J. Noncommut. Geom. 2 (2008), 353–404 **Journal of Noncommutative Geometry**

Cohomology of Yang–Mills algebras

Michael Movshev

Abstract. In this paper we compute cyclic and Hochschild homology of the universal envelope $U(YM)$ of the Yang–Mills Lie algebra YM. We also compute Hochschild cohomology with coefficients in $U(YM)$, considered as a bimodule over itself.

The result of the calculations depends on the number of generators n of YM. The semidirect product $\mathfrak{so}(n) \ltimes \mathbb{C}^n$ acts by derivations upon $U(YM)$. One of the important consequences of our results is that if $n \geq 3$ then the Lie algebra of outer derivations of $U(YM)$ coincides with $\mathfrak{so}(n) \ltimes \mathbb{C}^n$.

Mathematics Subject Classification (2000). 16E40, 55P62.

Keywords. Yang–Mills algebra, cohomology, derivations.

Contents

1. Introduction

This paper is an account of the preliminary material that appeared in the preprint [16].

Our main object of study is the Yang–Mills algebra YM as introduced in [6] and [19].

By definition the Lie algebra YM is a quotient of the free Lie algebra $\mathrm{Free}(V)$, where V is a complex n-dimensional linear space equipped with a symmetric non-degenerate inner product $(\cdot,\cdot)$. In a basis u_s the Gram matrix $g_{ss'}$ of the inner product has the inverse $g^{ss'}$. The relations defining the YM algebra

$$\sum_{s,s'=1}^{n} g^{ss'}[u_s,[u_{s'},u_t]] = 0, \quad t = 1,\dots,n,$$

depend only on the choice of the bilinear form $(\cdot,\cdot)$.

The purpose of this paper is to compute various kinds of homology of the universal enveloping algebra $U(YM)$. We study cyclic and Hochschild homology as well as Hochschild cohomology with coefficients in the adjoint bimodule.

Some words about the history of our subject are in order.

The associative algebra $U(YM)$ was explicitly mentioned as an algebraic object for the first time in [19], where some interesting representations of $U(YM)$ were presented. A thorough algebraic treatment of $U(YM)$ has been done in [6] and [7].

In [6] Connes and Dubois-Violette defined the Yang–Mills algebra as a cubic associative algebra. They observed the fact that it is the universal enveloping algebra of a graded Lie algebra and stated that it is a Koszul algebra of global dimension 3. Also they noticed that it has the Poincaré duality property: a property that, following Artin and Schelter [1], is referred to as the Gorenstein property. Moreover it is also in this paper that Connes and Dubois-Violette gave an explicit formula for the Poincaré series of this Yang–Mills algebra (and computed it by using the dimensions of the homogeneous components of the corresponding Lie algebra in terms of the Möbius function). We should acknowledge that most of our definitions and most of the elementary properties of the algebra $U(YM)$ discussed in Section 2 are already contained in [6], [7]. Also [7] describes some components of the Hochschild homology of $U(YM)$. Hence one can consider our work as a generalisation of the results in [7].

Our main motivation was to understand symmetries of the Yang–Mills equation over the flat Riemannian $\mathbb{R}^n$ that has been reduced to a point. More specifically we consider covariant differentiations $\nabla_s = \frac{\partial}{\partial x_s} + A_s(x_1,\dots,x_n)$, $1 \le s \le n$, that act on sections of a trivial N-dimensional Hermitian vector bundle over $\mathbb{R}^n$. We assume that ∇_s preserve the Hermitian structure. The space $\mathbb{R}^n$ is equipped with the metric $g_{ss'}dx^s dx^{s'}$. The system of Yang–Mills equations $g^{ss'}[\nabla_s,[\nabla_{s'},\nabla_t]] = 0$, $t = 1,\dots,n$, is compatible with the space-time shifts. Upon the restriction of this system to translation-invariant covariant differentiations it becomes a system of matrix equations

$$\sum_{s,s'=1}^{n} g^{ss'}[A_s,[A_{s'},A_t]] = 0, \quad t = 1,\dots,n, \tag{1.1}$$

where $A_1, \dots, A_n$ is an array of anti-hermitian matrices. The linear space of anti-hermitian matrices $\mathfrak{u}(N)$ is the Lie algebra of the unitary group $U(N)$-translation-invariant gauge transformations. This is an obvious symmetry group of the equation (1.1).

Solutions of this equation can be identified with critical points of the function $\sum_{1\le s<t\le n} g^{ss'}g^{tt'}\operatorname{tr}([A_s, A_t][A_{s'}, A_{t'}])$, defined on $\mathfrak{u}(N)^{\times n}$.

The complexification $\mathfrak{u}_{\mathbb{C}}(N)^{\times n}$ of the space $\mathfrak{u}(N)^{\times n}$ coincides with the space of n-tuples of complex $(N\times N)$-matrices $\mathrm{Mat}_N^{\times n}$. The latter has a system of coordinates x_{ij}^s, $1\le i, j\le N$, $1\le s\le n$. A $\mathrm{GL}(N,\mathbb{C})\cong U_{\mathbb{C}}(N)$-invariant vector field on $\mathrm{Mat}_N^{\times n}$ of degree $k-1$ can constructed by the formula

$$\sum c^{s_k}_{s_1,s_2,\dots,s_{k-1}} x^{s_1}_{i_1i_2}x^{s_2}_{i_2i_3}\dots x^{s_{k-1}}_{i_{k-2}i_{k-1}}\frac{\partial}{\partial x^{s_k}_{i_{k-1}i_1}}.$$

In this formula we perform summation over the repeated indices. The coefficients $c^{s_k}_{s_1,s_2,\dots,s_{k-1}}$ are arbitrary complex numbers. The above vector field can be written in terms of the matrix multiplication and the trace functional: $\sum c^{s_k}_{s_1,s_2,\dots,s_{k-1}}\operatorname{tr}(X^{s_1}X^{s_2}\dots X^{s_{k-1}}\frac{\partial}{\partial X^{s_k}})$; we use matrices X^s and $\frac{\partial}{\partial X^s}$ with entries x^s_{ij} and $\frac{\partial}{\partial x^s_{ij}}$ respectively. We shall call a non-homogeneous linear combination of such vector fields a noncommutative vector field. We would like to classify noncommutative vector fields that are tangent to the space of solutions of the complexified Euler–Lagrange equation (1.1) for all N, i.e., the corresponding derivations leave the ideal of the equations invariant. In our classification we identify two vector fields that coincide on the space of solutions, i.e., the difference of the two is a vector field with the coefficients in the ideal.

Inspired by the general intuition about the noncommutative structures, described in [12], it is not hard to see that there is a one-to-one correspondence between the classes of noncommutative vector fields and derivations of the algebra $U(YM)$. The standard method of the analysis of the space of derivations of an associative algebra A is through the computation of its first Hochschild cohomology $H^1(A, A)$ [14]. This explains our interest in $H^\bullet(U(YM), U(YM))$. So much then for motivations.

1.1. Formulation of the results. The Lie algebra *YM* and the universal enveloping $U(YM)$ are graded by the degree of monomials.

If $B=\bigoplus_{i>-\infty} B_i$ is a graded linear space then the generating function (Poincaré series) $B(t)=\sum_i \dim(B_i)t^i$ is a formal Laurent series.

Denote by $\mathrm{HH}^\bullet(U(YM), U(YM))$ and $\mathrm{HH}_\bullet(U(YM), U(YM))$ the Hochschild cohomology and the Hochschild homology of $U(YM)$ with coefficients in the adjoint bimodule $U(YM)$, respectively [14]. Let $\mathrm{HC}_\bullet(U(YM))$ be the cyclic homology of $U(YM)$ [13]. The grading on $U(YM)$ induces a grading on the (co)homology groups for which we will reserve bold indices.

Define generating functions

$$\mathrm{HH}^i(U(YM),U(YM))(t)=\sum_j \dim(\mathrm{HH}^{i,j}(U(YM),U(YM)))t^j,$$

$$\mathrm{HH}_i(U(YM),U(YM))(t)=\sum_j \dim(\mathrm{HH}_{i,j}(U(YM),U(YM)))t^j,$$

$$\mathrm{HC}_i(U(YM))(t)=\sum_j \dim(\mathrm{HC}_{i,j}(U(YM)))t^j.$$

Results of [7] can be easily adapted to prove of the following.

The generating functions $\mathrm{HH}^i(U(YM),U(YM))(t)$ *are well-defined formal Laurent series. The homological generating functions are given by*

$$\mathrm{HH}_i(U(YM),U(YM))(t)=\mathrm{HH}^{3-i}(U(YM),U(YM))(t)t^4. \tag{1.2}$$

The proof can be found in Section 2.2.

Introduce the formal power series

$$\mu_n(t)=-\sum_{k\geq 1}\ln(1-nt^k+nt^{3k}-t^{4k})^{\frac{\phi(k)}{k}}, \tag{1.3}$$

where the totient function $\phi(k)$ is defined as the number of positive integers $\leq k$ that are relatively prime to k.

The main result of this paper is a proof of Theorems 1.1 and 1.2.

Theorem 1.1. *If* $\dim(V)=n\geq 3$, *then*

$$\begin{aligned}
\mathrm{HH}^0(U(YM),U(YM))(t)&=1,\\
\mathrm{HH}^1(U(YM),U(YM))(t)&=\frac{n(n-1)}{2}+1+\frac{n}{t},\\
\mathrm{HH}^2(U(YM),U(YM))(t)&=\frac{\mu_n(t)}{t^4}+n(n-1)-1+\frac{2n}{t},\\
\mathrm{HH}^3(U(YM),U(YM))(t)&=\frac{\mu_n(t)}{t^4}+\frac{n(n-1)}{2}-1+\frac{n}{t}+\frac{1}{t^4},\\
\mathrm{HH}^i(U(YM),U(YM))(t)&=0,\quad i\geq 4,
\end{aligned}$$

and

$$\begin{aligned}
\mathrm{HC}_0(U(YM))(t)&=1+\mu_n(t)+\Big(\frac{n(n-1)}{2}-1\Big)t^4+nt^3,\\
\mathrm{HC}_1(U(YM))(t)&=\frac{n(n-1)}{2}t^4+nt^3,\\
\mathrm{HC}_2(U(YM))(t)&=1+t^4,\\
\mathrm{HC}_{3+2i}(U(YM))(t)&=0,\quad i\geq 0,\\
\mathrm{HC}_{4+2i}(U(YM))(t)&=1,\quad i\geq 0.
\end{aligned}$$

Theorem 1.2. *If* $\dim(V) = 2$*, then*

$$\begin{aligned}
\mathrm{HH}^0(U(YM), U(YM))(t) &= \frac{1}{1-t^2},\\
\mathrm{HH}^1(U(YM), U(YM))(t) &= 2\frac{1+t-t^2}{t(1-t^2)(1-t)},\\
\mathrm{HH}^2(U(YM), U(YM))(t) &= \frac{(2-t)(1+t^2)}{(1-t)^2t^3},\\
\mathrm{HH}^3(U(YM), U(YM))(t) &= \frac{1}{(1-t)^2t^4},\\
\mathrm{HH}^i(U(YM), U(YM))(t) &= 0, \quad i \geq 4,
\end{aligned} \tag{1.4}$$

and

$$\begin{aligned}
\mathrm{HC}_0(U(YM))(t) &= \frac{1}{(1-t)^2},\\
\mathrm{HC}_1(U(YM))(t) &= \frac{(2-t)t^3}{(1-t)^2},\\
\mathrm{HC}_2(U(YM))(t) &= 1 + \frac{t^4}{1-t^2},\\
\mathrm{HC}_{3+2i}(U(YM))(t) &= 0, \quad i \geq 0,\\
\mathrm{HC}_{4+2i}(U(YM))(t) &= 1, \quad i \geq 0.
\end{aligned} \tag{1.5}$$

1.2. An outline of the proofs of the main theorems. For the proofs of the main results – Theorems 1.1 and 1.2 – we need Connes' exact sequence for cyclic homology, adapted to graded algebras (see Section 2.1). We also use Poincaré duality in the homology of $U(YM)$.

Besides these the proof of Theorem 1.2 uses only elementary considerations, which can be found in Section 6.2.

The proof of Theorem 1.1 is much more involved.

The computation of $\mathrm{HH}_i(A, A)(t)$, $0 \leq i \leq 3$, where A is a graded associative algebra A with Poincaré duality in dimension 3, requires only knowledge of $A(t)$ and two of the three series $\mathrm{HC}_0(A)(t)$, $\mathrm{HC}_1(A)(t)$, $\mathrm{HC}_2(A)(t)$.

In addition, the equalities

$$\begin{aligned}
\mathrm{HC}_2(A)(t) &= \mathrm{HH}_3(A, A)(t) + 1,\\
\mathrm{HC}_2(A)(t) + \mathrm{HC}_1(A)(t) &= \mathrm{HH}_2(A, A)(t) + 1,
\end{aligned} \tag{1.6}$$

which are corollaries of Connes' exact sequence, imply that it suffices for our purposes to know $A(t)$, $\mathrm{HH}_2(A, A)(t)$ and $\mathrm{HH}_3(A, A)(t)$. We briefly review this material in Section 2.1.

The Poincaré series

$$U(YM)(t) = \frac{1}{1 - nt + nt^3 - t^4}, \quad \text{with } n = \dim(V), \tag{1.7}$$

was computed in [6]. Our task then is to determine $\mathrm{HH}_2(U(YM), U(YM))(t)$ and $\mathrm{HH}_3(U(YM), U(YM))(t)$. In view of (1.2) if we find $\mathrm{HH}^0(U(YM), U(YM))(t)$ and $\mathrm{HH}^1(U(YM), U(YM))(t)$, the formulas of Section 2.1 will enable us to prove Theorems 1.1 and 1.2. Thus from now on we concentrate on the groups

$$\mathrm{HH}_3(U(YM), U(YM)) \cong \mathrm{HH}^0(U(YM), U(YM)) = Z(U(YM)) \tag{1.8}$$

and

$$\mathrm{HH}_2(U(YM), U(YM)) \cong \mathrm{HH}^1(U(YM), U(YM)) = \mathrm{Out}(U(YM), \tag{1.9}$$

where $Z(U(YM))$ and $\mathrm{Out}(U(YM))$ denote the center and the Lie algebra of outer derivations of $U(YM)$, respectively.

The groups $\mathrm{HH}^i(U(YM), U(YM))$ are isomorphic to $H^i(YM, U(YM))$, the Lie algebra cohomology groups [5] of YM with coefficients in $U(YM)$, equipped with the adjoint action $\rho(a)b = [a, b]$.

A central fact about the Lie algebra $YM = \bigoplus_{i \geq 1} YM_i$ is that it contains a free Lie subalgebra (a Lie ideal)

$$TYM = \bigoplus_{i \geq 2} YM_i$$

(see [17] and also Section 4.1).

We evaluate $H^i(YM, U(YM))(t)$, $i = 0, 1$, in two steps.

(1) We determine $H^i(YM, U(TYM))$, $i = 0, 1$.

(2) Using our knowledge of the groups in (1) we calculate $Z(U(YM))$, $\mathrm{Out}(U(YM))$ and as corollary we find $\mathrm{HH}^i(U(YM), U(YM)))(t) = H^i(YM, U(YM)))(t)$, $i = 2, 3$.

For this we use a spectral sequence technique.

We denote by $\mathrm{Ker}\,\varepsilon \subset U(TYM)$ be the kernel of the canonical augmentation $\varepsilon\colon U(TYM) \to \mathbb{C}$.

The filtration

$$U(TYM) = F^0 \supset F^1 \supset \cdots \supset F^k \supset \tag{1.10}$$

of $U(TYM)$ is generated by powers of $\mathrm{Ker}\,\varepsilon$:

$$F^k = \mathrm{Ker}^{\times k}\,\varepsilon.$$

Let

$$M = TYM/[TYM, TYM].$$

The filtration $F^i(U(TYM))$ defines a filtration

$$F^i C^\bullet(YM, U(TYM)) = C^\bullet(YM, F^i(U(TYM)))$$

and a spectral sequence

$$E_1^{ij} = H^{i+j}(YM, M^{\otimes i}) \Rightarrow H^{i+j}(YM, U(TYM)). \tag{1.11}$$

We show the vanishing of $H^0(YM, M^{\otimes j})$, $j \geq 1$, in Section 4.3 and that of $H^1(YM, M^{\otimes j})$, $j \geq 2$, in Section 4.4.

We identify the cohomology of the higher differential of our spectral sequence in Sections 5.2, 5.4. From this we argue in Section 5.5 that $H^0(YM, U(TYM)) \cong \mathbb{C}$ and $H^1(YM, U(TYM)) \cong V + V$.

For N a module over a Lie algebra $\mathfrak{g}$ we denote by $\mathrm{Sym}(N)$ the symmetric (polynomial) algebra of N with multiplication $a \bullet b$.

In order to compute the cohomology $H^\bullet(YM, U(YM))$, which is isomorphic to $H^\bullet(YM, \mathrm{Sym}(YM))$, we introduce a filtration $F^i_{TYM} C^\bullet(YM, \mathrm{Sym}(YM))$. By definition

$$F^i_{TYM}(\mathrm{Sym}(YM)) = I^{\times i} \tag{1.12}$$
$$F^i_{TYM} C^\bullet(YM, \mathrm{Sym}(YM)) = C^\bullet(YM, I^{\times i}), \tag{1.13}$$

where I is the ideal generated by TYM (see Section 6.1). Again this leads to a spectral sequence

$$E_1^{ij} \cong H^{i+j}(YM, \mathrm{Sym}^i(TYM) \otimes \mathrm{Sym}^{k-i}(V)) \Rightarrow H^{i+j}(YM, \mathrm{Sym}^k(YM)). \tag{1.14}$$

We say that

$$f(t) = \sum_k a_k t^k \geq 0 \quad \text{iff} \quad a_k \geq 0 \quad \text{for all } k,$$

and also

$$f(t) \geq g(t) \quad \text{iff} \quad f(t) - g(t) \geq 0.$$

An estimate of the E_2-term enables us to assert that $Z(U(YM))(t)$ and $\mathrm{Out}(U(YM))(t)$ are not larger than the series stated in Theorem 1.1; the opposite inequality is obvious. From this we can completely determine the groups $Z(U(YM))$ and $\mathrm{Out}(U(YM))$. As a corollary we prove Theorem 1.1.

Let us rapidly review the contents of sections not explicitly mentioned above.

In Section 2 we collect known facts about the cohomology of the YM algebra and more generally about algebras with Poincaré duality in dimension 3. Most of these facts were discovered in [7] and [21].

The two basic spectral sequences are formally introduced in Section 3.

In Section 4 we prove vanishing results for the E_1-term of the first spectral sequence. The long exact sequence of Section 4.2 is our basic tool.

The methods of Section 5 complement the method of long exact sequences from Section 4.2. They constitute the core of the paper.

The technical Section 5.3 and the Appendices C and D deal with homologies of modules over the algebra of homogeneous polynomial functions on a nonsingular quadric. This material is needed only for the proof of Proposition 5.6 and can be skipped in a first reading.

Concerning the notations and terminology: the ground field is assumed to be the field of complex numbers $\mathbb{C}$. It is easy to check however that most of the proofs go through for any field of characteristic zero. In particular Theorems 1.1 and 1.2 remain valid when the ground field is $\mathbb{Q}$.

By a module we understand a left module. Let $K^\bullet$ be a cohomological complex; then $K_\bullet$ denotes the homological complex $K^i = K_{-i}$, $d^i = d_{-i}$. Fix an integer i. We let $(K^\bullet[l])^i$ be equal to K^{i+l}, with $d_{K[l]} = (-1)^l d_K$. Define $(K_\bullet[l])_i = K_{i-l}$ for the homological indexing (see [8]).

Acknowledgement. The author would like to thank IAS, IHES, MPI and SUNY at Stony Brook where this work has been done. He would also like to thank M. Kontsevich, N. Nekrasov, A. S. Schwarz, D. Sullivan for useful discussions, and M. Rocek for the opportunity to present this material at the "Simons Workshop in Mathematics and Physics 2005". The author is deeply indebted to Anthony Phillips, who read the manuscript and suggested numerous corrections and improvements.

2. Preliminaries

The material of this section is not new and is included in this article in a form convenient for us to have a quick reference. Some part of this material is already contained implicitly or explicitly in [7].

2.1. Algebras with Poincaré duality. The universal enveloping $U(YM)$ is an example of a graded associative algebra A with Poincaré duality in (co)homology of dimension 3 (see [7], [18]). We review general facts about the homology of such algebras. Missing definitions and constructions from this section can be found in the book [13].

The Connes periodicity long exact sequence

$$\to \mathrm{HC}_{i+1}(A) \to \mathrm{HC}_{i-1}(A) \to \mathrm{HH}_i(A, A) \to \mathrm{HC}_i(A) \to$$

becomes particularly simple for graded algebras. It is convenient to formulate it in terms of the reduced homology theories $\overline{\mathrm{HC}}_i(A) := \mathrm{HC}_i(A)/\mathrm{HC}_i(A_0)$ and $\overline{\mathrm{HH}}_i(A, A) := \mathrm{HH}_i(A, A)/\mathrm{HH}_i(A_0, A_0)$.

