

# Geometric cycles, index theory and twisted K-homology

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**Abstract.** We study twisted  $\text{Spin}^c$ -manifolds over a paracompact Hausdorff space  $X$  with a twisting  $\alpha : X \rightarrow K(\mathbb{Z}, 3)$ . We introduce the topological index and the analytical index on the bordism group of  $\alpha$ -twisted  $\text{Spin}^c$ -manifolds over  $(X, \alpha)$ , taking values in topological twisted K-homology and analytical twisted K-homology respectively. The main result of this article is to establish the equality between the topological index and the analytical index for closed smooth manifolds. We also define a notion of geometric twisted K-homology, whose cycles are geometric cycles of  $(X, \alpha)$  analogous to Baum–Douglas’s geometric cycles. As an application of our twisted index theorem, we discuss the twisted longitudinal index theorem for a foliated manifold  $(X, F)$  with a twisting  $\alpha : X \rightarrow K(\mathbb{Z}, 3)$ , which generalizes the Connes–Skandalis index theorem for foliations and the Atiyah–Singer families index theorem to twisted cases.

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## Contents

1	Introduction . . . . .	497
2	Review of twisted K-theory . . . . .	504
3	Twisted $\text{Spin}^c$ -manifolds and analytical index . . . . .	509
4	Twisted $\text{Spin}^c$ bordism and topological index . . . . .	516
5	Topological index = analytical index . . . . .	527
6	Geometric cycles and geometric twisted K-homology . . . . .	535
7	The twisted longitudinal index theorem for foliation . . . . .	540
8	Final remarks . . . . .	545
	References . . . . .	549

## 1. Introduction

According to work of Baum and Douglas [10], [11], the Atiyah–Singer index theorem ([6], [7]) for a closed smooth manifold  $X$  can be formulated as in the following

commutative triangle

$$\begin{array}{ccc}
 & K^0(T^*X) & \\
 \text{Index}_t \swarrow & & \searrow \text{Index}_a \\
 K_0^t(X) & \xrightarrow[\cong]{\mu} & K_0^a(X),
 \end{array} \tag{1.1}$$

whose arrows are all isomorphisms. Here  $K^0(T^*X)$  denotes the K-cohomology with compact supports of the cotangent bundle  $T^*X$ , corresponding to symbol classes of elliptic pseudo-differential operators on  $X$ .  $K_0^t(X)$  is the topological K-homology constructed in [10], and  $K_0^a(X)$  is the Kasparov’s analytical K-homology (see [30] and [26]) of the  $C^*$ -algebra  $C(X)$  of continuous complex-valued functions on  $X$ .

The topological index and the analytical index can defined on the level of cycles. The basic cycles for  $K_0^t(X)$  (resp.  $K_1^t(X)$ ) are triples  $(M, \iota, E)$  consisting of even-dimensional (resp. odd-dimensional) closed smooth manifolds  $M$  with a given  $\text{Spin}^c$  structure on the tangent bundle of  $M$  together with a continuous map  $\iota: M \rightarrow X$  and a complex vector bundle  $E$  over  $M$ . The equivalence relation on the set of all cycles is generated by the following three steps (see [10] for details):

- (i) bordism,
- (ii) direct sum and disjoint union,
- (iii) vector bundle modification.

Addition in  $K_{\text{ev/odd}}^t(X)$  is given by the disjoint union operation of topological cycles. In this paper, this K-homology will be called the geometric K-homology of  $X$ , while the notion of topological K-homology will be reserved for homotopy theoretically defined K-homology.

Recall that a symbol class in  $K^0(T^*X)$  of an elliptic pseudo-differential operator  $D$  on  $X$  is represented by

$$\sigma(D): \pi^* E_0 \rightarrow \pi^* E_1,$$

where  $\pi: T^*X \rightarrow X$  is the projection,  $E_0$  and  $E_1$  are complex vector bundles over  $X$ . Choose a Riemannian metric on  $X$ , let  $S(T^*X \oplus \mathbb{R})$  be the unit sphere bundle in  $T^*X \oplus \mathbb{R}$ , equipped with the natural  $\text{Spin}^c$  structure. Denote by  $\phi$  the projection  $S(T^*X \oplus \mathbb{R}) \rightarrow X$ . Let  $\hat{E}$  be the complex vector bundle over  $S(T^*X \oplus \mathbb{R})$  obtained by the clutch construction (see Section 10 in [10]): as  $S(T^*X \oplus \mathbb{R})$  consists of two copies of the unit ball bundle of  $T^*X$  glued together along the unit sphere bundle, one can use the symbol  $\sigma(D)$  to clutch  $\pi^* E_0$  and  $\pi^* E_1$  together along the unit sphere bundle  $S(T^*X)$ . The topological index  $\text{Index}_t([\sigma(D)])$  is represented by the following topological cycle

$$(S(T^*X \oplus \mathbb{R}), \hat{E}, \phi).$$

The Kasparov analytical K-homology  $K_{\text{ev/odd}}^a(X)$ , denoted  $KK^{\text{ev/odd}}(C(X), \mathbb{C})$  in the literature, is generated by unitary equivalence classes of multi-graded Fredholm modules over  $C(X)$  modulo operator homotopy relation [30]. Addition in  $KK^{\text{ev/odd}}(C(X), \mathbb{C})$  is defined using a natural notion of direct sum of Fredholm modules; see [26] for details. The analytical index  $\text{Index}_a([\sigma(D)])$  is defined in terms of Poincaré duality (cf. [31]):

$$\begin{aligned} K^0(T^*X) &\cong KK^0(\mathbb{C}, C_c(T^*X)) \\ &\cong KK^0(C(X), \mathbb{C}) && \text{(Kasparov's Poincaré duality)} \\ &= K_0^a(X). \end{aligned}$$

On the level of cycles, an even dimensional topological cycle  $(M, \iota, E)$  defines a canonical element  $[\not{D}_M^E]$  in  $K_0^a(M)$  determined by the Dirac operator

$$\not{D}_M^E : C^\infty(S^+ \otimes E) \rightarrow C^\infty(S^- \otimes E)$$

where  $S^\pm$  are the positive and negative spinor bundles (called reduced spinor bundles in [26]). Then the natural isomorphism

$$\mu : K_0^t(X) \rightarrow K_0^a(X)$$

is defined by the correspondence

$$(M, \iota, E) \mapsto \iota_*([\not{D}_M^E]),$$

where  $\iota_* : K_0^a(M) \rightarrow K_0^a(X)$  is the covariant homomorphism induced by  $\iota$ .

The commutative triangle (1.1) has played an important role in the understanding of the Atiyah–Singer index theorem and its various generalizations such as the Baum–Connes conjecture in [9]. In this article, we will generalize the Atiyah–Singer index theorem to the framework of twisted K-theory following ideas inspired from Baum–Douglas [10], [11] and Baum–Connes [9].

In this article, we aim to develop the index theorem in the framework of twisted K-theory which is a natural generalization of the Baum–Douglas commutative triangle (1.1). We need a notion of a twisting in complex K-theory, given by a continuous map

$$\alpha : X \rightarrow K(\mathbb{Z}, 3),$$

where  $K(\mathbb{Z}, 3)$  is an Eilenberg–MacLane space. We often choose a homotopy model of  $K(\mathbb{Z}, 3)$  as the classifying space of the projective unitary group  $\text{PU}(\mathcal{H})$  of an infinite dimensional, complex and separable Hilbert space  $\mathcal{H}$ , equipped with the norm topology. The norm topology could be too restrictive for some examples, one might have to use the compact-open topology instead as discussed in [5].

For any paracompact Hausdorff space  $X$  with a continuous map  $\alpha : X \rightarrow K(\mathbb{Z}, 3)$ , the corresponding principal  $K(\mathbb{Z}, 2)$ -bundle over  $X$  will be denoted by  $\mathcal{P}_\alpha$ . Then

any base-point preserving action of  $K(\mathbb{Z}, 2)$  on a spectrum defines an associated bundle of based spectra. In this article, we mainly consider two spectra, one is the complex K-theory spectrum  $\mathbb{K} = \{\Omega^n \mathbb{K}\}$ , the other is the  $\text{Spin}^c$  Thom spectrum  $M\text{Spin}^c = \{M\text{Spin}^c(n)\}$ . The corresponding bundle of based spectra over  $X$  will be denoted by

$$\mathcal{P}_\alpha(\mathbb{K}) \quad \text{and} \quad \mathcal{P}_\alpha(M\text{Spin}^c),$$

respectively.

Twisted K-cohomology groups of  $(X, \alpha)$  are defined to be

$$\pi_0(C_c(X, \mathcal{P}_\alpha(\Omega^n \mathbb{K}))),$$

the homotopy classes of compactly supported sections of  $\mathcal{P}_\alpha(\Omega^n \mathbb{K})$ .

Let  $\mathcal{K}$  be the  $C^*$ -algebra of compact operators on the Hilbert space  $\mathcal{H}$ , and  $\mathcal{P}_\alpha(\mathcal{K})$  be the associated bundle of compact operators corresponding to the  $\text{PU}(\mathcal{H})$ -action on  $\mathcal{K}$  by conjugation. An equivalent definition of twisted K-theory of  $(X, \alpha)$  is the algebraic K-cohomology groups of the continuous trace  $C^*$ -algebra over  $X$  of compactly supported sections of  $\mathcal{P}_\alpha(\mathcal{K})$ . The Bott periodicity of the K-theory spectrum implies that we only have two twisted K-groups, denoted by

$$K^0(X, \alpha) \quad \text{and} \quad K^1(X, \alpha),$$

or simply  $K^{\text{ev/odd}}(X, \alpha)$ . We will review twisted K-theory and its basic properties in Section 2.

We define topological twisted K-homology to be

$$K_n^t(X, \alpha) := \varinjlim_{k \rightarrow \infty} [S^{n+2k}, \mathcal{P}_\alpha(\Omega^{2k} \mathbb{K})/X],$$

the stable homotopy groups of  $\mathcal{P}_\alpha(\mathbb{K})/X$ . Due to Bott periodicity, we only have two different topological twisted K-homology groups denoted by  $K_{\text{ev/odd}}^t(X, \alpha)$ .

There is a notion of analytical twisted K-homology defined as Kasparov's analytical K-homology:

$$K_{\text{ev/odd}}^a(X, \alpha) := KK^{\text{ev/odd}}(C_c(X, \mathcal{P}_\alpha(\mathcal{K})), \mathbb{C}).$$

Now we can state the main theorem of this article, which should be thought of as the general index theorem in the framework of twisted K-theory.

