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Spectral action on noncommutative torus

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Dedicated to Alain Connes on the occasion of his 60th birthday

Abstract. The spectral action on the noncommutative torus is obtained using a Chamseddine–Connes formula via computations of zeta functions. The importance of a Diophantine condition is outlined. Several results on holomorphic continuation of series of holomorphic functions are obtained in this context.

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Contents

1. Introduction

The spectral action introduced by Chamseddine–Connes plays an important role [3] in noncommutative geometry. More precisely, given a spectral triple $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ where $\mathcal{A}$ is an algebra acting on the Hilbert space $\mathcal{H}$ and $\mathcal{D}$ is a Dirac-like operator (see [7], [23]), Chamseddine and Connes proposed a physical action depending only on the spectrum of the covariant Dirac operator

$$\mathcal{D}_A := \mathcal{D} + A + \epsilon JAJ^{-1}, \tag{1.1}$$

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where A is a 1-form represented on $\mathcal{H}$, so has the decomposition

$$A = \sum_i a_i[\mathcal{D}, b_i], \tag{1.2}$$

with $a_i, b_i \in \mathcal{A}$, J is a real structure on the triple corresponding to charge conjugation and $\epsilon \in \{1, -1\}$ depends on the dimension of this triple and comes from the commutation relation

$$J\mathcal{D} = \epsilon \mathcal{D} J. \tag{1.3}$$

This action is defined by

$$\mathcal{S}(\mathcal{D}_A, \Phi, \Lambda) := \mathrm{Tr}(\Phi(\mathcal{D}_A/\Lambda)), \tag{1.4}$$

where Φ is any even positive cut-off function which could be replaced by a step function up to some mathematical difficulties investigated in [16]. This means that $\mathcal{S}$ counts the spectral values of $|\mathcal{D}_A|$ that are less than the mass scale Λ (note that the resolvent of $\mathcal{D}_A$ is compact since, by assumption, the same is true for $\mathcal{D}$; see Lemma 3.1 below).

In [18], the spectral action on NC-tori has been computed only for operators of the form $\mathcal{D} + A$ and for $\mathcal{D}_A$ in [20]. It appears that the implementation of the real structure via J does change the spectral action up to a coefficient when the torus has dimension 4. Here we prove that this can be also directly obtained from the Chamseddine–Connes analysis in [11] that we follow quite closely. Actually,

$$\mathcal{S}(\mathcal{D}_A, \Phi, \Lambda) = \sum_{0<k\in \mathrm{Sd}^+} \Phi_k \Lambda^k \fint |D_A|^{-k} + \Phi(0)\zeta_{D_A}(0) + \mathcal{O}(\Lambda^{-1}), \tag{1.5}$$

where $D_A = \mathcal{D}_A + P_A$, P_A is the projection on $\mathrm{Ker}\,\mathcal{D}_A$, $\Phi_k = \frac{1}{2}\int_0^\infty \Phi(t)t^{k/2-1}dt$ and Sd^+ is the strictly positive part of the dimension spectrum of $(\mathcal{A}, \mathcal{H}, \mathcal{D})$. As we will see, $\mathrm{Sd}^+ = \{1, 2, \ldots, n\}$ and $\fint |D_A|^{-n} = \fint |D|^{-n}$. Moreover, the coefficient $\zeta_{D_A}(0)$ related to the constant term in (1.5) can be computed from the unperturbed spectral action since it has been proved in [11] (with an invertible Dirac operator and a 1-form A such that $\mathcal{D} + A$ is also invertible) that

$$\zeta_{\mathcal{D}+A}(0) - \zeta_{\mathcal{D}}(0) = \sum_{q=1}^{n} \frac{(-1)^q}{q} \fint (A\mathcal{D}^{-1})^q, \tag{1.6}$$

using $\zeta_X(s) = \mathrm{Tr}(|X|^{-s})$. We will see how this formula can be extended to the case of a noninvertible Dirac operator and noninvertible perturbation of the form $\mathcal{D} + \tilde{A}$ with $\tilde{A} := A + \varepsilon JAJ^{-1}$.

All these results on spectral action are quite important in physics, especially in quantum field theory and particle physics, where one adds to the effective action some

counterterms explicitly given by (1.6); see for instance [2], [3], [4], [11], [17], [18], [20], [22], [28], [34], [36], [37], [38].

Since the computation of zeta functions is crucial in the following, we investigate in Section 2 residues of series and integrals. This section contains results of independent interest on the holomorphy of series of holomorphic functions. In particular, the necessity of a Diophantine constraint is naturally emphasized.

In Section 3 we revisit the notions of pseudodifferential operators and their associated zeta functions and of dimension spectrum. The reality operator J is incorporated, and we pay a particular attention to kernels of operators that can play a role in the constant term of (1.5). This section concerns general spectral triples with simple dimension spectrum.

Section 4 is devoted to the example of the noncommutative torus. It is shown that it has a vanishing tadpole.

In Section 5 all previous technical points are then widely used for the computation of terms in (1.5) or (1.6).

Finally, the spectral action (1.6) is obtained in Section 6, and we conjecture that the noncommutative spectral action of $\mathcal{D}_A$ has terms proportional to the spectral action of $\mathcal{D} + A$ on the commutative torus.

2. Residues of series and integral, holomorphic continuation, etc.

Notations. In the following the prime in $\sum'$ means that we omit terms with division by zero in the summand. B^n (resp. S^{n-1}) is the closed ball (resp. the sphere) of $\mathbb{R}^n$ with center 0 and radius 1. The Lebesgue measure on S^{n-1} will be noted by dS.

For any $x = (x_1, \dots, x_n) \in \mathbb{R}^n$ we denote by $|x| = \sqrt{x_1^2 + \cdots + x_n^2}$ the Euclidean norm and $|x|_1 := |x_1| + \cdots + |x_n|$.

$\mathbb{N} = \{1, 2, \dots\}$ is the set of positive integers and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$ the set of non-negative integers.

By $f(x, y) \ll_y g(x)$ uniformly in x, we mean that, for some $a(y) > 0$, $|f(x, y)| \le a(y)|g(x)|$ for all x and y.

2.1. Residues of series and integral. In order to be able to compute later the residues of certain series, we now prove the following result.

Theorem 2.1. *Let $P(X) = \sum_{j=0}^{d} P_j(X) \in \mathbb{C}[X_1, \dots, X_n]$ be a polynomial function where P_j is the homogeneous part of P of degree j. The function*

$$\zeta^P(s) := \sum_{k \in \mathbb{Z}^n}{}' \frac{P(k)}{|k|^s}, \quad s \in \mathbb{C},$$

has a meromorphic continuation to the whole complex plane $\mathbb{C}$.

Also $\zeta^P(s)$ is not entire if and only if $\mathcal{P}_P := \{j \mid \int_{u\in S^{n-1}} P_j(u)dS(u) \neq 0\} \neq \varnothing$. In that case ζ^P has only simple poles at the points $j+n$, $j \in \mathcal{P}_P$, with

$$\operatorname*{Res}_{s=j+n} \zeta^P(s) = \int_{u\in S^{n-1}} P_j(u)\, dS(u).$$

The proof of this theorem is based on the following lemmas.

Lemma 2.2. *For any polynomial $P \in \mathbb{C}[X_1,\dots,X_n]$ of total degree $\delta(P) := \sum_{i=1}^n \deg_{X_i} P$ and any $\alpha \in \mathbb{N}_0^n$, we have*

$$\partial^\alpha(P(x)|x|^{-s}) \ll_{P,\alpha,n} (1+|s|)^{|\alpha|_1}|x|^{-\sigma-|\alpha|_1+\delta(P)}$$

uniformly in $x \in \mathbb{R}^n$, $|x| \geq 1$, where $\sigma = \mathrm{Re}(s)$.

Proof. By linearity, we may assume without loss of generality that $P(X) = X^\gamma$ is a monomial. It is easy to prove (for example by induction on $|\alpha|_1$) that for all $\alpha \in \mathbb{N}_0^n$ and $x \in \mathbb{R}^n \setminus \{0\}$:

$$\partial^\alpha(|x|^{-s}) = \alpha! \sum_{\substack{\beta,\mu\in\mathbb{N}_0^n\\ \beta+2\mu=\alpha}} \binom{-s/2}{|\beta|_1+|\mu|_1} \frac{(|\beta|_1+|\mu|_1)!}{\beta!\,\mu!} \frac{x^\beta}{|x|^{\sigma+2(|\beta|_1+|\mu|_1)}}.$$

It follows that for all $\alpha \in \mathbb{N}_0^n$, we have uniformly in $x \in \mathbb{R}^n$, $|x| \geq 1$,

$$\partial^\alpha(|x|^{-s}) \ll_{\alpha,n} (1+|s|)^{|\alpha|_1}|x|^{-\sigma-|\alpha|_1}. \tag{2.1}$$

By the Leibniz formula and (2.1), we have uniformly in $x \in \mathbb{R}^n$, $|x| \geq 1$,

$$\begin{aligned}\partial^\alpha(x^\gamma|x|^{-s}) &= \sum_{\beta\leq\alpha} \binom{\alpha}{\beta} \partial^\beta(x^\gamma)\,\partial^{\alpha-\beta}(|x|^{-s})\\ &\ll_{\gamma,\alpha,n} \sum_{\beta\leq\alpha;\beta\leq\gamma} x^{\gamma-\beta}(1+|s|)^{|\alpha|_1-|\beta|_1}|x|^{-\sigma-|\alpha|_1+|\beta|_1}\\ &\ll_{\gamma,\alpha,n} (1+|s|)^{|\alpha|_1}|x|^{-\sigma-|\alpha|_1+|\gamma|_1}.\end{aligned}$$

□

Lemma 2.3. *Let $P \in \mathbb{C}[X_1,\dots,X_n]$ be a polynomial of degree d. Then the difference*

$$\Delta_P(s) := \sum_{k\in\mathbb{Z}^n}{}' \frac{P(k)}{|k|^s} - \int_{\mathbb{R}^n\setminus B^n} \frac{P(x)}{|x|^s}\, dx,$$

which is defined for $\mathrm{Re}(s) > d+n$, extends holomorphically to the whole complex plane $\mathbb{C}$.

Proof. We fix in the sequel a function $\psi \in C^\infty(\mathbb{R}^n, \mathbb{R})$ such that for all $x \in \mathbb{R}^n$ we have

$$0 \le \psi(x) \le 1, \quad \psi(x) = 1 \text{ if } |x| \ge 1 \quad \text{and} \quad \psi(x) = 0 \text{ if } |x| \le 1/2.$$

The function $f(x,s) := \psi(x)P(x)|x|^{-s}$, $x \in \mathbb{R}^n$ and $s \in \mathbb{C}$, is in $\mathcal{C}^\infty(\mathbb{R}^n \times \mathbb{C})$ and depends holomorphically on s.

Lemma 2.2 above shows that f is a "gauged symbol" in the terminology of [24], p. 4. Thus [24], Theorem 2.1, implies that $\Delta_P(s)$ extends holomorphically to the whole complex plane $\mathbb{C}$. However, for completeness, we will give here a short proof of Lemma 2.3:

It follows from the classical Euler–Maclaurin formula that for any function $h\colon \mathbb{R} \to \mathbb{C}$ of class $\mathcal{C}^{N+1}$ satisfying $\lim_{|t|\to+\infty} h^{(k)}(t) = 0$ and $\int_{\mathbb{R}} |h^{(k)}(t)|\,dt < +\infty$ for any $k = 0, \dots, N+1$, we have

$$\sum_{k\in\mathbb{Z}} h(k) = \int_{\mathbb{R}} h(t) + \frac{(-1)^N}{(N+1)!} \int_{\mathbb{R}} B_{N+1}(t)\, h^{(N+1)}(t)\,dt,$$

where B_{N+1} is the Bernoulli function of order $N+1$ (it is a bounded periodic function).

Fix $m' \in \mathbb{Z}^{n-1}$ and $s \in \mathbb{C}$. Applying this to the function

$$h(t) := \psi(m',t)P(m',t)|(m',t)|^{-s}$$

(we use Lemma 2.2 to verify the hypothesis), we obtain that for any $N \in \mathbb{N}_0$,

$$\begin{aligned} &\sum_{m_n\in\mathbb{Z}} \psi(m',m_n)P(m',m_n)|(m',m_n)|^{-s} \\ &\qquad = \int_{\mathbb{R}} \psi(m',t)P(m',t)|(m',t)|^{-s}dt + \mathcal{R}_N(m';s) \end{aligned} \tag{2.2}$$

where $\mathcal{R}_N(m';s) := \frac{(-1)^N}{(N+1)!}\int_{\mathbb{R}} B_{N+1}(t)\frac{\partial^{N+1}}{\partial {x_n}^{N+1}}(\psi(m',t)P(m',t)|(m',t)|^{-s})\,dt$.

By Lemma 2.2,

$$\begin{aligned} &\int_{\mathbb{R}} \Big| B_{N+1}(t)\frac{\partial^{N+1}}{\partial {x_n}^{N+1}}(\psi(m',t)P(m',t)|(m',t)|^{-s})\Big|\,dt \\ &\qquad \ll_{P,n,N} (1+|s|)^{N+1}(|m'|+1)^{-\sigma-N+\delta(P)}. \end{aligned}$$

Thus $\sum_{m'\in\mathbb{Z}^{n-1}} \mathcal{R}_N(m';s)$ converges absolutely and defines a holomorphic function in the half plane $\{\sigma = \mathrm{Re}(s) > \delta(P) + n - N\}$.

Since N is an arbitrary integer, by letting $N \to \infty$ and using (2.2) above, we conclude that

$$s \mapsto \sum_{(m',m_n)\in\mathbb{Z}^{n-1}\times\mathbb{Z}} \psi(m',m_n)P(m',m_n)|(m',m_n)|^{-s} - \sum_{m'\in\mathbb{Z}^{n-1}} \int_{\mathbb{R}} \psi(m',t)P(m',t)|(m',t)|^{-s}dt$$

has a holomorphic continuation to the whole complex plane $\mathbb{C}$.

After n iterations, we obtain that

$$s \mapsto \sum\nolimits_{m\in\mathbb{Z}^n} \psi(m)P(m)|m|^{-s} - \int_{\mathbb{R}^n} \psi(x)P(x)|x|^{-s}dx$$

has a holomorphic continuation to the whole of $\mathbb{C}$.

To finish the proof of Lemma 2.3, it is enough to notice that

- $\psi(0) = 0$ and $\psi(m) = 1$ for all $m \in \mathbb{Z}^n \setminus \{0\}$;
- $s \mapsto \int_{B^n} \psi(x)P(x)|x|^{-s}dx = \int_{\{x\in\mathbb{R}^n|1/2\le|x|\le 1\}} \psi(x)P(x)|x|^{-s}dx$ is a holomorphic function in $\mathbb{C}$. □

Proof of Theorem 2.1. Using the polar decomposition of the volume form $dx = \rho^{n-1}d\rho\, dS$ in $\mathbb{R}^n$, we obtain for $\mathrm{Re}(s) > d + n$ that

$$\int_{\mathbb{R}^n\setminus B^n} \frac{P_j(x)}{|x|^s}dx = \int_1^\infty \frac{\rho^{j+n-1}}{\rho^s} \int_{S^{n-1}} P_j(u)\, dS(u) = \frac{1}{j+n-s} \int_{S^{n-1}} P_j(u)\, dS(u).$$

Lemma 2.3 now gives the result. □

2.2. Holomorphy of certain series. Before stating the main result of this section, we give first in the following some preliminaries from Diophantine approximation theory.

Definition 2.4. (i) Let $\delta > 0$. A vector $a \in \mathbb{R}^n$ is said to be δ-diophantine if there exists $c > 0$ such that $|q \cdot a - m| \ge c|q|^{-\delta}$ for all $q \in \mathbb{Z}^n \setminus \{0\}$ and for all $m \in \mathbb{Z}$.

We denote $\mathcal{BV}(\delta)$ the set of δ-diophantine vectors and $\mathcal{BV} := \bigcup_{\delta>0} \mathcal{BV}(\delta)$ the set of diophantine vectors.

(ii) A matrix $\Theta \in \mathcal{M}_n(\mathbb{R})$ (real $n \times n$ matrices) is said to be diophantine if there exists $u \in \mathbb{Z}^n$ such that ${}^t\Theta(u)$ is a diophantine vector of $\mathbb{R}^n$.

Remark 2.5. A classical result from Diophantine approximation asserts that for all $\delta > n$, the Lebesgue measure of $\mathbb{R}^n \setminus \mathcal{BV}(\delta)$ is zero (i.e., almost any element of $\mathbb{R}^n$ is δ-diophantine).

Let $\Theta \in \mathcal{M}_n(\mathbb{R})$. If its row of index i is a diophantine vector of $\mathbb{R}^n$ (i.e., if $L_i \in \mathcal{BV}$) then ${}^t\Theta(e_i) \in \mathcal{BV}$ and thus Θ is a diophantine matrix. It follows that almost any matrix of $\mathcal{M}_n(\mathbb{R}) \approx \mathbb{R}^{n^2}$ is diophantine.

The goal of this section is to show the following result.

Theorem 2.6. *Let $P \in \mathbb{C}[X_1, \dots, X_n]$ be a homogeneous polynomial of degree d and let b be in $\mathcal{S}(\mathbb{Z}^n \times \cdots \times \mathbb{Z}^n)$ (q times, $q \in \mathbb{N}$). Then the following holds:*

(i) *Let $a \in \mathbb{R}^n$. We define $f_a(s) := \sum'_{k \in \mathbb{Z}^n} \frac{P(k)}{|k|^s} e^{2\pi i k \cdot a}$.*

1. *If $a \in \mathbb{Z}^n$, then f_a has a meromorphic continuation to the whole complex plane $\mathbb{C}$.*
   Moreover if S is the unit sphere and dS its Lebesgue measure, then f_a is not entire if and only if $\int_{u \in S^{n-1}} P(u) dS(u) \neq 0$. In that case, f_a has only a simple pole at the point $d+n$ with $\operatorname*{Res}_{s=d+n} f_a(s) = \int_{u \in S^{n-1}} P(u) dS(u)$.
2. *If $a \in \mathbb{R}^n \setminus \mathbb{Z}^n$, then $f_a(s)$ extends holomorphically to the whole complex plane $\mathbb{C}$.*

(ii) *Suppose that $\Theta \in \mathcal{M}_n(\mathbb{R})$ is diophantine. For any $(\varepsilon_i)_i \in \{-1, 0, 1\}^q$, the function*

$$g(s) := \sum_{l \in (\mathbb{Z}^n)^q} b(l) f_{\Theta \sum_i \varepsilon_i l_i}(s)$$

extends meromorphically to the whole complex plane $\mathbb{C}$ with only one possible pole on $s = d + n$.

Moreover, if we set $\mathcal{Z} := \{l \in (\mathbb{Z}^n)^q \mid \sum_{i=1}^q \varepsilon_i l_i = 0\}$ and $V := \sum_{l \in \mathcal{Z}} b(l)$, then:

1. *If $V \int_{S^{n-1}} P(u)\, dS(u) \neq 0$, then $s = d + n$ is a simple pole of $g(s)$ and*

$$\operatorname*{Res}_{s=d+n} g(s) = V \int_{u \in S^{n-1}} P(u)\, dS(u).$$

2. *If $V \int_{S^{n-1}} P(u)\, dS(u) = 0$, then $g(s)$ extends holomorphically to the whole complex plane $\mathbb{C}$.*

(iii) *Suppose that $\Theta \in \mathcal{M}_n(\mathbb{R})$ is diophantine. For any $(\varepsilon_i)_i \in \{-1, 0, 1\}^q$, the function*

$$g_0(s) := \sum_{l \in (\mathbb{Z}^n)^q \setminus \mathcal{Z}} b(l) f_{\Theta \sum_{i=1}^q \varepsilon_i l_i}(s)$$

where $\mathcal{Z} := \{l \in (\mathbb{Z}^n)^q \mid \sum_{i=1}^q \varepsilon_i l_i = 0\}$ extends holomorphically to the whole complex plane $\mathbb{C}$.

Proof of Theorem 2.6. First we remark that if $a \in \mathbb{Z}^n$ then $f_a(s) = \sum'_{k\in\mathbb{Z}^n} \frac{P(k)}{|k|^s}$, so (i.1) follows from Theorem 2.1.

Further, $g(s) := \sum_{l\in(\mathbb{Z}^n)^q\setminus\mathbb{Z}} b(l) f_{\Theta\sum_i \varepsilon_i l_i}(s) + \left(\sum_{l\in\mathbb{Z}} b(l)\right) \sum'_{k\in\mathbb{Z}^n} \frac{P(k)}{|k|^s}$, so (ii) follows easily from (iii) and Theorem 2.1.

Therefore, to complete the proof it remains to show (i.2) and (iii).

The direct proof of (i.2) is easy, but it is not sufficient to deduce (iii), the proof of which is more delicate and requires a more precise (i.e., more effective) version of (i.2). The next lemma gives such crucial version, but before let us introduce some notations. Let

$$\mathcal{F} := \left\{ \frac{P(X)}{(X_1^2+\cdots+X_n^2+1)^{r/2}} \mid P(X) \in \mathbb{C}[X_1,\ldots,X_n] \text{ and } r \in \mathbb{N}_0 \right\},$$

and put $g = \deg(G) = \deg(P) - r \in \mathbb{Z}$, where $G = \frac{P(X)}{(X_1^2+\cdots+X_n^2+1)^{r/2}} \in \mathcal{F}$. Moreover, we set $\deg(0) = -\infty$.

Lemma 2.7. *Let $a \in \mathbb{R}^n$. We assume that $d\,(a\cdot u, \mathbb{Z}) := \inf_{m\in\mathbb{Z}} |a\cdot u - m| > 0$ for some $u \in \mathbb{Z}^n$. For all $G \in \mathcal{F}$, we define formally*

$$F_0(G;a;s) := \sum_{k\in\mathbb{Z}^n}{}' \frac{G(k)}{|k|^s} e^{2\pi i k\cdot a} \quad \textit{and} \quad F_1(G;a;s) := \sum_{k\in\mathbb{Z}^n} \frac{G(k)}{(|k|^2+1)^{s/2}} e^{2\pi i k\cdot a}.$$

Then for all $N \in \mathbb{N}$, all $G \in \mathcal{F}$ and all $i \in \{0,1\}$, there exist positive constants $C_i := C_i(G,N,u)$, $B_i := B_i(G,N,u)$ and $A_i := A_i(G,N,u)$ such that $s \mapsto F_i(G;\alpha;s)$ extends holomorphically to the half plane $\{\mathrm{Re}(s) > -N\}$ and satisfies

$$F_i(G;a;s) \le C_i(1+|s|)^{B_i}(d(a\cdot u,\mathbb{Z}))^{-A_i}.$$

Remark 2.8. The important point here is that we obtain an explicit bound of $F_i(G;\alpha;s)$ in $\{\mathrm{Re}(s) > -N\}$ which depends on the vector a only through $d(a\cdot u,\mathbb{Z})$, so depends on u and indirectly on a (in the sequel, a will vary). In particular, the constants $C_i := C_i(G,N,u)$, $B_i = B_i(G,N)$ and $A_i := A_i(G,N)$ do not depend on the vector a but only on u. This is crucial for the proof of (ii) and (iii) in Theorem 2.6.

2.2.1. Proof of Lemma 2.7 for $i = 1$. Let $N \in \mathbb{N}_0$ be a fixed integer and set $g_0 := n + N + 1$. We will prove Lemma 2.7 by induction on $g = \deg(G) \in \mathbb{Z}$. More precisely, in order to prove the case $i = 1$, it suffices to show that Lemma 2.7 is true for all $G \in \mathcal{F}$ with $\deg(G) \le -g_0$.

Let $g \in \mathbb{Z}$ with $g \ge -g_0 + 1$. If Lemma 2.7 is true for all $G \in \mathcal{F}$ such that $\deg(G) \le g - 1$, then it is also true for all $G \in \mathcal{F}$ satisfying $\deg(G) = g$.

Step 1: Checking Lemma 2.7 for $\deg(G) \le -g_0 := -(n+N+1)$.

Let $G(X) = \frac{P(X)}{(X_1^2+\cdots+X_n^2+1)^{r/2}} \in \mathcal{F}$ with $\deg(G) \le -g_0$. It is easy to see that we have uniformly in $s = \sigma + i\tau \in \mathbb{C}$ and in $k \in \mathbb{Z}^n$:

$$\begin{aligned}\frac{|G(k)e^{2\pi i k\cdot a}|}{(|k|^2+1)^{\sigma/2}} &= \frac{|P(k)|}{(|k|^2+1)^{(r+\sigma)/2}}\\ &\ll_G \frac{1}{(|k|^2+1)^{(r+\sigma-\deg(P))/2}}\\ &\ll_G \frac{1}{(|k|^2+1)^{(\sigma-\deg(G))/2}}\\ &\ll_G \frac{1}{(|k|^2+1)^{(\sigma+g_0)/2}}.\end{aligned}$$

It follows that $F_1(G;a;s) = \sum_{k\in\mathbb{Z}^n} \frac{G(k)}{(|k|^2+1)^{s/2}} e^{2\pi i k\cdot a}$ converges absolutely and defines a holomorphic function in the half plane $\{\sigma > -N\}$. Therefore, for any $s \in \{\mathrm{Re}(s) > -N\}$ we have

$$|F_1(G;a;s)| \ll_G \sum_{k\in\mathbb{Z}^n} \frac{1}{(|k|^2+1)^{(-N+g_0)/2}} \ll_G \sum_{k\in\mathbb{Z}^n} \frac{1}{(|k|^2+1)^{(n+1)/2}} \ll_G 1.$$

Thus, Lemma 2.7 is true when $\deg(G) \le -g_0$.

Step 2: Induction.

Now let $g \in \mathbb{Z}$ satisfy $g \ge -g_0 + 1$ and suppose that Lemma 2.7 is valid for all $G \in \mathcal{F}$ with $\deg(G) \le g-1$. Let $G \in \mathcal{F}$ with $\deg(G) = g$. We will prove that G also fulfills the conclusions of Lemma 2.7:

There exist $P \in \mathbb{C}[X_1,\dots,X_n]$ of degree $d \ge 0$ and $r \in \mathbb{N}_0$ such that $G(X) = \frac{P(X)}{(X_1^2+\cdots+X_n^2+1)^{r/2}}$ and $g = \deg(G) = d - r$. Since $G(k) \ll (|k|^2+1)^{g/2}$ uniformly in $k \in \mathbb{Z}^n$, we deduce that $F_1(G;a;s)$ converges absolutely in $\{\sigma = \mathrm{Re}(s) > n+g\}$.

Since $k \mapsto k+u$ is a bijection from $\mathbb{Z}^n$ into $\mathbb{Z}^n$, it follows that for $\mathrm{Re}(s) > n+g$ we also have

$$\begin{aligned}&F_1(G;a;s)\\ &= \sum_{k\in\mathbb{Z}^n} \frac{P(k)}{(|k|^2+1)^{(s+r)/2}} e^{2\pi i k\cdot a}\\ &= \sum_{k\in\mathbb{Z}^n} \frac{P(k+u)}{(|k+u|^2+1)^{(s+r)/2}} e^{2\pi i (k+u)\cdot a}\\ &= e^{2\pi i u\cdot a} \sum_{k\in\mathbb{Z}^n} \frac{P(k+u)}{(|k|^2+2k\cdot u+|u|^2+1)^{(s+r)/2}} e^{2\pi i k\cdot a}\\ &= e^{2\pi i u\cdot a} \sum_{\substack{\alpha\in\mathbb{N}_0^n\\ |\alpha|_1=\alpha_1+\cdots+\alpha_n\le d}} \frac{u^\alpha}{\alpha!} \sum_{k\in\mathbb{Z}^n} \frac{\partial^\alpha P(k)}{(|k|^2+2k\cdot u+|u|^2+1)^{(s+r)/2}} e^{2\pi i k\cdot a} =\end{aligned}$$

$$= e^{2\pi i u\cdot a} \sum_{|\alpha|_1 \le d} \frac{u^\alpha}{\alpha!} \sum_{k\in\mathbb{Z}^n} \frac{\partial^\alpha P(k)}{(|k|^2+1)^{(s+r)/2}} \left(1 + \frac{2k\cdot u + |u|^2}{(|k|^2+1)}\right)^{-(s+r)/2} e^{2\pi i k\cdot a}.$$

Let $M := \sup(N+n+g, 0) \in \mathbb{N}_0$. We have uniformly in $k \in \mathbb{Z}^n$:

$$\left(1 + \frac{2k\cdot u + |u|^2}{(|k|^2+1)}\right)^{-(s+r)/2} = \sum_{j=0}^{M} \binom{-(s+r)/2}{j} \frac{(2k\cdot u + |u|^2)^j}{(|k|^2+1)^j} + O_{M,u}\left(\frac{(1+|s|)^{M+1}}{(|k|^2+1)^{(M+1)/2}}\right).$$

Thus for $\sigma = \mathrm{Re}(s) > n + d$ it follows that

$$\begin{aligned} &F_1(G;a;s) \\ &= e^{2\pi i u\cdot a} \sum_{|\alpha|_1 \le d} \frac{u^\alpha}{\alpha!} \sum_{k\in\mathbb{Z}^n} \frac{\partial^\alpha P(k)}{(|k|^2+1)^{(s+r)/2}} \left(1 + \frac{2k\cdot u + |u|^2}{(|k|^2+1)}\right)^{-(s+r)/2} e^{2\pi i k\cdot a} \\ &= e^{2\pi i u\cdot a} \sum_{|\alpha|_1 \le d} \sum_{j=0}^{M} \frac{u^\alpha}{\alpha!} \binom{-(s+r)/2}{j} \sum_{k\in\mathbb{Z}^n} \frac{\partial^\alpha P(k)(2k\cdot u + |u|^2)^j}{(|k|^2+1)^{(s+r+2j)/2}} e^{2\pi i k\cdot a} \qquad (2.3) \\ &\qquad + O_{G,M,u}\Big((1+|s|)^{M+1} \sum_{k\in\mathbb{Z}^n} \frac{1}{(|k|^2+1)^{(\sigma+M+1-g)/2}}\Big). \end{aligned}$$

Set $I := \{(\alpha, j) \in \mathbb{N}_0^n \times \{0,\dots,M\} \mid |\alpha|_1 \le d\}$ and $I^* := I \setminus \{(0,0)\}$. Set also $G_{(\alpha,j);u}(X) := \frac{\partial^\alpha P(X)(2X\cdot u + |u|^2)^j}{(|X|^2+1)^{(r+2j)/2}} \in \mathcal{F}$ for all $(\alpha, j) \in I^*$.

Since $M \ge N + n + g$, it follows from (2.3) that

$$\begin{aligned} (1 - e^{2\pi i u\cdot a}) F_1(G;a;s) = e^{2\pi i u\cdot a} \sum_{(\alpha,j)\in I^*} \frac{u^\alpha}{\alpha!} \binom{-(s+r)/2}{j} F_1(G_{(\alpha,j);u};\alpha;s) \\ + R_N(G;a;u;s), \end{aligned} \qquad (2.4)$$

where the map $s \mapsto R_N(G;a;u;s)$ is a holomorphic function in the half plane $\{\sigma = \mathrm{Re}(s) > -N\}$ satisfying $R_N(G;a;u;s) \ll_{G,N,u} 1$.

Moreover it is easy to see that for any $(\alpha, j) \in I^*$,

$$\begin{aligned} \deg(G_{(\alpha,j);u}) &= \deg(\partial^\alpha P) + j - (r+2j) \\ &\le d - |\alpha|_1 + j - (r+2j) = g - |\alpha|_1 - j \le g - 1. \end{aligned}$$

Relation (2.4) and the induction hypothesis then imply that

$$(1 - e^{2\pi i u\cdot a}) F_1(G;a;s) \text{ verifies the conclusions of Lemma 2.7.} \qquad (2.5)$$

Since $|1 - e^{2\pi i u\cdot a}| = 2|\sin(\pi u\cdot a)| \ge d(u\cdot a, \mathbb{Z})$, (2.5) implies that $F_1(G;a;s)$ satisfies the conclusions of Lemma 2.7. This completes the induction and the proof for $i = 1$.

2.2.2. Proof of Lemma 2.7 for $i = 0$. Let $N \in \mathbb{N}$ be a fixed integer. Let $G(X) = \frac{P(X)}{(X_1^2+\cdots+X_n^2+1)^{r/2}} \in \mathcal{F}$ and $g = \deg(G) = d - r$ where $d \geq 0$ is the degree of the polynomial P. Put $M := \sup(N + g + n, 0) \in \mathbb{N}_0$.