Theorem 2.1 ([13], Theorem 4.1.13). *If A is a positively graded unital algebra with $\mathbb{Q} \subset A_0$, then the Connes sequence for $\overline{\mathrm{HC}}_i(A)$ splits into exact sequences*

$$0 \to \overline{\mathrm{HC}}_{i-1,\boldsymbol{j}}(A) \to \overline{\mathrm{HH}}_{i,\boldsymbol{j}}(A,A) \to \overline{\mathrm{HC}}_{i,\boldsymbol{j}}(A) \to 0. \tag{2.1}$$

The equalities (1.6) are direct corollaries of this theorem.

The linear space of the algebra A can be considered as an object in two different tensor categories: one is the category $\mathsf{Vect}^{\mathbb{Z}}$ of $\mathbb{Z}$-graded vector spaces with nontrivial commutativity morphism R (A.3), and the other is a category similar to $\mathsf{Vect}^{\mathbb{Z}}$ where the morphism R has no $\pm$ signs. If the latter is true then we shall call A even.

The Poincaré series $A(t)$ fully determines the generating function $\chi\overline{\mathrm{HC}}(A)(t)$ of Euler characteristics of graded components of $\overline{\mathrm{HC}}_\bullet(A)$. This follows from the next theorem.

Theorem 2.2. *Let A be a positively graded associative even algebra. The generating function $\chi\overline{\mathrm{HC}}(A)(t)$ giving the Euler characteristics of reduced cyclic homology can be computed using the formula*

$$\chi\overline{\mathrm{HC}}(A)(t) = -\sum_{m\geq 1} \ln(A(t^m))\frac{\phi(m)}{m},$$

with the totient function ϕ defined in (1.3).

Corollary 2.3. *If A is a positively graded associative algebra of homological dimension k, then*

$$1 - \sum_{m\geq 1} \ln(A(t^m))\frac{\phi(m)}{m} = \sum_{i=0}^{k-1}(-1)^i(k-i)\,\mathrm{HH}_i(A,A)(t) \tag{2.2}$$

$$\sum_{i=0}^{k}(-1)^i\,\mathrm{HH}_i(A,A)(t) = 0 \tag{2.3}$$

$$\overline{\mathrm{HC}}_i(A) = 0, \quad i \geq n. \tag{2.4}$$

Proof. This follows from Theorems 2.1 and 2.2. □

In the following we will be using A^{op} for the algebra with the multiplication defined by the rule $a \otimes b \to ba$, where $a \otimes b \to ab$ is the multiplication in A.

Let us assume that the algebra A satisfies Poincaré duality in dimension 3, i.e., there is a resolution of the diagonal

$$A \leftarrow A \otimes A^{\mathrm{op}} \leftarrow A \otimes W_1 \otimes A^{\mathrm{op}} \leftarrow A \otimes W_2 \otimes A^{\mathrm{op}} \leftarrow A \otimes W_3 \otimes A^{\mathrm{op}} = P_\bullet$$

and an $A \otimes A^{\mathrm{op}}$-isomorphism (duality)

$$\mathrm{Hom}_{A\otimes A^{\mathrm{op}}}(P_\bullet, A \otimes A^{\mathrm{op}}) \to P^\bullet[3]. \tag{2.5}$$

Here the subscript $A \otimes A^{\mathrm{op}}$ means that we are considering only $A \otimes A^{\mathrm{op}}$-homomorphisms of the modules in question. Then $W_1 \cong W_2^*$ and $W_3 \cong \langle c\rangle$, where $\langle c\rangle$ is a one-dimensional space spanned by a generator c, dual to $1 \in A \otimes A^{\mathrm{op}}$. The isomorphism $\mathrm{HH}^i(A, A) \cong \mathrm{HH}_{3-i}(A, A)$ and the vanishing

$$\mathrm{HH}_i(A, A) = 0, \quad i \neq 0, 1, 2, 3$$

are direct corollaries of (2.5).

The definition of duality has a refinement in the case of graded algebras, provided that (2.5) is compatible with the grading. In particular $H_{3,\boldsymbol{j}}(A, \mathbb{C}) \neq 0$ only for only for one value of j. By definition the invariant $j(A)$ is equal to this value. Also

$$\mathrm{HH}^{i,\boldsymbol{j}}(A, A) = \mathrm{HH}_{3-i,\boldsymbol{j}(A)+\boldsymbol{j}}(A, A). \tag{2.6}$$

The Poincaré duality (2.6), the exact sequence (2.1) and the formulas (2.2), (2.3), (2.4) allow us to express $\mathrm{HH}^3(A, A)(t)$ and $\mathrm{HH}^2(A, A)(t)$ in terms of $Z(A)(t)$, $\mathrm{Out}(A)(t)$ and $A(t)$. The relevant formulas are listed in the following result.

Proposition 2.4.

$$\begin{aligned}
\mathrm{HH}^3(A, A)(t) &= t^{-j(A)}\Big(1 - \sum_{m\geq 1} \ln(A(t^m))\tfrac{\phi(m)}{m}\Big) + \mathrm{Out}(t) - 2Z(A)(t),\\
\mathrm{HH}^2(A, A)(t) &= t^{-j(A)}\Big(- \sum_{m\geq 1} \ln(A(t^m))\tfrac{\phi(m)}{m}\Big) + 2\,\mathrm{Out}(t) - 3Z(A)(t),\\
\overline{\mathrm{HC}}_2(A)(t) &= Z(A)(t)t^{-j(A)},\\
\overline{\mathrm{HC}}_1(A)(t) &= (\mathrm{Out}(t) - Z(A)(t))t^{-j(A)}.
\end{aligned}$$

This proposition explains why we pay so much attention in this paper to the low-dimensional cohomology groups $Z(U(YM))$ and $\mathrm{Out}(U(YM))$.

2.2. Cohomology of *YM*: general facts. We first briefly recall the pertinent definitions from [5]. Let $\mathfrak{g}$ be a Lie algebra over the field $\mathbb{C}$. If N is a $\mathfrak{g}$-module, then we use the following notation for the action $(l, n) \to \rho_N(l)n$, $l \in \mathfrak{g}$, $n \in N$.

Definition 2.5. Let $C^\bullet(\mathfrak{g}, N)$ be the standard cochain complex of $\mathfrak{g}$, N:

$$C^k(\mathfrak{g}, N) = \mathrm{Hom}(\Lambda^k(\mathfrak{g}), N).$$

The differential $d\colon C^k(\mathfrak{g}, N) \to C^{k+1}(\mathfrak{g}, N)$ is given by

$$(dc)(l_{,} \dots, l_{k+1}) = \sum_{i=1}^{k+1} (-1)^i \rho_N(l_i) c(l_1, \dots, \hat{l}_i, \dots, l_{k+1}) \\ + \sum_{i<j} (-1)^{i+j-1} c([l_i, l_j], \dots, \hat{l}_i, \dots, \hat{l}_j, \dots, l_{k+1}).$$

By definition $H^\bullet(\mathfrak{g}, N)$ is the cohomology of $C^\bullet(\mathfrak{g}, N)$.

The chain complex $C_\bullet(\mathfrak{g}, N)$ is equal to $\sum_k \Lambda^k(\mathfrak{g}) \otimes N$. It is equipped with the differential

$$d(m \otimes l_1 \wedge \dots \wedge l_k) = \sum_{i=1}^{k+1} (-1)^{i+1} \rho_N(l_i) m \otimes l_1 \wedge \dots \wedge \hat{l}_i \wedge \dots \wedge l_k \\ + \sum_{i<j} (-1)^{i+j-1} m \otimes [l_i, l_j] \wedge \dots \wedge \hat{l}_i \wedge \dots \wedge \hat{l}_j \wedge \dots \wedge l_k$$

and $H_\bullet(\mathfrak{g}, N) := H_\bullet(C_\bullet(\mathfrak{g}, N))$.

Definition 2.6. Let $N = \bigoplus_i N_i$ be a graded module. Then $N(k)$ is a module with the shifted grading $[N(k)]_i = N_{k+i}$.

If the algebra $\mathfrak{g}$ and the module N are graded, then the complexes are also graded: $C^\bullet(\mathfrak{g}, N) = \bigoplus_{\mathbf{j}} C^{\bullet,\mathbf{j}}(\mathfrak{g}, N)$ and $C_\bullet(\mathfrak{g}, N) = \bigoplus_{\mathbf{j}} C_{\bullet,\mathbf{j}}(\mathfrak{g}, N)$. As it was mentioned earlier we are using bold Roman letters for the grading indices.

Let N be a bimodule over $U(\mathfrak{g})$. Define a new left module structure on N by the formula

$$\rho_{N^{\mathrm{ad}}}(l)n = ln - nl.$$

Theorem 2.7 ([4]). *There are canonical isomorphisms*

$$\mathrm{HH}^\bullet(U(\mathfrak{g}), N) \cong H^\bullet(\mathfrak{g}, N^{\mathrm{ad}}),$$
$$\mathrm{HH}_\bullet(U(\mathfrak{g}), N) \cong H_\bullet(\mathfrak{g}, N^{\mathrm{ad}}).$$

To simplify notations we shall work with an orthonormal basis $v_1, \dots, v_n$ of V. Then the relations of *YM* become

$$\sum_{s=1}^{n} [v_s, [v_s, v_t]] = r_t = 0, \quad t = 1, \dots, n. \tag{2.7}$$

If the Lie algebra $\mathfrak{g}$ is isomorphic to *YM* then a complex $C^\bullet(N)$, which is much smaller than $C^\bullet(YM, N)$, can be used for the computation of $H^\bullet(YM, N)$. In the case of a graded module N, this $C^\bullet(N)$ decomposes into a direct sum $C^\bullet(N) = \bigoplus_j C^{\bullet,j}(N)$ with

$$C^{\bullet,j}(N) = 0 \to N_j \xrightarrow{d^0} N_{j+1} \otimes V^* \xrightarrow{d^1} N_{j+3} \otimes V \xrightarrow{d^2} N_{j+4} \to 0. \tag{2.8}$$

The differentials in $C^{\bullet,j}(N)$ are defined by the formulas

$$\begin{aligned} d^0 w &= \sum_{1\le s\le n} \rho_N(v_s) w \otimes v^{*s}, \\ d^1 w \otimes v^{*i} &= \sum_{1\le s\le n} (\rho_N(v_s^2) w \otimes v_i - 2\rho_N(v_i v_s) w \otimes v_s + \rho_N(v_s v_i) w \otimes v_s), \\ d^2 w \otimes v_i &= \rho_N(v_i) w. \end{aligned}$$

The elements v^{*s} and v_s ($1 \le s \le n$) are elements of orthogonal bases of V^* and V, respectively.

Proposition 2.8. *There are isomorphisms*

$$\begin{aligned} H^{i,j}(YM, N) &\cong H^{i,j}(C(N)), \\ H_{3-i,j}(YM, N) &\cong H^{i,j-4}(C(N)). \end{aligned}$$

In particular $j(U(YM)) = 4$ (*the invariant* j *was defined in* (2.6)).

Proof. See [7]. □

The formula (1.2) is a direct consequence of this proposition.

For the sake of completeness we explain briefly how to construct a resolution of the diagonal $P_\bullet(U(YM))$. The reader can easily reconstruct the details.

The tensor product $U(YM) \otimes U(YM)$ is equipped with the left YM-action $\rho(v_s)(a \otimes b) = -av_s \otimes b + a \otimes v_s b$. Let the YM-action on $U(YM)$ be defined by left multiplication.

Note that in this case the multiplication map $C^\bullet(U(YM)\otimes U(YM)) \to U(YM)[-3]$ is a quasi-isomorphism. We define $P_\bullet(U(YM))$ to be equal to $C_\bullet(U(YM)\otimes U(YM))[3]$. The Poincaré duality isomorphism (2.5) can be easily verified.

3. The spectral sequences

Our aim in this section is to provide the missing details about the spectral sequences that were briefly defined in the introduction.

Let $F^i(X)$ be a decreasing filtration of a linear space X. We will be using the standard notation $\mathrm{Gr}^i_F(X)$ for the adjoint quotient $F^i(X)/F^{i+1}(X)$. To shorten the notations, when it does not lead to confusion, we will write $\mathrm{Gr}^i(X)$ or Gr^i for $\mathrm{Gr}^i_F(X)$.

The adjoint quotients of the filtration (1.13) in $C^\bullet(YM, \mathrm{Sym}^k(YM))$ are equal to

$$\begin{aligned} F^i_{TYM} \cap C^\bullet(YM, \mathrm{Sym}^k(YM)/F^{i+1}_{TYM} \cap C^\bullet(YM, \mathrm{Sym}^k(YM) \\ = C^\bullet(YM, \mathrm{Sym}^i(TYM) \otimes \mathrm{Sym}^{k-i}(V)). \end{aligned} \tag{3.1}$$

Proposition 3.1. *The filtration* (1.13) *enables us to define a spectral sequence* (1.14) *The action of YM on the tensor factor* $\mathrm{Sym}^{k-i}(V)$ *in the module of coefficients* (3.1) *is trivial.*

Proof. Even though the spectral sequence is located in the quadrants I and IV, there are no problems with convergence because the depth of the filtration in $\mathrm{Sym}^k(YM)$ is finite.

The construction of the spectral sequence is standard and can be found in [8]. □

The filtration $F^i = F^i(U(TYM))$ was introduced in (1.10).

Proposition 3.2. *There is a spectral sequence associated with the filtration* F^i*:*

$$E_1^{i,j} = H^{i+j}(YM, \mathrm{Gr}^i(U(TYM))) \Rightarrow H^{i+j}(YM, U(TYM)). \tag{3.2}$$

There is also a spectral sequence in homology:

$$E^1_{i,j} = H_{i+j}(YM, \mathrm{Gr}^{-i}(U(TYM))) \Rightarrow H_{i+j}(YM, U(TYM)). \tag{3.3}$$

Poincaré duality defines an isomorphism of the spectral sequences $E_r^{ij} \cong E^r_{-i,3-j}$.

Proof. The cohomological spectral sequence is located in the quadrants I and IV. Its homological counterpart is in the quadrants II and III. This might lead to convergence problems. We shall address this issue presently.

In view of the isomorphism

$$F^l\Big(\bigoplus_i U(YM)_i\Big) \cong \bigoplus_i F^l(U(YM)_i)$$

the filtration of $C^\bullet(U(TYM))$ is the (direct) sum of the filtrations of $C^{\bullet,j}(U(TYM))$. The complexes $C^{\bullet,j}(U(TYM))$ are formed by finite-dimensional spaces. Spectral sequences associated with finite filtrations always converge.

Other than that the proof is straightforward. □

4. Computation of $H^\bullet(YM, \mathrm{Gr}^j(U(TYM))$

Our main task in this section is to describe the E_1-term of the spectral sequence (3.2).

4.1. The structure of the Lie algebra *TYM*. This section contains preliminaries to Section 4.2. To get a good grip on the cohomology $H^\bullet(YM, \mathrm{Gr}^j(U(TYM))$ we need an understanding of the YM-action on $\mathrm{Gr}^j(U(TYM))$. The following theorem is the first step in this direction.

Theorem–Definition 4.1 (See also [18]). *The Lie algebra TYM is free. The space* $M = TYM/[TYM, TYM]$ *is a module over* YM_{ab} (*the subscript 'ab' stands for abelianization*).

The proof is given in Appendix B.

From the canonical isomorphism $YM_{\mathrm{ab}} \cong V$ any $\mathrm{Sym}(V)$-module is a $U(YM)$-module. Conversely if the action of TYM on a YM-module N is trivial then N can be considered as a $\mathrm{Sym}(V)$-module.

In view of Theorem 4.1 the following corollary becomes evident.

Corollary 4.2. *As a* $\mathrm{Sym}(V)$*-module,* $\mathrm{Gr}^i(U(TYM))$ *coincides with* $M^{\otimes i}$ *and the* E_1*-terms of the spectral sequences* (3.2)*,* (3.3) *are equal to* $H^{i+j}(YM, M^{\otimes i})$ *and* $H_{i+j}(YM, M^{\otimes(-i)})$*, respectively.*

We let x_s $(s = 1, \ldots, n)$ denote the image of the generator $v_s \in YM$ under the canonical map $U(YM) \to \mathrm{Sym}(V)$.

The next proposition is the key ingredient in the construction of the exact sequence (4.4).

Proposition 4.3. *The following isomorphisms hold*

$$\begin{aligned} H^0(C^\bullet(\mathrm{Sym}(V))) = H^1(C^\bullet(\mathrm{Sym}(V))) = 0,\\ H^{3,-\mathbf{4}}(C^\bullet(\mathrm{Sym}(V))) = \mathbb{C}. \end{aligned}$$

Define

$$M_j = H^{2,\boldsymbol{j}-\mathbf{4}}(C^\bullet(\mathrm{Sym}(V))).$$

The linear space $M = \bigoplus_j M_j$ *is a* $\mathrm{Sym}(V)$*-module. There is also an action of the abelian Lie algebra* YM_{ab} *on* M *that comes from the identification of* M *with* $TYM/[TYM, TYM]$*. These actions coincide.*

See Appendix B for the proof.

The next set of definitions will be used to identify $U(TYM)$ with a tensor algebra.

We start with a general construction. As before, we let $\mathfrak{g}$ be a Lie algebra. Let $r\colon H_1(\mathfrak{g}, \mathbb{C}) = \mathfrak{g}_{\mathrm{ab}} \to \mathfrak{g}$ be a splitting of the canonical homomorphism $\mathfrak{g} \to \mathfrak{g}_{\mathrm{ab}}$ of Lie algebras. If $\mathfrak{g}$ is positively graded then $\operatorname{Im} r$ defines a minimal set of algebraic generators of $\mathfrak{g}$. The map r can be lifted to a surjective map $\mathrm{Free}(\mathfrak{g}_{\mathrm{ab}}) \to \mathfrak{g}$.

In particular the surjective mapping

$$r: T(M) \to U(TYM) \tag{4.1}$$

is an isomorphism. This is a corollary of Proposition 4.3.

We finish this section with the construction of a certain exact triangle in the derived category of $\mathrm{Sym}(V)$-modules.

Namely there is a triangle

$$M \xrightarrow{u} C^\bullet(\mathrm{Sym}(V))[2](-4) \xrightarrow{v} \mathbb{C}[-1] \to M[1] \tag{4.2}$$

in the (bounded) derived category of graded $\mathrm{Sym}(V)$-modules.

The morphism u is represented by a "roof" (s, f) in the localized category; it has the following components. The map

$$f: \tau^{\leq 0} C^\bullet(\mathrm{Sym}(V))[2](-4) \to C^\bullet(\mathrm{Sym}(V))[2](-4)$$

is the inclusion (the operation $\tau^{\leq 0}$ is the truncation functor). The map

$$s: \tau^{\leq 0} C^\bullet(\mathrm{Sym}(V))[2](-4) \to M$$

is the projection to the zero cohomology of $\tau^{\leq 0} C^\bullet(\mathrm{Sym}(V))[2](-4)$.

The map v is the projection to the first cohomology $C^\bullet(\mathrm{Sym}(V))[2](-4)$.

4.2. A long exact sequence for $H_\bullet(YM, M^{\otimes j})$. From Corollary 4.2 we know that the E_1-term of the spectral sequence (3.3) coincides with the homology group $H_{i+j}(YM, M^{\otimes -i})$. The linear spaces $M^{\otimes j}$ are $\mathrm{Sym}(V)$- and $U(YM)$-modules at the same time. To understand these homology groups we introduce a long exact sequence that involves $H_\bullet(YM, M^{\otimes j})$, $H_\bullet(V, M^{\otimes j})$ and $H_\bullet(V, M^{\otimes (j+1)})$.

The following terminology is standard.

Definition 4.4. Let N denote a $\mathrm{Sym}(V)$-module. Let us interpret the linear space V as an abelian Lie algebra $\mathfrak{g}$. The complex $C_\bullet(V, N) := C_\bullet(\mathfrak{g}, N)$ is called the Koszul complex of N.

Remark 4.5. The algebra $\mathrm{Sym}(V)$ is a Hopf algebra with the diagonal

$$\Delta\colon\ \mathrm{Sym}(V) \to \mathrm{Sym}(V) \otimes \mathrm{Sym}(V); \quad \Delta(x_s) = x_s \otimes 1 + 1 \otimes x_s. \tag{4.3}$$

We use it to define a tensor product (over $\mathbb{C}$) of representations: $N_1 \otimes N_2$.

The map $\Delta^2\colon \mathrm{Sym}(V) \to \mathrm{Sym}(V)^{\otimes 3}$ is the composition $(\Delta \otimes \mathrm{id}) \circ \Delta = (\mathrm{id} \otimes \Delta) \circ \Delta$. There are similarly defined maps $\Delta^{j-1}\colon \mathrm{Sym}(V) \to \mathrm{Sym}(V)^{\otimes j}$. They do not depend on the way we present them as a composition of $\mathrm{id}^{\otimes k} \otimes \Delta \otimes \mathrm{id}^{\otimes l}$.