**Main Theorem** (cf. Theorem 5.1 and Remark 5.3). *Let  $X$  be a closed smooth manifold and  $\pi : T^*X \rightarrow X$  be the projection. Then there is a natural isomorphism*

$$\Phi : K_{\text{ev/odd}}^t(X, \alpha) \rightarrow K_{\text{ev/odd}}^a(X, \alpha),$$

*and there exist notions of the topological index and of the analytical index on*

$$K^{\text{ev/odd}}(T^*X, \alpha \circ \pi)$$

such that the following diagram

$$\begin{array}{ccc}
 & K^{\text{ev/odd}}(T^*X, \alpha \circ \pi) & \\
 \text{Index}_t \swarrow & & \searrow \text{Index}_a \\
 K^t_{\text{ev/odd}}(X, \alpha) & \xrightarrow[\cong]{\Phi} & K^a_{\text{ev/odd}}(X, \alpha)
 \end{array}$$

is commutative, and all arrows are isomorphisms.

We remark that topological and analytical twisted K-homology groups are well defined for any paracompact Hausdorff space  $X$  with a continuous map  $\alpha: X \rightarrow K(\mathbb{Z}, 3)$ . The above main theorem only holds for smooth manifolds, we believe that the isomorphism

$$\Phi: K^t_{\text{ev/odd}}(X, \alpha) \rightarrow K^a_{\text{ev/odd}}(X, \alpha),$$

should be true for more general spaces such as paracompact Hausdorff spaces with the homotopy type of finite CW complexes. We only establish this isomorphism for smooth manifolds by applying the Poincaré duality in twisted K-theory which requires differential structures, see the proof of Theorem 5.1 for details. It would be interesting to have this isomorphism for paracompact Hausdorff spaces with the homotopy type of finite CW complexes.

To prove the main theorem, we introduce a notion of  $\alpha$ -twisted  $\text{Spin}^c$  manifolds over any paracompact Hausdorff space  $X$  with a continuous map  $\alpha: X \rightarrow K(\mathbb{Z}, 3)$  in Section 3, which consists of quadruples  $(M, \nu, \iota, \eta)$ , where

- (1)  $M$  is a smooth, oriented and compact manifold together with a fixed classifying map of its stable normal bundle,

$$\nu: M \rightarrow BSO,$$

with  $BSO = \lim_{\rightarrow k} BSO(k)$  the classifying space of the stable normal bundle of  $M$ ;

- (2)  $\iota: M \rightarrow X$  is a continuous map;
- (3)  $\eta$  is an  $\alpha$ -twisted  $\text{Spin}^c$  structure on  $M$ , that is, a homotopy commutative diagram (see Definition 3.1 for details)

$$\begin{array}{ccc}
 M & \xrightarrow{\nu} & BSO \\
 \downarrow \iota & \searrow \eta & \downarrow W_3 \\
 X & \xrightarrow{\alpha} & K(\mathbb{Z}, 3).
 \end{array}$$

A manifold  $M$  admits an  $\alpha$ -twisted  $\text{Spin}^c$  structure if and only if there exists a continuous map  $\iota: M \rightarrow X$  such that

$$\iota^*([\alpha]) + W_3(M) = 0$$

in  $H^3(M, \mathbb{Z})$ . (This is known to physicists as the Freed–Witten anomaly cancellation condition for type II D-branes (cf. [25]).)

We then define an analytical index for each  $\alpha$ -twisted  $\text{Spin}^c$  manifold over  $X$  taking values in the analytical twisted K-homology  $K_{\text{ev/odd}}^a(X, \alpha)$  and establish its bordism invariance.

In Section 4, we study the geometric  $\alpha$ -twisted bordism groups  $\Omega_*^{\text{Spin}^c}(X, \alpha)$  and establish a generalized Pontrjagin–Thom isomorphism (cf. Theorem 4.4) between our geometric  $\alpha$ -twisted bordism groups and the homotopy theoretic definition of  $\alpha$ -twisted bordism groups

$$\Omega_n^{\text{Spin}^c}(X, \alpha) \cong \lim_{k \rightarrow \infty} \pi_{n+k}(\mathcal{P}_\alpha(M\text{Spin}^c(k))/X).$$

We also define a topological index on geometric  $\alpha$ -twisted bordism groups. Then the main theorem is proved in Section 5.

In Section 6, we explain the notion of geometric cycles for any paracompact Hausdorff space  $X$  with a continuous map  $\alpha: X \rightarrow K(\mathbb{Z}, 3)$ . Geometric cycles in this sense are called ‘D-branes’ in string theory. These consist of an  $\alpha$ -twisted  $\text{Spin}^c$  manifold  $M$  over  $X$  together with an ordinary K-class  $[E]$ . Following the work of Baum–Douglas, we impose an equivalence relation generated by

- (i) direct sum and disjoint union,
- (ii) bordism,
- (iii)  $\text{Spin}^c$  vector bundle modification

on the set of all geometric cycles to obtain the geometric twisted K-homology  $K_{\text{ev/odd}}^{\text{geo}}(X, \alpha)$ . Then we establish the commutative diagram (cf. Theorem 6.4) for a closed smooth manifold  $X$  with a twisting  $\alpha: X \rightarrow K(\mathbb{Z}, 3)$ ,

$$\begin{array}{ccc}
 & K_{\text{ev/odd}}^t(X, \alpha) & \\
 \Psi \swarrow & & \searrow \Phi \\
 K_{\text{ev/odd}}^{\text{geo}}(X, \alpha) & \xrightarrow[\mu]{\cong} & K_{\text{ev/odd}}^a(X, \alpha),
 \end{array} \tag{1.2}$$

whose arrows are all isomorphisms. One consequence of this commutative diagram is that every twisted K-class in  $K^{\text{ev/odd}}(X, \alpha)$  can be realized by appropriate geometric cycles (cf. Corollary 6.5).

Another application of the Main Theorem and the commutative diagram (1.2) is that the index pairing

$$K^{\text{ev/odd}}(X, \alpha) \times K^a_{\text{ev/odd}}(X, \alpha) \longrightarrow \mathbb{Z}$$

can be expressed in terms of the usual index pairing for geometric cycles.

We remark that the commutative diagram (1.2) of isomorphisms should hold for general paracompact Hausdorff spaces with the homotopy type of finite CW complexes. The restriction to smooth manifolds is due to the fact that we only establish the isomorphism  $\Phi$  in Theorem 5.1 for smooth manifolds. We expect that the equivalence of geometric, topological and analytical twisted K-homology exists for any finite CW complex. We will return to this equivalence and the corresponding index pairing in a sequel paper [13].

In Section 7, we study the twisted longitudinal index theorem (cf. Theorem 7.3) for a foliated manifold  $(X, F)$  with a twisting  $\alpha: X \rightarrow K(\mathbb{Z}, 3)$ , and show that this twisted longitudinal index theorem generalizes both the Atiyah–Singer families index theorem in [8] and Mathai–Melrose–Singer index theorem for projective families of elliptic operators associated to a torsion twisting in [34].

In Section 8, we introduce a notion of twisted Spin manifolds over a manifold  $X$  with a KO-twisting  $\alpha: X \rightarrow K(\mathbb{Z}_2, 2)$ . A smooth manifold  $M$  admits an  $\alpha$ -twisted Spin structure if and only if there exists a continuous map  $\iota: M \rightarrow X$  such that

$$\iota^*([\alpha]) + w_2(M) = 0$$

in  $H^2(M, \mathbb{Z}_2)$ . Here  $w_2(X)$  is the second Stiefel–Whitney class of  $TM$ . (This is the anomaly cancellation condition for type I D-branes (cf. [44]).) We also discuss a notion of twisted string manifolds over a manifold  $X$  with a string twisting  $\alpha: X \rightarrow K(\mathbb{Z}, 4)$ . A smooth manifold  $M$  admits an  $\alpha$ -twisted string structure if and only if there is a continuous map  $\iota: M \rightarrow X$  such that

$$\iota^*([\alpha]) + \frac{p_1(M)}{2} = 0$$

in  $H^4(M, \mathbb{Z})$ . Here  $p_1(X)$  is the first Pontrjagin class of  $TM$ . These notions could be useful in the study of twisted elliptic cohomology.

It would be interesting to establish a local index theorem in the framework of twisted K-theory in which differential twisted K-theory in [17], [28] will come into play. We will return to these problems in subsequent work. Finally, we like to point that, except in Sections 5 and 7 and Theorem 6.4 where  $X$  is smooth,  $X$  is assumed to be a paracompact Hausdorff topological space throughout this article.

**2. Review of twisted K-theory**

In this section, we briefly review some basic facts about twisted K-theory; the main references are [5] and [18] (see also [15], [24], [39]).

Let  $\mathcal{H}$  be an infinite dimensional, complex and separable Hilbert space. We shall consider locally trivial principal  $\text{PU}(\mathcal{H})$ -bundles over a paracompact Hausdorff topological space  $X$ . The structure group  $\text{PU}(\mathcal{H})$  is equipped with the norm topology. The projective unitary group  $\text{PU}(\mathcal{H})$  with the norm topology (cf. [32]) has the homotopy type of an Eilenberg–MacLane space  $K(\mathbb{Z}, 2)$ . The classifying space of  $\text{PU}(\mathcal{H})$ , as a classifying space of the principal  $\text{PU}(\mathcal{H})$ -bundle, is a  $K(\mathbb{Z}, 3)$ . Thus, the set of isomorphism classes of principal  $\text{PU}(\mathcal{H})$ -bundles over  $X$  is canonically identified with (Proposition 2.1 in [5])

$$[X, K(\mathbb{Z}, 3)] \cong H^3(X, \mathbb{Z}).$$

A twisting of complex K-theory on  $X$  is given by a continuous map  $\alpha: X \rightarrow K(\mathbb{Z}, 3)$ . For such a twisting, we can associate a canonical principal  $K(\mathbb{Z}, 2)$ -bundle  $\mathcal{P}_\alpha$  through the following pull-back construction:

$$\begin{array}{ccc} \mathcal{P}_\alpha & \longrightarrow & EK(\mathbb{Z}, 2) \\ \downarrow & & \downarrow \\ X & \xrightarrow{\alpha} & K(\mathbb{Z}, 3). \end{array} \tag{2.1}$$

Let  $\mathbb{K}$  be the 0-th space of the complex K-spectrum. In what follows we take  $\mathbb{K}$  to be  $\text{Fred}(\mathcal{H})$ , the space of Fredholm operators on  $\mathcal{H}$ . There is a base-point preserving action of  $K(\mathbb{Z}, 2)$  on the K-theory spectrum

$$K(\mathbb{Z}, 2) \times \mathbb{K} \rightarrow \mathbb{K}$$

which is represented by the action of complex line bundles on ordinary K-groups. As we identify  $K(\mathbb{Z}, 2)$  with  $\text{PU}(\mathcal{H})$  and  $\mathbb{K}$  with  $\text{Fred}(\mathcal{H})$ , the above base point preserving action is given by the conjugation action

$$\text{PU}(\mathcal{H}) \times \text{Fred}(\mathcal{H}) \rightarrow \text{Fred}(\mathcal{H}). \tag{2.2}$$

The action (2.2) defines an associated bundle of K-theory spectra over  $X$ . Denote by

$$\mathcal{P}_\alpha(\mathbb{K}) = \mathcal{P}_\alpha \times_{K(\mathbb{Z}, 2)} \mathbb{K}$$

the bundle of based spectra over  $X$  with fiber the K-theory spectrum, and by  $\{\Omega_X^n \mathcal{P}_\alpha(\mathbb{K}) = \mathcal{P}_\alpha \times_{K(\mathbb{Z}, 2)} \Omega^n \mathbb{K}\}$  the fiber-wise iterated loop spaces.

