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Some notes on the Feigin–Losev–Shoikhet integral conjecture

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Abstract. Given a holomorphic vector bundle $\mathcal{E}$ on a smooth connected compact complex manifold X, Feigin, Losev and Shoikhet [FLS] use a notion of completed Hochschild homology $\widehat{\mathrm{HH}}$ of $\mathcal{D}\mathrm{iff}(\mathcal{E})$ such that $\widehat{\mathrm{HH}}_0(\mathcal{D}\mathrm{iff}(\mathcal{E}))$ is isomorphic to $\mathrm{H}^{2n}(X,\mathbb{C})$. On the other hand, they construct a linear functional on $\widehat{\mathrm{HH}}_0(\mathcal{D}\mathrm{iff}(\mathcal{E}))$. This therefore gives rise to a linear functional $I_{\mathcal{E}}$ on $\mathrm{H}^{2n}(X,\mathbb{C})$. They show that this functional is $\int_X$ if $\mathcal{E}$ has non-zero Euler characteristic. They conjecture that this functional is $\int_X$ for all $\mathcal{E}$.

In this article it is proved that $I_{\mathcal{E}} = I_{\mathcal{F}}$ for any pair $(\mathcal{E},\mathcal{F})$ of holomorphic vector bundles on X. In particular, if X has one vector bundle with non-zero Euler characteristic, then $I_{\mathcal{E}} = \int_X$ for every vector bundle $\mathcal{E}$ on X.

In [FLS] there is also used a notion of completed cyclic homology $\widehat{\mathrm{HC}}$ of $\mathcal{D}\mathrm{iff}(\mathcal{E})$ such that $\widehat{\mathrm{HC}}_{-i}(\mathcal{D}\mathrm{iff}(\mathcal{E})) \simeq \mathrm{H}^{2n-i}(X,\mathbb{C}) \oplus \mathrm{H}^{2n-i+2}(X,\mathbb{C}) \oplus \cdots$. The construction yielding $I_{\mathcal{E}}$ generalizes to give linear functionals on $\widehat{\mathrm{HC}}_{-2i}(\mathcal{D}\mathrm{iff}(\mathcal{E}))$ for each $i \geq 0$. The linear functional thus obtained on $\widehat{\mathrm{HC}}_{-2i}(\mathcal{D}\mathrm{iff}(\mathcal{E}))$ yields a linear functional $I_{\mathcal{E},2i,2k}$ on $\mathrm{H}^{2n-2k}(X,\mathbb{C})$ for $0 \leq k \leq i$. It is conjectured in [FLS] that $I_{\mathcal{E},2,0} = \int_X$, and a further conjecture about $I_{\mathcal{E},2,2}$ is made.

In this article we prove that $I_{\mathcal{E},2i,0} = I_{\mathcal{E}}$ for all $i \geq 0$. In particular, if X has at least one vector bundle with non-zero Euler characteristic, then $I_{\mathcal{E},2i,0} = \int_X$. We also prove that $I_{\mathcal{E},2i,2k} = 0$ for $k > 0$. The latter is stronger than what is expected in [FLS] when $i = k = 1$.

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1. Introduction

Let X be a smooth connected compact complex manifold such that $\dim_{\mathbb{C}} X = n$. Let $\mathcal{E}$ be a holomorphic vector bundle on X. Let $\mathcal{D}\mathrm{iff}(\mathcal{E})$ be the sheaf of *holomorphic* differential operators on $\mathcal{E}$. Let $\mathrm{Diff}(\mathcal{E})$ be the algebra of global sections of $\mathcal{D}\mathrm{iff}(\mathcal{E})$. One is interested in understanding the Hochschild homology of $\mathrm{Diff}(\mathcal{E})$. This is, however, not an object that one can easily analyze.

Let $\mathrm{Dolb}^\bullet(X, \mathcal{O}_X)$ denote the Dolbeaux resolution of the sheaf $\mathcal{O}_X$ of holomorphic functions on X. Let $\mathcal{D}\mathrm{iff}^\bullet(\mathcal{E}) = \mathrm{Dolb}^\bullet(X, \mathcal{O}_X) \otimes_{\mathcal{O}_X} \mathcal{D}\mathrm{iff}(\mathcal{E})$. Following [FLS] we replace $\mathrm{Diff}(\mathcal{E})$ by

$$\mathrm{Diff}^\bullet(\mathcal{E}) := \Gamma(X, \mathcal{D}\mathrm{iff}^\bullet(\mathcal{E})).$$

Let $K^\bullet_{\mathcal{E}}$ denote the complex $\Gamma_{C^\infty}(X, \mathrm{Dolb}^\bullet(X, \mathcal{O}_X) \otimes_{\mathcal{O}_X} \mathcal{E})$ of smooth global sections of $\mathrm{Dolb}^\bullet(X, \mathcal{O}_X) \otimes_{\mathcal{O}_X} \mathcal{E}$. This is the Dolbeaux complex of $\mathcal{E}$. If $\mathcal{D}$ is a holomorphic differential operator on $\mathcal{E}$, then $\mathcal{D}$ gives rise to a $\mathbb{C}$-endomorphism of the complex $K^\bullet_{\mathcal{E}}$. It follows that $\mathcal{D}$ induces an endomorphism $\mathcal{D}_*\colon \mathrm{H}^i(X, \mathcal{E}) \to \mathrm{H}^i(X, \mathcal{E})$ for each i. The *supertrace* of $\mathcal{D}$ is given by

$$\mathrm{str}(\mathcal{D}) := \sum_i (-1)^i \, \mathrm{tr}_{\mathrm{H}^i(X,\mathcal{E})} \mathcal{D}_*.$$

Note that $\mathrm{Diff}(\mathcal{E})$ is a subalgebra of $\mathrm{Diff}^\bullet(\mathcal{E})$. It follows that there is a natural map from the i-th Hochschild homology of $\mathrm{Diff}(\mathcal{E})$ to that of $\mathrm{Diff}^\bullet(\mathcal{E})$ for all i. Also recall that if $\mathcal{A}$ is any associative $\mathbb{C}$-algebra concentrated in degree 0 then the 0-th Hochschild homology $\mathrm{HH}_0(\mathcal{A})$ of $\mathcal{A}$ is the Abelianization of $\mathcal{A}$. It follows that any trace on $\mathcal{A}$ is a linear functional on $\mathrm{HH}_0(\mathcal{A})$ and a Hochschild 0-cocycle of A. Note that the supertrace described in the previous paragraph is a Hochschild 0-cocycle of $\mathrm{Diff}(\mathcal{E})$.

In [FLS], a Hochschild 0-cocycle tr on $\mathrm{Diff}^\bullet(\mathcal{E})$ is constructed algebraically. This Hochschild cocycle is constructed in such a way that the following diagram commutes:

$$\begin{array}{ccc} \mathrm{HH}_0(\mathrm{Diff}(\mathcal{E})) & \longrightarrow & \mathrm{HH}_0(\mathrm{Diff}^\bullet(\mathcal{E})) \\ \Big\downarrow{\scriptstyle \mathrm{str}} & & \Big\downarrow{\scriptstyle \mathrm{tr}} \\ \mathbb{C} & \xrightarrow[\mathrm{id}]{} & \mathbb{C}. \end{array}$$

The top row of the above diagram is the natural map from $\mathrm{HH}_0(\mathrm{Diff}(\mathcal{E}))$ to $\mathrm{HH}_0(\mathrm{Diff}^\bullet(\mathcal{E}))$ arising out of the natural inclusion of $\mathrm{Diff}(\mathcal{E})$ as a subalgebra of $\mathrm{Diff}^\bullet(\mathcal{E})$.

The theory that is more related to the cohomology of X is, however, the *completed Hochschild homology* of the sheaf $\mathcal{D}\mathrm{iff}(\mathcal{E})$ of holomorphic differential operators on $\mathcal{E}$. It is defined in Section 3. Let $\widehat{\mathrm{HH}}_i(\mathcal{D}\mathrm{iff}(\mathcal{E}))$ denote the i-th completed Hochschild homology of $\mathcal{D}\mathrm{iff}(\mathcal{E})$. By Lemma 3 we have the following generalization of a theorem of Brylinski [Bryl]:

$$\widehat{\mathrm{HH}}_{-i}(\mathcal{D}\mathrm{iff}(\mathcal{E})) \simeq \mathrm{H}^{2n-i}(X, \mathbb{C}).$$

Call this isomorphism $\beta_{\mathcal{E}}$. This is the key property of $\widehat{\mathrm{HH}}_*(\mathcal{D}\mathrm{iff}(\mathcal{E}))$ that relates it to the cohomology of X with coefficients in $\mathbb{C}$. In [FLS] there is also a notion of completed Hochschild homology that has the same relation with $\mathrm{H}^*(X, \mathbb{C})$. However, as pointed out in [FLS], understanding the isomorphism $\beta_{\mathcal{E}}$ is easier with the definition of completed Hochschild homology used here.

In [FLS] it is shown that the Hochschild cocycle tr of $\mathrm{Diff}^\bullet(\mathcal{E})$ "extends" to yield a linear functional $\widehat{\mathrm{tr}}$ on $\widehat{\mathrm{HH}}_0(\mathcal{D}\mathrm{iff}(\mathcal{E}))$. In this article, we are forced to redo this very step in Section 3 since we use a definition of $\widehat{\mathrm{HH}}_*(\mathcal{D}\mathrm{iff}(\mathcal{E}))$ that is a priori different from that in [FLS]. Our definition of $\widehat{\mathrm{HH}}_*(\mathcal{D}\mathrm{iff}(\mathcal{E}))$ implies that there is a natural map from $\mathrm{HH}_0(\mathrm{Diff}(\mathcal{E}))$ to $\widehat{\mathrm{HH}}_0(\mathcal{D}\mathrm{iff}(\mathcal{E}))$ (see Section 3 for more details). The outcome of our "extending" tr to a linear functional $\widehat{\mathrm{tr}}$ on $\widehat{\mathrm{HH}}_0(\mathcal{D}\mathrm{iff}(\mathcal{E}))$ is the following commutative diagram:

$$\begin{array}{ccc} \mathrm{HH}_0(\mathrm{Diff}(\mathcal{E})) & \longrightarrow & \widehat{\mathrm{HH}}_0(\mathcal{D}\mathrm{iff}(\mathcal{E})) \\ \Big\downarrow{\scriptstyle \mathrm{str}} & & \Big\downarrow{\scriptstyle \widehat{\mathrm{tr}}} \\ \mathbb{C} & \xrightarrow[\mathrm{id}]{} & \mathbb{C}. \end{array}$$

The linear functional we construct on $\widehat{\mathrm{HH}}_0(\mathcal{D}\mathrm{iff}(\mathcal{E}))$ still uses the basic construction of [FLS] and, as will be clear later in this article, yields an algebraic construction of $\int_X$ when X has at least one vector bundle with non-zero Euler characteristic.

Denote the composite map $\widehat{\mathrm{tr}} \circ \beta_{\mathcal{E}}^{-1} \colon \mathrm{H}^{2n}(X, \mathbb{C}) \to \mathbb{C}$ by $I_{\mathcal{E}}$. $I_{\mathcal{E}}$ is a linear functional on $\mathrm{H}^{2n}(X, \mathbb{C})$.

The following theorem was proven in [FLS]. The proof goes through in our setup as well.

Theorem 1 ([FLS]). *If the Euler characteristic $\chi_{\mathcal{E}}$ of $\mathcal{E}$ is non-zero, then $I_{\mathcal{E}} = \int_X$.*

Further, in [FLS], it was conjectured that $I_{\mathcal{E}} = \int_X$ for all vector bundles $\mathcal{E}$, even those of zero Euler characteristic. Here we prove the following theorem.

Theorem 2. *If $\mathcal{E}$ and $\mathcal{G}$ are two vector bundles on X, then $I_{\mathcal{E}} = I_{\mathcal{G}}$.*

The following corollary is immediate from this and Theorem 1.

Corollary 1. *If X has a vector bundle $\mathcal{G}$ whose Euler characteristic $\chi_{\mathcal{G}}$ is non-zero, then $I_{\mathcal{E}} = \int_X$ for any vector bundle $\mathcal{E}$ on X.*

In particular, if X is a smooth separated scheme over $\mathbb{C}$, then the skyscraper sheaf supported at a point p of X has non-zero Euler characteristic. On the other hand, it is a coherent $\mathcal{O}_X$-module. It therefore has a finite resolution by (holomorphic) vector bundles on X. It follows that at least one vector bundle in such a resolution has non-zero Euler characteristic. Thus, if $\mathcal{E}$ is a vector bundle on a smooth compact complex manifold X that is a smooth separated scheme over $\mathbb{C}$, then $I_{\mathcal{E}} = \int_X$.

Note that if $\mathcal{D}$ is a global holomorphic differential operator on a vector bundle $\mathcal{E}$ on X, then $\mathcal{D}$ is a 0-cycle of the Hochschild chain complex of $\mathrm{Diff}(\mathcal{E})$. It follows that

$\mathcal{D}$ yields an element in $\widehat{\mathrm{HH}}_0(\mathcal{D}\mathrm{iff}(\mathcal{E}))$ and hence an element in $\mathrm{H}^{2n}(X,\mathbb{C})$, which will be denoted by $[\mathcal{D}]$. It follows from Corollary 1 that if X admits at least one vector bundle with non-zero Euler characteristic, then

$$\mathrm{str}(\mathcal{D}) = \int_X [\mathcal{D}].$$

The above statement is somewhat similar to a result (Corollary 5.6) in an article by P. Schapira and J.-P. Schneiders [S-S]. More recently, an entirely different approach has yielded a proof of a stronger version of this statement (the supertrace theorem) without the condition that X admits a vector bundle with non-zero Euler characteristic [EnFe]. In a sequel to this article, we shall develop our approach further to show that $I_{\mathcal{E}} = \int_X$ for any vector bundle $\mathcal{E}$ on any compact complex manifold X. This yields another proof of the supertrace theorem different from the one in [EnFe].

One also defines the completed cyclic homology $\widehat{\mathrm{HC}}_*(\mathcal{D}\mathrm{iff}(\mathcal{E}))$ of $\mathcal{D}\mathrm{iff}(\mathcal{E})$. Further, as in [FLS], the construction of the linear functional on the 0-th completed Hochschild homology of $\mathcal{D}\mathrm{iff}(\mathcal{E})$ is extended to the construction of multiple linear functionals $\widehat{\mathrm{tr}}_{2i} : \widehat{\mathrm{HC}}_{-2i}(\mathcal{D}\mathrm{iff}(\mathcal{E})) \to \mathbb{C}$. On the other hand, it is shown that one has an isomorphism

$$\widehat{\mathrm{HC}}_{-2i}(\mathcal{D}\mathrm{iff}(\mathcal{E})) \simeq \mathrm{H}^{2n}(X,\mathbb{C}) \oplus \mathrm{H}^{2n-2}(X,\mathbb{C}) \oplus \cdots \oplus \mathrm{H}^{2n-2i}(X,\mathbb{C}).$$

Denote the inverse of this isomorphism by $\mathcal{I}^i_{\mathcal{E}}$. Let $I_{\mathcal{E},2i} = \widehat{\mathrm{tr}}_{2i} \circ \mathcal{I}^i_{\mathcal{E}}$. Then

$$I_{\mathcal{E},2i} : \mathrm{H}^{2n}(X,\mathbb{C}) \oplus \mathrm{H}^{2n-2}(X,\mathbb{C}) \oplus \cdots \oplus \mathrm{H}^{2n-2i}(X,\mathbb{C}) \to \mathbb{C}.$$

For $0 \leq k \leq i$ let $I_{\mathcal{E},2i,2k} : \mathrm{H}^{2n-2k}(X,\mathbb{C}) \to \mathbb{C}$ be the composition of $I_{\mathcal{E},2i}$ with the inclusion of $\mathrm{H}^{2n-2k}(X,\mathbb{C})$ in $\mathrm{H}^{2n}(X,\mathbb{C}) \oplus \mathrm{H}^{2n-2}(X,\mathbb{C}) \oplus \cdots \oplus \mathrm{H}^{2n-2i}(X,\mathbb{C})$ as a direct summand. In [FLS] it is conjectured that $I_{\mathcal{E},2,0} = C \cdot \int_X$ where C is a constant independent of $\mathcal{E}$. Furthermore, there is a prediction in [FLS] that $I_{\mathcal{E},2,2}$ is the integral over a cycle related to the Chern classes of $\mathcal{E}$ and those of vector bundles intrinsic to X. Here we prove the following theorem.

Theorem 3. *$I_{\mathcal{E},2i,0} = I_{\mathcal{E}}$. Moreover, $I_{\mathcal{E},2i,2k} = 0$ if $k > 0$.*

The latter part is more than what is expected in [FLS], though a piece of circumstantial evidence for $I_{\mathcal{E},2,2}$ to be 0 is provided there. By this theorem and Corollary 1, $I_{\mathcal{E},2i,0} = \int_X$ whenever X has a vector bundle with non-zero Euler characteristic. In particular, this happens when X is a smooth separated scheme over $\mathbb{C}$.

1.1. Outline of the article. The proofs of Theorems 2 and 3 are simple and involve two easy "bookkeeping lemmas" (Lemma 1 and Lemma 4). This is done in Sections 3 and 4, respectively.

Section 2 recalls basic notions about Hochschild homology that we require. In addition, Section 2 recalls the definition of an A_∞-morphism between two differential graded algebras. It shows that an A_∞-morphism $\mathcal{F}$ between two dg algebras $\mathcal{A}$ and $\mathcal{B}$ induces a map $\mathcal{F}_{\text{Hoch}}$ of complexes from the complex of Hochschild chains of $\mathcal{A}$ to that of $\mathcal{B}$ (Proposition 2). None of the material presented in this section is new. Proposition 2 is from [FLS].

The first part of Section 3 recalls the basic set up for this note from [FLS]. In particular, we recall the construction of the Hochschild 0-cocycle tr of $\text{Diff}^\bullet(\mathcal{E})$ from [FLS]. The completed Hochschild homology $\widehat{\text{HH}}_*(\mathcal{D}\text{iff}(\mathcal{E}))$ is defined in Section 3. We show that the formula for the Hochschild cocycle tr "extends" to a formula for a linear functional $\widehat{\text{tr}}$ on $\widehat{\text{HH}}_0(\mathcal{D}\text{iff}(\mathcal{E}))$. In [FLS] something very similar is done. The definition of $\widehat{\text{HH}}_*(\mathcal{D}\text{iff}(\mathcal{E}))$ that is used in this article is however a priori somewhat different from that used in [FLS]. This forces us to redo the parts where in [FLS] the Hochschild cocycle tr of $\text{Diff}^\bullet(\mathcal{E})$ is extended to a linear functional on $\widehat{\text{HH}}_0(\mathcal{D}\text{iff}(\mathcal{E}))$.

The last part of Section 3 is devoted to the "bookkeeping lemma for $\widehat{\text{tr}}$" (Lemma 1).

Section 4 is devoted to understanding the relation between $\widehat{\text{HH}}_*(\mathcal{D}\text{iff}(\mathcal{E}))$ and $\text{H}^*(X, \mathbb{C})$. In particular, we prove that

$$\widehat{\text{HH}}_{-i}(\mathcal{D}\text{iff}(\mathcal{E})) \simeq \text{H}^{2n-i}(X, \mathbb{C})$$

(Corollary 6). For $\mathcal{E} = \mathcal{O}_X$, this was proven by Brylinski [Bryl]. Let $\beta_{\mathcal{E}}$ denote the isomorphism between $\widehat{\text{HH}}_{-i}(\mathcal{D}\text{iff}(\mathcal{E}))$ and $\text{H}^{2n-i}(X, \mathbb{C})$. Section 4 also proves the "bookkeeping lemma for β" (Lemma 4). Theorem 2 is an immediate consequence of Lemma 1 and Lemma 4.

Section 5 has four subsections. The first subsection recalls the basic notions of cyclic homology that we require. In the second subsection we carefully examine the linear functionals on the completed cyclic homology of $\mathcal{D}\text{iff}(\mathcal{E})$. The third subsection analyses the relation between the completed cyclic homology of $\mathcal{D}\text{iff}(\mathcal{E})$ and $\text{H}^*(X, \mathbb{C})$. In the final subsection we write down the final steps of the proof of Theorem 3.

Note. All complexes that appear in this article are cochain complexes by convention. The term *Hochschild cocycle* of a dg algebra shall only refer to cocycles of the complex of Hochschild cochains of that algebra as defined in Definition 2.

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2. Basic results about Hochschild homology

Conventions used in this section. The term "dg algebra" in this section refers to a differential graded $\mathbb{C}$-algebra with unit.

All complexes of $\mathbb{C}$-vector space are cochain complexes, i.e, the differential of any complex of $\mathbb{C}$-vector spaces has degree $+1$.

2.1. The Hochschild chain complex and bar complex of a dg algebra

Definition 1. If $\mathcal{A}$ is a differential graded $\mathbb{C}$-algebra with differential δ, the complex of Hochschild chains $C^\bullet(\mathcal{A})$ is the cochain complex obtained by equipping the graded $\mathbb{C}$-vector space $\bigoplus_{i\geq 1}\mathcal{A}[1]^{\otimes i}[-1]$ with the *Hochschild differential*. The Hochschild differential d is given by the formula

$$\begin{aligned} d(a_0\otimes\cdots\otimes a_n) = &\sum_{i=0}^{i=n-1}(-1)^{(d_0+\cdots+d_i+i+1)}a_0\otimes\cdots\otimes a_ia_{i+1}\otimes\cdots\otimes a_n\\ &+(-1)^{(d_n+1)(d_0+\cdots+d_{n-1}+n-1)}a_na_0\otimes a_1\otimes\cdots\otimes a_{n-1}\\ &+\sum_{j=0}^{j=n}(-1)^{(d_0+\cdots+d_{j-1}+j)}a_0\otimes\cdots\otimes\delta(a_j)\otimes\cdots\otimes a_n \end{aligned}$$

for homogenous elements $a_0,\ldots,a_n$ of $\mathcal{A}$ of degrees $d_0,\ldots,d_n$, respectively.

Let $\tau_n\colon\mathcal{A}^{\otimes n}\to\mathcal{A}^{\otimes n}$ denote the endomorphism given by the formula

$$a_1\otimes\cdots\otimes a_n\rightsquigarrow(-1)^{(d_n+1)(d_1+\cdots+d_{n-1}+n-1)}a_n\otimes a_1\otimes\cdots\otimes a_{n-1}$$

for homogenous elements $a_1,\ldots,a_n$ of $\mathcal{A}$ of degrees $d_1,\ldots,d_n$, respectively. Let $\partial_1\colon\mathcal{A}^{\otimes n}\to\mathcal{A}^{\otimes n-1}$ denote the morphism given by the formula

$$a_1\otimes\cdots\otimes a_n\rightsquigarrow(-1)^{d_1+1}a_1a_2\otimes\cdots\otimes a_n$$

for homogenous elements $a_1,\ldots,a_n$ of $\mathcal{A}$ of degrees $d_1,\ldots,d_n$, respectively. Then the Hochschild differential on $\mathcal{A}^{\otimes k}$ is also given by the formula

$$\sum_{i=1}^{i=k}\tau_{k-1}^{i-1}\circ\partial_1\circ\tau_k^{k-i+1}+\sum_{i=1}^{i=k}\tau_k^{i-1}\circ(\delta\otimes\mathrm{id}\otimes\cdots\otimes\mathrm{id})\circ\tau_k^{k-i+1}.$$

Contrary to the standard practice, we will refer to a degree $-n$ cocycle of $C^\bullet(\mathcal{A})$ as a *Hochschild n-cycle of* $\mathcal{A}$.

The Hochschild homology $\mathrm{HH}_{-n}(\mathcal{A})$ of $\mathcal{A}$ is the $-n$-th cohomology of the complex $C^\bullet(\mathcal{A})$. By the definition of the Hochschild complex, it is clear that if $\mathcal{A}$ is a $\mathbb{C}$-algebra (i.e., $\mathcal{A}$ is concentrated in degree 0), then $\mathrm{HH}_0(\mathcal{A}) = \frac{\mathcal{A}}{[\mathcal{A},\mathcal{A}]}$ where $[\mathcal{A},\mathcal{A}]$ is the commutator of $\mathcal{A}$. Also, if $\mathcal{A}$ is a $\mathbb{C}$-algebra, then $\mathrm{HH}_{-n}(\mathcal{A}) = 0$ if $n < 0$. And, if $\mathcal{A}$ is dg algebra, then elements of $\ker(\delta) \subset \mathcal{A}^0$ are 0-Hochschild cycles of $C^\bullet(\mathcal{A})$ where δ denotes the internal differential on $\mathcal{A}$. It follows that there is a canonical map $\ker(\delta) \to \mathrm{HH}_0(\mathcal{A})$.