We introduce the following notation. Suppose that N is a graded $\mathrm{Sym}(V)$-module. We denote bi-degree $(1,1)$ generators of $\Lambda(V)$ in the Koszul complex $C_\bullet(V,N)$ by $\varsigma_1,\dots,\varsigma_n, n=\dim(V)$. Also

$$C_i(V,N)=\bigoplus_{\boldsymbol{j}} C_{i,\boldsymbol{k}}(V,N)=\bigoplus_{\boldsymbol{k}} N_{k-i}\otimes\Lambda^i[V]$$

is the decomposition of $C_\bullet(V,N)$ into graded components.

Proposition 4.6. *For $j\geq 0$ there is an exact sequence*

$$\begin{aligned}\cdots &\xrightarrow{B^{i-1}_{M^{\otimes(j+1)},\boldsymbol{k}}} H_{i,\boldsymbol{k}}(YM,M^{\otimes j})\xrightarrow{I^i_{M^{\otimes j},\boldsymbol{k}}} H_{i,\boldsymbol{k}}(V,M^{\otimes j})\\ &\xrightarrow{S^i_{M^{\otimes j},\boldsymbol{k}}} H_{i-2,\boldsymbol{k}}(V,M^{\otimes(j+1)})\xrightarrow{B^{i-2}_{M^{\otimes(j+1)},\boldsymbol{k}}} H_{i-1,\boldsymbol{k}}(YM,M^{\otimes j})\longrightarrow\cdots.\end{aligned}\tag{4.4}$$

To avoid clutter in notations we will abbreviate $B^i_{M^{\otimes j},\boldsymbol{k}}$, $I^i_{M^{\otimes j},\boldsymbol{k}}$ and $S^i_{M^{\otimes j},\boldsymbol{k}}$ to $B^i_{j,\boldsymbol{k}}$, $I^i_{j,\boldsymbol{k}}$ and $S^i_{j,\boldsymbol{k}}$, respectively. Furthermore we might drop some of the remaining unused indices.

Proof. Apply the functor $A\Rightarrow C_\bullet(V,A\otimes M^{\otimes j})$ to the triangle (4.2).

Since $C^\bullet(\mathrm{Sym}(V))$ consists of free $\mathrm{Sym}(V)$-modules, if follows that the bicomplex $C_\bullet(V,C^\bullet(\mathrm{Sym}(V))\otimes M^{\otimes j})$ is quasi-isomorphic to $C^\bullet(M^{\otimes j})$. □

Proposition 4.7 is a related statement.

Proposition 4.7. *Let N be a YM-module. There is a long exact sequence in homology*

$$\begin{aligned}\cdots\to H_i(YM,N)&\xrightarrow{I^i_N} H_i(V,H_0(TYM,N))\\ &\to H_{i-2}(V,H_1(TYM,N))\to H_{i-1}(YM,N)\to\cdots,\end{aligned}\tag{4.5}$$

where the map I^i_N is induced by abelianization.

There is a graded version of the exact sequence (4.5).

The exact sequence (4.5) *reproduces* (4.4) *in the case that $N\cong M^{\otimes j}$.*

Proof. The E^2-term of the homological Hochschild–Serre spectral sequence [9] associated with a pair $TYM\subset YM$ has many trivial entries. Indeed, E^2_{ij} is equal to $H_i(YM/TYM,H_j(TYM,N))=H_i(V,H_j(TYM,N))$. As we know, TYM is free, thus $H_j(TYM,N)=0$ for $j\geq 2$. In fact, the Hochschild–Serre spectral sequence reduces to the long exact sequence (4.5). This proves the first statement.

The statement about the map I^i_N is a direct consequence of the construction.

For the proof of the last statement we assume that the action of YM on N factors through $V = YM_{\mathrm{ab}}$. Let us calculate $H_\bullet(YM, N)$ using the language of resolutions. We start with $C_\bullet(U(YM))[3]$, a free $U(YM)$resolution of $\mathbb{C}$.

Let Mod_A denote the category of left modules over an algebra A. Consider the spectral sequence of composition (see [8]) of two derived functors of the following right exact functors. The first one is $K \Rightarrow H_0(TYM, K \otimes N)$, which acts from $\mathsf{Mod}_{U(YM)}$ to $\mathsf{Mod}_{\mathrm{Sym}(V)}$. The second one is $L \Rightarrow H_0(V, L)$; it acts from $\mathsf{Mod}_{\mathrm{Sym}(V)}$ to the category of vector spaces Vect.

Notice that the complexes $H_0(TYM, C_\bullet(U(YM)[3] \otimes N) = C_\bullet(N)[3]$ and $C_\bullet(U(YM))[3]$ are adjusted (see [8] for the definition) to the first functor.

The E^r-term ($r \geq 1$) of the spectral sequence of the composition of our two functors is equal to the E^r-term of the spectral sequence of the bicomplex $C_\bullet(V, C_\bullet(\mathrm{Sym}(V)) \otimes N)$.

The long exact sequence (4.5) for $N \cong M^{\otimes j}$ is canonically isomorphic to (4.4) because of the equivalence of the classical and the derived approaches to the Hochschild–Serre sequence.

The graded version of the statement can be derived straightforwardly. □

Here are our first vanishing results.

Lemma 4.8. *The map*

$$I^3_{\mathbb{C},\boldsymbol{k}} : H_{3,\boldsymbol{k}}(YM, \mathbb{C}) \to H_{3,\boldsymbol{k}}(V, \mathbb{C})$$

is trivial.

If $\dim(V) = 2$ *then* $H_{3,\boldsymbol{k}}(YM, M^{\otimes j}) \cong \mathbb{C}$ *for* $\boldsymbol{k} = 2j + 4$, $j \geq 0$; *otherwise* $H_{3,\boldsymbol{k}}(YM, M^{\otimes j}) = 0$.

If $\dim(V) \geq 3$ *then* $H_3(YM, M^{\otimes j}) = 0$ *for* $j \geq 1$.

Proof. Representatives of cycles generating $H_1(YM, \mathbb{C})$ and $H_2(YM, \mathbb{C})$ are $v_s \in C_1(YM, \mathbb{C})$ and $\sum_{s=1}^{n} v_s \wedge [v_s, v_t] \in C_2(YM, \mathbb{C})$.

From this we find that a representative of the class $[c] \in H_3(YM, \mathbb{C})$ is

$$c = \sum_{1 \leq s < t \leq n} v_s \wedge v_t \wedge [v_s, v_t]. \tag{4.6}$$

The map $I = I^3_{\mathbb{C},\boldsymbol{k}}$ is induced by abelianization (see Proposition 4.7). The class $I([c])$ is zero because of the commutators in the formula (4.6).

The proof in the case of $\dim(V) = 2$ is straightforward and is omitted.

An identification of the formulas for differentials in complexes $C_\bullet(N)$ and $C_\bullet(V, N)$ shows that $H_3(YM, N) = H_n(V, N)$ for any $\mathrm{Sym}(V)$-module N. The result follows from the long exact sequence (4.4), the vanishing of $H_i(V, M^{\otimes j})$ for $i < 0$,

$\dim(V) < i$ and the vanishing of $H_k(YM, M^{\otimes j})$ for $k > 3$ and $k < 0$ (Proposition 2.8). $\square$

As was already mentioned in Section 2.2, the algebra $U(YM)$ is homologically three-dimensional. This lets us push our analysis of (4.4) a bit further.

Proposition 4.9. *For $j \geq 0$ there is an exact sequence*

$$\begin{aligned} 0 \to H_{3,\boldsymbol{k}}(V, M^{\otimes j}) &\xrightarrow{S^3_{j,\boldsymbol{k}}} H_{1,\boldsymbol{k}}(V, M^{\otimes(j+1)}) \xrightarrow{B^1_{j+1,\boldsymbol{k}}} H_{2,\boldsymbol{k}}(YM, M^{\otimes j}) \\ &\xrightarrow{I^2_{j,\boldsymbol{k}}} H_{2,\boldsymbol{k}}(V, M^{\otimes j}) \xrightarrow{S^2_{j,\boldsymbol{k}}} H_{0,\boldsymbol{k}}(V, M^{\otimes(j+1)}) \\ &\xrightarrow{B^0_{j+1,\boldsymbol{k}}} H_{1,\boldsymbol{k}}(YM, M^{\otimes j}) \xrightarrow{I^1_{j,\boldsymbol{k}}} H_{1,\boldsymbol{k}}(V, M^{\otimes j}) \to 0. \end{aligned} \tag{4.7}$$

There are also isomorphisms

$$\begin{aligned} H_{0,\boldsymbol{k}}(YM, M^{\otimes j}) &= H_{0,\boldsymbol{k}}(V, M^{\otimes j}), \\ H_{i+2,\boldsymbol{k}}(V, M^{\otimes j}) &= H_{i,\boldsymbol{k}}(V, M^{\otimes(j+1)}), \quad i \geq 2. \end{aligned}$$

If $\dim(V) = 2$ *then*

$$H_{s,\boldsymbol{k}}(V, M^{\otimes j}) \cong \begin{cases} \Lambda^{2j+s}[V] & \textit{if } k = 2j+s,\ s = 2, \\ 0 & \textit{if } s > 2 \textit{ or } s = 2,\ k \neq 2j+s. \end{cases}$$

If $\dim(V) \geq 3$ *then*

$$H_{s,\boldsymbol{k}}(V, M^{\otimes j}) \cong \begin{cases} \Lambda^{2j+s}[V] & \textit{if } k = 2j+s,\ s \geq 2,\ \textit{except } s = 2,\ j = 1, \\ 0 & \textit{if } k \neq 2j+s. \end{cases}$$

There is a $\mathrm{Sym}(V)$*-linear map*

$$\nabla^i_j \colon H_i(YM, M^{\otimes j}) \to H_{i+1}(YM, M^{\otimes j-1}), \quad j \geq 1,$$

which is equal to the composition $B^i_j \circ I^i_j$. Then $\nabla^{i+1}_{j-1} \circ \nabla^i_j = 0$ (we omit the index $\boldsymbol{k}$ to make the formulas more readable).

Proof. The proposition follows directly from Proposition 4.6 and Lemma 4.8. $\square$

In our cohomological computations we will need a presentation of the $\mathrm{Sym}(M)$-module M using generators and relations. Such presentation is given in the next two statements.

Lemma 4.10. *The module M is a subquotient of $C^2(\mathrm{Sym}(V))$. It is generated by elements*

$$\mathcal{F}_{ij} = x_i \otimes v_j - x_j \otimes v_i$$

of degree -2 ($+2$ in the homological complex). The generators satisfy

$$G_{ijk} = x_i \mathcal{F}_{jk} + x_k \mathcal{F}_{ij} + x_j \mathcal{F}_{ki} = 0 \tag{4.8}$$

and

$$\sum_{s=1}^{n} x_s \mathcal{F}_{sj} = 0.$$

The last formula is a consequence of the defining relations (2.7)*.*

Proof. The differential d^3 in $C^\bullet(\mathrm{Sym}(V))$ coincides with the differential d_0 in the Koszul complex $C_\bullet(V, \mathrm{Sym}(V))$. The module $\mathrm{Ker}\, d_0$ is generated by $\mathcal{F}_{ij}$; this follows from the acyclicity of $C_\bullet(V, \mathrm{Sym}(V))$. The relations (4.8) follow from explicit computations with the differential d^2 in $C^\bullet(\mathrm{Sym}(V))$. □

To establish the structure of M we need to show that the set of relations found in Lemma 4.8 is complete.

Proposition 4.11. *There are exact sequences and an isomorphism*

$$0 \to \Lambda^4 V \to H_{2,\mathbf{4}}(V, M) \to \mathbb{C} \to 0, \tag{4.9}$$

$$0 \to \Lambda^3 V \to H_{1,\mathbf{3}}(V, M) \to V \to 0, \tag{4.10}$$

$$\Lambda^2 V \cong H_{0,\mathbf{2}}(V, M) = H_0(V, M).$$

In particular all the relations in the module M follow from the relations described in Lemma 4.10*.*

Proof. Consider the exact sequence (4.7) for $j = 0$. Clearly the map $I^1_{\mathbb{C}}$ is an isomorphism. So the map $S^2_{\mathbb{C}}$ is surjective. The elements $\mathcal{F}_{ij}$, $1 \le i < j \le n$, define a minimal set of generators of the module M. This is because the equation $\sum_{1 \le i < j \le n} c_{ij} \mathcal{F}_{ij} = da$ has no solutions (the linear space $C^{1,\mathbf{2}}(\mathrm{Sym}(V))$ is zero). This implies that $I^2_{M \otimes j} = 0$.

From this we conclude that the segment of (4.7) containing $S^3_{M \otimes j}$, $B^1_{M \otimes (j+1)}$ and corresponding to (4.10) in fact coincides with (4.10).

Using Lemma 4.8 we find that the segment of (4.4) containing $S^4_{\mathbb{C}}$ and B^2_M coincides with (4.9). □

The elements $v_{(i_1} \otimes \cdots \otimes v_{i_k)}$ and $v_{[i_1} \otimes \cdots \otimes v_{i_k]}$ of $V^{\otimes k}$ are the symmetrization (resp. antisymmetrization) of $v_{i_1} \otimes \cdots \otimes v_{i_k}$.

The cocycles in $H_{2,\mathbf{4}}(V,M) = \Lambda^4(V) + \mathbb{C}$ are spanned by $\mathcal{F}_{[ij} \otimes \varsigma_k \wedge \varsigma_{l]}$ and $\sum_{1\leq i<j\leq n} \mathcal{F}_{ij} \otimes \varsigma_i \wedge \varsigma_j$. The S^2_M images of these cocycles in $H_{0,\mathbf{4}}(V, M^{\otimes 2})$ are $\mathcal{F}_{[ij} \otimes \mathcal{F}_{kl]}$ and $\sum_{1\leq i<j\leq n} \mathcal{F}_{ij} \otimes \mathcal{F}_{ij}$, which are clearly linearly independent elements of $H_{0,\mathbf{4}}(V, M^{\otimes 2})$. Thus S^2_M and $S^2_{M,\boldsymbol{k}}$ have no kernel. The following proposition generalises this observation.

Proposition 4.12. *The map $S^{i+2}_{M,\boldsymbol{k}}\colon H_{i+2,\boldsymbol{k}}(V,M) \to H_{i,\boldsymbol{k}}(V, M^{\otimes 2})$ is an embedding.*

Proof. For the remaining case of $i \geq 1$ this statement follows from Proposition 4.9. □

The previous proposition and its corollary, which we are about to formulate, will turn out useful in Sections 5.1 and 5.2.

Corollary 4.13. *The map $B^1_{M^{\otimes 2},\boldsymbol{k}}\colon H_{1,\boldsymbol{k}}(V, M^{\otimes 2}) \to H_{2,\boldsymbol{k}}(YM, M)$ is a surjection.*

Proof. Combine Propositions 4.12 and 4.9. □

4.3. Properties of the module M. The module M has a description in terms of the differential algebraic 1-forms on a quadric. This will complement its original description (Theorem 4.1). In this section we will also give a representation-theoretic characterisation of the graded components of M.

Let us start with a definition.

Definition 4.14. Let C be a commutative algebra and let $I \subset C \otimes C$ be the kernel of the multiplication map $C \otimes C \to C$. The C-module of Kähler differentials Ω_C is equal to I/I^2. There is a universal derivation $d\colon C \to \Omega_C$ defined by $d(a) = a \otimes 1 - 1 \otimes a$. Any derivation ∂ with values in a module N factors through d, i.e., there is a C-homomorphism $m_\partial\colon \Omega_C \to N$ such that $\partial = m_\partial \circ d$.

Let us introduce an algebra

$$A = \mathrm{Sym}(V)/(q), \tag{4.11}$$
$$q = \sum_{s=1}^{n} x_s^2. \tag{4.12}$$

We can define a derivation eu of A by the formula

$$\sum_{s=1}^{n} x_s \frac{\partial}{\partial x_s}.$$

The description of M briefly mentioned above is formally stated in the next lemma.

Lemma 4.15. *The module M coincides with the kernel of $\Omega_A \xrightarrow{m_{\mathrm{eu}}} A$.*

Proof. The module $\Omega(V)$ is equal to $\Omega_{\mathrm{Sym}(V)}$ as a free $\mathrm{Sym}(V)$-module generated by dx_s. Let d denote the de Rham differential $\mathrm{Sym}(V) \to \Omega(V)$. The module Ω_A is equal to $\Omega^1(V)/q\Omega^1(V) + Adq$.

The homomorphism $M \to \Omega_A$, $\mathcal{F}_{ij} \to x_i dx_j - x_j dx_i$, is unambiguously defined. For the proof the reader should use the description of M as the second cohomology group of $C^\bullet(\mathrm{Sym}(V))$. $\square$

Corollary 4.16. *Define a complex*

$$0 \to A(-2) \xrightarrow{d^0} A \otimes V(-1) \xrightarrow{d^1} A \to 0 = \widetilde{C}^\bullet. \tag{4.13}$$

Let dx_s, $s = 1, \dots, n$, denote a basis of V. For the differentials we then obtain $d^0(a) = \sum_{s=1}^n a x_s dx_s$, $d^1(a_s dx_s) = a_s x_s$. Then $H^0(\widetilde{C}) = 0$, $H^1(\widetilde{C}) = M$ and $H^2(\widetilde{C}) = \mathbb{C}$.

Proof. The isomorphism $\Omega_A \cong \Omega^1(V)/q\Omega^1(V) + Adq$ is equivalent to a short exact sequence $0 \to A(-2) \to A \otimes V(-1) \to \Omega_A \to 0$. After this observation the proof is a straightforward consequence of Lemma 4.15. $\square$

Let U be a multiplicatively closed subset of C. Denote by $N[U^{-1}]$ a localisation of N with respect to U, and for $g \in C$ denote by $\mathrm{Ann}_N(g)$ the kernel of the map $N \xrightarrow{g\times} N$.

Choose a basis in the space V, $\dim(V) \geq 3$ such that the tensor q has the form $q^{ij} x_i x_j = \tilde{x}_1 \tilde{x}_2 + \tilde{x}_3^2 + \dots + \tilde{x}_n^2$.

Proposition 4.17. *Suppose that* $\dim(V) \geq 3$. *Let $(\tilde{x}_1)$ denote the multiplicative system generated by $\tilde{x}_1$.*

The maps $A \to A[(\tilde{x}_1)^{-1}]$ and $M \to M[(\tilde{x}_1)^{-1}]$ have no kernel.

The module $M[(\tilde{x}_1)^{-1}]$ is free over $A[(\tilde{x}_1)^{-1}] = \mathbb{C}[\tilde{x}_1, \tilde{x}_1^{-1}] \otimes \mathbb{C}[\tilde{x}_3, \dots, \tilde{x}_n]$ of rank $\dim(V) - 2$.

Proof. If the groups $\mathrm{Ann}_A(\tilde{x}_1)$, $\mathrm{Ann}_M(\tilde{x}_1)$ are trivial then the first statement of the proposition is true.

The group $\mathrm{Ann}_A(\tilde{x}_1)$ is trivial because q is irreducible.

The short exact sequence

$$0 \to \mathrm{Sym}(V) \xrightarrow{\tilde{x}_1 \times} \mathrm{Sym}(V) \to \mathrm{Sym}(V)/\tilde{x}_1 \mathrm{Sym}(V) \to 0$$

induces a long exact sequence of *YM*-cohomology. Then $\mathrm{Ann}_M(\tilde{x}_1)$ coincides with $H^1(YM, \mathrm{Sym}(V)/\tilde{x}_1 \mathrm{Sym}(V)) = H(\mathrm{Sym}(V)/\tilde{x}_1 \mathrm{Sym}(V))$ (the reader should consult Appendix B for the definition and properties of the functor H).

It is obvious that $\mathrm{Ann}_{\mathrm{Sym}(V)/\tilde{x}_1 \mathrm{Sym}(V)}(q) = 0$ if $\dim(V) \geq 3$. The groups $H_i(V, \mathrm{Sym}(V)/\tilde{x}_1 \mathrm{Sym}(V))$ are nontrivial only for $i = 0, 1$ (we omit their computation). Thus the conditions of Lemma B.5 hold and $H(\mathrm{Sym}(V)/\tilde{x}_1 \mathrm{Sym}(V)) = 0$.

The last statement can be proved in a straightforward manner. □

Corollary 4.18. *If* $\dim(V) \geq 3$ *then* $H^0(YM, M^{\otimes j}) = 0$ *for* $j \geq 1$.

We finish this section with the analysis of the $\mathfrak{so}_n$ representation-theoretic content of the module M.