**Definition 2.1.** The twisted K-groups of  $(X, \alpha)$  are defined to be the set of homotopy classes of compactly supported sections of the bundle of K-spectra:

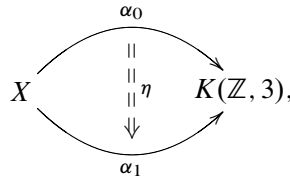
$$K^{-n}(X, \alpha) := \pi_0(C_c(X, \Omega_X^n \mathcal{P}_\alpha(\mathbb{K}))).$$



Due to Bott periodicity, we only have two different twisted K-groups  $K^0(X, \alpha)$  and  $K^1(X, \alpha)$ . Given a closed subspace  $A$  of  $X$ ,  $(X, A)$  is a pair of topological spaces, and we define relative twisted K-groups to be

$$K^{\text{ev/odd}}(X, A; \alpha) := K^{\text{ev/odd}}(X - A, \alpha).$$

**Remark 2.2.** Let  $\alpha_0, \alpha_1 : X \rightarrow K(\mathbb{Z}, 3)$  be a pair of twistings. If  $\eta : X \times [1, 0] \rightarrow K(\mathbb{Z}, 3)$  is a homotopy between  $\alpha_0$  and  $\alpha_1$ , written as



then there is a canonical isomorphism  $\mathcal{P}_{\alpha_0} \cong \mathcal{P}_{\alpha_1}$  induced by  $\eta$ . This canonical isomorphism determines a canonical isomorphism on twisted K-groups,

$$\eta_* : K^{\text{ev/odd}}(X, \alpha_0) \xrightarrow{\cong} K^{\text{ev/odd}}(X, \alpha_1). \tag{2.3}$$

This isomorphism  $\eta_*$  depends only on the homotopy class of  $\eta$ . The set of homotopy classes between  $\alpha_0$  and  $\alpha_1$  is an affine space modelled on  $[X, K(\mathbb{Z}, 2)]$ . Note that the first Chern class isomorphism is

$$\text{Vect}_1(X) \cong [X, K(\mathbb{Z}, 2)] \cong H^2(X, \mathbb{Z}),$$

where  $\text{Vect}_1(X)$  is the set of equivalence classes of complex line bundles on  $X$ . We remark that the isomorphisms induced by two different homotopies between  $\alpha_0$  and  $\alpha_1$  are related through an action of complex line bundles. This observation will play an important role in the local index theorem for twisted K-theory.

**Remark 2.3.** Let  $\mathcal{K}$  be the  $C^*$ -algebra of compact operators on  $\mathcal{H}$ . The isomorphism  $\text{PU}(\mathcal{H}) \cong \text{Aut}(\mathcal{K})$  via the conjugation action of the unitary group  $U(\mathcal{H})$  provides an action of  $K(\mathbb{Z}, 2)$  on the  $C^*$ -algebra  $\mathcal{K}$ . Hence, any  $K(\mathbb{Z}, 2)$ -principal bundle  $\mathcal{P}_\alpha$  defines a locally trivial bundle of compact operators, denoted by  $\mathcal{P}_\alpha(\mathcal{K}) = \mathcal{P}_\alpha \times_{K(\mathbb{Z}, 2)} \mathcal{K}$ . Let  $C_c(X, \mathcal{P}_\alpha(\mathcal{K}))$  be the  $C^*$ -algebra of the compact supported sections of  $\mathcal{P}_\alpha(\mathcal{K})$ . We remark that  $C_c(X, \mathcal{P}_\alpha(\mathcal{K}))$  is the (unique up to isomorphism) stable separable complex continuous-trace  $C^*$ -algebra over  $X$  with its Dixmier–Douady class  $[\alpha] \in H^3(X, \mathbb{Z})$ ; here we identify the Čech cohomology of  $X$  with its singular cohomology (cf. [39] and [38]). In [5] and [39], it was proved that twisted K-groups  $K^{\text{ev/odd}}(X, \alpha)$  are canonically isomorphic to the Kasparov KK-groups of the stable continuous trace  $C^*$ -algebra  $C_c(X, \mathcal{P}_\alpha(\mathcal{K}))$ :

$$K^{\text{ev/odd}}(X, \alpha) \cong KK^{\text{ev/odd}}(\mathbb{C}, C_c(X, \mathcal{P}_\alpha(\mathcal{K}))). \tag{2.4}$$

The twisted K-theory is a 2-periodic *generalized cohomology theory*: a contravariant functor on the category of pairs consisting of a pair of topological spaces  $A \subset X$  with a twisting  $\alpha: X \rightarrow K(\mathbb{Z}, 3)$  to the category of  $\mathbb{Z}_2$ -graded abelian groups. Note that a morphism between two pairs  $(X, \alpha)$  and  $(Y, \beta)$  is a continuous map  $f: X \rightarrow Y$  such that  $\beta \circ f = \alpha$ . The twisted K-theory satisfies the following three axioms whose proofs are rather standard for 2-periodic generalized cohomology theory.

- (I) (The homotopy axiom) If two morphisms  $f, g: (Y, B) \rightarrow (X, A)$  are homotopic through a map  $\eta: (Y \times [0, 1], B \times [0, 1]) \rightarrow (X, A)$ , written in terms of the following homotopy commutative diagram

$$\begin{array}{ccc}
 (Y, B) & \xrightarrow{f} & (X, A) \\
 g \downarrow & \swarrow \eta & \downarrow \alpha \\
 (X, A) & \xrightarrow{\alpha} & K(\mathbb{Z}, 3),
 \end{array}$$

then we have the following commutative diagram:

$$\begin{array}{ccc}
 & K^{\text{ev/odd}}(X, A; \alpha) & \\
 f^* \swarrow & & \searrow g^* \\
 K^{\text{ev/odd}}(Y, B; \alpha \circ f) & \xrightarrow{\eta_*} & K^{\text{ev/odd}}(Y, B; \alpha \circ g).
 \end{array}$$

Here  $\eta_*$  is the canonical isomorphism induced by the homotopy  $\eta$ .

- (II) (The exact axiom) For any pair  $(X, A)$  with a twisting  $\alpha: X \rightarrow K(\mathbb{Z}, 3)$ , there exists the following six-term exact sequence:

$$\begin{array}{ccccc}
 K^0(X, A; \alpha) & \longrightarrow & K^0(X, \alpha) & \longrightarrow & K^0(A, \alpha|_A) \\
 \uparrow & & & & \downarrow \\
 K^1(A, \alpha|_A) & \longleftarrow & K^1(X, \alpha) & \longleftarrow & K^1(X, A; \alpha).
 \end{array}$$

Here  $\alpha|_A$  is the composition of the inclusion and  $\alpha$ .

- (III) (The excision axiom) Let  $(X, A)$  be a pair of spaces and let  $U \subset A$  be a subspace such that the closure  $\bar{U}$  is contained in the interior of  $A$ . Then the inclusion  $\iota: (X - U, A - U) \rightarrow (X, A)$  induces, for all  $\alpha: X \rightarrow K(\mathbb{Z}, 3)$ , an isomorphism

$$K^{\text{ev/odd}}(X, A; \alpha) \xrightarrow{\cong} K^{\text{ev/odd}}(X - U, A - U; \alpha \circ \iota).$$

In addition, twisted K-theory satisfies the following basic properties (see [5], [18] for detailed proofs).

- (IV) (Multiplicative property) Let  $\alpha, \beta: X \rightarrow K(\mathbb{Z}, 3)$  be a pair of twistings on  $X$ . Denote by  $\alpha + \beta$  the new twisting defined by the map

$$\alpha + \beta: X \xrightarrow{(\alpha, \beta)} K(\mathbb{Z}, 3) \times K(\mathbb{Z}, 3) \xrightarrow{m} K(\mathbb{Z}, 3), \tag{2.5}$$

where  $m$  is defined as

$$BPU(\mathcal{H}) \times BPU(\mathcal{H}) \cong B(PU(\mathcal{H}) \times PU(\mathcal{H})) \rightarrow BPU(\mathcal{H})$$

for a fixed isomorphism  $\mathcal{H} \otimes \mathcal{H} \cong \mathcal{H}$ . Then there is a canonical multiplication

$$K^{\text{ev/odd}}(X, \alpha) \times K^{\text{ev/odd}}(X, \beta) \rightarrow K^{\text{ev/odd}}(X, \alpha + \beta),$$

which defines a  $K^0(X)$ -module structure on twisted K-groups  $K^{\text{ev/odd}}(X, \alpha)$ .

- (V) (Thom isomorphism) Let  $\pi: E \rightarrow X$  be an oriented real vector bundle of rank  $k$  over  $X$  with the classifying map denoted by  $\nu_E: X \rightarrow BSO(k)$ . Then, for any twisting  $\alpha: X \rightarrow K(\mathbb{Z}, 3)$ , there is a canonical isomorphism

$$K^{\text{ev/odd}}(X, \alpha + (W_3 \circ \nu_E)) \cong K^{\text{ev/odd}}(E, \alpha \circ \pi), \tag{2.6}$$

with the grading shifted by  $k \pmod{2}$ . Here  $W_3: BSO(k) \rightarrow K(\mathbb{Z}, 3)$  is the classifying map of the principal  $K(\mathbb{Z}, 2)$ -bundle  $BSpin^c(k) \rightarrow BSO(k)$ .

