a})$ for all $\hat{a} \in \hat{\mathcal{A}}_a(W)^\rho_\sigma$, $\rho, \sigma \in \mathrm{PAut}(B)$. If ${}_\beta E$ is decomposable, then $\hat{\epsilon}$ is non-degenerate. If additionally $\hat{\Delta}(\hat{\mathcal{B}}) \subseteq \mathcal{M}(\hat{\mathcal{B}} \otimes \hat{\mathcal{B}})$ and if we identify $B \otimes E \cong E \cong E_{\hat{\beta}} \otimes B$, then $(\hat{\epsilon} \otimes \mathrm{id}) \circ \hat{\Delta} = \mathrm{id} = (\mathrm{id} \otimes \hat{\epsilon}) \circ \hat{\Delta}$.*

(ii) *Assume that W is proper. Then there exists a morphism $\epsilon \colon \mathcal{B} \to \mathcal{L}(B)$ such that $\epsilon^\rho_\sigma(a) = \epsilon_a(a)$ for all $a \in \mathcal{A}_a(W)^\rho_\sigma$, $\rho, \sigma \in \mathrm{PAut}(B)$. If ${}_{\hat{\beta}}E$ is decomposable, then ϵ is non-degenerate. If additionally $\Delta(\mathcal{B}) \subseteq \mathcal{M}(\mathcal{B} \otimes \mathcal{B})$ and if we identify $B \otimes {}_\beta E \cong E \cong E \otimes B$, then $(\epsilon \otimes \mathrm{id}) \circ \Delta = \mathrm{id} = (\mathrm{id} \otimes \epsilon) \circ \Delta$.*

Proof. We only prove (i). Choose $\eta_0 \in \mathrm{Fix}(W)$ such that $\langle \eta_0|\eta_0\rangle = \mathrm{id}_B$, and define $\hat{\epsilon} \colon \hat{\mathcal{B}} \to \mathcal{L}(B)$ by $\hat{\epsilon}^\rho_\sigma(\hat{a}) = \langle \eta_0|\hat{a}|\eta_0\rangle$ for all $\hat{a} \in \hat{\mathcal{B}}^\rho_\sigma$, $\rho, \sigma \in \mathrm{PAut}(B)$. Let $\xi, \xi' \in {}_\beta E$ be homogeneous. Then $\hat{\epsilon}(\hat{a}_{(\xi',\xi)}) = \langle \eta_0|[\xi'|_2 W|\xi\rangle_2|\eta_0\rangle = \langle \eta_0|[\xi'|_2|\eta_0 \otimes \xi\rangle = [\xi'||\xi\rangle = \hat{\epsilon}_a(\hat{a}_{(\xi',\xi)})$ and evidently $\hat{\epsilon}(\hat{a}_{(\xi',\xi)})^* = \hat{\epsilon}(\hat{a}^*_{(\xi',\xi)})$. To prove that $\hat{\epsilon}$ is a morphism of C*-families, it is enough to show that $\hat{\epsilon}(\hat{a}_{(\xi',\xi)}\hat{a}_{(\zeta',\zeta)}) = \hat{\epsilon}(\hat{a}_{(\xi',\xi)})\hat{\epsilon}(\hat{a}_{(\zeta',\zeta)})$ for all homogeneous $\zeta, \zeta' \in {}_\beta E$. By the proof of Proposition 4.6, $\hat{\epsilon}(\hat{a}_{(\xi',\xi)}\hat{a}_{(\zeta',\zeta)}) = \langle \eta_0|\hat{a}_{(\xi',\xi)}\hat{a}_{(\zeta',\zeta)}|\eta\rangle_0$ is equal to

$$B \xrightarrow{|\eta_0\rangle} E \xrightarrow{|\xi \otimes \zeta\rangle_2} E_{\hat{\beta}} \otimes (E \otimes {}_\beta E) \xrightarrow{W_{23}W_{12}W_{23}^*} E \otimes {}_\beta E \otimes {}_\beta E \xrightarrow{[\xi' \otimes \zeta'|_2} E \xrightarrow{\langle \eta_0|} B.$$

Hence, $\hat{\epsilon}(\hat{a}_{(\xi',\xi)}\hat{a}_{(\zeta',\zeta)})^*$ and $\hat{\epsilon}(\hat{a}_{(\zeta',\zeta)})^*\hat{\epsilon}(\hat{a}_{(\xi',\xi)})^*$ act on each $b \in B$ by

$$\begin{aligned} b &\xmapsto{|\xi' \otimes \zeta']_2|\eta_0\rangle} \eta_0 b \otimes \xi' \otimes \zeta' = \eta_0 \otimes b\xi' \otimes \zeta' \\ &\xmapsto{W_{23}W_{12}^*W_{23}^*} \eta_0 \otimes (b\xi' \otimes \zeta') \xmapsto{\langle \eta_0|\langle \xi \otimes \zeta|_2} \langle \xi \otimes \zeta|b\xi' \otimes \zeta'\rangle \end{aligned}$$

and by

$$b \xrightarrow{\langle \xi||\xi']} \langle \xi|b\xi'\rangle \xrightarrow{\langle \zeta||\zeta']} \langle \zeta|\langle \xi|b\xi'\rangle\zeta'\rangle = \langle \xi \otimes \zeta|b\xi' \otimes \zeta'\rangle,$$

respectively (use the assumptions on η_0); so $\hat{\epsilon}(\hat{a}_{(\xi',\xi)}\hat{a}_{(\zeta',\zeta)})^* = \hat{\epsilon}(\hat{a}_{(\zeta',\zeta)})^*\hat{\epsilon}(\hat{a}_{(\xi',\xi)})^*$.