Definition 2. The *complex of Hochschild cochains* of $\mathcal{A}$ is the complex whose $-i$-th term is $\mathrm{Hom}_{\mathbb{C}}(C^i(\mathcal{A}), \mathbb{C})$ with differential induced by that on $C^\bullet(\mathcal{A})$. A *Hochschild cocycle* of $\mathcal{A}$ is a cocycle of the complex of Hochschild cochains of $\mathcal{A}$. Note that a Hochschild 0-cocycle of $\mathcal{A}$ induces a $\mathbb{C}$-linear functional on $\mathrm{HH}_0(\mathcal{A})$.

Definition 3. The *bar complex* $\mathrm{bar}^\bullet(\mathcal{A})$ of $\mathcal{A}$ is the cochain complex obtained by equipping the graded $\mathbb{C}$-vector space $\bigoplus_{i\geq 1} \mathcal{A}[1]^{\otimes i}[-1]$ with the *bar differential*. The bar differential is given by the formula

$$d(a_0 \otimes \cdots \otimes a_n) = \sum_{i=0}^{i=n-1} (-1)^{(d_0+\cdots+d_i+i+1)} a_0 \otimes \cdots \otimes a_i a_{i+1} \otimes \cdots \otimes a_n + \sum_{j=0}^{j=n} (-1)^{(d_0+\cdots+d_{j-1}+j)} a_0 \otimes \cdots \otimes \delta(a_j) \otimes \cdots \otimes a_n$$

for homogenous elements $a_0, \ldots, a_n$ of $\mathcal{A}$ of degrees $d_0, \ldots, d_n$, respectively.

The bar differential on $\mathcal{A}^{\otimes k}$ is also given by the formula

$$\sum_{i=1}^{i=k-1} \tau_{k-1}^{i-1} \circ \partial_1 \circ \tau_k^{k-i+1} + \sum_{i=1}^{i=k} \tau_k^{i-1} \circ (\delta \otimes \mathrm{id} \otimes \cdots \otimes \mathrm{id}) \circ \tau_k^{k-i+1}.$$

We recall that if $s_n \colon \mathrm{bar}^\bullet(\mathcal{A}) \to \mathrm{bar}^{\bullet-1}(\mathcal{A})$ is the map $a_0 \otimes \cdots \otimes a_n \rightsquigarrow -1 \otimes a_0 \otimes \cdots \otimes a_n$ then the maps $\{s_n\}$, $n \geq -1$, give a homotopy between $\mathrm{id} \colon \mathrm{bar}^\bullet(\mathcal{A}) \to \mathrm{bar}^\bullet(\mathcal{A})$ and $0 \colon \mathrm{bar}^\bullet(\mathcal{A}) \to \mathrm{bar}^\bullet(\mathcal{A})$. It follows that $\mathrm{bar}^\bullet(\mathcal{A})$ is acyclic.

If $V^\bullet$ is a finite dimensional graded $\mathbb{C}$-vector space then $\mathrm{End}(V^\bullet) = \bigoplus_{i,j} \mathrm{Hom}(V^i, V^j)$. Let $\pi_{i,j}$ denote the projection from $\mathrm{End}(V^\bullet)$ to $\mathrm{Hom}(V^i, V^j)$. If $M \in \mathrm{End}(V^\bullet)$, the supertrace of M is the alternating sum $\sum_i (-1)^i \mathrm{tr}(\pi_{i,i}(M))$. We recall the following proposition from [FLS].

Proposition 1. *Let $V^\bullet$ be a finite dimensional graded $\mathbb{C}$-vector space with zero differential. The the following holds:*

(1) $\mathrm{HH}_i(\mathrm{End}(V^\bullet)) = 0$ *for* $i \neq 0$.

(2) $\mathrm{HH}_0(\mathrm{End}(V^\bullet)) \simeq \mathbb{C}$.

(3) *The composite* $\mathrm{End}(V^\bullet)^0 \to \mathrm{HH}_0(\mathrm{End}(V^\bullet)) \to \mathbb{C}$ *takes an element of* $\mathrm{End}(V^\bullet)^0$ *to its supertrace.*

(4) *The isomorphism* $\mathrm{HH}_0(\mathrm{End}(V^\bullet)) \simeq \mathbb{C}$ *takes the class in* $\mathrm{HH}_0(\mathrm{End}(V^\bullet))$ *of a Hochschild* 0*-cycle in* $\mathrm{End}(V^\bullet)^{\otimes k}$ *to* 0 *for all* $k \geq 2$.

Proof. The proof is a trivial modification of the proof of the Morita invariance of Hochschild homology for matrices in Loday [2] (see Theorem 1.2.4 of [2]). □

We may therefore denote the isomorphism $\mathrm{HH}_0(\mathrm{End}(V^\bullet)) \simeq \mathbb{C}$ by str.

2.2. A_∞-morphisms between dg algebras. Let $\mathcal{A}$ and $\mathcal{B}$ be two dg algebras. Let $\mathcal{F}_i : \mathcal{A}^{\otimes i} \to \mathcal{B}[n_i]$ be $\mathbb{C}$-linear maps with $n_i \in \mathbb{Z}$ for each i. If $k_1, \ldots, k_l$ are positive integers such that $\sum_j k_j = k$ and if $\sum_j n_j = n$, then $\mathcal{F}_{k_1} \boxtimes \cdots \boxtimes \mathcal{F}_{k_l}$ will denote the $\mathbb{C}$-linear map from $\mathcal{A}^{\otimes k}$ to $\mathcal{B}^{\otimes l}[n]$ such that

$$\begin{aligned}\mathcal{F}_{k_1} \boxtimes \cdots \boxtimes \mathcal{F}_{k_l}&(a_1 \otimes \cdots \otimes a_k)\\ &= \mathcal{F}_{k_1}(a_1 \otimes \cdots \otimes a_{k_1}) \otimes \cdots \otimes \mathcal{F}_{k_l}(a_{k_1+\cdots+k_{l-1}+1} \otimes \cdots \otimes a_k).\end{aligned}$$

Definition 4. An A_∞*-morphism* $\mathcal{F}$ from a dg algebra $\mathcal{A}$ to a dg algebra $\mathcal{B}$ is a collection of maps

$$\mathcal{F}_k : \mathcal{A}^{\otimes k} \to \mathcal{B}[1-k]$$

for all $k \geq 1$ such that the map $\mathcal{F}_{\mathrm{bar}} : \mathrm{bar}^\bullet(\mathcal{A}) \to \mathrm{bar}^\bullet(\mathcal{B})$ defined by

$$\mathcal{F}_{\mathrm{bar}}(a_1 \otimes \cdots \otimes a_k) = \sum_{\substack{(k_1,\ldots,k_l)\\ l>0,\ \sum_j k_j = k}} \mathcal{F}_{k_1} \boxtimes \cdots \boxtimes \mathcal{F}_{k_l}(a_1 \otimes \cdots \otimes a_k)$$

is a morphism of complexes from $\mathrm{bar}^\bullet(\mathcal{A})$ to $\mathrm{bar}^\bullet(\mathcal{B})$. The maps $\mathcal{F}_k$ are called the *Taylor components* of $\mathcal{F}$.

The condition that $\mathcal{F}_{\mathrm{bar}}$ commutes with the differentials on the bar complexes of $\mathcal{A}$ and $\mathcal{B}$ respectively is equivalent to the condition that the maps $\mathcal{F}_k$ satisfy the following relations:

$$\begin{aligned}&\pm \mathcal{F}_{k-1}(a_1 \cdot a_2 \otimes a_3 \otimes \cdots \otimes a_k) \mp \mathcal{F}_{k-1}(a_1 \otimes a_2 \cdot a_3 \otimes \cdots \otimes a_k)\\ &\quad \pm \mathcal{F}_{k-1}(a_1 \otimes a_3 \otimes \cdots \otimes a_{k-1} \cdot a_k) \pm \mathcal{F}_k(\delta(a_1 \otimes a_2 \otimes \cdots \otimes a_k))\\ &\quad \pm \delta(\mathcal{F}_k(a_1 \otimes \cdots \otimes a_k))\\ &\quad \pm \sum_{l=1}^{l=k-1} \pm \mathcal{F}_l(a_1 \otimes \cdots \otimes a_l) \circ \mathcal{F}_{k-l}(a_{l+1} \otimes \cdots \otimes a_k) = 0, \quad k \geq 1.\end{aligned}$$

We recall the following proposition from [FLS].

Proposition 2. *An A_∞-morphism $\mathcal{F}$ from an associative dg algebra $\mathcal{A}$ to an associative dg algebra $\mathcal{B}$ induces a map $\mathcal{F}_{\mathrm{Hoch}}$ of complexes from the $C^\bullet(\mathcal{A})$ to $C^\bullet(\mathcal{B})$.*

We recall the proof of this proposition from [FLS].

Proof. Denote by $\tau\colon \mathcal{A}^{\otimes k} \to \mathcal{A}^{\otimes k}$ the map which takes $a_1 \otimes \cdots \otimes a_k$ to $(-1)^{(d_k+1)(d_1+\cdots+d_{k-1}+k-1)} a_k \otimes a_1 \otimes \cdots \otimes a_{k-1}$ for homogenous elements $a_1, \ldots, a_k$ of $\mathcal{A}$ of degrees $d_1, \ldots, d_k$, respectively. Consider the map $\mathcal{F}_{\mathrm{Hoch}}\colon C^\bullet(\mathcal{A}) \to C^\bullet(\mathcal{B})$ defined by the formula

$$\mathcal{F}_{\mathrm{Hoch}}(a_1 \otimes \cdots \otimes a_k) = \sum_{\substack{(k_1,\ldots,k_l),\\ l>0,\ \sum_j k_j = k}} [\, \mathcal{F}_{k_1} \boxtimes \cdots \boxtimes \mathcal{F}_{k_l}(a_1 \otimes \cdots \otimes a_k) + \sum_{j=1}^{j=k_l-1} \mathcal{F}_{k_l} \boxtimes \mathcal{F}_{k_1} \boxtimes \cdots \boxtimes \mathcal{F}_{k_{l-1}}(\tau^j(a_1 \otimes \cdots \otimes a_k))\,].$$

We leave the verification that $\mathcal{F}_{\mathrm{Hoch}}$ respects the Hochschild differential to the reader. □

Let $\mathcal{A}$ be a dg algebra. Let $V^\bullet$ be a finite dimensional graded $\mathbb{C}$-vector space with 0 differential. Suppose that $\mathcal{F}$ is an A_∞-morphism from $\mathcal{A}$ to $\mathrm{End}(V^\bullet)$ with Taylor components $\mathcal{F}_k$. Recall that for any $k > 0$, we have a map $\tau\colon \mathcal{A}^{\otimes k} \to \mathcal{A}^{\otimes k}$ such that $\tau(a_1 \otimes \cdots \otimes a_k) = (-1)^{(d_k+1)(d_1+\cdots+d_{k-1}+k-1)} a_k \otimes a_1 \otimes \cdots \otimes a_{k-1}$ for homogenous elements $a_1, \ldots, a_k$ of $\mathcal{A}$ of degrees $d_1, \ldots, d_k$, respectively. By Proposition 1, str is a Hochschild 0-cocycle of $\mathrm{End}(V^\bullet)$. We now have the following corollary of Proposition 2.

Corollary 2. *The supertrace on* $\mathrm{End}(V^\bullet)$ *pulls back to a Hochschild* 0*-cocycle* tr *of* $\mathcal{A}$*. On Hochschild* 0*-cycles of* $\mathcal{A}$ *that arise from elements on degree* $k-1$ *in* $\mathcal{A}^{\otimes k}$*, the Hochschild cocycle* tr *is given by the map from* $\mathcal{A}^{\otimes k}$ *to* $\mathbb{C}$ *defined by*

$$a_1 \otimes \cdots \otimes a_k \rightsquigarrow \sum_{j=0}^{j=k-1} \mathrm{str}(\mathcal{F}_k(\tau^j(a_1 \otimes \cdots \otimes a_k))).$$

Proof. The Hochschild cocycle tr is given by $\mathrm{tr}(x) = \mathrm{str}(\mathcal{F}_{\mathrm{Hoch}}(x))$ for any $x \in C^0(\mathcal{A})$. Note that by Proposition 2, $\mathcal{F}_{\mathrm{Hoch}}(x) \in C^0(\mathrm{End}(V^\bullet))$. The exact formula for tr given in this corollary is now immediate from the formula for $\mathcal{F}_{\mathrm{Hoch}}$ given in the proof of Proposition 2. □

3. The completed Hochschild homology of $\mathcal{D}\mathbf{iff}(\mathcal{E})$ and a linear functional on $\widehat{\mathbf{HH}_0}(\mathcal{D}\mathbf{iff}(\mathcal{E}))$

3.1. The basic construction in [FLS]. Let $K^\bullet_{\mathcal{E}}$ denote the Dolbeaux complex of $\mathcal{E}$, as in the introduction to this article. Then $K^\bullet_{\mathcal{E}}$ decomposes as the direct sum of a complex with zero differential and an acyclic complex, i.e., $K^\bullet_{\mathcal{E}} = K^\bullet_{0\mathcal{E}} \oplus K^\bullet_{1\mathcal{E}}$ where $\mathrm{H}^\bullet(K^\bullet_{0\mathcal{E}}) = \mathrm{H}^\bullet(K^\bullet_{\mathcal{E}})$, $K^\bullet_{0\mathcal{E}}$ has 0 differential and $K^\bullet_{1\mathcal{E}}$ is acyclic. This is a consequence of Hodge theory (see, for instance, Theorem 5.24 in [Vois]).

Recall that $\mathcal{D}\mathrm{iff}^\bullet(\mathcal{E}) = \mathrm{Dolb}^\bullet(X, \mathcal{O}_X) \otimes_{\mathcal{O}_X} \mathcal{D}\mathrm{iff}(\mathcal{E})$ and that

$$\mathrm{Diff}^\bullet(\mathcal{E}) := \Gamma(X, \mathcal{D}\mathrm{iff}^\bullet(\mathcal{E})).$$

The main construction in [FLS] *is that of an* A_∞*-morphism* $\mathcal{F}$ *from* $\mathrm{Diff}^\bullet(\mathcal{E})$ *to* $\mathrm{End}(K^\bullet_{0\mathcal{E}})$. Note that $K^\bullet_{0\mathcal{E}}$ is a finite dimensional $\mathbb{C}$-vector space with 0 differential. As $\mathbb{C}$-vector spaces, $\mathrm{K}^i_{0\mathcal{E}} \simeq \mathrm{H}^i(X, \mathcal{E})$. We may therefore apply Propositions 1, 2 and Corollary 2, with $\mathcal{A} = \mathrm{Diff}^\bullet(\mathcal{E})$ and $\mathcal{B} = \mathrm{End}(K^\bullet_{0\mathcal{E}})$. We obtain the following facts immediately.

Fact 1. By Proposition 1, $\mathrm{HH}_i(\mathrm{End}(K^\bullet_{0\mathcal{E}})) = 0$ for $i \neq 0$ and $\mathrm{HH}_0(\mathrm{End}(K^\bullet_{0\mathcal{E}})) \simeq \mathbb{C}$. This isomorphism is induced by the map taking a degree 0 element of $\mathrm{End}(K^\bullet_{0\mathcal{E}})$ to its supertrace.

Fact 2. By Proposition 2, the supertrace on $\mathrm{End}(K^\bullet_{0\mathcal{E}})$ pulls back to a Hochschild 0-cocycle tr on $\mathrm{Diff}^\bullet(\mathcal{E})$. If D is a Hochschild 0-cycle of $\mathrm{Diff}^\bullet(\mathcal{E})$ that arises out of a degree $k-1$ element of $\mathrm{Diff}^\bullet(\mathcal{E})^{\otimes k}$, and if $[D]$ denotes the class of D in $\mathrm{HH}_0(\mathrm{Diff}^\bullet(\mathcal{E}))$, then

$$\mathrm{tr}([D]) = \sum_{j=0}^{j=k-1} \mathrm{str}(\mathcal{F}_k(\tau^j(D))).$$

Notation. We shall also denote the map

$$D \mapsto \sum_{j=0}^{j=k-1} \mathrm{str}(\mathcal{F}_k(\tau^j(D)))$$

by I_{FLS}.

3.1.1. Construction of $\mathcal{F}$. We now recall the construction of $\mathcal{F}$ from [FLS]. Let C_k denote the configuration space $\{t_1 < \cdots < t_k \mid t_i \in \mathbb{R}\}/G^{(1)}$ where $G^{(1)}$ is the one-dimensional group of shifts $(t_1, \ldots, t_k) \to (t_1 + c, \ldots, t_k + c)$. This is a smooth $(k-1)$-dimensional manifold though it is *not* compact if $k \geq 2$. Let $\tau_i = t_{i+1} - t_i$ for $1 \leq i \leq k-1$. The map $(t_1, \ldots, t_k) \rightsquigarrow (\tau_1, \ldots, \tau_{k-1})$ is a diffeomorphism between C_k and the product $\prod_{i=1}^{i=k-1}\{\tau_i > 0\}$. Let $\overline{\{\tau_i > 0\}}$ denote the compactification of

$\{\tau_i \geq 0\}$ by a point at infinity. The cube $\overline{C_k} := \prod_{i=1}^{i=k-1} \overline{\{\tau_i > 0\}}$ is a compactification of C_k.

Let $D = D_1 \otimes \cdots \otimes D_k \in \mathrm{Diff}^\bullet(\mathcal{E})^{\otimes k}$. Recall that the D_i yield endomorphisms of $K^\bullet_{\mathcal{E}}$ as follows: Let $U \subset X$ be an open ball on which $\mathcal{E}$ is trivial. Let $z_1, \ldots, z_n$ be local holomorphic coordinates on U. A section of $K^\bullet_{\mathcal{E}}|_U$ is a linear combination of sections of the form $s \otimes \overline{dz_{i_1}} \wedge \cdots \wedge \overline{dz_{i_m}}$. $D_i(s \otimes \overline{dz_{i_1}} \wedge \cdots \wedge \overline{dz_{i_m}}) = D_i(s) \otimes \overline{dz_{i_1}} \wedge \cdots \wedge \overline{dz_{i_m}}$.

Let $\overline{\partial_{\mathcal{E}}}^*$ be the Hodge adjoint of $\overline{\partial_{\mathcal{E}}}$. Let $\Delta_{\mathcal{E}}$ denote the Laplacian of $\overline{\partial_{\mathcal{E}}}$. We also note that $\overline{\partial_{\mathcal{E}}}$, $\overline{\partial_{\mathcal{E}}}^*$ and $\Delta_{\mathcal{E}}$ yield endomorphisms of $K^\bullet_{\mathcal{E}}$.

Let Ω_D denote the differential form on C_k with values in $\mathrm{End}(K^\bullet_{0\mathcal{E}})$ given by the formula

$$\Omega_D = \Pi_{K^\bullet_{0\mathcal{E}}} \circ D_k \circ \exp[-d(t_k - t_{k-1})\overline{\partial_{\mathcal{E}}}^* - (t_k - t_{k-1})\Delta_{\mathcal{E}}] \circ \cdots \circ D_1 \circ \mathcal{I}_{K^\bullet_{0\mathcal{E}}}$$

where $\Pi_{K^\bullet_{0\mathcal{E}}}$ and $\mathcal{I}_{K^\bullet_{0\mathcal{E}}}$ denote the projection from $K^\bullet_{\mathcal{E}}$ to $K^\bullet_{0\mathcal{E}}$ and the inclusion from $K^\bullet_{0\mathcal{E}}$ to $K^\bullet_{\mathcal{E}}$ respectively.

As noted in [FLS], to write Ω_D this way, we require that $\Delta_{\mathcal{E}}$ have discrete non negative spectrum (which is the case for a compact complex manifold). Further, as noted in [FLS], Ω_D extends to a $\mathrm{End}(K^\bullet_{0\mathcal{E}})$-valued (non-homogenous) differential form on $\overline{C_k}$.

We define

$$\mathcal{F}_k(D) = \int_{C_k} \Omega_D = \int_{\overline{C_k}} \Omega_D.$$

By the integral over $\overline{C_k}$ of a non-homogenous differential form we mean the integral over $\overline{C_k}$ of its component of top de Rham degree. It helps to view the differential forms above as differential forms on $\overline{C_k}$ rather than on C_k as that will ensure that the integrals defining the $\mathcal{F}_k$'s converge.

That the $\mathcal{F}_k$'s defined in this manner are the Taylor coefficients of an A_∞-morphism is shown in [FLS].

Before we proceed further, we note that if $k = 1$, then C_1 is a point. The formula for $\mathcal{F}_k$ given here yields that

$$\mathcal{F}_1(D) = \Pi_{K^\bullet_{0\mathcal{E}}} \circ D \circ \mathcal{I}_{K^\bullet_{0\mathcal{E}}}$$

for $D \in \mathrm{Diff}^\bullet(\mathcal{E})$. In particular, if $\mathcal{D} \in \mathrm{Diff}(\mathcal{E})$ and if $\mathcal{D}_* \in \mathrm{End}(\mathrm{H}^*(X, \mathcal{E}))$ is the endomorphism of $\mathrm{H}^*(X, \mathcal{E})$ induced by $\mathcal{D}$, then $\mathcal{F}_1(\mathcal{D}) = \mathcal{D}_*$. It follows from Fact 2 on p. 414 that $\mathrm{tr}([\mathcal{D}]) = \mathrm{str}(\mathcal{D}_*)$. This proves that the following diagram commutes:

$$\begin{array}{ccc} \mathrm{HH}_0(\mathrm{Diff}(\mathcal{E})) & \longrightarrow & \mathrm{HH}_0(\mathrm{Diff}^\bullet(\mathcal{E})) \\ \Big\downarrow{\scriptstyle \mathrm{str}} & & \Big\downarrow{\scriptstyle \mathrm{tr}} \\ \mathbb{C} & \xrightarrow[\mathrm{id}]{} & \mathbb{C}. \end{array}$$

3.1.2. Rewriting the formula for $\mathcal{F}_k$. Given an endomorphism φ of $K_{\mathcal{E}}^{\bullet}$, let $[\varphi]_i$ denote the endomorphism $\mathrm{id} \otimes \cdots \otimes \varphi \otimes \cdots \otimes \mathrm{id}$ of $K_{\mathcal{E}}^{\bullet \otimes k}$ with φ acting on the i-th factor from the right.

Let

$$\Phi = [\mathrm{id}]_k \circ [\exp[-d(t_k - t_{k-1})\overline{\partial_{\mathcal{E}}}^* - (t_k - t_{k-1})\Delta_{\mathcal{E}}]]_{k-1} \circ \cdots \\ \cdots \circ [\exp[-d(t_2 - t_1)\overline{\partial_{\mathcal{E}}}^* - (t_2 - t_1)\Delta_{\mathcal{E}}]]_1.$$

This is a differential form on C_k with values in $\mathrm{End}(K_{\mathcal{E}}^{\bullet \otimes k})$, though it is not a differential operator. Similarly, if $D = D_1 \otimes \cdots \otimes D_k \in \mathrm{Diff}^{\bullet}(\mathcal{E})^{\otimes k}$, D yields an endomorphism $D := [D_1]_k \circ \cdots \circ [D_k]_1$ of $\mathrm{End}(K_{\mathcal{E}}^{\bullet \otimes k})$.