Since $P(k) \ll |k|^d$ for $k \in \mathbb{Z}^n \setminus \{0\}$, it follows that $F_0(G; a; s)$ and $F_1(G; a; s)$ converge absolutely in the half plane $\{\sigma = \mathrm{Re}(s) > n + g\}$. Moreover, we have for $s = \sigma + i\tau \in \mathbb{C}$ with $\sigma > n + g$:

$$\begin{aligned}
F_0(G; a; s) &= \sum_{k\in\mathbb{Z}^n\setminus\{0\}} \frac{G(k)}{(|k|^2+1-1)^{s/2}} e^{2\pi i k\cdot a} \\
&= \sum_{k\in\mathbb{Z}^n}{}' \frac{G(k)}{(|k|^2+1)^{s/2}} \left(1 - \frac{1}{|k|^2+1}\right)^{-s/2} e^{2\pi i k\cdot a} \\
&= \sum_{k\in\mathbb{Z}^n}{}' \sum_{j=0}^{M} \binom{-s/2}{j} (-1)^j \frac{G(k)}{(|k|^2+1)^{(s+2j)/2}} e^{2\pi i k\cdot a} \\
&\quad + O_M\left((1+|s|)^{M+1} \sum_{k\in\mathbb{Z}^n}{}' \frac{|G(k)|}{(|k|^2+1)^{(\sigma+2M+2)/2}}\right) \\
&= \sum_{j=0}^{M} \binom{-s/2}{j} (-1)^j F_1(G; a; s+2j) \\
&\quad + O_M\left[(1+|s|)^{M+1}\left(1 + \sum_{k\in\mathbb{Z}^n}{}' \frac{|G(k)|}{(|k|^2+1)^{(\sigma+2M+2)/2}}\right)\right].
\end{aligned} \tag{2.6}$$

In addition we have uniformly in $s = \sigma + i\tau \in \mathbb{C}$ with $\sigma > -N$,

$$\begin{aligned}
\sum_{k\in\mathbb{Z}^n}{}' \frac{|G(k)|}{(|k|^2+1)^{(\sigma+2M+2)/2}} &\ll \sum_{k\in\mathbb{Z}^n}{}' \frac{|k|^g}{(|k|^2+1)^{(-N+2M+2)/2}} \\
&\ll \sum_{k\in\mathbb{Z}^n}{}' \frac{1}{|k|^{n+1}} < +\infty.
\end{aligned}$$

So (2.6) and Lemma 2.7 for $i = 1$ imply that Lemma 2.7 is also true for $i = 0$. This completes the proof of Lemma 2.7.

2.2.3. Proof of item (i.2) of Theorem 2.6. Since $a \in \mathbb{R}^n \setminus \mathbb{Z}^n$, there exists $i_0 \in \{1, \ldots, n\}$ such that $a_{i_0} \notin \mathbb{Z}$. In particular $d(a \cdot e_{i_0}, \mathbb{Z}) = d(a_{i_0}, \mathbb{Z}) > 0$. Therefore, a satisfies the assumption of Lemma 2.7 with $u = e_{i_0}$. Thus, for all $N \in \mathbb{N}$, $s \mapsto f_a(s) = F_0(P; a; s)$ has a holomorphic continuation to the half plane $\{\mathrm{Re}(s) > -N\}$. It follows, by letting $N \to \infty$, that $s \mapsto f_a(s)$ has a holomorphic continuation to the whole complex plane $\mathbb{C}$.

2.2.4. Proof of item (iii) of Theorem 2.6. Let $\Theta \in \mathcal{M}_n(\mathbb{R})$, $(\varepsilon_i)_i \in \{-1,0,1\}^q$ and $b \in \mathcal{S}(\mathbb{Z}^n \times \mathbb{Z}^n)$. We assume that Θ is a diophantine matrix. Set $\mathcal{Z} := \{l = (l_1,\dots,l_q) \in (\mathbb{Z}^n)^q \mid \sum_i \varepsilon_i l_i = 0\}$ and $P \in \mathbb{C}[X_1,\dots,X_n]$ of degree $d \geq 0$.

It is easy to see that for $\sigma > n+d$:

$$\sum_{l\in(\mathbb{Z}^n)^q\setminus\mathcal{Z}} |b(l)| \sideset{}{'}\sum_{k\in\mathbb{Z}^n} \frac{|P(k)|}{|k|^\sigma} \left|e^{2\pi i k\cdot\Theta\sum_i \varepsilon_i l_i}\right| \ll_P \sum_{l\in(\mathbb{Z}^n)^q\setminus\mathcal{Z}} |b(l)| \sideset{}{'}\sum_{k\in\mathbb{Z}^n} \frac{1}{|k|^{\sigma-d}}$$
$$\ll_{P,\sigma} \sum_{l\in(\mathbb{Z}^n)^q\setminus\mathcal{Z}} |b(l)| < +\infty.$$

So

$$g_0(s) := \sum_{l\in(\mathbb{Z}^n)^q\setminus\mathcal{Z}} b(l) f_{\Theta\sum_i \varepsilon_i l_i}(s) = \sum_{l\in(\mathbb{Z}^n)^q\setminus\mathcal{Z}} b(l) \sideset{}{'}\sum_{k\in\mathbb{Z}^n} \frac{P(k)}{|k|^s} e^{2\pi i k\cdot\Theta\sum_i \varepsilon_i l_i}$$

converges absolutely in the half plane $\{\mathrm{Re}(s) > n+d\}$.

Moreover with the notations of Lemma 2.7, we have for all $s = \sigma + i\tau \in \mathbb{C}$ with $\sigma > n+d$:

$$g_0(s) = \sum_{l\in(\mathbb{Z}^n)^q\setminus\mathcal{Z}} b(l) f_{\Theta\sum_i \varepsilon_i l_i}(s) = \sum_{l\in(\mathbb{Z}^n)^q\setminus\mathcal{Z}} b(l) F_0(P;\Theta\textstyle\sum_i \varepsilon_i l_i; s). \tag{2.7}$$

But Θ is diophantine, so there exist $u \in \mathbb{Z}^n$ and $\delta, c > 0$ such

$$|q\cdot {}^t\Theta u - m| \geq c(1+|q|)^{-\delta} \quad \text{for all } q \in \mathbb{Z}^n \setminus \{0\} \text{ and for all } m \in \mathbb{Z}.$$

We deduce that for all $l \in (\mathbb{Z}^n)^q \setminus \mathcal{Z}$,

$$|(\Theta\textstyle\sum_i \varepsilon_i l_i)\cdot u - m| = |(\sum_i \varepsilon_i l_i)\cdot {}^t\Theta u - m| \geq c(1+|\sum_i \varepsilon_i l_i|)^{-\delta} \geq c(1+|l|)^{-\delta}.$$

It follows that there exist $u \in \mathbb{Z}^n$, $\delta > 0$ and $c > 0$ such that

$$d((\Theta\textstyle\sum_i \varepsilon_i l_i)\cdot u;\mathbb{Z}) \geq c(1+|l|)^{-\delta} \quad \text{for all } l \in (\mathbb{Z}^n)^q \setminus \mathcal{Z}. \tag{2.8}$$

Therefore, for any $l \in (\mathbb{Z}^n)^q \setminus \mathcal{Z}$, the vector $a = \Theta\sum_i \varepsilon_i l_i$ verifies the assumption of Lemma 2.7 with the same u. Moreover δ and c in (2.8) are also independent on l.

We fix now $N \in \mathbb{N}$. Lemma 2.7 implies that there exist positive constants $C_0 := C_0(P,N,u)$, $B_0 := B_i(P,N,u)$ and $A_0 := A_0(P,N,u)$ such that for all $l \in (\mathbb{Z}^n)^q \setminus \mathcal{Z}$, the map $s \mapsto F_0(P;\Theta\sum_i \varepsilon_i l_i; s)$ extends *holomorphically* to the half plane $\{\mathrm{Re}(s) > -N\}$ satisfying

$$F_0(P;\Theta\textstyle\sum_i \varepsilon_i l_i; s) \leq C_0(1+|s|)^{B_0} d((\Theta\sum_i \varepsilon_i l_i)\cdot u;\mathbb{Z})^{-A_0}.$$

Together with (2.8) this implies that for any compact set K included in the half plane $\{\mathrm{Re}(s) > -N\}$, there exist two constants $C := C(P, N, c, \delta, u, K)$ and $D := D(P, N, c, \delta, u)$ (independent of $l \in (\mathbb{Z}^n)^q \setminus \mathcal{Z}$) such that

$$F_0(P; \Theta \textstyle\sum_i \varepsilon_i l_i; s) \leq C(1 + |l|)^D \quad \text{for all } s \in K \text{ and for all } l \in (\mathbb{Z}^n)^q \setminus \mathcal{Z}. \quad (2.9)$$

It follows that $s \mapsto \sum_{l \in (\mathbb{Z}^n)^q \setminus \mathcal{Z}} b(l) F_0(P; \Theta \sum_i \varepsilon_i l_i; s)$ has a holomorphic continuation to the half plane $\{\mathrm{Re}(s) > -N\}$.

This and (2.7) imply that $s \mapsto g_0(s) = \sum_{l \in (\mathbb{Z}^n)^q \setminus \mathcal{Z}} b(l) f_{\Theta \sum_i \varepsilon_i l_i}(s)$ has a holomorphic continuation to $\{\mathrm{Re}(s) > -N\}$. Since N is an arbitrary integer, by letting $N \to \infty$, it follows that $s \mapsto g_0(s)$ has a holomorphic continuation to the whole complex plane $\mathbb{C}$, which completes the proof of the theorem. □

Remark 2.9. From equation (2.5) we see that a Diophantine condition is sufficient to get Lemma 2.7. Our Diophantine condition appears also (in equivalent form) in Connes [6], Prop. 49 (see Remark 4.2). The following heuristic argument shows that our condition seems to be necessary in order to get the result of Theorem 2.6:

For simplicity we assume $n = 1$ (but the argument extends easily to any n). Let $\theta \in \mathbb{R} \setminus \mathbb{Q}$. We know (see this reflection formula in [15], p. 6) that for any $l \in \mathbb{Z} \setminus \{0\}$,

$$g_{\theta l}(s) := \sum_{k \in \mathbb{Z}}{}' \frac{e^{2\pi i \theta l k}}{|k|^s} = \frac{\pi^{s-1/2}}{\Gamma(\frac{1-s}{2})} \Gamma(\tfrac{s}{2}) h_{\theta l}(1-s)$$

where $h_{\theta l}(s) := \sum'_{k \in \mathbb{Z}} \frac{1}{|\theta l + k|^s}$. So, for any $(a_l) \in \mathcal{S}(\mathbb{Z})$, the existence of a meromorphic continuation of $g_0(s) := \sum'_{l \in \mathbb{Z}} a_l g_{\theta l}(s)$ is equivalent to the existence of a meromorphic continuation of

$$h_0(s) := \sum_{l \in \mathbb{Z}}{}' a_l h_{\theta l}(s) = \sum_{l \in \mathbb{Z}}{}' a_l \sum_{k \in \mathbb{Z}}{}' \frac{1}{|\theta l + k|^s}.$$

So, for at least one $\sigma_0 \in \mathbb{R}$, we must have $\frac{|a_l|}{|\theta l + k|^{\sigma_0}} = O(1)$ uniformly in $k, l \in \mathbb{Z}^*$.

It follows that for any $(a_l) \in \mathcal{S}(\mathbb{Z})$, $|\theta l + k| \gg |a_l|^{1/\sigma_0}$ uniformly in $k, l \in \mathbb{Z}^*$. Therefore, our Diophantine condition seems to be necessary.

2.2.5. Commutation between sum and residue. Let $p \in \mathbb{N}$. Recall that $\mathcal{S}((\mathbb{Z}^n)^p)$ is the set of the Schwartz sequences on $(\mathbb{Z}^n)^p$. In other words, $b \in \mathcal{S}((\mathbb{Z}^n)^p)$ if and only if $(1 + |l_1|^2 + \ldots |l_p|^2)^r |b(l_1, \ldots, l_p)|^2$ is bounded on $(\mathbb{Z}^n)^p$ for all $r \in \mathbb{N}_0$. We note that if $Q \in \mathbb{R}[X_1, \ldots, X_{np}]$ is a polynomial, $(a_j) \in \mathcal{S}(\mathbb{Z}^n)^p$, $b \in \mathcal{S}(\mathbb{Z}^n)$ and ϕ is a real-valued function, then $l := (l_1, \ldots, l_p) \mapsto \tilde{a}(l) b(-\hat{l}_p) Q(l) e^{i\phi(l)}$ is a Schwartz sequence on $(\mathbb{Z}^n)^p$, where

$$\tilde{a}(l) := a_1(l_1) \ldots a_p(l_p), \quad \hat{l}_i := l_1 + \cdots + l_i.$$

In the following, we will use several times the fact that for any $(k,l) \in (\mathbb{Z}^n)^2$ such that $k \neq 0$ and $k \neq -l$, we have

$$\frac{1}{|k+l|^2} = \frac{1}{|k|^2} - \frac{2k \cdot l + |l|^2}{|k|^2|k+l|^2}. \tag{2.10}$$

Lemma 2.10. *There exists a polynomial $P \in \mathbb{R}[X_1, \dots, X_p]$ of degree $4p$ and with positive coefficients such that for any $k \in \mathbb{Z}^n$ and $l := (l_1, \dots, l_p) \in (\mathbb{Z}^n)^p$ such that $k \neq 0$ and $k \neq -\hat{l}_i$ for all $1 \leq i \leq p$, the following holds:*

$$\frac{1}{|k+\hat{l}_1|^2 \dots |k+\hat{l}_p|^2} \leq \frac{1}{|k|^{2p}} P(|l_1|, \dots, |l_p|).$$

Proof. Let us fix i such that $1 \leq i \leq p$. Using two times (2.10), the Cauchy–Schwarz inequality and the fact that $|k+\hat{l}_i|^2 \geq 1$, we get

$$\begin{aligned}\frac{1}{|k+\hat{l}_i|^2} &\leq \frac{1}{|k|^2} + \frac{2|k||\hat{l}_i| + |\hat{l}_i|^2}{|k|^4} + \frac{(2|k||\hat{l}_i| + |\hat{l}_i|^2)^2}{|k|^4|k+\hat{l}_i|^2} \\ &\leq \frac{1}{|k|^2} + \frac{2}{|k|^3}|\hat{l}_i| + \left(\frac{1}{|k|^4} + \frac{4}{|k|^2}\right)|\hat{l}_i|^2 + \frac{4}{|k|^3}|\hat{l}_i|^3 + \frac{1}{|k|^4}|\hat{l}_i|^4.\end{aligned}$$

Since $|k| \geq 1$, and $|\hat{l}_i|^j \leq |\hat{l}_i|^4$ if $1 \leq j \leq 4$, we find

$$\frac{1}{|k+\hat{l}_i|^2} \leq \frac{5}{|k|^2}\sum_{j=0}^{4}|\hat{l}_i|^j \leq \frac{5}{|k|^2}(1+4|\hat{l}_i|^4) \leq \frac{5}{|k|^2}\Big(1+4\Big(\sum_{j=1}^{p}|l_j|\Big)^4\Big),$$

$$\frac{1}{|k+\hat{l}_1|^2 \dots |k+\hat{l}_p|^2} \leq \frac{5^p}{|k|^{2p}}\Big(1+4\Big(\sum_{j=1}^{p}|l_j|\Big)^4\Big)^p.$$

Taking $P(X_1, \dots, X_p) := 5^p(1+4(\sum_{j=1}^{p} X_j)^4)^p$ now gives the result. □

Lemma 2.11. *Let $b \in \mathcal{S}((\mathbb{Z}^n)^p)$, $p \in \mathbb{N}$. Let $P_j \in \mathbb{R}[X_1, \dots, X_n]$ be a homogeneous polynomial function of degree j, and let $k \in \mathbb{Z}^n$, $l := (l_1, \dots, l_p) \in (\mathbb{Z}^n)^p$, $r \in \mathbb{N}_0$. Moreover, let ϕ be a real-valued function on $\mathbb{Z}^n \times (\mathbb{Z}^n)^p$ and put*

$$h(s,k,l) := \frac{b(l)P_j(k)e^{i\phi(k,l)}}{|k|^{s+r}|k+\hat{l}_1|^2 \dots |k+\hat{l}_p|^2},$$

with $h(s,k,l) := 0$ if, for $k \neq 0$, one of the denominators is zero.

Then for all $s \in \mathbb{C}$ such that $\mathrm{Re}(s) > n + j - r - 2p$, the series

$$H(s) := \sum\nolimits'_{(k,l)\in(\mathbb{Z}^n)^{p+1}} h(s,k,l)$$

is absolutely summable. In particular,

$$\sum_{k\in\mathbb{Z}^n}{}' \sum_{l\in(\mathbb{Z}^n)^p} h(s,k,l) = \sum_{l\in(\mathbb{Z}^n)^p} \sum_{k\in\mathbb{Z}^n}{}' h(s,k,l).$$

Proof. Let $s = \sigma + i\tau \in \mathbb{C}$ such that $\sigma > n + j - r - 2p$. By Lemma 2.10 we obtain, for $k \neq 0$, that

$$|h(s,k,l)| \leq |b(l)P_j(k)|\,|k|^{-r-\sigma-2p}P(l),$$

where $P(l) := P(|l_1|,\dots,|l_p|)$ and P is a polynomial of degree $4p$ with positive coefficients. Thus, $|h(s,k,l)| \leq F(l)G(k)$ where $F(l) := |b(l)|P(l)$ and $G(k) := |P_j(k)||k|^{-r-\sigma-2p}$. That $\sum_{l\in(\mathbb{Z}^n)^p} F(l)$ is summable follows from the fact that $b \in \mathcal{S}((\mathbb{Z}^n)^p)$. The summability of $\sum'_{k\in\mathbb{Z}^n} G(k)$ is a consequence of the fact that $\sigma > n + j - r - 2p$. Finally, as a product of two summable series, $\sum_{k,l} F(l)G(k)$ is a summable series, which proves that $\sum_{k,l} h(s,k,l)$ is also absolutely summable. □

Definition 2.12. Let f be a function on $D\times(\mathbb{Z}^n)^p$ where D is an open neighborhood of 0 in $\mathbb{C}$. We say that f satisfies (H1) if and only if there exists $\rho > 0$ such that

(i) for any l, $s \mapsto f(s,l)$ extends as a holomorphic function in U_ρ, where U_ρ is the open disk of center 0 and radius ρ;

(ii) the series $\sum_{l\in(\mathbb{Z}^n)^p} \|H(\cdot,l)\|_{\infty,\rho}$ is summable, where $\|H(\cdot,l)\|_{\infty,\rho} := \sup_{s\in U_\rho} |H(s,l)|$.

We say that f satisfies (H2) if and only if there exists $\rho > 0$ such that

(i) for any l, $s \mapsto f(s,l)$ extends as a holomorphic function to $U_\rho - \{0\}$;

(ii) for any δ such that $0 < \delta < \rho$, the series $\sum_{l\in(\mathbb{Z}^n)^p} \|H(\cdot,l)\|_{\infty,\delta,\rho}$ is summable, where $\|H(\cdot,l)\|_{\infty,\delta,\rho} := \sup_{\delta<|s|<\rho} |H(s,l)|$.

Remark 2.13. Note that (H1) implies (H2). Moreover, if f satisfies (H1) (resp. (H2)) for $\rho > 0$, then it is straightforward to check that $f : s \mapsto \sum_{l\in(\mathbb{Z}^n)^p} f(s,l)$ extends as an holomorphic function to U_ρ (resp. to $U_\rho \setminus \{0\}$).

Corollary 2.14. *With the same notations as in Lemma* 2.11*, suppose that* $r+2p-j > n$*. Then the function* $H(s,l) := \sum'_{k\in\mathbb{Z}^n} h(s,k,l)$ *satisfies* (H1).

Proof. (i) Fix $\rho > 0$ such that $\rho < r + 2p - j - n$. Since $r + 2p - j > n$, U_ρ is inside the half plane of absolute convergence of the series defined by $H(s,l)$. Thus, $s \mapsto H(s,l)$ is holomorphic in U_ρ.

(ii) Since $\left||k|^{-s}\right| \leq |k|^\rho$ for all $s \in U_\rho$ and $k \in \mathbb{Z}^n \setminus \{0\}$, we get as in the above proof

$$|h(s,k,l)| \leq |b(l)P_j(k)|\,|k|^{-r+\rho-2p}P(|l_1|,\dots,|l_p|).$$

Since $\rho < r + 2p - j - n$, the series $\sum'_{k\in\mathbb{Z}^n} |P_j(k)||k|^{-r+\rho-2p}$ is summable.

Thus, $\|H(\cdot\,,l)\|_{\infty,\rho} \le KF(l)$ where $K := \sum_k{}'|P_j(k)||k|^{-r+\rho-2p} < \infty$. We have already seen that the series $\sum_l F(l)$ is summable, so the result follows. □

We note that if f and g both satisfy (H1) (or (H2)), then so does $f + g$. In the following we will use the equivalence relation

$$f \sim g \iff f - g \text{ satisfies (H1)}.$$

Lemma 2.15. *Let f and g be two functions on $D \times (\mathbb{Z}^n)^p$, where D is an open neighborhood of* 0*, in $\mathbb{C}$ such that $f \sim g$ and g satisfies* (H2)*. Then*

$$\operatorname*{Res}_{s=0} \sum_{l\in(\mathbb{Z}^n)^p} f(s,l) = \sum_{l\in(\mathbb{Z}^n)^p} \operatorname*{Res}_{s=0} g(s,l).$$

Proof. Since $f \sim g$, f satisfies (H2) for a certain $\rho > 0$. Fix η such that $0 < \eta < \rho$ and define C_η as the circle of center 0 and radius η. Then

$$\operatorname*{Res}_{s=0} g(s,l) = \operatorname*{Res}_{s=0} f(s,l) = \frac{1}{2\pi i}\oint_{C_\eta} f(s,l)\,ds = \int_I u(t,l)\,dt,$$

where $I = [0, 2\pi]$ and $u(t,l) := \frac{1}{2\pi}\eta e^{it} f(\eta e^{it}, l)$. The fact that f satisfies (H2) entails that the series $\sum_{l\in(\mathbb{Z}^n)^p} \|f(\cdot\,,l)\|_{\infty,C_\eta}$ is summable. Thus, since $\|u(\cdot\,,l)\|_\infty = \frac{1}{2\pi}\eta\|f(\cdot\,,l)\|_{\infty,C_\eta}$, the series $\sum_{l\in(\mathbb{Z}^n)^p} \|u(\cdot\,,l)\|_\infty$ is summable, and consequently $\int_I \sum_{l\in(\mathbb{Z}^n)^p} u(t,l)dt = \sum_{l\in(\mathbb{Z}^n)^p} \int_I u(t,l)dt$, which gives the result. □

2.3. Computation of residues of zeta functions. Since we will have to compute residues of series, let us introduce the following notation.

Definition 2.16.

$$\zeta(s) := \sum_{n=1}^{\infty} n^{-s},$$
$$Z_n(s) := \sum_{k\in\mathbb{Z}^n}{}' |k|^{-s},$$
$$\zeta_{p_1,\dots,p_n}(s) := \sum_{k\in\mathbb{Z}^n}{}' \frac{k_1^{p_1}\dots k_n^{p_n}}{|k|^s} \quad \text{for } p_i \in \mathbb{N},$$

where $\zeta(s)$ is the Riemann zeta function (see [25] or [14]).

By the symmetry $k \to -k$, it is clear that the functions $\zeta_{p_1,\dots,p_n}$ all vanish for odd values of p_i.

Let us now compute $\zeta_{0,\dots,0,1_i,0\dots,0,1_j,0\dots,0}(s)$ in terms of $Z_n(s)$:

Since $\zeta_{0,\dots,0,1_i,0\dots,0,1_j,0\dots,0}(s) = A_i(s)\delta_{ij}$, exchanging the components k_i and k_j we get

$$\zeta_{0,\dots,0,1_i,0\dots,0,1_j,0\dots,0}(s) = \frac{\delta_{ij}}{n} Z_n(s-2).$$

Similarly,

$$\sum_{\mathbb{Z}^n}{}' \frac{k_1^2 k_2^2}{|k|^{s+8}} = \frac{1}{n(n-1)} Z_n(s+4) - \frac{1}{n-1} \sum_{\mathbb{Z}^n}{}' \frac{k_1^4}{|k|^{s+8}},$$

but it is difficult to write explicitly $\zeta_{p_1,\dots,p_n}(s)$ in terms of $Z_n(s-4)$ and other $Z_n(s-m)$ when at least four indices p_i are non-zero.

When all p_i are even, $\zeta_{p_1,\dots,p_n}(s)$ is a non-zero series of fractions $\frac{P(k)}{|k|^s}$ where P is a homogeneous polynomial of degree $p_1 + \cdots + p_n$. Theorem 2.1 now gives us the following result.

Proposition 2.17. *The function $\zeta_{p_1,\dots,p_n}$ has a meromorphic extension to the whole plane with a unique pole at $n + p_1 + \cdots + p_n$. This pole is simple and the residue at this pole is*

$$\operatorname*{Res}_{s=n+p_1+\cdots+p_n} \zeta_{p_1,\dots,p_n}(s) = 2\,\frac{\Gamma(\frac{p_1+1}{2})\dots\Gamma(\frac{p_n+1}{2})}{\Gamma(\frac{n+p_1+\cdots+p_n}{2})} \tag{2.11}$$

when all p_i are even or this residue is zero otherwise.

In particular, for $n = 2$,

$$\operatorname*{Res}_{s=0} \sum_{k\in\mathbb{Z}^2}{}' \frac{k_i k_j}{|k|^{s+4}} = \delta_{ij}\pi, \tag{2.12}$$

and for $n = 4$,

$$\begin{aligned} \operatorname*{Res}_{s=0} \sum_{k\in\mathbb{Z}^4}{}' \frac{k_i k_j}{|k|^{s+6}} &= \delta_{ij}\tfrac{\pi^2}{2}, \\ \operatorname*{Res}_{s=0} \sum_{k\in\mathbb{Z}^4}{}' \frac{k_i k_j k_l k_m}{|k|^{s+8}} &= (\delta_{ij}\delta_{lm} + \delta_{il}\delta_{jm} + \delta_{im}\delta_{jl})\tfrac{\pi^2}{12}. \end{aligned} \tag{2.13}$$

Proof. Equation (2.11) follows from Theorem 2.1,

$$\operatorname*{Res}_{s=n+p_1+\cdots+p_n} \zeta_{p_1,\dots,p_n}(s) = \int_{k\in S^{n-1}} k_1^{p_1}\dots k_n^{p_n}\, dS(k),$$

and standard formulae (see for instance [32], VIII, 1; 22). Equation (2.12) is a straightforward consequence of equation (2.11). Equation (2.13) can be checked for the cases $i = j \neq l = m$ and $i = j = l = m$. □

Note that $Z_n(s)$ is an Epstein zeta function associated to the quadratic form $q(x) := x_1^2 + \cdots + x_n^2$, so Z_n satisfies the following functional equation

$$Z_n(s) = \pi^{s-n/2}\Gamma(n/2 - s/2)\Gamma(s/2)^{-1} Z_n(n-s).$$

Since $\pi^{s-n/2}\Gamma(n/2 - s/2)\Gamma(s/2)^{-1} = 0$ for any negative even integer n and $Z_n(s)$ is meromorphic on $\mathbb{C}$ with only one pole at $s = n$ with residue $2\pi^{n/2}\Gamma(n/2)^{-1}$ according to the previous proposition, it follows that $Z_n(0) = -1$. We have proved that

$$\operatorname*{Res}_{s=0} Z_n(s+n) = 2\pi^{n/2}\Gamma(n/2)^{-1}, \tag{2.14}$$

$$Z_n(0) = -1. \tag{2.15}$$

2.4. Meromorphic continuation of a class of zeta functions. Let $n, q \in \mathbb{N}, q \geq 2$, and $p = (p_1, \ldots, p_{q-1}) \in \mathbb{N}_0^{q-1}$. Set $I := \{i \mid p_i \neq 0\}$ and assume that $I \neq \emptyset$ and

$$\mathcal{I} := \{\alpha = (\alpha_i)_{i\in I} \mid \alpha_i = (\alpha_{i,1}, \ldots, \alpha_{i,p_i}) \in \mathbb{N}_0^{p_i} \text{ for all } i \in I\} = \prod_{i\in I} \mathbb{N}_0^{p_i}.$$

We will use in the sequel also the following notations.

For $x = (x_1, \ldots, x_t) \in \mathbb{R}^t$ recall that $|x|_1 = |x_1| + \cdots + |x_t|$ and $|x| = \sqrt{x_1^2 + \cdots + x_t^2}$;

for all $\alpha = (\alpha_i)_{i\in I} \in \mathcal{I} = \prod_{i\in I} \mathbb{N}_0^{p_i}$ we have

$$|\alpha|_1 = \sum_{i\in I} |\alpha_i|_1 = \sum_{i\in I}\sum_{j=1}^{p_i} |\alpha_{i,j}| \text{ and } \binom{1/2}{\alpha} = \prod_{i\in I}\binom{1/2}{\alpha_i} = \prod_{i\in I}\prod_{j=1}^{p_i}\binom{1/2}{\alpha_{i,j}}.$$

2.4.1. A family of polynomials. In this paragraph we define a family of polynomials which plays an important role later. Consider first the variables:

- for $X_1, \ldots, X_n$ we set $X = (X_1, \ldots, X_n)$;
- for any $i = 1, \ldots, 2q$, we regard the variables $Y_{i,1}, \ldots, Y_{i,n}$ and put $Y_i := (Y_{i,1}, \ldots, Y_{i,n})$ and $Y := (Y_1, \ldots, Y_{2q})$;
- for $Y = (Y_1, \ldots, Y_{2q})$ we set $\widetilde{Y}_j := Y_1 + \cdots + Y_j + Y_{q+1} + \cdots + Y_{q+j}$ for any $1 \leq j \leq q$.

We define for all $\alpha = (\alpha_i)_{i\in I} \in \mathcal{I} = \prod_{i\in I} \mathbb{N}_0^{p_i}$ the polynomial

$$P_\alpha(X, Y) := \prod_{i\in I}\prod_{j=1}^{p_i} (2\langle X, \widetilde{Y}_i\rangle + |\widetilde{Y}_i|^2)^{\alpha_{i,j}}. \tag{2.16}$$

It is clear that $P_\alpha(X, Y) \in \mathbb{Z}[X, Y]$, $\deg_X P_\alpha \leq |\alpha|_1$ and $\deg_Y P_\alpha \leq 2|\alpha|_1$.

Let us fix a polynomial $Q \in \mathbb{R}[X_1, \dots, X_n]$ and write $d := \deg Q$. For $\alpha \in \mathcal{I}$ we want to expand $P_\alpha(X,Y)Q(X)$ in homogeneous polynomials in X and Y. Define

$$L(\alpha) := \{\beta \in \mathbb{N}_0^{(2q+1)n} \mid |\beta|_1 - d_\beta \le 2|\alpha|_1 \text{ and } d_\beta \le |\alpha|_1 + d\}$$

where $d_\beta := \sum_1^n \beta_i$, and set

$$\binom{1/2}{\alpha} P_\alpha(X,Y)Q(X) =: \sum_{\beta \in L(\alpha)} c_{\alpha,\beta} X^\beta Y^\beta$$

where $c_{\alpha,\beta} \in \mathbb{R}$, $X^\beta := X_1^{\beta_1} \dots X_n^{\beta_n}$ and $Y^\beta := Y_{1,1}^{\beta_{n+1}} \dots Y_{2q,n}^{\beta_{(2q+1)n}}$. By definition, X^β is a homogeneous polynomial of degree in X equals to d_β. Write

$$M_{\alpha,\beta}(Y) := c_{\alpha,\beta} Y^\beta.$$

2.4.2. Residues of a class of zeta functions. In this section we prove the following result that will be used in Proposition 5.4 for the computation of the spectrum dimension of the noncommutative torus.

Theorem 2.18. (i) *Let $\frac{1}{2\pi}\Theta$ be a diophantine matrix, and $\tilde{a} \in \mathcal{S}((\mathbb{Z}^n)^{2q})$. Then*

$$s \mapsto f(s) := \sum_{l \in [(\mathbb{Z}^n)^q]^2} \tilde{a}_l \sideset{}{'}\sum_{k \in \mathbb{Z}^n} \prod_{i=1}^{q-1} |k + \tilde{l}_i|^{p_i} |k|^{-s} Q(k) e^{ik\cdot\Theta \sum_1^q l_j}$$

has a meromorphic continuation to the whole complex plane $\mathbb{C}$ with at most simple possible poles at the points $s = n + d + |p|_1 - m$ where $m \in \mathbb{N}_0$.

(ii) *Let $m \in \mathbb{N}_0$ and set $I(m) := \{(\alpha,\beta) \in \mathcal{I} \times \mathbb{N}_0^{(2q+1)n} \mid \beta \in L(\alpha)$ and $m = 2|\alpha|_1 - d_\beta + d\}$. Then $I(m)$ is a finite set and $s = n + d + |p|_1 - m$ is a pole of f if and only if*

$$C(f,m) := \sum_{l \in Z} \tilde{a}_l \sum_{(\alpha,\beta) \in I(m)} M_{\alpha,\beta}(l) \int_{u \in S^{n-1}} u^\beta \, dS(u) \neq 0,$$

with $Z := \{l \mid \sum_1^q l_j = 0\}$ and the convention $\sum_\emptyset = 0$. In that case $s = n + d + |p|_1 - m$ is a simple pole of residue $\operatorname*{Res}_{s=n+d+|p|_1-m} f(s) = C(f,m)$.

In order to prove the theorem above we need the following result.

Lemma 2.19. *For all $N \in \mathbb{N}$ we have*

$$\prod_{i=1}^{q-1} |k + \tilde{l}_i|^{p_i} = \sum_{\alpha = (\alpha_i)_{i \in I} \in \prod_{i \in I} \{0,\dots,N\}^{p_i}} \binom{1/2}{\alpha} \frac{P_\alpha(k,l)}{|k|^{2|\alpha|_1 - |p|_1}} + \mathcal{O}_N(|k|^{|p|_1 - (N+1)/2})$$

uniformly in $k \in \mathbb{Z}^n$ and $l \in (\mathbb{Z}^n)^{2q}$ such that $|k| > U(l) := 36\big(\sum_{i=1, i\neq q}^{2q-1} |l_i|\big)^4$.