We need to introduce notations for the complex finite-dimensional irreducible representations of the Lie algebra of the algebraic complex Lie group $\mathrm{SO}(n)$

Definition 4.19. Let $[w_1, w_2, \ldots, w_{[n/2]}]$ denote an irreducible complex representation of $\mathfrak{so}_n = \mathrm{Lie}(\mathrm{SO}(n))$, $n \geq 5$. It has the highest weight with coordinates $(w_1, w_2, \ldots, w_{[n/2]})$ in a standardly ordered basis of fundamental weights. For example $[1, 0, \ldots, 0]$ stands for the defining representation in V. The representation $[0, \ldots, 1, \ldots, 0]$ where the unit is in the i-th place corresponds to $\Lambda^i(V)$ $(i \leq n/2)$ and so on. Every such representation comes from a representation of a finite cover of $\mathrm{SO}(n)$ (see [20] for details).

The Lie groups $\mathrm{SO}(2)$, $\mathrm{SO}(3)$, $\mathrm{SO}(4)$ are exceptional here.

An irreducible complex representation of $\mathrm{SO}(2)$ is classified by an integer.

The Lie algebra of $\mathrm{SO}(3)$ is isomorphic to $\mathfrak{sl}_2$. We adopt the notation $[w]$ for $\mathrm{Sym}^w(W)$, where W is the two-dimensional representation of $\mathfrak{sl}_2$.

The Lie algebra $\mathrm{Lie}(\mathrm{SO}(4))$ of $\mathrm{SO}(4)$ is isomorphic to $\mathfrak{sl}_2 \times \mathfrak{sl}_2$. The irreducible representations, in this case $\mathrm{Sym}^{w_1}(W_1) \otimes \mathrm{Sym}^{w_2}(W_2)$, are denoted by $[w_1][w_2]$ for $w_1, w_2 \in \mathbb{Z}_{\geq 0}$.

Lemma 4.10 implies that if $\dim(V) \geq 5$ then $M_2 = [0, 1, 0, \ldots, 0]$. Suppose that a_i and m_2 are highest weight vectors in A_i and M_2. Using the multiplication map $A_i \otimes M_2 \to M_{2+i}$ we can construct the highest weight vector $a_i \times m_2 \in M_{2+i}$ (Proposition 4.17 implies that $a_i \times m_2$ is nonzero). Then a subrepresentation

$$[i, 1, 0, \ldots, 0] \subset M_{i+2} \tag{4.14}$$

is generated by the action of $\mathfrak{so}_n$ on $a_i \times m_2$.

If $\dim(V) = 2$ then using the defining relations (2.7) we can see that

$$M_{2+i} = 0, \quad i \geq 1.$$

If $\dim(V) = 3$ then $M_2 = [2]$ and

$$M_{2+i} \supset [2 + 2i], \quad i \geq 0. \tag{4.15}$$

If $\dim(V) = 4$ then $M_2 = [2][0] + [0][2]$ and

$$M_{2+i} \supset [2+i][i] + [i][2+i], \quad i \geq 0. \tag{4.16}$$

We need to make a short digression into Borel–Weyl–Bott theory adapted to the Lie algebra $\mathfrak{so}(n)$. This is a theory about cohomology of invertible sheaves on the manifold of full isotropic flags $\mathrm{Fl} = \{0 = F_0 \subset F_1 \subset \dots F_{[n/2]} \subset V\}$, the basic compact Kähler homogeneous space of $\mathrm{SO}(n)$. The total space of a line bundle $\mathcal{T}_i$, $1 \leq i \leq [n/2]$, is defined as $\{(F_1 \subset \cdots \subset F_{[n/2]} \subset V, l \in F_i/F_{i-1})\}$. According to [15] the duals $\mathcal{T}_i^*$ are free very ample generators of $\mathrm{Pic}(\mathrm{Fl})$. A cover of $\mathrm{SO}(n)$ that simultaneously acts on all $\mathcal{T}_i$ is $\mathrm{Spin}(n)$. Let $\mathcal{O}(w_1, \dots, w_{[n/2]})$ denote the tensor product $\bigotimes_{i=1}^{[n/2]} \mathcal{T}_i^{*\otimes w_i}$. By functoriality the cohomology $H^j(\mathrm{Fl}, \mathcal{O}(w_1, \dots, w_{[n/2]}))$ becomes a $\mathrm{Spin}(n)$-representation.

The Borel–Weyl–Bott theory (see [15]) applied to the space of isotropic flags Fl among other things asserts that the cohomology group $H^j(\mathrm{Fl}, \mathcal{O}(w_1, \dots, w_{[n/2]}))$ can be nonzero for only one value of j. At this value the group is an irreducible $\mathrm{Spin}(n)$-representation. In particular, if all $w_i \geq 0$ then $H^j(\mathrm{Fl}, \mathcal{O}(w_1, \dots, w_{[n/2]}) = 0$ for $j > 0$ and $H^0(\mathrm{Fl}, \mathcal{O}(w_1, \dots, w_{[n/2]})$ is the irreducible representation $[w_1, \dots, w_{[n/2]}]$. For example the algebra A is equal to $\bigoplus_{w \geq 0} H^0(\mathrm{Fl}, \mathcal{O}(w, 0, \dots, 0))$.

Proposition 4.17 implies that the embedding $M_2 \to H^0(\mathrm{Fl}, \mathcal{O}(0, 1, \dots, w_{[n/2]})$ extends to a homomorphism $t\colon M(2) \to \bigoplus_{w \geq 0} H^0(\mathrm{Fl}, \mathcal{O}(w, 1, 0, \dots, 0)$, where all graded components t_j of t are nonzero.

Definition of a quadratic algebra. Let $C = \mathbb{C} + \bigoplus_{i \geq 1} C_i$ be a graded algebra. We say that C is quadratic if

(1) C is generated by C_1;

(2) the ideal of relations of C is quadratic, i.e., if we let τ denote the natural map $\tau\colon T(C_1) \to C$, then $\operatorname{Ker} \tau = \bigoplus_{i,j \geq 0} C_1^{\otimes i} \otimes W \otimes C_1^{\otimes j}$, where $W = \operatorname{Ker} \tau \cap C_1^{\otimes 2}$.

Definition of a quadratic module over a quadratic algebra. Let C be a quadratic algebra. A graded module $N = \bigoplus_{i \geq 0} N_i$ over C is quadratic if

(1) N is generated by N_0;

(2) the submodule of relations of N is quadratic, i.e., if we let π denote the natural map $\pi\colon C \otimes N_0 \to N$ then $\operatorname{Ker} \pi = \bigoplus_{i \geq 0} C_i \otimes U$, where $U = \operatorname{Ker} \pi \cap C_1 \otimes N_0$.

Let B be a Borel subgroup of a reductive algebraic group G and let $\mathcal{L}_1 \dots \mathcal{L}_l$ denote an array of ample line bundles on G/B. Kempf and Ramanathan proved in [12] that the equations defining the embedding of G/B in the complete linear system of $\mathcal{L}_1 \dots \mathcal{L}_l$ are quadratic.

This implies that $\bigoplus_{w_1,w_2\geq 0} H^0(\mathrm{Fl}, \mathcal{O}(w_1, w_2, 0, \dots, 0))$ is a quadratic algebra and consequently $\bigoplus_{w\geq 0} H^0(\mathrm{Fl}, \mathcal{O}(w, 1, 0, \dots, 0))$ is a quadratic A-module. In these considerations $n \geq 5$.

It is a matter of a simple check using the classical invariant theory of H. Weyl or alternatively the results of [10] that $V \otimes \Lambda^2(V) \cong V + \Lambda^3(V) + [1, 1, 0, \dots, 0]$. Thus $V + \Lambda^3(V)$ is a maximal subrepresentation of $V \otimes \Lambda^2(V)$.

We conclude that the module $M(2)$ has the same relations as the module $\bigoplus_{w\geq 0} H^0(\mathrm{Fl}, \mathcal{O}(w, 1, \dots, 0))$ and the map t is an isomorphism. For $n \geq 5$ we have proved the following.

Proposition 4.20. *Inclusions* (4.14)*,* (4.15)*,* (4.16) *are isomorphisms. The* $\mathfrak{so}_n$ *representations* M_i *are self-dual.*

Proof. The representation M_{i+2} (non-canonically) is a subrepresentation of $V^{\otimes(n+2)}$, whose elements satisfy some symmetry conditions with respect to the natural symmetry group action.

By definition $V^{\otimes(n+2)}$ is a complexification of a real representation, equipped with a positive-definite bilinear form. Therefore all subrepresentations of $V^{\otimes(n+2)}$ are self-dual.

The exceptional cases $3 \leq n \leq 4$ can be treated similarly to the generic case $n \geq 5$, with a minor modification in dimension 4. □

4.4. Computation of $H^1(YM, M^{\otimes j})$. It is convenient to introduce the following notations. There are canonical projections of algebras $l_j\colon \mathrm{Sym}(V)^{\otimes j} \to A^{\otimes j}$. By abuse of notation we denote the composition $l_j \circ \Delta^{j-1}$ by Δ^{j-1}. By definition $\Delta^0 = l$.

The element q acts on $M^{\otimes j}$ through multiplication on $\Delta^{j-1}(q)$. Though $\Delta^0 q$ is zero, the element $\Delta(q)$ is nonzero and is equal to $2\sum_s x_s \otimes x_s$. Using the axioms of Hopf algebra it is not hard to see that $\epsilon \otimes \dots \otimes \epsilon \otimes \mathrm{id} \otimes \mathrm{id} \circ \Delta^{j-1}(q) = \Delta^1(q)$, where $\epsilon\colon A \to \mathbb{C}$ is the standard augmentation. This implies that $\Delta^{j-1}(q) \neq 0$.

Lemma 4.21. *If* $\dim(V) \geq 3$ *and* $j \geq 2$ *then* $\mathrm{Ann}_{M^{\otimes j}}\, \Delta^{j-1}(q) = 0$.

Proof. The module $M^{\otimes j}$ is an $A^{\otimes j}$ submodule of $(M[(\tilde{x}_1)^{-1}])^{\otimes j}$. The latter is free over $(A[(\tilde{x}_1)^{-1}])^{\otimes j}$ (Proposition 4.17). □

Proposition 4.22. (1) $H^1(YM, M^{\otimes j}) = H_2(YM, M^{\otimes j}) = 0$ *if* $\dim(V) \geq 3$ *and* $j \geq 2$.

(2) $H^1(YM, M^{\otimes j}) = V^* \otimes M^{\otimes j}$ *for* $j \geq 0$ *if* $\dim(V) = 2$.

Proof. The proof follows from Lemmas B.3 and B.5 and Propositions 4.9, 4.21.

The case of $\dim(V)$ equal to 2 is self-evident. □

5. Analysis of the E_1-differential

Proposition 4.22 shows that E_1-terms of the spectral sequences (3.2), (3.3) have significant numbers of zeros. However there is no reason to believe that $E_1^{1,0} = H^1(YM, M) \cong H_2(YM, M) = E^1_{-1,3}$ is trivial. In fact it is not (the interested reader may consult [16], which gives a decomposition of $H^1(YM, M)$ into SO(n)-irreducible components). It follows immediately from our previous computations that $E_1^{1,0}$ is the only nontrivial component of E_1 that can contribute to the limiting cohomology $H^1(YM, U(TYM))$.

In this section we choose the homological framework for a computation of the groups (1.8), (1.9). We shall analyze the higher differential d of the spectral sequence (3.3) for which we reserve a separate notation:

$$\delta\colon H_2(YM, M) = E^1_{-1,3} \to H_1(YM, M^{\otimes 2}) = E^1_{-2,3}. \tag{5.1}$$

There is a more down-to-earth description of the differential δ in terms of a short exact sequence of YM-modules:

$$0 \to M^{\otimes 2} \to F^1(U(TYM))/F^3(U(TYM)) \to M \to 0.$$

The differential δ is the connecting homomorphism in the corresponding sequence of homology. The proof of this fact is a simple exercise in understanding the definition of a spectral sequence.

5.1. Basic properties of the differential δ. The differential (5.1) is in fact a composition:

$$\delta\colon H_2(YM, M) \to H_1(YM, \Lambda^2(M)) \hookrightarrow H_1(YM, M^{\otimes 2}).$$

Indeed, by anti-commutativity of the Lie bracket in TYM only the $\Lambda^2(M)$ tensor component of the direct sum $\Lambda^2(M) + \mathrm{Sym}^2(M) = M^{\otimes 2}$ appears in the range of δ.

Let us consider the surjective maps $B^1_{M^{\otimes 2}}\colon H_1(V, M^{\otimes 2}) \to H_2(YM, M)$ (Corollary 4.13) and $I^1_{M^{\otimes 2}}\colon H_1(YM, M^{\otimes 2}) \to H_1(V, M^{\otimes 2})$ (Proposition 4.6).

Definition 5.1. The space $H_2(YM, M)$ can be decomposed into a direct sum

$$H_2(YM, M) = H_2(YM, M)^s + H_2(YM, M)^a,$$

where by definition $H_2(YM, M)^s = B^1_{M^{\otimes 2}}(H_1(V, \mathrm{Sym}^2(M)))$ and $H_2(YM, M)^a = B^1_{M^{\otimes 2}}(H_1(V, \Lambda^2(M)))$. We call elements of $H_2(YM, M)^s$ symmetric and those of $H_2(YM, M)^a$ antisymmetric.

We denote the restrictions of δ to $H_2(YM, M)^a$ and to $H_2(YM, M)^s$ by δ^a and δ^s, respectively.

We now turn to the explicit construction of the maps $B^1_{M^{\otimes 2}}\colon H_1(V, M^{\otimes 2}) \to H_2(YM, M)$ and $B^0_{M^{\otimes 3}}\colon H_0(V, \Lambda^3(M)) \to H_1(YM, \Lambda^2(M))$ in terms of chains of $C_\bullet(YM, M)$. The map r (4.1) enables us to lift elements of M into TYM. Pick a cycle $z = \sum d^{ijs} a_i \otimes a_j \otimes \varsigma_s$ in the class $[z] \in H_1(V, M^{\otimes 2})$.

Proposition 5.2. *There is a procedure for constructing elements $c_k, \tilde{c}_{k'} \in TYM$ and coefficients $f^{kk'}_{sj}$ such that an element*

$$\tilde{z} = \sum d^{ijs} a_i \otimes (ra_j \wedge v_s) + \sum d^{ijs} f^{kk'}_{sj} a_i \otimes (c_k \wedge \tilde{c}_{k'}) \in M \otimes \Lambda^2(YM), \quad (5.2)$$

$c_k, \tilde{c}_{k'} \in TYM$, is a cycle in $C_2(YM, M)$ representing $B^1_{M^{\otimes 2}}(z)$.

The map $B^0_{M^{\otimes 3}}\colon H_0(V, \Lambda^3(M)) \to H_1(YM, \Lambda^2(M)) \subset H_1(YM, M^{\otimes 2})$ is defined on chains by the formula

$$a \wedge b \wedge c \to a \wedge b \otimes c + c \wedge a \otimes b + b \wedge c \otimes a \in \Lambda^2(M) \otimes YM.$$

Proof. The proof follows easily from the description of the maps $B^1_{M^{\otimes 2}}$ and $B^0_{M^{\otimes 3}}$ in terms of Hochschild–Serre spectral sequence given in Proposition 4.7. □

We finish this section with a formula for the differential δ.

Lemma 5.3. *Under the differential δ the cycle* (5.2) *transforms into*

$$\begin{aligned}\delta\tilde{z} = 2\sum d^{ijs}(a_i \wedge a_j \otimes v_s &+ f^{kk'}_{sj} c_k \wedge \tilde{c}_{k'} \otimes ra_j \\ &- f^{kk'}_{sj} c_k \wedge a_i \otimes \tilde{c}_{k'} + f^{kk'}_{sj} \tilde{c}_{k'} \wedge a_i \otimes c_k) \in \Lambda^2[M] \otimes \Lambda^1(YM).\end{aligned} \quad (5.3)$$

By abuse of notation we write c_k and $\tilde{c}_{k'}$ for the image of c_k and $\tilde{c}_{k'}$ in $M = TYM/[TYM, TYM]$.

Proof. We lift $\tilde{z}$ to an element of $F^1/F^3 \otimes \Lambda^2(YM)$ and apply the homology differential d. The result belongs to $C_1(YM, F^2/F^3) = C_1(YM, \Lambda^2(M))$. In the formula for the boundary we identify the commutators $[ra, rb] \in F^2/F^3$, $a, b \in M$, which are the coefficients of the chain, with the monomials $2a \wedge b$.

We leave details to the reader. □

5.2. Properties of δ^a.

To simplify our analysis of (5.1) we decompose the differential δ into the sum

$$\delta = \delta^a + \delta^s \quad (5.4)$$

as in Definition 5.1. In this section we study the kernel of δ^a.

Proposition 5.4. *Consider the map ∇ that has been defined in Proposition* 4.9. *The composition $\nabla^2_1 \circ \delta^a\colon H_2(YM, M)^a \to H_2(YM, M)$ transforms $H_2(YM, M)^a$ to itself. As an operator acting on $H_2(YM, M)^a$ the map $\nabla^2_1 \circ \delta^a$ is equal to* $2 \times \mathrm{id}$. *In particular $H_2(YM, M)^a \cap \operatorname{Ker} \delta = 0$.*

Proof. We may assume that every homology class in $H_2(YM, M)$ has a representative (5.3) for suitable $\tilde{z}$ (use Corollary 4.13 and Proposition 5.2).

The map I_M^2 transforms all terms of (5.3) to zero, except the ones containing a $\otimes v_s$-factor. This follows from the description of the map I in Proposition 4.7 as a map in homology induced by abelianization.

This implies that

$$I_M^2 \circ \delta \circ B^1_{M^{\otimes 2}}\Big(\sum d^{ijs} a_i \otimes a_j \otimes v_s\Big) = 2\sum d^{ijs} a_i \wedge a_j \otimes v_s, \tag{5.5}$$

where we identify $[a_i, a_j]$ (an element in F^2/F^3) with $2a_i \wedge a_j \in \Lambda^2[M] \subset M^{\otimes 2}$.

Let $z = \sum d^{ijs} a_i \otimes a_j \otimes v_s$ satisfy $\sum d^{ijs} a_i \otimes a_j \otimes v_s = -\sum d^{ijs} a_j \otimes a_i \otimes v_s$, i.e., $[z] \in H_2(YM, M)^a$. It follows from (5.5) that $2B^1_{M^{\otimes 2}}(z) = \nabla_1^2 \circ \delta \circ B^1_{M^{\otimes 2}}(z)$. □

5.3. $\mathbf{Tor}_{\boldsymbol{i}}^{\boldsymbol{A}}(\boldsymbol{N_1}, \boldsymbol{N_2})$ and related functors. In this section we formulate the rather technical Proposition 5.6. It will be used once, in Section 5.4. Besides this, the main line of exposition is independent of this material.

There is a resolution

$$A \leftarrow \mathrm{Sym}(V) \xleftarrow{\times q} \mathrm{Sym}(V)(-2). \tag{5.6}$$

A cocycle κ, which will be introduced in the following lemma, is needed for the formulation of the main proposition of this section.

The next lemma easily follows from a standard computation with the resolution (5.6) and degree counting.

Lemma 5.5. *We have* $\mathrm{Tor}_{0,\boldsymbol{i}}(A, A) = A_i$, $\mathrm{Tor}_{1,\boldsymbol{i}}(A, A) = A_{i-2}$.

A cocycle in $\boldsymbol{C}_\bullet(V, A \otimes A)$ that represents an A-generator in $\mathrm{Tor}_{1,\mathbf{2}}(A, A)$ *is given by*

$$\kappa = \sum_{s=1}^{n} (x_s \otimes 1 - 1 \otimes x_s) \otimes \varsigma_s. \tag{5.7}$$

Next a map of $\mathrm{Sym}(V)$-modules $N_1 \otimes N_2 \to N_3$ defines a map of complexes $\boldsymbol{C}_\bullet(V, N_1) \otimes \boldsymbol{C}_\bullet(V, N_2) \to \boldsymbol{C}_\bullet(V, N_3)$.

We want to apply this construction to $N_1 = A \otimes A$, $N_2 = M \otimes M$, $N_3 = M \otimes M$, where the module map is multiplication. In this context any cocycle $a \in \boldsymbol{C}_\bullet(V, A \otimes A)$ defines a multiplication map $a\times\colon \boldsymbol{C}_\bullet(V, M \otimes M) \to \boldsymbol{C}_\bullet(V, M \otimes M)$. By abuse of notation we write a for $a\times$.

Proposition 5.6. *There is a right exact sequence*

$$H_{0,\boldsymbol{j-2}}(V, M \otimes M) \xrightarrow{\kappa} H_{1,\boldsymbol{j}}(V, M \otimes M) \to \mathrm{Coker}_j \to 0, \tag{5.8}$$

where the element κ is defined in equation (5.7). *The group* Coker_j *is equal to zero for* $j \neq 5$ *and* $\mathrm{Coker}_5 = \Lambda^5(V) + \Lambda^3(V) + V$. *The group* $H_{0,3}(V, M \otimes M)$ *is equal to zero.*

The proof is given in Appendix D.