We have a composition map $m\colon \mathrm{End}(K_{\mathcal{E}}^{\bullet})^{\otimes k} \to \mathrm{End}(K_{\mathcal{E}}^{\bullet})$. Identifying $\mathrm{End}(K_{\mathcal{E}}^{\bullet \otimes k})$ with $\mathrm{End}(K_{\mathcal{E}}^{\bullet})^{\otimes k}$, we obtain a composition map $m\colon \mathrm{End}(K_{\mathcal{E}}^{\bullet \otimes k}) \to \mathrm{End}(K_{\mathcal{E}}^{\bullet})$. We recall from [FLS] that the formula for Ω_D can be rewritten as follows:

$$\Omega_D = \Pi_{K_{0\mathcal{E}}^{\bullet}} \circ m(\Phi \circ D) \circ \mathcal{I}_{K_{0\mathcal{E}}^{\bullet}}.$$

Thus

$$\mathcal{F}_k(D) = \int_{C_k} \Pi_{K_{0\mathcal{E}}^{\bullet}} \circ m(\Phi \circ D) \circ \mathcal{I}_{K_{0\mathcal{E}}^{\bullet}}.$$

3.2. Extending the supertrace – I. Recall that $\mathcal{D}\mathrm{iff}(\mathcal{E})$ denotes the sheaf of holomorphic differential operators on $\mathcal{E}$. Let $\mathcal{D}\mathrm{iff}(\mathcal{E})(U)$ denote $\Gamma(U, \mathcal{D}\mathrm{iff}(\mathcal{E}))$ for any open $U \subset X$. Let $\mathcal{E}^{\boxtimes k}$ denote the k-fold exterior tensor power of $\mathcal{E}$ on $X^k := X \times \cdots \times X$. We observe that the differential on the Hochschild complex $C^{\bullet}(\mathcal{D}\mathrm{iff}(\mathcal{E})(U))$ extends to a differential on the graded vector space $\bigoplus_{k \geq 1} \mathcal{D}\mathrm{iff}(\mathcal{E}^{\boxtimes k}(U^k))[k-1]$. The resulting complex is called the completed Hochschild complex of $\mathcal{D}\mathrm{iff}(\mathcal{E})(U)$ and denoted by $\widehat{C^{\bullet}(\mathcal{D}\mathrm{iff}(\mathcal{E})(U))}$.

Definition 5. The completed Hochschild complex of $\mathcal{D}\mathrm{iff}(\mathcal{E})$ is the sheaf of complexes associated to the presheaf $U \rightsquigarrow \widehat{C^{\bullet}(\mathcal{D}\mathrm{iff}(\mathcal{E})(U))}$ of complexes of $\mathbb{C}$-vector spaces.

The completed Hochschild complex of $\mathcal{D}\mathrm{iff}(\mathcal{E})$ is a sheaf of complexes of $\mathbb{C}$-vector spaces on X. It is denoted in this article by $\widehat{\mathrm{Hoch}}(\mathcal{D}\mathrm{iff}(\mathcal{E}))$.

Definition 6. The i-th completed Hochschild homology $\widehat{\mathrm{HH}}_i(\mathcal{D}\mathrm{iff}(\mathcal{E}))$ of $\mathcal{D}\mathrm{iff}(\mathcal{E})$ is the hypercohomology $\mathbb{H}^i(X, \widehat{\mathrm{Hoch}}(\mathcal{D}\mathrm{iff}(\mathcal{E})))$ of $\widehat{\mathrm{Hoch}}(\mathcal{D}\mathrm{iff}(\mathcal{E}))$.

Note that the A_∞-map $\mathcal{F}$ whose construction we recalled in Section 3.1 enables us to pull back the supertrace on $\mathrm{End}(K_{\mathcal{E}}^{\bullet})$ to a Hochschild 0-cocycle of $\mathrm{Diff}^{\bullet}(\mathcal{E})$.

Unfortunately, the A_∞-map $\mathcal{F}$ does not automatically enable us to directly pull back the supertrace to a $\mathbb{C}$-linear functional on $\widehat{\mathrm{HH}_0}(\mathcal{D}\mathrm{iff}(\mathcal{E}))$. This subsection is devoted to an important intermediate step that enables us to construct a $\mathbb{C}$-linear functional on $\widehat{\mathrm{HH}_0}(\mathcal{D}\mathrm{iff}(\mathcal{E}))$ which extends the supertrace on $\mathrm{Diff}(\mathcal{E})$.

3.2.1. "Estimating" I_{FLS}. We now use a homotopy very similar to the homotopy in Proposition 3.1 [Tel] to show that the FLS functional of any 0-cycle in $C^\bullet(\mathrm{Diff}^\bullet(\mathcal{E}))$ depends only on its component in $\mathrm{Diff}^0(\mathcal{E})$.

Construction 1. Let $\phi\colon X \times X \to [0,\infty)$ be a Riemannian distance. Let t be any positive real number. We can choose a finite cover of X by open sets U_i, $1 \le i \le m$, such that $\phi(x,y) < t$ for all $x, y \in U_i$ for any i. Choose a partition of unity $\{f_i\}$ by compactly supported (nonnegative valued) smooth functions subordinate to the cover $\{U_i\}$. Let g_i be a compactly supported smooth function on U_i with values in $[0,1]$ that is identically 1 on the support of f_i. Then $f := \sum_{i=1}^{i=m} f_i \otimes g_i$ is a smooth function on $X \times X$ whose restriction to the diagonal is identically 1. Also f vanishes outside the subset $\{\phi(x,y) \le t\}$ of $X \times X$, and the maximum value of f on $X \times X$ is 1.

Somewhat as in [Tel], let $E_f\colon \mathrm{Diff}^\bullet(\mathcal{E})^{\otimes k} \to \mathrm{Diff}^\bullet(\mathcal{E})^{\otimes k+1}$ be the map

$$D_1 \otimes \cdots \otimes D_k \mapsto -\sum_{i=1}^{i=m} f_i \otimes g_i D_1 \otimes D_2 \otimes \cdots \otimes D_k.$$

Let d be the differential of the complex $C^\bullet(\mathrm{Diff}^\bullet(\mathcal{E}))$. Then

$$dE_f + E_f d = 1 - N_f,$$

where

$$\begin{aligned} N_f(D_1 \otimes \cdots \otimes D_k) &= \pm \sum_{i=1}^{i=m} \bar\partial f_i \otimes g_i D_1 \otimes \cdots \otimes D_k \pm \sum_{i=1}^{i=m} f_i \otimes (\bar\partial g_i) D_1 \otimes \cdots \otimes D_k \\ &\pm \sum_{i=1}^{i=m} f_i \otimes g_i D_k D_1 \otimes \cdots \otimes D_{k-1} \pm \sum_{i=1}^{i=m} D_k f_i \otimes g_i D_1 \otimes \cdots \otimes D_{k-1} \end{aligned} \tag{1}$$

for all $k \ge 2$.

Basic argument. Note that if α is a cycle in $C^\bullet(\mathrm{Diff}^\bullet(\mathcal{E}))$, then α is homotopic to $N_f\alpha$. This is true for all $t > 0$. Also the FLS linear functional I_{FLS} is a Hochschild 0-cocycle of $\mathrm{Diff}^\bullet(\mathcal{E})$. It follows that

$$I_{\mathrm{FLS}}(\alpha) = I_{\mathrm{FLS}}(N_f\alpha).$$

Let $\alpha = \alpha_1 + \cdots + \alpha_p$ with $\alpha_i \in \mathrm{Diff}^\bullet(\mathcal{E})^{\otimes i}$. We then show that

$$\sum_{i\geq 2} |I_{\mathrm{FLS}}(N_f \alpha_i)| \leq C\epsilon(t),$$

where C is a constant that only depends on α and $\epsilon(t)\cdot \mathrm{vol}(X\times X)$ is the volume of the subset $\{\phi(x,y)\leq t\}$ of $X\times X$. Similarly, we show that $|I_{\mathrm{FLS}}(\alpha_1) - I_{\mathrm{FLS}}(N_f(\alpha_1))| \leq C'\epsilon(t)$ where C' depends only on α. Since the construction of f as in Construction 1 is possible for all positive t, letting t approach 0 we see that $I_{\mathrm{FLS}}(\alpha) = I_{\mathrm{FLS}}(\alpha_1)$.

Remark. Recall that $K^\bullet_{0\mathcal{E}}$ can be identified with the kernel of the Laplacian $\Delta_{\mathcal{E}}$. Denote by $K^\bullet_{L^2\mathcal{E}}$ the Hilbert space of square integrable sections of $\mathrm{Dolb}^\bullet(X,\mathcal{O}_X)\otimes_{\mathcal{O}_X}\mathcal{D}\mathrm{iff}(\mathcal{E})$. Then $\mathcal{I}_{K^\bullet_{0\mathcal{E}}}\circ\Pi_{K^\bullet_{0\mathcal{E}}}$ is an integral operator on $K^\bullet_{L^2\mathcal{E}}$ with smooth kernel that projects onto the image of $\mathcal{I}_{K^\bullet_{0\mathcal{E}}}$ (see [BGV], Chapter 2). We will denote this operator by $\Pi_{0,\mathcal{E}}$ or Π_0 when there is no confusion regarding the vector bundle being used. Also one can check that if $M\in\mathrm{End}(K^\bullet_{\mathcal{E}})$, then $\Pi_0 M\Pi_0$ makes sense as a trace class operator on $K^\bullet_{L^2\mathcal{E}}$ and

$$\mathrm{str}(\Pi_0 M\Pi_0) = \mathrm{str}(\Pi_{K^\bullet_{0\mathcal{E}}}\circ M\circ \mathcal{I}_{K^\bullet_{0\mathcal{E}}}).$$

Proposition 3. *Let $D_1,\ldots,D_k\in\mathrm{Diff}^\bullet(\mathcal{E})$. Then*

$$|I_{\mathrm{FLS}}(N_f(D_1\otimes\cdots\otimes D_k))| \leq C\epsilon(t)\,.$$

The constant C above depends only on $D_1,\ldots,D_k$.

Proof. Part 1: The setup.

Recall that

$$\begin{aligned} &N_f(D_1\otimes\cdots\otimes D_k)\\ &\quad= \pm\sum_{i=1}^{i=m}\bar\partial f_i\otimes g_i D_1\otimes\cdots\otimes D_k \pm\sum_{i=1}^{i=m} f_i\otimes(\bar\partial g_i)D_1\otimes\cdots\otimes D_k\\ &\qquad\pm\sum_{i=1}^{i=m} f_i\otimes g_i D_k D_1\otimes\cdots\otimes D_{k-1}\pm\sum_{i=1}^{i=m} D_k f_i\otimes g_i D_1\otimes\cdots\otimes D_{k-1}\end{aligned}$$

for all $k\geq 2$.

We estimate the FLS functional of each of the summands on the right-hand side separately. Let $\mathcal{F}_k$ denote the k-th Taylor component of the FLS A_∞-map. Let C_k denote the configuration space $\prod_{i=1}^{k-1}\{\tau_i>0\}$. Then if $\alpha_i\in\mathrm{Diff}^\bullet(\mathcal{E})$, it follows that

$$\mathcal{F}_k(\alpha_1\otimes\cdots\otimes\alpha_k) = \int_{C_k}\Pi_0\alpha_1\bar\partial^* e^{-\tau_1\Delta}\alpha_2\ldots e^{-\tau_{k-1}\Delta}\alpha_k\Pi_0\,d\tau_1\ldots d\tau_{k-1}.$$

Let $Q_\tau(x,y) \in \Gamma(X \times X, \mathcal{E} \otimes \Omega^{0,\bullet} \boxtimes (\Omega^{0,\bullet})^* \otimes \mathcal{E}^*)$ denote the (smooth) kernel of the operator $\bar\partial^* e^{-\tau\Delta}$. Let $p_\infty(x,y)$ denote the (smooth) kernel of Π_0. Then the kernel of $\mathcal{F}_k(\alpha_1 \otimes \cdots \otimes \alpha_k)$ is

$$\int_{C_k} \int_{X^k} p_\infty(y_0,x_1)\alpha_{1,x_1} Q_{\tau_1}(x_1,x_2) \\ \ldots Q_{\tau_{k-1}} \alpha_{k,x_k} p_\infty(x_k,y_0')\,|dx_1|\ldots|dx_k|\,d\tau_1\ldots d\tau_{k-1}.$$

Recall that

$$I_{\mathrm{FLS}}(\alpha_1 \otimes \cdots \otimes \alpha_k) = \sum_{s=0}^{s=k-1} \mathrm{str}(\mathcal{F}_k(\sigma^s(\alpha))).$$

Here σ is a (signed) cyclic permutation. Each of the k cyclic permutations of $\alpha_1 \otimes \cdots \otimes \alpha_k$ yields exactly one summand contributing towards $I_{\mathrm{FLS}}(\alpha_1 \otimes \cdots \otimes \alpha_k)$.

Part 2: Estimating $I_{\mathrm{FLS}}(\pm \sum_{i=1}^{i=m} \bar\partial f_i \otimes g_i D_1 \otimes \cdots \otimes D_k)$, I.

The cyclic permutations of $\sum_{i=1}^{i=m} \bar\partial f_i \otimes g_i D_1 \otimes \cdots \otimes D_k$ are as follows:

$$\pm \sum_{i=1}^{i=m} \bar\partial f_i \otimes g_i D_1 \otimes \cdots \otimes D_k,$$

$$\pm \sum_{l=1}^{l=m} D_{k-i+1} \otimes \cdots \otimes D_k \otimes \bar\partial f_l \otimes g_l D_1 \otimes \cdots \otimes D_{k-i}, \quad 1 \le i \le k-1,$$

$$\pm \sum_{l=1}^{l=m} g_l D_1 \otimes \cdots \otimes D_k \otimes \bar\partial f_l.$$

We shall henceforth denote the kernel of an integral operator T on $\Gamma(X, \mathcal{E} \otimes \Omega^{0,\bullet}(X))$ by $\overline{T}$. Let D^a denote the formal adjoint of a differential operator D on $\Gamma(X, \mathcal{E} \otimes \Omega^{0,\bullet}(X))$. This is a differential operator on $\Gamma(X, \Omega^{0,\bullet}(X)^* \otimes \mathcal{E}^*)$ since the bundle of densities on X has a canonical trivialization (see [BGV], Chapter 2). With this in mind, we have

$$\overline{\mathcal{F}_{k+1}}\Big(\sum_{i=1}^{i=m} \bar\partial f_i \otimes g_i D_1 \otimes \cdots \otimes D_k\Big) \\ = \int_{C_{k+1}} \int_{X^{k+1}} \sum_{i=1}^{i=m} p_\infty(y_0,x_0)(\bar\partial f_i)(x_0) Q_{\tau_1}(x_0,x_1) g_i(x_1) D_{1,x_1} Q_{\tau_2}(x_1,x_2)\ldots \\ \ldots p_\infty(x_k,y_0')|dx_0|\ldots|dx_k|\,d\tau_1\ldots d\tau_k$$

$$
\begin{aligned}
&= \pm \int_{C_{k+1}} \int_{X^{k+1}} \textstyle\sum_{i=1}^{i=m} p_\infty(y_0, x_0) \bar\partial_{x_0} f_i(x_0) Q_{\tau_1}(x_0, x_1) g_i(x_1) D_{1,x_1} Q_{\tau_2}(x_1, x_2) \dots \\
&\qquad \dots p_\infty(x_k, y_0') |dx_0| \dots |dx_k| \, d\tau_1 \dots d\tau_k \\
&\quad \pm \int_{C_{k+1}} \int_{X^{k+1}} \textstyle\sum_{i=1}^{i=m} p_\infty(y_0, x_0) f_i(x_0) \bar\partial_{x_0} Q_{\tau_1}(x_0, x_1) g_i(x_1) D_{1,x_1} Q_{\tau_2}(x_1, x_2) \dots \\
&\qquad \dots p_\infty(x_k, y_0') |dx_0| \dots |dx_k| \, d\tau_1 \dots d\tau_k \\
&= \pm \int_{C_{k+1}} \int_{X^{k+1}} [\bar\partial^a_{x_0} p_\infty(y_0, x_0)][\textstyle\sum_{i=1}^{i=m} f_i(x_0) g_i(x_1)][Q_{\tau_1}(x_0, x_1)][D_{1,x_1} Q_{\tau_2}(x_1, x_2)] \dots \\
&\qquad \dots [\dots p_\infty(x_k, y_0')] |dx_0| \dots |dx_k| \, d\tau_1 \dots d\tau_k \\
&\quad \pm \int_{C_{k+1}} \int_{X^{k+1}} [p_\infty(y_0, x_0)][\textstyle\sum_{i=1}^{i=m} f_i(x_0) g_i(x_1)][\bar\partial_{x_0} Q_{\tau_1}(x_0, x_1)][D_{1,x_1} Q_{\tau_2}(x_1, x_2)] \dots \\
&\qquad \dots [\dots p_\infty(x_k, y_0')] |dx_0| \dots |dx_k| \, d\tau_1 \dots d\tau_k. \qquad (2)
\end{aligned}
$$

The last equality in (2) is valid since X is compact. Moreover, we also use the fact that if $h(x, y) \in \Gamma(X \times X, \mathcal{E} \otimes \Omega^{0,\bullet} \boxtimes \Omega^{0,\bullet *} \otimes \mathcal{E}^*)$, then it follows that $a(x)h(x, y)b(y) = \pm a(x)b(y)h(x, y)$ for smooth functions a, b on X. We now note that if p_τ denotes the kernel of $e^{-\tau\Delta}$ then

$$
Q_\tau(x, y) = \bar\partial^*_x p_\tau(x, y) = \bar\partial^* p_{(0,\infty)}(x) p_\tau(x, y) p_{(0,\infty)}(y),
$$

where $p_{(0,\infty)}$ is the projection to the span of the eigenvectors of Δ corresponding to positive eigenvalues as in [BGV], Proposition 2.37. This is because $\bar\partial^*$ kills the kernel of Δ. It follows from [BGV], Proposition 2.37, that

$$
\| Q_\tau(x, y) \|_l \le C(\| \cdot \|_l) \mathrm{e}^{-\frac{1}{2}\lambda_1 \tau} \qquad (3)
$$

where $\| \cdot \|_l$ is any C^l norm on $\Gamma(X \times X, \mathcal{E} \otimes \Omega^{0,\bullet} \boxtimes \Omega^{0,\bullet *} \otimes \mathcal{E}^*)$ and λ_1 is the first positive eigenvalue of $\Delta_{\mathcal{E}}$.

It follows from (3) that the sup-norm of each term within a square bracket in each integrand that involves a Q_{τ_i} is bounded above by $C_i \mathrm{e}^{-\tau_i \frac{\lambda_1}{2}}$ for some constant C_i depending only on $D_1 \otimes \cdots \otimes D_k$. The remaining terms have finite sup-norm. The sup-norm of $\sum_{i=1}^{i=m} f_i(x_0) g_i(x_1)$ is 1. Hence there exists a positive constant C depending only on $D_1 \otimes \cdots \otimes D_k$ such that

$$
\begin{aligned}
&\big\| \overline{\mathcal{F}_{k+1}} \big(\textstyle\sum_{i=1}^{i=m} \bar\partial f_i \otimes g_i D_1 \otimes \cdots \otimes D_k \big) \big\|_0 \\
&\le C \int_{C_{k+1}} \mathrm{e}^{-\tau_1 \frac{\lambda_1}{2}} \dots \mathrm{e}^{-\tau_k \frac{\lambda_k}{2}} \epsilon(t) \operatorname{vol}(X)^{k+1} d\tau_1 \dots d\tau_k = \frac{2^k}{\lambda_1^k} C \epsilon(t) \operatorname{vol}(X)^{k+1},
\end{aligned} \qquad (4)
$$

where $\epsilon(t)\,\mathrm{vol}(X\times X)$ is the volume of the support of $\{\phi(x,y)\le t\}$ in $X\times X$. Now

$$\begin{aligned}&\mathrm{str}(\mathcal{F}_{k+1}(\pm\textstyle\sum_{i=1}^{i=m}\bar\partial f_i\otimes g_i D_1\otimes\cdots\otimes D_k))\\&\quad=\pm\int_X \mathrm{str}(\overline{\mathcal{F}_{k+1}}\big(\sum_{i=1}^{i=m}\bar\partial f_i\otimes g_i D_1\otimes\cdots\otimes D_k\big)(y_0,y_0)\,|dy_0|.\end{aligned}$$

It follows from (4) that

$$|\mathrm{str}(\mathcal{F}_{k+1}(\pm\textstyle\sum_{i=1}^{i=m}\bar\partial f_i\otimes g_i D_1\otimes\cdots\otimes D_k))|\le C_1\epsilon(t)$$

for some constant C_1 depending on $D_1\otimes\cdots\otimes D_k$ only.

The same method is used to estimate the other summands that contribute to $I_{\mathrm{FLS}}(\pm\sum_{i=1}^{i=m}\bar\partial f_i\otimes g_i D_1\otimes\cdots\otimes D_k))$ and show that those contributions are at most $C'\epsilon(t)$ as well - in fact the FLS functionals of the other summands of $N_f(D_1\otimes\cdots\otimes D_k)$ are estimated in the same way. We however, present the detailed calculations in the next two parts to be extra careful.

Part 3: Estimating $I_{\mathrm{FLS}}(\pm\sum_{i=1}^{i=m}\bar\partial f_i\otimes g_i D_1\otimes\cdots\otimes D_k))$, II.