Proof. For $i = 1, \dots, q-1$, we have uniformly in $k \in \mathbb{Z}^n$ and $l \in (\mathbb{Z}^n)^{2q}$ with $|k| > U(l)$,

$$\frac{\left|2\langle k, \tilde{l}_i\rangle + |\tilde{l}_i|^2\right|}{|k|^2} \leq \frac{\sqrt{U(l)}}{2|k|} < \frac{1}{2\sqrt{|k|}}. \tag{2.17}$$

In that case,

$$\begin{aligned} |k + \tilde{l}_i| &= \left(|k|^2 + 2\langle k, \tilde{l}_i\rangle + |\tilde{l}_i|^2\right)^{1/2} = |k|\left(1 + \frac{2\langle k, \tilde{l}_i\rangle + |\tilde{l}_i|^2}{|k|^2}\right)^{1/2} \\ &= \sum_{u=0}^{\infty} \binom{1/2}{u} \frac{1}{|k|^{2u-1}} P_u^i(k,l), \end{aligned}$$

where

$$P_u^i(k,l) := \left(2\langle k, \tilde{l}_i\rangle + |\tilde{l}_i|^2\right)^u$$

for all $i = 1, \dots, q-1$ and for all $u \in \mathbb{N}_0$, and $P_0^i(k,l) := 1$.

In particular $P_u^i(k,l) \in \mathbb{Z}[k,l]$, $\deg_k P_u^i \leq u$ and $\deg_l P_u^i \leq 2u$. Inequality (2.17) implies that for all $i = 1, \dots, q-1$ and for all $u \in \mathbb{N}$,

$$\frac{1}{|k|^{2u}} |P_u^i(k,l)| \leq (2\sqrt{|k|})^{-u}$$

uniformly in $k \in \mathbb{Z}^n$ and $l \in (\mathbb{Z}^n)^{2q}$ such that $|k| > U(l)$.

Let $N \in \mathbb{N}$. We deduce from the previous that for any $k \in \mathbb{Z}^n$ and $l \in (\mathbb{Z}^n)^{2q}$ with $|k| > U(l)$ and for all $i = 1, \dots, q-1$, we have

$$\begin{aligned} |k + \tilde{l}_i| &= \sum_{u=0}^{N} \binom{1/2}{u} \frac{1}{|k|^{2u-1}} P_u^i(k,l) + \mathcal{O}\Big(\sum_{u>N} |k|\,\Big|\binom{1/2}{u}\Big|\,(2\sqrt{|k|})^{-u}\Big) \\ &= \sum_{u=0}^{N} \binom{1/2}{u} \frac{1}{|k|^{2u-1}} P_u^i(k,l) + \mathcal{O}_N\Big(\frac{1}{|k|^{(N-1)/2}}\Big). \end{aligned}$$

It follows that for any $N \in \mathbb{N}$, we have uniformly in $k \in \mathbb{Z}^n$ and $l \in (\mathbb{Z}^n)^{2q}$ with $|k| > U(l)$ and for all $i \in I$,

$$|k + \tilde{l}_i|^{p_i} = \sum_{\alpha_i \in \{0,\dots,N\}^{p_i}} \binom{1/2}{\alpha_i} \frac{1}{|k|^{2|\alpha_i|_1 - p_i}} P_{\alpha_i}^i(k,l) + \mathcal{O}_N\Big(\frac{1}{|k|^{(N+1)/2 - p_i}}\Big),$$

where $P_{\alpha_i}^i(k,l) = \prod_{j=1}^{p_i} P_{\alpha_{i,j}}^i(k,l)$ for all $\alpha_i = (\alpha_{i,1}, \dots, \alpha_{i,p_i}) \in \{0, \dots, N\}^{p_i}$ and

$$\begin{aligned} \prod_{i \in I} |k + \tilde{l}_i|^{p_i} = \sum_{\alpha = (\alpha_i) \in \prod_{i \in I} \{0,\dots,N\}^{p_i}} &\binom{1/2}{\alpha} \frac{1}{|k|^{2|\alpha|_1 - |p|_1}} P_\alpha(k,l) \\ &+ \mathcal{O}_N\Big(\frac{1}{|k|^{(N+1)/2 - |p|_1}}\Big) \end{aligned}$$

with $P_\alpha(k,l) = \prod_{i\in I} P^i_{\alpha_i}(k,l) = \prod_{i\in I}\prod_{j=1}^{p_i} P^i_{\alpha_{i,j}}(k,l)$. □

Proof of Theorem 2.18. (i) All n, q, $p = (p_1,\dots,p_{q-1})$ and $\tilde{a} \in \mathcal{S}\left((\mathbb{Z}^n)^{2q}\right)$ are fixed as above and we define formally for any $l \in (\mathbb{Z}^n)^{2q}$

$$F(l,s) := \sum_{k\in\mathbb{Z}^n}{}' \prod_{i=1}^{q-1} |k+\tilde{l}_i|^{p_i} Q(k) e^{ik\cdot\Theta\sum_1^q l_j} |k|^{-s}. \tag{2.18}$$

Thus, still formally,

$$f(s) := \sum_{l\in(\mathbb{Z}^n)^{2q}} \tilde{a}_l F(l,s). \tag{2.19}$$

It is clear that $F(l,s)$ converges absolutely in the half plane $\{\sigma = \mathrm{Re}(s) > n+d+|p|_1\}$ where $d = \deg Q$.

Let $N \in \mathbb{N}$. Lemma 2.19 implies that for any $l \in (\mathbb{Z}^n)^{2q}$ and for $s \in \mathbb{C}$ such that $\sigma > n + |p|_1 + d$,

$$\begin{aligned} F(l,s) = &\sum_{|k|\le U(l)}{}' \prod_{i=1}^{q-1} |k+\tilde{l}_i|^{p_i} Q(k) e^{ik\cdot\Theta\sum_1^q l_j} |k|^{-s} \\ &+ \sum_{\substack{\alpha=(\alpha_i)_{i\in I}\in \\ \prod_{i\in I}\{0,\dots,N\}^{p_i}}} \binom{1/2}{\alpha} \sum_{|k|>U(l)} \frac{1}{|k|^{s+2|\alpha|_1-|p|_1}} P_\alpha(k,l) Q(k) e^{ik\cdot\Theta\sum_1^q l_j} \\ &+ G_N(l,s), \end{aligned}$$

where the map $s \mapsto G_N(l,s)$ is a holomorphic function in the half plane $D_N := \{\sigma > n+d+|p|_1 - \frac{N+1}{2}\}$ and verifies in it the bound $G_N(l,s) \ll_{N,\sigma} 1$ uniformly in l.

It follows that

$$F(l,s) = \sum_{\alpha=(\alpha_i)_{i\in I}\in\prod_{i\in I}\{0,\dots,N\}^{p_i}} H_\alpha(l,s) + R_N(l,s), \tag{2.20}$$

where

$$\begin{aligned} H_\alpha(l,s) &:= \sum_{k\in\mathbb{Z}^n}{}' \binom{1/2}{\alpha} \frac{1}{|k|^{s+2|\alpha|_1-|p|_1}} P_\alpha(k,l) Q(k) e^{ik\cdot\Theta\sum_1^q l_j}, \\ R_N(l,s) &:= \sum_{|k|\le U(l)}{}' \prod_{i=1}^{q-1} |k+\tilde{l}_i|^{p_i} Q(k) e^{ik\cdot\Theta\sum_1^q l_j} |k|^{-s} \\ &\quad - \sum_{|k|\le U(l)}{}' \sum_{\substack{\alpha=(\alpha_i)_{i\in I}\in \\ \prod_{i\in I}\{0,\dots,N\}^{p_i}}} \binom{1/2}{\alpha} \frac{P_\alpha(k,l)}{|k|^{s+2|\alpha|_1-|p|_1}} Q(k) e^{ik\cdot\Theta\sum_1^q l_j} + G_N(l,s). \end{aligned}$$

In particular there exists $A(N) > 0$ such that $s \mapsto R_N(l,s)$ extends holomorphically to the half plane D_N and satisfies $R_N(l,s) \ll_{N,\sigma} 1 + |l|^{A(N)}$ uniformly in l.

Write formally

$$h_\alpha(s) := \sum_l \tilde{a}_l H_\alpha(l,s).$$

Equation (2.20) and $R_N(l,s) \ll_{N,\sigma} 1 + |l|^{A(N)}$ imply that

$$f(s) \sim_N \sum_{\alpha=(\alpha_i)_{i\in I} \in \prod_{i\in I}\{0,\dots,N\}^{p_i}} h_\alpha(s), \tag{2.21}$$

where $\sim_N$ means modulo a holomorphic function in D_N.

Recall the decomposition $\binom{1/2}{\alpha} P_\alpha(k,l)Q(k) = \sum_{\beta\in L(\alpha)} M_{\alpha,\beta}(l)k^\beta$. Similarly we decompose $h_\alpha(s) = \sum_{\beta\in L(\alpha)} h_{\alpha,\beta}(s)$. Theorem 2.6 now implies that for all $\alpha = (\alpha_i)_{i\in I} \in \prod_{i\in I}\{0,\dots,N\}^{p_i}$ and $\beta \in L(\alpha)$

– the map $s \mapsto h_{\alpha,\beta}(s)$ has a meromorphic continuation to the whole complex plane $\mathbb{C}$ with only one simple possible pole at $s = n + |p|_1 - 2|\alpha|_1 + d_\beta$,

– the residue at this point is equal to

$$\operatorname*{Res}_{s=n+|p|_1-2|\alpha|_1+d_\beta} h_{\alpha,\beta}(s) = \sum_{l\in\mathcal{Z}} \tilde{a}_l M_{\alpha,\beta}(l) \int_{u\in S^{n-1}} u^\beta \, dS(u) \tag{2.22}$$

where $\mathcal{Z} := \{l \in (\mathbb{Z})^n)^{2q} \mid \sum_1^q l_j = 0\}$. If the right hand side is zero, $h_{\alpha,\beta}(s)$ is holomorphic in $\mathbb{C}$.

By (2.21), we deduce therefore that $f(s)$ has a meromorphic continuation to the half plane D_N, with only simple possible poles in the set $\{n + |p|_1 + k \mid -2N|p|_1 \le k \le d\}$. Taking now $N \to \infty$ yields the result.

(ii) Let $m \in \mathbb{N}_0$ and set $I(m) := \{(\alpha,\beta) \in \mathcal{I} \times \mathbb{N}_0^{(2q+1)n} \mid \beta \in L(\alpha) \text{ and } m = 2|\alpha|_1 - d_\beta + d\}$. If $(\alpha,\beta) \in I(m)$, then $|\alpha|_1 \le m$ and $|\beta|_1 \le 3m + d$, so $I(m)$ is finite. With N chosen such that $2N|p|_1 + d > m$, we get by (2.21) and (2.22)

$$\operatorname*{Res}_{s=n+d+|p|_1-m} f(s) = \sum_{l\in\mathcal{Z}} \tilde{a}_l \sum_{(\alpha,\beta)\in I(m)} M_{\alpha,\beta}(l) \int_{u\in S^{n-1}} u^\beta \, dS(u) = C(f,m)$$

with the convention $\sum_\emptyset = 0$. Thus, $n + d + |p|_1 - m$ is a pole of f if and only if $C(f,m) \neq 0$. □

3. Noncommutative integration on a simple spectral triple

In this section we revisit the notion of noncommutative integral pioneered by Alain Connes, paying particular attention to the reality (Tomita–Takesaki) operator J and to kernels of perturbed Dirac operators by symmetrized 1-forms.

3.1. Kernel dimension. We will have to compare here the kernels of $\mathcal{D}$ and $\mathcal{D}_A$ which are both finite dimensional:

Lemma 3.1. *Let* $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ *be a spectral triple with a reality operator* J *and chirality* χ. *If* $A \in \Omega^1_{\mathcal{D}}$ *is a* 1*-form, then the fluctuated Dirac operator*

$$\mathcal{D}_A := \mathcal{D} + A + \epsilon JAJ^{-1}$$

(*where* $\mathcal{D}J = \epsilon J\mathcal{D}$, $\epsilon = \pm 1$) *is an operator with compact resolvent, and in particular its kernel* $\operatorname{Ker} \mathcal{D}_A$ *is a finite dimensional space. This space is invariant by* J *and* χ.

Proof. Let T be a bounded operator and let z, z' be in the resolvent of $\mathcal{D} + T$ and $\mathcal{D}$, respectively. Then

$$(\mathcal{D} + T - z)^{-1} = (\mathcal{D} - z')^{-1}[1 - (T + z' - z)(\mathcal{D} + T - z)^{-1}].$$

Since $(\mathcal{D} - z')^{-1}$ is compact by hypothesis and since the term in bracket is bounded, $\mathcal{D} + T$ has a compact resolvent. Applying this to $T = A + \epsilon JAJ^{-1}$, $\mathcal{D}_A$ has a finite dimensional kernel (see for instance [27], Theorem 6.29).

Since according to the dimension, $J^2 = \pm 1$, J commutes or anticommutes with χ, χ commutes with the elements in the algebra $\mathcal{A}$ and $\mathcal{D}\chi = -\chi\mathcal{D}$ (see [9] or [23], p. 405), we obtain that $\mathcal{D}_A\chi = -\chi\mathcal{D}_A$ and $\mathcal{D}_A J = \pm J\mathcal{D}_A$, which gives the result. □

3.2. Pseudodifferential operators. Let $(A, \mathcal{D}, \mathcal{H})$ be a given real regular spectral triple of dimension n. Let P_0 be the projection on $\operatorname{Ker} \mathcal{D}$ and let P_A be the projection on $\operatorname{Ker} \mathcal{D}_A$. Write $D := \mathcal{D} + P_0$, $D_A := \mathcal{D}_A + P_A$. The maps P_0 and P_A are thus finite-rank selfadjoint bounded operators. We remark that D and D_A are selfadjoint invertible operators with compact inverses.

Remark 3.2. Since we only need to compute the residues and the value at 0 of the ζ_D, ζ_{D_A} functions, it is not necessary to define the operators $\mathcal{D}^{-1}$ or $\mathcal{D}_A^{-1}$ and the associated zeta functions. However, we note that all the work presented here could be done using the process of Higson in [26] which proves that we can add any smoothing operator to $\mathcal{D}$ or $\mathcal{D}_A$ such that the result is invertible without changing anything in the computation of residues.

Define for any $\alpha \in \mathbb{R}$,

$$OP^0 := \{T \mid t \mapsto F_t(T) \in C^\infty(\mathbb{R}, \mathcal{B}(\mathcal{H}))\},$$
$$OP^\alpha := \{T \mid T|D|^{-\alpha} \in OP^0\},$$

where $F_t(T) := e^{it|D|}Te^{-it|D|} = e^{it|\mathcal{D}|}Te^{-it|\mathcal{D}|}$ since $|D| = |\mathcal{D}| + P_0$. Define

$$\begin{aligned}\delta(T) &:= [|D|, T],\\ \nabla(T) &:= [\mathcal{D}^2, T],\\ \sigma_s(T) &:= |D|^s T|D|^{-s}, \quad s \in \mathbb{C}.\end{aligned}$$

It has been shown in [13] that $OP^0 = \bigcap_{p\geq 0} \mathrm{Dom}(\delta^p)$. In particular, OP^0 is a subalgebra of $\mathcal{B}(\mathcal{H})$ (while elements of OP^α are not necessarily bounded for $\alpha > 0$) and $\mathcal{A} \subseteq OP^0$, $J\mathcal{A}J^{-1} \subseteq OP^0$, $[\mathcal{D}, \mathcal{A}] \subseteq OP^0$. Note that $P_0 \in OP^{-\infty}$ and $\delta(OP^0) \subseteq OP^0$.

For any $t > 0$, $\mathcal{D}^t$ and $|\mathcal{D}|^t$ are in OP^t, and for any $\alpha \in \mathbb{R}$, D^α and $|D|^\alpha$ are in OP^α. By hypothesis, $|D|^{-n} \in \mathcal{L}^{(1,\infty)}(\mathcal{H})$, so $OP^{-\alpha} \subseteq \mathcal{L}^1(\mathcal{H})$ for any $\alpha > n$.

Lemma 3.3 ([13]). (i) *For any $T \in OP^0$ and $s \in \mathbb{C}$, $\sigma_s(T) \in OP^0$.*
(ii) *For any $\alpha, \beta \in \mathbb{R}$, $OP^\alpha OP^\beta \subseteq OP^{\alpha+\beta}$.*
(iii) *If $\alpha \leq \beta$, $OP^\alpha \subseteq OP^\beta$.*
(iv) *For any α, $\delta(OP^\alpha) \subseteq OP^\alpha$.*
(v) *For any α and $T \in OP^\alpha$, $\nabla(T) \in OP^{\alpha+1}$.*

Proof. See the appendix. □

Remark 3.4. Any operator in OP^α, $\alpha \in \mathbb{R}$, extends as a continuous linear operator from $\mathrm{Dom}\,|D|^{\alpha+1}$ to $\mathrm{Dom}\,|D|$, where the $\mathrm{Dom}\,|D|^\alpha$ spaces have their natural norms; see [13], [26].

We now introduce a definition of pseudodifferential operators in a slightly different way than in [13], [8], [26], which in particular pays attention to the reality operator J and the kernel of $\mathcal{D}$ and allows $\mathcal{D}$ and $|D|^{-1}$ to be pseudodifferential operators. It is more in the spirit of [11].

Definition 3.5. Let us define $\mathcal{D}(\mathcal{A})$ as the polynomial algebra generated by $\mathcal{A}$, $J\mathcal{A}J^{-1}$, $\mathcal{D}$ and $|\mathcal{D}|$. A pseudodifferential operator is an operator T so that there is an element $d \in \mathbb{Z}$ such that for any $N \in \mathbb{N}$, there exist $p \in \mathbb{N}_0$, $P \in \mathcal{D}(\mathcal{A})$ and $R \in OP^{-N}$ (p, P and R may depend on N) with $PD^{-2p} \in OP^d$ and

$$T = PD^{-2p} + R.$$

Define $\Psi(\mathcal{A})$ as the set of pseudodifferential operators and $\Psi(\mathcal{A})^k := \Psi(\mathcal{A}) \cap OP^k$.

Note that if A is a 1-form, then A and JAJ^{-1} are in $\mathcal{D}(\mathcal{A})$ and, moreover, it follows that $\mathcal{D}(\mathcal{A}) \subseteq \bigcup_{p\in\mathbb{N}_0} OP^p$. Since $|\mathcal{D}| \in \mathcal{D}(\mathcal{A})$ by construction and P_0 is a pseudodifferential operator, $|D|^p$ is a pseudodifferential operator (in OP^p) for any $p \in \mathbb{Z}$. We also remark that $\mathcal{D}(\mathcal{A}) \subseteq \Psi(\mathcal{A}) \subseteq \bigcup_{k\in\mathbb{Z}} OP^k$.

Lemma 3.6 ([13], [8]). *The set of all pseudodifferential operators $\Psi(\mathcal{A})$ is an algebra. Moreover, if $T \in \Psi(\mathcal{A})^d$ and $T \in \Psi(\mathcal{A})^{d'}$, then $TT' \in \Psi(\mathcal{A})^{d+d'}$.*

Proof. See the appendix. □

Due to the little difference of behavior between scalar and nonscalar pseudodifferential operators (i.e., when coefficients like $[\mathcal{D}, a]$, $a \in \mathcal{A}$, appear in P of Definition 3.5), it is convenient to introduce the following notation.

Definition 3.7. Let $\mathcal{D}_1(\mathcal{A})$ be the algebra generated by $\mathcal{A}$, $J\mathcal{A}J^{-1}$ and $\mathcal{D}$, and let $\Psi_1(\mathcal{A})$ be the set of pseudodifferential operators constructed as before, with $\mathcal{D}_1(\mathcal{A})$ instead of $\mathcal{D}(\mathcal{A})$. Note that $\Psi_1(\mathcal{A})$ is subalgebra of $\Psi(\mathcal{A})$.

Observe that $\Psi_1(\mathcal{A})$ does not necessarily contain operators such as $|D|^k$ where $k \in \mathbb{Z}$ is odd. This algebra is similar to the one defined in [11].

3.3. Zeta functions and dimension spectrum. For any operator B and if X is either D or D_A, we define

$$\zeta_X^B(s) := \mathrm{Tr}(B|X|^{-s}),$$
$$\zeta_X(s) := \mathrm{Tr}(|X|^{-s}).$$

The *dimension spectrum* $\mathrm{Sd}(\mathcal{A}, \mathcal{H}, \mathcal{D})$ of a spectral triple has been defined in [8], [13]. It is extended here to pay attention to the operator J and to our definition of pseudodifferential operator.

Definition 3.8. The spectrum dimension of the spectral triple is the subset $\mathrm{Sd}(\mathcal{A}, \mathcal{H}, \mathcal{D})$ of all poles of the functions $\zeta_D^P := s \mapsto \mathrm{Tr}(P|D|^{-s})$ where P is any pseudodifferential operator in OP^0. The spectral triple $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ is simple when these poles are all simple.

Remark 3.9. If $\mathrm{Sp}(\mathcal{A}, \mathcal{H}, \mathcal{D})$ denotes the set of all poles of the functions $s \mapsto \mathrm{Tr}(P|D|^{-s})$, where P is any pseudodifferential operator, then $\mathrm{Sd}(\mathcal{A}, \mathcal{H}, \mathcal{D}) \subseteq \mathrm{Sp}(\mathcal{A}, \mathcal{H}, \mathcal{D})$.

If $\mathrm{Sp}(\mathcal{A}, \mathcal{H}, \mathcal{D}) = \mathbb{Z}$, then $\mathrm{Sd}(\mathcal{A}, \mathcal{H}, \mathcal{D}) = \{n - k \mid k \in \mathbb{N}_0\}$: indeed, if P is a pseudodifferential operator in OP^0, and $q \in \mathbb{N}$ is such that $q > n$, then $P|D|^{-s}$ is in $OP^{-\mathrm{Re}(s)}$, so is trace-class for s in a neighborhood of q; hence q cannot be a pole of $s \mapsto \mathrm{Tr}(P|D|^{-s})$.

Remark 3.10. The set $\mathrm{Sp}(\mathcal{A}, \mathcal{H}, \mathcal{D})$ is also the set of all poles of functions $s \mapsto \mathrm{Tr}(B|D|^{-s-2p})$ where $p \in \mathbb{N}_0$ and $B \in \mathcal{D}(\mathcal{A})$.

3.4. The noncommutative integral $⨍$. We already defined the one parameter group $\sigma_z(T) := |D|^z T|D|^{-z}$, $z \in \mathbb{C}$.

Introducing the notation (recall that $\nabla(T) = [\mathcal{D}^2, T]$) for an operator T,

$$\varepsilon(T) := \nabla(T)D^{-2},$$

we get from [11], (2.44), the following expansion for $T \in OP^q$:

$$\sigma_z(T) \sim \sum_{r=0}^{N} g(z,r)\varepsilon^r(T) \mod OP^{-N-1+q}, \tag{3.1}$$

where $g(z,r) := \frac{1}{r!}\left(\frac{z}{2}\right)\dots\left(\frac{z}{2}-(r-1)\right) = \binom{z/2}{r}$ with the convention $g(z,0) := 1$.

We define the noncommutative integral by

$$⨍ T := \operatorname*{Res}_{s=0} \zeta_D^T(s) = \operatorname*{Res}_{s=0} \operatorname{Tr}(T|D|^{-s}).$$

Proposition 3.11 ([13]). *If the spectral triple is simple, then $⨍$ is a trace on $\Psi(\mathcal{A})$.*

Proof. See the appendix. □

4. Residues of ζ_{D_A} for a spectral triple with simple dimension spectrum

We fix a regular spectral triple $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ of dimension n and a selfadjoint 1-form A.

Recall that

$$\mathcal{D}_A := \mathcal{D} + \tilde{A},$$
$$D_A := \mathcal{D}_A + P_A,$$

where $\tilde{A} := A + \varepsilon JAJ^{-1}$ and P_A is the projection on $\operatorname{Ker}\mathcal{D}_A$. Observe that $\tilde{A} \in \mathcal{D}(\mathcal{A}) \cap OP^0$ and $\mathcal{D}_A \in \mathcal{D}(\mathcal{A}) \cap OP^1$.

Write

$$V_A := P_A - P_0.$$

As the following lemma shows, V_A is a smoothing operator:

Lemma 4.1. (i) $\bigcap_{k\geq 1} \operatorname{Dom}(\mathcal{D}_A)^k \subseteq \bigcap_{k\geq 1} \operatorname{Dom}|D|^k$.

(ii) $\operatorname{Ker}\mathcal{D}_A \subseteq \bigcap_{k\geq 1} \operatorname{Dom}|D|^k$.

(iii) *For any $\alpha, \beta \in \mathbb{R}$, $|D|^\beta P_A |D|^\alpha$ is bounded.*

(iv) $P_A \in OP^{-\infty}$.

Proof. (i) For any $p \in \mathbb{N}$ let us define $R_p := (\mathcal{D}_A)^p - \mathcal{D}^p$, so $R_p \in OP^{p-1}$ and $R_p(\mathrm{Dom}\,|D|^p) \subseteq \mathrm{Dom}\,|D|$ (see Remark 3.4).

Fix $k \in \mathbb{N}$, $k \geq 2$. Since $\mathrm{Dom}\,\mathcal{D}_A = \mathrm{Dom}\,\mathcal{D} = \mathrm{Dom}\,|D|$, we have

$$\mathrm{Dom}(\mathcal{D}_A)^k = \{\phi \in \mathrm{Dom}\,|D| \mid (\mathcal{D}^j + R_j)\phi \in \mathrm{Dom}\,|D| \text{ for all } 1 \leq j \leq k-1\}.$$

Let $\phi \in \mathrm{Dom}(\mathcal{D}_A)^k$. We prove by recurrence that $\phi \in \mathrm{Dom}\,|D|^{j+1}$ for any $j \in \{1, \ldots, k-1\}$. We have $\phi \in \mathrm{Dom}\,|D|$ and $(\mathcal{D} + R_1)\phi \in \mathrm{Dom}\,|D|$. Thus, since $R_1\phi \in \mathrm{Dom}\,|D|$, it follows that $\mathcal{D}\phi \in \mathrm{Dom}\,|D|$, which proves that $\phi \in \mathrm{Dom}\,|D|^2$. Hence the case $j = 1$ is done.

Suppose now that $\phi \in \mathrm{Dom}\,|D|^{j+1}$ for a $j \in \{1, \ldots, k-2\}$. Since $(\mathcal{D}^{j+1} + R_{j+1})\phi \in \mathrm{Dom}\,|D|$, and $R_{j+1}\phi \in \mathrm{Dom}\,|D|$, we get $\mathcal{D}^{j+1}\phi \in \mathrm{Dom}\,|D|$, which proves that $\phi \in \mathrm{Dom}\,|D|^{j+2}$.

Finally, if we set $j = k-1$, we get $\phi \in \mathrm{Dom}\,|D|^k$, so $\mathrm{Dom}(\mathcal{D}_A)^k \subseteq \mathrm{Dom}\,|D|^k$.

(ii) follows from $\mathrm{Ker}\,\mathcal{D}_A \subseteq \bigcap_{k\geq 1} \mathrm{Dom}(\mathcal{D}_A)^k$ and (i).

(iii) Let us first check that $|D|^\alpha P_A$ is bounded. We define D_0 as the operator with domain $\mathrm{Dom}\,D_0 = \mathrm{Im}\,P_A \cap \mathrm{Dom}\,|D|^\alpha$ and such that $D_0\phi = |D|^\alpha\phi$. Since $\mathrm{Dom}\,D_0$ is finite dimensional, D_0 extends to a bounded operator on $\mathcal{H}$ with finite rank. We have

$$\sup_{\substack{\phi \in \mathrm{Dom}\,|D|^\alpha P_A \\ \|\phi\| \leq 1}} \||D|^\alpha P_A\phi\| \leq \sup_{\substack{\phi \in \mathrm{Dom}\,D_0 \\ \|\phi\| \leq 1}} \||D|^\alpha\phi\| = \|D_0\| < \infty,$$

so $|D|^\alpha P_A$ is bounded. Note that by (ii), $\mathrm{Dom}\,D_0 = \mathrm{Im}\,P_A$ and $\mathrm{Dom}\,|D|^\alpha P_A = \mathcal{H}$. We now show that $P_A|D|^\alpha$ is bounded: Let $\phi \in \mathrm{Dom}\,P_A|D|^\alpha = \mathrm{Dom}\,|D|^\alpha$. By (ii), we have $\mathrm{Im}\,P_A \subseteq \mathrm{Dom}\,|D|^\alpha$, so we get

$$\begin{aligned}\|P_A|D|^\alpha\phi\| &\leq \sup_{\substack{\psi \in \mathrm{Im}\,P_A \\ \|\psi\| \leq 1}} |\langle\psi, |D|^\alpha\phi\rangle| \leq \sup_{\substack{\psi \in \mathrm{Im}\,P_A \\ \|\psi\| \leq 1}} |\langle |D|^\alpha\psi, \phi\rangle| \\ &\leq \sup_{\substack{\psi \in \mathrm{Im}\,P_A \\ \|\psi\| \leq 1}} \||D|^\alpha\psi\|\|\phi\| = \|D_0\|\|\phi\|.\end{aligned}$$

(iv) For any $k \in \mathbb{N}_0$ and $t \in \mathbb{R}$, $\delta^k(P_A)|D|^t$ is a linear combination of terms of the form $|D|^\beta P_A|D|^\alpha$, so the result follows from (iii). □

Remark 4.2. We will see later in the noncommutative torus example how important the difference between $\mathcal{D}_A$ and $\mathcal{D} + A$ is. In particular, the inclusion $\mathrm{Ker}\,\mathcal{D} \subseteq \mathrm{Ker}\,\mathcal{D} + A$ is not satisfied since A, in contrast to $\tilde{A}$, does not preserve $\mathrm{Ker}\,\mathcal{D}$.

The coefficient of the nonconstant term Λ^k $(k > 0)$ in the expansion (1.5) of the spectral action $S(\mathcal{D}_A, \Phi, \Lambda)$ is equal to the residue of $\zeta_{\mathcal{D}_A}(s)$ at k. We will see in this section how we can compute these residues in terms of the noncommutative integral of certain operators.

Define for any operator T, $p \in \mathbb{N}$, $s \in \mathbb{C}$,

$$K_p(T,s) := (-s/2)^p \int_{0 \le t_1 \le \cdots \le t_p \le 1} \sigma_{-st_1}(T) \dots \sigma_{-st_p}(T)\, dt$$

with $dt := dt_1 \dots dt_p$.

Note that if $T \in OP^\alpha$, then $\sigma_z(T) \in OP^\alpha$ for $z \in \mathbb{C}$ and $K_p(T,s) \in OP^{\alpha p}$.

Let us define

$$\begin{aligned} X &:= \mathcal{D}_A^2 - \mathcal{D}^2 = \tilde{A}\mathcal{D} + \mathcal{D}\tilde{A} + \tilde{A}^2, \\ X_V &:= X + V_A; \end{aligned}$$

then $X \in \mathcal{D}_1(\mathcal{A}) \cap OP^1$ and by Lemma 4.1,

$$X_V \sim X \mod OP^{-\infty}. \tag{4.1}$$

We will use

$$Y := \log(D_A^2) - \log(D^2),$$

which makes sense since $D_A^2 = \mathcal{D}_A^2 + P_A$ is invertible for any A.

By definition of X_V, we see that

$$Y = \log(D^2 + X_V) - \log(D^2).$$

Lemma 4.3 ([11]). (i) *The map Y is a pseudodifferential operator in OP^{-1} with the following expansion for any $N \in \mathbb{N}$:*

$$Y \sim \sum_{p=1}^{N} \sum_{k_1,\dots,k_p=0}^{N-p} \frac{(-1)^{|k|_1+p+1}}{|k|_1+p} \nabla^{k_p}(X\nabla^{k_{p-1}}(\dots X\nabla^{k_1}(X)\dots))D^{-2(|k|_1+p)}$$

mod OP^{-N-1}.

(ii) *For any $N \in \mathbb{N}$ and $s \in \mathbb{C}$ we have*

$$|D_A|^{-s} \sim |D|^{-s} + \sum_{p=1}^{N} K_p(Y,s)|D|^{-s} \mod OP^{-N-1-\mathrm{Re}(s)}. \tag{4.2}$$

Proof. (i) We follow [11], Lemma 2.2. By functional calculus, $Y = \int_0^\infty I(\lambda)\, d\lambda$ where

$$I(\lambda) \sim \sum_{p=1}^{N} (-1)^{p+1}\big((D^2+\lambda)^{-1}X_V\big)^p (D^2+\lambda)^{-1} \mod OP^{-N-3}.$$

By (4.1), $((D^2+\lambda)^{-1}X_V)^p \sim ((D^2+\lambda)^{-1}X)^p \mod OP^{-\infty}$ and we get

$$I(\lambda) \sim \sum_{p=1}^{N}(-1)^{p+1}\big((D^2+\lambda)^{-1}X\big)^p(D^2+\lambda)^{-1} \mod OP^{-N-3}.$$

We set $A_p(X) := ((D^2+\lambda)^{-1}X)^p(D^2+\lambda)^{-1}$ and $L := (D^2+\lambda)^{-1} \in OP^{-2}$ for a fixed λ. Since $[D^2+\lambda, X] \sim \nabla(X) \mod OP^{-\infty}$, a recurrence proves that if T is an operator in OP^r, then for $q \in \mathbb{N}_0$,

$$A_1(T) = LTL \sim \sum_{k=0}^{q}(-1)^k\nabla^k(T)L^{k+2} \mod OP^{r-q-5}.$$

With $A_p(X) = LXA_{p-1}(X)$ another recurrence gives for any $q \in \mathbb{N}_0$,

$$A_p(X) \sim \sum_{k_1,\dots,k_p=0}^{q}(-1)^{|k|_1}\nabla^{k_p}(X\nabla^{k_{p-1}}(\dots X\nabla^{k_1}(X)\dots))L^{|k|_1+p+1}$$

mod OP^{-q-p-3}, which implies that

$$I(\lambda) \sim \sum_{p=1}^{N}(-1)^{p+1}\sum_{k_1,\dots,k_p=0}^{N-p}(-1)^{|k|_1}\nabla^{k_p}(X\nabla^{k_{p-1}}(\dots X\nabla^{k_1}(X)\dots))L^{|k|_1+p+1}$$

mod OP^{-N-3}. With $\int_0^\infty(D^2+\lambda)^{-(|k|_1+p+1)}d\lambda = \frac{1}{|k|_1+p}D^{-2(|k|_1+p)}$ we get the result provided we control the remainders. Such a control is given in [11], (2.27).