- (VI) (The push-forward map) For any differentiable map  $f: X \rightarrow Y$  between two smooth manifolds  $X$  and  $Y$ , let  $\alpha: Y \rightarrow K(\mathbb{Z}, 3)$  be a twisting. Then there is a canonical push-forward homomorphism

$$f_!: K^{\text{ev/odd}}(X, (\alpha \circ f) + (W_3 \circ \nu_f)) \rightarrow K^{\text{ev/odd}}(Y, \alpha), \tag{2.7}$$

with the grading shifted by  $n \pmod{2}$  for  $n = \dim(X) + \dim(Y)$ . Here  $\nu_f$  is the classifying map

$$X \rightarrow BSO(n)$$

corresponding to the bundle  $TX \oplus f^*TY$  over  $X$ .

- (VII) (Mayer–Vietoris sequence) If  $X$  is covered by two open subsets  $U_1$  and  $U_2$  with a twisting  $\alpha: X \rightarrow K(\mathbb{Z}, 3)$ , then there is a Mayer–Vietoris exact sequence

$$\begin{array}{ccccccc} K^0(X, \alpha) & \longrightarrow & K^1(U_1 \cap U_2, \alpha_{12}) & \longrightarrow & K^1(U_1, \alpha_1) \oplus K^1(U_2, \alpha_2) & & \\ & & \uparrow & & \downarrow & & \\ K^0(U_1, \alpha_1) \oplus K^0(U_2, \alpha_2) & \longleftarrow & K^0(U_1 \cap U_2, \alpha_{12}) & \longleftarrow & K^1(X, \alpha) & & \end{array}$$

where  $\alpha_1, \alpha_2$  and  $\alpha_{12}$  are the restrictions of  $\alpha$  to  $U_1, U_2$  and  $U_1 \cap U_2$ , respectively.

**Remark 2.4.** (1) Note that  $[\alpha + (W_3 \circ \nu_E)] = [\alpha] + W_3(E)$ . Our Thom isomorphism agrees with the Thom isomorphism in [18] and [20], where the notation

$$K^{\text{ev/odd}}(X, [\alpha] + W_3(E)) \cong K^{\text{ev/odd}}(E, \pi^*([\alpha]))$$

is used.

(2) The push-forward map constructed in [18] is established in the following form

$$f_! : K^{\text{ev/odd}}(X, f^*[\alpha] + W_3(TX \oplus f^*TY)) \rightarrow K^{\text{ev/odd}}(Y, [\alpha]),$$

which is obtained by applying the Thom isomorphism and Bott periodicity as follows.

Choose an embedding  $i : X \rightarrow \mathbb{R}^{2k}$ . Then  $x \mapsto (f(x), i(x))$  defines an embedding of  $X \rightarrow Y \times \mathbb{R}^{2k}$  whose normal bundle  $N$  is identified with a tubular neighborhood of  $X$ . Let  $\nu_N : X \rightarrow BSO$  be the classifying map of the normal bundle  $N$ , let  $\iota : N \rightarrow Y \times \mathbb{R}^{2k}$  be the inclusion map, and  $\pi : Y \times \mathbb{R}^{2k} \rightarrow Y$  be the projection. We use the commutative diagram

$$\begin{array}{ccccc} N & \xrightarrow{\iota} & Y \times \mathbb{R}^{2k} & & \\ \downarrow & \nearrow (f,i) & \downarrow \pi & & \\ X & \xrightarrow{f} & Y & \xrightarrow{\alpha} & K(\mathbb{Z}, 3) \end{array}$$

to illustrate induced twistings  $\alpha \circ f$ ,  $\alpha \circ \pi \circ \iota$  and  $\alpha \circ \pi$  on  $X$ ,  $N$  and  $Y \times \mathbb{R}^{2n}$ , respectively. Notice the isomorphism, as bundles over  $X$ ,

$$N \oplus TX \oplus TX \cong TX \oplus f^*TY \oplus \underline{\mathbb{R}}^{2n},$$

and the canonical  $\text{Spin}^c$  structure on  $TX \oplus TX$  determines a canonical homotopy between  $W_3 \circ \nu_N$  and  $W_3 \circ \nu_f$ , which in turn induces a canonical isomorphism

$$K^{\text{ev/odd}}(X, (\alpha \circ f) + (W_3 \circ \nu_f)) \cong K^{\text{ev/odd}}(X, (\alpha \circ f) + (W_3 \circ \nu_N)).$$

Applying the Thom isomorphism (2.6), we have

$$K^{\text{ev/odd}}(X, (\alpha \circ f) + (W_3 \circ \nu_N)) \cong K^{\text{ev/odd}}(N, \alpha \circ \pi \circ \iota),$$

with the grading shifted by  $n \pmod{2}$  for  $n = \dim(X) + \dim(Y)$ . The inclusion map  $\iota : N \rightarrow Y \times \mathbb{R}^{2k}$  induces a natural push-forward map

$$\iota_! : K^{\text{ev/odd}}(N, \alpha \circ \pi \circ \iota) \rightarrow K^{\text{ev/odd}}(Y \times \mathbb{R}^{2n}, \alpha \circ \pi).$$

The Bott periodicity gives a canonical isomorphism

$$K^{\text{ev/odd}}(Y \times \mathbb{R}^{2n}, \alpha \circ \pi) \cong K^{\text{ev/odd}}(Y, \alpha).$$

The composition of the above isomorphisms and the map  $\iota_!$  gives rise to the canonical push-forward map (2.7).

### 3. Twisted $\text{Spin}^c$ -manifolds and analytical index

**Definition 3.1.** Let  $(X, \alpha)$  be a paracompact Hausdorff topological space with a twisting  $\alpha$ . An  $\alpha$ -twisted  $\text{Spin}^c$  manifold over  $X$  is a quadruple  $(M, \nu, \iota, \eta)$ , where

- (1)  $M$  is a smooth, oriented and compact manifold together with a fixed classifying map of its stable normal bundle

$$\nu: M \rightarrow BSO$$

with  $BSO = \varinjlim_k BSO(k)$  the classifying space of stable normal bundle of  $M$ ;

- (2)  $\iota: M \rightarrow X$  is a continuous map;
- (3)  $\eta$  is an  $\alpha$ -twisted  $\text{Spin}^c$  structure on  $M$ , that is a homotopy commutative diagram

$$\begin{array}{ccc} M & \xrightarrow{\nu} & BSO \\ \iota \downarrow & \eta \swarrow \text{dashed} & \downarrow W_3 \\ X & \xrightarrow{\alpha} & K(\mathbb{Z}, 3), \end{array}$$

where  $W_3$  is the classifying map of the principal  $K(\mathbb{Z}, 2)$ -bundle  $B\text{Spin}^c \rightarrow BSO$  associated to the third integral Stiefel–Whitney class and  $\eta$  is a homotopy between  $W_3 \circ \nu$  and  $\alpha \circ \iota$ .

Two  $\alpha$ -twisted  $\text{Spin}^c$  structures  $\eta$  and  $\eta'$  on  $M$  are called equivalent if there is a homotopy between  $\eta$  and  $\eta'$ .

**Remark 3.2.** (1) The definition of twisted  $\text{Spin}^c$  manifolds over  $X$  was previously given by Douglas in [21] using Hopkins–Singer’s differential cochains developed in [28]. Here in Definition 3.1, we define an  $\alpha$ -twisted  $\text{Spin}^c$  structure on  $M$  to be a homotopy between  $W_3 \circ \nu$  and  $\alpha \circ \iota$  since it induces a canonical isomorphism  $\eta_*$  (3.5) that will play an important role in our definition of the analytical index.

(2) Let  $(W, \nu, \iota, \eta)$  be an  $\alpha$ -twisted  $\text{Spin}^c$  manifold with boundary over  $X$ . Then there is a natural  $\alpha$ -twisted  $\text{Spin}^c$  structure on the boundary  $\partial W$  with outer normal orientation, which is the restriction of the  $\alpha$ -twisted  $\text{Spin}^c$  structure on  $W$ :

$$\begin{array}{ccc} \partial W & \xrightarrow{\nu|_{\partial W}} & BSO \\ \iota|_{\partial W} \downarrow & \eta|_{\partial W} \swarrow \text{dashed} & \downarrow W_3 \\ X & \xrightarrow{\alpha} & K(\mathbb{Z}, 3). \end{array} \tag{3.1}$$

(3) Given an oriented real vector bundle  $E$  of rank  $k$  over a smooth manifold  $M$ , the classifying map of  $E$

$$\nu_E: M \rightarrow BSO(k)$$

and the principal  $K(\mathbb{Z}, 2)$ -bundle  $B\text{Spin}^c(k) \rightarrow BSO(k)$  define an associated twisting

$$W_3 \circ \nu_E : M \rightarrow BSO(k) \rightarrow K(\mathbb{Z}, 3).$$

**Proposition 3.3.** *Let  $M$  be a smooth, oriented and compact  $n$ -dimensional manifold and let  $X$  be a paracompact space with a twisting  $\alpha : X \rightarrow K(\mathbb{Z}, 3)$ . Then the following holds:*

- (1)  $M$  admits an  $\alpha$ -twisted  $\text{Spin}^c$  structure if and only if there exists a continuous map  $\iota : M \rightarrow X$  such that

$$\iota^*([\alpha]) + W_3(M) = 0 \tag{3.2}$$

in  $H^3(M, \mathbb{Z})$ . Here  $W_3(M)$  is the third integral Stiefel–Whitney class,

$$W_3(M) = \beta(w_2(M))$$

with  $\beta : H^2(M, \mathbb{Z}_2) \rightarrow H^3(M, \mathbb{Z})$  the Bockstein homomorphism and  $w_2(M)$  the second Stiefel–Whitney class of  $TM$ . (Condition (3.2) is the Freed–Witten anomaly cancellation condition for type II D-branes; cf. [25].)

- (2) If  $\iota^*([\alpha]) + W_3(M) = 0$ , then the set of equivalence classes of  $\alpha$ -twisted  $\text{Spin}^c$  structures on  $M$  is an affine space modelled on  $H^2(M, \mathbb{Z})$ .

*Proof.* If  $M$  admits an  $\alpha$ -twisted  $\text{Spin}^c$  structure, then  $W_3 \circ \nu$  and  $\alpha \circ \iota$  are homotopic as maps from  $M$  to  $K(\mathbb{Z}, 3)$ . This means that the third integral Stiefel–Whitney class of the stable normal bundle is equal to  $\iota^*([\alpha])$ . As  $M$  is compact, we can find an embedding

$$i_k : M^n \rightarrow \mathbb{R}^{n+k}$$

for a sufficiently large  $k$ . Denote by  $\nu(i_k)$  the normal bundle of  $i_k$ . Then we know that  $W_3(\nu(i_k)) = \iota^*([\alpha])$ , and

$$\nu(i_k) \oplus TM \cong i_k^*(T\mathbb{R}^{n+k})$$

is a trivial bundle, which implies  $W_3(M) + W_3(\nu(i_k)) = 0$ . So  $\iota^*([\alpha]) + W_3(M) = 0$ .