We show the equivalent of the calculation (2) for the other summands contributing to $I_{\mathrm{FLS}}(\pm\sum_{i=1}^{i=m}\bar\partial f_i\otimes g_i D_1\otimes\cdots\otimes D_k)$. At the end of each calculation, the integrand in the integral computing the kernel of the operator whose supertrace we need is written as a product of terms marked by square brackets. The argument in part 2 then works almost word for word to show that the corresponding contribution to $I_{\mathrm{FLS}}(N_f(D_1\otimes\cdots\otimes D_k))$ is bounded by a constant depending on the D_i's times $\epsilon(t)$. We have

$$\begin{aligned}&\overline{\mathcal{F}_{k+1}}(\textstyle\sum_{l=1}^{l=m}D_{k-i+1}\otimes\cdots\otimes D_k\otimes\bar\partial f_l\otimes g_l D_1\otimes\cdots\otimes D_{k-i})\\&=\int_{C_{k+1}}\int_{X^{k+1}}\textstyle\sum_{l=1}^{l=m}p_\infty(y_0,x_0)D_{k-i+1,x_0}Q_{\tau_1}(x_0,x_1)\ldots\\&\qquad\ldots D_{k,x_{i-1}}Q_{\tau_i}(x_{i-1},x_i)(\bar\partial f_l)(x_i)Q_{\tau_{i+1}}(x_i,x_{i+1})g_l(x_{i+1})D_{1,x_{i+1}}\ldots\\&\qquad\ldots p_\infty(x_k,y_0')|dx_0|\ldots|dx_k|\,d\tau_1\ldots d\tau_k\\&=\pm\int_{C_{k+1}}\int_{X^{k+1}}\textstyle\sum_{l=1}^{l=m}p_\infty(y_0,x_0)D_{k-i+1,x_0}Q_{\tau_1}(x_0,x_1)\ldots\\&\qquad\ldots D_{k,x_{i-1}}Q_{\tau_i}(x_{i-1},x_i)\bar\partial_{x_i}f_l(x_i)Q_{\tau_{i+1}}(x_i,x_{i+1})g_l x_{i+1})D_{1,x_{i+1}}\ldots\\&\qquad\ldots p_\infty(x_k,y_0')|dx_0|\ldots|dx_k|\,d\tau_1\ldots d\tau_k\\&\quad\pm\int_{C_{k+1}}\int_{X^{k+1}}\textstyle\sum_{l=1}^{l=m}p_\infty(y_0,x_0)D_{k-i+1,x_0}Q_{\tau_1}(x_0,x_1)\ldots.\\&\qquad\ldots D_{k,x_{i-1}}Q_{\tau_i}(x_{i-1},x_i)f_l(x_i)\bar\partial_{x_i}Q_{\tau_{i+1}}(x_i,x_{i+1})g_l(x_{i+1})D_{1,x_{i+1}}\ldots\\&\qquad\ldots p_\infty(x_k,y_0')|dx_0|\ldots|dx_k|\,d\tau_1\ldots d\tau_k\end{aligned}$$

$$
\begin{aligned}
&= \pm \int_{C_{k+1}} \int_{X^{k+1}} [p_\infty(y_0, x_0)][D_{k-i+1,x_0} Q_{\tau_1}(x_0, x_1)] \dots \\
&\qquad \dots [D_{k,x_{i-1}} \bar{\partial}^a_{x_i} Q_{\tau_i}(x_{i-1}, x_i)][\textstyle\sum_{l=1}^{l=m} f_l(x_i) g_l(x_{i+1})][Q_{\tau_{i+1}}(x_i, x_{i+1})] \\
&\qquad \cdot [D_{1,x_{i+1}} \dots][\dots p_\infty(x_k, y_0')]|dx_0| \dots |dx_k|\, d\tau_1 \dots d\tau_k \\
&\quad \pm \int_{C_{k+1}} \int_{X^{k+1}} [p_\infty(y_0, x_0)][D_{k-i+1,x_0} Q_{\tau_1}(x_0, x_1)] \dots \\
&\qquad \dots [D_{k,x_{i-1}} Q_{\tau_i}(x_{i-1}, x_i)][\textstyle\sum_{l=1}^{l=m} f_l(x_i) g_l(x_{i+1})][\bar{\partial}_{x_i} Q_{\tau_{i+1}}(x_i, x_{i+1})] \\
&\qquad \cdot [D_{1,x_{i+1}} \dots][\dots p_\infty(x_k, y_0')]|dx_0| \dots |dx_k|\, d\tau_1 \dots d\tau_k \qquad (5)
\end{aligned}
$$

and

$$
\begin{aligned}
&\mathcal{F}_{k+1}(\textstyle\sum_{l=1}^{l=m} g_l D_1 \otimes \dots \otimes D_k \otimes \bar{\partial} f_l) \\
&\quad = \int_{C_{k+1}} \int_{X^{k+1}} \textstyle\sum_{l=1}^{l=m} p_\infty(y_0, x_0) g_l(x_0) D_{1,x_0} Q_{\tau_1}(x_0, x_1) \dots \\
&\qquad \dots D_k Q_{\tau_k}(x_{k-1}, x_k)(\bar{\partial} f_l)(x_k) p_\infty(x_k, y_0')|dx_0| \dots |dx_k|\, d\tau_1 \dots d\tau_k \\
&\quad = \pm \int_{C_{k+1}} \int_{X^{k+1}} \textstyle\sum_{l=1}^{l=m} p_\infty(y_0, x_0) g_l(x_0) D_{1,x_0} Q_{\tau_1}(x_0, x_1) \dots \\
&\qquad \dots D_k Q_{\tau_k}(x_{k-1}, x_k) \bar{\partial}_{x_k} f_l(x_k) p_\infty(x_k, y_0')|dx_0| \dots |dx_k| d\tau_1 \dots d\tau_k \\
&\qquad \pm \int_{C_{k+1}} \int_{X^{k+1}} \textstyle\sum_{l=1}^{l=m} p_\infty(y_0, x_0) g_l(x_0) D_{1,x_0} Q_{\tau_1}(x_0, x_1) \dots \\
&\qquad \dots D_k Q_{\tau_k}(x_{k-1}, x_k) f_l(x_k) \bar{\partial}_{x_k} p_\infty(x_k, y_0')|dx_0| \dots |dx_k| d\tau_1 \dots d\tau_k \\
&\quad = \pm \int_{C_{k+1}} \int_{X^{k+1}} [p_\infty(y_0, x_0)][\textstyle\sum_{l=1}^{l=m} f_l(x_k) g_l(x_0)][D_{1,x_0} Q_{\tau_1}(x_0, x_1)] \dots \\
&\qquad \dots [D_k \bar{\partial}^a_{x_k} Q_{\tau_k}(x_{k-1}, x_k)][p_\infty(x_k, y_0')]|dx_0| \dots |dx_k| d\tau_1 \dots d\tau_k \\
&\qquad \pm \int_{C_{k+1}} \int_{X^{k+1}} [p_\infty(y_0, x_0)][\textstyle\sum_{l=1}^{l=m} f_l(x_k) g_l(x_0)][D_{1,x_0} Q_{\tau_1}(x_0, x_1)] \dots \\
&\qquad \dots [D_k Q_{\tau_k}(x_{k-1}, x_k)][\bar{\partial}_{x_k} p_\infty(x_k, y_0')]|dx_0| \dots |dx_k|\, d\tau_1 \dots d\tau_k. \qquad (6)
\end{aligned}
$$

Part 4: Estimating other summands contributing to $I_{\mathrm{FLS}}(N_f(D_1 \otimes \dots \otimes D_k))$.

Recall that

$$\begin{aligned} N_f(D_1 \otimes \cdots \otimes D_k) &= \pm \sum_{i=1}^{i=m} \bar{\partial} f_i \otimes g_i D_1 \otimes \cdots \otimes D_k \pm \sum_{i=1}^{i=m} f_i \otimes (\bar{\partial} g_i) D_1 \otimes \cdots \otimes D_k \\ &\pm \sum_{i=1}^{i=m} f_i \otimes g_i D_k D_1 \otimes \cdots \otimes D_{k-1} \pm \sum_{i=1}^{i=m} D_k f_i \otimes g_i D_1 \otimes \cdots \otimes D_{k-1} \end{aligned}$$

for all $k \geq 2$.

The cyclic permutations of $\sum_{l=1}^{l=m} f_l \otimes g_l D_k D_1 \otimes \cdots \otimes D_{k-1}$ that contribute to its FLS functional are as follows:

$$\begin{gathered} \sum_{l=1}^{l=m} f_l \otimes g_l D_k D_1 \otimes \cdots \otimes D_{k-1}, \\ D_{k-i} \otimes \cdots \otimes D_{k-1} \otimes \sum_{l=1}^{l=m} f_l \otimes g_l D_k D_1 \otimes \cdots \otimes D_{k-i-1}, \quad 1 \leq i \leq k-2, \\ \sum_{l=1}^{l=m} g_l D_k D_1 \otimes \cdots \otimes D_{k-1} \otimes f_l. \end{gathered}$$

As before we have

$$\begin{aligned} &\overline{\mathcal{F}_k}\left(\textstyle\sum_{l=1}^{l=m} f_l \otimes g_l D_k D_1 \otimes \cdots \otimes D_{k-1}\right) \\ &\quad = \int_{C_k} \int_{X^k} [p_\infty(y_0, x_1)][\textstyle\sum_{l=1}^{l=m} f_l(x_1) g_l(x_2)][Q_{\tau_1}(x_1, x_2)][D_{k,x_2} D_{1,x_2} \ldots] \ldots \\ &\qquad \ldots [D_{k-1} p_\infty(x_k, y_0')] |dx_1| \ldots |dx_k| \, d\tau_1 \ldots d\tau_{k-1}, \end{aligned} \tag{7}$$

$$\begin{aligned} &\overline{\mathcal{F}_k}\left(D_{k-i} \otimes \cdots \otimes D_{k-1} \otimes \textstyle\sum_{l=1}^{l=m} f_l \otimes g_l D_k D_1 \otimes \cdots \otimes D_{k-i-1}\right) \\ &\quad = \int_{C_k} \int_{X^k} [p_\infty(y_0, x_1)][D_{k-i,x_1} Q_{\tau_1}(x_1, x_2)] \ldots \\ &\qquad \ldots [\textstyle\sum_{l=1}^{l=m} f_l(x_{i+1}) g_l(x_{i+2})][Q_{\tau_{i+1}}(x_{i+1}, x_{i+2})][D_{k,x_{i+2}} D_{1,x_{i+2}} \ldots] \ldots \\ &\qquad [\ldots p_\infty(x_k, y_0')] |dx_1| \ldots |dx_k| \, d\tau_1 \ldots d\tau_{k-1} \end{aligned} \tag{8}$$

and

$$\overline{\mathcal{F}_k}\Big(\sum_{l=1}^{l=m} g_l D_k D_1 \otimes \cdots \otimes D_{k-1} \otimes f_l\Big) = \int_{C_k}\int_{X^k} [p_\infty(y_0, x_1)]\Big[\sum_{l=1}^{l=m} f_l(x_k) g_l(x_1)\Big][D_{k,x_1} D_{1,x_1} Q_{\tau_1}(x_1, x_2)] \ldots \ldots [p_\infty(x_k, y_0')]|dx_1| \ldots |dx_k|\, d\tau_1 \ldots d\tau_{k-1}. \tag{9}$$

For each of the summands dealt with by equations (7), (8) and (9) respectively, one can argue exactly as we did after (2).

The cyclic permutations of $\sum_{l=1}^{l=m} D_k f_l \otimes g_l D_1 \otimes \cdots \otimes D_{k-1}$ that contribute to its FLS functional are

$$\sum_{l=1}^{l=m} D_k f_l \otimes g_l D_1 \otimes \cdots \otimes D_{k-1},$$

$$\sum_{l=1}^{l=m} D_{k-i} \otimes \cdots \otimes D_{k-1} \otimes D_k f_l \otimes g_l D_1 \otimes \cdots \otimes D_{k-i-1}, \quad 1 \le i \le k-2,$$

$$\sum_{l=1}^{l=m} g_l D_1 \otimes \cdots \otimes D_k f_l.$$

We have

$$\overline{\mathcal{F}_k}(D_k f_l \otimes g_l D_1 \otimes \cdots \otimes D_{k-1}) = \int_{C_k}\int_{X^k} \sum_{l=1}^{l=m} p_\infty(y_0, x_1) D_{k,x_1} f_l(x_1) Q_{\tau_1}(x_1, x_2) g_l(x_2) D_{1,x_2} \ldots \ldots D_{k-1,x_k} p_\infty(x_k, y_0')|dx_1| \ldots |dx_k|\, d\tau_1 \ldots d\tau_{k-1}$$

$$= \int_{C_k}\int_{X^k} [D^a_{k,x_1} p_\infty(y_0, x_1)]\Big[\sum_{l=1}^{l=m} f_l(x_1) g_l(x_2)\Big][Q_{\tau_1}(x_1, x_2)][D_{1,x_2} \ldots] \ldots \ldots [D_{k-1,x_k} p_\infty(x_k, y_0')]|dx_1| \ldots |dx_k|\, d\tau_1 \ldots d\tau_{k-1}, \tag{10}$$

$$\overline{\mathcal{F}_k}\Big(\sum_{l=1}^{l=m} D_{k-i} \otimes \cdots \otimes D_{k-1} \otimes D_k f_l \otimes g_l D_1 \otimes \cdots \otimes D_{k-i-1}\Big) = \int_{C_k}\int_{X^k} \sum_{l=1}^{l=m} p_\infty(y_0, x_1) D_{k-i,x_1} Q_{\tau_1}(x_1, x_2) \ldots \ldots D_{k-1,x_i} Q_{\tau_i}(x_i, x_{i+1}) D_{k,x_{i+1}} f_l(x_{i+1}) Q_{\tau_{i+1}}(x_{i+1}, x_{i+2}) g_l(x_{i+2}) \cdot D_{1,x_{i+2}} \ldots p_\infty(x_k, y_0')|dx_1| \ldots |dx_k|\, d\tau_1 \ldots d\tau_{k-1}$$

$$= \int_{C_k} \int_{X^k} [p_\infty(y_0, x_1)][D_{k-i,x_1} Q_{\tau_1}(x_1, x_2)] \dots$$
$$\dots [D_{k-1,x_i} D^a_{k,x_{i+1}} Q_{\tau_i}(x_i, x_{i+1})][\textstyle\sum_{l=1}^{l=m} f_l(x_{i+1}) g_l(x_{i+2})]$$
$$\cdot [Q_{\tau_{i+1}}(x_{i+1}, x_{i+2})][D_{1,x_{i+2}} \dots] \dots [\dots p_\infty(x_k, y_0')]$$
$$\cdot |dx_1| \dots |dx_k|\, d\tau_1 \dots d\tau_{k-1} \tag{11}$$

and

$$\overline{\mathcal{F}_k}(\textstyle\sum_{l=1}^{l=m} g_l D_1 \otimes \dots \otimes D_k f_l)$$
$$= \int_{C_k} \int_{X^k} \textstyle\sum_{l=1}^{l=m} p_\infty(y_0, x_1) g_l(x_1) D_{1,x_1} Q_{\tau_1}(x_1, x_2) \dots$$
$$\dots D_{k-1,x_{k-1}} Q_{\tau_{k-1}}(x_{k-1}, x_k) D_{k,x_k} f_l(x_k)$$
$$\cdot p_\infty(x_k, y_0') |dx_1| \dots |dx_k|\, d\tau_1 \dots d\tau_{k-1}$$
$$= \int_{C_k} \int_{X^k} [p_\infty(y_0, x_1)][\textstyle\sum_{l=1}^{l=m} f_l(x_k) g_l(x_1)][D_{1,x_1} Q_{\tau_1}(x_1, x_2)] \dots$$
$$\dots [D_{k-1,x_{k-1}} D^a_{k,x_k} Q_{\tau_{k-1}}(x_{k-1}, x_k)]$$
$$\cdot [p_\infty(x_k, y_0')] |dx_1| \dots |dx_k|\, d\tau_1 \dots d\tau_{k-1}. \tag{12}$$

For each of the summands dealt with by equations (10), (11) and (12) respectively, one can argue exactly as we did after (2).

The cyclic permutations of $\sum_{l=1}^{l=m} f_l \otimes (\bar\partial g_l) D_1 \otimes \dots \otimes D_k$ that contribute to its FLS functional are as follows:

$$\sum_{l=1}^{l=m} f_l \otimes (\bar\partial g_l) D_1 \otimes \dots \otimes D_k,$$

$$\sum_{l=1}^{l=m} D_{k-i} \otimes \dots \otimes D_k \otimes f_l \otimes (\bar\partial g_l) D_1 \otimes \dots \otimes D_{k-i-1}, \quad 0 \le i \le k-2,$$

$$\sum_{l=1}^{l=m} (\bar\partial g_l) D_1 \otimes \dots \otimes D_k \otimes f_l.$$

We have

$$\overline{\mathcal{F}_{k+1}}(\textstyle\sum_{l=1}^{l=m} f_l \otimes (\bar\partial g_l) D_1 \otimes \dots \otimes D_k)$$
$$= \int_{C_{k+1}} \int_{X^{k+1}} \textstyle\sum_{l=1}^{l=m} p_\infty(y_0, x_0) f_l(x_0) Q_{\tau_1}(x_0, x_1) (\bar\partial g_l)(x_1) D_{1,x_1} \dots$$
$$\dots p_\infty(x_k, y_0') |dx_0| \dots |dx_k| d\tau_1 \dots d\tau_k$$

$$
\begin{aligned}
&= \pm \int_{C_{k+1}} \int_{X^{k+1}} \textstyle\sum_{l=1}^{l=m} p_\infty(y_0, x_0) f_l(x_0) Q_{\tau_1}(x_0, x_1) \bar{\partial}_{x_1} g_l(x_1) D_{1,x_1} \dots \\
&\qquad \dots p_\infty(x_k, y_0')|dx_0| \dots |dx_k|\, d\tau_1 \dots d\tau_k \\
&\quad \pm \int_{C_{k+1}} \int_{X^{k+1}} \textstyle\sum_{l=1}^{l=m} p_\infty(y_0, x_0) f_l(x_0) Q_{\tau_1}(x_0, x_1) g_l(x_1) \bar{\partial}_{x_1} D_{1,x_1} \dots \\
&\qquad \dots p_\infty(x_k, y_0')|dx_0| \dots |dx_k| d\tau_1 \dots d\tau_k \\
&= \pm \int_{C_{k+1}} \int_{X^{k+1}} [p_\infty(y_0, x_0)][\textstyle\sum_{l=1}^{l=m} f_l(x_0) g_l(x_1)][\bar{\partial}^a_{x_1} Q_{\tau_1}(x_0, x_1)] \\
&\qquad \cdot [D_{1,x_1} \dots] \dots [\dots p_\infty(x_k, y_0')]|dx_0| \dots |dx_k|\, d\tau_1 \dots d\tau_k \\
&\quad \pm \int_{C_{k+1}} \int_{X^{k+1}} [p_\infty(y_0, x_0)][\textstyle\sum_{l=1}^{l=m} f_l(x_0) g_l(x_1)][Q_{\tau_1}(x_0, x_1)][\bar{\partial}_{x_1} \\
&\qquad \cdot D_{1,x_1} \dots] \dots [\dots p_\infty(x_k, y_0')]|dx_0| \dots |dx_k|\, d\tau_1 \dots d\tau_k, \quad (13)
\end{aligned}
$$

$$
\begin{aligned}
&\overline{\mathcal{F}_{k+1}}(\textstyle\sum_{l=1}^{l=m} D_{k-i} \otimes \dots \otimes D_k \otimes f_l \otimes (\bar{\partial} g_l) D_1 \otimes \dots \otimes D_{k-i-1}) \\
&\quad = \int_{C_{k+1}} \int_{X^{k+1}} \textstyle\sum_{l=1}^{l=m} p_\infty(y_0, x_0) \dots Q_{\tau_{i+1}}(x_i, x_{i+1}) f_l(x_{i+1}) \\
&\qquad \cdot Q_{\tau_{i+2}}(x_{i+1}, x_{i+2})(\bar{\partial} g_l)(x_{i+2}) D_{1,x_{i+2}} \dots \\
&\qquad \dots p_\infty(x_k, y_0')|dx_0| \dots |dx_k|\, d\tau_1 \dots d\tau_k \\
&\quad = \pm \int_{C_{k+1}} \int_{X^{k+1}} \textstyle\sum_{l=1}^{l=m} p_\infty(y_0, x_0) \dots Q_{\tau_{i+1}}(x_i, x_{i+1}) f_l(x_{i+1}) \\
&\qquad \cdot Q_{\tau_{i+2}}(x_{i+1}, x_{i+2}) \bar{\partial}_{x_{i+2}} g_l(x_{i+2}) D_{1,x_{i+2}} \dots \\
&\qquad \dots p_\infty(x_k, y_0')|dx_0| \dots |dx_k|\, d\tau_1 \dots d\tau_k \\
&\qquad \pm \int_{C_{k+1}} \int_{X^{k+1}} \textstyle\sum_{l=1}^{l=m} p_\infty(y_0, x_0) \dots Q_{\tau_{i+1}}(x_i, x_{i+1}) f_l(x_{i+1}) \\
&\qquad \cdot Q_{\tau_{i+2}}(x_{i+1}, x_{i+2}) g_l(x_{i+2}) \bar{\partial}_{x_{i+2}} D_{1,x_{i+2}} \dots \\
&\qquad \dots p_\infty(x_k, y_0')|dx_0| \dots |dx_k|\, d\tau_1 \dots d\tau_k \\
&\quad = \pm \int_{C_{k+1}} \int_{X^{k+1}} [p_\infty(y_0, x_0)] \dots [\dots Q_{\tau_{i+1}}(x_i, x_{i+1})][\textstyle\sum_{l=1}^{l=m} f_l(x_{i+1}) g_l(x_{i+2})] \\
&\qquad \cdot [\bar{\partial}^a_{x_{i+2}} Q_{\tau_{i+2}}(x_{i+1}, x_{i+2})][D_{1,x_{i+2}} \dots] \dots [\dots p_\infty(x_k, y_0')]
\end{aligned}
$$

$$\cdot |dx_0|\dots|dx_k|\,d\tau_1\dots d\tau_k$$
$$\pm\int_{C_{k+1}}\int_{X^{k+1}}[p_\infty(y_0,x_0)]\dots[\dots Q_{\tau_{i+1}}(x_i,x_{i+1})]$$
$$\cdot[\textstyle\sum_{l=1}^{l=m} f_l(x_{i+1})g_l(x_{i+2})]$$
$$\cdot[Q_{\tau_{i+2}}(x_{i+1},x_{i+2})][\bar\partial_{x_{i+2}}D_{1,x_{i+2}}\dots]\dots[\dots p_\infty(x_k,y_0')]$$
$$\cdot|dx_0|\dots|dx_k|d\tau_1\dots d\tau_k \tag{14}$$

and

$$\begin{aligned}
&\overline{\mathcal{F}_{k+1}}(\textstyle\sum_{l=1}^{l=m}(\bar\partial g_l)D_1\otimes\dots\otimes D_k\otimes f_l)\\
&\quad=\int\limits_{C_{k+1}}\int\limits_{X^{k+1}}\textstyle\sum_{l=1}^{l=m}p_\infty(y_0,x_0)(\bar\partial g_l)(x_0)D_{1,x_0}Q_{\tau_1}(x_0,x_1)\dots\\
&\qquad\qquad\dots f_l(x_k)p_\infty(x_k,y_0')|dx_0|\dots|dx_k|\,d\tau_1\dots d\tau_k\\
&\quad=\pm\int_{C_{k+1}}\int_{X^{k+1}}\textstyle\sum_{l=1}^{l=m}p_\infty(y_0,x_0)\bar\partial_{x_0}g_l(x_0)D_{1,x_0}Q_{\tau_1}(x_0,x_1)\dots\\
&\qquad\qquad\dots f_l(x_k)p_\infty(x_k,y_0')|dx_0|\dots|dx_k|\,d\tau_1\dots d\tau_k\\
&\qquad\pm\int_{C_{k+1}}\int_{X^{k+1}}\textstyle\sum_{l=1}^{l=m}p_\infty(y_0,x_0)g_l(x_0)\bar\partial_{x_0}D_{1,x_0}Q_{\tau_1}(x_0,x_1)\dots\\
&\qquad\qquad\dots f_l(x_k)p_\infty(x_k,y_0')|dx_0|\dots|dx_k|\,d\tau_1\dots d\tau_k\\
&\quad=\pm\int_{C_{k+1}}\int_{X^{k+1}}[\bar\partial^a_{x_0}p_\infty(y_0,x_0)][\textstyle\sum_{l=1}^{l=m}f_l(x_k)g_l(x_0)][D_{1,x_0}Q_{\tau_1}(x_0,x_1)]\dots\\
&\qquad\qquad\dots[p_\infty(x_k,y_0')]|dx_0|\dots|dx_k|\,d\tau_1\dots d\tau_k\\
&\qquad\pm\int_{C_{k+1}}\int_{X^{k+1}}[p_\infty(y_0,x_0)][\textstyle\sum_{l=1}^{l=m}f_l(x_k)g_l(x_0)][\bar\partial_{x_0}D_{1,x_0}Q_{\tau_1}(x_0,x_1)]\dots\\
&\qquad\qquad\dots[p_\infty(x_k,y_0')]|dx_0|\dots|dx_k|\,d\tau_1\dots d\tau_k.
\end{aligned}\tag{15}$$

For each of the summands dealt with by equations (13), (14) and (15) respectively, one can argue exactly as we did after (2).

This finally proves Proposition 3. □

Proposition 4. *Let* $\beta=\beta_1+\dots+\beta_k$ *be a* 0*-cycle in* $C^\bullet(\mathrm{Diff}^\bullet(\mathcal{E}))$ *with* $\beta_k\in\mathrm{Diff}^\bullet(\mathcal{E})^{\otimes k}$. *Then*

$$I_{\mathrm{FLS}}(\beta)=I_{\mathrm{FLS}}(\beta_1).$$

Proof. Pick any $t>0$. If f is as in Construction 1 (see p. 417), then the cycle β is homotopic to the cycle $N_f(\beta)$. Since I_{FLS} is a Hochschild 0-cocycle of $\mathrm{Diff}^\bullet(\mathcal{E}))$, we have

$$I_{\mathrm{FLS}}(\beta)=I_{\mathrm{FLS}}(N_f(\beta))=I_{\mathrm{FLS}}(N_f(\beta_1))+\dots+I_{\mathrm{FLS}}(N_f(\beta_k)).$$

By Proposition 4,

$$\sum_{j=2}^{j=k} |I_{\mathrm{FLS}}(N_f(\beta_j))| \leq C\epsilon(t)$$

for some constant C depending on β only. Also

$$N_f(\beta_1) = \beta_1 \pm \sum_{l=1}^{l=m} (\bar{\partial} f_l) \otimes g_l \beta_1 \pm \sum_{l=1}^{l=m} f_l \otimes (\bar{\partial} g_l)\beta_1.$$

The proof of Proposition 4 can also be used to verify that

$$\left| I_{\mathrm{FLS}}\left(\textstyle\sum_{l=1}^{l=m} (\bar{\partial} f_l) \otimes g_l \beta_1 \pm \sum_{l=1}^{l=m} f_l \otimes (\bar{\partial} g_l)\beta_1 \right) \right| \leq C'\epsilon(t)$$

for some constant C' depending only on β_1. It follows that

$$|I_{\mathrm{FLS}}(\beta) - I_{\mathrm{FLS}}(\beta_1)| = |I_{\mathrm{FLS}}(N_f(\beta) - I_{\mathrm{FLS}}(\beta_1)| \leq (C + C')\epsilon(t)\,.$$

Now t may be chosen to be arbitrarily small. In this case, $\epsilon(t)$ becomes arbitrarily small as well. As $C + C'$ depends only on β, the proposition follows. □

Proposition 3 implies the following (surprising) statement.