5.4. Properties of δ^s. The computation of the kernel of δ^s (5.4) is the most difficult part of the present paper. It will be carried out in Proposition 5.13, which is the main result of this section. Proposition 5.8 lets us decrease by one the homological degree of the cycles involved in the computations. This greatly simplifies the proof of Proposition 5.13. In the proof of the main assertion, Proposition 5.10 is technically central.

Fix a Lie algebra $\mathfrak{g}$. The group

$$D(\mathfrak{g}) = H_0(\mathfrak{g}, \mathrm{Sym}^2(\mathfrak{g})) \tag{5.9}$$

has already been found useful in [11], where Kontsevich studied noncommutative analogs of formal symplectic geometry.

We would like to specialize the construction (5.9) to the case of the algebra $\mathfrak{h}$ defined as the quotient

$$0 \to [TYM, [TYM, TYM]] \to TYM \xrightarrow{\gamma} \mathfrak{h} \to 0.$$

The algebra $\mathfrak{h}$ is the universal central extension of the abelian Lie algebra M:

$$0 \to \Lambda^2(M) \to \mathfrak{h} \to M \to 0. \tag{5.10}$$

The algebra YM acts on TYM by commutators. Let $\rho_{TYM}\colon YM \to \mathrm{End}(TYM)$ denote the corresponding representation. The action ρ_{TYM} preserves the Lie ideal $[TYM, [TYM, TYM]]$. Denote the representation of YM in $\mathrm{End}(\mathfrak{h})$ by $\rho_{\mathfrak{h}}$.

To analyze $D(\mathfrak{h})$ it will be convenient to choose a linear splitting of the extension (5.10) and to identify the linear space $\mathfrak{h}$ with $\Lambda^2(M) + M$.

The linear space $D(\mathfrak{h})$ is a quotient of $\mathrm{Sym}^2(M + \Lambda^2(M)) \cong \mathrm{Sym}^2(M) + \Lambda^2(M) \otimes M + \mathrm{Sym}^2(\Lambda^2(M))$. The image of the last summand in $D(\mathfrak{h})$ is zero because of the equivalence $[a,b] \bullet [c,d] \sim -b \bullet [a,[c,d]] = 0$. The linear subspace $\mathrm{Sym}^2(M)$ injects into $D(\mathfrak{h})$. The space $\Lambda^3(M) \subset \Lambda^2(M) \otimes M$ maps isomorphically to $\mathrm{Ker}\,\gamma$ because of the relation $[a,b] \bullet c \sim -[a,c] \bullet b$.

The space $D(\mathfrak{h})$ is naturally a YM-module. By definition the commutators $[v_i, v_j] \in TYM$ act trivially on $D(\mathfrak{h})$. This means that the action factors through YM_{ab}. We have proved the following.

Proposition 5.7. *The linear space* $D(\mathfrak{h})$ *is a* $\mathrm{Sym}(V)$*-module. There is a short exact sequence of modules*

$$0 \to \Lambda^3(M) \to D(\mathfrak{h}) \xrightarrow{\gamma} \mathrm{Sym}^2(M) \to 0.$$

Proposition 5.8. *There is a commutative diagram*

$$\begin{tikzcd} H_2(YM, M) \arrow[r, "\delta"] & H_1(YM, M^{\otimes 2}) \\ H_1(V, \mathrm{Sym}^2(M)) \arrow[u, "B^1_{M^{\otimes 2}}"] \arrow[r, "\delta_{D(\mathfrak{h})}"] & H_0(V, \Lambda^3(M)). \arrow[u, "B^0_{M^{\otimes 3}}"'] \end{tikzcd} \tag{5.11}$$

The map $\delta_{D(\mathfrak{h})}$ is the boundary differential in homology corresponding to the extension $D(\mathfrak{h})$.

Proof. This follows from Lemma 5.3 and Proposition 5.2. □

Our plan is to replace computations that involve δ by the corresponding simpler computations that use $\delta_{D(\mathfrak{h})}$.

To study δ we need information about the kernels of the maps $B^1_{M^{\otimes 2}}$ and $B^0_{M^{\otimes 3}}$ in the commutative diagram (5.11). This we will attend to do presently.

Lemma 5.9. *The following holds:*

$$\operatorname{Ker} B^0_{M^{\otimes 3}} = 0, \tag{5.12}$$

$$H_1(V, \mathrm{Sym}^2(M)) \cap \operatorname{Ker} B^1_{M^{\otimes 2}} \cong \Lambda^5(V) \subset H_{1,\mathbf{5}}(V, \mathrm{Sym}^2(M)). \tag{5.13}$$

Proof. The long exact sequence (4.7) implies that $\operatorname{Ker} B^0_{M^{\otimes 3}} = \operatorname{Im} S^2_{M^{\otimes 2}}$. The group $H_2(V, M^{\otimes 2}) = H_{2,\mathbf{6}}(V, M^{\otimes 2}) \cong \Lambda^6(V)$, $\dim(V) \geq 3$, has been computed in Proposition 4.7. Nontrivial cocycle representatives that span this group are $\mathcal{F}_{[ij} \otimes \mathcal{F}_{kl} \otimes \varsigma_s \wedge \varsigma_{t]}$. The map $S^2_{M^{\otimes 2}}$ transforms them into $\mathcal{F}_{[ij} \otimes \mathcal{F}_{kl} \otimes \mathcal{F}_{st]}$, which are elements of $\mathrm{Sym}^3(M)$. Considered as zero-cycles these elements are clearly not equivalent to any of $\Lambda^3(M)$. This proves (5.12) in the case $\dim(V) \geq 3$, because $\operatorname{Ker} B^0_{M^{\otimes 3}} = 0$. The remaining case, $\dim(V)$ is equal to 2, is straightforward.

The generators of $H_3(V, M)$ are $\mathcal{F}_{[ij} \otimes \varsigma_s \wedge \varsigma_t \wedge \varsigma_{u]}$. They are mapped by S^3_M to $\mathcal{F}_{[ij} \otimes \mathcal{F}_{st} \otimes \varsigma_{u]} \in C_1(V, \mathrm{Sym}^2(M))$. The isomorphism (5.13) holds because of the equality $\operatorname{Ker} B^1_{M^{\otimes 2}} = \operatorname{Im} S^3_M$. □

Proposition 5.10. *Suppose that* $\dim(V) \geq 3$ *holds. Then the kernel of* $\delta_{D(\mathfrak{h})}$ *is* $\Lambda^5(V) + V \cong H_{1,\mathbf{5}}(V, \mathrm{Sym}^2(M))$.

Proof. The isomorphism $H_{1,\mathbf{5}}(V, M^{\otimes 2}) \cong \Lambda^5(V) + \Lambda^3(V) + V$ follows from Proposition 5.6.

The subspaces $\Lambda^5(V)$ and V of $H_{1,\mathbf{5}}(V, M^{\otimes 2})$ are spanned by the cocycles

$$\mathcal{F}_{[ij} \otimes \mathcal{F}_{kl} \otimes \varsigma_{m]}$$

and

$$\sum_{1\le i<j\le n}^{n} \mathcal{F}_{ij}\otimes\mathcal{F}_{ij}\otimes\varsigma_k+\sum_{i,j=1}^{n}(\mathcal{F}_{ki}\otimes\mathcal{F}_{ij}+\mathcal{F}_{ij}\otimes\mathcal{F}_{ki})\otimes\varsigma_j,$$

respectively.

The subspace $\Lambda^3(V)$ is spanned by elements

$$\sum_{s=1}^{n}(\mathcal{F}_{[ij}\otimes\mathcal{F}_{ks}\otimes\varsigma_{s]}+2\mathcal{F}_{[is}\otimes\mathcal{F}_{js}\otimes\varsigma_{k]}),$$

which belong to $H_{1,\mathbf{5}}(V,\Lambda^2(M))$. So $\Lambda^3(V)\subset H_{1,\mathbf{5}}(V,\Lambda^2(M))$

The boundary differential $\delta_{D(\mathfrak{h})}$ preserves the bold grading (see Section 2.2). Thus

$$\delta_{D(\mathfrak{h})}H_{1,\mathbf{5}}(V,\mathrm{Sym}^2(M))\subset H_{0,\mathbf{5}}(V,\Lambda^3(M)).$$

On the other hand the lowest degree component of $\Lambda^3(M)$ is $\Lambda^3(M)_{\mathbf{6}}$ because M is generated by elements of degree two. Hence $\delta_{D(\mathfrak{h})}H_{1,\mathbf{5}}(V,\mathrm{Sym}^2(M))=0$.

Using Proposition 5.6 we conclude that

$$\operatorname{Ker}\delta_{D(\mathfrak{h})}=\operatorname{Ker}\delta_{D(\mathfrak{h})}\cap\operatorname{Im}\kappa+H_{1,\mathbf{5}}(V,\mathrm{Sym}^2(M)),$$

and that there are surjective maps $H_{0,\boldsymbol{j}}(V,\Lambda^2(M))\xrightarrow{\kappa}H_{1,\boldsymbol{j}+\mathbf{2}}(V,\mathrm{Sym}^2(M))$, $\boldsymbol{j}\ge 4$. We define x as

$$x=\mathcal{F}_{ij}\otimes x^{\alpha}\mathcal{F}_{kl}-x^{\alpha}\mathcal{F}_{kl}\otimes\mathcal{F}_{ij}\in\Lambda^2(M)\subset M^{\otimes 2}. \tag{5.14}$$

The linear space $\operatorname{Im}\kappa$ is spanned by elements

$$\begin{aligned}\kappa(x)=&\sum_{s=1}^{n}(x_s\mathcal{F}_{ij}\otimes x^{\alpha}\mathcal{F}_{kl}-\mathcal{F}_{ij}\otimes x_s x^{\alpha}\mathcal{F}_{kl})\otimes\varsigma_s\\&+\sum_{s=1}^{n}(x^{\alpha}\mathcal{F}_{kl}\otimes x_s\mathcal{F}_{ij}-x_s x^{\alpha}\mathcal{F}_{kl}\otimes\mathcal{F}_{ij})\otimes\varsigma_s.\end{aligned}$$

In the following part of the proof we shall show that $\operatorname{Ker}\delta_{D(\mathfrak{h})}\cap\operatorname{Im}\kappa=0$.

We do it by proving that the composition $\delta_{D(\mathfrak{h})}\circ\kappa$ has a trivial kernel.

A choice of the isomorphism r in (4.1) enables us to transfer the YM-action from TYM to $\mathrm{Free}(M)$.

If $m\in M\subset\mathrm{Free}(M)$ then

$$\rho_{\mathrm{Free}(M)}(v_s)m=x_s m+\psi_s^2(m)+\psi_s^3(m)+\cdots.$$

In this formula $\psi_s^k\colon M\to\mathrm{Free}^k(M)$, for $k=2,\ldots,$ are linear maps, where $\mathrm{Free}^k(M):=\mathrm{Free}(M)\cap M^{\otimes n}$ is the linear subspace of $\mathrm{Free}(M)$ spanned by k repeated commutators of generators.

The algebra $TYM \subset YM$ acts on $\mathrm{Free}(M)$ by inner derivations. As before $\mathcal{F}_{ij}$ are $\mathrm{Sym}(V)$-generators of M. Then by definition

$$[\rho_{\mathrm{Free}(M)}(v_i), \rho_{\mathrm{Free}(M)}(v_j)]m = [\mathcal{F}_{ij}, m] \in \mathrm{Free}(M).$$

Since $\psi_s^k \in [TYM, [TYM, TYM]]$ for $k \geq 3$, the formula for the YM-action on $\mathfrak{h}$ involves only ψ_i^2.

As a result we can write the differential $\delta_{D(\mathfrak{h})}$ using only $\psi_s := \psi_s^2$:

$$\begin{aligned}
\delta_{D(\mathfrak{h})} \circ \kappa(x) = {} & \overbrace{\sum_{s=1}^{n} \psi_s(x_s \mathcal{F}_{ij}) \otimes x^\alpha \mathcal{F}_{kl}}^{a_1} + \overbrace{\sum_{s=1}^{n} x_s \mathcal{F}_{ij} \otimes \psi_s(x^\alpha \mathcal{F}_{kl})}^{b_1} \\
& + \overbrace{\sum_{s=1}^{n} -\psi_s(\mathcal{F}_{ij}) \otimes x_s x^\alpha \mathcal{F}_{kl}}^{a_2} + \overbrace{\sum_{s=1}^{n} -\mathcal{F}_{ij} \otimes \psi_s(x_s x^\alpha \mathcal{F}_{kl})}^{b_2} \\
& + \overbrace{\sum_{s=1}^{n} \psi_s(x^\alpha \mathcal{F}_{kl}) \otimes x_s \mathcal{F}_{ij}}^{a_3} + \overbrace{\sum_{s=1}^{n} x^\alpha \mathcal{F}_{kl} \otimes \psi_s(x_s \mathcal{F}_{ij})}^{b_3} \\
& + \overbrace{\sum_{s=1}^{n} -\psi_s(x_s x^\alpha \mathcal{F}_{kl}) \otimes \mathcal{F}_{ij}}^{a_4} + \overbrace{\sum_{s=1}^{n} -x_s x^\alpha \mathcal{F}_{kl} \otimes \psi_s(\mathcal{F}_{ij})}^{b_4} .
\end{aligned}$$

The element $\delta_{D(\mathfrak{h})} \circ \kappa(x)$ belongs to the zero homology group $H_0(V, \Lambda^3(M))$. The equivalence relation $\sim$ for zero cycles gives some freedom for algebraic manipulations. In particular

$$a_1 + a_2 \sim \Big(\sum_{s=1}^{n} \psi_s(x_s \mathcal{F}_{ij}) + \Delta(x_s)\psi_s(\mathcal{F}_{ij}) \Big) \otimes x^\alpha \mathcal{F}_{kl}. \tag{5.15}$$

There are similar formulas for $a_3 + a_4$, $b_1 + b_2$ and $b_3 + b_4$. The diagonal map Δ from (4.3) enables us to define the action of $x_s \in \mathrm{Sym}(V)$ on $\Lambda^2(M)$.

Consider the operator $\mathbb{L} \colon M \to \mathrm{Free}(M)$ defined by the formula

$$\mathbb{L}(m) := \sum_{s=1}^{n} \rho_{\mathrm{Free}(M)}(v_s)\rho_{\mathrm{Free}(M)}(v_s)m.$$

By abuse of notation we write $\mathbb{L}$ for the composition $\gamma \circ \mathbb{L} \colon M \to \mathfrak{h}$. It can be written in terms of the operators ψ_s and x_s:

$$\mathbb{L}(m) = \sum_{s=1}^{n} \Delta(x_s)\psi_s(m) + \psi_s(x_s m) \in \mathfrak{h}.$$

Observe that the left tensor factor in (5.15) has the same form.

To make our computations more tractable we need further simplifications. For this we map $H_0(V, \Lambda^3(M))$ to $\Lambda^2(V) \otimes H_0(V, \Lambda^2(M))$ via the map p, which we define in the next paragraph. The composition $p \circ \delta_{D(\mathfrak{h})} \circ \kappa$ has a simpler structure then $\delta_{D(\mathfrak{h})} \circ \kappa$. We will manage to prove that $\mathrm{Ker}(p \circ \delta_{D(\mathfrak{h})} \circ \kappa) = 0$. This will imply injectivity of $\delta_{D(\mathfrak{h})} \circ \kappa$. Let us equip $\Lambda^2(V)$ with a trivial $\mathrm{Sym}(V)$-module structure. The module M is generated by the linear space $M_0 = \Lambda^2(V)$ (Lemma 4.10), and there is a canonical map of $\mathrm{Sym}(V)$-modules $M \to M_0$. We denote elements of the generating set of M as in Lemma 4.10 by $\mathcal{F}_{ij}$, and their images in $\Lambda^2(V)$ by $\widehat{\mathcal{F}}_{ij}$.

There is a map of $\mathrm{Sym}(V)$-modules $p\colon \Lambda^3(M) \to \Lambda^2(V) \otimes \Lambda^2(M)$, defined by the formula

$$\begin{aligned} p(a\mathcal{F}_{ij} \wedge b\mathcal{F}_{kl} \wedge c\mathcal{F}_{st}) &= a(0)\widehat{\mathcal{F}}_{ij} \wedge b\mathcal{F}_{kl} \wedge c\mathcal{F}_{st} \\ &\quad + a\mathcal{F}_{ij} \wedge b(0)\widehat{\mathcal{F}}_{kl} \wedge c\mathcal{F}_{st} + a\mathcal{F}_{ij} \wedge b\mathcal{F}_{kl} \wedge c(0)\widehat{\mathcal{F}}_{st}. \end{aligned}$$

Here the map $a \to a(0)$ is the standard augmentation of $\mathrm{Sym}(V)$. It is easy to check that this p is unambiguously defined. The homomorphism induces a map in homology $p\colon H_0(V, \Lambda^3(M)) \to \Lambda^2(V) \otimes H_0(V, \Lambda^2(M))$.

Under the $\mathfrak{so}_n$-isomorphism $\mathrm{Ad}(\mathfrak{so}_n) \cong \Lambda^2(V)$ an element $\widehat{\mathcal{F}}_{ij} \in \Lambda^2(V)$ transforms into a generator $\xi_{ij} \in \mathfrak{so}_n$. The linear space $H_0(V, \Lambda^2(M))$ is an $\mathfrak{so}_n$-representation, so it makes sense to talk about the generators ξ_{ij} acting on an element $x \in H_0(V, \Lambda^2(M))$.

Lemma 5.11. *Let x be as in* (5.14). *Then the following formula holds:*

$$p \circ \delta_{D(\mathfrak{h})} \circ \kappa(x) = 4 \sum_{1 \le s < t \le n} \widehat{\mathcal{F}}_{st} \otimes \xi_{st} x.$$

Proof. The simplification pointed out in (5.15) leads to

$$\delta_{D(\mathfrak{h})} \circ \kappa(x) = \mathbb{L}(\mathcal{F}_{ij}) \otimes x^\alpha \mathcal{F}_{kl} - \mathbb{L}(x^\alpha \mathcal{F}_{kl}) \otimes \mathcal{F}_{ij} - \mathcal{F}_{ij} \otimes \mathbb{L}(x^\alpha \mathcal{F}_{kl}) + x^\alpha \mathcal{F}_{kl} \otimes \mathbb{L}(\mathcal{F}_{ij}).$$

Let us write $x^\alpha \mathcal{F}_{ij}$ as $x_{i_1} \dots x_{i_k} \mathcal{F}_{ij}$, where $|\alpha| = k$. The element

$$a = \sum_{s=1}^{n} \rho_{\mathfrak{h}}(v_s)\rho_{\mathfrak{h}}(v_s)\rho_{\mathfrak{h}}(v_{i_1}) \dots \rho_{\mathfrak{h}}(v_{i_k})\mathcal{F}_{ij} \in \mathfrak{h}$$

satisfies

$$a = \mathbb{L}(x^\alpha \mathcal{F}_{ij}) + \sum_{t=1}^{k} \sum_{s=1}^{n} \Delta(x_s^2 x_{i_1} \dots x_{i_{t-1}})\psi_{i_t}(x_{i_{t+1}} \dots x_{i_k} \mathcal{F}_{ij}) \in \Lambda^2(M). \tag{5.16}$$

On the other hand

$$
\begin{aligned}
a &= \sum_{t=1}^{k}\sum_{s=1}^{n} \rho_{\mathfrak{h}}(v_s)\dots\rho_{\mathfrak{h}}(v_{i_{t-1}})[\mathcal{F}_{si_t}, \rho_{\mathfrak{h}}(v_{i_{t+1}})\dots\mathcal{F}_{ij}] \\
&\quad + \sum_{s=1}^{n} \rho_{\mathfrak{h}}(v_s)\rho_{\mathfrak{h}}(v_1)\dots\rho_{\mathfrak{h}}(v_s)\mathcal{F}_{ij} \qquad (5.17)\\
&= 2\sum_{t=1}^{k}\sum_{s=1}^{n} \Delta(x_s x_{i_1}\dots x_{i_{t-1}})([\mathcal{F}_{si_t}, x_{i_{t+1}}\dots\mathcal{F}_{ij}]) + 2\sum_{s}\Delta(x^{\alpha})[\mathcal{F}_{si}, \mathcal{F}_{sj}].
\end{aligned}
$$

In the proof of (5.17) we use the identities

$$
\sum_{s=1}^{n} \rho_{\mathfrak{h}}(v_s)\rho_{\mathfrak{h}}(v_s)\mathcal{F}_{ij} = 2\sum_{s=1}^{n}[\mathcal{F}_{si}, \mathcal{F}_{sj}] \quad \text{and} \quad \sum_{s=1}^{n} x_s\mathcal{F}_{sj} = 0.
$$

The first holds in $\mathfrak{h}$, the second in M.