Conversely, if  $\iota^*([\alpha]) + W_3(M) = 0$ , then  $W_3(\nu(i_k))$  agrees with  $\iota^*([\alpha])$ . Hence the classifying map  $\nu_k : M \rightarrow BSO(k)$  makes the following diagram homotopy commutative for some homotopy  $\eta$ :

$$\begin{array}{ccc} M & \xrightarrow{\nu_k} & BSO(k) \\ \iota \downarrow & \eta \dashrightarrow & \downarrow W_3 \\ X & \xrightarrow{\alpha} & K(\mathbb{Z}, 3). \end{array}$$

This defines an  $\alpha$ -twisted  $\text{Spin}^c$  structure on  $M$  by letting  $k \rightarrow \infty$ .

The set of equivalence classes of  $\alpha$ -twisted  $\text{Spin}^c$  structures on  $M$  corresponds to the set of homotopy classes of homotopies between  $W_3 \circ \nu$  and  $\alpha \circ \iota$ . The latter is an affine space over

$$[\Sigma M, K(\mathbb{Z}, 3)] \cong [M, K(\mathbb{Z}, 2)].$$

Here  $\Sigma$  denotes the suspension. Since  $[M, K(\mathbb{Z}, 2)] \cong H^2(M, \mathbb{Z})$ ,  $H^2(M, \mathbb{Z})$  acts freely and transitively on the set of equivalence classes of  $\alpha$ -twisted  $\text{Spin}^c$  structures on  $M$ . □

**Remark 3.4.** (1) If the twisting  $\alpha: X \rightarrow K(\mathbb{Z}, 3)$  is homotopic to the trivial map, then an  $\alpha$ -twisted  $\text{Spin}^c$  structure on  $M$  is equivalent to a  $\text{Spin}^c$  structure on  $M$ .

(2) Let  $\tau_X: X \rightarrow BSO$  be a classifying map of the stable tangent bundle of  $X$ , then a  $W_3 \circ \tau_X$ -twisted  $\text{Spin}^c$  structure on  $M$  is equivalent to a K-oriented map from  $M$  to  $X$ .

(3) Let  $(M, \nu, \iota, \eta)$  be an  $\alpha$ -twisted  $\text{Spin}^c$  manifold over  $X$ . Any K-oriented map  $f: M' \rightarrow M$  defines a canonical  $\alpha$ -twisted  $\text{Spin}^c$  structure on  $M'$ .

Recall that for  $k \in \{0, 1, 2, \dots\}$  and a separable  $C^*$ -algebra  $A$ , Kasparov’s K-homology group

$$KK^k(A, \mathbb{C}) \cong KK(A, \text{Cliff}(\mathbb{C}^k))$$

is the abelian group generated by unitary equivalence classes of  $\text{Cliff}(\mathbb{C}^k)$ -graded Fredholm modules over  $A$  modulo certain relations (see [26] for details). Then  $KK^{\text{ev}}(A, \mathbb{C})$  and  $KK^{\text{odd}}(A, \mathbb{C})$  denote the direct limits under the periodicity maps

$$KK^{\text{ev}}(A, \mathbb{C}) = \varinjlim_k KK^{2k}(A, \mathbb{C}) \quad \text{and} \quad KK^{\text{odd}}(A, \mathbb{C}) = \varinjlim_k KK^{2k+1}(A, \mathbb{C}).$$

**Definition 3.5.** Suppose that  $X$  be a paracompact Hausdorff space with a twisting  $\alpha: X \rightarrow K(\mathbb{Z}, 3)$ . Let  $\mathcal{P}_\alpha(\mathcal{K})$  be the associated bundle of compact operators on  $X$ . Analytical twisted K-homology, denoted by  $K_{\text{ev/odd}}^a(X, \alpha)$ , is defined to be

$$K_{\text{ev/odd}}^a(X, \alpha) := KK^{\text{ev/odd}}(C_c(X, \mathcal{P}_\alpha(\mathcal{K})), \mathbb{C}),$$

the Kasparov  $\mathbb{Z}_2$ -graded K-homology of the  $C^*$ -algebra  $C_c(X, \mathcal{P}_\alpha(\mathcal{K}))$ . Given a closed subspace  $A$  of  $X$ , the relative twisted K-homology  $K_{\text{ev/odd}}^a(X, A; \alpha)$  is defined to be

$$KK^{\text{ev/odd}}(C_c(X - A, \mathcal{P}_\alpha(\mathcal{K})), \mathbb{C}).$$

Analytical twisted K-homology is a 2-periodic generalized homology theory.

We first discuss the relationship between the stable normal bundle of  $M$  and its stable tangent bundle, and apply it to study the corresponding twisted K-homology groups. Note that the classifying space of  $\text{SO}(k)$  is given by the direct limit

$$BSO(k) = \varinjlim_m \text{Gr}(k, m + k),$$

where  $\text{Gr}(k, m+k)$  is the Grassmann manifold of oriented  $k$ -planes in  $\mathbb{R}^{k+m}$ . The classifying space of the stable special orthogonal group is  $\varinjlim_k \text{BSO}(k)$ , and will be denoted by  $\text{BSO}$ .

The map  $I_{k,m}: \text{Gr}(k, m+k) \rightarrow \text{Gr}(m, k+m)$  of assigning to each oriented  $k$ -plane in  $\mathbb{R}^{k+m}$  its orthogonal  $m$ -plane induces a map

$$I: \text{BSO} \rightarrow \text{BSO},$$

with  $I^2$  the identity map.

For a compact  $n$ -dimensional manifold  $M^n$ , the stable normal bundle is represented by the normal bundle of an embedding  $i_k: M^n \rightarrow \mathbb{R}^{n+k}$  for any sufficiently large  $k$ . The normal bundle  $\nu(i_k)$  of  $i_k$  is the quotient of the pull-back of the tangent bundle  $T\mathbb{R}^{n+k} = \mathbb{R}^{n+k} \times \mathbb{R}^{n+k}$  by the tangent bundle  $TM$ . Then the normal map

$$\nu_k: M \rightarrow \text{Gr}(k, k+n)$$

and the tangent map

$$\tau_k: M \rightarrow \text{Gr}(n, k+n)$$

are related to each other by  $\tau_k = I_{k,n} \circ \nu_k$ . So the classifying map for the stable normal bundle

$$\nu: M \rightarrow \text{BSO}$$

and the classifying map of the stable tangent bundle

$$\tau: M \rightarrow \text{BSO}$$

are related by  $\tau = I \circ \nu$ . Thus, we have a natural isomorphism

$$I^*: \tau^* \text{BSpin}^c(\mathcal{K}) \rightarrow \nu^* \text{BSpin}^c(\mathcal{K}). \quad (3.3)$$

on the associated bundles of compact operators. This determines an isomorphism

$$I_*: K_{\text{ev/odd}}^a(M, W_3 \circ \tau) \cong K_{\text{ev/odd}}^a(M, W_3 \circ \nu). \quad (3.4)$$

on the corresponding twisted K-homology groups.

**Remark 3.6.** Given an embedding  $i_k: M \rightarrow \mathbb{R}^{n+k}$  with the normal bundle  $N$ , the natural isomorphism

$$TM \oplus N \oplus N \cong \underline{\mathbb{R}}^{n+k} \oplus N$$

and the canonical  $\text{Spin}^c$  structure on  $N \oplus N$  define a canonical homotopy between  $W_3 \circ \tau$  and  $W_3 \circ \nu$ . The isomorphism (3.4) is induced by this canonical homotopy.



For a Riemannian manifold  $M$ , denote by  $\text{Cliff}(TM)$  the bundle of complex Clifford algebras of  $TM$  over  $M$ . As algebras of the sections,  $C(M, \text{Cliff}(TM))$  is Morita equivalent to  $C(M, \tau^* B\text{Spin}^c(\mathcal{K}))$ . Hence, we have a canonical isomorphism

$$K_{\text{ev/odd}}^a(M, W_3 \circ \tau) \cong KK^{\text{ev/odd}}(C(M, \text{Cliff}(M)), \mathbb{C}),$$

with the degree shift by  $\dim M \pmod{2}$ . Applying Kasparov's Poincaré duality (cf. [31])

$$KK^{\text{ev/odd}}(\mathbb{C}, C(M)) \cong KK^{\text{ev/odd}}(C(M, \text{Cliff}(M)), \mathbb{C}),$$

we obtain a canonical isomorphism

$$PD: K^0(M) \cong K_{\text{ev/odd}}^a(M, W_3 \circ \tau),$$

with the degree shift by  $\dim M \pmod{2}$ . The Poincaré dual of the unit element in  $K^0(M)$  is the fundamental class  $[M] \in K_{\text{ev/odd}}^a(M, W_3 \circ \tau)$ . Note that  $[M] \in K_{\text{ev}}^a(M, W_3(M))$  if  $M$  is even dimensional, and  $[M] \in K_{\text{odd}}^a(M, W_3(M))$  if  $M$  is odd dimensional. The cap product

$$\frown: K_{\text{ev/odd}}^a(M, W_3 \circ \tau) \otimes K^0(M) \rightarrow K_{\text{ev/odd}}^a(M, W_3 \circ \tau)$$

is defined by the Kasparov product. We remark that the cap product of the fundamental class  $[M]$  and  $[E] \in K^0(M)$  is given by

$$[M] \frown [E] = PD([E]).$$

Given an  $\alpha$ -twisted  $\text{Spin}^c$  manifold  $(M, \nu, \iota, \eta)$  over  $X$ , the homotopy  $\eta$  induces an isomorphism  $\nu^* B\text{Spin}^c \cong \iota^* \mathcal{P}_\alpha$  as principal  $K(\mathbb{Z}, 2)$ -bundles on  $M$ , hence defines an isomorphism

$$\nu^* B\text{Spin}^c(\mathcal{K}) \xrightarrow[\cong]{\eta^*} \iota^* \mathcal{P}_\alpha(\mathcal{K})$$

as bundles of  $C^*$ -algebras on  $M$ . This isomorphism determines a canonical isomorphism

$$C(M, \nu^* B\text{Spin}^c(\mathcal{K})) \cong C(M, \iota^* \mathcal{P}_\alpha(\mathcal{K}))$$

between the corresponding continuous trace  $C^*$ -algebras. Therefore, we have a canonical isomorphism

$$\eta_*: K_{\text{ev/odd}}^a(M, W_3 \circ \nu) \cong K_{\text{ev/odd}}^a(M, \alpha \circ \iota). \tag{3.5}$$

Notice that the natural push-forward map in analytic K-homology theory is

$$\iota_*: K_{\text{ev/odd}}^a(M, \alpha \circ \iota) \rightarrow K_{\text{ev/odd}}^a(X, \alpha). \tag{3.6}$$

We can introduce a notion of analytical index for any  $\alpha$ -twisted  $\text{Spin}^c$  manifold over  $X$ , taking values in analytical twisted K-homology of  $(X, \alpha)$ .