Proposition 5. *The linear functional*

$$D \mapsto \mathrm{str}(\Pi_0 D \Pi_0)$$

vanishes on 0*-chains of* $\mathrm{Diff}^\bullet(\mathcal{E})$ *that are commutators of elements in* $\mathrm{Diff}^\bullet(\mathcal{E})$.

Proof. Let $D_1, D_2 \in \mathrm{Diff}^0(\mathcal{E})$. This proposition is immediate from Proposition 4 and the following two facts: $I_{\mathrm{FLS}}(d(D_1 \otimes D_2)) = 0$ since I_{FLS} is a Hochschild 0-cocycle, and the component of $d(D_1 \otimes D_2)$ in $\mathrm{Diff}^\bullet(\mathcal{E})$ is precisely $-D_1 D_2 + D_2 D_1$. □

Of course, when D is the commutator of two purely holomorphic differential operators on $\mathcal{E}$, the above proposition is standard. The above proposition in general is however very counter-intuitive, and the author does not see any other way of proving it.

3.3. Extending the supertrace – II. Let $\mathcal{D}\mathrm{iff}^\bullet(\mathcal{E})(U)$ denote $\Gamma(U, \mathcal{D}\mathrm{iff}^\bullet(\mathcal{E}))$. Note that the differential on $C^\bullet(\mathcal{D}\mathrm{iff}^\bullet(\mathcal{E})(U))$ extends to a differential on the graded vector space $\bigoplus_k \mathcal{D}\mathrm{iff}^\bullet(\mathcal{E}^{\boxtimes k})(U^k)[k-1]$. We denote the resulting complex by $\widehat{C^\bullet(\mathcal{D}\mathrm{iff}^\bullet(\mathcal{E})(U))}$.

Let $\widetilde{\mathrm{Hoch}(\mathcal{D}\mathrm{iff}(\mathcal{E}))}$ denote the sheaf associated to the pre-sheaf

$$U \mapsto \widetilde{C^\bullet(\mathcal{D}\mathrm{iff}^\bullet(\mathcal{E})(U))}$$

of complexes of $\mathbb{C}$-vector spaces. Note that $\widetilde{\mathrm{Hoch}(\mathcal{D}\mathrm{iff}(\mathcal{E}))}$ is a complex whose terms are modules over the sheaf of smooth functions on X. It follows that

$$\mathrm{H}^\bullet(\Gamma(X, \widetilde{\mathrm{Hoch}(\mathcal{D}\mathrm{iff}(\mathcal{E}))})) \simeq \mathbb{H}^\bullet(X, \widetilde{\mathrm{Hoch}(\mathcal{D}\mathrm{iff}(\mathcal{E}))}).$$

The following proposition follows from Proposition 4.

Proposition 6. *I_{FLS} extends to a linear functional on $\Gamma(X, \widetilde{\mathrm{Hoch}(\mathcal{D}\mathrm{iff}(\mathcal{E}))})$ that vanishes on the image of the differential d of $\Gamma(X, \widetilde{\mathrm{Hoch}(\mathcal{D}\mathrm{iff}(\mathcal{E}))})$.*

Proof. Part 1: Constructing $\widehat{\mathrm{str}}$:

Let $\widetilde{\mathrm{Hoch}^k(\mathcal{D}\mathrm{iff}(\mathcal{E}))}$ denote the sheaf associated to the pre-sheaf

$$U \mapsto \mathcal{D}\mathrm{iff}^\bullet(\mathcal{E}^{\boxtimes k})(U^k)[k-1]$$

of graded $\mathbb{C}$-vector spaces. Then any 0-cycle in $\Gamma(X, \widetilde{\mathrm{Hoch}(\mathcal{D}\mathrm{iff}(\mathcal{E}))})$ is given by $\beta_1 + \cdots + \beta_k$ where $\beta_i \in \Gamma(X, \widetilde{\mathrm{Hoch}^i(\mathcal{D}\mathrm{iff}(\mathcal{E}))})$. Note that $\widetilde{\mathrm{Hoch}^1(\mathcal{D}\mathrm{iff}(\mathcal{E}))} = \mathcal{D}\mathrm{iff}^\bullet(\mathcal{E})$. It follows that β_1 is an element of $\mathrm{Diff}^0(\mathcal{E})$.

Let $\widehat{\mathrm{str}}$ be the linear functional on $\Gamma(X, \widetilde{\mathrm{Hoch}(\mathcal{D}\mathrm{iff}(\mathcal{E}))})$ that vanishes on p-chains whenever $p \neq 0$. For a 0-chain $\beta = \beta_1 + \cdots + \beta_k$ with $\beta_i \in \Gamma(X, \widetilde{\mathrm{Hoch}^i(\mathcal{D}\mathrm{iff}(\mathcal{E}))})$, set

$$\widehat{\mathrm{str}}(\beta) = I_{\mathrm{FLS}}(\beta_1).$$

This is a well-defined linear functional on the space of 0-chains of $\Gamma(X, \widetilde{\mathrm{Hoch}(\mathcal{D}\mathrm{iff}(\mathcal{E}))})$.

Part 2: We now need to show that $\widehat{\mathrm{str}}$ vanishes on the image of d.

Claim A: The natural map from $\mathrm{Diff}^\bullet(\mathcal{E}^{\boxtimes 2})[1]$ to $\Gamma(X, \widetilde{\mathrm{Hoch}^2(\mathcal{D}\mathrm{iff}(\mathcal{E}))})$ is a surjection of graded $\mathbb{C}$-vector spaces.

We postpone the proof of Claim A for the next part of the proof. Note that the differential on $C^\bullet(\mathrm{Diff}^\bullet(\mathcal{E}))$ extends to a differential d on $\bigoplus_k \mathrm{Diff}^\bullet(\mathcal{E}^{\boxtimes k})[k-1]$. This differential is a sum of the differentials $d_{\mathrm{H}} \colon \mathrm{Diff}^\bullet(\mathcal{E}^{\boxtimes k}) \to \mathrm{Diff}^\bullet(\mathcal{E}^{\boxtimes k-1})$ and $\bar{\partial}$. By Proposition 4 and Claim A, to show that $\widehat{\mathrm{str}}$ vanishes on the image of d, it suffices to show that the map $\mathrm{Diff}^0(\mathcal{E}) \to \mathbb{C}$,

$$D \mapsto \mathrm{str}(\Pi_0 D \Pi_0),$$

vanishes on the image of $d_{\mathrm{H}} \colon \mathrm{Diff}^0(\mathcal{E}^{\boxtimes 2}) \to \mathrm{Diff}^0(\mathcal{E})$.

Let $\mathrm{Diff}^{\leq k,\bullet}(\mathcal{E})$ denote $\Gamma(X, \mathrm{Dolb}^\bullet(X, \mathcal{O}_X) \otimes \mathcal{D}\mathrm{iff}^{\leq k}(\mathcal{E}))$ where $\mathcal{D}\mathrm{iff}^{\leq k}(\mathcal{E})$ denotes the sheaf of differential operators on $\mathcal{E}$ of order $\leq k$. Equip $\mathrm{Diff}^{\leq k,\bullet}(\mathcal{E})$ with the topology generated by the family of seminorms $\{\|\cdot\|_{K,s} \mid K \subset X \text{ compact}, s \in \Gamma(K, \mathcal{E} \otimes \Omega^\bullet)\}$ given by

$$\|D\|_{K,s} = \sup\{\|D(s)(x)\| \mid x \in K\}.$$

The topology on $\mathrm{Diff}^\bullet(\mathcal{E})$ is the direct limit of the topologies on the $\mathrm{Diff}^{\leq k,\bullet}(\mathcal{E})$. The topology on $\mathrm{Diff}^\bullet(\mathcal{E}^{\boxtimes 2})$ is defined analogously.

Note that $\mathrm{Diff}^\bullet(\mathcal{E})^{\otimes 2}$ is dense in $\mathrm{Diff}^\bullet(\mathcal{E}^{\boxtimes 2})$. Furthermore, it follows that $d_{\mathrm{H}}\colon \mathrm{Diff}^\bullet(\mathcal{E}^{\boxtimes 2}) \to \mathrm{Diff}^\bullet(\mathcal{E})$ is continuous. The restriction of d_{H} to $\mathrm{Diff}^0(\mathcal{E})^{\otimes 2}$ is just the map

$$D_1 \otimes D_2 \mapsto D_2 D_1 - D_1 D_2.$$

The proposition now follows from Proposition 5.

Part 3: Proof of Claim A.

Given any element α of $\Gamma(X, \widetilde{\mathrm{Hoch}^2(\mathcal{D}\mathrm{iff}(\mathcal{E}))})$, pick a finite cover $X = \bigcup_i U_i$ of open sets such that $\alpha|_{U_i}$ is an element of $\mathrm{Diff}^\bullet(\mathcal{E}^{\boxtimes 2})(U_i \times U_i)[1]$. Pick a partition of unity $\{f_i\}$ by compactly supported smooth functions subordinate to he cover $\bigcup_i U_i$. Let g_i be a compactly supported smooth function supported on a subset of U_i that is identically 1 on the support of f_i. Then

$$\sum_i (f_i \boxtimes g_i)\alpha|_{U_i}$$

is an element of $\mathrm{Diff}^\bullet(\mathcal{E}^{\boxtimes 2})[1]$ whose image in $\Gamma(X, \widetilde{\mathrm{Hoch}^2(\mathcal{D}\mathrm{iff}(\mathcal{E}))})$ is α. □

Next we note that we have a natural (degree preserving) map

$$\gamma\colon \widehat{\mathrm{Hoch}}(\mathcal{D}\mathrm{iff}(\mathcal{E})) \to \widetilde{\mathrm{Hoch}(\mathcal{D}\mathrm{iff}(\mathcal{E}))}$$

of complexes of sheaves of $\mathbb{C}$-vector spaces on X. This induces a map

$$\gamma_*\colon \mathbb{H}^\bullet(X, \widehat{\mathrm{Hoch}}(\mathcal{D}\mathrm{iff}(\mathcal{E}))) \to \mathbb{H}^\bullet(X, \widetilde{\mathrm{Hoch}(\mathcal{D}\mathrm{iff}(\mathcal{E}))}).$$

Recall that $\mathrm{H}^0(\Gamma(X, \widetilde{\mathrm{Hoch}(\mathcal{D}\mathrm{iff}(\mathcal{E}))})) = \mathbb{H}^0(X, \widetilde{\mathrm{Hoch}(\mathcal{D}\mathrm{iff}(\mathcal{E}))})$. The following corollary of Proposition 6 is immediate.

Corollary 3. *The Hochschild cocycle* $\mathrm{str}\colon \mathrm{HH}_0(\mathrm{Diff}^\bullet(\mathcal{E})) \to \mathbb{C}$ *extends to a* $\mathbb{C}$*-linear functional* $\widehat{\mathrm{str}}\colon \mathbb{H}^0(X, \widehat{\mathrm{Hoch}(\mathcal{D}\mathrm{iff}(\mathcal{E}))}) \to \mathbb{C}$.

It is obvious that $\widehat{\mathrm{str}} \circ \gamma_*$ yields a linear functional on $\mathbb{H}^0(X, \widehat{\mathrm{Hoch}}(\mathcal{D}\mathrm{iff}(\mathcal{E})))$. We state this as a corollary for emphasis.

Corollary 4. *The linear functional* $\widehat{\mathrm{str}}$ *in the previous corollary gives us a linear functional* $\widehat{\mathrm{tr}}\colon \mathbb{H}^0(X, \widehat{\mathrm{Hoch}}(\mathcal{D}\mathrm{iff}(\mathcal{E}))) \to \mathbb{C}$.

We denote $\mathbb{H}^{-i}(X, \widehat{\mathrm{Hoch}}(\mathcal{D}\mathrm{iff}(\mathcal{E})))$ by $\widehat{\mathrm{HH}}_{-i}(\mathcal{D}\mathrm{iff}(\mathcal{E}))$.

Now consider a global holomorphic differential operator D on $\mathcal{E}$. D can be thought of as an element of $\Gamma(X, \mathcal{D}\mathrm{iff}^\bullet(\mathcal{E}))$ as well. Thus, D gives us a 0-cocycle of the cochain complex $\Gamma(X, \widetilde{\mathrm{Hoch}(\mathcal{D}\mathrm{iff}(\mathcal{E}))})$. Let $\bar{D}$ denote this cocycle.

One the other hand, D can be thought of as a global section of the degree 0 term of the complex $\widehat{\mathrm{Hoch}}(\mathcal{D}\mathrm{iff}(\mathcal{E}))$. It follows that D yields a 0-cycle in the bi-complex computing $\mathbb{H}^\bullet(X, \widehat{\mathrm{Hoch}}(\mathcal{D}\mathrm{iff}(\mathcal{E})))$. Call this 0-cycle $\widetilde{D}$. It is easy to check that $\gamma(\widetilde{D}) = \bar{D}$. We have thus proven the following proposition.

Proposition 7. *The following diagram commutes:*

$$\begin{array}{ccc} \mathrm{HH}_0(\mathrm{Diff}(\mathcal{E})) & \longrightarrow & \widehat{\mathrm{HH}}_0(\mathcal{D}\mathrm{iff}(\mathcal{E})) \\ \downarrow{\scriptstyle \mathrm{str}} & & \downarrow{\scriptstyle \mathrm{tr}} \\ \mathbb{C} & \xrightarrow[\mathrm{id}]{} & \mathbb{C}. \end{array}$$

3.4. The first bookkeeping lemma. Consider $\mathcal{E}$ as a sub bundle of $\mathcal{E} \oplus \mathcal{F}$ where $\mathcal{F}$ is another vector bundle on X. We then have a map $\iota\colon \mathcal{D}\mathrm{iff}(\mathcal{E}) \to \mathcal{D}\mathrm{iff}(\mathcal{E} \oplus \mathcal{F})$ of sheaves of $\mathbb{C}$-vector spaces whose restriction to U, $\iota\colon \Gamma(U, \mathcal{D}\mathrm{iff}(\mathcal{E})) \to \Gamma(U, \mathcal{D}\mathrm{iff}(\mathcal{E} \oplus \mathcal{F}))$, is an injection that preserves addition and multiplication for each open $U \subset X$. This also induces an map $\bar{\iota}\colon \mathcal{D}\mathrm{iff}^\bullet(\mathcal{E}) \to \mathcal{D}\mathrm{iff}^\bullet(\mathcal{E} \oplus \mathcal{F})$ of sheaves of $\mathbb{C}$-vector spaces whose restriction to U, $\bar{\iota}\colon \mathcal{D}\mathrm{iff}^\bullet(\mathcal{E})(U) \to \mathcal{D}\mathrm{iff}^\bullet(\mathcal{E} \oplus \mathcal{F})(U)$, is an injection for each open $U \subset X$.

The map $\bar{\iota}$ preserves addition and multiplication though it does not preserve the identity. Thus $\bar{\iota}$ induces a map from $\widetilde{\mathrm{Hoch}(\mathcal{D}\mathrm{iff}(\mathcal{E}))}$ to $\widetilde{\mathrm{Hoch}(\mathcal{D}\mathrm{iff}(\mathcal{E} \oplus \mathcal{F}))}$, which we will denote by $\tilde{\iota}$. Let $\tilde{\iota}_*$ denote the map from $\mathrm{H}^*(\Gamma(X, \widetilde{\mathrm{Hoch}(\mathcal{D}\mathrm{iff}(\mathcal{E}))}))$ to $\mathrm{H}^*(\Gamma(X, \widetilde{\mathrm{Hoch}(\mathcal{D}\mathrm{iff}(\mathcal{E} \oplus \mathcal{F}))}))$ induced by $\tilde{\iota}$. The following "bookkeeping lemma" holds.

Lemma 1. *The following diagram commutes:*

$$\begin{array}{ccc} \mathrm{H}^0(\Gamma(X, \widetilde{\mathrm{Hoch}(\mathcal{D}\mathrm{iff}(\mathcal{E}))})) & \xrightarrow{\tilde{\iota}_*} & \mathrm{H}^*(\Gamma(X, \widetilde{\mathrm{Hoch}(\mathcal{D}\mathrm{iff}(\mathcal{E} \oplus \mathcal{F}))})) \\ \downarrow{\scriptstyle \widehat{\mathrm{str}}} & & \downarrow{\scriptstyle \widehat{\mathrm{str}}} \\ \mathbb{C} & \xrightarrow[\mathrm{id}]{} & \mathbb{C}. \end{array}$$

Proof. By Proposition 5 and the proof of Proposition 6, it is enough to show that

$$\mathrm{str}(\Pi_{K^\bullet_{0\mathcal{E}}} D\, \mathcal{I}_{K^\bullet_{0\mathcal{E}}}) = \mathrm{str}(\Pi_{K^\bullet_{0\mathcal{E}\oplus\mathcal{F}}} \tilde{\iota}(D) \mathcal{I}_{K^\bullet_{0\mathcal{E}\oplus\mathcal{F}}})$$

for any $D \in \mathrm{Diff}^0(\mathcal{E})$.

The above equality holds since the following diagrams commute:

$$\begin{array}{ccc} K^\bullet_{\mathcal{E}} & \longrightarrow & K^\bullet_{\mathcal{E}\oplus\mathcal{F}} \\ {\scriptstyle D}\downarrow & & \downarrow{\scriptstyle \tilde{\iota}(D)} \\ K^\bullet_{\mathcal{E}} & \longrightarrow & K^\bullet_{\mathcal{E}\oplus\mathcal{F}}, \end{array} \qquad \begin{array}{ccc} K^\bullet_{\mathcal{E}} & \longrightarrow & K^\bullet_{\mathcal{E}\oplus\mathcal{F}} \\ {\scriptstyle \Pi_{K^\bullet_{0\mathcal{E}}}}\downarrow & & \downarrow{\scriptstyle \Pi_{K^\bullet_{0\mathcal{E}\oplus\mathcal{F}}}} \\ K^\bullet_{0\mathcal{E}} & \longrightarrow & K^\bullet_{0\mathcal{E}\oplus\mathcal{F}}, \end{array} \qquad \begin{array}{ccc} K^\bullet_{0\mathcal{E}} & \longrightarrow & K^\bullet_{0\mathcal{E}\oplus\mathcal{F}} \\ {\scriptstyle \mathcal{I}_{K^\bullet_{0\mathcal{E}}}}\downarrow & & \downarrow{\scriptstyle \mathcal{I}_{K^\bullet_{0\mathcal{E}\oplus\mathcal{F}}}} \\ K^\bullet_{\mathcal{E}} & \longrightarrow & K^\bullet_{\mathcal{E}\oplus\mathcal{F}}. \end{array}$$

□

The map ι induces a map $\hat{\iota}\colon \widehat{\mathrm{Hoch}}(\mathcal{D}\mathrm{iff}(\mathcal{E})) \to \widehat{\mathrm{Hoch}}(\mathcal{D}\mathrm{iff}(\mathcal{E}\oplus\mathcal{F}))$. Let $\hat{\iota}_*$ denote the map induced by $\hat{\iota}$ from $\widehat{\mathrm{HH}}_*(\mathcal{D}\mathrm{iff}(\mathcal{E}))$ to $\widehat{\mathrm{HH}}_*(\mathcal{D}\mathrm{iff}(\mathcal{E}\oplus\mathcal{F}))$. The following corollary holds.

Corollary 5. *The following diagram commutes:*

$$\begin{array}{ccc} \widehat{\mathrm{HH}}_0(\mathcal{D}\mathrm{iff}(\mathcal{E})) & \xrightarrow{\hat{\iota}_*} & \widehat{\mathrm{HH}}_0(\mathcal{D}\mathrm{iff}(\mathcal{E}\oplus\mathcal{F})) \\ {\scriptstyle \hat{\mathrm{tr}}}\downarrow & & \downarrow{\scriptstyle \hat{\mathrm{tr}}} \\ \mathbb{C} & \xrightarrow[\mathrm{id}]{} & \mathbb{C}. \end{array}$$

Proof. For this we only need to recall that by the definition of ι at the beginning of this section the following diagram commutes:

$$\begin{array}{ccc} \Gamma(U^k, \mathcal{D}\mathrm{iff}(\mathcal{E}^{\boxtimes k})) & \xrightarrow{\hat{\iota}} & \Gamma(U^k, \mathcal{D}\mathrm{iff}((\mathcal{E}\oplus\mathcal{F})^{\boxtimes k})) \\ {\scriptstyle \gamma}\downarrow & & \downarrow{\scriptstyle \gamma} \\ \Gamma(U^k, \mathcal{D}\mathrm{iff}^\bullet(\mathcal{E}^{\boxtimes k})) & \xrightarrow{\tilde{\iota}} & \Gamma(U^k, \mathcal{D}\mathrm{iff}^\bullet((\mathcal{E}\oplus\mathcal{F})^{\boxtimes k})). \end{array}$$

Taking hypercohomology we see that $\gamma_* \circ \hat{\iota}_* = \tilde{\iota}_* \circ \gamma_*$. The corollary now follows immediately from Lemma 1. □

4. The completed Hochschild homology of $\mathcal{D}\mathrm{iff}(\mathcal{E})$ and the cohomology of X

4.1. Preliminaries. Recall the definition of $\widehat{\mathrm{Hoch}}(\mathcal{D}\mathrm{iff}(\mathcal{E}))$ from Section 3.2. Let $\widehat{\mathrm{Hoch}}(\mathrm{Diff}(X))$ denote $\widehat{\mathrm{Hoch}}(\mathcal{D}\mathrm{iff}(\mathcal{O}_X))$. Let $\underline{\mathbb{C}}$ denote the constant sheaf on X such that $\Gamma(U, \underline{\mathbb{C}}) \simeq \mathbb{C}$ for every open $U \subset X$ and whose restriction maps are all identity.

Lemma 2 ([Bryl]). $\widehat{\mathrm{Hoch}}(\mathrm{Diff}(X))$ *is quasi-isomorphic to* $\underline{\mathbb{C}}[2n]$ *where* n *is the dimension of* X.

Sketch of the proof recalled from [Bryl]. It is enough to check the above fact for any open ball $U \subset X$ at the level of pre-sheaves. In other words, it is enough to show that the complex $\mathrm{Diff}(U^\bullet)$ with Hochschild differential is quasi-isomorphic to $\mathbb{C}[2n]$ as complexes of $\mathbb{C}$-vector spaces.

We filter $\mathrm{Diff}(U)$ by degree. More precisely, set $F^{-n}(\mathrm{Diff}(U)) := \mathrm{Diff}^{\leq n}(U)$. The associated graded $\mathrm{gr}(\mathrm{Diff}(U))$ is the space of functions $\mathcal{O}_{T^*U}$ on the cotangent bundle T^*U that are algebraic(polynomial) along the fibres.

This filtration of $\mathrm{Diff}(U)$ is exhaustive. It yields a filtration of the complex $\mathrm{Diff}(U^\bullet)$ equipped with Hochschild differential. This filtration yields a spectral sequence converging to the cohomology of the complex $\mathrm{Diff}(U^\bullet)$. The $E_1^m := \bigoplus_{p+q=m} E_1^{p,q}$-term is m-th cohomology of the completed Hochschild complex of $\mathcal{O}_{T^*U}$. This is isomorphic to the space of $-m$-holomorphic forms on T^*U that are algebraic along the fibres (see [Bryl]). Given local coordinates $z_1, \ldots z_n$ on U and $y_1, \ldots, y_n$ on the fibre of T^*U, putting the weight of dz_i equal to 0 and that of dy_i equal to 1 for all i enables us to define the notion of the weight of a holomorphic form on T^*U. In the spectral sequence of this proof-sketch, $E_1^{p,m-p}$ is simply the space of $-m$-forms on T^*U of weight $-p$. Therefore, $E_1^{-n,-n}$ is the only non-zero summand of E_1^{-2n}. By Theorem 3.1.1 and Corollary 2.2.2 of [Bryl], the $E_2^m = E_\infty^m$-term of this spectral sequence is the $2n + m$-th de Rham cohomology of T^*U. This is 0 if $m \neq -2n$ and $\mathbb{C}$ otherwise. This proves the lemma. Moreover, $E_2^{-n,-n}$ is the only nontrivial summand of E_2^{-2n}. Therefore, the $-2n$-th cohomology of this complex can be identified with $E_2^{-n,-n}$. □

Lemma 3. $\widehat{\mathrm{Hoch}}(\mathcal{D}\mathrm{iff}(\mathcal{E}))$ *is quasi-isomorphic to* $\mathbb{C}[2n]$ *where* n *is the dimension of* X.