We use the formulas (5.16), (5.17) to compute $p \circ \delta_{D(\mathfrak{h})} \circ \kappa(x)$. By manipulations similar to (5.15) we obtain

$$
\begin{aligned}
p(a_1 + a_2) &\sim 2\widehat{\mathcal{F}}_{si} \otimes \mathcal{F}_{sj} \wedge x^{\alpha}\mathcal{F}_{kl} - 2\widehat{\mathcal{F}}_{sj} \otimes \mathcal{F}_{si} \wedge x^{\alpha}\mathcal{F}_{kl},\\
p(a_3 + a_4) &\sim 2\widehat{\mathcal{F}}_{sk} \otimes \mathcal{F}_{sj} \wedge x^{\alpha}\mathcal{F}_{sl} - 2\widehat{\mathcal{F}}_{sl} \otimes \mathcal{F}_{ij} \wedge x^{\alpha}\mathcal{F}_{sk},\\
p(b_1 + b_2) &\sim 2\widehat{\mathcal{F}}_{sk} \otimes \mathcal{F}_{ij} \wedge x^{\alpha}\mathcal{F}_{sl} - 2\widehat{\mathcal{F}}_{sl} \otimes \mathcal{F}_{ij} \wedge x^{\alpha}\mathcal{F}_{sk}\\
&\qquad + 2\widehat{\mathcal{F}}_{st} \otimes \mathcal{F}_{ij} \wedge (\xi_{st}x^{\alpha})\mathcal{F}_{kl}, \qquad (5.18)\\
p(b_3 + b_4) &\sim 2\widehat{\mathcal{F}}_{st} \otimes \mathcal{F}_{ij} \wedge (\xi_{st}x^{\alpha})\mathcal{F}_{kl} + 2\widehat{\mathcal{F}}_{si} \otimes \mathcal{F}_{sj} \wedge x^{\alpha}\mathcal{F}_{kl}\\
&\qquad - 2\widehat{\mathcal{F}}_{sj} \otimes \mathcal{F}_{si} \wedge x^{\alpha}\mathcal{F}_{kl}.
\end{aligned}
$$

The formulas (5.16), (5.17) imply that $\mathbb{L}(x^{\alpha}\mathcal{F}_{ij})$ represents a trivial element in $H_0(V, \Lambda^2(M))$, which is used in the proof of formulas (5.18).

From this we conclude that

$$
p(a_1 + a_2 + a_2 + a_4 + b_1 + b_3 + b_2 + b_4) \sim 4 \sum_{1\le s<t\le n} \widehat{\mathcal{F}}_{st} \otimes \xi_{st}(\mathcal{F}_{ij} \wedge x^{\alpha}\mathcal{F}_{kl}). \quad \square
$$

Lemma 5.11 implies that the composed map $p\circ\delta\circ\kappa$ has its kernel precisely equal to the $\mathfrak{so}_n$-invariant elements in $H_0(V, \Lambda^2(M))$. The space $\Lambda^2(\Lambda^2(V)) = \Lambda^2(M_2) \cong H_{0,\mathbf{4}}(V, \Lambda^2(M)) \subset H_0(V, \Lambda^2(M))$ contains no invariants since $\Lambda^2(V)$ is isomorphic to the adjoint representation, which can have only a symmetric invariant bilinear form. For $j \ge 5$ the components $H_{0,\boldsymbol{j}}(V, \Lambda^2(M))$ are quotients of $M_{j-2} \otimes M_2$. The latter contains no invariants, because by Lemma 4.20 the M_j are mutually nonisomorphic self-dual representations. From this we conclude that $p\circ\delta\circ\kappa$ has no kernel. Therefore $\delta \circ \kappa$ has no kernel, which is equivalent to saying that $\operatorname{Ker}\delta \cap \operatorname{Im}\kappa = 0$. $\square$

The following lemma will be used only in Section 5.5.

Lemma 5.12. *The composition* $\nabla_1^1 \circ \delta^s\colon H_2(YM, M)^s \to H_2(YM, M)$ *is zero.*

Proof. The proof follows the lines of the proof of Proposition 5.4 and uses the formula (5.5). $\square$

Finally we are in a position to prove the central statement of this section.

Proposition 5.13. *The kernel of the map* δ *in* (5.11) *is equal to* V.

Proof. Suppose that $x \in \operatorname{Ker}\delta$. Then $\nabla_1^1 \circ \delta(x) = 0$. Decompose $x = x^a + x^s$ according to Definition 5.1. Proposition 5.4 and Lemma 5.12 imply that $x^a = 0$. This means that $x \in H_2(YM, M)^s$. Proposition 5.8, Lemma 5.9 and Proposition 5.10 then yield that $x \in V \subset H_{2,\mathbf{5}}(YM, M)$. $\square$

5.5. Computation of the cohomology $H^1(YM, U(TYM))$

Proposition 5.14. *Suppose that* $\dim(V) \geq 3$. *Then the following isomorphisms hold:*

(1) $H^0(YM, U(TYM)) \cong H_3(YM, U(TYM)) = \mathbb{C}$.

(2) $H^1(YM, U(TYM)) \cong H_2(YM, U(TYM)) \cong V + V$.

If $\dim(V) = 2$ *then*

(3) $H^0(YM, U(TYM)) \cong H_3(YM, U(TYM)) \cong U(TYM)$.

(4) $H^1(YM, U(TYM)) \cong H_2(YM, U(TYM)) \cong V \otimes U(TYM)$.

Proof. Suppose that $\dim(V) \geq 3$. Proposition 4.17 implies that $H^0(YM, M^{\otimes n}) = H^0(V, M^{\otimes n})$. That $H^0(YM, U(TYM)) \cong H^0(YM, \mathbb{C}) + H^0(YM, F^1(U(TYM))) \cong \mathbb{C}+0$ follows from the spectral sequence (3.2) and the definition of the filtration (1.10).

The groups $H^i(YM, \mathbb{C})$ can be easily computed with the complex $C^\bullet(\mathbb{C})$. This observation will be used in the following.

Let N be the submodule of $U(YM)$ generated by $V = U(YM)_1$ and $F^1(U(TYM))$. Since $\rho_N(v_s)V \subset F^1(U(TYM))$ we have a short exact sequence

$$0 \to F^1(U(TYM)) \to N \to V \to 0$$

with δ_N the boundary map in cohomology. The YM-action on V is trivial. The boundary differential δ_N maps $V = H^{0,\mathbf{1}}(YM, V)$ into $H^{1,\mathbf{1}}(YM, F^1(U(TYM)))$. On the level of chains of the cochain complex (2.8) it maps $x_k \in C^0(V)$ to $\sum_{s=1}^n [v_s, v_k] \otimes v^{*s} \in C^1(F^1U(TYM))$. This last cocycle represents a nontrivial cohomology class because the group $C^{0,\mathbf{1}}(YM, F^1(U(TYM)))$ is zero. From this we conclude that $H^1(YM, F^1(U(TYM)))$ contains a nontrivial subspace.

We can also compute $H^1(YM, F^1(U(TYM)))$ using the spectral sequence of Proposition 3.2.

Corollary 4.2, Propositions 4.22 and 5.13 together imply that $E^2_{-j,2+j} = 0$ for $j \geq 2$. Furthermore, it follows that only $E^2_{-1,3} = V$ can contribute to $H_2(YM, F^1(U(TYM))) \cong H^1(YM, F^1(U(TYM)))$. From the above considerations we know that the space V survives all the higher differentials. Therefore we have $H^1(YM, F^1(U(TYM))) \cong V$.

Combining all the previous considerations together we finally conclude that $H^1(YM, U(TYM)) = H^1(YM, F^1(U(TYM)))H^1(YM, \mathbb{C}) \cong V + V$.

In the case of $\dim(V) = 2$ the arguments are completely straightforward. □

6. The center and the Lie algebra of outer derivations of $U(YM)$

In this section we shall determine $Z(U(YM))$ and $\mathrm{Out}(U(YM))$, the main ingredients of the formulas of Proposition 2.4.

To get an idea how derivations of $U(YM)$ might look like we list some examples in the following lemma.

Lemma 6.1. *The following holds:*

$$H^1(YM, U(YM)) \supset \mathbb{C} + V + \Lambda^2(V), \tag{6.1}$$

$$H^1(YM, YM) \supset \mathbb{C} + \Lambda^2(V). \tag{6.2}$$

Proof. Let us make some simple observations. The algebra $U(YM)$ is graded, hence we have a derivation corresponding to the grading. This explains the $\mathbb{C}$ in (6.1).

The linear space spanned by the relations (2.7) of YM is invariant with respect to the action of the group of symmetries of the bilinear form $(\cdot,\cdot)$. This Lie group coincides with $O(n)$ and has the Lie algebra $\mathfrak{so}_n$. The adjoint representation of $\mathfrak{so}_n$ coincides with $\Lambda^2(V)$. This explains the last summand in (6.1).

The formulas $D_s(v_t) = \delta_{st}$ define derivations of $U(YM)$. The linear space generated by D_s is equipped with the $O(n)$-action. As an $O(n)$-representation this space is isomorphic to the fundamental representation V. The derivation D_s does not preserve the Lie algebra YM inside $U(YM)$. This explains why V is not present in (6.2).

A linear combination of these derivations cannot belong to $\mathrm{Inn}(U(YM))$ because it would have degree strictly less than one, the degree of the generators v_s. □

From now on it will be convenient to consider the cases $\dim(V) \geq 3$ and $\dim(V) = 2$ separately.

6.1. A generic *YM* algebra. We assume throughout this section that $\dim(V) \geq 3$. This entire section is devoted to the proof of the following result.

Proposition 6.2. *The center and the Lie algebra of outer derivations of $U(YM)$ and of YM are given in the next formulas:*

$$\begin{aligned}
\mathrm{Out}(U(YM)) &= H^{1,\mathbf{-1}}(YM, U(YM)) + H^{1,\mathbf{0}}(YM, U(YM)),\\
H^{1,\mathbf{0}}(YM, U(YM)) &\cong \Lambda^2(V) + \mathbb{C},\ H^{1,\mathbf{-1}}(YM, U(YM)) \cong V,\\
\mathrm{Out}(U(YM)) &= H^{1,\mathbf{0}}(YM, YM),\\
H^{1,\mathbf{0}}(YM, YM) &\cong \mathbb{C} + \Lambda^2(V),\\
Z(U(YM)) &= H^{1,\mathbf{0}}(YM, U(YM)), \cong \mathbb{C}\\
Z(YM) &= 0.
\end{aligned}$$

Recall that for calculating the cohomology of *YM* with coefficients in the adjoint module $U(YM)$ we use the isomorphic module $\mathrm{Sym}(YM)$. So

$$\begin{aligned}
Z(U(YM)) &\cong H^0(YM, \mathrm{Sym}(YM)),\\
Z(YM) &\cong H^0(YM, YM),\\
\mathrm{Out}(U(YM)) &\cong H^1(YM, \mathrm{Sym}(YM)),\\
\mathrm{Out}(YM) &\cong H^1(YM, YM).
\end{aligned}$$

We use the spectral sequence (1.14) to compute these groups. Our prime interest will be the fragment E_r^{ij}, $i + j = 0, 1$, because it is the only part contributing to $H^k(YM, \mathrm{Sym}(YM))$, $k = 0, 1$.

The results of the previous section provide us with the necessary information about E_1^{ij}. They are reformulated in the next two statements in a more convenient form.

Lemma 6.3. *The following isomorphisms hold:*

(1) $H^1(YM, \mathrm{Sym}^i(TYM)) = 0$, *$i \geq 2$, and* $H^1(YM, \mathrm{Sym}^i(TYM)) \cong V$, $i = 0, 1$.

(2) $H^0(YM, \mathrm{Sym}^i(TYM)) = 0$, $i \geq 1$.

Proof. Set

$$\boldsymbol{F}_{ij} = [v_i, v_j] \in YM. \tag{6.3}$$

The cocycles $\sum_s \boldsymbol{F}_{ks} \otimes v^{*s}$, $k = 1, \ldots, n$, of the complex $C^1(\mathrm{Sym}(TYM))$ span $H^1(YM, \mathrm{Sym}(TYM))$ by Proposition 5.14. The latter is a direct summand of $\mathrm{Sym}(TYM)$. From this we conclude that $H^1(YM, \mathrm{Sym}^j(TYM)) = 0$, $j \geq 2$. The case $i = 0$ is obvious. The statement about $H^0(YM, \mathrm{Sym}^j(TYM))$ follows from Proposition 5.14. $\square$

This lemma implies that $E_1^{j,1-j} = 0$ for $j \geq 2$ in the spectral sequence (1.14).

Corollary 6.4. *The group* $H^1(YM, \mathrm{Sym}^i(V) \otimes \mathrm{Sym}^k(TYM))$ *is equal to* $\mathrm{Sym}^i(V) \otimes V$ *if* $k = 0, 1$, *and is equal to zero if* $k > 1$.

Also $H^0(YM, \mathrm{Sym}(V))$ *is isomorphic to* $\mathrm{Sym}(V)$ (*this is obvious*).

Proof. This follows from (6.3). □

We would like to compute the differential

$$d\colon E_1^{i,j} \to E_1^{i+1,j}$$

in the range $i + j \leq 2$. We introduce a special notation for the segments

$$E^{0,0} \to E^{1,0} \to E^{2,0} \quad \text{and} \quad E^{0,1} \to E^{1,1}$$

of $E_1^{i,j}$ that takes into account the isomorphism (1.11):

$$\begin{aligned} &H^0(YM, \mathrm{Sym}^k(V)) \xrightarrow{d_{\mathrm{I}}^k} H^1(YM, \mathrm{Sym}^{k-1}(V) \otimes TYM) \\ &\xrightarrow{d_{\mathrm{II}}^{k-1}} H^2(YM), \mathrm{Sym}^{k-2}(V) \otimes \mathrm{Sym}^2(TYM)) \end{aligned} \tag{6.4}$$

and

$$H^1(YM, \mathrm{Sym}^k(V)) \xrightarrow{d_{\mathrm{III}}^k} H^2(YM), \mathrm{Sym}^{k-1}(V) \otimes TYM). \tag{6.5}$$

The analogs of groups that appeared in (6.4), (6.5) with higher symmetric powers of TYM are not present in E_1-term because they vanish (Lemma 6.3).

Information about the differentials d_{I}^k, d_{II}^k and d_{III}^k is contained in the extensions

$$\begin{gathered} TYM \otimes \mathrm{Sym}^{k-1}(V) \to Q_k \to \mathrm{Sym}^k(V), \\ Q_k = F^0_{TYM} \cap \mathrm{Sym}^k(YM)/F^2_{TYM} \cap \mathrm{Sym}^k(YM), \end{gathered} \tag{6.6}$$

and

$$\begin{gathered} \mathrm{Sym}^2(TYM) \otimes \mathrm{Sym}^{k-2}(V) \to R_k \to TYM \otimes \mathrm{Sym}^{k-1}(V), \\ R_k = F^1_{TYM} \cap \mathrm{Sym}^k(YM)/F^3_{TYM} \cap \mathrm{Sym}^k(YM), \end{gathered} \tag{6.7}$$

with $F^i_{TYM}(YM)$ as in (1.12).

More precisely d_{I}, d_{II} and d_{III} are the boundary maps

$$\begin{aligned} d_I^k\colon H^0(YM, \mathrm{Sym}^k(V)) &\to H^1(YM, TYM \otimes \mathrm{Sym}^{k-1}(V)), \\ d_{\mathrm{II}}^k\colon H^1(YM, \mathrm{Sym}^k(V)) &\to H^2(YM, TYM \otimes \mathrm{Sym}^{k-1}(V)), \\ d_{\mathrm{III}}^k\colon H^1(YM, TYM \otimes \mathrm{Sym}^k(V)) &\to H^2(YM, \mathrm{Sym}^2(TYM) \otimes \mathrm{Sym}^{k-1}(V)) \end{aligned}$$

in the long exact sequences of cohomology associated with (6.6), (6.7).

Lemma 6.5. *The space $d^1_{\mathrm{III}}(H^1(YM, V \otimes TYM))$ is equal to*

$$\Lambda^2(V) \subset H^{2,\mathbf{2}}(YM, V \otimes TYM).$$

Proof. We already know that $d^1_I\colon H^0(YM, V) \to H^1(YM, TYM)$ is an isomorphism (Lemma 6.3); hence $d^2_{\mathrm{II}}\colon \mathrm{Sym}^2(V) = H^0(YM, \mathrm{Sym}^2(V)) \to H^1(YM, V \otimes TYM)$ is an embedding. In view of Lemma 6.3, $H^1(YM, V \otimes TYM)$ decomposes as $\mathrm{Sym}^2(V) + \Lambda^2(V)$.

Equation $d^1_{\mathrm{III}} d^2_{\mathrm{I}} = 0$ is a consequence of the equation $(d)^2 = 0$ satisfied by the spectral sequence differential. From this the equality $d^1_{\mathrm{III}}|_{\mathrm{Sym}^2(V)} = 0$, with $\mathrm{Sym}^2(V) \subset H^1(YM, V \otimes TYM)$ described above, follows immediately. Thus the map d^1_{III} factors through the projection on $\Lambda^2(V) \subset H^1(YM, V \otimes TYM)$.

Suppose that a cocycle $a \in C^1(TYM \otimes V)$ is defined by the formula

$$a = \sum_{s=1}^{n} (v_1 \bullet \boldsymbol{F}_{2,s} - v_2 \bullet \boldsymbol{F}_{1,s}) \otimes v^{*s}.$$

Let a also denote the lift of a to a cochain in $C^1(R_2)$. Then

$$\begin{aligned} d^2_{II}(a) = &-2 \sum_{s,t=1}^{n} (\boldsymbol{F}_{s,1} \bullet [v_t, \boldsymbol{F}_{s,2}] - \boldsymbol{F}_{s,2} \bullet [v_t, \boldsymbol{F}_{s,1}]) \otimes v_t \\ &+ 4 \sum_{s,t=1}^{n} (\boldsymbol{F}_{s,1} \bullet [v_2, \boldsymbol{F}_{s,t}] - \boldsymbol{F}_{s,2} \bullet [v_1, \boldsymbol{F}_{s,t}]) \otimes v_t. \end{aligned}$$

This is a nontrivial cocycle in $H^{2,\mathbf{2}}(\mathrm{Sym}^2(TYM))$ since $C^{1,\mathbf{2}}(\mathrm{Sym}^2(TYM)) = 0$. □

We would like to generalize the statement of Lemma 6.5. As in the case $k = 2$ we prove that the map $d^k_{\mathrm{I}}\colon H^0(YM, \mathrm{Sym}^k(V)) \to H^1(YM, \mathrm{Sym}^{k-1}(V) \otimes TYM)$ is an embedding. We interpret d^k_{I} as the de Rham differential d_{DR}, mapping polynomial functions to polynomial 1-forms, via the isomorphisms $H^0(YM, \mathrm{Sym}^k(V)) \cong \mathrm{Sym}^k(V)$ and $H^1(YM, \mathrm{Sym}^{k-1}(V) \otimes TYM) \cong \mathrm{Sym}^{k-1}(V) \otimes V$.

With $a \in H^0(YM, \mathrm{Sym}^m(V))$, $b \in H^1(YM, \mathrm{Sym}^k(V) \otimes TYM)$ the differential d_{II} satisfies

$$d_{\mathrm{II}}(ab) = d_{\mathrm{II}}(a)b + a d_{\mathrm{II}}(b)$$

for suitable m, k. This is one of the standard properties of the differential in the spectral sequence defined by a multiplicative filtration.

The proof of the following lemma is left to the reader.

Lemma 6.6. *Let $d_{\mathrm{II}}\colon \mathrm{Sym}(V) \otimes \Lambda^1(V) \to \mathrm{Sym}(V) \otimes \Lambda^2(V)$ be a linear map such that $d_{\mathrm{II}}(ab) = d_{\mathrm{DR}} ab + a d_{\mathrm{II}} b$, $a \in \mathrm{Sym}(V)$, $d_{\mathrm{II}} d_{\mathrm{DR}} = 0$ and $d_{\mathrm{II}}(x_i d_{\mathrm{DR}} x_j) = d_{\mathrm{DR}} x_i \wedge d_{\mathrm{DR}} x_j$. Then $d_{\mathrm{II}} = d_{\mathrm{DR}}$.*

Proposition 6.7. *The complex* (6.7) *is acyclic in the middle term. The zeroth cohomology is equal to* $\mathbb{C}$.

Proof. Using Lemma 6.5 we identify $\Lambda^2(V)$ with a subspace of $H^{2,\mathbf{2}}(YM, V \otimes TYM)$. The subcomplex of (6.4)

$$H^0(YM,\mathrm{Sym}^k(V)) \xrightarrow{d_{\mathrm{I}}^k} H^1(YM,\mathrm{Sym}^{k-1}(V)\otimes TYM) \xrightarrow{d_{\mathrm{II}}^{k-1}} \Lambda^2(V)\otimes \mathrm{Sym}^{k-2}(V)$$

can be interpreted as a truncated polynomial De Rham complex on the space V, which is acyclic in the middle term. □

We can rephrase the statement of this proposition as that no subquotients of $E_1^{1,0}$ contribute to $H^1(YM, \mathrm{Sym}^k(YM))$; also only $H^0(YM, \mathbb{C})$ from all subquotients contributes $H^0(YM, U(YM))$. This observation proves the following.