**Definition 3.7.** Given an  $\alpha$ -twisted  $\text{Spin}^c$  closed manifold  $(M, \nu, \iota, \eta)$  and  $[E] \in K^0(M)$ , we define its analytical index

$$\text{Index}_a((M, \nu, \iota, \eta), [E]) \in K_{\text{ev/odd}}^a(X, \alpha)$$

to be the image of the cap product  $[M] \cap [E] \in K_{\text{ev/odd}}^a(M, W_3 \circ \tau)$  under the maps (3.4), (3.5) and (3.6):

$$K_{\text{ev/odd}}^a(M, W_3 \circ \tau) \xrightarrow{\iota_* \circ \eta_* \circ I_*} K_{\text{ev/odd}}^a(X, \alpha).$$

The analytical index enjoys the following properties.

**Proposition 3.8.** (1) *The analytical index  $\text{Index}_a((M, \nu, \iota, \eta), [E])$  depends only on the equivalence class of the  $\alpha$ -twisted  $\text{Spin}^c$  structure  $\eta$ .*

(2) *(Disjoint union and direct sum) Let  $(M_1, \nu_1, \iota_1, \eta_1)$  and  $(M_2, \nu_2, \iota_2, \eta_2)$  be a pair of  $\alpha$ -twisted  $\text{Spin}^c$  manifolds, and let  $[E_i] \in K^0(M_i)$ . Then*

$$\begin{aligned} &\text{Index}_a((M_2, \nu_2, \iota_2, \eta_2) \sqcup (M_1, \nu_1, \iota_1, \eta_1), [E_1] \sqcup [E_2]) \\ &= \text{Index}_a((M_1, \nu_1, \iota_1, \eta_1), [E_1]) + \text{Index}_a((M_2, \nu_2, \iota_2, \eta_2), [E_2]). \end{aligned}$$

(3) *(Bordism invariance) If  $(W, \nu, \iota, \eta)$  is an  $\alpha$ -twisted  $\text{Spin}^c$  manifold with boundary over  $X$  and  $[E] \in K^0(W)$ , then*

$$\text{Index}_a((\partial W, \partial \nu, \partial \iota, \partial \eta), [E|_{\partial W}]) = 0.$$

*Proof.* In the definition of  $\text{Index}_a((M, \nu, \iota, \eta), [E])$  the  $\alpha$ -twisted  $\text{Spin}^c$  structure  $\eta$  shows up only through

$$\eta_* : K_{\text{ev/odd}}^a(M, W_3 \circ \nu) \cong K_{\text{ev/odd}}^a(M, \alpha \circ \iota).$$

This isomorphism depends only on the homotopy class of  $\eta$ . So claim (1) is obvious.

Claim (2) follows from the disjoint union and direct sum property of the fundamental classes and the cap product.

To establish claim (3), let  $(W, \nu, \iota, \eta)$  be an  $\alpha$ -twisted  $\text{Spin}^c$  manifold with boundary over  $X$  and denote its boundary by  $M = \partial W$  with the induced  $\alpha$ -twisted  $\text{Spin}^c$  structure  $(\partial \nu, \partial \iota, \partial \eta)$ . Let  $i : M \rightarrow W$  be the boundary inclusion map. The exact sequence in topological K-theory and analytical K-homology are related through Poincaré duality (assume that  $W$  is odd dimensional) as in the following commutative diagram of exact sequences:

$$\begin{array}{ccccc} K^1(W, M) & \longleftarrow & K^0(M) & \xleftarrow{i^*} & K^0(W) \\ \downarrow PD & & \downarrow PD & & \downarrow PD \\ K_0^a(W, W_3 \circ \tau_W) & \xleftarrow{i_*} & K_0^a(M, W_3 \circ \tau_M) & \xleftarrow{\partial} & K_1^a(W, M; W_3 \circ \tau_W), \end{array} \tag{3.7}$$

where  $\tau_M$  and  $\tau_W$  are classifying maps of the stable tangent bundles of  $M$  and  $W$ , respectively. One could get this K-homology exact sequence by applying the Kasparov KK-functor to the following short exact sequence of  $C^*$ -algebras:

$$0 \rightarrow C_0(W, \text{Cliff}(W)) \rightarrow C(W, \text{Cliff}(W)) \rightarrow C(M, \text{Cliff}(M)) \rightarrow 0.$$

$C_0(W, \text{Cliff}(W))$  is the  $C^*$ -algebra of continuous sections of  $\text{Cliff}(W)$  vanishing at the boundary  $M$ . The relative analytical K-homology  $KK^{\text{ev/odd}}(C_0(W, \text{Cliff}(W)), \mathbb{C})$  is isomorphic to  $K_{\text{ev/odd}}^a(W, M; W_3 \circ \tau)$ , and hence isomorphic to  $K_{\text{ev/odd}}^a(W, M; W_3 \circ \nu)$  under (3.4).

It follows from (3.7) that the Poincaré dual of

$$[i^* E] \in K^0(M)$$

is mapped to zero in  $K_0^a(W, W_3 \circ \tau_W)$  for  $[E] \in K^0(W)$  under the map

$$i_* \circ PD \circ i^*([E]) = i_* \circ \partial \circ PD([E]) = 0. \tag{3.8}$$

Notice that  $\text{Index}_a((M, \partial\nu, \partial\iota, \partial\eta), [E|_M])$  is image of the class  $PD(i^*[E])$  under the sequence of maps

$$K_0^a(M, W_3 \circ \tau_M) \xrightarrow{I_*} K_0^a(M, W_3 \circ \partial\nu) \xrightarrow{(\partial\eta)_*} K_0^a(M, \alpha \circ \partial\iota) \xrightarrow{(\partial\iota)_*} K_0^a(X, \alpha),$$

and the inclusion map  $i : M \rightarrow W$  induces the following commutative diagram:

$$\begin{array}{ccccccc} K_0^a(M, W_3 \circ \tau_M) & \xrightarrow{I_*} & K_0^a(M, W_3 \circ \partial\nu) & \xrightarrow{(\partial\eta)_*} & K_0^a(M, \alpha \circ \partial\iota) & & \\ \downarrow i_* & & \downarrow i_* & & \downarrow i_* & \searrow (\partial\iota)_* & \\ K_0^a(W, W_3 \circ \tau_W) & \xrightarrow{I_*} & K_0^a(W, W_3 \circ \nu) & \xrightarrow{\eta_*} & K_0^a(W, \alpha \circ \nu) & \xrightarrow{\iota_*} & K_0^a(X, \alpha). \end{array}$$

Therefore, we conclude that

$$\begin{aligned} & \text{Index}_a((M, \partial\nu, \partial\iota, \partial\eta), [E|_M]) \\ &= (\partial\iota)_* \circ (\partial\eta)_* \circ I_* \circ PD(i^*[E]) \quad (\text{Definition 3.7}) \\ &= \iota_* \circ \eta_* \circ I_* \circ i_* \circ PD(i^*[E]) \quad (\text{the above commutative diagram}) \\ &= 0 \quad (3.8). \end{aligned} \quad \square$$

**Remark 3.9.** Given an  $\alpha$ -twisted  $\text{Spin}^c$  structure  $\eta$  on  $(M, \nu, \iota)$  and a complex line bundle  $L$  over  $M$ , denote by  $c_1 \cdot [\eta]$  the action of the first Chern class  $c_1 = c_1(L) \in H^2(M, \mathbb{Z})$  on the homotopy class of  $\eta$ . Then the analytical index depends on the choice of equivalence classes of  $\alpha$ -twisted  $\text{Spin}^c$  structures through the following formula

$$\text{Index}_a((M, \nu, \iota, c_1 \cdot [\eta]), [E]) = \text{Index}_a((M, \nu, \iota, [\eta]), ([L] \otimes [E])).$$

**4. Twisted  $\text{Spin}^c$  bordism and topological index**

Given a manifold  $X$  with a twisting  $\alpha : X \rightarrow K(\mathbb{Z}, 3)$ ,  $\alpha$ -twisted  $\text{Spin}^c$  manifolds over  $X$  form a bordism category, called the  $\alpha$ -twisted  $\text{Spin}^c$  bordism over  $(X, \alpha)$ , whose objects are compact smooth manifolds over  $X$  with an  $\alpha$ -twisted  $\text{Spin}^c$  structure as in Definition 3.1. A morphism between  $\alpha$ -twisted  $\text{Spin}^c$  manifolds  $(M_1, \nu_1, \iota_1, \eta_1)$  and  $(M_2, \nu_2, \iota_2, \eta_2)$  is a boundary preserving continuous map  $f : M_1 \rightarrow M_2$  and the following diagram

$$\begin{array}{ccccc}
 M_1 & \xrightarrow{\nu_1} & & & BSO \\
 \downarrow f & \searrow & \xrightarrow{\nu_2} & & \downarrow W_3 \\
 & M_2 & & & \\
 \downarrow \iota_1 & \downarrow \iota_2 & \swarrow \eta_2 & & \downarrow W_3 \\
 & X & \xrightarrow{\alpha} & K(\mathbb{Z}, 3) & \\
 & & & & \downarrow W_3
 \end{array} \tag{4.1}$$

is a homotopy commutative diagram such that

- (1)  $\nu_1$  is homotopic to  $\nu_2 \circ f$  through a continuous map  $\nu : M_1 \times [0, 1] \rightarrow BSO$ ;
- (2)  $\iota_2 \circ f$  is homotopic to  $\iota_1$  through continuous map  $\iota : M_1 \times [0, 1] \rightarrow X$ ;
- (3) the composition of homotopies  $(\alpha \circ \iota) * (\eta_2 \circ (f \times \text{Id})) * (W_3 \circ \nu)$  is homotopic to  $\eta_1$ .