Proof. This is again something that needs to be verified locally. We imitate the proof of the Morita invariance of Hochschild homology in [Lod], Section 1.2, here.

Part 1: Recalling the proof of Morita invariance of the Hochschild homology of a $\mathbb{C}$-algebra.

Recall from [Lod] that if A is any $\mathbb{C}$-algebra, and if $\mathrm{Mat}_r(A)$ denotes the algebra of $r \times r$ matrices with entries in A, then we have a map tr from the Hochschild complex of $\mathrm{Mat}_r(A)$ to that of A. The Hochschild chain $M_1 \otimes \cdots \otimes M_k$ is mapped to $\mathrm{tr}(M_1 \odot \cdots \odot M_k)$ where $\odot \colon \mathrm{Mat}_r(A) \otimes \mathrm{Mat}_r(B) \to \mathrm{Mat}_r(A \otimes B)$ is an exterior multiplication. There is also a map inc of complexes into the opposite direction which is induced by the inclusion of A in $\mathrm{Mat}_r(A)$ taking an element a of A to the matrix with $a \cdot E_{11}$. Then $\mathrm{tr} \circ \mathrm{inc} = \mathrm{id}$ and there is a presimplicial homotopy h from $\mathrm{inc} \circ \mathrm{tr}$ to id. This homotopy is given by $h = \sum_i (-1)^i h_i$ where $h_i \colon \mathrm{Mat}_r(A)^{\otimes k+1} \to$

$\mathrm{Mat}_r(A)^{\otimes k+2}$ is defined by the formula

$$h_i(\alpha^0 \otimes \cdots \otimes \alpha^k) = \sum E_{j1}(\alpha^0_{jr}) \otimes E_{11}(\alpha^1_{rm}) \otimes \cdots \otimes E_{11}(\alpha^i_{pq}) \otimes E_{1q}(1) \otimes \alpha^{i+1} \otimes \cdots \otimes \alpha^k.$$

The sum here is over all possible tuples of indices $(j, r, \ldots, p, q)$, see [Lod], Section 1.2, and $E_{ij}(x) = x \cdot E_{ij}$, where E_{ij} is the elementary matrix whose only non-zero entry is a 1 at the ij position.

Let U be an open ball contained in X.

Part 2: Morita invariance of the completed Hochschild homology of $\mathrm{Diff}(U)$.

Recall that the completed Hochschild complex $\widehat{C^\bullet(\mathrm{Diff}(U))}$ is obtained by equipping the graded vector space $\bigoplus_{k\geq 1} \mathrm{Diff}(U^k)[k-1]$ with the Hochschild differential. In this complex, $\mathrm{Diff}(U^k)$ should be viewed as the k-th completed tensor power of $\mathrm{Diff}(U)$. With this in mind, the k-th completed tensor power of $\mathrm{Mat}_r(\mathrm{Diff}(U))$ will be $\mathrm{Mat}_r(\mathbb{C})^{\otimes k} \otimes \mathrm{Diff}(U^k)$. One may verify that the Hochschild differential on $C^\bullet(\mathrm{Mat}_r(\mathrm{Diff}(U)))$ extends to a differential on the graded vector space $\bigoplus_{k\geq 1} \mathrm{Mat}_r(\mathbb{C})^{\otimes k} \otimes \mathrm{Diff}(U^k)[k-1]$. The resulting complex is the completed Hochschild complex $\widehat{C^\bullet(\mathrm{Mat}_r(\mathrm{Diff}(U)))}$.

One can verify without much difficulty that tr and inc extend to maps of complexes

$$\mathrm{tr}\colon \widehat{C^\bullet(\mathrm{Mat}_r(\mathrm{Diff}(U)))} \to \widehat{C^\bullet(\mathrm{Diff}(U))}$$

and

$$\mathrm{inc}\colon \widehat{C^\bullet(\mathrm{Diff}(U))} \to \widehat{C^\bullet(\mathrm{Mat}_r(\mathrm{Diff}(U)))}$$

respectively such that $\mathrm{tr} \circ \mathrm{inc} = \mathrm{id}$. It is also useful for us to note explicitly that if $m_1, \ldots, m_k \in \mathrm{Mat}_r(\mathbb{C})$ and $\alpha \in \mathrm{Diff}(U^k)$, then

$$\mathrm{tr}(m_1 \otimes \cdots \otimes m_k \otimes \alpha) = \mathrm{tr}(m_1 \circ \cdots \circ m_k)\alpha.$$

Moreover h extends to a map $h\colon \widehat{C^k(\mathrm{Mat}_r(\mathrm{Diff}(U)))} \to \widehat{C^{k+1}(\mathrm{Mat}_r(\mathrm{Diff}(U)))}$ for all k with $dh + hd = \mathrm{id} - \mathrm{inc} \circ \mathrm{tr}$. Thus $\mathrm{tr}\colon \widehat{C^\bullet(\mathrm{Mat}_r(\mathrm{Diff}(U)))} \to \widehat{C^\bullet(\mathrm{Diff}(U))}$ is a quasi-isomorphism. It follows from Lemma 2 that $\widehat{C^\bullet(\mathrm{Mat}_r(\mathrm{Diff}(U)))}$ is quasi-isomorphic to $\mathbb{C}[2n]$.

Part 3: Proof of the lemma.

Let r be the rank of $\mathcal{E}$. If U is an open ball of X, on which $\mathcal{E}$ is trivial, then $\mathrm{Diff}(\mathcal{E}|_U)$ is isomorphic to $\mathrm{Mat}_r(\mathrm{Diff}(U))$ as topological algebras. The actual isomorphism depends on the choice of (holomorphic) trivialization of $\mathcal{E}$. For a holomorphic trivialization ϕ of $\mathcal{E}$ over U, let $\phi_\circ$ denote the isomorphism between $\mathrm{Diff}(\mathcal{E}|_U)$ and $\mathrm{Mat}_r(\mathrm{Diff}(U))$. Then $\phi_\circ$ induces an isomorphism of complexes $C^\bullet(\mathcal{D}\mathrm{iff}(\mathcal{E})(U)) \to C^\bullet(\mathrm{Mat}_r(\mathrm{Diff}(U)))$. This extends to an isomorphism

of complexes $\phi_*\colon \widehat{C^\bullet(\mathcal{D}\mathrm{iff}(\mathcal{E})(U))} \to \widehat{C^\bullet(\mathrm{Mat}_r(\mathrm{Diff}(U)))}$. We know however that $\widehat{C^\bullet(\mathrm{Mat}_r(\mathrm{Diff}(U)))}$ is quasi-isomorphic to $\mathbb{C}[2n]$.

The lemma will follow provided that we check that the quasi-isomorphism $\mathrm{tr}\circ\phi_*$ is independent of the choice of trivialization ϕ. We will show that if ψ is another holomorphic trivialization of $\mathcal{E}$ over U, then the maps $\mathrm{tr}\circ\psi_* = \mathrm{tr}\circ\phi_*$ of complexes from $\widehat{C^\bullet(\mathrm{Diff}(\mathcal{E}|_U))}$ to $\widehat{C^\bullet(\mathrm{Diff}(U))}$ induce the same map on cohomology.

Note that $\phi_*\circ\psi_*^{-1}\colon \widehat{C^\bullet(\mathrm{Mat}_r(\mathrm{Diff}(U)))} \to \widehat{C^\bullet(\mathrm{Mat}_r(\mathrm{Diff}(U)))}$ is induced by the map $\mathrm{Mat}_r(\mathrm{Diff}(U)) \to \mathrm{Mat}_r(\mathrm{Diff}(U))$ taking an element of $\mathrm{Mat}_r(\mathrm{Diff}(U))$ to its conjugate by an $(r\times r)$-matrix N of holomorphic functions on U. Denote this map of completed Hochschild complexes by c_N. We only need to show that the maps $\mathrm{tr}\circ c_N$ and tr of complexes of $\mathbb{C}$-vector spaces induce the same map on cohomology.

Recall that $\mathrm{Diff}(U)$ is a filtered algebra with $F_{-m}(\mathrm{Diff}(U)) = \mathrm{Diff}^{\le m}(U)$, the space of (holomorphic) differential operators on U of order at most m. This yields a filtration on $\mathrm{Mat}_r(\mathrm{Diff}(U))$ with $F_{-m}(\mathrm{Mat}_r(\mathrm{Diff}(U))) = \mathrm{Mat}_r(\mathrm{Diff}^{\le m}(U))$. We have $\mathrm{gr}(\mathrm{Mat}_r(\mathrm{Diff}(U))) = \mathrm{Mat}_r(\mathrm{gr}(\mathrm{Diff}(U)))$.

This filtration gives us a spectral sequence with $E_1^n = \widehat{\mathrm{HH}}_{-n}(\mathrm{gr}(\mathrm{Mat}_r(\mathrm{Diff}(U))))$ and $E_\infty^n = \widehat{\mathrm{HH}}_{-n}(\mathrm{Mat}_r(\mathrm{Diff}(U)))$. The endomorphism induced by the endomorphism $\alpha \rightsquigarrow N\alpha N^{-1}$, where N is a matrix of holomorphic functions, preserves the filtration on $\mathrm{Mat}_r(\mathrm{Diff}(U))$ and thus induces an endomorphism of a spectral sequence on the spectral sequence described above.

Note that $\mathrm{gr}(\mathrm{Mat}_r(\mathrm{Diff}(U))) = \mathrm{Mat}_r(\mathrm{gr}(\mathrm{Diff}(U)))$ and that the endomorphism induced by conjugation by N is still conjugation by N. Denote this endomorphism by c_N as well. We now claim that the following diagram commutes up to cohomology:

$$\begin{array}{ccc}
\widehat{C^\bullet(\mathrm{Mat}_r(\mathrm{gr}(\mathrm{Diff}(U))))} & \xrightarrow{c_N} & \widehat{C^\bullet(\mathrm{Mat}_r(\mathrm{gr}(\mathrm{Diff}(U))))} \\
\Big\downarrow{\scriptstyle \mathrm{tr}} & & \Big\downarrow{\scriptstyle \mathrm{tr}} \\
\widehat{C^\bullet(\mathrm{gr}(\mathrm{Diff}(U)))} & \xrightarrow[\mathrm{id}]{} & \widehat{C^\bullet(\mathrm{gr}(\mathrm{Diff}(U)))}.
\end{array}$$

To see this, note that if $\alpha \in \widehat{C^k(\mathrm{gr}(\mathrm{Diff}(U)))}$ is a cocycle, then it follows that $\alpha = \mathrm{tr}\left(\frac{1}{r}\mathrm{id}\otimes\cdots\otimes\mathrm{id}\otimes\alpha\right)$. In other words, α is the trace of a "scalar" matrix whose diagonal elements are α up to a scalar factor. Now

$$\begin{aligned} c_N(\mathrm{id}\otimes\cdots\otimes\mathrm{id}\otimes\alpha) &= (N\otimes\cdots\otimes N)(\mathrm{id}\otimes\cdots\otimes\mathrm{id}\otimes\alpha)(N^{-1}\otimes\cdots\otimes N^{-1}) \\ &= (\mathrm{id}\otimes\cdots\otimes\mathrm{id}\otimes\alpha), \end{aligned}$$

so $(\mathrm{tr}\circ c_N)_*(\overline{\mathrm{id}\otimes\cdots\otimes\mathrm{id}\otimes\alpha}) = \mathrm{tr}_*(\overline{\mathrm{id}\otimes\cdots\otimes\mathrm{id}\otimes\alpha})$ where $\overline{\mathrm{id}\otimes\cdots\otimes\mathrm{id}\otimes\alpha}$ denotes the class in cohomology of the cocycle $\mathrm{id}\otimes\cdots\otimes\mathrm{id}\otimes\alpha$.

Therefore, the map of spectral sequences induced by $\mathrm{tr}\circ c_N$ coincides with that induced by tr at the E_1-level, and hence at the $E_2 = E_\infty$-level. Finally, by part 2 of

this proof and by the proof-sketch for Lemma 2, $E_2^{-n,-n}$ is the only non-zero E_2-term in this spectral sequence, and $E_2^{-n,-n} \simeq \mathbb{C}$. It follows that the endomorphisms on cohomology induced by tr and tr $\circ\, c_N$ are indeed the endomorphisms they induce on $E_2^{-n,-n}$. We have just shown that the maps induced tr and tr $\circ\, c_N$ on E_r-terms coincide for any $r \geq 1$. This proves the lemma. □

We can now state the following immediate corollary to Lemma 3.

Corollary 6. $\widehat{\mathrm{HH}}_{-i}(\mathcal{D}\mathrm{iff}(\mathcal{E})) \simeq \mathrm{H}^{2n-i}(X, \mathbb{C})$.

We denote the isomorphism described in this corollary by $\beta_{\mathcal{E}}$.

4.2. The second "bookkeeping" lemma. We once more look at the situation where $\mathcal{E}$ is a direct summand of $\mathcal{E} \oplus \mathcal{F}$. Notation is as in Section 3.4 of this article. We have a map $\iota\colon \mathcal{D}\mathrm{iff}(\mathcal{E}) \to \mathcal{D}\mathrm{iff}(\mathcal{E} \oplus \mathcal{F})$. This induces a map, denoted by $\hat{\iota}$, from $\widehat{\mathrm{Hoch}}(\mathcal{D}\mathrm{iff}(\mathcal{E}))$ to $\widehat{\mathrm{Hoch}}(\mathcal{D}\mathrm{iff}(\mathcal{E} \oplus \mathcal{F}))$. The following lemma holds.

Lemma 4. *The following diagram commutes:*

$$\begin{array}{ccc} \widehat{\mathrm{HH}}_{-i}(\mathcal{D}\mathrm{iff}(\mathcal{E})) & \xrightarrow{\hat{\iota}_*} & \widehat{\mathrm{HH}}_{-i}(\mathcal{D}\mathrm{iff}(\mathcal{E} \oplus \mathcal{F})) \\ \downarrow{\scriptstyle \beta_{\mathcal{E}}} & & \downarrow{\scriptstyle \beta_{\mathcal{E}\oplus\mathcal{F}}} \\ \mathrm{H}^{2n-i}(X, \mathbb{C}) & \xrightarrow[\mathrm{id}]{} & \mathrm{H}^{2n-i}(X, \mathbb{C}). \end{array}$$

Proof. Step 1: Let $\mathrm{D}(\mathrm{Sh}_{\mathbb{C}}(X))$ denote the derived category of sheaves of $\mathbb{C}$-vector spaces on X.

Recall that in the proof of Lemma 3 we showed that the complex $\widehat{\mathrm{Hoch}}(\mathcal{D}\mathrm{iff}(\mathcal{E}))$ of sheaves of $\mathbb{C}$-vector spaces is quasi-isomorphic to $\widehat{\mathrm{Hoch}}(\mathcal{D}\mathrm{iff}(\mathcal{O}_X))$. Denote this quasi-isomorphism by $i_{\mathcal{E}}$. Now $\widehat{\mathrm{Hoch}}(\mathcal{D}\mathrm{iff}(\mathcal{O}_X))$ is quasi-isomorphic to $\underline{\mathbb{C}}[2n]$. Let i denote this quasi-isomorphism for this proof.

It suffices to verify that the following diagram commutes in $\mathrm{D}(\mathrm{Sh}_{\mathbb{C}}(X))$:

$$\begin{array}{ccc} \widehat{\mathrm{Hoch}}(\mathcal{D}\mathrm{iff}(\mathcal{E})) & \xrightarrow{\hat{\iota}} & \widehat{\mathrm{Hoch}}(\mathcal{D}\mathrm{iff}(\mathcal{E} \oplus \mathcal{F})) \\ \downarrow{\scriptstyle i \circ i_{\mathcal{E}}} & & \downarrow{\scriptstyle i \circ i_{\mathcal{E}\oplus\mathcal{F}}} \\ \underline{\mathbb{C}}[2n] & \xrightarrow[\mathrm{id}]{} & \underline{\mathbb{C}}[2n]. \end{array} \tag{16}$$

Since a sheaf of $\mathbb{C}$-vector spaces is injective iff it is flabby (see [Riet], Lemma 3.3), the constant sheaf $\underline{\mathbb{C}}$ is an injective object in the category of sheaves of $\mathbb{C}$-vector spaces on X. It follows from this that

$$\mathrm{Hom}_{\mathrm{D}(\mathrm{Sh}_{\mathbb{C}}(X))}(\underline{\mathbb{C}}, \underline{\mathbb{C}}) \simeq \mathbb{C}.$$

The diagram (16) thus commutes in $\mathrm{D}(\mathrm{Sh}_{\mathbb{C}}(X))$ up to a scalar factor. Checking that that scalar factor is 1 can be "done locally". It therefore suffices to verify that there exists a neighbourhood U of every point in X such that the following diagram commutes in $\mathrm{D}(\mathrm{Sh}_{\mathbb{C}}(U))$:

$$\begin{array}{ccc} \widehat{\mathrm{Hoch}}(\mathcal{D}\mathrm{iff}(\mathcal{E}))|_U & \xrightarrow{\hat{\imath}|_U} & \widehat{\mathrm{Hoch}}(\mathcal{D}\mathrm{iff}(\mathcal{E}\oplus\mathcal{F}))|_U \\ \downarrow{\scriptstyle i\circ i_{\mathcal{E}}|_U} & & \downarrow{\scriptstyle i\circ i_{\mathcal{E}\oplus\mathcal{F}}|_U} \\ \underline{\mathbb{C}}[2n] & \xrightarrow[\mathrm{id}]{} & \underline{\mathbb{C}}[2n]. \end{array} \tag{17}$$

Step 2: Verifying (17).

It suffices to verify (17) at the level of pre-sheaves. We must therefore prove that the following diagram commutes in the category of complexes of $\mathbb{C}$-vector spaces up to cohomology:

$$\begin{array}{ccc} \widehat{C^\bullet(\mathcal{D}\mathrm{iff}(\mathcal{E})(U))} & \xrightarrow{\hat{\imath}} & \widehat{C^\bullet(\mathcal{D}\mathrm{iff}(\mathcal{E}\oplus\mathcal{F})(U))} \\ \downarrow{\scriptstyle i\circ i_{\mathcal{E}}|_U} & & \downarrow{\scriptstyle i\circ i_{\mathcal{E}\oplus\mathcal{F}}|_U} \\ \mathbb{C}[2n] & \xrightarrow[\mathrm{id}]{} & \mathbb{C}[2n]. \end{array}$$

To verify that the above diagram commutes up to cohomology, it suffices to verify that the diagram below commutes up to cohomology in the category of complexes of $\mathbb{C}$-vector spaces:

$$\begin{array}{ccc} \widehat{C^\bullet(\mathcal{D}\mathrm{iff}(\mathcal{E})(U))} & \xrightarrow{\hat{\imath}} & \widehat{C^\bullet(\mathcal{D}\mathrm{iff}(\mathcal{E}\oplus\mathcal{F})(U))} \\ \downarrow{\scriptstyle i_{\mathcal{E}}|_U} & & \downarrow{\scriptstyle i_{\mathcal{E}\oplus\mathcal{F}}|_U} \\ \widehat{C^\bullet(\mathcal{D}\mathrm{iff}(U))} & \xrightarrow[\mathrm{id}]{} & \widehat{C^\bullet(\mathcal{D}\mathrm{iff}(U))}. \end{array} \tag{18}$$

Most of the hard work necessary for this step has already been done. Let $\phi_{\mathcal{E}}$ and $\phi_{\mathcal{F}}$ denote holomorphic trivializations over U of $\mathcal{E}$ and $\mathcal{F}$, respectively. Then $\phi := \phi_{\mathcal{E}} \oplus \phi_{\mathcal{F}}$ is a holomorphic trivialization of $\mathcal{E}\oplus\mathcal{F}$ over U.

This yields the following commutative diagram, all of whose morphisms are continuous. Here r and s denote the ranks of $\mathcal{E}$ and $\mathcal{F}$, respectively:

$$\begin{array}{ccc} \mathcal{D}\mathrm{iff}(\mathcal{E})(U) & \xrightarrow{\iota} & \mathcal{D}\mathrm{iff}(\mathcal{E}\oplus\mathcal{F})(U) \\ \downarrow{\scriptstyle \phi_{\mathcal{E}}} & & \downarrow{\scriptstyle \phi} \\ \mathrm{Mat}_r(\mathrm{Diff}(U)) & \xrightarrow[\iota_{r,s}]{} & \mathrm{Mat}_{r+s}(\mathrm{Diff}(U)). \end{array}$$

This gives the following commutative diagram:

$$\begin{array}{ccc} \widehat{C^\bullet(\mathcal{D}\mathrm{iff}(\mathcal{E})(U))} & \xrightarrow{\hat{\iota}_*} & \widehat{C^\bullet(\mathcal{D}\mathrm{iff}(\mathcal{E}\oplus\mathcal{F})(U))} \\ \Big\downarrow{\scriptstyle \phi_{\mathcal{E},*}} & & \Big\downarrow{\scriptstyle \phi_*} \\ \widehat{C^\bullet(\mathrm{Mat}_r(\mathrm{Diff}(U)))} & \xrightarrow[\hat{\iota}_{r,s,*}]{} & \widehat{C^\bullet(\mathrm{Mat}_{r+s}(\mathrm{Diff}(U)))}. \end{array}$$

Here $\iota_{r,s}$ denotes the embedding from $\mathrm{Mat}_r(\mathrm{Diff}(U))$ to $\mathrm{Mat}_{r+s}(\mathrm{Diff}(U))$ which takes a matrix $\alpha \in \mathrm{Mat}_r(\mathrm{Diff}(U))$ to $\bar{\alpha} \in \mathrm{Mat}_{r+s}(\mathrm{Diff}(U))$, where $\bar{\alpha}_{ij} = \alpha_{ij}$ if $(i, j) \in \{1, \ldots, r\} \times \{1, \ldots, r\}$ and $\bar{\alpha}_{ij} = 0$ otherwise.

Now we showed that $i_{\mathcal{E}}|_U = \mathrm{tr}_r \circ \phi_{\mathcal{E},*}$ in part 3 of the proof Lemma 3. Here tr_r is the map $\mathrm{tr}\colon \widehat{C^\bullet(\mathrm{Mat}_r(\mathrm{Diff}(U)))} \to \widehat{C^\bullet(\mathrm{Diff}(U))}$ described in the proof of Lemma 3. Similarly, $i_{(\mathcal{E}\oplus\mathcal{F})|_U} = \mathrm{tr}_{r+s} \circ \phi_*$.