(ii) We have $|D_A|^{-s} = e^{B-(s/2)Y}e^{-B}|D|^{-s}$ where $B := (-s/2)\log(D^2)$. Following [11], Theorem 2.4, we obtain that

$$|D_A|^{-s} = |D|^{-s} + \sum_{p=1}^{\infty}K_p(Y,s)|D|^{-s} \tag{4.3}$$

and each $K_p(Y,s)$ is in OP^{-p}. □

Corollary 4.4. *For any $p \in \mathbb{N}$ and $r_1,\dots,r_p \in \mathbb{N}_0$ we have $\varepsilon^{r_1}(Y)\dots\varepsilon^{r_p}(Y) \in \Psi_1(\mathcal{A})$.*

Proof. If for any $q \in \mathbb{N}$ and $k = (k_1,\dots,k_q) \in \mathbb{N}_0^q$,

$$\Gamma_q^k(X) := \tfrac{(-1)^{|k|_1+q+1}}{|k|_1+q}\nabla^{k_q}(X\nabla^{k_{q-1}}(\dots X\nabla^{k_1}(X)\dots)),$$

then $\Gamma_q^k(X) \in OP^{|k|_1+q}$. For any $N \in \mathbb{N}$,

$$Y \sim \sum_{q=1}^{N}\sum_{k_1,\dots,k_q=0}^{N-q}\Gamma_q^k(X)D^{-2(|k|_1+q)} \mod OP^{-N-1}. \tag{4.4}$$

Note that the $\Gamma_q^k(X)$ are in $\mathcal{D}_1(\mathcal{A})$, which in view of (4.4) shows that Y and thus $\varepsilon^r(Y) = \nabla^r(Y)D^{-2r}$ are also in $\Psi_1(\mathcal{A})$. □

We remark, as in [10], that the fluctuations leave invariant the first term of the spectral action (1.5). This is a generalization of the fact that in the commutative case, the noncommutative integral depends only on the principal symbol of the Dirac operator $\mathcal{D}$ and this symbol is stable by adding a gauge potential like in $\mathcal{D} + A$. Note however that the symmetrized gauge potential $A + \epsilon JAJ^{-1}$ is always zero in this case for any selfadjoint 1-form A.

Lemma 4.5. *If the spectral triple is simple, then formula* (1.6) *can be extended as*

$$\zeta_{D_A}(0) - \zeta_D(0) = \sum_{q=1}^{n} \frac{(-1)^q}{q} \fint (\tilde{A}D^{-1})^q. \tag{4.5}$$

Proof. Since the spectral triple is simple, equation (4.3) entails that

$$\zeta_{D_A}(0) - \zeta_D(0) = \mathrm{Tr}(K_1(Y,s)|D|^{-s})_{|s=0}.$$

Thus, with (3.1), we get $\zeta_{D_A}(0) - \zeta_D(0) = -\frac{1}{2}\fint Y$. Replacing A by $\tilde{A}$, the same proof as in [11] gives

$$-\frac{1}{2}\fint Y = \sum_{q=1}^{n} \frac{(-1)^q}{q} \fint (\tilde{A}D^{-1})^q. \qquad \square$$

Lemma 4.6. *For any $k \in \mathbb{N}_0$ we have*

$$\begin{aligned}&\operatorname*{Res}_{s=n-k} \zeta_{D_A}(s)\\ &\quad = \operatorname*{Res}_{s=n-k} \zeta_D(s) + \sum_{p=1}^{k} \sum_{r_1,\dots,r_p=0}^{k-p} \operatorname*{Res}_{s=n-k} h(s,r,p)\,\mathrm{Tr}(\varepsilon^{r_1}(Y)\dots\varepsilon^{r_p}(Y)|D|^{-s}),\end{aligned}$$

where

$$h(s,r,p) := (-s/2)^p \int_{0\le t_1\le\cdots\le t_p\le 1} g(-st_1,r_1)\dots g(-st_p,r_p)\,dt.$$

Proof. From Lemma 4.3 (ii) it follows that $|D_A|^{-s} \sim |D|^{-s} + \sum_{p=1}^{k} K_p(Y,s)|D|^{-s}$ mod $OP^{-(k+1)-\mathrm{Re}(s)}$, where the convention $\sum_\emptyset = 0$ is used. Thus, we get for s in a neighborhood of $n-k$,

$$|D_A|^{-s} - |D|^{-s} - \sum_{p=1}^{k} K_p(Y,s)|D|^{-s} \in OP^{-(k+1)-\mathrm{Re}(s)} \subseteq \mathcal{L}^1(\mathcal{H}),$$

which gives

$$\operatorname*{Res}_{s=n-k} \zeta_{D_A}(s) = \operatorname*{Res}_{s=n-k} \zeta_D(s) + \sum_{p=1}^{k} \operatorname*{Res}_{s=n-k} \operatorname{Tr}(K_p(Y,s)|D|^{-s}). \tag{4.6}$$

Let us fix $1 \le p \le k$ and $N \in \mathbb{N}$. By (3.1) we get

$$\begin{aligned} K_p(Y,s) \sim (-s/2)^p \int_{0\le t_1\le \dots t_p \le 1} \sum_{r_1,\dots,r_p=0}^{N} g(-st_1,r_1)\dots g(-st_p,r_p) \\ \varepsilon^{r_1}(Y)\dots\varepsilon^{r_p}(Y)\,dt \mod OP^{-N-p-1}. \end{aligned} \tag{4.7}$$

If we now take $N = k - p$, we obtain that

$$K_p(Y,s)|D|^{-s} - \sum_{r_1,\dots,r_p=0}^{k-p} h(s,r,p)\varepsilon^{r_1}(Y)\dots\varepsilon^{r_p}(Y)|D|^{-s} \in OP^{-k-1-\mathrm{Re}(s)} \subseteq \mathcal{L}^1(\mathcal{H})$$

for s in a neighborhood of $n-k$, so (4.6) gives the result. □

Our operators $|D_A|^k$ are pseudodifferential operators:

Lemma 4.7. *For any $k \in \mathbb{Z}$ we have $|D_A|^k \in \Psi^k(\mathcal{A})$.*

Proof. Using (4.7), we see that $K_p(Y,s)$ is a pseudodifferential operator in OP^{-p}, so (4.2) shows that $|D_A|^k$ is a pseudodifferential operator in OP^k. □

The following result is quite important since it shows that one can use $⨍$ for D or D_A:

Proposition 4.8. *If the spectral triple is simple, then $\operatorname*{Res}_{s=0} \operatorname{Tr}(P|D_A|^{-s}) = ⨍ P$ for any pseudodifferential operator P. In particular, for any $k \in \mathbb{N}_0$,*

$$⨍ |D_A|^{-(n-k)} = \operatorname*{Res}_{s=n-k} \zeta_{D_A}(s).$$

Proof. Suppose that $P \in OP^k$ with $k \in \mathbb{Z}$ and fix $p \ge 1$. With (4.7) we see that

$$\begin{aligned} &PK_p(Y,s)|D|^{-s} \\ &\quad\sim \sum_{r_1,\dots,r_p=0}^{N} h(s,r,p)P\varepsilon^{r_1}(Y)\dots\varepsilon^{r_p}(Y)|D|^{-s} \mod OP^{-N-p-1+k-\mathrm{Re}(s)} \end{aligned}$$

for any $N \in \mathbb{N}$. Thus if we take $N = n - p + k$, we get

$$\operatorname*{Res}_{s=0} \operatorname{Tr}(PK_p(Y,s)|D|^{-s}) = \sum_{r_1,\dots,r_p=0}^{n-p+k} \operatorname*{Res}_{s=0} h(s,r,p) \operatorname{Tr}(P\varepsilon^{r_1}(Y)\dots\varepsilon^{r_p}(Y)|D|^{-s}).$$

Now $s = 0$ is a zero of the analytic function $s \mapsto h(s,r,p)$. Moreover, the map $s \mapsto \operatorname{Tr} P\varepsilon^{r_1}(Y)\dots\varepsilon^{r_p}(Y)|D|^{-s}$ has only simple poles. So, by hypothesis, we see that $\operatorname*{Res}_{s=0} h(s,r,p) \operatorname{Tr}(P\varepsilon^{r_1}(Y)\dots\varepsilon^{r_p}(Y)|D|^{-s}) = 0$ and

$$\operatorname*{Res}_{s=0} \operatorname{Tr}(PK_p(Y,s)|D|^{-s}) = 0. \tag{4.8}$$

Using (4.2), $P|D_A|^{-s} \sim P|D|^{-s} + \sum_{p=1}^{k+n} PK_p(Y,s)|D|^{-s} \mod OP^{-n-1-\mathrm{Re}(s)}$ and thus

$$\operatorname*{Res}_{s=0} \operatorname{Tr}(P|D_A|^{-s}) = ⨍ P + \sum_{p=1}^{k+n} \operatorname*{Res}_{s=0} \operatorname{Tr}(PK_p(Y,s)|D|^{-s}). \tag{4.9}$$

The result now follows from (4.8) and (4.9). To obtain the last equality, one uses the pseudodifferential operator $|D_A|^{-(n-k)}$. □

Proposition 4.9. *If the spectral triple is simple, then*

$$⨍ |D_A|^{-n} = ⨍ |D|^{-n}. \tag{4.10}$$

Proof. This follows from Lemma 4.6 and the previous proposition for $k = 0$. □

Lemma 4.10. *If the spectral triple is simple, then*

(i) $⨍ |D_A|^{-(n-1)} = ⨍ |D|^{-(n-1)} - \left(\frac{n-1}{2}\right) ⨍ X|D|^{-n-1}$,

(ii) $⨍ |D_A|^{-(n-2)} = ⨍ |D|^{-(n-2)} + \frac{n-2}{2}\left(- ⨍ X|D|^{-n} + \frac{n}{4} ⨍ X^2|D|^{-2-n}\right)$.

Proof. (i) By (4.2),

$$\begin{aligned}\operatorname*{Res}_{s=n-1} \zeta_{D_A}(s) - \zeta_D(s) &= \operatorname*{Res}_{s=n-1} (-s/2) \operatorname{Tr}(Y|D|^{-s}) \\ &= -\tfrac{n-1}{2} \operatorname*{Res}_{s=0} \operatorname{Tr}(Y|D|^{-(n-1)}|D|^{-s})\end{aligned}$$

where for the last equality we use the simple dimension spectrum hypothesis. Lemma 4.3 (i) yields $Y \sim XD^{-2} \mod OP^{-2}$ and $Y|D|^{-(n-1)} \sim X|D|^{-n-1} \mod OP^{-n-1} \subseteq \mathcal{L}^1(\mathcal{H})$. Thus,

$$\operatorname*{Res}_{s=0} \operatorname{Tr}(Y|D|^{-(n-1)}|D|^{-s}) = \operatorname*{Res}_{s=0} \operatorname{Tr}(X|D|^{-n-1}|D|^{-s}) = ⨍ X|D|^{-n-1}.$$

(ii) Lemma 4.6 (ii) gives

$$\operatorname*{Res}_{s=n-2} \zeta_{D_A}(s) = \operatorname*{Res}_{s=n-2} \zeta_D(s) + \operatorname*{Res}_{s=n-2} \sum_{r=0}^{1} h(s,r,1)\operatorname{Tr}(\varepsilon^r(Y)|D|^{-s}) + h(s,0,2)\operatorname{Tr}(Y^2|D|^{-s}).$$

We have $h(s,0,1) = -\frac{s}{2}$, $h(s,1,1) = \frac{1}{2}\left(\frac{s}{2}\right)^2$ and $h(s,0,2) = \frac{1}{2}\left(\frac{s}{2}\right)^2$. Using again Lemma 4.3 (i), we obtain that

$$Y \sim XD^{-2} - \tfrac{1}{2}\nabla(X)D^{-4} - \tfrac{1}{2}X^2D^{-4} \mod OP^{-3}.$$

Thus,

$$\operatorname*{Res}_{s=n-2} \operatorname{Tr}\left(Y|D|^{-s}\right) = ⨍ X|D|^{-n} - \frac{1}{2}⨍ (\nabla(X) + X^2)|D|^{-2-n}.$$

Moreover, using $⨍ \nabla(X)|D|^{-k} = 0$ for any $k \geq 0$ since $⨍$ is a trace,

$$\operatorname*{Res}_{s=n-2} \operatorname{Tr}\left(\varepsilon(Y)|D|^{-s}\right) = \operatorname*{Res}_{s=n-2} \operatorname{Tr}\left(\nabla(X)D^{-4}|D|^{-s}\right) = ⨍ \nabla(X)|D|^{-2-n} = 0.$$

Similarly, since $Y \sim XD^{-2} \mod OP^{-2}$ and $Y^2 \sim X^2D^{-4} \mod OP^{-3}$, we get

$$\operatorname*{Res}_{s=n-2} \operatorname{Tr}(Y^2|D|^{-s}) = \operatorname*{Res}_{s=n-2} \operatorname{Tr}(X^2D^{-4}|D|^{-s}) = ⨍ X^2|D|^{-2-n}.$$

Thus,

$$\begin{aligned}\operatorname*{Res}_{s=n-2} \zeta_{D_A}(s) = {} & \operatorname*{Res}_{s=n-2} \zeta_D(s) \\ & + \left(-\frac{n-2}{2}\right)\left(⨍ X|D|^{-n} - \frac{1}{2}⨍ (\nabla(X) + X^2)|D|^{-2-n}\right) \\ & + \frac{1}{2}\left(\frac{n-2}{2}\right)^2 ⨍ \nabla(X)|D|^{-2-n} + \frac{1}{2}\left(\frac{n-2}{2}\right)^2 ⨍ X^2|D|^{-2-n}.\end{aligned}$$

Finally,

$$\begin{aligned}\operatorname*{Res}_{s=n-2} \zeta_{D_A}(s) = {} & \operatorname*{Res}_{s=n-2} \zeta_D(s) + \left(-\frac{n-2}{2}\right)\left(⨍ X|D|^{-n} - \frac{1}{2}⨍ X^2|D|^{-2-n}\right) \\ & + \frac{1}{2}\left(\frac{n-2}{2}\right)^2 ⨍ X^2|D|^{-2-n},\end{aligned}$$

and the result follows from Proposition 4.8. □

Corollary 4.11. *If the spectral triple is simple and satisfies the identity* $⨍ |D|^{-(n-2)} = ⨍ \tilde{A}\mathcal{D}|D|^{-n} = ⨍ \mathcal{D}\tilde{A}|D|^{-n} = 0$, *then*

$$⨍ |D_A|^{-(n-2)} = \frac{n(n-2)}{4}\Big(⨍ \tilde{A}\mathcal{D}\tilde{A}\mathcal{D}|D|^{-n-2} + \frac{n-2}{n} ⨍ \tilde{A}^2|D|^{-n}\Big).$$

Proof. By the previous lemma, we have

$$\operatorname*{Res}_{s=n-2} \zeta_{D_A}(s) = \frac{n-2}{2}\Big(- ⨍ \tilde{A}^2|D|^{-n} + \frac{n}{4} ⨍ (\tilde{A}\mathcal{D}\tilde{A}\mathcal{D} + \mathcal{D}\tilde{A}\mathcal{D}\tilde{A} + \tilde{A}\mathcal{D}^2\tilde{A} + \mathcal{D}\tilde{A}^2\mathcal{D})|D|^{-n-2}\Big).$$

Since $\nabla(\tilde{A}) \in OP^1$, the trace property of $⨍$ yields the result. □

5. The noncommutative torus

5.1. Notations. Let $C^\infty(\mathbb{T}^n_\Theta)$ be the smooth noncommutative n-torus associated to a non-zero skew-symmetric deformation matrix $\Theta \in M_n(\mathbb{R})$ (see [5], [30]). This means that $C^\infty(\mathbb{T}^n_\Theta)$ is the algebra generated by n unitaries u_i, $i = 1, \dots, n$, subject to the relations

$$u_i u_j = e^{i\Theta_{ij}} u_j u_i, \tag{5.1}$$

and with Schwartz coefficients: an element $a \in C^\infty(\mathbb{T}^n_\Theta)$ can be written as $a = \sum_{k\in\mathbb{Z}^n} a_k U_k$, where $\{a_k\} \in \mathcal{S}(\mathbb{Z}^n)$ with the Weyl elements defined by $U_k := e^{-\frac{i}{2}k\cdot\chi k} u_1^{k_1} \dots u_n^{k_n}$, $k \in \mathbb{Z}^n$. Relation (5.1) reads

$$U_k U_q = e^{-\frac{i}{2}k\cdot\Theta q} U_{k+q} \quad \text{and} \quad U_k U_q = e^{-ik\cdot\Theta q} U_q U_k \tag{5.2}$$

where χ is the matrix restriction of Θ to its upper triangular part. Thus unitary operators U_k satisfy $U_k^* = U_{-k}$ and $[U_k, U_l] = -2i \sin\big(\frac{1}{2}k\cdot\Theta l\big)U_{k+l}$.

Let τ be the trace on $C^\infty(\mathbb{T}^n_\Theta)$ defined by $\tau\big(\sum_{k\in\mathbb{Z}^n} a_k U_k\big) := a_0$ and $\mathcal{H}_\tau$ be the GNS Hilbert space obtained by completion of $C^\infty(\mathbb{T}^n_\Theta)$ with respect to the norm induced by the scalar product $\langle a, b\rangle := \tau(a^*b)$. On $\mathcal{H}_\tau = \{\sum_{k\in\mathbb{Z}^n} a_k U_k \mid \{a_k\}_k \in l^2(\mathbb{Z}^n)\}$ we consider the left and right regular representations of $C^\infty(\mathbb{T}^n_\Theta)$ by bounded operators, which we denote by $L(\cdot)$ and $R(\cdot)$, respectively.

Let also δ_μ, $\mu \in \{1, \dots, n\}$, be the n (pairwise commuting) canonical derivations defined by

$$\delta_\mu(U_k) := ik_\mu U_k. \tag{5.3}$$

We need to fix notations: let $\mathcal{A}_\Theta := C^\infty(\mathbb{T}^n_\Theta)$ acting on $\mathcal{H} := \mathcal{H}_\tau \otimes \mathbb{C}^{2^m}$ with $n = 2m$ or $n = 2m+1$ (i.e., $m = \lfloor \frac{n}{2} \rfloor$ is the integer part of $\frac{n}{2}$), the square integrable sections of the trivial spin bundle over $\mathbb{T}^n$.

Each element of $\mathcal{A}_\Theta$ is represented on $\mathcal{H}$ as $L(a) \otimes 1_{2^m}$ where L (resp. R) is the left (resp. right) multiplication. The Tomita conjugation $J_0(a) := a^*$ satisfies $[J_0, \delta_\mu] = 0$ and we define $J := J_0 \otimes C_0$ where C_0 is an operator on $\mathbb{C}^{2^m}$. The Dirac operator is given by

$$\mathcal{D} := -i\delta_\mu \otimes \gamma^\mu, \tag{5.4}$$

where we use hermitian Dirac matrices γ. It is defined and symmetric on the dense subset of $\mathcal{H}$ given by $C^\infty(\mathbb{T}^n_\Theta) \otimes \mathbb{C}^{2^m}$. We still denote $\mathcal{D}$ its selfadjoint extension. This implies that

$$C_0\gamma^\alpha = -\varepsilon\gamma^\alpha C_0, \tag{5.5}$$

and

$$\mathcal{D}\, U_k \otimes e_i = k_\mu U_k \otimes \gamma^\mu e_i,$$

where (e_i) is the canonical basis of $\mathbb{C}^{2^m}$. Moreover, $C_0^2 = \pm 1_{2^m}$, depending on the parity of m. Finally, one introduces the chirality (which in the even case is $\chi := \mathrm{id} \otimes (-i)^m \gamma^1 \dots \gamma^n$) and this yields that $(\mathcal{A}_\Theta, \mathcal{H}, \mathcal{D}, J, \chi)$ satisfies all axioms of a spectral triple; see [7], [23].

The Dirac operator $V_u \mathcal{D} V_u^*$ perturbed by the unitary

$$V_u := (L(u) \otimes 1_{2^m}) J (L(u) \otimes 1_{2^m}) J^{-1},$$

defined by $uu^* = u^*u = U_0$ for every unitary $u \in \mathcal{A}$, must satisfy condition (1.3) (which is equivalent to $\mathcal{H}$ being endowed with a structure of $\mathcal{A}_\Theta$-bimodule). This implies the existence of a symmetrized covariant Dirac operator,

$$\mathcal{D}_A := \mathcal{D} + A + \epsilon J A J^{-1},$$

since $V_u \mathcal{D} V_u^* = \mathcal{D}_{L(u)\otimes 1_{2^m}[\mathcal{D}, L(u^*)\otimes 1_{2^m}]}$: in fact, for $a \in \mathcal{A}_\Theta$, using $J_0 L(a) J_0^{-1} = R(a^*)$, we obtain that

$$\epsilon J(L(a) \otimes \gamma^\alpha) J^{-1} = -R(a^*) \otimes \gamma^\alpha,$$

and the representation L and the anti-representation R are $\mathbb{C}$-linear, commute and satisfy

$$[\delta_\alpha, L(a)] = L(\delta_\alpha a), \quad [\delta_\alpha, R(a)] = R(\delta_\alpha a).$$

This induces some covariance property for the Dirac operator; one checks that

$$L(U_k) \otimes 1_{2^m}[\mathcal{D}, L(U_k^*) \otimes 1_{2^m}] = 1 \otimes (-k_\mu \gamma^\mu), \tag{5.6}$$

for all $k \in \mathbb{Z}^n$. Thus, with (5.5), we get $U_k[\mathcal{D}, U_k^*] + \epsilon J U_k [\mathcal{D}, U_k^*] J^{-1} = 0$ and

$$V_{U_k} \mathcal{D} V_{U_k}^* = \mathcal{D} = \mathcal{D}_{L(U_k)\otimes 1_{2^m}[\mathcal{D}, L(U_k^*)\otimes 1_{2^m}]}. \tag{5.7}$$

Moreover, we obtain the gauge transformation

$$V_u \mathcal{D}_A V_u^* = \mathcal{D}_{\gamma_u(A)}, \tag{5.8}$$

where the gauged transform 1-form of A is

$$\gamma_u(A) := u[\mathcal{D}, u^*] + uAu^*, \tag{5.9}$$

with the shorthand $L(u) \otimes 1_{2^m} \to u$.

As a consequence, the spectral action is gauge invariant:

$$\mathcal{S}(\mathcal{D}_A, \Phi, \Lambda) = \mathcal{S}(\mathcal{D}_{\gamma_u(A)}, \Phi, \Lambda).$$

An arbitrary selfadjoint 1-form A can be written as

$$A = L(-iA_\alpha) \otimes \gamma^\alpha \quad \text{with } A_\alpha = -A_\alpha^* \in \mathcal{A}_\Theta, \tag{5.10}$$

thus

$$\mathcal{D}_A = -i(\delta_\alpha + L(A_\alpha) - R(A_\alpha)) \otimes \gamma^\alpha. \tag{5.11}$$

Defining

$$\tilde{A}_\alpha := L(A_\alpha) - R(A_\alpha)$$

we get $\mathcal{D}_A^2 = -g^{\alpha_1\alpha_2}(\delta_{\alpha_1} + \tilde{A}_{\alpha_1})(\delta_{\alpha_2} + \tilde{A}_{\alpha_2}) \otimes 1_{2^m} - \frac{1}{2}\Omega_{\alpha_1\alpha_2} \otimes \gamma^{\alpha_1\alpha_2}$, where

$$\begin{aligned} \gamma^{\alpha_1\alpha_2} &:= \tfrac{1}{2}(\gamma^{\alpha_1}\gamma^{\alpha_2} - \gamma^{\alpha_2}\gamma^{\alpha_1}), \\ \Omega_{\alpha_1\alpha_2} &:= [\delta_{\alpha_1} + \tilde{A}_{\alpha_1}, \delta_{\alpha_2} + \tilde{A}_{\alpha_2}] = L(F_{\alpha_1\alpha_2}) - R(F_{\alpha_1\alpha_2}) \end{aligned}$$

with

$$F_{\alpha_1\alpha_2} := \delta_{\alpha_1}(A_{\alpha_2}) - \delta_{\alpha_2}(A_{\alpha_1}) + [A_{\alpha_1}, A_{\alpha_2}]. \tag{5.12}$$

In summary,

$$\begin{aligned} \mathcal{D}_A^2 = -\delta^{\alpha_1\alpha_2}&(\delta_{\alpha_1} + L(A_{\alpha_1}) - R(A_{\alpha_1}))(\delta_{\alpha_2} + L(A_{\alpha_2}) - R(A_{\alpha_2})) \\ &\otimes 1_{2^m} - \tfrac{1}{2}(L(F_{\alpha_1\alpha_2}) - R(F_{\alpha_1\alpha_2})) \otimes \gamma^{\alpha_1\alpha_2}. \end{aligned} \tag{5.13}$$

5.2. Kernels and dimension spectrum. We now compute the kernel of the perturbed Dirac operator.

Proposition 5.1. (i) $\operatorname{Ker} \mathcal{D} = U_0 \otimes \mathbb{C}^{2^m}$, *so* $\dim \operatorname{Ker} \mathcal{D} = 2^m$.

(ii) *For any selfadjoint* 1*-form* A *we have* $\operatorname{Ker} \mathcal{D} \subseteq \operatorname{Ker} \mathcal{D}_A$.

(ii) *For any unitary* $u \in \mathcal{A}$ *we have* $\operatorname{Ker} \mathcal{D}_{\gamma_u(A)} = V_u \operatorname{Ker} \mathcal{D}_A$.

Proof. (i) Let $\psi = \sum_{k,j} c_{k,j} U_k \otimes e_j \in \operatorname{Ker} \mathcal{D}$. Then $0 = \mathcal{D}^2\psi = \sum_{k,i} c_{k,j}|k|^2 U_k \otimes e_j$, which implies that $c_{k,j}|k|^2 = 0$ for any $k \in \mathbb{Z}^n$ and $1 \le j \le 2^m$. The result follows.

(ii) Let $\psi \in \operatorname{Ker} \mathcal{D}$. Then $\psi = U_0 \otimes v$ with $v \in \mathbb{C}^{2^m}$, and from (5.11) we get

$$\mathcal{D}_A\psi = \mathcal{D}\psi + (A + \epsilon JAJ^{-1})\psi = (A + \epsilon JAJ^{-1})\psi = -i[A_\alpha, U_0] \otimes \gamma^\alpha v = 0$$

since U_0 is the unit of the algebra, which proves that $\psi \in \operatorname{Ker} \mathcal{D}_A$.

(iii) This is a direct consequence of (5.8). □

Corollary 5.2. *Let A be a selfadjoint 1-form. Then* $\operatorname{Ker} \mathcal{D}_A = \operatorname{Ker} \mathcal{D}$ *in the following cases:*

(i) $A_u := L(u) \otimes 1_{2^m}[\mathcal{D}, L(u^*) \otimes 1_{2^m}]$ *when u is a unitary in $\mathcal{A}$.*

(ii) $\|A\| < \frac{1}{2}$.

(iii) *The matrix $\frac{1}{2\pi}\Theta$ has only integral coefficients.*

Proof. (i) This follows from previous result because $V_u(U_0 \otimes v) = U_0 \otimes v$ for any $v \in \mathbb{C}^{2^m}$.

(ii) Let $\psi = \sum_{k,j} c_{k,j} U_k \otimes e_j$ be in $\operatorname{Ker} \mathcal{D}_A$ (so $\sum_{k,j} |c_{k,j}|^2 < \infty$) and $\phi := \sum_j c_{0,j} U_0 \otimes e_j$. Thus $\psi' := \psi - \phi \in \operatorname{Ker} \mathcal{D}_A$ since $\phi \in \operatorname{Ker} \mathcal{D} \subseteq \operatorname{Ker} \mathcal{D}_A$ and

$$\begin{aligned}\Big\| \sum_{0 \neq k \in \mathbb{Z}^n, j} c_{k,j} k_\alpha U_k \otimes \gamma^\alpha e_j \Big\|^2 &= \|\mathcal{D}\psi'\|^2 \\ &= \|-(A + \epsilon JAJ^{-1})\psi'\|^2 \le 4\|A\|^2\|\psi'\|^2 < \|\psi'\|^2.\end{aligned}$$

Let $X_k := \sum_\alpha k_\alpha \gamma_\alpha$. Then $X_k^2 = \sum_\alpha |k_\alpha|^2 1_{2^m}$ is invertible and the vectors $\{U_k \otimes X_k e_j\}_{0 \neq k \in \mathbb{Z}^n, j}$ are orthogonal in $\mathcal{H}$, so

$$\sum_{0 \neq k \in \mathbb{Z}^n, j} \Big(\sum_\alpha |k_\alpha|^2\Big)|c_{k,j}|^2 < \sum_{0 \neq k \in \mathbb{Z}^n, j} |c_{k,j}|^2,$$

which is possible only if $c_{k,j} = 0$ for all k, j, that is, $\psi' = 0$ and $\psi = \phi \in \operatorname{Ker} \mathcal{D}$.

(iii) This is a consequence of the fact that the algebra is commutative, and hence $A + \epsilon JAJ^{-1} = 0$. □

Note that if $\tilde{A}_u := A_u + \epsilon JA_uJ^{-1}$, then, by (5.6), $\tilde{A}_{U_k} = 0$ for all $k \in \mathbb{Z}^n$ and $\|A_{U_k}\| = |k|$, but for an arbitrary unitary $u \in \mathcal{A}$, $\tilde{A}_u \neq 0$ and hence $\mathcal{D}_{A_u} \neq \mathcal{D}$.

Naturally the above result is also a direct consequence of the fact that the eigenspace of an isolated eigenvalue of an operator is not modified by small perturbations. However, it is interesting to compute the last result directly to emphasize the difficulty of the general case.

So let $\psi = \sum_{l\in\mathbb{Z}^n, 1\le j\le 2^m} c_{l,j} U_l \otimes e_j \in \operatorname{Ker}\mathcal{D}_A$. Then $\sum_{l\in\mathbb{Z}^n, 1\le j\le 2^m} |c_{l,j}|^2 < \infty$. We have to show that $\psi \in \operatorname{Ker}\mathcal{D}$ that is $c_{l,j} = 0$ when $l \neq 0$.

Taking the scalar product of $\langle U_k \otimes e_i|$ with

$$0 = \mathcal{D}_A\psi = \sum_{l,\alpha,j} c_{l,j}(l^\alpha U_l - i[A_\alpha, U_l]) \otimes \gamma^\alpha e_j,$$

we obtain that

$$0 = \sum_{l,\alpha,j} c_{l,j}(l^\alpha \delta_{k,l} - i\langle U_k, [A_\alpha, U_l]\rangle)\langle e_i, \gamma^\alpha e_j\rangle.$$

If $A_\alpha = \sum_{\alpha,l} a_{\alpha,l} U_l \otimes \gamma^\alpha$, $\{a_{\alpha,l}\}_l \in \mathcal{S}(\mathbb{Z}^n)$, then it follows that $[U_l, U_m] = -2i \sin\left(\frac{1}{2} l\cdot\Theta m\right) U_{l+m}$ and

$$\begin{aligned}\langle U_k, [A_\alpha, U_l]\rangle &= \sum_{l'\in\mathbb{Z}^n} a_{\alpha,l'}\left(-2i\sin\left(\tfrac{1}{2}l'\cdot\Theta l\right)\right)\langle U_k, U_{l'+l}\rangle \\ &= -2i\, a_{\alpha,k-l}\sin\left(\tfrac{1}{2}k\cdot\Theta l\right).\end{aligned}$$

Thus

$$0 = \sum_{l\in\mathbb{Z}^n}\sum_{\alpha=1}^{n}\sum_{j=1}^{2^m} c_{l,j}\left(l^\alpha\delta_{k,l} - 2a_{\alpha,k-l}\sin\left(\tfrac{1}{2}k\cdot\Theta l\right)\right)\langle e_i, \gamma^\alpha e_j\rangle \tag{5.14}$$

for all $k \in \mathbb{Z}^n$ and for all i with $1 \le i \le 2^m$.

We conjecture that $\operatorname{Ker}\mathcal{D} = \operatorname{Ker}\mathcal{D}_A$ *at least for generic* Θ*'s*: the constraints (5.14) should imply $c_{l,j} = 0$ for all j and all $l \neq 0$, which means that $\psi \in \operatorname{Ker}\mathcal{D}$. If $\frac{1}{2\pi}\Theta$ has only integer coefficients, then the sin part of these constraints disappears, which gives the result.