Corollary 6.8. $H^0(YM, U(YM)) \cong \mathbb{C}$.

An alternative spot in the E_1-term of (1.14) that can potentially contribute to $H^1(YM, \mathrm{Sym}^n(YM))$ is $E_1^{0,1} = H^1(YM, \mathrm{Sym}^n(V))$. We proceed with the analysis of this remaining case.

Lemma 6.9. *The boundary map* $d_{\mathrm{I}}^1\colon H^1(YM, V) \to H^2(YM, TYM)$ *has its image equal to* $\mathrm{Sym}^2(V)/\mathbb{C}q$ ($\mathbb{C}q$ *is a linear subspace spanned by the tensor* $q \in \mathrm{Sym}^2(V)$).

Proof. The proof follows from direct inspection of the complex $C^\bullet(Q_1)$ and is conceptually similar to the proof of Lemma 6.5. □

Using the Leibniz rule and Lemma 6.9 we factor the differential d_{III} (6.5) into a composition

$$\begin{aligned} H^1(YM, \mathrm{Sym}^k(V)) \cong \mathrm{Sym}^n(V)\otimes V &\xrightarrow{d_{\mathrm{DR}}\otimes 1} \mathrm{Sym}^{k-1}(V)\otimes V^{\otimes 2} \\ &\xrightarrow{1\otimes p} \mathrm{Sym}^{n-1}(V)\otimes \mathrm{Sym}^2(V)/\mathbb{C}q \subset H^2(YM, \mathrm{Sym}^{k-1}(V)\otimes TYM). \end{aligned} \tag{6.8}$$

The map p is the obvious projection $p\colon V^{\otimes 2} \to \mathrm{Sym}^2(V)/\mathbb{C}q$. The differential d_{III} has the following geometric interpretation. The linear space $\mathrm{Sym}(V)\otimes V$ can be identified with the space of polynomial vector fields on $\mathbb{C}^n$, and the space $\mathrm{Sym}(V)\otimes \mathrm{Sym}^2(V)/\mathbb{C}q$ with the space of polynomial traceless (with respect to the metric g defined by the bilinear form $(\cdot,\cdot)$) symmetric two-tensors. In this setup $d_{\mathrm{III}}(\xi)$, $\xi \in \mathrm{Sym}(V)\otimes V$, is the traceless part of the Lie derivative $L_\xi g_q$.

Lemma 6.10. *The kernel of the map* d_{III} *in* (6.8) *is the Lie algebra of the conformal Lie group.*

Proof. The statement about Ker d_{III} can be considered as a definition of the mentioned Lie algebra. If $\dim(V) \geq 3$ then the following vector fields form a basis of Ker d_{III}: $\frac{\partial}{\partial x_i}$-shifts, $x_i \frac{\partial}{\partial x_j} - x_j \frac{\partial}{\partial x_i}$-rotations, $\sum_{i=1}^n (x_k x_i \frac{\partial}{\partial x_i} - 1/2 x_i^2 \frac{\partial}{\partial x_k}) = \sum_{s=1}^n a_{ks} \frac{\partial}{\partial x_s}$-conformal vector fields, $\sum_{i=1}^n x_i \frac{\partial}{\partial x_i}$-dilation. □

This lemma fits into the framework of our spectral sequence as follows. The term $E_2^{0,1}$ of the spectral sequence is equal to the Lie algebra of the conformal group. All other entries of $E_2^{i,j}$ with $i + j = 1$ are equal to zero. From this we conclude that $H^1(YM, \mathrm{Sym}(YM))$ is a subspace of the conformal algebra.

We already know that translations, rotations and dilations are symmetries of YM; they represent nontrivial cocycles in $H^1(YM, \mathrm{Sym}(YM))$ and survive all the spectral sequence differentials. A possibility to construct similar cocycles for conformal vector fields is explored in the remaining part of the proof.

First we associate with a conformal vector field $\sum_{s=1}^n a_{ks} \frac{\partial}{\partial x_s}$ the 1-cocycle $a = \sum_{s=1}^n a_{ks} \otimes v^{*s} \in C^1(\mathrm{Sym}^2(V))$; for example

$$v^{*i}, \tag{6.9}$$

$$x_i \otimes v^{*j} - x_j \otimes v^{*i}, \tag{6.10}$$

$$\sum_{s=1}^n x_s v^{*s} \tag{6.11}$$

are the cocycles corresponding to shifts, rotations and dilations. Let d_2 denote the differential in E_2. The fact that $d_2(a) = 0$ means that the cocycle can be lifted to a cocycle with values in $E = F^0/F^2$. We would like to know if it is possible to lift it to a cocycle with values in $\mathrm{Sym}^2(YM)$. To do that it is sufficient to evaluate on a the boundary operator $\delta_{\mathrm{Sym}^2(YM)}$ associated with the extension

$$\mathrm{Sym}^2(TYM) \to \mathrm{Sym}^2(YM) \to Q_2.$$

Lemma 6.11. *The conformal vector field a cannot be lifted to a* 1*-cocycle in* $C^{\bullet}(\mathrm{Sym}^2(YM))$.

Proof. Otherwise the cohomology class of $\delta_{\mathrm{Sym}^2(YM)}(a)$ would be trivial.

On the other hand explicit computation shows that $\delta_{\mathrm{Sym}^2(YM)}(a)$ is equal to

$$\sum_{s,t=1}^n 4[v_s, v_k] \bullet [v_s, v_t] \otimes v_t - [v_s, v_i] \bullet [v_s, v_i] \otimes v_k \in C^{2,\mathbf{1}}(\mathrm{Sym}^2(TYM)).$$

This is a nonzero cocycle. It cannot be a coboundary because $C^{1,\mathbf{1}}(\mathrm{Sym}^2(TYM))$ is zero. We conclude that $\delta_{\mathrm{Sym}^2(YM)}(a)$ is a nontrivial cohomology class. □

The cocycles (6.9), (6.10), (6.11) can be lifted to cocycles in $C^1(\mathrm{Sym}(YM))$. To do this it suffices to make a substitution $x_s \to v_s$ in the formulas (6.9), (6.10), (6.11).

This completes the proof of Proposition 6.2

6.2. The YM_2 algebra. Throughout this section the dimension of the vector space V is two. Let YM_2 denote the YM algebra in this context.

Set $\boldsymbol{F} = \boldsymbol{F}_{1,2}$, with $\boldsymbol{F}_{1,2}$ as in (6.3). The Lie algebra YM_2 is isomorphic to the classical Heisenberg algebra, a nontrivial central extension of the two-dimensional abelian Lie algebra. This follows easily from the relations $[v_1, \boldsymbol{F}] = [v_2, \boldsymbol{F}] = 0$.

The close relation of $U(YM_2)$ to the algebra $\mathrm{Diff}(\mathbb{R})$ of differential operators with polynomial coefficients on the real line is not very surprising. This is the content of the following lemma.

Proposition 6.12. *There is an embedding* $\psi\colon U(YM_2) \to \mathrm{Diff}(\mathbb{R}) \otimes \mathbb{C}[h]$ *defined by* $\psi(v_1) = x$, $\psi(v_2) = h\frac{\partial}{\partial x}$.

Proof. The degree in v_2 defines an increasing filtration of the space $U(YM_2)$. The algebra $\mathrm{Diff}(\mathbb{R}) \otimes \mathbb{C}[h]$ is equipped with a similar filtration by degree in $\frac{\partial}{\partial x}$. The map ψ is compatible with the filtrations. The sums of adjoint quotients of these filtrations are the polynomial algebras $\mathbb{C}[x_1, x_2, F]$ and $\mathbb{C}[x, y, h]$. The map between them defined by $x_1 \to x$, $x_2 \to yh$, $F \to h$ is obviously an embedding. □

The following is a corollary of the proposition we have just proved.

Proposition 6.13. *Any element of the center of* $U(YM_2)$ *is a polynomial in the variable* $\boldsymbol{F}$. *Moreover,* $\psi(\boldsymbol{F}) = h$.

Let us describe some derivations of $U(YM_2)$.

The algebra $U(YM_2)$ is equipped with derivation eu corresponding to the grading. Fix a central element $P(\boldsymbol{F})$. A new derivation can be constructed by multiplying eu with $P(\boldsymbol{F})$. We shall call it a derivation of the first kind.

Let a be an element of $U(YM_2)$. It is easy to check using the embedding ψ that $[a, v_1]$ and $[a, v_2]$ are divisible on $\boldsymbol{F}$. Define a derivation ∂_a of $U(YM_2)$ by the formula $\partial_a(x) = [a, x]/\boldsymbol{F}$. We shall call it a derivation of the second kind.

Outer derivations of $U(YM_2)$ are characterised as follows.

Proposition 6.14.

$$\mathrm{HH}^1(U(YM_2), U(YM_2)) \cong H^1(YM_2, U(YM_2)) \cong \mathbb{C}[x_1, x_2]/\mathbb{C} + \mathbb{C}[h].$$

Proof. Let ∂ be a derivation of $U(YM_2)$. Subtracting from ∂ a suitable derivation of the first kind we will make $\partial\boldsymbol{F}$ a constant.

The space of derivations splits according to eigenvalue decomposition with respect to the operator eu: $\mathrm{Der}(U(YM)) \to \mathrm{Der}(U(YM))$, defined by $\mathrm{eu}(\partial) = [\mathrm{eu}, \partial]$. We call eigenvectors of eu homogeneous derivations and the corresponding eigenvalues their degree.

It is easy to see that the minimal possible degree of a derivation is equal to minus one, whereas a homogeneous derivation ∂, with $\partial \boldsymbol{F}$ being equal to a nontrivial constant, has degree minus two. Hence the constant is zero.

We conclude that every derivation can be decomposed into a sum of an $\boldsymbol{F}$-linear derivation and a derivation $P(\boldsymbol{F})\mathrm{eu}$ of the first kind.

It is easy to see that an $\boldsymbol{F}$-linear derivation $\partial_{U(YM_2)}$ can be extended to an h-linear derivation ∂ of $\mathrm{Diff}(\mathbb{R}) \otimes \mathbb{C}[h, h^{-1}]$ via the embedding ψ.

The equality $[x, \partial(h\frac{\partial}{\partial x})] = [h\frac{\partial}{\partial x}, \partial(x)]$, which follows from the commutation relations in $\mathrm{Diff}(\mathbb{R}) \otimes \mathbb{C}[h, h^{-1}]$, implies that there is an element $a \in \mathrm{Diff}(\mathbb{R}) \otimes \mathbb{C}[h, h^{-1}]$ such that $\partial(y) = [a, y]$, $y \in \mathrm{Diff}(\mathbb{R}) \otimes \mathbb{C}[h, h^{-1}]$. The inclusion $[a, \mathrm{Im}\,\psi] \subset \mathrm{Im}\,\psi$ implies that a can be chosen to be an element of $h^{-1}\,\mathrm{Im}\,\psi$. So $\partial_{U(YM_2)}$ is a derivation of the second kind.

Consider a short exact sequence of YM_2-modules

$$U(YM_2) \xrightarrow{h} U(YM_2) \to U(YM_2)/(F) = \mathbb{C}[x_1, x_2]. \tag{6.12}$$

The action of YM_2 on $\mathbb{C}[x_1, x_2]$ is trivial. The sequence (6.12) gives rise to a long exact cohomology sequence

$$\begin{aligned} 0 \to Z(U(YM_2)) &\xrightarrow{h} Z(U(YM_2)) \to \mathbb{C}[x_1, x_2] \\ \to \mathrm{Out}(U(YM_2)) &\xrightarrow{F} \mathrm{Out}(U(YM_2)) \to V \otimes \mathbb{C}[x_1, x_2] \to \cdots . \end{aligned}$$

From this we conclude that the space of nonequivalent derivations of the second kind is isomorphic to $\mathbb{C}[x_1, x_2]/\mathbb{C}$. □

7. Proofs of the main theorems

Proof of Theorem 1.1. Recall that $U(YM)(t)$ is given by the formula (1.7); also $j(U(YM)) = 4$ (Proposition 2.8).

The equalities

$$\begin{aligned} \chi\overline{\mathrm{HC}}(U(YM))(t) &= -\sum_{k\geq 1} \ln(1 - nt^k + nt^{3k} - t^{4k})\tfrac{\phi(k)}{k}, \\ Z(U(YM))(t) &= 1, \\ \mathrm{Out}(U(YM))(t) &= \frac{n(n-1)}{2} + 1 + \frac{n}{t} \end{aligned}$$

follow from Theorem 2.2 and Proposition 6.2. Then Proposition 2.4 yields Theorem 1.1.

Proof of Theorem 1.2. The generating functions

$$Z(U(YM_2))(t) = \frac{1}{1-t^2},$$
$$\mathrm{Out}(U(YM_2))(t) = \frac{1}{(1-t)^2} - 1 + \frac{1}{1-t^2}$$

can be easily found from Propositions 6.13 and 6.14.

To compute $\chi\overline{\mathrm{HC}}(U(YM_2))(t)$ explicitly one can use the following considerations. The series $\chi\overline{\mathrm{HC}}(U(YM_2))(t)$ depends only upon the formal function $U(YM_2)(t)$. Then for a computation of $\chi\overline{\mathrm{HC}}(U(YM_2))(t)$ we are free to choose any A, as long as $A(t) = U(YM_2)(t)$. One of the options is to take $A = \mathbb{C}[x_1, x_2, F]$, with $\deg(x_1) = \deg(x_2) = 1, \deg(F) = 2$. A simple application of Corollary 2.3 and the Hochschild–Kostant–Rosenberg theorem [13] yields

$$\chi\overline{\mathrm{HC}}(U(YM_2))(t) = \chi\overline{\mathrm{HC}}(A)(t) = \frac{(3t+2)t}{1-t^2}.$$

From this using Proposition 2.4 we arrive at the formulas (1.4) and (1.5). This finishes the proof of Theorem 1.2.

A. Cyclic homology of a multi graded free algebra

The formulas presented in this section are not new and should be well known to specialists. They are hard to locate in print or online and are presented here for the readers' convenience.

A positively multi-graded vector space $W = \bigoplus W_\alpha$ has Poincaré series $W(z) = W(z_1, \dots, z_k) = \sum_\alpha \dim(W_\alpha) z^\alpha$. For a multi-index $\alpha = (i_1, \dots, i_k)$ we define $|\alpha| = \sum_{j=1}^k i_j$.

Let $T(W)$ be a free algebra on W and let $\mathrm{Cyc}(W)$ denote the linear space spanned by cyclic words over the alphabet make up of elements of some basis of W. While performing cyclic identification of words the sign rule must be applied. There is an obvious isomorphism $\mathrm{Cyc}(W) \cong T(W)/[T(W), T(W)]$, where $[T(W), T(W)]$ the linear space spanned by the graded commutators. The linear space $\overline{\mathrm{Cyc}}(W)$ by definition is equal to $\mathrm{Cyc}(W)/\mathbb{C}$, this is the reduced version of $\mathrm{Cyc}(W)$.

Theorem A.1. *The reduced cyclic homology groups* $\overline{\mathrm{HC}}_i(T(W))$ *are equal to zero for* $i > 0$ *and*

$$\begin{aligned}\overline{\mathrm{HC}}_0(T(W))(z_1,\dots,z_k) &= \overline{\mathrm{Cyc}}(W)(z_1,\dots,z_k)\\ &= -\sum_{t\geq 0}\tfrac{\phi(t)}{t}\ln(1-W((-1)^{t+1}z_1^t,\dots,(-1)^{t+1}z_k^t))\end{aligned}\tag{A.1}$$

where $\phi(t)$ *is as in* (1.3).

Proof. See [16]. □

In the following part of this section we discuss some applications of formula (A.1).

Let A be a positively graded associative algebra with $\bar{A} = \bigoplus_{i\geq 1} A_i$. The isomorphism $\overline{\mathrm{HC}}_\bullet(A) \cong \mathrm{HC}_\bullet(\bar{A})$ is proved in [13]. The Connes complex $C_\bullet^\lambda \bar{A}$ [13] is isomorphic to $\overline{\mathrm{Cyc}}(\bar{A}[-1])$. In this context the linear space $\bar{A}$ is bi-graded: it carries both the original and the cohomological gradation $c\deg$ (by definition $c\deg(\bar{A}) = 1$). The whole space $\overline{\mathrm{Cyc}}(\bar{A}[-1])$ inherits the bi-grading. This enables us to define $\overline{\mathrm{Cyc}}(\bar{A}[-1])(z_1,z_2)$. The generating function for Euler characteristics $\chi\,\mathrm{HC}_\bullet(\bar{A})(z)$ is equal to $\overline{\mathrm{Cyc}}(\bar{A}[-1])(z,-1)$.

Now we are in a position to apply formula (A.1). Since $\bar{A}(z_1,z_2) = (A(z_1)-1)z_2$, after simple manipulations we conclude that

$$\overline{\mathrm{Cyc}}(\bar{A}[-1])(z,-1) = -\sum_{n\geq 1}\ln A((-1)^{n+1}z^n)\tfrac{\phi(n)}{n}.$$

We have just proved the following.

Proposition A.2. *Let A be a positively graded algebra. Then*

$$\chi\overline{\mathrm{HC}}(A)(z) = -\sum_{n\geq 1}\ln A((-1)^{n+1}z^n)\tfrac{\phi(n)}{n}.\tag{A.2}$$

Proof of Theorem 2.2. The grading of an associative algebra is an additional structure on which cyclic homology depends.

In the statement of Proposition A.2 it is assumed that the algebra A belongs to the tensor category $\mathsf{Vect}^{\mathbb{Z}}$ of $\mathbb{Z}$-graded linear spaces with commutativity morphism $R\colon W_1\otimes W_2 \to W_2\otimes W_1$ satisfying

$$R(w_1\otimes w_2) = (-1)^{\deg(w_1)\deg(w_2)}w_2\otimes w_1.\tag{A.3}$$

If we wish for no $\pm$ sign to appear in (A.3) (the algebra A in Theorem 2.2 is an even object), this can be achieved by scaling the grading of A by the factor of two. In abstract terms this is the same as to say that the category of graded vector spaces with trivial commutativity morphism is the subcategory of $\mathsf{Vect}^{\mathbb{Z}}$ formed by even objects. This results in the substitution $A(t) \to A(t^2)$ in the generating function and eliminates all $\pm$ signs occurring in formula (A.2). After this the inverse substitution $\chi\overline{\mathrm{HC}}(T(V), d)(t) \to \chi\overline{\mathrm{HC}}(T(V), d)(\sqrt{t})$ yields the generating function we are looking for.

B. Proof of Theorem 4.1

First we compute the Lie algebra homology of TYM. We choose to follow an indirect approach.

One of the examples of representations of YM is in $\mathrm{Sym}(V)$, which we will now describe.

We will follow convention of Section 4.1 where the symbols x_s, $s = 1, \dots, n$, denote the generators of the abelianization $YM_{\mathrm{ab}} = V$.

Notice that $\mathrm{Sym}(V) \cong U(YM) \otimes_{U(TYM)} \mathbb{C}$. This isomorphism means that $\mathrm{Sym}(V)$ is induced from the trivial TYM-module.

Consider the homology $H_{\bullet}(YM, \mathrm{Sym}(V))$.

Lemma B.1. $H_{\bullet,j}(YM, \mathrm{Sym}(V))[3] = H_{\bullet,j}(TYM, \mathbb{C})$.

Proof. The module $\mathrm{Sym}(V)$ is induced from the trivial representation of TYM. The result follows from the Shapiro lemma. □

Let q denote the quadric (4.12). The tensor q is the inverse to the bilinear form used in the definition of the algebra YM. We introduce the following notation.

Definition B.2. Let N be a $\mathrm{Sym}(V)$-module.

$$\begin{aligned}
\mathrm{Ann}(N) &= \{m \in N \mid am = 0 \text{ for } a \in \textstyle\bigoplus_{k\geq 1} \mathrm{Sym}^k(V)\},\\
Z(N) &= \{m_i \in N,\ i = 1, \dots, n \mid \textstyle\sum_{s=1}^{n} x_k x_s m_s = q m_k\},\\
B(N) &= \{m_i = x_i m \mid m \in N\}, \text{ it is easy to see that } B(N) \subset Z(N),\\
H(N) &= Z(N)/B(N).
\end{aligned}$$

Define a map $\nu\colon H(N) \to \mathrm{Ann}(N/qN)$ by the formula

$$\nu\colon m_1, \dots, m_n \to \sum_{s=1}^{n} s x_s m_s. \tag{B.1}$$

The next lemma is self-evident.