It therefore suffices to prove that the following diagram commutes:

$$\begin{array}{ccc} \widehat{C^\bullet(\mathrm{Mat}_r(\mathrm{Diff}(U)))} & \xrightarrow{\hat{\iota}_{r,s,*}} & \widehat{C^\bullet(\mathrm{Mat}_{r+s}(\mathrm{Diff}(U)))} \\ \Big\downarrow{\scriptstyle \mathrm{tr}_r} & & \Big\downarrow{\scriptstyle \mathrm{tr}_{r+s}} \\ \widehat{C^\bullet(\mathrm{Diff}(U))} & \xrightarrow[\mathrm{id}]{} & \widehat{C^\bullet(\mathrm{Diff}(U))}. \end{array}$$

This follows immediately from the explicit formula for tr_r recalled in part 2 of the proof of Lemma 3. This finally verifies (18), thus proving the lemma. □

4.3. Proof of Theorem 2

Proof of Theorem 2. Corollary 5 states that the following diagram commutes:

$$\begin{array}{ccc} \widehat{\mathrm{HH}}_0(\mathcal{D}\mathrm{iff}(\mathcal{E})) & \xrightarrow{\hat{\iota}_*} & \widehat{\mathrm{HH}}_0(\mathcal{D}\mathrm{iff}(\mathcal{E}\oplus\mathcal{F})) \\ \Big\downarrow{\scriptstyle \hat{\mathrm{tr}}} & & \Big\downarrow{\scriptstyle \hat{\mathrm{tr}}} \\ \mathbb{C} & \xrightarrow[\mathrm{id}]{} & \mathbb{C}. \end{array}$$

Lemma 4 for $i = 0$ now states that the following diagram commutes:

$$\begin{array}{ccc} \widehat{\mathrm{HH}}_0(\mathcal{D}\mathrm{iff}(\mathcal{E})) & \xrightarrow{\hat{\iota}_*} & \widehat{\mathrm{HH}}_0(\mathcal{D}\mathrm{iff}(\mathcal{E}\oplus\mathcal{F})) \\ \Big\downarrow{\scriptstyle \beta_{\mathcal{E}}} & & \Big\downarrow{\scriptstyle \beta_{\mathcal{E}\oplus\mathcal{F}}} \\ \mathrm{H}^{2n}(X, \mathbb{C}) & \xrightarrow[\mathrm{id}]{} & \mathrm{H}^{2n}(X, \mathbb{C}). \end{array}$$

It follows that $\widehat{\mathrm{tr}} \circ {\beta_{\mathcal{E}}}^{-1} = \widehat{\mathrm{tr}} \circ {\beta_{\mathcal{E}\oplus\mathcal{F}}}^{-1}$. In the notation of Theorem 2, this says that $I_{\mathcal{E}} = I_{\mathcal{E}\oplus\mathcal{F}}$. A symmetric argument shows that $I_{\mathcal{F}} = I_{\mathcal{E}\oplus\mathcal{F}}$. This completes the proof of Theorem 2. □

5. The completed cyclic homology of Diff($\mathcal{E}$)

5.1. Recollections on cyclic homology

5.1.1. Tsygan's double complex. Let $\mathcal{A}$ be a dg $\mathbb{C}$-algebra such that $\mathcal{A}^n = 0$ for almost all n. Let d_{bar} and d_{Hoch} denote the differentials of $\mathrm{bar}^\bullet(\mathcal{A})$ and $C^\bullet(\mathcal{A})$ respectively.

For homogenous elements $a_1, \dots, a_k$ of $\mathcal{A}$ of degrees $d_1, \dots, d_k$, respectively, let the map $a_1 \otimes \cdots \otimes a_k \rightsquigarrow (-1)^{(d_k+1)(d_1+\cdots+d_{k-1}+k-1)} a_k \otimes a_1 \otimes \cdots \otimes a_{k-1}$ be denoted by $\tau\colon \mathcal{A}^{\otimes k} \to \mathcal{A}^{\otimes k}$. Let $N\colon \mathcal{A}^{\otimes k} \to \mathcal{A}^{\otimes k}$ be the map $N = 1 + \tau + \cdots + \tau^{k-1}$.

The maps τ and N induce maps from $\mathcal{A}^{\otimes k}$ to $\mathcal{A}^{\otimes k}$ for any k. It follows that τ and N induce maps from $\mathrm{bar}^{-n}(\mathcal{A})$ to $C^{-n}(\mathcal{A})$ and from $C^{-n}(\mathcal{A})$ to $\mathrm{bar}^{-n}(\mathcal{A})$, respectively, for any n.

Consider the following double complex the degree of whose non-zero columns is non-positive:

$$\begin{array}{ccccccccc}
\cdots & \longrightarrow & \cdots & \longrightarrow & \cdots & \longrightarrow & \cdots & \longrightarrow & \cdots \\
\uparrow{\scriptstyle d_{\mathrm{Hoch}}} & & \uparrow{\scriptstyle -d_{\mathrm{bar}}} & & \uparrow{\scriptstyle d_{\mathrm{Hoch}}} & & \uparrow{\scriptstyle -d_{\mathrm{bar}}} & & \uparrow{\scriptstyle d_{\mathrm{Hoch}}} \\
\cdots & \xrightarrow{N} & \mathrm{bar}^0(\mathcal{A}) & \xrightarrow{\mathrm{id}-\tau} & C^0(\mathcal{A}) & \xrightarrow{N} & \mathrm{bar}^0(\mathcal{A}) & \xrightarrow{\mathrm{id}-\tau} & C^0(\mathcal{A}) \\
\uparrow{\scriptstyle d_{\mathrm{Hoch}}} & & \uparrow{\scriptstyle -d_{\mathrm{bar}}} & & \uparrow{\scriptstyle d_{\mathrm{Hoch}}} & & \uparrow{\scriptstyle -d_{\mathrm{bar}}} & & \uparrow{\scriptstyle d_{\mathrm{Hoch}}} \\
\cdots & \xrightarrow{N} & \mathrm{bar}^{-1}(\mathcal{A}) & \xrightarrow{\mathrm{id}-\tau} & C^{-1}(\mathcal{A}) & \xrightarrow{N} & \mathrm{bar}^{-1}(\mathcal{A}) & \xrightarrow{\mathrm{id}-\tau} & C^{-1}(\mathcal{A}) \\
\uparrow{\scriptstyle d_{\mathrm{Hoch}}} & & \uparrow{\scriptstyle -d_{\mathrm{bar}}} & & \uparrow{\scriptstyle d_{\mathrm{Hoch}}} & & \uparrow{\scriptstyle -d_{\mathrm{bar}}} & & \uparrow{\scriptstyle d_{\mathrm{Hoch}}} \\
\cdots & \xrightarrow{N} & \mathrm{bar}^{-2}(\mathcal{A}) & \xrightarrow{\mathrm{id}-\tau} & C^{-2}(\mathcal{A}) & \xrightarrow{N} & \mathrm{bar}^{-2}(\mathcal{A}) & \xrightarrow{\mathrm{id}-\tau} & C^{-2}(\mathcal{A}) \\
\uparrow{\scriptstyle d_{\mathrm{Hoch}}} & & \uparrow{\scriptstyle -d_{\mathrm{bar}}} & & \uparrow{\scriptstyle d_{\mathrm{Hoch}}} & & \uparrow{\scriptstyle -d_{\mathrm{bar}}} & & \uparrow{\scriptstyle d_{\mathrm{Hoch}}} \\
\cdots & \longrightarrow & \cdots & \longrightarrow & \cdots & \longrightarrow & \cdots & \longrightarrow & \cdots.
\end{array}$$

We denote the above double complex by $\mathrm{CC}^{\bullet,\bullet}(\mathcal{A})$. Then $\mathrm{CC}^{p,q}(\mathcal{A}) = C^q(\mathcal{A})$ if p is even and non-positive, $\mathrm{CC}^{p,q}(\mathcal{A}) = \mathrm{bar}^q(\mathcal{A})$ if p is odd and negative, and $\mathrm{CC}^{p,q}(\mathcal{A}) = 0$ otherwise.

The vertical differential $d_v\colon \mathrm{CC}^{p,q}(\mathcal{A}) \to \mathrm{CC}^{p,q+1}(\mathcal{A})$ is d_{Hoch} if p is even and non-positive, and $-d_{\mathrm{bar}}$ if p is odd and negative.

The horizontal differential $d_h\colon \mathrm{CC}^{p,q}(\mathcal{A}) \to \mathrm{CC}^{p+1,q}(\mathcal{A})$ is given by $\mathrm{id} - \tau$ if p is odd and negative and N if p is even and negative.

The double complex $\mathrm{CC}^{\bullet,\bullet}(\mathcal{A})$ is called the *Tsygan double complex* of $\mathcal{A}$.

Definition 7. The *cyclic complex* of $\mathcal{A}$ is the total complex of $\mathrm{CC}^{\bullet,\bullet}(\mathcal{A})$. It is denoted by $\mathrm{Cycl}^{\bullet}(\mathcal{A})$.

Definition 8. The *cyclic homology* $\mathrm{HC}_{-i}(\mathcal{A})$ is the $-i$-th cohomology of $\mathrm{Cycl}^{\bullet}(\mathcal{A})$.

As in Section 2.1, contrary to the standard practice, we refer to a $-i$ cocycle of $\mathrm{Cycl}^{\bullet}(\mathcal{A})$ as an i-cycle in $\mathrm{Cycl}^{\bullet}(\mathcal{A})$. The following standard propositions are important to us.

Proposition 8. (i) *If $\mathcal{B} = \mathrm{End}(V^{\bullet})$, where $V^{\bullet}$ is a finite dimensional graded $\mathbb{C}$-vector space with zero differential, then*

$$\mathrm{HC}_{-2i}(\mathcal{B}) \simeq \mathbb{C}$$

for all $i \geq 0$. All other cyclic homologies of $\mathcal{B}$ vanish.

(ii) *The map tr_{2i} yielding the above isomorphism is obtained by mapping the class in $\mathrm{HC}_{-2i}(\mathcal{B})$ of a $2i$-cycle in $\mathrm{Cycl}^{\bullet}(\mathcal{B})$ given by a tuple $(b_{2i}, b_{2i-1}, \ldots, b_0)$ to $\mathrm{str}(b_0)$, where $b_k \in \mathrm{CC}^{-2i+k,-k}(\mathcal{B})$.*

(iii) *The map tr_{2i} yielding the above isomorphism maps the class in $\mathrm{HC}_{-2i}(\mathcal{B})$ of a $2i$-cycle in $\mathrm{Cycl}^{\bullet}(\mathcal{B})$ given by a tuple $(b_{2i}, b_{2i-1}, \ldots, b_0, b_{-1}, \ldots, b_{-2j})$ ($j > 0$, $b_k \in \mathrm{CC}^{-2i+k,-k}(\mathcal{B})$) to $\mathrm{str}(b_0)$.*

Proof. There is a spectral sequence, namely the one that arises out of the filtration of the double complex $\mathrm{CC}^{\bullet,\bullet}(\mathcal{B})$ by columns, converging to $\mathrm{HC}_{\bullet}(\mathcal{B})$ such that $E_1^{p,q} = \mathcal{H}^q(\mathrm{CC}^{p,\bullet}(\mathcal{B}))$. Since $\mathcal{B}$ has a unit, $\mathrm{bar}^{\bullet}(\mathcal{B})$ is acyclic (see p. 411). Now $C^{\bullet}(\mathcal{B})$ is quasi-isomorphic to $\mathbb{C}$ concentrated in degree 0 by Proposition 1. It follows that $E_1^{-2i,0} \simeq \mathbb{C}$ for all $i \geq 0$ and $E_1^{p,q} = 0$ for all other (p, q). This spectral sequence therefore collapses at E_1. It follows that $\mathrm{HC}_{-2i}(\mathcal{B}) \simeq E_1^{-2i,0} \simeq \mathbb{C}$ for all $i \geq 0$. This proves part (i).

Let $F^{\bullet}$ be the filtration on $\mathrm{Cycl}^{\bullet}(\mathcal{B})$ yielding the spectral sequence in part (i). Consider a tuple $(b_{2i}, b_{2i-1}, \ldots, b_0)$ with $b_k \in \mathrm{CC}^{-2i+k,-k}(\mathcal{B})$ that yields a cyclic cycle. Then $(b_{2i}, b_{2i-1}, \ldots, b_0) \in F^{-2i}\,\mathrm{Cycl}^{\bullet}(\mathcal{B})$. It follows that the image of $(b_{2i}, b_{2i-1}, \ldots, b_0)$ in $\frac{F^{-2i}\,\mathrm{Cycl}^{\bullet}(\mathcal{B})}{F^{-2i+1}\,\mathrm{Cycl}^{\bullet}(\mathcal{B})} \simeq C^{\bullet}(\mathcal{B})$ is the Hochschild 0-cycle b_0. Note that $E_1^{-2i+j,-j} = 0$ for $j \neq 0$ and $E_1^{-2i,0} = \mathrm{H}^0(\frac{F^{-2i}\,\mathrm{Cycl}^{\bullet}(\mathcal{B})}{F^{-2i+1}\,\mathrm{Cycl}^{\bullet}(\mathcal{B})}) \simeq \mathrm{HH}_0(\mathcal{B})$. It follows that the image of the tuple $(b_{2i}, b_{2i-1}, \ldots, b_0)$ in $E_1^{-2i,0}$, and hence in $\mathrm{HC}_{-2i}(\mathcal{B})$, is the image of b_0 in $\mathrm{HH}_0(\mathcal{B})$. By Proposition 1, this is precisely $\mathrm{str}(b_0)$. This shows that $\mathrm{tr}_{2i}(\widetilde{(b_{2i}, \ldots, b_0)}) = \mathrm{str}(b_0)$ where $\widetilde{(b_{2i}, \ldots, b_0)}$ is the class in $\mathrm{HC}_{-2i}(\mathcal{B})$ of the cycle obtained from $(b_{2i}, b_{2i-1}, \ldots, b_0)$. This proves part (ii).

To prove part (iii), note that if $(b_{2i}, b_{2i-1}, \ldots, b_0, b_{-1}, \ldots, b_{-2j})$ $(j > 0)$ is a cyclic cycle, then b_{-2j} is a Hochschild $-2j$-cycle. It follows from Proposition 1 that $b_{-2j} = d_{\mathrm{Hoch}} c_{-2j+1}$ for some $c_{-2j+1} \in C^{2j-1}(\mathcal{B})$. Now consider $c_{-2j+1} \in \mathrm{CC}^{-2i-2j,2j-1}(\mathcal{B})$ as an element of $\mathrm{Cycl}^{-2i-1}(\mathcal{B})$. Then the cycle $(b_{2i}, b_{2i-1}, \ldots, b_0, b_{-1}, \ldots, b_{-2j}) - d_{\mathrm{Cycl}} c_{-2j+1}$ arises out of the cycle $(b_{2i}, b_{2i-1}, \ldots, b_0, b_{-1}, \ldots, b_{-2j+1} + Nc_{-2j+1}, 0)$ of $\mathrm{CC}^{\bullet,\bullet}(\mathcal{B})$. Therefore the class of $(b_{2i}, b_{2i-1}, \ldots, b_0, b_{-1}, \ldots, b_{-2j})$ in $\mathrm{HC}_{-2i}(\mathcal{B})$ is the same as the class of $(b_{2i}, b_{2i-1}, \ldots, b_0, b_{-1}, \ldots, b_{-2j+1} + Nc_{-2j+1}, 0)$ in $\mathrm{HC}_{-2i}(\mathcal{B})$. Since the bar complex of $\mathcal{B}$ is acyclic, it follows that $b_{-2j+1} + Nc_{-2j+1} = d_{\mathrm{bar}} c_{-2j+2}$ for some element c_{-2j+2} of $\mathrm{bar}^{2j-2}(\mathcal{B})$. The previous step can be repeated to show that the class of $(b_{2i}, b_{2i-1}, \ldots, b_0, b_{-1}, \ldots, b_{-2j})$ in $\mathrm{HC}_{-2i}(\mathcal{B})$ is the same as the class of $(b_{2i}, b_{2i-1}, \ldots, b_0, b_{-1}, \ldots, b_{-2j+2} + (\mathrm{id} - \tau)c_{-2j+2}, 0, 0)$ in $\mathrm{HC}_{-2i}(\mathcal{B})$. This process can be continued to show that the class of $(b_{2i}, b_{2i-1}, \ldots, b_0, b_{-1}, \ldots, b_{-2j})$ in $\mathrm{HC}_{-2i}(\mathcal{B})$ is the same as the class of $(b_{2i}, b_{2i-1}, \ldots, b_0 + (\mathrm{id} - \tau)c_0)$ in $\mathrm{HC}_{-2i}(\mathcal{B})$ for some element c_0 of $\mathrm{bar}^0(\mathcal{B})$. By part (ii) of this proposition, this is equal to $\mathrm{str}(b_0 + (\mathrm{id} - \tau)c_0)$. But $\mathrm{str}((\mathrm{id} - \tau)c_0) = 0$ for any element c_0 of $\mathrm{bar}^0(\mathcal{B})$. This proves (iii). □

If $\mathcal{A}$ is a (unital) dg $\mathbb{C}$-algebra, let $\mathrm{CC}^{\{2\},\bullet}(\mathcal{A})$ denote the bi-complex consisting of the columns $\mathrm{CC}^{-1,\bullet}(\mathcal{A})$ and $\mathrm{CC}^{0,\bullet}(\mathcal{A})$ with differentials as in $\mathrm{CC}^{\bullet,\bullet}(\mathcal{A})$.

We recall that we have an exact sequence of complexes

$$0 \to \mathrm{Tot}(\mathrm{CC}^{\{2\},\bullet}(\mathcal{A})) \xrightarrow{I} \mathrm{Cycl}^{\bullet}(\mathcal{A}) \xrightarrow{S} \mathrm{Cycl}^{\bullet+2}(\mathcal{A}) \to 0.$$

We recall that $\mathrm{CC}^{1,\bullet}(\mathcal{A})$ is acyclic (see p. 411). It follows that $\mathrm{CC}^{\{2\},\bullet}(\mathcal{A})$ is quasi-isomorphic to $C^{\bullet}(\mathcal{A})$. This quasi-isomorphism is realized by the map of complexes taking $a_k \in C^{-k}(\mathcal{A})$ to $(a_k, 0) \in C^{-k}(\mathcal{A}) \oplus \mathrm{bar}^{1-k}(\mathcal{A})$. We denote the composite of I with this quasi-isomorphism by $\mathcal{I}$. The map S is obtained by projection to the double complex obtained from $\mathrm{CC}^{\bullet,\bullet}(\mathcal{A})$ by truncating the columns $\mathrm{CC}^{i,\bullet}(\mathcal{A})$ for $i = 0, -1$. We now obtain the following proposition.

Proposition 9. *If $\mathcal{B} = \mathrm{End}(V^{\bullet})$, where $V^{\bullet}$ is a finite dimensional graded $\mathbb{C}$-vector space with zero differential, then the following diagrams commute:*

$$\begin{array}{ccc} \mathrm{HC}_{-2i}(\mathcal{B}) & \xrightarrow{S} & \mathrm{HC}_{-2i+2}(\mathcal{B}) \\ {\scriptstyle \mathrm{tr}_{2i}}\downarrow & & \downarrow{\scriptstyle \mathrm{tr}_{2i-2}} \\ \mathbb{C} & \xrightarrow[\mathrm{id}]{} & \mathbb{C}, \end{array} \qquad \begin{array}{ccc} \mathrm{HH}_0(\mathcal{B}) & \xrightarrow{\mathcal{I}} & \mathrm{HC}_0(\mathcal{B}) \\ {\scriptstyle \mathrm{str}}\downarrow & & \downarrow{\scriptstyle \mathrm{tr}_0} \\ \mathbb{C} & \xrightarrow[\mathrm{id}]{} & \mathbb{C}. \end{array}$$

Proof. Let $(b_{2i}, b_{2i-1}, \ldots, b_0)$, $b_j \in \mathrm{CC}^{-2i+j,-j}(\mathcal{B})$, be a tuple yielding a cyclic cycle. We already demonstrated while proving part (iii) of Proposition 8 that any class in $\mathrm{HC}_{-2i}(\mathcal{B})$ can be represented by a cycle coming from a tuple of this form.

Let $\widetilde{(b_{2i}, b_{2i-1}, \ldots, b_0)}$ denote the class of $(b_{2i}, b_{2i-1}, \ldots, b_0)$ in $\mathrm{HC}_{-2i}(\mathcal{B})$. Then $\mathrm{tr}_{2i}(\widetilde{(b_{2i}, b_{2i-1}, \ldots, b_0)}) = \mathrm{str}(b_0)$ by Proposition 8 and $S((b_{2i}, b_{2i-1}, \ldots, b_0)) = (b_{2i-2}, b_{2i-3}, \ldots, b_0)$ by the definition of S. Finally, by Proposition 8, $\mathrm{tr}_{2i-2}(\widetilde{(b_{2i-2}, b_{2i-3}, \ldots, b_0)}) = \mathrm{str}(b_0)$. This shows that the first diagram commutes.

Let b_0 be a Hochschild 0-cycle of $\mathcal{B}$. Then $\mathcal{I}(b_0) = b_0 \in \mathrm{CC}^{0,0}(\mathcal{B})$. Let $\bar{b_0}$ denote the class of b_0 in $\mathrm{HH}_0(\mathcal{B})$. Then, $\bar{b_0} = \mathrm{str}(b_0)$. However, $\mathcal{I}(\bar{b_0}) = \widetilde{b_0}$. Now $\mathrm{tr}_0(\widetilde{b_0}) = \mathrm{str}(b_0)$ by Proposition 8. This proves that the second diagram commutes. □

Moreover, if $\mathcal{F}: \mathcal{A} \to \mathcal{B}$ is an A_∞-morphism with Taylor components $\mathcal{F}_k$, we have the following result.

Proposition 10. *The map $\mathcal{F}_{\mathrm{Hoch}}$ mentioned in Proposition* 2 *extends to a map $\mathcal{F}_{\mathrm{Cycl}}$ of complexes from* $\mathrm{Cycl}^\bullet(\mathcal{A})$ *to* $\mathrm{Cycl}^\bullet(\mathcal{B})$.

Proof. It suffices to check that $\mathcal{F}_{\mathrm{Hoch}}$ extends to a map of $\mathcal{F}_{\mathrm{Tsyg}}$ bi-complexes from $\mathrm{CC}^{\bullet,\bullet}(\mathcal{A})$ to $\mathrm{CC}^{\bullet,\bullet}(\mathcal{B})$.

Let $\mathcal{F}_{\mathrm{Tsyg}}: \mathrm{CC}^{p,q}(\mathcal{A}) \to \mathrm{CC}^{p,q}(\mathcal{B})$ be $\mathcal{F}_{\mathrm{Hoch}}$ if p is even and non-positive and $\mathcal{F}_{\mathrm{bar}}$ if p is odd and negative.

By Proposition 2, $\mathcal{F}_{\mathrm{Hoch}}: \mathrm{CC}^{p,\bullet}(\mathcal{A}) \to \mathrm{CC}^{p,\bullet}(\mathcal{B})$ is a map of complexes of $\mathbb{C}$-vector spaces if p is even and non-positive. By the definition of an A_∞-morphism, $\mathcal{F}_{\mathrm{bar}}: \mathrm{CC}^{p,\bullet}(\mathcal{A}) \to \mathrm{CC}^{p,\bullet}(\mathcal{B})$ is a map of complexes if p is odd and negative.

The following verifications, which we leave to the reader, complete the proof that $\mathcal{F}_{\mathrm{Tsyg}}$ is a map of bi-complexes, and thus yields a map $\mathcal{F}_{\mathrm{Cycl}}: \mathrm{Cycl}^\bullet(\mathcal{A}) \to \mathrm{Cycl}^\bullet(\mathcal{B})$ of complexes:

$$(\mathrm{id} - \tau) \circ \mathcal{F}_{\mathrm{bar}} = \mathcal{F}_{\mathrm{Hoch}} \circ (1 - \tau),$$
$$N \circ \mathcal{F}_{\mathrm{Hoch}} = \mathcal{F}_{\mathrm{bar}} \circ N.$$

Here N and τ are as in the definitions of $\mathrm{CC}^{\bullet,\bullet}(\mathcal{A})$ and $\mathrm{CC}^{\bullet,\bullet}(\mathcal{B})$. □

We also state the following consequence of Propositions 10 and 8 as a proposition.

Proposition 11. *Let $\mathcal{A}$ be a dg $\mathbb{C}$ algebra. Let $\mathcal{B}$ be as in Proposition* 9. *Suppose that $\mathcal{F}$ is an A_∞-morphism from $\mathcal{A}$ to $\mathcal{B}$. Let $\widetilde{\mathcal{F}_{\mathrm{Cycl}}}$ denote the map from* $\mathrm{HC}_\bullet(\mathcal{A})$ *to* $\mathrm{HC}_\bullet(\mathcal{B})$ *induced by $\mathcal{F}_{\mathrm{Cycl}}$. Then the following holds:*

(1) $\mathrm{tr}_{2i} \circ \widetilde{\mathcal{F}_{\mathrm{Cycl}}}(\widetilde{(a_{2i}, \ldots, a_0, a_{-1}, \ldots, a_{-l})}) = \mathrm{str}(\mathcal{F}_{\mathrm{Hoch}}(a_0))$.