Lemma 5.3. *If* $\frac{1}{2\pi}\Theta$ *is diophantine, then* $\operatorname{Sp}(C^\infty(\mathbb{T}^n_\Theta), \mathcal{H}, \mathcal{D}) = \mathbb{Z}$*, and all these poles are simple.*

Proof. Let $B \in \mathcal{D}(\mathcal{A})$ and $p \in \mathbb{N}_0$. Suppose that B is of the form

$$B = a_r b_r \mathcal{D}^{q_{r-1}}|\mathcal{D}|^{p_{r-1}} a_{r-1} b_{r-1} \dots \mathcal{D}^{q_1}|\mathcal{D}|^{p_1} a_1 b_1$$

where $r \in \mathbb{N}$, $a_i \in \mathcal{A}$, $b_i \in J\mathcal{A}J^{-1}$, $q_i, p_i \in \mathbb{N}_0$. We denote $a_i =: \sum_l a_{i,l} U_l$ and $b_i =: \sum_i b_{i,l} U_l$. With the shorthand $k_{\mu_1,\mu_{q_i}} := k_{\mu_1}\dots k_{\mu_{q_i}}$ and $\gamma^{\mu_1,\mu_{q_i}} = \gamma^{\mu_1}\dots\gamma^{\mu_{q_i}}$, we get

$$\begin{aligned}&\mathcal{D}^{q_1}|\mathcal{D}|^{p_1} a_1 b_1 U_k \otimes e_j \\ &\quad = \sum_{l_1,l_1'} a_{1,l_1} b_{1,l_1'} U_{l_1} U_k U_{l_1'} |k + l_1 + l_1'|^{p_1} (k + l_1 + l_1')_{\mu_1,\mu_{q_1}} \otimes \gamma^{\mu_1,\mu_{q_1}} e_j,\end{aligned}$$

which gives after r iterations

$$BU_k \otimes e_j = \sum_{l,l'} \tilde{a}_l \tilde{b}_l U_{l_r} \dots U_{l_1} U_k U_{l'_1} \dots U_{l'_r}$$
$$\prod_{i=1}^{r-1} |k + \hat{l}_i + \hat{l}'_i|^{p_i} (k + \hat{l}_i + \hat{l}'_i)_{\mu^i_1, \mu^i_{q_i}} \otimes \gamma^{\mu^{r-1}_1, \mu^{r-1}_{q_{r-1}}} \dots \gamma^{\mu^1_1, \mu^1_{q_1}} e_j,$$

where $\tilde{a}_l := a_{1,l_1} \dots a_{r,l_r}$ and $\tilde{b}_{l'} := b_{1,l'_1} \dots b_{r,l'_r}$.

Let us write $F_\mu(k,l,l') := \prod_{i=1}^{r-1} |k + \hat{l}_i + \hat{l}'_i|^{p_i} (k + \hat{l}_i + \hat{l}'_i)_{\mu^i_1, \mu^i_{q_i}}$ and $\gamma^\mu := \gamma^{\mu^{r-1}_1, \mu^{r-1}_{q_{r-1}}} \dots \gamma^{\mu^1_1, \mu^1_{q_1}}$. Thus, with the shortcut $\sim_c$ meaning modulo a constant function in the variable s, we have

$$\begin{aligned} &\mathrm{Tr}(B|D|^{-2p-s}) \\ &\quad \sim_c \sum_k{}' \sum_{l,l'} \tilde{a}_l \tilde{b}_{l'} \tau(U_{-k} U_{l_r} \dots U_{l_1} U_k U_{l'_1} \dots U_{l'_r}) \frac{F_\mu(k,l,l')}{|k|^{s+2p}} \mathrm{Tr}(\gamma^\mu). \end{aligned}$$

Since $U_{l_r} \dots U_{l_1} U_k = U_k U_{l_r} \dots U_{l_1} e^{-i \sum_1^r l_i . \Theta k}$ we get

$$\tau(U_{-k} U_{l_r} \dots U_{l_1} U_k U_{l'_1} \dots U_{l'_r}) = \delta_{\sum_1^r l_i + l'_i, 0} e^{i\phi(l,l')} e^{-i \sum_1^r l_i \cdot \Theta k},$$

where ϕ is a real valued function. Thus,

$$\begin{aligned} \mathrm{Tr}(B|D|^{-2p-s}) &\sim_c \sum_k{}' \sum_{l,l'} e^{i\phi(l,l')} \delta_{\sum_1^r l_i + l'_i, 0} \tilde{a}_l \tilde{b}_{l'} \frac{F_\mu(k,l,l') e^{-i \sum_1^r l_i \cdot \Theta k}}{|k|^{s+2p}} \mathrm{Tr}(\gamma^\mu) \\ &\sim_c f_\mu(s) \mathrm{Tr}(\gamma^\mu). \end{aligned}$$

The function $f_\mu(s)$ can be decomposed into a linear combination of zeta functions of the type described in Theorem 2.18 (resp. Theorem 2.6 if $r = 1$ or all the p_i are zero). Thus, $s \mapsto \mathrm{Tr}(B|D|^{-2p-s})$ has only poles in $\mathbb{Z}$ and each pole is simple. Finally, by linearity, the result follows. □

The dimension spectrum of the noncommutative torus is simple:

Proposition 5.4. (i) *If $\frac{1}{2\pi}\Theta$ is diophantine, then the dimension spectrum of the triple $(C^\infty(\mathbb{T}^n_\Theta), \mathcal{H}, \mathcal{D})$ is equal to the set $\{n - k \mid k \in \mathbb{N}_0\}$, and all these poles are simple.*
(ii) $\zeta_D(0) = 0$.

Proof. (i) This follows from Lemma 5.3 and Remark 3.9.
(ii) We have $\zeta_D(s) = \sum_{k \in \mathbb{Z}^n} \sum_{1 \le j \le 2^m} \langle U_k \otimes e_j, |D|^{-s} U_k \otimes e_j \rangle = 2^m \big(\sum'_{k \in \mathbb{Z}^n} \frac{1}{|k|^s} + 1 \big) = 2^m (Z_n(s) + 1)$. The result follows from (2.15). □

We have computed $\zeta_D(0)$ relatively easy, but the main difficulty of the present work is essentially to calculate $\zeta_{D_A}(0)$.

5.3. Noncommutative integral computations. We fix a selfadjoint 1-form A on the noncommutative torus of dimension n.

Proposition 5.5. *If $\frac{1}{2\pi}\Theta$ is diophantine, then the first elements of the expansion* (1.5) *are given by*

$$\begin{aligned} \fint |D_A|^{-n} &= \fint |D|^{-n} = 2^{m+1}\pi^{n/2}\Gamma(\tfrac{n}{2})^{-1}, \\ \fint |D_A|^{n-k} &= 0 \quad \textit{for } k \textit{ odd}, \\ \fint |D_A|^{n-2} &= 0. \end{aligned} \tag{5.15}$$

We need few technical lemmas.

Lemma 5.6. *On the noncommutative torus, for any $t \in \mathbb{R}$,*

$$\fint \tilde{A}\mathcal{D}|D|^{-t} = \fint \mathcal{D}\tilde{A}|D|^{-t} = 0.$$

Proof. Using the notation of (5.10), we have

$$\begin{aligned} \mathrm{Tr}(\tilde{A}\mathcal{D}|D|^{-s}) &\sim_c \sum\nolimits_j \sum_k{}' \langle U_k \otimes e_j, -ik_\mu |k|^{-s}[A_\alpha, U_k] \otimes \gamma^\alpha\gamma^\mu e_j\rangle \\ &\sim_c -i\,\mathrm{Tr}(\gamma^\alpha\gamma^\mu) \sum_k{}' k_\mu |k|^{-s}\langle U_k, [A_\alpha, U_k]\rangle = 0 \end{aligned}$$

since $\langle U_k, [A_\alpha, U_k]\rangle = 0$. Similarly

$$\begin{aligned} &\mathrm{Tr}(\mathcal{D}\tilde{A}|D|^{-s}) \\ &\quad\sim_c \sum_j \sum_k{}' \langle U_k \otimes e_j, |k|^{-s} \textstyle\sum_l a_{\alpha,l} 2\sin\frac{k\cdot\Theta l}{2}(l+k)_\mu U_{l+k} \otimes \gamma^\mu\gamma^\alpha e_j\rangle \\ &\quad\sim_c 2\,\mathrm{Tr}(\gamma^\mu\gamma^\alpha) \sum_k{}' \textstyle\sum_l a_{\alpha,l} \sin\frac{k\cdot\Theta l}{2}(l+k)_\mu |k|^{-s}\langle U_k, U_{l+k}\rangle = 0. \end{aligned}$$ □

Any element h in the algebra generated by $\mathcal{A}$ and $[\mathcal{D}, \mathcal{A}]$ can be written as a linear combination of terms of the form ${a_1}^{p_1} \dots {a_n}^{p_r}$, where a_i are elements of $\mathcal{A}$ or $[\mathcal{D}, \mathcal{A}]$. Such a term can be written as a series $b := \sum a_{1,\alpha_1,l_1} \dots a_{q,\alpha_q,l_q} U_{l_1} \dots U_{l_q} \otimes \gamma^{\alpha_1} \dots \gamma^{\alpha_q}$, where a_{i,α_i} are Schwartz sequences and when $a_i =: \sum_l a_l U_l \in \mathcal{A}$ we set $a_{i,\alpha,l} = a_{i,l}$ with $\gamma^\alpha = 1$. We define

$$L(b) := \tau\big(\textstyle\sum_l a_{1,\alpha_1,l_1} \dots a_{q,\alpha_q,l_q} U_{l_1} \dots U_{l_q}\big)\,\mathrm{Tr}(\gamma^{\alpha_1} \dots \gamma^{\alpha_q}).$$

By linearity, L is defined as a linear form on the whole algebra generated by $\mathcal{A}$ and $[\mathcal{D}, \mathcal{A}]$.

Lemma 5.7. *If h is an element of the algebra generated by $\mathcal{A}$ and $[\mathcal{D}, \mathcal{A}]$, then*

$$\operatorname{Tr}(h|D|^{-s}) \sim_c L(h)Z_n(s).$$

In particular, $\operatorname{Tr}(h|D|^{-s})$ *has at most one pole at $s = n$.*

Proof. With b of the form $\sum a_{1,\alpha_1,l_1} \dots a_{q,\alpha_q,l_q} U_{l_1} \dots U_{l_q} \otimes \gamma^{\alpha_1} \dots \gamma^{\alpha_q}$, we get

$$\begin{aligned}
&\operatorname{Tr}(b|D|^{-s}) \\
&\quad \sim_c \sum_{k\in\mathbb{Z}^n}{}' \langle U_k, \textstyle\sum_l a_{1,\alpha_1,l_1} \dots a_{q,\alpha_q,l_q} U_{l_1} \dots U_{l_q} U_k\rangle \operatorname{Tr}(\gamma^{\alpha_1} \dots \gamma^{\alpha_q})|k|^{-s} \\
&\quad \sim_c \tau\big(\textstyle\sum_l a_{1,\alpha_1,l_1} \dots a_{q,\alpha_q,l_q} U_{l_1} \dots U_{l_q}\big) \operatorname{Tr}(\gamma^{\alpha_1} \dots \gamma^{\alpha_q}) Z_n(s) = L(b)Z_n(s).
\end{aligned}$$

The result now follows from linearity of the trace. □

Lemma 5.8. *If $\frac{1}{2\pi}\Theta$ is diophantine, the function $s \mapsto \operatorname{Tr}(\varepsilon JAJ^{-1}A|D|^{-s})$ extends meromorphically on the whole plane with only one possible pole at $s = n$. Moreover, this pole is simple and*

$$\operatorname*{Res}_{s=n} \operatorname{Tr}(\varepsilon JAJ^{-1}A|D|^{-s}) = a_{\alpha,0}a_0^{\alpha} 2^{m+1}\pi^{n/2}\Gamma(n/2)^{-1}.$$

Proof. With $A = L(-iA_\alpha) \otimes \gamma^\alpha$, we get $\epsilon JAJ^{-1} = R(iA_\alpha) \otimes \gamma^\alpha$, and by multiplication $\varepsilon JAJ^{-1}A = R(A_\beta)L(A_\alpha) \otimes \gamma^\beta\gamma^\alpha$. Thus,

$$\begin{aligned}
\operatorname{Tr}(\varepsilon JAJ^{-1}A|D|^{-s}) &\sim_c \sum_{k\in\mathbb{Z}^n}{}' \langle U_k, A_\alpha U_k A_\beta\rangle |k|^{-s} \operatorname{Tr}(\gamma^\beta\gamma^\alpha) \\
&\sim_c \sum_{k\in\mathbb{Z}^n}{}' \sum_l a_{\alpha,l}a_{\beta,-l}e^{ik\cdot\Theta l}|k|^{-s} \operatorname{Tr}(\gamma^\beta\gamma^\alpha) \\
&\sim_c 2^m \sum_{k\in\mathbb{Z}^n}{}' \sum_l a_{\alpha,l}a_{-l}^{\alpha}e^{ik\cdot\Theta l}|k|^{-s}.
\end{aligned}$$

Theorem 2.6 (ii) entails that $\sum'_{k\in\mathbb{Z}^n} \sum_l a_{\alpha,l}a_{-l}^{\alpha}e^{ik\cdot\Theta l}|k|^{-s}$ extends meromorphically to the whole plane $\mathbb{C}$ with only one possible pole at $s = n$. Moreover, this pole is simple and we have

$$\operatorname*{Res}_{s=n} \sum_{k\in\mathbb{Z}^n}{}' \sum_l a_{\alpha,l}a_{-l}^{\alpha}e^{ik\cdot\Theta l}|k|^{-s} = a_{\alpha,0}a_0^{\alpha} \operatorname*{Res}_{s=n} Z_n(s).$$

Equation (2.14) now gives the result. □

Lemma 5.9. *If $\frac{1}{2\pi}\Theta$ is diophantine, then for any $t \in \mathbb{R}$,*

$$⨍ X|D|^{-t} = \delta_{t,n}2^{m+1}(-\textstyle\sum_l a_{\alpha,l}a_{-l}^{\alpha} + a_{\alpha,0}a_0^{\alpha})\, 2\pi^{n/2}\Gamma(n/2)^{-1}.$$

where $X = \tilde{A}\mathcal{D} + \mathcal{D}\tilde{A} + \tilde{A}^2$ and $A =: -i\sum_l a_{\alpha,l}U_l \otimes \gamma^\alpha$.

Proof. By Lemma 5.6, we get $⨍ X|D|^{-t} = \operatorname{Res}_{s=0} \operatorname{Tr}(\tilde{A}^2|D|^{-s-t})$. Since A and εJAJ^{-1} commute, we have $\tilde{A}^2 = A^2 + JA^2J^{-1} + 2\varepsilon JAJ^{-1}A$. Thus,

$$\operatorname{Tr}(\tilde{A}^2|D|^{-s-t}) = \operatorname{Tr}(A^2|D|^{-s-t}) + \operatorname{Tr}(JA^2J^{-1}|D|^{-s-t}) + 2\operatorname{Tr}(\varepsilon JAJ^{-1}A|D|^{-s-t}).$$

Since $|D|$ and J commute, we have with Lemma 5.7,

$$\operatorname{Tr}(\tilde{A}^2|D|^{-s-t}) \sim_c 2L(A^2)Z_n(s+t) + 2\operatorname{Tr}(\varepsilon JAJ^{-1}A|D|^{-s-t}).$$

Thus Lemma 5.8 entails that $\operatorname{Tr}(\tilde{A}^2|D|^{-s-t})$ is holomorphic at 0 if $t \neq n$. When $t = n$,

$$\operatorname*{Res}_{s=0} \operatorname{Tr}(\tilde{A}^2|D|^{-s-t}) = 2^{m+1}(-\textstyle\sum_l a_{\alpha,l}a^{\alpha}_{-l} + a_{\alpha,0}a^{\alpha}_0)2\pi^{n/2}\Gamma(n/2)^{-1}, \tag{5.16}$$

which gives the result. □

Lemma 5.10. *If* $\frac{1}{2\pi}\Theta$ *is diophantine, then*

$$⨍ \tilde{A}\mathcal{D}\tilde{A}\mathcal{D}|D|^{-2-n} = -\frac{n-2}{n} ⨍ \tilde{A}^2|D|^{-n}.$$

Proof. With $\mathcal{D}J = \varepsilon J\mathcal{D}$, we get

$$⨍ \tilde{A}\mathcal{D}\tilde{A}\mathcal{D}|D|^{-2-n} = 2⨍ A\mathcal{D}A\mathcal{D}|D|^{-2-n} + 2⨍ \varepsilon JAJ^{-1}\mathcal{D}A\mathcal{D}|D|^{-2-n}.$$

Let us first compute $⨍ A\mathcal{D}A\mathcal{D}|D|^{-2-n}$. We have, with $A =: -iL(A_\alpha) \otimes \gamma^\alpha =: -i\sum_l a_{\alpha,l}U_l \otimes \gamma^\alpha$,

$$\begin{aligned}&\operatorname{Tr}(A\mathcal{D}A\mathcal{D}|D|^{-s-2-n})\\ &\quad\sim_c -\sum_k{}' \sum_{l_1,l_2} a_{\alpha_2,l_2}a_{\alpha_1,l_1}\tau(U_{-k}U_{l_2}U_{l_1}U_k)\frac{k_{\mu_1}(k+l_1)_{\mu_2}}{|k|^{s+2+n}}\operatorname{Tr}(\gamma^{\alpha,\mu})\end{aligned}$$

where $\gamma^{\alpha,\mu} := \gamma^{\alpha_2}\gamma^{\mu_2}\gamma^{\alpha_1}\gamma^{\mu_1}$. Thus,

$$⨍ A\mathcal{D}A\mathcal{D}|D|^{-2-n} = -\sum_l a_{\alpha_2,-l}a_{\alpha_1,l}\operatorname*{Res}_{s=0}\left(\textstyle\sum_k{}' \frac{k_{\mu_1}k_{\mu_2}}{|k|^{s+2+n}}\right)\operatorname{Tr}(\gamma^{\alpha,\mu}).$$

We have also, with $\varepsilon JAJ^{-1} = iR(A_\alpha) \otimes \gamma^a$,

$$\begin{aligned}&\operatorname{Tr}(\varepsilon JAJ^{-1}\mathcal{D}A\mathcal{D}|D|^{-s-2-n})\\ &\quad\sim_c \sum_k{}' \sum_{l_1,l_2} a_{\alpha_2,l_2}a_{\alpha_1,l_1}\tau(U_{-k}U_{l_1}U_kU_{l_2})\frac{k_{\mu_1}(k+l_1)_{\mu_2}}{|k|^{s+2+n}}\operatorname{Tr}(\gamma^{\alpha,\mu}).\end{aligned}$$

which gives

$$\fint \varepsilon JAJ^{-1}\mathcal{D}A\mathcal{D}|D|^{-2-n} = a_{\alpha_2,0}a_{\alpha_1,0} \operatorname*{Res}_{s=0}\big(\textstyle\sum_k' \frac{k_{\mu_1}k_{\mu_2}}{|k|^{s+2+n}}\big)\operatorname{Tr}(\gamma^{\alpha,\mu}).$$

Thus,

$$\begin{aligned}\frac{1}{2}\fint \tilde{A}\mathcal{D}\tilde{A}\mathcal{D}|D|^{-2-n}\\ = \big(a_{\alpha_2,0}a_{\alpha_1,0} - \textstyle\sum_l a_{\alpha_2,-l}a_{\alpha_1,l}\big)\operatorname*{Res}_{s=0}\big(\textstyle\sum_k' \frac{k_{\mu_1}k_{\mu_2}}{|k|^{s+2+n}}\big)\operatorname{Tr}(\gamma^{\alpha,\mu}).\end{aligned}$$

With $\sum_k' \frac{k_{\mu_1}k_{\mu_2}}{|k|^{s+2+n}} = \frac{\delta_{\mu_1\mu_2}}{n} Z_n(s+n)$ and $C_n := \operatorname{Res}_{s=0} Z_n(s+n) = 2\pi^{n/2}\Gamma(n/2)^{-1}$ we obtain that

$$\frac{1}{2}\fint \tilde{A}\mathcal{D}\tilde{A}\mathcal{D}|D|^{-2-n} = (a_{\alpha_2,0}a_{\alpha_1,0} - \textstyle\sum_l a_{\alpha_2,-l}a_{\alpha_1,l})\frac{C_n}{n}\operatorname{Tr}(\gamma^{\alpha_2}\gamma^{\mu}\gamma^{\alpha_1}\gamma_{\mu}).$$

Since $\operatorname{Tr}(\gamma^{\alpha_2}\gamma^{\mu}\gamma^{\alpha_1}\gamma_{\mu}) = 2^m(2-n)\delta^{\alpha_2,\alpha_1}$, we get

$$\frac{1}{2}\fint \tilde{A}\mathcal{D}\tilde{A}\mathcal{D}|D|^{-2-n} = 2^m(-a_{\alpha,0}a_0^{\alpha} + \textstyle\sum_l a_{\alpha,-l}a_l^{\alpha})\frac{C_n(n-2)}{n}.$$

Equation (5.16) now proves the lemma. □

Lemma 5.11. *If $\frac{1}{2\pi}\Theta$ is diophantine, then for any $P \in \Psi_1(\mathcal{A})$ and $q \in \mathbb{N}$, q odd,*

$$\fint P|D|^{-(n-q)} = 0.$$

Proof. There exist $B \in \mathcal{D}_1(\mathcal{A})$ and $p \in \mathbb{N}_0$ such that $P = BD^{-2p} + R$ where R is in OP^{-q-1}. Consequently, $\fint P|D|^{-(n-q)} = \fint B|D|^{-n-2p+q}$. Assume that $B = a_r b_r \mathcal{D}^{q_{r-1}} a_{r-1} b_{r-1} \dots \mathcal{D}^{q_1} a_1 b_1$, where $r \in \mathbb{N}, a_i \in \mathcal{A}, b_i \in J\mathcal{A}J^{-1}, q_i \in \mathbb{N}$. If we prove that $\fint B|D|^{-n-2p+q} = 0$, then the general case will follow by linearity. Write $a_i =: \sum_l a_{i,l}U_l$ and $b_i =: \sum_l b_{i,l}U_l$. With $k_{\mu_1,\mu_{q_i}} := k_{\mu_1}\dots k_{\mu_{q_i}}$ and $\gamma^{\mu_1,\mu_{q_i}} = \gamma^{\mu_1}\dots\gamma^{\mu_{q_i}}$, we get

$$\mathcal{D}^{q_1}a_1b_1U_k \otimes e_j = \sum_{l_1,l_1'} a_{1,l_1}b_{1,l_1'}U_{l_1}U_kU_{l_1'}(k+l_1+l_1')_{\mu_1,\mu_{q_1}} \otimes \gamma^{\mu_1,\mu_{q_1}}e_j,$$

which gives after iteration

$$\begin{aligned}BU_k \otimes e_j = \sum_{l,l'} \tilde{a}_l\tilde{b}_l U_{l_r}\dots U_{l_1}U_kU_{l_1'}\dots U_{l_r'}\\ \cdot\prod_{i=1}^{r-1}(k+\hat{l}_i+\hat{l}_i')_{\mu_1^i,\mu_{q_i}^i} \otimes \gamma^{\mu_1^{r-1},\mu_{q_{r-1}}^{r-1}}\dots\gamma^{\mu_1^1,\mu_{q_1}^1}e_j,\end{aligned}$$

where $\tilde{a}_l := a_{1,l_1} \dots a_{r,l_r}$ and $\tilde{b}_{l'} := b_{1,l'_1} \dots b_{r,l'_r}$. We denote $Q_\mu(k,l,l') := \prod_{i=1}^{r-1}(k+\hat{l}_i+\hat{l}'_i)_{\mu^i_1,\mu^i_{q_i}}$ and $\gamma^\mu := \gamma^{\mu^{r-1}_1,\mu^{r-1}_{q_{r-1}}} \dots \gamma^{\mu^1_1,\mu^1_{q_1}}$. Then

$$\fint B|D|^{-n-2p+q} = \operatorname*{Res}_{s=0} \sum_k{}' \sum_{l,l'} \tilde{a}_l \tilde{b}_{l'} \tau(U_{-k}U_{l_r} \dots U_{l_1}U_kU_{l'_1} \dots U_{l'_r}) \cdot \frac{Q_\mu(k,l,l')}{|k|^{s+2p+n-q}} \operatorname{Tr}(\gamma^\mu).$$

Since $U_{l_r} \dots U_{l_1}U_k = U_kU_{l_r} \dots U_{l_1}e^{-i\sum_1^r l_i.\Theta k}$, we get

$$\tau(U_{-k}U_{l_r} \dots U_{l_1}U_kU_{l'_1} \dots U_{l'_r}) = \delta_{\sum_1^r l_i+l'_i,0} e^{i\phi(l,l')} e^{-i\sum_1^r l_i\cdot\Theta k}$$

where ϕ is a real valued function. Thus,

$$\begin{aligned}\fint & B|D|^{-n-2p+q} \\ &= \operatorname*{Res}_{s=0} \sum_k{}' \sum_{l,l'} e^{i\phi(l,l')} \delta_{\sum_1^r l_i+l'_i,0} \tilde{a}_l \tilde{b}_{l'} \frac{Q_\mu(k,l,l')e^{-i\sum_1^r l_i.\Theta k}}{|k|^{s+2p+n-q}} \operatorname{Tr}(\gamma^\mu) \\ &=: \operatorname*{Res}_{s=0} f_\mu(s) \operatorname{Tr}(\gamma^\mu).\end{aligned}$$

We decompose $Q_\mu(k,l,l')$ as a sum $\sum_{h=0}^r M_{h,\mu}(l,l')Q_{h,\mu}(k)$ where $Q_{h,\mu}$ is a homogeneous polynomial in $(k_1,\dots,k_n)$ and $M_{h,\mu}(l,l')$ is a polynomial in $\big((l_1)_1,\dots,(l_r)_n,(l'_1)_1,\dots,(l'_r)_n\big)$.

Similarly, we decompose $f_\mu(s)$ as $\sum_{h=0}^r f_{h,\mu}(s)$. Theorem 2.6 (ii) implies that $f_{h,\mu}(s)$ extends meromorphically to the whole complex plane $\mathbb{C}$ with only one possible pole for $s+2p+n-q = n+d$, where $d := \deg Q_{h,\mu}$. In other words, if $d+q-2p \neq 0$, then $f_{h,\mu}(s)$ is holomorphic at $s=0$. Suppose now that $d+q-2p = 0$ (note that this implies that d is odd since q is odd by hypothesis). Then it follows from Theorem 2.6 (ii) that

$$\operatorname*{Res}_{s=0} f_{h,\mu}(s) = V \int_{u\in S^{n-1}} Q_{h,\mu}(u)\, dS(u),$$

with $V := \sum_{l,l'\in Z} M_{h,\mu}(l,l')e^{i\phi(l,l')}\delta_{\sum_1^r l_i+l'_i,0}\tilde{a}_l\tilde{b}_{l'}$ and $Z := \{l,l' \mid \sum_{i=1}^r l_i = 0\}$. Since d is odd, $Q_{h,\mu}(-u) = -Q_{h,\mu}(u)$ and $\int_{u\in S^{n-1}} Q_{h,\mu}(u)\, dS(u) = 0$. Thus, $\operatorname*{Res}_{s=0} f_{h,\mu}(s) = 0$ in any case, which gives the result. □

As we have seen, the crucial point of the preceding lemma is the decomposition of the numerator of the series $f_\mu(s)$ as polynomials in k. This is possible because we restrict our pseudodifferential operators to $\Psi_1(\mathcal{A})$.

Proof of Proposition 5.5. The first equality follows from Proposition 4.9 and according to (2.14) we have

$$\fint |D|^{-n} = \operatorname*{Res}_{s=0} \operatorname{Tr}\left(|D|^{-s-n}\right) = 2^m \operatorname*{Res}_{s=0} Z_n(s+n) = \frac{2^{m+1}\pi^{n/2}}{\Gamma(n/2)}.$$

For the second equality, we get from Lemmas 5.7 and 4.6

$$\operatorname*{Res}_{s=n-k} \zeta_{D_A}(s) = \sum_{p=1}^{k} \sum_{r_1,\dots,r_p=0}^{k-p} h(n-k,r,p) \fint \varepsilon^{r_1}(Y)\dots\varepsilon^{r_p}(Y)|D|^{-(n-k)}.$$

Corollary 4.4 and Lemma 5.11 imply that $\fint \varepsilon^{r_1}(Y)\dots\varepsilon^{r_p}(Y)|D|^{-(n-k)} = 0$, which gives the result.

The last equality follows from Lemma 5.10 and Corollary 4.11. □

6. The spectral action

Here is the main result of this section.

Theorem 6.1. *Consider the n-NC-torus* $(C^\infty(\mathbb{T}^n_\Theta), \mathcal{H}, \mathcal{D})$*, where* $n \in \mathbb{N}$ *and* $\frac{1}{2\pi}\Theta$ *is a real* $n \times n$ *skew-symmetric diophantine matrix, and a selfadjoint* 1*-form* $A = L(-iA_\alpha) \otimes \gamma^\alpha$*. Then the full spectral action of* $\mathcal{D}_A = \mathcal{D} + A + \epsilon JAJ^{-1}$ *is*

(i) $\mathcal{S}(\mathcal{D}_A, \Phi, \Lambda) = 4\pi\Phi_2\Lambda^2 + \mathcal{O}(\Lambda^{-2})$ *for* $n = 2$,

(ii) $\mathcal{S}(\mathcal{D}_A, \Phi, \Lambda) = 8\pi^2\Phi_4\Lambda^4 - \frac{4\pi^2}{3}\Phi(0)\tau(F_{\mu\nu}F^{\mu\nu}) + \mathcal{O}(\Lambda^{-2})$ *for* $n = 4$.

(iii) *More generally, in* $\mathcal{S}(\mathcal{D}_A, \Phi, \Lambda) = \sum_{k=0}^{n} \Phi_{n-k}c_{n-k}(A)\Lambda^{n-k} + \mathcal{O}(\Lambda^{-1})$ *we have* $c_{n-2}(A) = 0$ *and* $c_{n-k}(A) = 0$ *for* k *odd. In particular,* $c_0(A) = 0$ *when* n *is odd.*

This result (for $n = 4$) has also been obtained in [20] using the heat kernel method. It is however of interest to obtain it via direct computations of (1.5) since it shows how efficient this formula is. As we will see, the computation of all the noncommutative integrals requires a lot of technical steps. One of the main points, namely to isolate where the Diophantine condition on Θ comes into play, is outlined here.

Remark 6.2. Note that all terms must be gauge invariants, that is, according to (5.9), they are invariant under $A_\alpha \to \gamma_u(A_\alpha) = uA_\alpha u^* + u\delta_\alpha(u^*)$. A special case is $u = U_k$ where $U_k\delta_\alpha(U_k^*) = -ik_\alpha U_0$.

In the same way, note that there is no contradiction to the commutative case where, for any selfadjoint 1-form A, $\mathcal{D}_A = \mathcal{D}$ (so A is equivalent to 0!), since we assume in Theorem 6.1 that Θ is diophantine, so $\mathcal{A}$ cannot be commutative.

Conjecture 6.3. The constant term of the spectral action of $\mathcal{D}_A$ on the noncommutative n-torus is proportional to the constant term of the spectral action of $\mathcal{D} + A$ on the commutative n-torus.

Remark 6.4. The appearance of a Diophantine condition for Θ has been characterized in dimension 2 by Connes [6], Prop. 49, where in this case, $\Theta = \theta \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$ with $\theta \in \mathbb{R}$. In fact, the Hochschild cohomology $H(\mathcal{A}_\Theta, \mathcal{A}_\Theta{}^*)$ satisfies $\dim H^j(\mathcal{A}_\Theta, \mathcal{A}_\Theta{}^*) = 2$ (or 1) for $j = 1$ (or $j = 2$) if and only if the irrational number θ satisfies a Diophantine condition like $|1 - e^{i2\pi n\theta}|^{-1} = \mathcal{O}(n^k)$ for some k.

Recall that when the matrix Θ is quite irrational (see [23], Cor. 2.12), then the C*-algebra generated by $\mathcal{A}_\Theta$ is simple.

Remark 6.5. One can generalize the above theorem to the case $\mathcal{D} = -i g^\mu{}_\nu \delta_\mu \otimes \gamma^\nu$ instead of (5.4) when g is a positive definite constant matrix. The formulae in Theorem 6.1 are still valid, up to obvious modifications due to volume variation.

6.1. Computations of $⨍$. In order to get this theorem, let us prove a few technical lemmas.

We suppose from now on that Θ is a skew-symmetric matrix in $\mathcal{M}_n(\mathbb{R})$. No other hypothesis is assumed for Θ, except when it is explicitly stated.

When A is a selfadjoint 1-form, we define for $n \in N$, $q \in \mathbb{N}$, $2 \le q \le n$ and $\sigma \in \{-, +\}^q$:

$$\begin{aligned} \mathbb{A}^+ &:= A\mathcal{D}D^{-2}, \\ \mathbb{A}^- &:= \epsilon JAJ^{-1}\mathcal{D}D^{-2}, \\ \mathbb{A}^\sigma &:= \mathbb{A}^{\sigma_q} \dots \mathbb{A}^{\sigma_1}. \end{aligned}$$

Lemma 6.6. *For any $q \in \mathbb{N}$ we have*

$$⨍ (\tilde{A}D^{-1})^q = ⨍ (\tilde{A}\mathcal{D}D^{-2})^q = \sum_{\sigma \in \{+,-\}^q} ⨍ \mathbb{A}^\sigma.$$

Proof. Since P_0 belongs to $OP^{-\infty}$, it follows that $D^{-1} = \mathcal{D}D^{-2} \mod OP^{-\infty}$ and $⨍(\tilde{A}D^{-1})^q = ⨍(\tilde{A}\mathcal{D}D^{-2})^q$. □

Lemma 6.7. *Let A be a selfadjoint 1-form, $n \in \mathbb{N}$, $q \in \mathbb{N}$ with $2 \le q \le n$, and $\sigma \in \{-, +\}^q$. Then*

$$⨍ \mathbb{A}^\sigma = ⨍ \mathbb{A}^{-\sigma}.$$

Proof. Let us first check that $JP_0 = P_0 J$. Since $\mathcal{D}J = \varepsilon J\mathcal{D}$, we get $\mathcal{D}JP_0 = 0$ and so $JP_0 = P_0 JP_0$. Since J is an antiunitary operator, it follows that $P_0 J = P_0 JP_0$

and $P_0 J = J P_0$. Consequently, $JD^2 = D^2 J$, $J\mathcal{D}D^{-2} = \varepsilon \mathcal{D} D^{-2} J$, $J\mathbb{A}^+ J^{-1} = \mathbb{A}^-$ and $J\mathbb{A}^- J^{-1} = \mathbb{A}^+$. In summary, $J\mathbb{A}^{\sigma_i} J^{-1} = \mathbb{A}^{-\sigma_i}$.