Lemma B.3. *There are isomorphisms* $H^0(C(N)) \cong \mathrm{Ann}(N)$*,* $H^1(C(N)) \cong H(N)$*,* $H^3(C(N)) \cong N \otimes_{\mathrm{Sym}(V)} \mathbb{C}$*. In the last formula* $\mathbb{C} = \mathrm{Sym}^0(V)$ *is a* $\mathrm{Sym}(V)$*-module via the standard augmentation* ε*.*

Lemma B.4. *If the map "multiplication on* q*" in* N *has no kernel then the map* (B.1) *is an isomorphism.*

Proof. We check first that the map (B.1) is correctly defined. Since $x_k \sum_{s=1}^n x_s m_s = q m_k \in qN$ the element $\sum_{s=1}^n x_s m_s$ belongs to $\mathrm{Ann}(N/qN)$. The array (m_s) such that $m_s = x_s n$ maps via ν into $qn \in qN$ is identically zero in $\mathrm{Ann}(N/qN)$.

Suppose that $m \in \mathrm{Ann}(N/qN)$. Then by definition of $\mathrm{Ann}(N/qN)$ we can find elements $m_s \in N$ such that $x_s m = q m_s$. This implies that $\sum_{s=1}^n x_s q m_s = qm$ and $q\nu(m_1, \ldots, m_n) = qm$. Assume that multiplication by q has a trivial kernel. Then $\nu(m_1, \ldots, m_n) = m$. This proves that map ν is surjective.

Suppose that $\nu(m_1, \ldots, m_n) = 0 \in \mathrm{Ann}(N/qN)$. This means that there is an element $m \in N$ such that $qm = \sum_{s=1}^n x_s m_s$. By assumption $x_k \sum_{s=1}^n x_s m_s = q m_k$. Hence we have $x_k q m = q m_k$. After dividing by q we see that $(m_1, \ldots, m_n)$ is the trivial element in $H(N)$. □

It is easy to interpret the group $\mathrm{Ann}(N)$ in terms of the Koszul complex $C_\bullet(V, N)$. A direct inspection shows that the map $m \to m\varsigma_1 \wedge \cdots \wedge \varsigma_n$ defines an isomorphism of groups $\mathrm{Ann}(N) \cong H_n(V, N)$.

Lemma B.5. *Suppose the map "multiplication by* q*" in the module* N *has no kernel and that* $H_n(V, N) = H_{n-1}(V, N) = 0$*. Then the group* $H(N)$ *is trivial.*

Proof. The short exact sequence $0 \to N \xrightarrow{q} N \to N/qN \to 0$ defines a long exact sequence in homology, whose terminal segment is of interest to us:

$$0 \to H_n(V, N) \to H_n(V, N) \to H_n(V, N/qN) \to H_{n-1}(V, N) \to \cdots .$$

The group $H_n(V, N/qN)$ is trivial because both $H_{n-1}(V, N)$ and $H_{n-1}(V, N)$ are. Together with Lemma B.4, this implies that $H(N) = 0$. □

The algebra $\mathrm{Sym}(V)$ has no zero divisors, so $\mathrm{Ann}(\mathrm{Sym}(V))) = 0$. The module $\mathrm{Sym}(V)$ is cyclic, hence $\mathrm{Sym}(V) \otimes_{\mathrm{Sym}(V)} \mathbb{C} = \mathbb{C}$.

The homology $H_\bullet(V, \mathrm{Sym}(V))$ is trivial because the module of coefficients is free. If $\dim(V) \geq 2$ then the conditions of Lemma B.5 are met, so $H(\mathrm{Sym}(V)) = 0$. By Lemma B.3 the complex $C^\bullet(\mathrm{Sym}(V))$ has cohomology in degree 2 and 3. In degree 3 it is one-dimensional. Denote the two-dimensional cohomology by M. From Proposition 2.8 it follows that $H_{0,\mathbf{0}}(TYM, \mathbb{C}) = \mathbb{C}$ and $H_{1,\boldsymbol{j}}(TYM, \mathbb{C}) = M_{\boldsymbol{j}}$.

The combination of these observations gives a proof of Proposition 4.3. The natural action of YM on the homology $H_\bullet(YM, \mathrm{Sym}(V))$ is trivial. This enables us to identify $x_s m \otimes l_1 \wedge \cdots \wedge l_k$ with $\sum_i m \otimes l_1 \wedge \cdots \wedge [v_s, l_i] \wedge \cdots \wedge l_k$.

It is easy to compute $M(t)$ using the generating function for Euler characteristics of graded components of $C^\bullet(\mathrm{Sym}(V))$. Indeed,

$$t^{-4}(M(t)-1) = \frac{1-nt^{-1}+nt^{-3}-t^{-4}}{(1-t)^n} = \chi C^\bullet(\mathrm{Sym}(V))(t).$$

We use the equality $\mathrm{Sym}(V)(t) = \frac{1}{(1-t)^n}$. The free algebra $T(M)$ has Poincaré series $T(M)(t) = \frac{1}{1-M(t)} = \frac{(1-t)^n}{1-nt+nt^3-t^4}$.

The isomorphism of graded spaces $U(YM) \cong \mathrm{Sym}(V) \otimes U(TYM)$ and formula (1.7) imply that $U(TYM)(t) = \frac{(1-t)^n}{1-nt+nt^3-t^4}$.

The surjective graded map (4.1) is an isomorphism since $T(M)(t) = U(TYM)(t)$.

C. The graded Heisenberg Lie algebra

The algebra (4.11) is Koszul (see for example [3]). A resolution of the trivial module $\mathbb{C}$ can be constructed by means of a quite general and powerful theory of such algebras [2]. We have tried to avoid the general theory as much as possible so as to keep this paper self-contained. Thus we give an ad hoc construction of the resolution in the setup of the algebra A (4.11).

Definition C.1. The graded Heisenberg algebra Heis has two components $\mathrm{Heis}_{-1} + \mathrm{Heis}_{-2}$. The odd Heis_{-1} has a basis $\langle \xi^1, \dots, \xi^n \rangle$, the even Heis_{-2} is spanned by a generator l. The commutation relations are $[\xi^i, \xi^j] = \delta^{ij} l$.

The algebra Heis admits two commuting actions L, R by derivations on the graded commutative algebra $\Lambda(V) \otimes \mathbb{C}[h] \cong \Lambda(\varsigma_1, \dots, \varsigma_n) \otimes \mathbb{C}[h]$:

$$L(\xi^s) = \frac{\partial}{\partial \varsigma_s} + \varsigma_s \frac{\partial}{\partial h}, \qquad L(l) = 2\frac{\partial}{\partial h},$$
$$R(\xi^s) = -\frac{\partial}{\partial \varsigma_s} + \varsigma_s \frac{\partial}{\partial h}, \quad R(l) = -2\frac{\partial}{\partial h}.$$

Consider the tensor product $A \otimes \Lambda(V) \otimes \mathbb{C}[h]$, and introduce operators $d_1 = \sum_{s=1}^n x_s L(\xi^s)$, $d_2 = \sum_{s=1}^n x_s R(\xi^s)$. Define a multiplicative bi-grading on $A \otimes \Lambda(V) \otimes \mathbb{C}[h] = \bigoplus_{i,\boldsymbol{j}} A \otimes \Lambda(V) \otimes \mathbb{C}[h]_{i,\boldsymbol{j}}$ as follows: assign to generators x_s the bi-degree $(0,1)$, to ς_s the bi-degree $(1,1)$, and to h the bi-degree $(2,2)$. It is easy to see that $A \otimes \Lambda(V) \otimes \mathbb{C}[h]$ decomposes into a direct sum of subcomplexes $A \otimes \Lambda(V) \otimes \mathbb{C}[h]_{\boldsymbol{j}} = \bigoplus_i A \otimes \Lambda(V) \otimes \mathbb{C}[h]_{i,\boldsymbol{j}}$.

Proposition C.2. *The operators d_i satisfy $d_i^2 = 0$, $i = 1, 2$. The complexes $(A \otimes \Lambda(V) \otimes \mathbb{C}[h], d_i)$, $i = 1, 2$, have only one nontrivial one-dimensional homology group in degree zero.*

Proof. We leave to the reader a verification of the equation $d_i^2 = 0$.

The universal enveloping algebra $U(\mathrm{Heis}) = \bigoplus_{i \le 0} U(\mathrm{Heis})_i$ has the structure of a graded co-commutative Hopf algebra. The values of the diagonal on algebraic generators are equal to $\Delta(\xi^s) = \xi^s \otimes 1 + 1 \otimes \xi^s$. By our definition the dual $U(\mathrm{Heis})^* := \bigoplus_{i \ge 0} U(\mathrm{Heis})_i^*$ with $U(\mathrm{Heis})_{-i}^* = (U(\mathrm{Heis})_i)^*$ has the structure of a graded commutative algebra, induced by Δ. As an associative algebra $U(\mathrm{Heis})^*$ is isomorphic to $\Lambda(V) \otimes \mathbb{C}[h]$.

The algebra $U(\mathrm{Heis})$ is a free right module over itself. Its graded dual $U(\mathrm{Heis})^*$ is a left $U(\mathrm{Heis})$-module. Consider the chain complex $C^\bullet(\mathrm{Heis}, U(\mathrm{Heis})^*)$. The module $U(\mathrm{Heis})^*$ is co-induced. It has cohomology equal to $\mathbb{C}$ because of the Shapiro lemma.

It will be convenient during the proof to use a modified cohomological grading in $A \otimes \Lambda(V) \otimes \mathbb{C}[h]$. The degrees of h, ς_s remain unchanged, but the degree of x^s becomes equal to two.

There is a map of differential graded algebras

$$\psi\colon (C^\bullet(\mathrm{Heis}, U(\mathrm{Heis})^*), d) \to (A \otimes \Lambda(V) \otimes \mathbb{C}[h], d_1).$$

The map ψ establishes the obvious isomorphism of $U(\mathrm{Heis})^* \subset C^\bullet(\mathrm{Heis}, U(\mathrm{Heis})^*)$ with $\Lambda(V) \otimes \mathbb{C}[h]$. Also ψ surjects $\mathrm{Sym}(\mathrm{Heis}^*[-1]) \subset C^\bullet(\mathrm{Heis}, U(\mathrm{Heis})^*)$ onto A.

The reader should check the compatibility of this map with the differentials.

Let us introduce a filtration $F^j(U(\mathrm{Heis})^*) = \bigoplus_{j \ge i} U(\mathrm{Heis})_j^*$ and a similar filtration on $\Lambda(V) \otimes \mathbb{C}[h]$. The map ψ is compatible with the filtrations and induces a map of spectral sequences. It is straightforward to check that ψ defines an isomorphism of E_1-terms. It is easy to construct a decomposition of $C^\bullet(\mathrm{Heis}, U(\mathrm{Heis})^*) = \bigoplus_{\boldsymbol{k}} C^\bullet(\mathrm{Heis}, U(\mathrm{Heis})^*)_{\boldsymbol{k}}$ into a sum of finite-dimensional complexes in complete analogy with the decomposition of $A \otimes \Lambda(V) \otimes \mathbb{C}[h]$. The filtrations agree with the direct sum decomposition. The pair of spectral sequences converges in both cases because they can be reduced to spectral sequences defined by finite filtrations. This proves the proposition for $(A \otimes \Lambda(V) \otimes \mathbb{C}[h], d_1)$.

One can construct a right $U(\mathrm{Heis})$-module using the left multiplication in $U(\mathrm{Heis})$ composed with the antipodal map. From this we get a map of differential graded algebras $\psi'\colon (C^\bullet(\mathrm{Heis}, U(\mathrm{Heis})^*), d') \to (A \otimes \Lambda(V) \otimes \mathbb{C}[h], d_2)$. To establish the desired homological properties of $(A \otimes \Lambda(V) \otimes \mathbb{C}[h], d_2)$, we can use the same technique. $\square$

Suppose that N is a finitely generated graded A-module. Define a differential on $A \otimes N \otimes \Lambda(V) \otimes \mathbb{C}[h]$ as the sum of two differentials $d_1 = \sum_{s=1}^n (x_s \otimes 1) L(\xi^s)$, $d_2 = \sum_{s=1}^n (1 \otimes x_s) R(\xi^s)$. The complex $A \otimes N \otimes \Lambda(V) \otimes \mathbb{C}[h]$ is assembled from free A-modules.

Proposition C.3. *Suppose that a graded A-module N satisfies $N = \bigoplus_{i \ge k} N_i$ and $\dim(N_i) < \infty$. Then the complex $A \otimes N \otimes \Lambda(V) \otimes \mathbb{C}[h]$ is a free resolution of N.*

Proof. Define a filtration on N by $F^j(N) = \bigoplus_{i \geq j} N_i$. Consider the filtration of the complex defined by $F^j(A \otimes N \otimes \Lambda(V) \otimes \mathbb{C}[h]) = A \otimes F^j(N) \otimes \Lambda(V) \otimes \mathbb{C}[h]$. From Proposition C.2 we deduce that the E_1-term of the associated spectral sequence is isomorphic to N (placed in zero degree).

The complex $A \otimes N \otimes \Lambda(V) \otimes \mathbb{C}[h] = \bigoplus_{\boldsymbol{k}} A \otimes N \otimes \Lambda(V) \otimes \mathbb{C}[h]_{\boldsymbol{k}}$ (constructed in the same way as for $A \otimes \Lambda(V) \otimes \mathbb{C}[h]$) decomposes into a sum of finite-dimensional subcomplexes compatible with the filtration; hence we do not have to deal with convergence. □

Corollary C.4. *The cohomology of the complex*

$$C_\bullet(V, \langle h \rangle, N_1 \otimes N_2) := N_1 \otimes N_2 \otimes \Lambda(V) \otimes \mathbb{C}[h] \tag{C.1}$$

is equal to $\mathrm{Tor}^A_\bullet(N_1, N_2)$ *if the modules* N_1, N_2 *satisfy the finiteness properties that have been formulated in Proposition C.3.*

Proof. This follows from definition of Tor and Proposition C.3. □

D. Proof of Proposition 5.6

In this section we assume that all A-modules satisfy the conditions of Proposition C.3.

We use the complex $C_\bullet(V, \langle h \rangle, N_1 \otimes N_2)$ from (C.1) for the computation of $\mathrm{Tor}^A_{i,\boldsymbol{j}}(N_1, N_2)$. $C_\bullet(V, N_1 \otimes N_2)$ is a subcomplex of $C_\bullet(V, \langle h \rangle, N_1 \otimes N_2)$. Together they define a short exact sequence of complexes:

$$0 \to C_\bullet(V, N_1 \otimes N_2) \xrightarrow{\iota} C_\bullet(V, \langle h \rangle, N_1 \otimes N_2) \xrightarrow{h^{-1}} C_\bullet(V, \langle h \rangle, N_1 \otimes N_2)[-2] \to 0. \tag{D.1}$$

In the last formula we denote the embedding map by ι.

Lemma D.1. *For any A-modules N_1, N_2 there is a long exact sequence*

$$\begin{aligned}\cdots \to \mathrm{Tor}^A_{i+1,\boldsymbol{j}}(N_1, N_2) \to \mathrm{Tor}^A_{i-1,\boldsymbol{j}-\boldsymbol{2}}(N_1, N_2) \xrightarrow{\delta} \mathrm{Tor}_{i,\boldsymbol{j}}(N_1, N_2)\\ \xrightarrow{\iota} \mathrm{Tor}^A_{i,\boldsymbol{j}}(N_1, N_2) \to \mathrm{Tor}^A_{i-2,\boldsymbol{j}-\boldsymbol{2}}(N_1, N_2) \to \cdots\end{aligned}$$

with the terminal segment

$$\begin{aligned}\cdots \to \mathrm{Tor}^A_{2,\boldsymbol{j}}(N_1, N_2) \to \mathrm{Tor}^A_{0,\boldsymbol{j}-\boldsymbol{2}}(N_1, N_2)\\ \xrightarrow{\delta} \mathrm{Tor}_{1,\boldsymbol{j}}(N_1, N_2) \xrightarrow{\iota} \mathrm{Tor}^A_{1,\boldsymbol{j}}(N_1, N_2) \to 0,\end{aligned} \tag{D.2}$$

$$\mathrm{Tor}_{0,\boldsymbol{j}}(N_1, N_2) = \mathrm{Tor}^A_{0,\boldsymbol{j}}(N_1, N_2).$$

As usual $\mathrm{Tor}_{i,\boldsymbol{j}}(N_1, N_2)$ *stands for the i-th* Tor *functor over* $\mathrm{Sym}(V)$.

Proof. This is the long exact sequence of cohomology associated with the short exact sequence (D.1). □

Lemma D.1 has also a proof that uses the language of derived categories, which the interested reader may try to reconstruct.

Lemma D.2. *With the notations of Lemma* 5.5, *we have* $\kappa(a) = \delta \circ \iota(a)$ *for any cycle* $a \in C_i(V, N_1 \otimes N_2)$.

Proof. The complex $C_\bullet(V, \langle h\rangle, A \otimes A)$ contains an element h such that $dh = \kappa \in C_1(V, A \otimes A)$. The proof readily follows from this observation. □

Lemma D.3. $\mathrm{Coker}_j \cong \mathrm{Tor}^A_{1,j}(M, M)$.

Proof. The module M satisfies conditions of Proposition C.3. The proof follows from (D.2) and Lemma D.2. □

Lemma D.4. *Let* N *be an* A*-module. Then there is an isomorphism*

$$\mathrm{Tor}^A_{i,j}(M, N) \cong \mathrm{Tor}^A_{i+2,j}(\mathbb{C}, N) \quad \textit{for } i \geq 2.$$

If $H^n(V, N) = 0$ *then the above isomorphism holds for* $i \geq 1$.

Proof. The complex (4.13) can be used to define an exact triangle

$$\mathbb{C}[-2] \to M \to \widetilde{C}^\bullet[1] \to \mathbb{C}[-1]$$

in the (bounded) derived category of graded A-modules. If we apply the functor of tensor multiplication by N and pass to hypercohomology then the triangle will give us a long exact sequence of Tor^A groups.

The complex $\widetilde{C}^\bullet$ consists of free A-modules, adapted in the derived sense to the functor $K^\bullet \Rightarrow K^\bullet \otimes^L_A N$ (L stands for left derived functor). Thus the hypercohomology of the complex $(\widetilde{C}^\bullet) \otimes^L_A N$ coincides with $H^\bullet(\widetilde{C} \otimes_A N)$.

The claim of the lemma for $i \geq 2$ follows from the long exact sequence for the functor Tor because $H^j(\widetilde{C}^\bullet \otimes N) = 0$ for $j \leq -1$.

If $H^n(V, N) = 0$ then the map $N \to N \otimes V$, which is defined by the formula $n \to \sum_{s=1}^n x_s n \otimes v_s$, has no kernel. Thus $H^0(\widetilde{C}^\bullet \otimes N)$ is zero. This is sufficient for the extension of the previous result to the case $i \geq 1$. □

Corollary D.5. *There is an isomorphism* $\mathrm{Tor}^A_{1,j}(M, M) \cong \mathrm{Tor}^A_{5,j}(\mathbb{C}, \mathbb{C})$.

Proof. The isomorphisms $\mathrm{Tor}^A_{1,j}(M, M) \cong \mathrm{Tor}^A_{3,j}(\mathbb{C}, M) \cong \mathrm{Tor}^A_{5,j}(\mathbb{C}, \mathbb{C})$ follow from Lemma D.4. □

Lemma D.6. *There is an isomorphism* $\mathrm{Tor}^A_{i,\boldsymbol{i}}(\mathbb{C},\mathbb{C}) \cong \bigoplus_{k\geq 0} \Lambda^{i-2k}(V^*)$. *Also* $\mathrm{Tor}^A_{i,\boldsymbol{j}}(\mathbb{C},\mathbb{C}) = 0$ *for* $i \neq j$.

Proof. Use $C_\bullet(V, \langle h\rangle, \mathbb{C}\otimes\mathbb{C}) \cong \Lambda(V)\otimes\mathbb{C}[h]$ with zero differential for the computation of $\mathrm{Tor}^A_{i,\boldsymbol{i}}(\mathbb{C},\mathbb{C})$. □

Corollary D.7. $\mathrm{Tor}_{1,\boldsymbol{5}}(M,M) \cong \bigoplus_{k\geq 0}(\Lambda^{5-2k}(V)h^k)^*$.

Proof. The proof follows from Lemma D.6, the long exact sequence (D.2) and the isomorphism $\mathrm{Tor}^A_{0,\boldsymbol{3}}(M,M) = 0$. □

The sequence (5.8) is a part of the long exact sequence (D.2), with $N_1 = N_2 = M$, under the identifications $H_{0,\boldsymbol{j-2}}(V, M\otimes M) \cong \mathrm{Tor}^A_{0,\boldsymbol{j-2}}(M,M)$ and $H_{1,\boldsymbol{j}}(V, M\otimes M) \cong \mathrm{Tor}^A_{1,\boldsymbol{j}}(M,M)$. There is an isomorphism $\mathrm{Coker}_5 \cong \mathrm{Tor}_{1,\boldsymbol{5}}(M,M)$ (because $H_{0,\boldsymbol{3}}(V, M\otimes M) = 0$).

The proof of Proposition 5.6 follows from the results of the previous lemmas.

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Received March 12, 2008; revised April 15, 2008

M. Movshev, Stony Brook University, Stony Brook, NY, 11794-3651, U.S.A.
E-mail: mmovshev@math.sunysb.edu