(2) *If a_0 is a Hochschild* 0- *cycle arising out of a degree $k-1$ element of $\mathcal{A}^{\otimes k}$ then*

$$\mathrm{tr}_{2i} \circ \widetilde{\mathcal{F}_{\mathrm{Cycl}}}(\widetilde{(a_{2i}, \ldots, a_0, a_{-1}, \ldots, a_{-l})}) = \sum_{s=0}^{s=k-1} \mathrm{str}(\mathcal{F}_k(\tau^s(a_0))),$$

where τ is as in Corollary 2.

Proof. By the proof of Proposition 10, we have

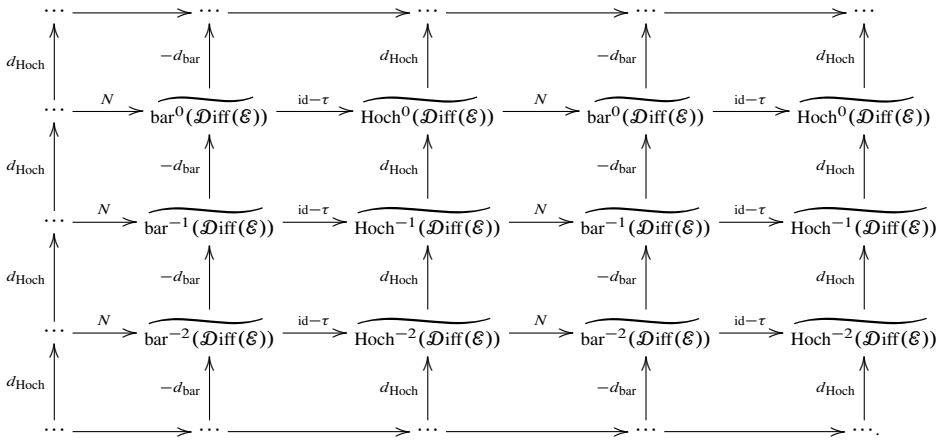
$$\mathcal{F}_{\mathrm{Tsyg}}(a_{2i},\dots,a_0,a_{-1},\dots,a_{-l}) = (\mathcal{F}_{\mathrm{Hoch}}(a_{2i}),\mathcal{F}_{\mathrm{bar}}(a_{2i-1}),\dots,\mathcal{F}_{\mathrm{bar}}(a_1),\mathcal{F}_{\mathrm{Hoch}}(a_0),\dots,\mathcal{F}_{\mathrm{bar/hoch}}(a_{-l})),$$

where $\mathcal{F}_{\mathrm{bar/Hoch}}(a_{-l}) = \mathcal{F}_{\mathrm{Hoch}}(a_{-l})$ if l is even and $\mathcal{F}_{\mathrm{bar/Hoch}}(a_{-l}) = \mathcal{F}_{\mathrm{bar}}(a_{-l})$ if l is odd. Part (1) now follows immediately from Proposition 8, part (iii), and part (2) is a consequence of part (1) and Corollary 2. □

5.2. The complex $\widetilde{\mathrm{Cycl}^\bullet(\mathcal{D}\mathrm{iff}(\mathcal{E}))}$. Let $\underline{\mathbb{C}}^\infty$ denote the sheaf of smooth functions on X. For each open $U \subset X$, we can consider the completed Hochschild complex of $\mathcal{D}\mathrm{iff}^\bullet(\mathcal{E})(U)$ as in Section 3.3. Denote this complex by $\widehat{C^\bullet(\mathcal{D}\mathrm{iff}^\bullet(\mathcal{E}))(U)}$. Consider the sheafification of the pre-sheaf $U \rightsquigarrow \widehat{C^\bullet(\mathcal{D}\mathrm{iff}^\bullet(\mathcal{E}))(U)}$. Recall that this sheaf of complexes of $\underline{\mathbb{C}}^\infty$-modules is denoted by $\widetilde{\mathrm{Hoch}(\mathcal{D}\mathrm{iff}(\mathcal{E}))}$ in Section 3.3. Unlike in Section 3.3, let $\widetilde{\mathrm{Hoch}^n(\mathcal{D}\mathrm{iff}(\mathcal{E}))}$ denote the degree n component of $\widetilde{\mathrm{Hoch}(\mathcal{D}\mathrm{iff}(\mathcal{E}))}$.

One can also consider the completed bar complex $\widehat{\mathrm{bar}^\bullet(\mathcal{D}\mathrm{iff}^\bullet(\mathcal{E})(U))}$. This is defined as in Section 3.3: the underlying graded $\mathbb{C}$-vector space of the complex $\widehat{\mathrm{bar}^\bullet(\mathcal{D}\mathrm{iff}^\bullet(\mathcal{E})(U))}$ is the same as that of $\widehat{C^\bullet(\mathcal{D}\mathrm{iff}^\bullet(\mathcal{E}))(U)}$, but the differential on $\widehat{\mathrm{bar}^\bullet(\mathcal{D}\mathrm{iff}^\bullet(\mathcal{E})(U))}$ is the bar differential. We will denote the sheafification of the pre-sheaf $U \rightsquigarrow \widehat{\mathrm{bar}^\bullet(\mathcal{D}\mathrm{iff}^\bullet(\mathcal{E})(U))}$ by $\widetilde{\mathrm{bar}^\bullet(\mathcal{D}\mathrm{iff}(\mathcal{E}))}$. We have the following result.

Proposition 12. *The differentials in Tsygan's double complex for $\mathcal{D}\mathrm{iff}^\bullet(\mathcal{E})(U)$ extend to yield differentials for the following double complex of $\underline{\mathbb{C}}^\infty$-modules on X:*



Proof. This only needs to be checked at the level of pre-sheaves. The double complex mentioned is the sheafification of the pre-sheaf $U \rightsquigarrow \widehat{\mathrm{CC}^{\bullet,\bullet}(\mathcal{D}\mathrm{iff}^{\bullet}(\mathcal{E})(U))}$. Here, $\widehat{\mathrm{CC}^{\bullet,\bullet}(\mathcal{D}\mathrm{iff}^{\bullet}(\mathcal{E})(U))}$ is the double complex obtained by replacing the Hochschild and bar complexes that make up the columns of Tsygan's double complex of $\mathcal{D}\mathrm{iff}^{\bullet}(\mathcal{E})(U)$ with their completed versions. It is also clear that $\mathrm{id} - \tau$ and N extend to horizontal differentials on the double complex in this proposition. □

We denote the total complex of this double complex by $\widetilde{\mathrm{Cycl}}^{\bullet}(\mathcal{D}\mathrm{iff}(\mathcal{E}))$. Since it is a complex of $\underline{\mathbb{C}}^{\infty}$ modules, its hypercohomology is computed by the complex $\Gamma(X, \widetilde{\mathrm{Cycl}}^{\bullet}(\mathcal{D}\mathrm{iff}(\mathcal{E})))$.

The following cyclic analog of Proposition 6 follows from Proposition 11 and the fact that $\sum_s \tau^s(\mathrm{id} - \tau) = 0$. We denote by $\widetilde{\mathrm{HC}}_{-2i}(\mathcal{D}\mathrm{iff}(\mathcal{E}))$ the $-2i$-th cohomology of $\Gamma(X, \widetilde{\mathrm{Cycl}}^{\bullet}(\mathcal{D}\mathrm{iff}(\mathcal{E})))$.

Proposition 13. *The formula described in Proposition* 11 *extends to yield us a* $\mathbb{C}$*-linear functional* $\tilde{\mathrm{tr}}_{2i} : \widetilde{\mathrm{HC}}_{-2i}(\mathcal{D}\mathrm{iff}(\mathcal{E})) \to \mathbb{C}$.

Proof. The fact that the formula makes sense follows directly from Proposition 11 and Proposition 6. To show that it vanishes on coboundaries, we recall that Proposition 6 also tells us that it vanishes on the image of the d_{Hoch} differential. We only need to verify that it vanishes on the image of the $\mathrm{id} - \tau$ differential. This is a consequence of the fact that $\sum_s \tau^s(\mathrm{id} - \tau) = 0$. □

Recall that we had a map $S : \mathrm{Cycl}^{\bullet}(\mathrm{Diff}^{\bullet}(\mathcal{E})) \to \mathrm{Cycl}^{\bullet+2}(\mathrm{Diff}^{\bullet}(\mathcal{E}))$. It is easy to verify that this yields a map $S : \widetilde{\mathrm{Cycl}}^{\bullet}(\mathrm{Diff}^{\bullet}(\mathcal{E})) \to \widetilde{\mathrm{Cycl}}^{\bullet+2}(\mathrm{Diff}^{\bullet}(\mathcal{E}))$. Similarly, $\mathcal{I}$ can be seen to extend to a map of complexes $\mathcal{I} : \widehat{\mathrm{Hoch}(\mathcal{D}\mathrm{iff}(\mathcal{E}))} \to \widetilde{\mathrm{Cycl}}^{\bullet}(\mathrm{Diff}^{\bullet}(\mathcal{E}))$. A direct consequence of the formula in Proposition 11 is the following proposition.

Proposition 14. *The following diagrams commute:*

$$\begin{array}{ccc}
\widetilde{\mathrm{HC}}_{-2i}(\mathcal{D}\mathrm{iff}(\mathcal{E})) & \xrightarrow{S} & \widetilde{\mathrm{HC}}_{-2i+2}(\mathcal{D}\mathrm{iff}(\mathcal{E})) \\
\downarrow{\scriptstyle \tilde{\mathrm{tr}}_{2i}} & & \downarrow{\scriptstyle \tilde{\mathrm{tr}}_{2i-2}} \\
\mathbb{C} & \xrightarrow[\mathrm{id}]{} & \mathbb{C},
\end{array}$$

$$\begin{array}{ccc}
\mathrm{H}^*(\Gamma(X, \widehat{\mathrm{Hoch}(\mathcal{D}\mathrm{iff}(\mathcal{E}))})) & \xrightarrow{\mathcal{I}} & \widetilde{\mathrm{HC}}_0(\mathcal{D}\mathrm{iff}(\mathcal{E})) \\
\downarrow{\scriptstyle \widehat{\mathrm{str}}} & & \downarrow{\scriptstyle \tilde{\mathrm{tr}}_0} \\
\mathbb{C} & \xrightarrow[\mathrm{id}]{} & \mathbb{C}.
\end{array}$$

5.3. The completed cyclic homology of $\mathcal{D}\mathrm{iff}(\mathcal{E})$ and the cohomology of X

5.3.1. The completed cyclic homology of $\mathcal{D}\mathrm{iff}(\mathcal{E})$. We define the completed bar complex of $\mathcal{D}\mathrm{iff}(\mathcal{E})$ to be the sheafification of the pre-sheaf $U \to \widehat{\mathrm{bar}^\bullet(\mathcal{D}\mathrm{iff}(\mathcal{E})(U))}$. The terms of this complex are the same as in $\widehat{\mathrm{Hoch}}(\mathcal{D}\mathrm{iff}(\mathcal{E}))$, but the differential is the bar differential. We denote this by $\widehat{\mathrm{bar}}(\mathcal{D}\mathrm{iff}(\mathcal{E}))$.

We can obtain the completed Tsygan double complex of $\mathcal{D}\mathrm{iff}(\mathcal{E})$. This is a double complex of sheaves of $\mathbb{C}$-vector spaces on X, whose non-positive even columns are $\widehat{\mathrm{Hoch}}^\bullet(\mathcal{D}\mathrm{iff}(\mathcal{E}))$ and whose negative odd columns are $\widehat{\mathrm{bar}}(\mathcal{D}\mathrm{iff}(\mathcal{E}))$. The horizontal differentials in Tsygan's double complex extend to this situation to give us well-defined horizontal differentials. We will denote the total complex of this double complex by $\widehat{\mathrm{Cycl}}^\bullet(\mathcal{D}\mathrm{iff}(\mathcal{E}))$.

Before we proceed, we note that as before we have a (degree preserving) natural map of complexes $\gamma\colon \widehat{\mathrm{Cycl}}^\bullet(\mathcal{D}\mathrm{iff}(\mathcal{E})) \to \widetilde{\mathrm{Cycl}}^\bullet(\mathcal{D}\mathrm{iff}(\mathcal{E}))$. This induces a map $\gamma_*\colon \mathbb{H}^\bullet(X, \widehat{\mathrm{Cycl}}^\bullet(\mathcal{D}\mathrm{iff}(\mathcal{E}))) \to \mathbb{H}^\bullet(X, \widetilde{\mathrm{Cycl}}^\bullet(\mathcal{D}\mathrm{iff}(\mathcal{E})))$. Note that the map S extends to a map of complexes $\widehat{\mathrm{Cycl}}^\bullet(\mathcal{D}\mathrm{iff}(\mathcal{E})) \to \widehat{\mathrm{Cycl}}^{\bullet+2}(\mathcal{D}\mathrm{iff}(\mathcal{E}))$. Similarly, the map $\mathcal{I}$ extends to a map of complexes $\mathcal{I}\colon \widehat{\mathrm{Hoch}}(\mathcal{D}\mathrm{iff}(\mathcal{E})) \to \widehat{\mathrm{Cycl}}^\bullet(\mathcal{D}\mathrm{iff}(\mathcal{E}))$. Furthermore, we have the following commutative diagrams:

$$\begin{array}{ccc}
\widehat{\mathrm{Hoch}}(\mathcal{D}\mathrm{iff}(\mathcal{E})) & \xrightarrow{\mathcal{I}} & \widehat{\mathrm{Cycl}}^\bullet(\mathcal{D}\mathrm{iff}(\mathcal{E})) \\
\downarrow{\scriptstyle\gamma} & & \downarrow{\scriptstyle\gamma} \\
\widetilde{\mathrm{Hoch}}(\mathcal{D}\mathrm{iff}(\mathcal{E})) & \xrightarrow[\mathcal{I}]{} & \widetilde{\mathrm{Cycl}}^\bullet(\mathcal{D}\mathrm{iff}(\mathcal{E})),
\end{array}$$

$$\begin{array}{ccc}
\widehat{\mathrm{Cycl}}^\bullet(\mathcal{D}\mathrm{iff}(\mathcal{E})) & \xrightarrow{S} & \widehat{\mathrm{Cycl}}^{\bullet+2}(\mathcal{D}\mathrm{iff}(\mathcal{E})) \\
\downarrow{\scriptstyle\gamma} & & \downarrow{\scriptstyle\gamma} \\
\widetilde{\mathrm{Cycl}}^\bullet(\mathcal{D}\mathrm{iff}(\mathcal{E})) & \xrightarrow[S]{} & \widetilde{\mathrm{Cycl}}^{\bullet+2}(\mathcal{D}\mathrm{iff}(\mathcal{E})).
\end{array}$$

We denote by $\widehat{\mathrm{HC}}_{-j}(\mathcal{D}\mathrm{iff}(\mathcal{E}))$ the hypercohomology $\mathbb{H}^{-j}(X, \widehat{\mathrm{Cycl}}^\bullet(\mathcal{D}\mathrm{iff}(\mathcal{E})))$.

Again we note that the complex $\widetilde{\mathrm{Cycl}}^\bullet(\mathcal{D}\mathrm{iff}(\mathcal{E}))$ is a complex of $\underline{\mathbb{C}}^\infty$ modules. Thus, its hypercohomology is precisely $\widetilde{\mathrm{HC}}_\bullet(\mathcal{D}\mathrm{iff}(\mathcal{E}))$. It follows that we have the following corollaries of Propositions 13 and 14, respectively.

Corollary 7 (Corollary to Proposition 13). *There exist traces*

$$\hat{\mathrm{tr}}_{2i}\colon \widehat{\mathrm{HC}}_{-2i}(\mathcal{D}\mathrm{iff}(\mathcal{E})) \to \mathbb{C}.$$

Proof. This follows from $\hat{\mathrm{tr}}_{2i} = \tilde{\mathrm{tr}}_{2i} \circ \gamma_*$. □

Corollary 8 (Corollary to Proposition 14). *The following diagrams commute:*

$$\begin{array}{ccc} \widehat{\mathrm{HH}}_0(\mathcal{D}\mathrm{iff}(\mathcal{E})) & \xrightarrow{I} & \widehat{\mathrm{HC}}_0(\mathcal{D}\mathrm{iff}(\mathcal{E})) \\ \downarrow{\scriptstyle \widehat{\mathrm{tr}}} & & \downarrow{\scriptstyle \widehat{\mathrm{tr}}_0} \\ \mathbb{C} & \xrightarrow[\mathrm{id}]{} & \mathbb{C}, \end{array} \qquad \begin{array}{ccc} \widehat{\mathrm{HC}}_{-2i}(\mathcal{D}\mathrm{iff}(\mathcal{E})) & \xrightarrow{S} & \widehat{\mathrm{HC}}_{-2i+2}(\mathcal{D}\mathrm{iff}(\mathcal{E})) \\ \downarrow{\scriptstyle \widehat{\mathrm{tr}}_{2i}} & & \downarrow{\scriptstyle \widehat{\mathrm{tr}}_{2i-2}} \\ \mathbb{C} & \xrightarrow[\mathrm{id}]{} & \mathbb{C}. \end{array}$$

Proof. This is an immediate consequence of Proposition 14 and the two commutative diagrams shown at the top. □

We have the following cyclic homology analog of Lemma 3.

Lemma 5. $\widehat{\mathrm{Cycl}}^{\bullet}(\mathcal{D}\mathrm{iff}(\mathcal{E}))$ *is quasi-isomorphic to* $\underline{\mathbb{C}}[2n] \oplus \underline{\mathbb{C}}[2n+2] \oplus \cdots$.

Proof. It suffices to verify this assertion locally at the level of pre-sheaves. In other words, we need to look at the completed Tsygans double complex for $\mathcal{D}\mathrm{iff}(\mathcal{E})(U)$ where U is an open set so that $\mathcal{E}$ is trivial on U. Of course, all tensor products here are completed tensor products.

The completed bar complex of $\mathcal{D}\mathrm{iff}(\mathcal{E})(U)$ is acyclic since the homotopy of recollection 1, Section 2.1 between the identity endomorphism and the 0 endomorphism of the bar complex of $\mathcal{D}\mathrm{iff}(\mathcal{E})(U)$ can be shown to "extend to" a homotopy between the identity and 0 endomorphisms of the completed bar complex of $\mathcal{D}\mathrm{iff}(\mathcal{E})(U)$. The complex $\widehat{\mathrm{Hoch}(\mathcal{D}\mathrm{iff}(\mathcal{E})(U))}$ is quasi-isomorphic to $\mathbb{C}[2n]$ by Lemma 3. It follows that the spectral sequence converging to the completed cyclic homology of $\mathcal{D}\mathrm{iff}(\mathcal{E})(U)$ that arises out of the filtration of the completed Tsygan double complex for $\mathcal{D}\mathrm{iff}(\mathcal{E})(U)$ by columns satisfies:

$$E_1^{-2i,-2n} \simeq \mathbb{C} \text{ for all } i \geq 0 \quad \text{and} \quad E_1^{p,q} = 0 \text{ otherwise.}$$

The lemma now follows from this. □

Corollary 9. $\widehat{\mathrm{HC}}_{-2i}(\mathcal{D}\mathrm{iff}(\mathcal{E})) = \mathrm{H}^{2n}(X,\mathbb{C}) \oplus \mathrm{H}^{2n-2}(X,\mathbb{C}) \oplus \cdots \oplus \mathrm{H}^{2n-2i}(X,\mathbb{C})$.

Denote the isomorphism in the above corollary by J_{2i}. Then, following the proof of Lemma 5, we obtain the following result.

Proposition 15. *The following diagrams commute:*

$$\begin{array}{ccc} \widehat{\mathrm{HC}}_{-2i}(\mathcal{D}\mathrm{iff}(\mathcal{E})) & \xrightarrow{J_{2i}} & \mathrm{H}^{2n}(X,\mathbb{C}) \oplus \mathrm{H}^{2n-2}(X,\mathbb{C}) \oplus \cdots \oplus \mathrm{H}^{2n-2i}(X,\mathbb{C}) \\ \downarrow{\scriptstyle S} & & \downarrow \\ \widehat{\mathrm{HC}}_{-2i+2}(\mathcal{D}\mathrm{iff}(\mathcal{E})) & \xrightarrow[J_{2i-2}]{} & \mathrm{H}^{2n}(X,\mathbb{C}) \oplus \mathrm{H}^{2n-2}(X,\mathbb{C}) \oplus \cdots \oplus \mathrm{H}^{2n-2i+2}(X,\mathbb{C}), \end{array}$$

$$\begin{array}{ccc} \widehat{\mathrm{HH}}_0(\mathcal{D}\mathrm{iff}(\mathcal{E})) & \xrightarrow{I} & \widehat{\mathrm{HC}}_0(\mathcal{D}\mathrm{iff}(\mathcal{E})) \\ \downarrow{\scriptstyle \beta_{\mathcal{E}}} & & \downarrow{\scriptstyle J_0} \\ \mathrm{H}^{2n}(X,\mathbb{C}) & \xrightarrow[\mathrm{id}]{} & \mathrm{H}^{2n}(X,\mathbb{C}). \end{array}$$

The vertical arrow in the right column of the first diagram is the obvious projection.

5.4. Proof of Theorem 3

Proof of Theorem 3. We apply the second diagram in Corollary 8 i times to obtain the following diagram:

$$\begin{array}{ccc} \widehat{\mathrm{HC}}_{-2i}(\mathcal{D}\mathrm{iff}(\mathcal{E})) & \xrightarrow{S^i} & \widehat{\mathrm{HC}}_0(\mathcal{D}\mathrm{iff}(\mathcal{E})) \\ \downarrow{\scriptstyle \widehat{\mathrm{tr}}_{2i}} & & \downarrow{\scriptstyle \widehat{\mathrm{tr}}_0} \\ \mathbb{C} & \xrightarrow[\mathrm{id}]{} & \mathbb{C}. \end{array}$$

We apply the first diagram Proposition 15 i times to obtain the following diagram:

$$\begin{array}{ccc} \widehat{\mathrm{HC}}_{-2i}(\mathcal{D}\mathrm{iff}(\mathcal{E})) & \xrightarrow{\qquad S^i \qquad} & \widehat{\mathrm{HC}}_0(\mathcal{D}\mathrm{iff}(\mathcal{E})) \\ \downarrow{\scriptstyle J_{2i}} & & \downarrow{\scriptstyle J_0} \\ \mathrm{H}^{2n}(X,\mathbb{C}) \oplus \mathrm{H}^{2n-2}(X,\mathbb{C}) \oplus \cdots \oplus \mathrm{H}^{2n-2i}(X,\mathbb{C}) & \longrightarrow & \mathrm{H}^{2n}(X,\mathbb{C}). \end{array}$$

In the notation used to state Theorem 3, this tells us that $I_{\mathcal{E},2i,0} = I_{\mathcal{E},0,0}$ and $I_{\mathcal{E},2i,2k} = 0$ for $k > 0$. Lastly, the second diagram of Proposition 15 and the first diagram of Corollary 8 together imply that $\widehat{\mathrm{tr}} \circ \beta_{\mathcal{E}}^{-1} = \widehat{\mathrm{tr}}_0 \circ J_0^{-1}$. This shows that $I_{\mathcal{E},0,0} = I_{\mathcal{E}}$, which completes the proof. □

Remark 1. Our notion of completed Hochschild and cyclic homologies seems to differ in detail from that used in [FLS]. This forced us to rework some steps of [FLS] – in particular, Proposition 6 and related matters – in our situation.

Remark 2. The key step in the proof of Theorem 1 in [FLS] consists in showing, on the one hand, that the linear functional defined in [FLS] applied to the operator id in $\mathrm{Diff}(\mathcal{E})$ is indeed its supertrace, i.e., the Euler characteristic of $\mathcal{E}$, and showing that the image of id in $\widehat{\mathrm{HH}}_0(\mathcal{D}\mathrm{iff}(\mathcal{E})$ gives the class $\mathrm{Ch}(\mathcal{E}) \cdot \mathrm{Td}(T_X)_{2n}$ (this is done by citing [NT1] and [NT2]) after passing to $\mathrm{H}^{2n}(X,\mathbb{C})$ and then applying the Hirzebruch R–R theorem. In spite of the differences in detail between our construction of completed Hochschild homology and that of [FLS] it may be checked that these two key steps go through, thus maintaining Theorem 1 in this situation.

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