The trace property of $⨍$ now gives

$$⨍ \mathbb{A}^{\sigma} = ⨍ \mathbb{A}^{\sigma_q} \dots \mathbb{A}^{\sigma_1} = ⨍ J\mathbb{A}^{\sigma_q} J^{-1} \dots J\mathbb{A}^{\sigma_1} J^{-1} ⨍ \mathbb{A}^{-\sigma_q} \dots \mathbb{A}^{-\sigma_1} = ⨍ \mathbb{A}^{-\sigma}.$$

□

Definition 6.8. In [11] the vanishing tadpole hypothesis was introduced:

$$⨍ AD^{-1} = 0 \quad \text{for all } A \in \Omega^1_{\mathcal{D}}(\mathcal{A}). \tag{6.1}$$

By the following lemma, this condition is satisfied for the noncommutative torus, a fact more or less already known within the noncommutative community [35].

Lemma 6.9. *Let $n \in \mathbb{N}$ and let $A = L(-iA_\alpha) \otimes \gamma^\alpha = -i \sum_{l \in \mathbb{Z}^n} a_{\alpha,l} U_l \otimes \gamma^\alpha$ be a hermitian 1-form with $A_\alpha \in \mathcal{A}_\Theta$, $\{a_{\alpha,l}\}_l \in \mathcal{S}(\mathbb{Z}^n)$. Then*

(i) $⨍ A^p D^{-q} = ⨍ (\epsilon J A J^{-1})^p D^{-q} = 0$ *for $p \geq 0$ and $1 \leq q < n$ (the case $p = q = 1$ is the tadpole hypothesis);*

(ii) *if $\frac{1}{2\pi}\Theta$ is diophantine, then $⨍ BD^{-q} = 0$ for $1 \leq q < n$ and any B in the algebra generated by $\mathcal{A}$, $[\mathcal{D}, \mathcal{A}]$, $J\mathcal{A}J^{-1}$ and $J[\mathcal{D}, \mathcal{A}]J^{-1}$.*

Proof. (i) Let us compute

$$⨍ A^p (\epsilon J A J^{-1})^{p'} D^{-q}.$$

With $A = L(-iA_\alpha) \otimes \gamma^\alpha$ and $\epsilon J A J^{-1} = R(iA_\alpha) \otimes \gamma^\alpha$, we get

$$A^p = L(-iA_{\alpha_1}) \dots L(-iA_{\alpha_p}) \otimes \gamma^{\alpha_1} \dots \gamma^{\alpha_p}$$

and

$$(\epsilon J A J^{-1})^{p'} = R(iA_{\alpha'_1}) \dots R(iA_{\alpha'_{p'}}) \otimes \gamma^{\alpha'_1} \dots \gamma^{\alpha'_{p'}}.$$

Write $\tilde{a}_{\alpha,l} := a_{\alpha_1,l_1} \dots a_{\alpha_p,l_p}$. Since

$$\begin{aligned} &L(-iA_{\alpha_1}) \dots L(-iA_{\alpha_p}) R(iA_{\alpha'_1}) \dots R(iA_{\alpha'_{p'}}) U_k \\ &\quad = (-i)^p i^{p'} \sum_{l,l'} \tilde{a}_{\alpha,l} \tilde{a}_{\alpha',l'} U_{l_1} \dots U_{l_p} U_k U_{l'_{p'}} \dots U_{l'_1}, \end{aligned}$$

and

$$U_{l_1}\dots U_{l_p}U_k = U_kU_{l_1}\dots U_{l_p}e^{-i(\sum_i l_i)\cdot\Theta k},$$

we get

$$\begin{aligned}&A^p(\epsilon JAJ^{-1})^{p'}D^{-q}|D|^{-s}U_k\otimes e_i\\&\quad\sim_c (-i)^p i^{p'}\sum_{l,l'} g_{\mu,\alpha,\alpha'}(s,k,l,l')U_kU_{l,l'}\otimes\gamma^{\alpha,\alpha',\mu}e_i,\end{aligned}$$

where we use the abbreviations $U_{l,l'} := U_{l_1}\dots U_{l_p}U_{l'_{p'}}\dots U_{l'_1}$ $g_{\mu,\alpha,\alpha'}(s,k,l,l') := e^{ik\cdot\Theta\sum_j l_j}\frac{k_{\mu_1}\dots k_{\mu_q}}{|k|^{s+2q}}\tilde{a}_{\alpha,l}\tilde{a}_{\alpha',l'}$ and $\gamma^{\alpha,\alpha',\mu} := \gamma^{\alpha_1}\dots\gamma^{\alpha_p}\gamma^{\alpha'_1}\dots\gamma^{\alpha'_{p'}}\gamma^{\mu_1}\dots\gamma^{\mu_q}$.

Thus, $⨍ A^p(\epsilon JAJ^{-1})^{p'}D^{-q} = \operatorname*{Res}_{s=0} f(s)$ where

$$\begin{aligned}f(s) :&= \operatorname{Tr}(A^p(\epsilon JAJ^{-1})^{p'}D^{-q}|D|^{-s})\\&\sim_c (-i)^p i^{p'}\sum_{k\in\mathbb{Z}^n}{}' \langle U_k\otimes e_i, \textstyle\sum_{l,l'} g_{\mu,\alpha,\alpha'}(s,k,l,l')U_kU_{l,l'}\otimes\gamma^{\alpha,\alpha',\mu}e_i\rangle\\&\sim_c (-i)^p i^{p'}\sum_{k\in\mathbb{Z}^n}{}'\tau\Big(\sum_{l,l'} g_{\mu,\alpha,\alpha'}(s,k,l,l')U_{l,l'}\Big)\operatorname{Tr}(\gamma^{\mu,\alpha,\alpha'})\\&\sim_c (-i)^p i^{p'}\sum_{k\in\mathbb{Z}^n}{}'\sum_{l,l'} g_{\mu,\alpha,\alpha'}(s,k,l,l')\tau(U_{l,l'})\operatorname{Tr}(\gamma^{\mu,\alpha,\alpha'}).\end{aligned}$$

It is straightforward to check that the series $\sum'_{k,l,l'} g_{\mu,\alpha,\alpha'}(s,k,l,l')\tau(U_{l,l'})$ is absolutely summable if $\operatorname{Re}(s) > R$ for a $R > 0$. Thus, we can exchange the summation over k and l,l', which gives

$$f(s)\sim_c (-i)^p i^{p'}\sum_{l,l'}\sum_{k\in\mathbb{Z}^n}{}' g_{\mu,\alpha,\alpha'}(s,k,l,l')\tau\big(U_{l,l'}\big)\operatorname{Tr}(\gamma^{\mu,\alpha,\alpha'}).$$

If we suppose now that $p' = 0$, we see that

$$f(s)\sim_c (-i)^p\sum_{l}\sum_{k\in\mathbb{Z}^n}{}'\frac{k_{\mu_1}\dots k_{\mu_q}}{|k|^{s+2q}}\tilde{a}_{\alpha,l}\delta_{\sum_{i=1}^p l_i,0}\operatorname{Tr}(\gamma^{\mu,\alpha,\alpha'}),$$

which is, by Proposition 2.17, analytic at 0. In particular, for $p = q = 1$, we see that $⨍ AD^{-1} = 0$, i.e., the vanishing tadpole hypothesis is satisfied. Similarly, if we suppose that $p = 0$, we get

$$f(s)\sim_c (-i)^{p'}\sum_{l'}\sum_{k\in\mathbb{Z}^n}{}'\frac{k_{\mu_1}\dots k_{\mu_q}}{|k|^{s+2q}}\tilde{a}_{\alpha,l'}\delta_{\sum_{i=1}^{p'} l'_i,0}\operatorname{Tr}(\gamma^{\mu,\alpha,\alpha'})$$

which is holomorphic at 0.

(ii) Adapting the proof of Lemma 5.11 to our setting (taking $q_i = 0$ and adding gamma matrices components), we see that

$$\fint BD^{-q}$$
$$= \operatorname*{Res}_{s=0} \sum_k{}' \sum_{l,l'} e^{i\phi(l,l')} \delta_{\sum_1^r l_i + l'_i, 0} \tilde{a}_{\alpha,l} \tilde{b}_{\beta,l'} \frac{k_{\mu_1} \dots k_{\mu_q} e^{-i\sum_1^r l_i \cdot \Theta k}}{|k|^{s+2q}} \operatorname{Tr}(\gamma^{(\mu,\alpha,\beta)}),$$

where $\gamma^{(\mu,\alpha,\beta)}$ is a complicated product of gamma matrices. By Theorem 2.6 (ii), since we suppose here that $\frac{1}{2\pi}\Theta$ is diophantine, this residue is 0. □

6.1.1. Even dimensional case

Corollary 6.10. *Let the same hypotheses be satisfied as in Lemma* 6.9. *Then:*

(i) *case* $n = 2$: $\fint A^q D^{-q} = -\delta_{q,2} 4\pi \tau(A_\alpha A^\alpha)$;

(ii) *case* $n = 4$:

$$\fint A^q D^{-q} = \delta_{q,4} \frac{\pi^2}{12} \tau\big(A_{\alpha_1} \dots A_{\alpha_4}\big) \operatorname{Tr}(\gamma^{\alpha_1} \dots \gamma^{\alpha_4} \gamma^{\mu_1} \dots \gamma^{\mu_4}) \delta_{\mu_1,\dots,\mu_4},$$

where $\delta_{\mu_1,\dots,\mu_4} := \delta_{\mu_1\mu_2}\delta_{\mu_3\mu_4} + \delta_{\mu_1\mu_3}\delta_{\mu_2\mu_4} + \delta_{\mu_1\mu_4}\delta_{\mu_2\mu_3}$.

Proof. (i), (ii) The same computation as in Lemma 6.9 (i) (with $p' = 0$, $p = q = n$) gives

$$\fint A^n D^{-n} = \operatorname*{Res}_{s=0} (-i)^n \Big(\sum_{k \in \mathbb{Z}^n}{}' \frac{k_{\mu_1} \dots k_{\mu_n}}{|k|^{s+2n}} \Big)$$
$$\cdot \tau\Big(\sum_{l \in (\mathbb{Z}^n)^n} \tilde{a}_{\alpha,l} U_{l_1} \dots U_{l_n} \Big) \operatorname{Tr}(\gamma^{\alpha_1} \dots \gamma^{\alpha_n} \gamma^{\mu_1} \dots \gamma^{\mu_n}),$$

and the result follows from Proposition 2.17. □

We will need some notations:

Let $n \in \mathbb{N}$, $q \geq 2$. Let $l := (l_1, \dots, l_{q-1}) \in (\mathbb{Z}^n)^{q-1}$, $\alpha := (\alpha_1, \dots, \alpha_q) \in \{1, \dots, n\}^q$, $k \in \mathbb{Z}^n \setminus \{0\}$, $\sigma \in \{-,+\}^q$, $(a_i)_{1 \leq i \leq n} \in (\mathcal{S}(\mathbb{Z}^n))^n$, and let

$$l_q := -\sum_{1 \leq j \leq q-1} l_j, \quad \lambda_\sigma := (-i)^q \prod_{j=1\dots q} \sigma_j, \quad \tilde{a}_{\alpha,l} := a_{\alpha_1,l_1} \dots a_{\alpha_q,l_q},$$
$$\phi_\sigma(k,l) := \sum_{1 \leq j \leq q-1} (\sigma_j - \sigma_q) k \cdot \Theta l_j + \sum_{2 \leq j \leq q-1} \sigma_j (l_1 + \dots + l_{j-1}) \cdot \Theta l_j,$$
$$g_\mu(s,k,l) := \frac{k_{\mu_1}(k+l_1)_{\mu_2} \dots (k+l_1+\dots+l_{q-1})_{\mu_q}}{|k|^{s+2} |k+l_1|^2 \dots |k+l_1+\dots+l_{q-1}|^2}.$$

Here we use the convention $\sum_{2\le j\le q-1} = 0$ when $q = 2$, and $g_\mu(s,k,l) = 0$ whenever $\hat{l}_i = -k$ for a $1 \le i \le q-1$.

Lemma 6.11. *Let $A = L(-iA_\alpha)\otimes\gamma^\alpha = -i\sum_{l\in\mathbb{Z}^n} a_{\alpha,l}U_l\otimes\gamma^\alpha$ be a hermitian 1-form, where $A_\alpha = -A_\alpha^* \in \mathcal{A}_\Theta$ and $\{a_{\alpha,l}\}_l \in \mathcal{S}(\mathbb{Z}^n)$ with $n\in\mathbb{N}$, and let $2\le q\le n$, $\sigma\in\{-,+\}^q$.*

Then $\displaystyle⨍ \mathbb{A}^\sigma = \operatorname*{Res}_{s=0} f(s)$ *where*

$$f(s) := \sum_{l\in(\mathbb{Z}^n)^{q-1}} \sum_{k\in\mathbb{Z}^n}{}' \lambda_\sigma e^{\frac{i}{2}\phi_\sigma(k,l)} g_\mu(s,k,l)\,\tilde{a}_{\alpha,l}\,\mathrm{Tr}(\gamma^{\alpha_q}\gamma^{\mu_q}\dots\gamma^{\alpha_1}\gamma^{\mu_1}).$$

Proof. By definition, $⨍ \mathbb{A}^\sigma = \operatorname*{Res}_{s=0} f(s)$ where

$$\mathrm{Tr}(\mathbb{A}^{\sigma_q}\dots\mathbb{A}^{\sigma_1}|D|^{-s}) \sim_c \sum_{k\in\mathbb{Z}^n}{}' \langle U_k\otimes e^i, |k|^{-s}\mathbb{A}^{\sigma_q}\dots\mathbb{A}^{\sigma_1}U_k\otimes e_i\rangle =: f(s).$$

Let $r\in\mathbb{Z}^n$ and $v\in\mathbb{C}^{2^m}$. Since $A = L(-iA_\alpha)\otimes\gamma^\alpha$ and $\epsilon JAJ^{-1} = R(iA_\alpha)\otimes\gamma^\alpha$, we get

$$\begin{aligned}
\mathbb{A}^+U_r\otimes v &= A\mathcal{D}D^{-2}U_r\otimes v\\
&= A\frac{r_\mu}{|r|^2+\delta_{r,0}}U_r\otimes\gamma^\mu v\\
&= -i\frac{r_\mu}{|r|^2+\delta_{r,0}}A_\alpha U_r\otimes\gamma^\alpha\gamma^\mu v,\\
\mathbb{A}^-U_r\otimes v &= \epsilon JAJ^{-1}\mathcal{D}D^{-2}U_r\otimes v\\
&= \epsilon JAJ^{-1}\frac{r_\mu}{|r|^2+\delta_{r,0}}U_r\otimes\gamma^\mu v\\
&= i\frac{r_\mu}{|r|^2+\delta_{r,0}}U_rA_\alpha\otimes\gamma^\alpha\gamma^\mu v.
\end{aligned}$$

With $U_lU_r = e^{\frac{i}{2}r\cdot\Theta l}U_{r+l}$ and $U_rU_l = e^{-\frac{i}{2}r\cdot\Theta l}U_{r+l}$ we obtain that

$$\mathbb{A}^{\sigma_j}U_r\otimes v = \sum_{l\in\mathbb{Z}^n}(-\sigma_j)ie^{\sigma_j\frac{i}{2}r\cdot\Theta l}\frac{r_\mu}{|r|^2+\delta_{r,0}}a_{\alpha,l}U_{r+l}\otimes\gamma^\alpha\gamma^\mu v.$$

for any $1\le j\le q$. We now apply q times this formula to get

$$\begin{aligned}
&|k|^{-s}\mathbb{A}^{\sigma_q}\dots\mathbb{A}^{\sigma_1}U_k\otimes e_i\\
&\quad= \lambda_\sigma\sum_{l\in(\mathbb{Z}^n)^q} e^{\frac{i}{2}\phi_\sigma(k,l)}g_\mu(s,k,l)\,\tilde{a}_{\alpha,l}U_{k+\sum_j l_j}\otimes\gamma^{\alpha_q}\gamma^{\mu_q}\dots\gamma^{\alpha_1}\gamma^{\mu_1}e_i
\end{aligned}$$

with

$$\phi_\sigma(k,l) := \sigma_1 k\cdot\Theta l_1 + \sigma_2(k+l_1)\cdot\Theta l_2 + \cdots + \sigma_q(k+l_1+\cdots+l_{q-1})\cdot\Theta l_q.$$

Thus

$$\begin{aligned} f(s) &= \sum_{k\in\mathbb{Z}^n}{}' \tau\big(\lambda_\sigma \sum_{l\in(\mathbb{Z}^n)^q} e^{\frac{i}{2}\phi_\sigma(k,l)} g_\mu(s,k,l)\tilde{a}_{\alpha,l} U_{\sum_j l_j} e^{\frac{i}{2}k\cdot\Theta\sum_j l_j}\big) \\ &\qquad \operatorname{Tr}(\gamma^{\alpha_q}\gamma^{\mu_q}\dots\gamma^{\alpha_1}\gamma^{\mu_1}) \\ &= \sum_{k\in\mathbb{Z}^n}{}' \lambda_\sigma \sum_{l\in(\mathbb{Z}^n)^q} e^{\frac{i}{2}\phi_\sigma(k,l)} g_\mu(s,k,l)\tilde{a}_{\alpha,l}\delta\big(\sum_j l_j\big)\operatorname{Tr}(\gamma^{\alpha_q}\gamma^{\mu_q}\dots\gamma^{\alpha_1}\gamma^{\mu_1}) \\ &= \sum_{k\in\mathbb{Z}^n}{}' \lambda_\sigma \sum_{l\in(\mathbb{Z}^n)^{q-1}} e^{\frac{i}{2}\phi_\sigma(k,l)} g_\mu(s,k,l)\tilde{a}_{\alpha,l}\operatorname{Tr}(\gamma^{\alpha_q}\gamma^{\mu_q}\dots\gamma^{\alpha_1}\gamma^{\mu_1}), \end{aligned}$$

where in the last sum l_q is fixed to $-\sum_{1\le j\le q-1} l_j$. Hence

$$\phi_\sigma(k,l) = \sum_{1\le j\le q-1}(\sigma_j-\sigma_q)k\cdot\Theta l_j + \sum_{2\le j\le q-1}\sigma_j(l_1+\cdots+l_{j-1})\cdot\Theta l_j.$$

By Lemma 2.11 there exists $R>0$ such that for any $s\in\mathbb{C}$ with $\operatorname{Re}(s)>R$ the family

$$(e^{\frac{i}{2}\phi_\sigma(k,l)} g_\mu(s,k,l)\tilde{a}_{\alpha,l})_{(k,l)\in(\mathbb{Z}^n\setminus\{0\})\times(\mathbb{Z}^n)^{q-1}}$$

is absolutely summable as a linear combination of families of the type considered in that lemma. As a consequence, we can exchange the summations over k and l, which gives the result. □

In the following we will use the abbreviation

$$c := \frac{4\pi^2}{3}.$$

Lemma 6.12. *Suppose that $n=4$ and let the same hypotheses be satisfied as in Lemma* 6.11. *Then:*

(i) $\frac{1}{2}⨍(\mathbb{A}^+)^2 = \frac{1}{2}⨍(\mathbb{A}^-)^2 = c\sum_{l\in\mathbb{Z}^4} a_{\alpha_1,l}a_{\alpha_2,-l}(l^{\alpha_1}l^{\alpha_2}-\delta^{\alpha_1\alpha_2}|l|^2).$

(ii) $-\frac{1}{3}⨍(\mathbb{A}^+)^3 = -\frac{1}{3}⨍(\mathbb{A}^-)^3 = 4c\sum_{l_i\in\mathbb{Z}^4} a_{\alpha_3,-l_1-l_2}a^{\alpha_1}_{l_2}a_{\alpha_1,l_1}\sin\frac{l_1\cdot\Theta l_2}{2}l_1^{\alpha_3}.$

(iii) $\frac{1}{4}⨍(\mathbb{A}^+)^4 = \frac{1}{4}⨍(\mathbb{A}^-)^4$
$= 2c\sum_{l_i\in\mathbb{Z}^4} a_{\alpha_1,-l_1-l_2-l_3}a_{\alpha_2,l_3}a^{\alpha_1}_{l_2}a^{\alpha_2}_{l_1}\sin\frac{l_1\cdot\Theta(l_2+l_3)}{2}\sin\frac{l_2\cdot\Theta l_3}{2}.$

(iv) *Suppose that* $\frac{1}{2\pi}\Theta$ *is diophantine. Then the crossed terms in* $⨍(\mathbb{A}^+ + \mathbb{A}^-)^q$ *vanish: if* C *is the set of all* $\sigma \in \{-,+\}^q$ *with* $2 \le q \le 4$ *such that there exist* i, j *satisfying* $\sigma_i \neq \sigma_j$*, then we have* $\sum_{\sigma\in C} ⨍ \mathbb{A}^\sigma = 0$.

Proof. (i) Lemma 6.11 implies that $⨍ \mathbb{A}^{++} = \operatorname*{Res}_{s=0} \sum_{l\in\mathbb{Z}^n} -f(s,l)$ where

$$f(s,l) := \sum_{k\in\mathbb{Z}^n}{}' \frac{k_{\mu_1}(k+l)_{\mu_2}}{|k|^{s+2}|k+l|^2}\tilde{a}_{\alpha,l}\operatorname{Tr}(\gamma^{\alpha_2}\gamma^{\mu_2}\gamma^{\alpha_1}\gamma^{\mu_1}) \quad \text{and} \quad \tilde{a}_{\alpha,l} := a_{\alpha_1,l}a_{\alpha_2,-l}.$$

We will now reduce the computation of the residue of an expression involving terms like $|k+l|^2$ in the denominator to the computation of residues of zeta functions. To proceed, we use (2.10) in an expression like the one appearing in $f(s,l)$. We see that the last term on the right-hand side yields a $Z_n(s)$, while the first one is less divergent by one power of k. If this is not enough, we repeat this operation for the new factor of $|k+l|^2$ in the denominator. For $f(s,l)$, which is quadratically divergent at $s=0$, we have to repeat this operation three times before ending with a convergent result. All the remaining terms are expressible in terms of Z_n functions. We get, using three times (2.10),

$$\frac{1}{|k+l|^2} = \frac{1}{|k|^2} - \frac{2k\cdot l + |l|^2}{|k|^4} + \frac{(2k\cdot l + |l|^2)^2}{|k|^6} - \frac{(2k\cdot l + |l|^2)^3}{|k|^6|k+l|^2}. \tag{6.2}$$

Put

$$f_{\alpha,\mu}(s,l) := \sum_{k\in\mathbb{Z}^n}{}' \frac{k_{\mu_1}(k+l)_{\mu_2}}{|k|^{s+2}|k+l|^2}\,\tilde{a}_{\alpha,l}$$

so that $f(s,l) = f_{\alpha,\mu}(s,l)\operatorname{Tr}(\gamma^{\alpha_2}\gamma^{\mu_2}\gamma^{\alpha_1}\gamma^{\mu_1})$. Equation (6.2) gives

$$f_{\alpha,\mu}(s,l) = f_1(s,l) - f_2(s,l) + f_3(s,l) - r(s,l),$$

with the obvious identifications. Note that the function

$$r(s,l) = \sum_{k\in\mathbb{Z}^n}{}' \frac{k_{\mu_1}(k+l)_{\mu_2}(2kl+|l|^2)^3}{|k|^{s+8}|k+l|^2}\,\tilde{a}_{\alpha,l}$$

is a linear combination of functions of the type $H(s,l)$ satisfying the hypothesis of Corollary 2.14. Thus $r(s,l)$ satisfies (H1), and with the previously seen equivalence relation modulo functions satisfying this hypothesis we get $f_{\alpha,\mu}(s,l) \sim f_1(s,l) - f_2(s,l) + f_3(s,l)$.

We now compute $f_1(s,l)$:

$$f_1(s,l) = \sum_{k\in\mathbb{Z}^n}{}' \frac{k_{\mu_1}(k+l)_{\mu_2}}{|k|^{s+4}}\tilde{a}_{\alpha,l} = \tilde{a}_{\alpha,l}\sum_{k\in\mathbb{Z}^n}{}' \frac{k_{\mu_1}k_{\mu_2}}{|k|^{s+4}} + 0.$$

Proposition 2.1 gives that $s \mapsto \sum_{k\in\mathbb{Z}^n}{}' \frac{k_{\mu_1}k_{\mu_2}}{|k|^{s+4}}$ is holomorphic at 0. Thus $f_1(s,l)$ satisfies (H1) and $f_{\alpha,\mu}(s,l) \sim -f_2(s,l) + f_3(s,l)$.

Let us now compute $f_2(s,l)$ modulo (H1). Invoking several times Proposition 2.1 we obtain that

$$\begin{aligned} f_2(s,l) &= \sum_{k\in\mathbb{Z}^n}{}' \frac{k_{\mu_1}(k+l)_{\mu_2}(2kl+|l|^2)}{|k|^{s+6}}\tilde{a}_{\alpha,l} \\ &= \sum_{k\in\mathbb{Z}^n}{}' \frac{(2kl)k_{\mu_1}k_{\mu_2} + (2kl)k_{\mu_1}l_{\mu_2} + |l|^2k_{\mu_1}k_{\mu_2} + l_{\mu_2}|l|^2k_{\mu_1}}{|k|^{s+6}}\tilde{a}_{\alpha,l} \\ &\sim 0 + \sum_{k\in\mathbb{Z}^n}{}' \frac{(2kl)k_{\mu_1}l_{\mu_2}}{|k|^{s+6}}\tilde{a}_{\alpha,l} + \sum_{k\in\mathbb{Z}^n}{}' \frac{|l|^2k_{\mu_1}k_{\mu_2}}{|k|^{s+6}}\tilde{a}_{\alpha,l} + 0. \end{aligned}$$

Recall that $\sum'_{k\in\mathbb{Z}^n} \frac{k_ik_j}{|k|^{s+6}} = \frac{\delta_{ij}}{n} Z_n(s+4)$. Thus,

$$f_2(s,l) \sim 2l^il_{\mu_2}\tilde{a}_{\alpha,l}\frac{\delta_{i\mu_1}}{n}Z_n(s+4) + |l|^2\tilde{a}_{\alpha,l}\frac{\delta_{\mu_1\mu_2}}{n}Z_n(s+4).$$

We compute $f_3(s,l)$ modulo (H1) following the same principles:

$$\begin{aligned} f_3(s,l) &= \sum_{k\in\mathbb{Z}^n}{}' \frac{k_{\mu_1}(k+l)_{\mu_2}(2kl+|l|^2)^2}{|k|^{s+8}}\tilde{a}_{\alpha,l} \\ &= \sum_{k\in\mathbb{Z}^n}{}' \frac{(2kl)^2k_{\mu_1}k_{\mu_2} + (2kl)^2k_{\mu_1}l_{\mu_2} + |l|^4k_{\mu_1}k_{\mu_2} + |l|^4k_{\mu_1}l_{\mu_2}}{|k|^{s+8}} \\ &\quad + \frac{(4kl)|l|^2k_{\mu_1}k_{\mu_2} + (4kl)|l|^2k_{\mu_1}l_{\mu_2}}{|k|^{s+8}}\tilde{a}_{\alpha,l} \\ &\sim 4l^il^j\sum_{k\in\mathbb{Z}^n}{}' \frac{k_ik_jk_{\mu_1}k_{\mu_2}}{|k|^{s+8}}\tilde{a}_{\alpha,l} + 0. \end{aligned}$$

Finally we have

$$\begin{aligned} &f_{\alpha,\mu}(s,l) \\ &\quad \sim -\frac{1}{4}(2l_{\mu_1}l_{\mu_2} + |l|^2\delta_{\mu_1\mu_2})\tilde{a}_{\alpha,l}Z_n(s+4) + 4l^il^j\tilde{a}_{\alpha,l}\sum_{k\in\mathbb{Z}^n}{}' \frac{k_ik_jk_{\mu_1}k_{\mu_2}}{|k|^{s+8}} \\ &\quad =: g_{\alpha,\mu}(s,l). \end{aligned}$$

Proposition 2.1 implies that $Z_n(s+4)$ and $s \mapsto \sum'_{k\in\mathbb{Z}^n} \frac{k_ik_jk_{\mu_1}k_{\mu_2}}{|k|^{s+8}}$ extend holomorphically in a punctured open disk centered at 0. Thus, $g_{\alpha,\mu}(s,l)$ satisfies (H2) and

we can apply Lemma 2.15 to get

$$\begin{aligned}-\fint(\mathbb{A}^+)^2 &= \operatorname*{Res}_{s=0}\sum_{l\in\mathbb{Z}^n} f(s,l)\\ &= \sum_{l\in\mathbb{Z}^n}\operatorname*{Res}_{s=0} g_{\alpha,\mu}(s,l)\operatorname{Tr}(\gamma^{\alpha_2}\gamma^{\mu_2}\gamma^{\alpha_1}\gamma^{\mu_1}) =: \sum_{l\in\mathbb{Z}^n}\operatorname*{Res}_{s=0} g(s,l).\end{aligned}$$

In this way the problem is now reduced to the computation of $\operatorname*{Res}_{s=0} g(s,l)$. Recall that $\operatorname*{Res}_{s=0} Z_4(s+4) = 2\pi^2$ by (2.14) or (2.11) and

$$\operatorname*{Res}_{s=0}\sum_{k\in\mathbb{Z}^n}{}' \frac{k_ik_jk_lk_m}{|k|^{s+8}} = (\delta_{ij}\delta_{lm}+\delta_{il}\delta_{jm}+\delta_{im}\delta_{jl})\tfrac{\pi^2}{12}.$$

Thus,

$$\operatorname*{Res}_{s=0} g_{\alpha,\mu}(s,l) = -\frac{\pi^2}{3}\tilde{a}_{\alpha,l}(l_{\mu_1}l_{\mu_2}+\tfrac{1}{2}|l|^2\delta_{\mu_1\mu_2}).$$

We will use

$$\operatorname{Tr}(\gamma^{\mu_1}\dots\gamma^{\mu_{2j}}) = \operatorname{Tr}(1)\sum_{\text{all pairings of }\{1\dots 2j\}} s(P)\delta_{\mu_{P_1}\mu_{P_2}}\delta_{\mu_{P_3}\mu_{P_4}}\dots\delta_{\mu_{P_{2j-1}}\mu_{P_{2j}}} \tag{6.3}$$

where $s(P)$ is the signature of the permutation P when $P_{2m-1} < P_{2m}$ for $1\le m\le n$. This gives

$$\operatorname{Tr}(\gamma^{\alpha_2}\gamma^{\mu_2}\gamma^{\alpha_1}\gamma^{\mu_1}) = 2^m(\delta^{\alpha_2\mu_2}\delta^{\alpha_1\mu_1}-\delta^{\alpha_1\alpha_2}\delta^{\mu_2\mu_1}+\delta^{\alpha_2\mu_1}\delta^{\mu_2\alpha_1}). \tag{6.4}$$

Hence

$$\begin{aligned}&\operatorname*{Res}_{s=0} g(s,l)\\ &\quad= -c\tilde{a}_{\alpha,l}(l_{\mu_1}l_{\mu_2}+\tfrac{1}{2}|l|^2\delta_{\mu_1\mu_2})(\delta^{\alpha_2\mu_2}\delta^{\alpha_1\mu_1}-\delta^{\alpha_1\alpha_2}\delta^{\mu_2\mu_1}+\delta^{\alpha_2\mu_1}\delta^{\mu_2\alpha_1})\\ &\quad= -2c\tilde{a}_{\alpha,l}(l^{\alpha_1}l^{\alpha_2}-\delta^{\alpha_1\alpha_2}|l|^2).\end{aligned}$$

Finally,

$$\frac{1}{2}\fint(\mathbb{A}^+)^2 = \frac{1}{2}\fint(\mathbb{A}^-)^2 = c\sum_{l\in\mathbb{Z}^n} a_{\alpha_1,l}a_{\alpha_2,-l}(l^{\alpha_1}l^{\alpha_2}-\delta^{\alpha_1\alpha_2}|l|^2).$$

(ii) By Lemma 6.11 we have $\fint \mathbb{A}^{+++} = \operatorname*{Res}_{s=0}\sum_{(l_1,l_2)\in(\mathbb{Z}^n)^2} f(s,l)$ where

$$\begin{aligned}f(s,l) &:= \sum_{k\in\mathbb{Z}^n}{}' i e^{\frac{i}{2}l_1\Theta l_2}\frac{k_{\mu_1}(k+l_1)_{\mu_2}(k+\hat{l}_2)_{\mu_3}}{|k|^{s+2}|k+l_1|^2|k+\hat{l}_2|^2}\,\tilde{a}_{\alpha,l}\operatorname{Tr}(\gamma^{\alpha_3}\gamma^{\mu_3}\gamma^{\alpha_2}\gamma^{\mu_2}\gamma^{\alpha_1}\gamma^{\mu_1})\\ &=: f_{\alpha,\mu}(s,l)\operatorname{Tr}(\gamma^{\alpha_3}\gamma^{\mu_3}\gamma^{\alpha_2}\gamma^{\mu_2}\gamma^{\alpha_1}\gamma^{\mu_1})\end{aligned}$$

and $\tilde{a}_{\alpha,l} := a_{\alpha_1,l_1} a_{\alpha_2,l_2} a_{\alpha_3,-\hat{l}_2}$ with $\hat{l}_2 := l_1 + l_2$.