Assume that ${}_\beta E$ is decomposable. Since $\hat{\epsilon}(\hat{a}_{(\xi',\xi)})^*b = \langle \xi|b\xi'\rangle$ for all homogeneous $\xi, \xi' \in {}_\beta E$ and all $b \in B$, and since E is full and β non-degenerate, we have $[\hat{\epsilon}(\hat{\mathcal{B}})B] = B$.

Let us prove the last assertion in (i). Since $\hat{\Delta}(\hat{a}_{(\xi',\xi)}) = [\xi'|_3 W_{13} W_{23} |\xi\rangle_3$ (Lemma 6.1), $(\mathrm{id} \odot \hat{\epsilon})(\hat{\Delta}(\hat{a}_{(\xi',\xi)}))$ and $(\hat{\epsilon} \odot \mathrm{id})(\hat{\Delta}(\hat{a}_{(\xi',\xi)})^*)$ act as follows:

$$\zeta \overset{|\eta_0\rangle_2}{\longmapsto} \zeta \odot \eta_0 \overset{|\xi\rangle_3}{\longmapsto} \zeta \odot \eta_0 \odot \xi \overset{W_{23}}{\longmapsto} \zeta \odot (\eta_0 \odot \xi) \overset{[\xi'|_3 W_{13}}{\longmapsto} \hat{a}_{(\xi',\xi)}\zeta \odot \eta_0 \overset{\langle \eta_0|_2}{\longmapsto} \hat{a}_{(\xi',\xi)}\zeta$$

and

$$\zeta \overset{|\eta_0]_1}{\longmapsto} \eta_0 \odot \zeta \overset{|\xi]_3}{\longmapsto} (\eta_0 \odot \zeta) \odot \xi \overset{W_{13}^*}{\longmapsto} \eta_0 \odot (\zeta \odot \xi) \overset{\langle \xi|_3 W_{23}^*}{\longmapsto} \eta_0 \odot \hat{a}^*_{(\xi',\xi)}\zeta \overset{[\eta_0|_1}{\longmapsto} \hat{a}^*_{(\xi',\xi)}\zeta$$

for all $\zeta \in E$, respectively. □

Example 7.17. Let us consider the pseudo-multiplicative unitary W_G of a decomposable groupoid G (see Example 2.5 and Sections 4, 6), and determine the counits on its legs.

Let $\xi, \xi' \in C_c(G)$. Then $\hat{a}_{(\xi',\xi)} = m(\bar{\xi'} \star \bar{\xi}^*)$ by Proposition 4.8, and

$$(\hat{\epsilon}_a(\hat{a}_{(\xi',\xi)})h)(u) = \int_{G^u} \overline{\xi'(x)} h(r_G(x))\xi(x)\, d\lambda^u(x) = (\bar{\xi'} \star \bar{\xi}^*)(u)h(u)$$

for all $h \in C_0(G^0)$, $u \in G^0$. If G is r-discrete, Theorem 7.16 applies and $\hat{\epsilon}_a$ extends to a morphism of C*-families (see Theorem 7.12).

Let $\eta, \eta' \in C_c(G) \subseteq {}_sL^2(G,\lambda)$ be homogeneous. Then $a_{(\eta',\eta)} = L(\bar{\eta'}\eta)$ by Proposition 4.14, and

$$(\epsilon_a(a_{(\eta',\eta)})h)(u) = \int_{G^u} (\bar{\eta'}\eta)(x)h(s_G(x))d\lambda^u(x)$$

for all $h \in C_0(G^0)$, $u \in G^0$. If G is proper, then Theorem 7.16 applies and ϵ_a extends to a morphism of C*-families; see Theorem 7.12.

Example 7.18. Let us consider the pseudo-multiplicative unitary W_τ of a center-valued conditional expectation τ, see Example 2.6 and Section 4, and determine the counit on its left leg. Recall from (the proof of) Proposition 4.17 that $\hat{\mathcal{A}}_a(W_\tau)^\rho_\sigma = \mathcal{O}^\rho_\sigma({}_{\hat{\beta}}E)$ for all $\rho, \sigma \in \mathrm{PAut}(B)$.

We shall need to distinguish the operators $o_{d,d''} \in \mathcal{O}^\rho_\sigma({}_\beta E)$ and $o_{d,d''} \in \mathcal{O}^\rho_\sigma(B)$, where $\rho, \sigma \in \mathrm{PAut}(B)$, $d \in \mathcal{H}_\rho(B)$, $d'' \in \mathcal{H}_{\sigma^*}(B)$, and therefore adorn them by upper indices E or B, respectively.

We claim that $\hat{\epsilon}_a(o^E_{d,d''}) = o^B_{d,d''}$ for all homogeneous $d, d'' \in B$. Indeed, by Lemma 4.16, $o^E_{d,d''} = \hat{a}_{(\xi',\xi)}$ for $\xi := 1 \odot d$, $\xi' := 1 \odot d''^*$, and by definition, $\hat{\epsilon}_a(o^E_{d,d''})^*c = \hat{\epsilon}_a(\hat{a}_{(\xi',\xi)})^*c = \langle\xi||\xi']c = \langle\xi|c\xi'\rangle = \langle 1 \odot d|1 \odot cd''^*\rangle = d^*\tau(1)cd''^* = (o^B_{d,d''})^*c$ for all $c \in B$.

If τ is of index-finite type, then Theorem 7.16 applies and $\hat{\epsilon}_a$ extends to a morphism $\hat{\epsilon}\colon \mathcal{O}({}_\beta E) \to \mathcal{O}(B)$; see Proposition 7.15.

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