The boundary functor  $\partial$  applied to an  $\alpha$ -twisted  $\text{Spin}^c$  manifold  $(M, \nu, \iota, \eta)$  is the manifold  $\partial M$  with outer normal orientation and the restriction of the  $\alpha$ -twisted  $\text{Spin}^c$  structure to  $M$ .

Two  $\alpha$ -twisted  $\text{Spin}^c$  manifolds  $(M_1, \nu_1, \iota_1, \eta_1)$  and  $(M_2, \nu_2, \iota_2, \eta_2)$  are called isomorphic if there exists a diffeomorphism  $f : M_1 \rightarrow M_2$  such that the diagram (4.1) is a homotopy commutative diagram.

**Definition 4.1.** We say that an  $\alpha$ -twisted  $\text{Spin}^c$  manifold  $(M, \nu, \iota, \eta)$  is null-bordant if there exists an  $\alpha$ -twisted  $\text{Spin}^c$  manifold  $W$  whose boundary is  $(M, \nu, \iota, \eta)$  in the sense of (3.1). We define the  $\alpha$ -twisted  $\text{Spin}^c$  bordism group of  $X$ , denoted by  $\Omega^{\text{Spin}^c}(X, \alpha)$ , to be the set of all isomorphism classes of closed  $\alpha$ -twisted  $\text{Spin}^c$  manifolds over  $X$  modulo null-bordism, with the sum given by the disjoint union.

The subgroup of isomorphism classes of  $n$ -dimensional closed  $\alpha$ -twisted  $\text{Spin}^c$  manifolds over  $X$  will be denoted  $\Omega_n^{\text{Spin}^c}(X, \alpha)$ . Set

$$\begin{aligned}
 \Omega_{\text{ev}}^{\text{Spin}^c}(X, \alpha) &= \bigoplus_k \Omega_{2k}^{\text{Spin}^c}(X, \alpha), \\
 \Omega_{\text{odd}}^{\text{Spin}^c}(X, \alpha) &= \bigoplus_k \Omega_{2k+1}^{\text{Spin}^c}(X, \alpha).
 \end{aligned}$$

**Proposition 4.2.** *The analytical index defined in the previous section induces a homomorphism*

$$\text{Index}_a : \Omega_{\text{ev/odd}}^{\text{Spin}^c}(X, \alpha) \rightarrow K_{\text{ev/odd}}^a(X, \alpha). \tag{4.2}$$

*Proof.* Let  $(M, \iota, \nu, \eta)$  be  $\alpha$ -twisted  $\text{Spin}^c$  manifold over  $X$  representing an element in the  $\alpha$ -twisted  $\text{Spin}^c$  bordism group  $\Omega_{\text{ev/odd}}^{\text{Spin}^c}(X, \alpha)$ . Define

$$\text{Index}_a(M, \iota, \nu, \eta) = \text{Index}_a((M, \iota, \nu, \eta), [\underline{\mathbb{C}}]) \in K_{\text{ev/odd}}^a(X, \alpha),$$

where  $\underline{\mathbb{C}}$  denotes the trivial line bundle over  $M$  representing the unit element in  $K^0(M)$ . We need to show that for a pair of isomorphic objects

$$(M_1, \iota_1, \nu_1, \eta_1) \quad \text{and} \quad (M_2, \iota_2, \nu_2, \eta_2)$$

in the  $\alpha$ -twisted  $\text{Spin}^c$  bordism category over  $X$ , we have

$$\text{Index}_a(M_1, \iota_1, \nu_1, \eta_1) = \text{Index}_a(M_2, \iota_2, \nu_2, \eta_2).$$

Let  $f$  be a diffeomorphism from  $M_1$  to  $M_2$  such that (4.1) is a homotopy commutative diagram. Let  $\tau_1$  and  $\tau_2$  be classifying maps of the stable tangent bundles of  $M_1$  and  $M_2$ , respectively. The homotopy between  $\nu_1$  and  $\nu_2 \circ f$  implies that  $\tau_1$  and  $\tau_2 \circ f$  are homotopy equivalent. This defines a canonical  $\text{Spin}^c$  structure on  $TM_1 \oplus f^*TM_2$ . Hence, there is a canonical Morita equivalence

$$C(M_1, \text{Cliff}(M_1)) \sim C(M_1, f^* \text{Cliff}(M_2)).$$

This Morita equivalence defines a canonical isomorphism

$$K_{\text{ev/odd}}^a(M_1, W_3 \circ \tau_1) \cong K_{\text{ev/odd}}^a(M_1, W_3 \circ \tau_2 \circ f).$$

Recall that natural push-forward map in analytical K-homology is related to the K-theoretical push-forward map  $f_!$  in topological K-theory via the Poincaré duality (PD):

$$\begin{array}{ccc} K^{\text{ev/odd}}(M_1) & \xrightarrow{f_!} & K^{\text{ev/odd}}(M_2) \\ PD \downarrow \cong & & PD \downarrow \cong \\ K_{\text{ev/odd}}^a(M_1, W_3 \circ \tau_1) & \xrightarrow{f_*} & K_{\text{ev/odd}}^a(M_2, W_3 \circ \tau_2), \end{array}$$

where the Poincaré duality shifts the degree by the dimension of the underlying manifold. Applying the natural push-forward map in analytical K-homology, we obtain that

$$f_* : K_{\text{ev/odd}}^a(M_1, W_3 \circ \tau_1) \cong K_{\text{ev/odd}}^a(M_1, W_3 \circ \tau_2 \circ f) \rightarrow K_{\text{ev/odd}}^a(M_2, W_3 \circ \tau_2),$$

with the degree shifted by  $d(f) = \dim M_1 - \dim M_2 \pmod{2}$ . The homotopy between  $W_3 \circ \nu_1$  and  $W_3 \circ \nu_2 \circ f$  defines a canonical homomorphism

$$f_* : K_{\text{ev/odd}}^a(M_1, W_3 \circ \nu_1) \cong K_{\text{ev/odd}}^a(M_1, W_3 \circ \nu_2 \circ f) \rightarrow K_{\text{ev/odd}}^a(M_2, W_3 \circ \nu_2)$$

such that the following diagram commutes:

$$\begin{array}{ccc} K_{\text{ev/odd}}^a(M_1, W_3 \circ \tau_1) & \xrightarrow{I_*^{M_1}} & K_{\text{ev/odd}}^a(M_1, W_3 \circ \nu_1) \\ f_* \downarrow & & \downarrow f_* \\ K_{\text{ev/odd}}^a(M_2, W_3 \circ \tau_2) & \xrightarrow{I_*^{M_2}} & K_{\text{ev/odd}}^a(M_2, W_3 \circ \nu_2). \end{array} \tag{4.3}$$

Similarly, the homotopy between  $(\alpha \circ \iota) * (\eta_2 \circ (f \times \text{Id})) * (W_3 \circ \nu)$  and  $\eta_1$  induces a commutative diagram

$$\begin{array}{ccc} K_{\text{ev/odd}}^a(M_1, W_3 \circ \nu_1) & \xrightarrow{(\eta_1)_*} & K_{\text{ev/odd}}^a(M_1, \alpha \circ \iota_1) \\ f_* \downarrow & & \downarrow f_* \\ K_{\text{ev/odd}}^a(M_2, W_3 \circ \nu_2) & \xrightarrow{(\eta_2)_*} & K_{\text{ev/odd}}^a(M_2, \alpha \circ \iota_2). \end{array} \tag{4.4}$$

The homotopy between  $\alpha \circ \iota_2 \circ f$  and  $\alpha \circ \iota_1$  induces the following commutative triangle

$$\begin{array}{ccc} K_{\text{ev/odd}}^a(M_1, \alpha \circ \iota_1) & & \\ \downarrow f_* & \searrow (\iota_1)_* & \\ & & K_{\text{ev/odd}}^a(X, \alpha) \\ & \nearrow (\iota_2)_* & \\ K_{\text{ev/odd}}^a(M_2, \alpha \circ \iota_2) & & \end{array} \tag{4.5}$$

These commutative diagrams (4.3), (4.4) and (4.5) imply that

$$\begin{aligned} \text{Index}_a(M_2, \iota_2, \nu_2, \eta_2) &= (\iota_2)_* \circ (\eta_2)_* \circ I_*^{M_2}([M_2]) \\ &= (\iota_2)_* \circ (\eta_2)_* \circ I_*^{M_2} \circ f_*([M_1]) \\ &= (\iota_1)_* \circ (\eta_1)_* \circ I_*^{M_1}([M_1]) \\ &= \text{Index}_a(M_1, \iota_1, \nu_1, \eta_1). \end{aligned}$$

Now the bordism invariance in Proposition 3.8 tells us that  $\text{Index}_a$  is a well-defined homomorphism from  $\Omega_{\text{ev/odd}}^{\text{Spin}^c}(X, \alpha)$  to  $K_{\text{ev/odd}}^a(X, \alpha)$ . □

We recall the construction of Thom spectrum of  $\text{Spin}^c$  bordism. Let  $\xi_k$  be the universal bundle over  $B\text{SO}(k)$ . The pull-back bundle over  $B\text{Spin}^c(k)$  is given by

$$\tilde{\xi}_k = E \text{Spin}^c(k) \times_{\text{Spin}^c(k)} \mathbb{R}^k.$$

Denote by  $M\text{Spin}^c(k)$  the Thom space of  $\tilde{\xi}_k$ . The inclusion map  $j_k$  induces a pull-back diagram

$$\begin{array}{ccc} j_k^* \tilde{\xi}_{k+1} & \longrightarrow & \tilde{\xi}_{k+1} \\ \downarrow & & \downarrow \\ B\text{Spin}^c(k) & \xrightarrow{j_k} & B\text{Spin}^c(k+1) \end{array}$$

with  $j_k^* \tilde{\xi}_{k+1} \cong \tilde{\xi}_k \oplus \mathbb{R}$ , where  $\mathbb{R}$  denotes the trivial real line bundle. Then the Thom space of  $j_k^* \tilde{\xi}_{k+1}$  can be identified with  $\Sigma M\text{Spin}^c(k)$  (the suspension of  $M\text{Spin}^c(k)$ ). Thus we have a sequence of continuous maps

$$\text{Th}(j_k): \Sigma M\text{Spin}^c(k) \rightarrow M\text{Spin}^c(k+1),$$

i.e.,  $\{M\text{Spin}^c(k)\}_k$  is the Thom spectrum associated to  $B\text{Spin}^c = \varinjlim_k B\text{Spin}^c(k)$ .