We use the same technique as in (i):

$$\frac{1}{|k+l_1|^2} = \frac{1}{|k|^2} - \frac{2k\cdot l_1 + |l_1|^2}{|k|^4} + \frac{(2k\cdot l_1 + |l_1|^2)^2}{|k|^4|k+l_1|^2},$$
$$\frac{1}{|k+\hat{l}_2|^2} = \frac{1}{|k|^2} - \frac{2k\cdot \hat{l}_2 + |\hat{l}_2|^2}{|k|^4} + \frac{(2k\cdot \hat{l}_2 + |\hat{l}_2|^2)^2}{|k|^4|k+\hat{l}_2|^2}$$

and thus

$$\frac{1}{|k+l_1|^2|k+\hat{l}_2|^2} = \frac{1}{|k|^4} - \frac{2k\cdot l_1}{|k|^6} - \frac{2k\cdot \hat{l}_2}{|k|^6} + R(k,l), \tag{6.5}$$

where the remain $R(k,l)$ is a term of order at most -6 in k. Equation (6.5) gives

$$f_{\alpha,\mu}(s,l) = f_1(s,l) + r(s,l)$$

where $r(s,l)$ corresponds to $R(k,l)$. Note that the function

$$r(s,l) = \sum_{k\in\mathbb{Z}^n}{}' \, i e^{\frac{i}{2} l_1 \Theta l_2} \frac{k_{\mu_1}(k+l)_{\mu_2}(k+\hat{l}_2)_{\mu_3} R(k,l)}{|k|^{s+2}} \tilde{a}_{\alpha,l}$$

is a linear combination of functions of the type $H(s,l)$ satisfying the hypothesis of Corollary 2.14. Thus, $r(s,l)$ satisfies (H1) and $f_{\alpha,\mu}(s,l) \sim f_1(s,l)$.

Let us compute $f_1(s,l)$ modulo (H1):

$$\begin{aligned}
f_1(s,l) &= \sum_{k\in\mathbb{Z}^n}{}' \, i e^{\frac{i}{2} l_1 \Theta l_2} \frac{k_{\mu_1}(k+l_1)_{\mu_2}(k+\hat{l}_2)_{\mu_3}}{|k|^{s+6}} \tilde{a}_{\alpha,l} \\
&\quad - \sum_{k\in\mathbb{Z}^n}{}' \, i e^{\frac{i}{2} l_1 \Theta l_2} \frac{k_{\mu_1}(k+l_1)_{\mu_2}(k+\hat{l}_2)_{\mu_3}(2k\cdot l_1 + 2k\cdot \hat{l}_2)}{|k|^{s+8}} \tilde{a}_{\alpha,l} \\
&\sim \sum_{k\in\mathbb{Z}^n}{}' \, i e^{\frac{i}{2} l_1 \Theta l_2} \frac{k_{\mu_1} k_{\mu_2} \widehat{l_2}_{\mu_3} + k_{\mu_1} k_{\mu_3} l_{1\mu_2}}{|k|^{s+6}} \tilde{a}_{\alpha,l} \\
&\quad - \sum_{k\in\mathbb{Z}^n}{}' \, i e^{\frac{i}{2} l_1 \Theta l_2} \frac{k_{\mu_1} k_{\mu_2} k_{\mu_3}(2k\cdot l_1 + 2k\cdot \hat{l}_2)}{|k|^{s+8}} \tilde{a}_{\alpha,l} \\
&= i e^{\frac{i}{2} l_1 \Theta l_2} \tilde{a}_{\alpha,l} \Big((l_{1\mu_2}\delta_{\mu_1\mu_3} + \widehat{l_2}_{\mu_3}\delta_{\mu_1\mu_2}) \tfrac{1}{4} Z_4(s+4) \\
&\quad - 2(l_1^i + \hat{l}_2^i) \sum_{k\in\mathbb{Z}^n}{}' \frac{k_{\mu_1} k_{\mu_2} k_{\mu_3} k_i}{|k|^{s+8}} \Big) \\
&=: g_{\alpha,\mu}(s,l).
\end{aligned}$$

Since $g_{\alpha,\mu}(s,l)$ satisfies (H2), we can apply Lemma 2.15 to get

$$\begin{aligned}\fint(\mathbb{A}^+)^3 &= \operatorname*{Res}_{s=0}\sum_{(l_1,l_2)\in(\mathbb{Z}^n)^2} f(s,l)\\ &= \sum_{(l_1,l_2)\in(\mathbb{Z}^n)^2}\operatorname*{Res}_{s=0} g_{\alpha,\mu}(s,l)\operatorname{Tr}(\gamma^{\alpha_3}\gamma^{\mu_3}\gamma^{\alpha_2}\gamma^{\mu_2}\gamma^{\alpha_1}\gamma^{\mu_1}) =: \sum_l X_l.\end{aligned}$$

Recall that $l_3 := -l_1 - l_2 = -\hat{l}_2$. By (2.11) and (2.13),

$$\begin{aligned}\operatorname*{Res}_{s=0} g_{\alpha,\mu}(s,l) i e^{\frac{i}{2}l_1\Theta l_2}\tilde{a}_{\alpha,l}\big(&2(-l_1^i + l_3^i)\tfrac{\pi^2}{12}(\delta_{\mu_1\mu_2}\delta_{\mu_3 i} + \delta_{\mu_1\mu_3}\delta_{\mu_2 i} + \delta_{\mu_1 i}\delta_{\mu_2\mu_3})\\ &+ (l_{1\mu_2}\delta_{\mu_1\mu_3} - l_{3\mu_3}\delta_{\mu_1\mu_2})\tfrac{\pi^2}{2}\big).\end{aligned}$$

We decompose X_l in five terms: $X_l = 2^m\,\frac{\pi^2}{2} i e^{\frac{i}{2}l_1\Theta l_2}\tilde{a}_{\alpha,l}(T_1+T_2+T_3+T_4+T_5)$ where

$$\begin{aligned}
T_0 &:= \tfrac{1}{3}(-l_1^i + l_3^i)(\delta_{\mu\nu}\delta_{\rho i} + \delta_{\mu\rho}\delta_{\nu i} + \delta_{\mu i}\delta_{\nu\rho}) + l_{1\nu}\delta_{\mu\rho} - l_{3\rho}\delta_{\mu\nu},\\
T_1 &:= (\delta^{\alpha_3\rho}\delta^{\alpha_2\nu}\delta^{\alpha_1\mu} - \delta^{\alpha_3\rho}\delta^{\alpha_2\alpha_1}\delta^{\mu\nu} + \delta^{\alpha_3\rho}\delta^{\alpha_2\mu}\delta^{\alpha_1\nu})T_0,\\
T_2 &:= (-\delta^{\alpha_2\alpha_3}\delta^{\rho\nu}\delta^{\alpha_1\mu} + \delta^{\alpha_2\alpha_3}\delta^{\alpha_1\rho}\delta^{\mu\nu} - \delta^{\alpha_2\alpha_3}\delta^{\rho\mu}\delta^{\alpha_1\nu})T_0,\\
T_3 &:= (\delta^{\alpha_3\nu}\delta^{\alpha_2\rho}\delta^{\alpha_1\mu} - \delta^{\alpha_3\nu}\delta^{\alpha_1\rho}\delta^{\alpha_2\mu} + \delta^{\alpha_3\nu}\delta^{\rho\mu}\delta^{\alpha_1\alpha_2})T_0,\\
T_4 &:= (-\delta^{\alpha_1\alpha_3}\delta^{\alpha_2\rho}\delta^{\mu\nu} + \delta^{\alpha_1\alpha_3}\delta^{\rho\nu}\delta^{\alpha_2\mu} - \delta^{\alpha_1\alpha_3}\delta^{\rho\mu}\delta^{\alpha_2\nu})T_0,\\
T_5 &:= (\delta^{\alpha_3\mu}\delta^{\alpha_2\rho}\delta^{\alpha_1\nu} - \delta^{\alpha_3\mu}\delta^{\rho\nu}\delta^{\alpha_1\alpha_2} + \delta^{\alpha_3\mu}\delta^{\alpha_1\rho}\delta^{\alpha_2\nu})T_0.
\end{aligned}$$

Let $p := -l_1 - 2l_3$, $q := 2l_1 + l_3$, $r := -p - q = -l_1 + l_3$. We compute each T_i and find that

$$\begin{aligned}
3T_1 &= \delta^{\alpha_1\alpha_2}(2-2^m)p^{\alpha_3} + \delta^{\alpha_3\alpha_1}q^{\alpha_2} - \delta^{\alpha_2\alpha_1}q^{\alpha_3} + \delta^{\alpha_3\alpha_2}q^{\alpha_1} + \delta^{\alpha_3\alpha_2}r^{\alpha_1}\\
&\quad - \delta^{\alpha_2\alpha_1}r^{\alpha_3} + \delta^{\alpha_3\alpha_1}r^{\alpha_2},\\
3T_2 &= (2^m-2)\delta^{\alpha_2\alpha_3}p^{\alpha_1} - 2^m\delta^{\alpha_2\alpha_3}q^{\alpha_1} - 2^m\delta^{\alpha_2\alpha_3}r^{\alpha_1},\\
3T_3 &= \delta^{\alpha_1\alpha_3}p^{\alpha_2} - \delta^{\alpha_2\alpha_3}p^{\alpha_1} + \delta^{\alpha_1\alpha_2}p^{\alpha_3} + 2^m\delta^{\alpha_2\alpha_1}q^{\alpha_3} + \delta^{\alpha_3\alpha_2}r^{\alpha_1}\\
&\quad - \delta^{\alpha_3\alpha_1}r^{\alpha_2} + \delta^{\alpha_1\alpha_2}r^{\alpha_3},\\
3T_4 &= -\delta^{\alpha_1\alpha_3}2^m p^{\alpha_2} - \delta^{\alpha_1\alpha_3}2^m q^{\alpha_2} + \delta^{\alpha_1\alpha_3}(2^m-2)r^{\alpha_2},\\
3T_5 &= \delta^{\alpha_1\alpha_3}p^{\alpha_2} - \delta^{\alpha_1\alpha_2}p^{\alpha_3} + \delta^{\alpha_3\alpha_2}p^{\alpha_1} + \delta^{\alpha_3\alpha_2}q^{\alpha_1} - \delta^{\alpha_1\alpha_2}q^{\alpha_3} + \delta^{\alpha_3\alpha_1}q^{\alpha_2}\\
&\quad + (2-2^m)\delta^{\alpha_1\alpha_2}r^{\alpha_3}.
\end{aligned}$$

Thus,

$$X_l = 2^m\tfrac{2\pi^2}{3} i e^{\frac{i}{2}l_1\cdot\Theta l_2}\tilde{a}_{\alpha,l}(q^{\alpha_3}\delta^{\alpha_1\alpha_2} + r^{\alpha_2}\delta^{\alpha_1\alpha_3} + p^{\alpha_1}\delta^{\alpha_2\alpha_3}) \tag{6.6}$$

and

$$\fint (\mathbb{A}^+)^3 = i2c(S_1 + S_2 + S_3),$$

where S_1, S_2 and S_3 correspond to $q^{\alpha_3}\delta^{\alpha_1\alpha_2}$, $r^{\alpha_2}\delta^{\alpha_1\alpha_3}$ and $p^{\alpha_1}\delta^{\alpha_2\alpha_3}$, respectively. In S_1 we permute the l_i variables the following way: $l_1 \mapsto l_3$, $l_2 \mapsto l_1$, $l_3 \mapsto l_2$. Therefore, $l_3 \cdot \Theta l_1 \mapsto l_3 \cdot \Theta l_1$ and $q \mapsto r$. With a similar permutation of the α_i, we see that $S_1 = S_2$. We apply the same principles to prove that $S_1 = S_3$ (using permutation $l_1 \mapsto l_2$, $l_2 \mapsto l_3$, $l_3 \mapsto l_1$). Thus,

$$\frac{1}{3}\fint (\mathbb{A}^+)^3 = i2c \sum_{l_i} \tilde{a}_{\alpha,l} e^{\frac{i}{2} l_1 \cdot \Theta l_2} (l_1 - l_2)^{\alpha_3} \delta^{\alpha_1\alpha_2} = S_4 - S_5,$$

where S_4 correspond to l_1 and S_5 to l_2. We permute the l_i variables in S_5 as follows: $l_1 \mapsto l_2$, $l_2 \mapsto l_1$, $l_3 \mapsto l_3$, with a similar permutation on the α_i. Since $l_1 \cdot \Theta l_2 \mapsto -l_1 \cdot \Theta l_2$, we finally get

$$\frac{1}{3}\fint (\mathbb{A}^+)^3 = -4c \sum_{l_i} a_{\alpha_1,l_1} a_{\alpha_2,l_2} a_{\alpha_3,-l_1-l_2} \sin \tfrac{l_1 \cdot \Theta l_2}{2}\, l_1^{\alpha_3} \delta^{\alpha_1\alpha_2}.$$

(iii) Lemma 6.11 gives that $\fint \mathbb{A}^{++++} = \operatorname*{Res}_{s=0} \sum_{(l_1,l_2,l_3)\in(\mathbb{Z}^n)^3} f_{\mu,\alpha}(s,l) \operatorname{Tr} \gamma^{\mu,\alpha}$ where

$$\begin{aligned}
&\theta := l_1.\Theta l_2 + l_1 \cdot \Theta l_3 + l_2 \cdot \Theta l_3,\\
&\operatorname{Tr} \gamma^{\mu,\alpha} := \operatorname{Tr}(\gamma^{\alpha_4}\gamma^{\mu_4}\gamma^{\alpha_3}\gamma^{\mu_3}\gamma^{\alpha_2}\gamma^{\mu 2}\gamma^{\alpha_1}\gamma^{\mu_1}),\\
&f_{\mu,\alpha}(s,l) := \sum_{k\in\mathbb{Z}^n}{}' e^{\frac{i}{2}\theta} \frac{k_{\mu_1}(k+l_1)_{\mu_2}(k+\hat{l}_2)_{\mu_3}(k+\hat{l}_3)_{\mu_4}}{|k|^{s+2}|k+l_1|^2|k+\hat{l}_2|^2|k+\hat{l}_3|^2} \tilde{a}_{\alpha,l},\\
&\tilde{a}_{\alpha,l} := a_{\alpha_1,l_1} a_{\alpha_2,l_2} a_{\alpha_3,l_3} a_{\alpha_4,-l_1-l_2-l_3}.
\end{aligned}$$

Using (2.10) and Corollary 2.14 successively, we find

$$\begin{aligned}
f_{\mu,\alpha}(s,l) &\sim \sum_{k\in\mathbb{Z}^n}{}' e^{\frac{i}{2}\theta} \frac{k_{\mu_1}k_{\mu_2}k_{\mu_3}k_{\mu_4}}{|k|^{s+2}|k+l_1|^2|k+l_1+l_2|^2|k+l_1+l_2+l_3|^2} \tilde{a}_{\alpha,l}\\
&\sim \sum_{k\in\mathbb{Z}^n}{}' e^{\frac{i}{2}\theta} \frac{k_{\mu_1}k_{\mu_2}k_{\mu_3}k_{\mu_4}}{|k|^{s+8}} \tilde{a}_{\alpha,l}.
\end{aligned}$$

Since the function $\sum'_{k\in\mathbb{Z}^n} e^{\frac{i}{2}\theta} \frac{k_{\mu_1}k_{\mu_2}k_{\mu_3}k_{\mu_4}}{|k|^{s+8}} \tilde{a}_{\alpha,l}$ satisfies (H2), Lemma 2.15 implies that

$$\fint (\mathbb{A}^+)^4 = \sum_{(l_1,l_2,l_3)\in(\mathbb{Z}^n)^3} e^{\frac{i}{2}\theta} \tilde{a}_{\alpha,l} \operatorname*{Res}_{s=0} \sum_{k\in\mathbb{Z}^n}{}' \frac{k_{\mu_1}k_{\mu_2}k_{\mu_3}k_{\mu_4}}{|k|^{s+8}} \operatorname{Tr} \gamma^{\mu,\alpha} =: \sum_l X_l.$$

Therefore, with (2.13), we get $X_l = \frac{\pi^2}{12}\tilde{a}_{\alpha,l}e^{\frac{i}{2}\theta}(A+B+C)$, where

$$\begin{aligned} A &:= \mathrm{Tr}(\gamma^{\alpha_4}\gamma^{\mu_4}\gamma^{\alpha_3}\gamma_{\mu_4}\gamma^{\alpha_2}\gamma^{\mu_2}\gamma^{\alpha_1}\gamma_{\mu_2}),\\ B &:= \mathrm{Tr}(\gamma^{\alpha_4}\gamma^{\mu_4}\gamma^{\alpha_3}\gamma^{\mu_2}\gamma^{\alpha_2}\gamma_{\mu_4}\gamma^{\alpha_1}\gamma_{\mu_2}),\\ C &:= \mathrm{Tr}(\gamma^{\alpha_4}\gamma^{\mu_4}\gamma^{\alpha_3}\gamma_{\mu_2}\gamma^{\alpha_2}\gamma^{\mu_2}\gamma^{\alpha_1}\gamma_{\mu_4}). \end{aligned}$$

Using successively $\{\gamma^\mu,\gamma^\nu\} = 2\delta^{\mu\nu}$ and $\gamma^\mu\gamma_\mu = 2^m\ 1_{2^m}$, we find that

$$\begin{aligned} A &= C = 4\,\mathrm{Tr}(\gamma^{\alpha_4}\gamma^{\alpha_3}\gamma^{\alpha_2}\gamma^{\alpha_1}),\\ B &= -4(\mathrm{Tr}(\gamma^{\alpha_4}\gamma^{\alpha_3}\gamma^{\alpha_1}\gamma^{\alpha_2}) + \mathrm{Tr}(\gamma^{\alpha_4}\gamma^{\alpha_2}\gamma^{\alpha_3}\gamma^{\alpha_1})). \end{aligned}$$

Thus, $A+B+C = 8\cdot 2^m(\delta^{\alpha_4\alpha_3}\delta^{\alpha_2\alpha_1} + \delta^{\alpha_4\alpha_1}\delta^{\alpha_3\alpha_2} - 2\delta^{\alpha_4\alpha_2}\delta^{\alpha_3\alpha_1})$ and

$$X_l = \frac{2\pi^2}{3}2^m\ e^{\frac{i}{2}\theta}\tilde{a}_{\alpha,l}(\delta^{\alpha_4\alpha_3}\delta^{\alpha_2\alpha_1} + \delta^{\alpha_4\alpha_1}\delta^{\alpha_3\alpha_2} - 2\delta^{\alpha_4\alpha_2}\delta^{\alpha_3\alpha_1}). \tag{6.7}$$

By (6.7) we get

$$⨍(\mathbb{A}^+)^4 = 2c(-2T_1 + T_2 + T_3),$$

where

$$\begin{aligned} T_1 &:= \sum_{l_1,\dots,l_4} a_{\alpha_4,l_4}a_{\alpha_3,l_3}a_{\alpha_2,l_2}a_{\alpha_1,l_1}e^{\frac{i}{2}\theta}\delta_{0,\sum_i l_i}\delta^{\alpha_4\alpha_2}\delta^{\alpha_3\alpha_1},\\ T_2 &:= \sum_{l_1,\dots,l_4} a_{\alpha_4,l_4}a_{\alpha_3,l_3}a_{\alpha_2,l_2}a_{\alpha_1,l_1}e^{\frac{i}{2}\theta}\delta_{0,\sum_i l_i}\ \delta^{\alpha_4\alpha_3}\delta^{\alpha_2\alpha_1},\\ T_3 &:= \sum_{l_1,\dots,l_4} a_{\alpha_4,l_4}a_{\alpha_3,l_3}a_{\alpha_2,l_2}a_{\alpha_1,l_1}e^{\frac{i}{2}\theta}\delta_{0,\sum_i l_i}\delta^{\alpha_4\alpha_1}\delta^{\alpha_3\alpha_2}. \end{aligned}$$

We now proceed to the following permutations of the l_i variables in the T_1 term: $l_1 \mapsto l_2$, $l_2 \mapsto l_1$, $l_3 \mapsto l_4$, $l_4 \mapsto l_3$. While $\sum_i l_i$ is invariant, θ is modified: $\theta \mapsto l_2\cdot\Theta l_1 + l_2\cdot\Theta l_4 + l_1\cdot\Theta l_4$. With $\delta_{0,\sum_i l_i}$ in factor, we can let l_4 be $-l_1-l_2-l_3$ so that $\theta \mapsto -\theta$. We also permute the α_i in the same way. Thus,

$$T_1 = \sum_{l_1,\dots,l_4} a_{\alpha_3,l_3}a_{\alpha_4,l_4}a_{\alpha_1,l_1}a_{\alpha_2,l_2}e^{-\frac{i}{2}\theta}\delta_{0,\sum_i l_i}\delta^{\alpha_3\alpha_1}\delta^{\alpha_4\alpha_2}.$$

Therefore,

$$2T_1 = 2\sum_{l_1,\dots,l_4} a_{\alpha_4,l_4}a_{\alpha_3,l_3}a_{\alpha_2,l_2}a_{\alpha_1,l_1}\cos\tfrac{\theta}{2}\delta_{0,\sum_i l_i}\delta^{\alpha_4\alpha_2}\delta^{\alpha_3\alpha_1}. \tag{6.8}$$

The same principles are applied to T_2 and T_3. Namely, the permutation $l_1 \mapsto l_1$, $l_2 \mapsto l_3$, $l_3 \mapsto l_2$, $l_4 \mapsto l_4$ in T_2 and the permutation $l_1 \mapsto l_2$, $l_2 \mapsto l_3$, $l_3 \mapsto l_1$,

$l_4 \mapsto l_4$ in T_3 (the α_i variables are permuted the same way) give

$$T_2 = \sum_{l_1,\dots,l_4} a_{\alpha_4,l_4} a_{\alpha_3,l_3} a_{\alpha_2,l_2} a_{\alpha_1,l_1} e^{\frac{i}{2}\phi} \delta_{0,\sum_i l_i} \delta^{\alpha_4\alpha_2} \delta^{\alpha_3\alpha_1},$$
$$T_3 = \sum_{l_1,\dots,l_4} a_{\alpha_4,l_4} a_{\alpha_3,l_3} a_{\alpha_2,l_2} a_{\alpha_1,l_1} e^{-\frac{i}{2}\phi} \delta_{0,\sum_i l_i} \delta^{\alpha_4\alpha_2} \delta^{\alpha_3\alpha_1},$$

where $\phi := l_1 \cdot \Theta\, l_2 + l_1 \cdot \Theta l_3 - l_2 \cdot \Theta\, l_3$. Finally we get

$$\begin{aligned} ⨍ (\mathbb{A}^+)^4 &= 4c \sum_{l_1,\dots,l_4} a_{\alpha_1,l_4} a_{\alpha_2,l_3} a^{\alpha_1}_{l_2} a^{\alpha_2}_{l_1} \delta_{0,\sum_i l_i} \left(\cos\tfrac{\phi}{2} - \cos\tfrac{\theta}{2}\right) \\ &= 8c \sum_{l_1,\dots,l_3} a_{\alpha_1,-l_1-l_2-l_3} a_{\alpha_2,l_3} a^{\alpha_1}_{l_2} a^{\alpha_2}_{l_1} \sin\tfrac{l_1\cdot\Theta(l_2+l_3)}{2} \sin\tfrac{l_2\cdot\Theta l_3}{2}. \end{aligned} \tag{6.9}$$

(iv) Suppose that $q = 2$. By Lemma 6.11 we get

$$⨍ \mathbb{A}^\sigma = \operatorname*{Res}_{s=0} \sum_{l\in\mathbb{Z}^n} \lambda_\sigma f_{\alpha,\mu}(s,l) \operatorname{Tr}(\gamma^{\alpha_2}\gamma^{\mu_2}\gamma^{\alpha_1}\gamma^{\mu_1})$$

where

$$f_{\alpha,\mu}(s,l) := \sum_{k\in\mathbb{Z}^n}{}' \frac{k_{\mu_1}(k+l)_{\mu_2}}{|k|^{s+2}|k+l|^2} e^{i\eta k\cdot\Theta l} \tilde{a}_{\alpha,l}$$

and $\eta := \frac{1}{2}(\sigma_1 - \sigma_2) \in \{-1, 1\}$. As in the proof of (i), since the presence of the phase does not change the fact that $r(s,l)$ satisfies (H1), we get

$$f_{\alpha,\mu}(s,l) \sim f_1(s,l) - f_2(s,l) + f_3(s,l)$$

where

$$f_1(s,l) = \sum_{k\in\mathbb{Z}^n}{}' \frac{k_{\mu_1}(k+l)_{\mu_2}}{|k|^{s+4}} e^{i\eta k\cdot\Theta l} \tilde{a}_{\alpha,l},$$
$$f_2(s,l) = \sum_{k\in\mathbb{Z}^n}{}' \frac{k_{\mu_1}(k+l)_{\mu_2}(2k\cdot l + |l|^2)}{|k|^{s+6}} e^{i\eta k\cdot\Theta l} \tilde{a}_{\alpha,l},$$
$$f_3(s,l) = \sum_{k\in\mathbb{Z}^n}{}' \frac{k_{\mu_1}(k+l)_{\mu_2}(2k\cdot l + |l|^2)^2}{|k|^{s+8}} e^{i\eta k\cdot\Theta l} \tilde{a}_{\alpha,l}.$$

Suppose that $l = 0$. Then $f_2(s,0) = f_3(s,0) = 0$ and Proposition 2.1 implies that

$$f_1(s,0) = \sum_{k\in\mathbb{Z}^n}{}' \frac{k_{\mu_1}k_{\mu_2}}{|k|^{s+4}} \tilde{a}_{\alpha,0}$$

is holomorphic at 0 and so is $f_{\alpha,\mu}(s,0)$.

Since $\frac{1}{2\pi}\Theta$ is diophantine, Theorem 2.6 3 gives us the result.

Suppose that $q = 3$. Then Lemma 6.11 implies that

$$\fint \mathbb{A}^\sigma = \operatorname*{Res}_{s=0} \sum_{l\in(\mathbb{Z}^n)^2} f_{\mu,\alpha}(s,l)\operatorname{Tr}(\gamma^{\mu_3}\gamma^{\alpha_3}\dots\gamma^{\mu_1}\gamma^{\alpha_1}),$$

where

$$f_{\mu,\alpha}(s,l) := \sum_{k\in\mathbb{Z}^n}{}' \lambda_\sigma e^{ik\cdot\Theta(\varepsilon_1 l_1+\varepsilon_2 l_2)} e^{\frac{i}{2}\sigma_2 l_1\cdot\Theta l_2} \frac{k_{\mu_1}(k+l_1)_{\mu_2}(k+l_1+l_2)_{\mu_3}}{|k|^{s+2}|k+l_1|^2|k+l_1+l_2|^2}\tilde{a}_{\alpha,l}$$

and $\varepsilon_i := \frac{1}{2}(\sigma_i - \sigma_3) \in \{-1,0,1\}$. By hypothesis $(\varepsilon_1,\varepsilon_2) \neq (0,0)$. There are six possibilities for the values of $(\varepsilon_1,\varepsilon_2)$ corresponding to the six possibilities for the values of σ: $(-,-,+)$, $(-,+,+)$, $(+,-,+)$, $(+,+,-)$, $(-,+,-)$, and $(+,-,-)$. As in (ii), we see that

$$\begin{aligned} &f_{\mu,\alpha}(s,l) \\ &\sim \Big(\sum_{k\in\mathbb{Z}^n}{}' \frac{e^{ik\cdot\Theta(\varepsilon_1 l_1+\varepsilon_2 l_2)}k_{\mu_1}(k+l_1)_{\mu_2}(k+\hat{l}_2)_{\mu_3}}{|k|^{s+6}} \\ &- \sum_{k\in\mathbb{Z}^n}{}' \frac{e^{ik\cdot\Theta(\varepsilon_1 l_1+\varepsilon_2 l_2)}k_{\mu_1}(k+l_1)_{\mu_2}(k+\hat{l}_2)_{\mu_3}(2k\cdot l_1+2k\cdot\hat{l}_2)}{|k|^{s+8}}\lambda_\sigma\tilde{a}_{\alpha,l}e^{\frac{i}{2}\sigma_2 l_1\cdot\Theta l_2}\Big). \end{aligned}$$

With $Z := \{(l_1,l_2) \mid \varepsilon_1 l_1 + \varepsilon_2 l_2 = 0\}$, it follows from Theorem 2.6 (iii) that $\sum_{l\in(\mathbb{Z}^n)^2\setminus Z} f_{\mu,\alpha}(s,l)$ is holomorphic at 0. To conclude we need to prove that

$$\sum_\sigma g(\sigma) := \sum_\sigma\sum_{l\in Z} f_{\mu,\alpha}(s,l)\operatorname{Tr}(\gamma^{\mu_3}\gamma^{\alpha_3}\dots\gamma^{\mu_1}\gamma^{\alpha_1})$$

is holomorphic at 0. By definition, $\lambda_\sigma = i\sigma_1\sigma_2\sigma_3$ and as a consequence, we check that

$$\begin{aligned} g(-,-,+) &= -g(+,+,-), \\ g(+,-,+) &= -g(+,-,-), \\ g(-,+,+) &= -g(-,+,-), \end{aligned}$$

which implies that $\sum_\sigma g(\sigma) = 0$. The result follows.

Suppose finally that $q = 4$. Again Lemma 6.11 implies that

$$\fint \mathbb{A}^\sigma = \operatorname*{Res}_{s=0} \sum_{l\in(\mathbb{Z}^n)^3} f_{\mu,\alpha}(s,l)\operatorname{Tr}(\gamma^{\mu_4}\gamma^{\alpha_4}\dots\gamma^{\mu_1}\gamma^{\alpha_1}),$$

where

$$f_{\mu,\alpha}(s,l) := \sum_{k\in\mathbb{Z}^n}{}' \lambda_\sigma e^{ik\cdot\Theta\sum_{i=1}^3 \varepsilon_i l_i} e^{\frac{i}{2}(\sigma_2 l_1\cdot\Theta l_2+\sigma_3(l_1+l_2)\cdot\Theta l_3)} \frac{k_{\mu_1}(k+l_1)_{\mu_2}(k+l_1+l_2)_{\mu_3}(k+l_1+l_2+l_3)_{\mu_4}}{|k|^{s+2}|k+l_1|^2|k+l_1+l_2|^2|k+l_1+l_2+l_3|^2}\tilde{a}_{\alpha,l}$$

and $\varepsilon_i := \frac{1}{2}(\sigma_i - \sigma_4) \in \{-1,0,1\}$. By hypothesis, $(\varepsilon_1,\varepsilon_2,\varepsilon_3) \neq (0,0,0)$. There are fourteen possibilities for the values of $(\varepsilon_1,\varepsilon_2,\varepsilon_3)$ corresponding to the fourteen possibilities for the values of σ: $(-,-,-,+)$, $(-,-,+,+)$, $(-,+,-,+)$, $(+,-,-,+)$, $(-,+,+,+)$, $(+,-,+,+)$, $(+,+,-,+)$, $(+,+,+,-)$, $(-,-,+,-)$, $(-,+,-,-)$, $(+,-,-,-)$, $(-,+,+,-)$, $(+,-,+,-)$ and $(+,+,-,-)$. As in (ii), we see that, with $\theta_\sigma := \sigma_2 l_1\cdot\Theta l_2 + \sigma_3(l_1+l_2)\cdot\Theta l_3$,

$$f_{\mu,\alpha}(s,l) \sim \sum_{k\in\mathbb{Z}^n}{}' \lambda_\sigma e^{ik\cdot\Theta\sum_{i=1}^3 \varepsilon_i l_i} e^{\frac{i}{2}\theta_\sigma} \frac{k_{\mu_1}k_{\mu_2}k_{\mu_3}k_{\mu_4}}{|k|^{s+8}}\tilde{a}_{\alpha,l} =: g_{\mu,\alpha}(s,l).$$

With $Z_\sigma := \{(l_1,l_2,l_3) \mid \sum_{i=1}^3 \varepsilon_i l_i = 0\}$, it follows from Theorem 2.6 (iii) that the series $\sum_{l\in(\mathbb{Z}^n)^3\setminus Z_\sigma} f_{\mu,\alpha}(s,l)$ is holomorphic at 0. To conclude, we need to prove that

$$\sum_\sigma g(\sigma) := \sum_\sigma \operatorname*{Res}_{s=0} \sum_{l\in Z_\sigma} g_{\mu,\alpha}(s,l)\operatorname{Tr}(\gamma^{\mu_4}\gamma^{\alpha_4}\dots\gamma^{\mu_1}\gamma^{\alpha_1}) = 0.$$

Let C be the set of the fourteen values of σ and C_7 be the set of the seven first values of σ given above. Lemma 6.7 implies

$$\sum_{\sigma\in C} g(\sigma) = 2\sum_{\sigma\in C_7} g(\sigma).$$

Thus, in the following, we restrict to these seven values. Denote $\sum_{k\in\mathbb{Z}^n}' \frac{k_{\mu_1}k_{\mu_2}k_{\mu_3}k_{\mu_4}}{|k|^{s+8}}$ by $F_\mu(s)$ so that

$$g(\sigma) = \operatorname*{Res}_{s=0} F_\mu(s)\lambda_\sigma \sum_{l\in Z_\sigma} e^{\frac{i}{2}\theta_\sigma}\tilde{a}_{\alpha,l}\operatorname{Tr}(\gamma^{\mu_4}\gamma^{\alpha_4}\dots\gamma^{\mu_1}\gamma^{\alpha_1}).$$

Recall from (6.7) that

$$\operatorname*{Res}_{s=0} F_\mu(s)\operatorname{Tr}(\gamma^{\mu_4}\gamma^{\alpha_4}\dots\gamma^{\mu_1}\gamma^{\alpha_1}) = 2c(\delta^{\alpha_4\alpha_3}\delta^{\alpha_2\alpha_1} + \delta^{\alpha_4\alpha_1}\delta^{\alpha_3\alpha_2} - 2\delta^{\alpha_4\alpha_2}\delta^{\alpha_3\alpha_1}).$$

As a consequence, we get, with $\tilde{a}_{\alpha,l} := a_{\alpha_1,l_1}\dots a_{\alpha_4,l_4}$,

$$\begin{aligned} g(\sigma) &= 2c\lambda_\sigma \sum_{l\in(\mathbb{Z}^n)^4} e^{\frac{i}{2}\theta_\sigma}\tilde{a}_{\alpha,l}\delta_{\sum_{i=1}^4 l_i,0}\delta_{\sum_{i=1}^3 \varepsilon_i l_i,0} \\ &\qquad\cdot(\delta^{\alpha_4\alpha_3}\delta^{\alpha_2\alpha_1} + \delta^{\alpha_4\alpha_1}\delta^{\alpha_3\alpha_2} - 2\delta^{\alpha_4\alpha_2}\delta^{\alpha_3\alpha_1}) \\ &=: 2c\lambda_\sigma(T_1 + T_2 - 2T_3). \end{aligned}$$

We proceed to the following change of variables in T_1: $l_1 \mapsto l_1$, $l_2 \mapsto l_3$, $l_3 \mapsto l_2$, $l_4 \mapsto l_4$. Thus, we get $\theta_\sigma \mapsto \psi_\sigma := \sigma_2 l_1 \cdot \Theta l_3 + \sigma_3(l_1 + l_3) \cdot \Theta l_2$ and $\sum_{i=1}^3 \varepsilon_i l_i \mapsto \varepsilon_1 l_1 + \varepsilon_3 l_2 + \varepsilon_2 l_3 =: u_\sigma(l)$. With a similar permutation of the α_i, we obtain that

$$T_1 = \sum_{l \in (\mathbb{Z}^n)^4} e^{\frac{i}{2}\psi_\sigma} \tilde{a}_{\alpha,l} \delta_{\sum_{i=1}^4 l_i,0} \delta_{\varepsilon_1 l_1 + \varepsilon_3 l_2 + \varepsilon_2 l_3,0} \delta^{\alpha_4\alpha_2} \delta^{\alpha_3\alpha_1}.$$

Next we proceed to the following change of variables in T_2: $l_1 \mapsto l_2$, $l_2 \mapsto l_3$, $l_3 \mapsto l_1$, $l_4 \mapsto l_4$. Thus, we get $\theta_\sigma \mapsto \phi_\sigma := \sigma_2 l_2 \cdot \Theta l_3 + \sigma_3(l_2 + l_3) \cdot \Theta l_1$ and $\sum_{i=1}^3 \varepsilon_i l_i \mapsto \varepsilon_3 l_1 + \varepsilon_1 l_2 + \varepsilon_2 l_3 =: v_\sigma(l)$. After a similar permutation of the α_i, we have

$$T_2 = \sum_{l \in (\mathbb{Z}^n)^4} e^{\frac{i}{2}\phi_\sigma} \tilde{a}_{\alpha,l} \delta_{\sum_{i=1}^4 l_i,0} \delta_{\varepsilon_3 l_1 + \varepsilon_1 l_2 + \varepsilon_2 l_3,0} \delta^{\alpha_4\alpha_2} \delta^{\alpha_3\alpha_1}.$$

Finally, we proceed to the following change of variables in T_3: $l_1 \mapsto l_2$, $l_2 \mapsto l_1$, $l_3 \mapsto l_4$, $l_4 \mapsto l_3$. Therefore we obtain $\theta_\sigma \mapsto -\theta_\sigma$ and $\sum_{i=1}^3 \varepsilon_i l_i \mapsto (\varepsilon_2 - \varepsilon_3) l_1 + (\varepsilon_1 - \varepsilon_3) l_2 - \varepsilon_3 l_3 =: w_\sigma(l)$. With a similar permutation of the α_i, we have

$$T_3 = \sum\nolimits_{l \in (\mathbb{Z}^n)^4} e^{-\frac{i}{2}\theta_\sigma} \tilde{a}_{\alpha,l} \delta_{\sum_{i=1}^4 l_i,0} \delta_{(\varepsilon_2 - \varepsilon_3) l_1 + (\varepsilon_1 - \varepsilon_3) l_2 - \varepsilon_3 l_3,0} \delta^{\alpha_4\alpha_2} \delta^{\alpha_3\alpha_1}.$$

As a consequence, we get

$$g(\sigma) = 2c \sum\nolimits_{l \in (\mathbb{Z}^n)^4} K_\sigma(l_1, l_2, l_3) \tilde{a}_{\alpha,l} \delta_{\sum_{i=1}^4 l_i,0} \delta^{\alpha_4\alpha_2} \delta^{\alpha_3\alpha_1},$$

where

$$\begin{aligned} &K_\sigma(l_1, l_2, l_3) \\ &\quad = \lambda_\sigma (e^{\frac{i}{2}\psi_\sigma} \delta_{u_\sigma(l),0} + e^{\frac{i}{2}\phi_\sigma} \delta_{v_\sigma(l),0} - e^{\frac{i}{2}\theta_\sigma} \delta_{\sum_{i=1}^3 \varepsilon_i l_i,0} - e^{-\frac{i}{2}\theta_\sigma} \delta_{w_\sigma(l),0}). \end{aligned}$$

The computation of $K_\sigma(l_1, l_2, l_3)$ for the seven values of σ yields

$$\begin{aligned} K_{--++}(l_1, l_2, l_3) &= \delta_{l_1+l_3,0} + \delta_{l_2+l_3,0} - \delta_{l_1+l_2,0} - \delta_{l_1+l_2,0}, \\ K_{-+-+}(l_1, l_2, l_3) &= \delta_{l_1+l_2,0} + \delta_{l_1+l_2,0} - \delta_{l_1+l_3,0} - \delta_{l_1+l_3,0}, \\ K_{--++}(l_1, l_2, l_3) &= \delta_{l_2+l_3,0} + \delta_{l_1+l_3,0} - \delta_{l_2+l_3,0} - \delta_{l_2+l_3,0}, \\ K_{---+}(l_1, l_2, l_3) &= -(e^{\frac{i}{2} l_1 \cdot \Theta l_2} \delta_{\sum_{i=1}^3 l_i,0} + e^{\frac{i}{2} l_2 \cdot \Theta l_1} \delta_{\sum_{i=1}^3 l_i,0} \\ &\qquad - e^{\frac{i}{2} l_2 \cdot \Theta l_1} \delta_{\sum_{i=1}^3 l_i,0} - e^{\frac{i}{2} l_1 \cdot \Theta l_2} \delta_{l_3,0}), \\ K_{-+++}(l_1, l_2, l_3) &= -(e^{\frac{i}{2} l_3 \cdot \Theta l_2} \delta_{l_1,0} + e^{\frac{i}{2} l_3 \cdot \Theta l_1} \delta_{l_2,0} \\ &\qquad - e^{\frac{i}{2} l_2 \cdot \Theta l_3} \delta_{l_1,0} - e^{\frac{i}{2} l_3 \cdot \Theta l_1} \delta_{l_2,0}), \end{aligned}$$

$$K_{+-++}(l_1,l_2,l_3) = -(e^{\frac{i}{2}l_1\cdot\Theta l_2}\delta_{l_3,0} + e^{\frac{i}{2}l_2\cdot\Theta l_1}\delta_{l_3,0} - e^{\frac{i}{2}l_1\cdot\Theta l_3}\delta_{l_2,0} - e^{\frac{i}{2}l_3\cdot\Theta l_2}\delta_{l_1,0}),$$

$$K_{++-+}(l_1,l_2,l_3) = -(e^{\frac{i}{2}l_1\cdot\Theta l_3}\delta_{l_2,0} + e^{\frac{i}{2}l_2\cdot\Theta l_3}\delta_{l_1,0} - e^{\frac{i}{2}l_1\cdot\Theta l_2}\delta_{l_3,0} - e^{\frac{i}{2}l_2\cdot\Theta l_1}\delta_{\sum_{i=1}^3 l_i,0}).$$

Thus,

$$\sum_{\sigma\in C_7} K_\sigma(l_1,l_2,l_3) = 2i(\delta_{l_3,0} - \delta_{\sum_{i=1}^3 l_i,0})\sin\tfrac{l_1\cdot\Theta l_2}{2}$$

and

$$\sum_{\sigma\in C_7} g(\sigma) = i4c\sum_{l\in(\mathbb{Z}^n)^4}(\delta_{l_3,0} - \delta_{\sum_{i=1}^3 l_i,0})\sin\tfrac{l_1\cdot\Theta l_2}{2}\tilde{a}_{\alpha,l}\delta_{\sum_{i=1}^4 l_i,0}\delta^{\alpha_4\alpha_2}\delta^{\alpha_3\alpha_1}.$$

The following change of variables $l_1\mapsto l_2$, $l_1\mapsto l_2$, $l_3\mapsto l_4$, $l_4\mapsto l_3$ gives

$$\sum_{l\in(\mathbb{Z}^n)^4}\delta_{\sum_1^3 l_i,0}\sin\tfrac{l_1\cdot\Theta l_2}{2}\tilde{a}_{\alpha,l}\delta_{\sum_1^4 l_i,0}\delta^{\alpha_4\alpha_2}\delta^{\alpha_3\alpha_1} = -\sum_{l\in(\mathbb{Z}^n)^4}\delta_{l_3,0}\sin\tfrac{l_1\cdot\Theta l_2}{2}\tilde{a}_{\alpha,l}\delta_{\sum_1^4 l_i,0}\delta^{\alpha_4\alpha_2}\delta^{\alpha_3\alpha_1},$$

and hence

$$\sum_{\sigma\in C_7} g(\sigma) = i8c\sum_{l\in(\mathbb{Z}^n)^4}\delta_{l_3,0}\sin\tfrac{l_1\cdot\Theta l_2}{2}\tilde{a}_{\alpha,l}\delta_{\sum_1^4 l_i,0}\delta^{\alpha_4\alpha_2}\delta^{\alpha_3\alpha_1}.$$

Finally, the change of variables $l_2\mapsto l_4$, $l_4\mapsto l_2$ gives

$$\sum_{l\in(\mathbb{Z}^n)^4}\delta_{l_3,0}\sin\tfrac{l_1\cdot\Theta l_2}{2}\tilde{a}_{\alpha,l}\delta_{\sum_1^4 l_i,0}\delta^{\alpha_4\alpha_2}\delta^{\alpha_3\alpha_1} = -\sum_{l\in(\mathbb{Z}^n)^4}\delta_{l_3,0}\sin\tfrac{l_1\cdot\Theta l_2}{2}\tilde{a}_{\alpha,l}\delta_{\sum_1^4 l_i,0}\delta^{\alpha_4\alpha_2}\delta^{\alpha_3\alpha_1},$$

which implies that $\sum_{\sigma\in C_7} g(\sigma) = 0$. □

Lemma 6.13. *Suppose that $n = 4$ and $\frac{1}{2\pi}\Theta$ is diophantine. For any selfadjoint 1-form A we have*

$$\zeta_{D_A}(0) - \zeta_D(0) = -c\tau(F_{\alpha_1,\alpha_2}F^{\alpha_1\alpha_2}).$$

Proof. By (4.5) and Lemma 6.6 we get

$$\zeta_{D_A}(0) - \zeta_D(0) = \sum_{q=1}^n \frac{(-1)^q}{q}\sum_{\sigma\in\{+,-\}^q}\fint \mathbb{A}^\sigma.$$

By Lemma 6.12 (iv), we see that the crossed terms all vanish. Thus, with Lemma 6.7, we get

$$\zeta_{D_A}(0) - \zeta_D(0) = 2\sum_{q=1}^{n} \frac{(-1)^q}{q} \fint (\mathbb{A}^+)^q. \tag{6.10}$$

By definition,

$$\begin{aligned} F_{\alpha_1\alpha_2} &= i\sum_k (a_{\alpha_2,k}k_{\alpha_1} - a_{\alpha_1,k}k_{\alpha_2})U_k + \sum_{k,l} a_{\alpha_1,k}a_{\alpha_2,l}[U_k, U_l] \\ &= i\sum_k \Big[(a_{\alpha_2,k}k_{\alpha_1} - a_{\alpha_1,k}k_{\alpha_2}) - 2\sum_l a_{\alpha_1,k-l}a_{\alpha_2,l}\sin\big(\tfrac{k\cdot\Theta l}{2}\big)\Big]U_k. \end{aligned}$$

Thus

$$\begin{aligned} &\tau(F_{\alpha_1\alpha_2}F^{\alpha_1\alpha_2}) \\ &\quad = \sum_{\alpha_1,\alpha_2=1}^{2^m} \sum_{k\in\mathbb{Z}^4} \Big[(a_{\alpha_2,k}k_{\alpha_1} - a_{\alpha_1,k}k_{\alpha_2}) - 2\sum_{l'\in\mathbb{Z}^4} a_{\alpha_1,k-l'}a_{\alpha_2,l'}\sin\big(\tfrac{k\cdot\Theta l'}{2}\big)\Big] \\ &\qquad \Big[(a_{\alpha_2,-k}k_{\alpha_1} - a_{\alpha_1,-k}k_{\alpha_2}) - 2\sum_{l''\in\mathbb{Z}^4} a_{\alpha_1,-k-l''}a_{\alpha_2,l''}\sin\big(\tfrac{k\cdot\Theta l''}{2}\big)\Big]. \end{aligned}$$

One checks that the term in a^q of $\tau(F_{\alpha_1\alpha_2}F^{\alpha_1\alpha_2})$ corresponds to the term $\fint(\mathbb{A}^+)^q$ given by Lemma 6.12. For $q = 2$, this is

$$-2\sum_{l\in\mathbb{Z}^4,\alpha_1,\alpha_2} a_{\alpha_1,l}a_{\alpha_2,-l}(l_{\alpha_1}l_{\alpha_2} - \delta_{\alpha_1\alpha_2}|l|^2).$$

For $q = 3$, we compute the crossed terms

$$i\sum_{k,k',l} (a_{\alpha_2,k}k_{\alpha_1} - a_{\alpha_1,k}k_{\alpha_2})a^{\alpha_1}_{k'}a^{\alpha_2}_{l}(U_k[U_{k'}, l] + [U_{k'}, U_l]U_k),$$

which gives the following a^3-term in $\tau(F_{\alpha_1\alpha_2}F^{\alpha_1\alpha_2})$:

$$-8\sum_{l_i} a_{\alpha_3,-l_1-l_2}a^{\alpha_1}_{l_2}a_{\alpha_1,l_1}\sin\tfrac{l_1\cdot\Theta l_2}{2}l_1^{\alpha_3}.$$

For $q = 4$, this is

$$-4\sum_{l_i} a_{\alpha_1,-l_1-l_2-l_3}a_{\alpha_2,l_3}a^{\alpha_1}_{l_2}a^{\alpha_2}_{l_1}\sin\tfrac{l_1\cdot\Theta(l_2+l_3)}{2}\sin\tfrac{l_2\cdot\Theta l_3}{2},$$

which corresponds to the term $\fint(\mathbb{A}^+)^4$. We finally obtain that

$$\sum_{q=1}^{n} \frac{(-1)^q}{q} \fint (\mathbb{A}^+)^q = -\frac{c}{2}\tau(F_{\alpha_1,\alpha_2}F^{\alpha_1\alpha_2}). \tag{6.11}$$

Equations (6.10) and (6.11) yield the result. □

Lemma 6.14. *Suppose that $n = 2$. Then, with the same hypothesis as in Lemma* 6.11*, the following holds:*

(i) $\fint (\mathbb{A}^+)^2 = \fint (\mathbb{A}^-)^2 = 0.$

(ii) *Suppose that $\frac{1}{2\pi}\Theta$ is diophantine. Then*

$$\fint \mathbb{A}^+\mathbb{A}^- = \fint \mathbb{A}^-\mathbb{A}^+ = 0.$$

Proof. (i) Lemma 6.11 implies that $\fint \mathbb{A}^{++} = \operatorname*{Res}_{s=0} \sum_{l\in\mathbb{Z}^2} -f(s,l)$, where

$$\begin{aligned} f(s,l) &:= \sum_{k\in\mathbb{Z}^2}{}' \frac{k_{\mu_1}(k+l)_{\mu_2}}{|k|^{s+2}|k+l|^2}\tilde{a}_{\alpha,l}\operatorname{Tr}(\gamma^{\alpha_2}\gamma^{\mu_2}\gamma^{\alpha_1}\gamma^{\mu_1}) \\ &=: f_{\mu,\alpha}(s,l)\operatorname{Tr}(\gamma^{\alpha_2}\gamma^{\mu_2}\gamma^{\alpha_1}\gamma^{\mu_1}) \end{aligned}$$

and $\tilde{a}_{\alpha,l} := a_{\alpha_1,l}a_{\alpha_2,-l}$. This time, since $n = 2$, it is enough to apply just once (2.10) to obtain an absolutely convergent series. Indeed, with (2.10) we get

$$f_{\mu,\alpha}(s,l) = \sum_{k\in\mathbb{Z}^2}{}' \frac{k_{\mu_1}(k+l)_{\mu_2}}{|k|^{s+4}}\tilde{a}_{\alpha,l} - \sum_{k\in\mathbb{Z}^2}{}' \frac{k_{\mu_1}(k+l)_{\mu_2}(2k\cdot l+|l|^2)}{|k|^{s+4}|k+l|^2}\tilde{a}_{\alpha,l}.$$

The function $r(s,l) := \sum'_{k\in\mathbb{Z}^2} \frac{k_{\mu_1}(k+l)_{\mu_2}(2k\cdot l+|l|^2)}{|k|^{s+4}|k+l|^2}\tilde{a}_{\alpha,l}$ is a linear combination of functions of the type $H(s,l)$ satisfying the hypothesis of Corollary 2.14. As a consequence, $r(s,l)$ satisfies (H1) and

$$f_{\mu,\alpha}(s,l) \sim \sum_{k\in\mathbb{Z}^2}{}' \frac{k_{\mu_1}(k+l)_{\mu_2}}{|k|^{s+4}}\tilde{a}_{\alpha,l} \sim \sum_{k\in\mathbb{Z}^2}{}' \frac{k_{\mu_1}k_{\mu_2}}{|k|^{s+4}}\tilde{a}_{\alpha,l}.$$

Note that the function $(s,l) \mapsto h_{\mu,\alpha}(s,l) := \sum'_{k\in\mathbb{Z}^2} \frac{k_{\mu_1}k_{\mu_2}}{|k|^{s+4}}\tilde{a}_{\alpha,l}$ satisfies (H2). Thus, Lemma 2.15 yields that

$$\operatorname*{Res}_{s=0} f(s,l) = \sum_{l\in\mathbb{Z}^2} \operatorname*{Res}_{s=0} h_{\mu,\alpha}(s,l)\operatorname{Tr}(\gamma^{\alpha_2}\gamma^{\mu_2}\gamma^{\alpha_1}\gamma^{\mu_1}).$$

By Proposition 2.17, we get $\operatorname*{Res}_{s=0} h_{\mu,\alpha}(s,l) = \delta_{\mu_1\mu_2}\pi\tilde{a}_{\alpha,l}$. Therefore,

$$\fint \mathbb{A}^{++} = -\pi \sum_{l\in\mathbb{Z}^2} \tilde{a}_{\alpha,l}\operatorname{Tr}(\gamma^{\alpha_2}\gamma^{\mu}\gamma^{\alpha_1}\gamma_{\mu}) = 0$$

according to (6.4).

(ii) By Lemma 6.11, we obtain that

$$\fint \mathbb{A}^{-+} = \operatorname*{Res}_{s=0} \sum_{l\in\mathbb{Z}^2} \lambda_\sigma f_{\alpha,\mu}(s,l)\operatorname{Tr}(\gamma^{\alpha_2}\gamma^{\mu_2}\gamma^{\alpha_1}\gamma^{\mu_1}),$$

where $\lambda_\sigma = -(-i)^2 = 1$,

$$f_{\alpha,\mu}(s,l) := \sum_{k\in\mathbb{Z}^2}{}' \frac{k_{\mu_1}(k+l)_{\mu_2}}{|k|^{s+2}|k+l|^2} e^{i\eta k\cdot\Theta l}\tilde{a}_{\alpha,l}$$

and $\eta := \frac{1}{2}(\sigma_1 - \sigma_2) = -1$. As in the proof of (i), since the presence of the phase does not change the fact that $r(s,l)$ satisfies (H1), we get

$$f_{\alpha,\mu}(s,l) \sim \sum_{k\in\mathbb{Z}^2}{}' \frac{k_{\mu_1}(k+l)_{\mu_2}}{|k|^{s+4}} e^{i\eta k\cdot\Theta l}\tilde{a}_{\alpha,l} := g_{\alpha,\mu}(s,l).$$

Since $\frac{1}{2\pi}\Theta$ is diophantine, the functions $s \mapsto \sum_{l\in\mathbb{Z}^2\setminus\{0\}} g_{\alpha,\mu}(s,l)$ are holomorphic at $s = 0$ by Theorem 2.6 3. As a consequence,

$$\begin{aligned}\fint \mathbb{A}^{-+} &= \operatorname*{Res}_{s=0} g_{\alpha,\mu}(s,0)\operatorname{Tr}(\gamma^{\alpha_2}\gamma^{\mu_2}\gamma^{\alpha_1}\gamma^{\mu_1})\\ &= \operatorname*{Res}_{s=0} \sum_{k\in\mathbb{Z}^2}{}' \frac{k_{\mu_1}k_{\mu_2}}{|k|^{s+4}}\tilde{a}_{\alpha,0}\operatorname{Tr}(\gamma^{\alpha_2}\gamma^{\mu_2}\gamma^{\alpha_1}\gamma^{\mu_1}).\end{aligned}$$

Recall from Proposition 2.1 that $\operatorname{Res}_{s=0}\sum'_{k\in\mathbb{Z}^2}\frac{k_ik_j}{|k|^{s+4}} = \delta_{ij}\pi$. Thus, again with (6.4),

$$\fint \mathbb{A}^{-+} = \tilde{a}_{\alpha,0}\pi\operatorname{Tr}(\gamma^{\alpha_2}\gamma^{\mu}\gamma^{\alpha_1}\gamma_\mu) = 0. \qquad \square$$

Lemma 6.15. *Suppose that $n = 2$ and $\frac{1}{2\pi}\Theta$ is diophantine. Then for any selfadjoint 1-form A,*

$$\zeta_{D_A}(0) - \zeta_D(0) = 0.$$

Proof. As in Lemma 6.13, we use (4.5) and Lemma 6.6 so the result follows from Lemma 6.14. $\square$

6.1.2. Odd dimensional case

Lemma 6.16. *Suppose that n is odd and $\frac{1}{2\pi}\Theta$ is diophantine. Then for any selfadjoint 1-form A and $\sigma \in \{-,+\}^q$ with $2 \le q \le n$,*

$$\fint \mathbb{A}^{\sigma} = 0.$$

Proof. Since $\mathbb{A}^\sigma \in \Psi_1(\mathcal{A})$, Lemma 5.11 with $k = n$ gives the result. □

Corollary 6.17. *Let the same hypothesis be satisfied as in Lemma* 6.16*. Then for any selfadjoint* 1*-form* A *we have* $\zeta_{D_A}(0) - \zeta_D(0) = 0$.

Proof. As in Lemma 6.13, we use (4.5) and Lemma 6.6, so the result follows from Lemma 6.16. □

6.2. Proof of the main result

Proof of Theorem 6.1. (i) From (1.5) and Proposition 5.5 we get

$$\mathcal{S}(\mathcal{D}_A, \Phi, \Lambda) = 4\pi\Phi_2\Lambda^2 + \Phi(0)\zeta_{D_A}(0) + \mathcal{O}(\Lambda^{-2}),$$

where $\Phi_2 = \frac{1}{2}\int_0^\infty \Phi(t)\,dt$. By Lemma 6.15, $\zeta_{D_A}(0) - \zeta_D(0) = 0$ and from Proposition 5.4, $\zeta_D(0) = 0$, so the result follows.

(ii) Similarly, $\mathcal{S}(\mathcal{D}_A, \Phi, \Lambda) = 8\pi^2\Phi_4\Lambda^4 + \Phi(0)\zeta_{D_A}(0) + \mathcal{O}(\Lambda^{-2})$ with $\Phi_4 = \frac{1}{2}\int_0^\infty \Phi(t)t\,dt$. Lemma 6.13 implies that $\zeta_{D_A}(0) - \zeta_D(0) = -c\tau(F_{\mu\nu}F^{\mu\nu})$, and by Proposition 5.4 we obtain that $\zeta_{D_A}(0) = -c\tau(F_{\mu\nu}F^{\mu\nu})$ and hence the result.

(iii) This is a direct consequence of (1.5), Propositions 5.4 and 5.5, and Corollary 6.17. □

A. Appendix

A.1. Proof of Lemma 3.3. (i) We have $|D|T|D|^{-1} = T + \delta(T)|D|^{-1}$ and $|D|^{-1}T|D| = T - |D|^{-1}\delta(T)$. By recurrence, $|D|^kT|D|^{-k} = \sum_{q=0}^k \binom{q}{k}\delta^q(T)|D|^{-q}$ for any $k \in \mathbb{N}$ and so $|D|^{-k}T|D|^k = \sum_{q=0}^k (-1)^q\binom{q}{k}|D|^{-q}\delta^q(T)$.

As a consequence, since T, $|D|^{-q}$ and $\delta^q(T)$ are in OP^0 for any $q \in \mathbb{N}$, for any $k \in \mathbb{Z}$, $|D|^kT|D|^{-k} \in OP^0$. Let us fix $p \in \mathbb{N}_0$ and define $F_p(s) := \delta^p(|D|^sT|D|^{-s})$ for $s \in \mathbb{C}$. Since for $k \in \mathbb{Z}$, $F_p(k)$ is bounded, a complex interpolation proves that $F_p(s)$ is bounded, which gives $|D|^sT|D|^{-s} \in OP^0$.

(ii) Let $T \in OP^\alpha$ and $T' \in OP^\beta$. Thus, $T|D|^{-\alpha}$, $T'|D|^{-\beta}$ are in OP^0. By (i) we get $|D|^\beta T|D|^{-\alpha}|D|^{-\beta} \in OP^0$, so $T'|D|^{-\beta}|D|^\beta T|D|^{-\beta-\alpha} \in OP^0$. Thus, $T'T|D|^{-(\alpha+\beta)} \in OP^0$.

(iii) For $T \in OP^\alpha$, $|D|^{\alpha-\beta}$ and contained in $T|D|^{-\alpha}$ are in OP^0 and therefore $T|D|^{-\beta} = T|D|^{-\alpha}|D|^{\alpha-\beta} \in OP^0$.

(iv) follows from $\delta(OP^0) \subseteq OP^0$.

(v) Since $\nabla(T) = \delta(T)|D| + |D|\delta(T) - [P_0, T]$, the result follows from (ii), (iv) and the fact that P_0 is in $OP^{-\infty}$.

A.2. Proof of Lemma 3.6. The non-trivial part of the proof is the stability under the product of operators. Let $T, T' \in \Psi(\mathcal{A})$. There are $d, d' \in \mathbb{Z}$ such that for any $N \in \mathbb{N}$, $N > |d| + |d'|$, there exist P, P' in $\mathcal{D}(\mathcal{A})$, $p, p' \in \mathbb{N}_0$, $R \in OP^{-N-d'}$, $R' \in OP^{-N-d}$ such that $T = PD^{-2p} + R$, $T' = P'D^{-2p'} + R'$, $PD^{-2p} \in OP^d$ and $P'D^{-2p'} \in OP^{d'}$.

Thus, $TT' = PD^{-2p}P'D^{-2p'} + RP'D^{-2p'} + PD^{-2p}R' + RR'$.

We also have $RP'D^{-2p'} \in OP^{-N-d'+d'} = OP^{-N}$ and, similarly, $PD^{-2p}R' \in OP^{-N}$. Since $RR' \in OP^{-2N}$, it follows that

$$TT' \sim PD^{-2p}P'D^{-2p'} \mod OP^{-N}.$$

If $p = 0$, then $TT' \sim QD^{-2p'} \mod OP^{-N}$, where $Q = PP' \in \mathcal{D}(\mathcal{A})$ and $QD^{-2p'} \in OP^{d+d'}$. Suppose that $p \neq 0$. By recurrence one shows that

$$D^{-2}P' \sim \sum_{k=0}^{q} (-1)^k \nabla^k(P')D^{-2k-2} + (-1)^{q+1}D^{-2}\nabla^{q+1}(P')D^{-2q-2} \mod OP^{-\infty}.$$

for any $q \in \mathbb{N}_0$. By Lemma 3.3 (v), the remainder is in $OP^{d'+2p'-q-3}$ since $P' \in OP^{d'+2p'}$. Another recurrence gives

$$D^{-2p}P' \sim \sum_{k_1,\dots,k_p=0}^{q} (-1)^{|k|_1}\nabla^{|k|_1}(P')D^{-2|k|_1-2p} \mod OP^{d'+2p'-q-1-2p}.$$

for any $q \in \mathbb{N}_0$. Thus, with $q_N = N + d + d' - 1$,

$$TT' \sim \sum_{k_1,\dots,k_p=0}^{q_N} (-1)^{|k|_1} P\nabla^{|k|_1}(P')D^{-2|k|_1-2(p+p')} \mod OP^{-N}.$$

The last sum can be written $Q_N D^{-2r_N}$ where $r_N := pq_N + (p + p')$. Since $Q_N \in \mathcal{D}(\mathcal{A})$ and $Q_N D^{-2r_N} \in OP^{d+d'}$, the result follows.

A.3. Proof of Proposition 3.11. Let $P \in OP^{k_1}$ and $Q \in OP^{k_2} \in \Psi(\mathcal{A})$. With $[Q, |D|^{-s}] = (Q - \sigma_{-s}(Q))|D|^{-s}$ and $Q - \sigma_{-s}(Q) \sim -\sum_{r=1}^{N} g(-s, r)\varepsilon^r(Q) \mod OP^{-N-1+k_2}$, we get

$$P[Q, |D|^{-s}] \sim -\sum_{r=1}^{N} g(-s, r)P\varepsilon^r(Q)|D|^{-s} \mod OP^{-N-1+k_1+k_2-\mathrm{Re}(s)},$$

which gives, if we choose $N = n + k_1 + k_2$,

$$\operatorname*{Res}_{s=0} \mathrm{Tr}(P[Q, |D|^{-s}]) = -\sum_{r=1}^{n+k_1+k_2} \operatorname*{Res}_{s=0} g(-s, r)\,\mathrm{Tr}(P\varepsilon^r(Q)|D|^{-s}).$$

By hypothesis $s \mapsto \operatorname{Tr}(P\varepsilon^r(Q)|D|^{-s})$ has only simple poles. Thus, because $s = 0$ is a zero of the analytic function $s \mapsto g(-s, r)$ for any $r \geq 1$, it follows that $\operatorname*{Res}_{s=0} g(-s, r) \operatorname{Tr}(P\varepsilon^r(Q)|D|^{-s}) = 0$, which implies that $\operatorname*{Res}_{s=0} \operatorname{Tr}(P[Q, |D|^{-s}]) = 0$ and hence

$$\fint PQ = \operatorname*{Res}_{s=0} \operatorname{Tr}(P|D|^{-s}Q).$$

When $s \in \mathbb{C}$ with $\operatorname{Re}(s) > 2\max(k_1 + n + 1, k_2)$, the operator $P|D|^{-s/2}$ is trace-class, while $|D|^{-s/2}Q$ is bounded, so $\operatorname{Tr}(P|D|^{-s}Q) = \operatorname{Tr}(|D|^{-s/2}QP|D|^{-s/2}) = \operatorname{Tr}(\sigma_{-s/2}(QP)|D|^{-s})$. Thus, using (3.1) again, we obtain that

$$\operatorname*{Res}_{s=0} \operatorname{Tr}(P|D|^{-s}Q) = \fint QP + \sum_{r=1}^{n+k_1+k_2} \operatorname*{Res}_{s=0} g(-s/2, r) \operatorname{Tr}(\varepsilon^r(QP)|D|^{-s}).$$

As before, for any $r \geq 1$, $\operatorname*{Res}_{s=0} g(-s/2, r) \operatorname{Tr}\left(\varepsilon^r(QP)|D|^{-s}\right) = 0$ since $g(0, r) = 0$ and the spectral triple is simple. Finally,

$$\operatorname*{Res}_{s=0} \operatorname{Tr}(P|D|^{-s}Q) = \fint QP.$$

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