Since  $B\text{Spin}^c(k)$  is a principal  $K(\mathbb{Z}, 2)$ -bundle over  $B\text{SO}(k)$ , we have a base point preserving action of  $K(\mathbb{Z}, 2)$  on the Thom spectrum  $\{M\text{Spin}^c(k)\}$ , written as

$$K(\mathbb{Z}, 2)_+ \wedge M\text{Spin}^c(k) = \frac{K(\mathbb{Z}, 2) \times M\text{Spin}^c(k)}{K(\mathbb{Z}, 2) \times *} \rightarrow M\text{Spin}^c(k),$$

which is compatible with the base-point action of  $K(\mathbb{Z}, 2)$  on the K-theory spectrum  $\mathbb{K}$  in the sense that there exists a  $K(\mathbb{Z}, 2)$ -equivariant map, called the index map

$$\text{Ind}: M\text{Spin} \rightarrow \mathbb{K}.$$

This  $K(\mathbb{Z}, 2)$ -equivariant map has been constructed in [21] and [43]. Here we provide a more geometric construction. Write the principal  $BU(1)$ -bundle  $B\text{Spin}^c(2k)$  as the pull-back bundle

$$\begin{array}{ccc} B\text{Spin}^c(2k) & \longrightarrow & EK(\mathbb{Z}, 2) \\ \downarrow & & \downarrow \\ B\text{SO}(2k) & \xrightarrow{W_3} & K(\mathbb{Z}, 3), \end{array}$$

which induces a natural  $\text{PU}(\mathcal{H})$ -action

$$\text{PU}(\mathcal{H}) \times B\text{Spin}^c(2k) \rightarrow B\text{Spin}^c(2k).$$

This action corresponds to the action of the set of complex line bundles on the set of  $\text{Spin}^c$  structures. The  $\text{PU}(\mathcal{H})$ -action on  $B\text{Spin}^c(2k)$  can be lifted to a base point preserving action

$$\text{PU}(\mathcal{H}) \times M\text{Spin}^c(2k) \rightarrow M\text{Spin}^c(2k)$$

of  $\text{PU}(\mathcal{H})$  on  $M\text{Spin}^c(2k)$ . Note that there is a fundamental  $\mathbb{Z}_2$ -graded spinor bundle  $S^+ \oplus S^-$  over  $B\text{Spin}^c(2k)$  – see Theorem C.9 in [33] –, which defines a canonical Thom class in  $K^0(M\text{Spin}^c(2k))$ . This canonical Thom class determines a  $\text{PU}(\mathcal{H})$ -equivariant map

$$\text{Ind} : M\text{Spin}(2k) \rightarrow \text{Fred}(\mathcal{H}).$$

Hence we have associated bundles of Thom spectra over  $X$ ,

$$\mathcal{P}_\alpha(M\text{Spin}^c(k)) = \mathcal{P}_\alpha \times_{K(\mathbb{Z},2)} M\text{Spin}^c(k),$$

and natural maps

$$\text{Ind} : \mathcal{P}_\alpha(M\text{Spin}^c(k)) \rightarrow \mathcal{P}_\alpha(\mathbb{K}) = \mathcal{P}_\alpha \times_{K(\mathbb{Z},2)} \mathbb{K}$$

to the associated bundle of K-theory spectra.

**Remark 4.3.** The  $\text{Spin}^c$  bordism groups over a pointed space  $X$ , denoted by  $\Omega_*^{\text{Spin}^c}(X)$  as in [41], can be identified (via the Pontrjagin–Thom isomorphism) as the stable homotopy groups of  $M\text{Spin}^c \wedge X$

$$\Omega_n^{\text{Spin}^c}(X) \cong \pi_n^S(M\text{Spin}^c \wedge X) := \lim_{\substack{\longrightarrow \\ k}} \pi_{n+k}(M\text{Spin}^c(k) \wedge X). \quad (4.6)$$

The index map  $\text{Ind} : M\text{Spin} \rightarrow \mathbb{K}$  determines a natural transformation from the even or odd dimensional  $\text{Spin}^c$  bordism group of  $X$  to K-homology of  $X$ , which is called the topological index:

$$\Omega_{\text{ev/odd}}^{\text{Spin}^c}(X) \rightarrow K_{\text{ev/odd}}^t(X).$$

The following theorem is the twisted version of the Pontrjagin–Thom isomorphism (4.6).

**Theorem 4.4.** *The bordism group  $\Omega_n^{\text{Spin}^c}(X, \alpha)$  of  $n$ -dimensional  $\alpha$ -twisted  $\text{Spin}^c$  manifolds over  $X$  is isomorphic to the stable homotopy group*

$$\Theta : \Omega_n^{\text{Spin}^c}(X, \alpha) \xrightarrow{\cong} \pi_n^S(\mathcal{P}_\alpha(M\text{Spin}^c)/X).$$

Here we denote  $\pi_n^S(\mathcal{P}_\alpha(M\text{Spin}^c)/X) := \lim_{k \rightarrow \infty} \pi_{n+k}(\mathcal{P}_\alpha(M\text{Spin}^c(k))/X)$ .

*Proof.* The proof is modeled on the proof of the classical Pontrjagin–Thom isomorphism (cf. [41])

*Step 1.* Definition of the homomorphism  $\Theta$ .

Let  $\sigma$  be an element in  $\Omega_n^{\text{Spin}^c}(X, \alpha)$  represented by an  $n$ -dimensional  $\alpha$ -twisted  $\text{Spin}^c$  manifold  $(M, \iota, \nu, \eta)$  over  $X$ . Let  $i_k : M \rightarrow \mathbb{R}^{n+k}$  be an embedding with the



classifying map of the normal bundle denoted by  $\nu_k$ . Then we have the following pull-back diagram:

$$\begin{array}{ccc}
 N & \xrightarrow{\tilde{\nu}_k} & \xi_k \\
 \pi \downarrow & & \downarrow \\
 M & \xrightarrow{\nu_k} & BSO(k).
 \end{array} \tag{4.7}$$

Here the total space  $N$  of the normal bundle of  $i_k$  can be thought of as a subspace of  $\mathbb{R}^{n+k} \times \mathbb{R}^{n+k}$ . Under the addition map  $\mathbb{R}^{n+k} \times \mathbb{R}^{n+k} \rightarrow \mathbb{R}^{n+k}$ , for some sufficiently small  $\varepsilon > 0$ , the  $\varepsilon$ -neighborhood  $N_\varepsilon$  of the zero section  $M \times \{0\}$  of  $N$  is an embedding  $\varepsilon|_{N_\varepsilon} : N_\varepsilon \rightarrow \mathbb{R}^{n+k}$ , whose restriction to the zero section  $M \times \{0\}$  is the embedding  $i_k : M \rightarrow \mathbb{R}^{n+k}$ .

Consider  $S^{n+k}$  as  $\mathbb{R}^{n+k} \cup \{\infty\}$  (the one point compactification), so we have an embedding  $N_\varepsilon \rightarrow S^{n+k}$ . Define

$$c : S^{n+r} \rightarrow N_\varepsilon / \partial N_\varepsilon$$

by collapsing all points of  $S^{n+k}$  outside and on the boundary of  $N_\varepsilon$  to a point. Note that  $N_\varepsilon / \partial N_\varepsilon$  is homeomorphic to the Thom space  $\text{Th}(N)$  of the normal bundle of  $i_k$ , induced by multiplication by  $\varepsilon^{-1}$ . Denote this homeomorphism by

$$\varepsilon^{-1} : N_\varepsilon / \partial N_\varepsilon \rightarrow \text{Th}(N).$$

The pull-back diagram

$$\begin{array}{ccc}
 \mathcal{P}_{W_3 \circ \nu_k} & \longrightarrow & EK(\mathbb{Z}, 2) \\
 \downarrow & & \downarrow \\
 M & \xrightarrow{W_3 \circ \nu_k} & K(\mathbb{Z}, 3)
 \end{array}$$

induces a homotopy pull-back

$$\begin{array}{ccc}
 \mathcal{P}_{W_3 \circ \nu_k}(B\text{Spin}^c(k)) & \longrightarrow & EK(\mathbb{Z}, 2)(B\text{Spin}^c(k)) \\
 \downarrow & & \downarrow \\
 M & \xrightarrow{W_3 \circ \nu_k} & K(\mathbb{Z}, 3).
 \end{array}$$

Since  $EK(\mathbb{Z}, 2)$  is contractible,  $EK(\mathbb{Z}, 2)(B\text{Spin}^c(k))$  is homotopy equivalent to  $BSO(k)$ . This implies that the diagram

$$\begin{array}{ccc}
 \mathcal{P}_{W_3 \circ \nu_k}(B\text{Spin}^c(k)) & \longrightarrow & BSO(k) \\
 \downarrow & & \downarrow \\
 M & \xrightarrow{W_3 \circ \nu_k} & K(\mathbb{Z}, 3)
 \end{array}$$

is a homotopy pull-back. Notice that the diagram

$$\begin{array}{ccc}
 M & \xrightarrow{\nu_k} & BSO(k) \\
 \text{Id} \downarrow & & \downarrow W_3 \\
 M & \xrightarrow{W_3 \circ \nu_k} & K(\mathbb{Z}, 3)
 \end{array}$$

is commutative. Thus there exists a unique map (up to homotopy)

$$h: M \rightarrow \mathcal{P}_{W_3 \circ \nu_k}(B\text{Spin}^c(k))$$

such that the diagram

$$\begin{array}{ccccc}
 M & & & & \\
 \downarrow \text{Id} & \searrow h & & \searrow \nu_k & \\
 & \mathcal{P}_{W_3 \circ \nu_k}(B\text{Spin}^c(k)) & \longrightarrow & BSO(k) & \\
 & \downarrow & \dashrightarrow & \downarrow W_3 & \\
 & M & \xrightarrow{W_3 \circ \nu_k} & K(\mathbb{Z}, 3) & 
 \end{array}$$

is homotopy commutative. Together with the pull-back diagram

$$\begin{array}{ccc}
 \tilde{\xi}_k & \longrightarrow & \xi_k \\
 \downarrow & & \downarrow \\
 B\text{Spin}^c(k) & \longrightarrow & BSO(k),
 \end{array}$$

we obtain a homotopy commutative diagram

$$\begin{array}{ccc}
 N & \longrightarrow & \mathcal{P}_{W_3 \circ \nu_k}(\tilde{\xi}_k) \\
 \downarrow & & \downarrow \\
 M & \longrightarrow & \mathcal{P}_{W_3 \circ \nu_k}(B\text{Spin}^c(k)),
 \end{array}$$

which in turn determines a canonical map

$$h_*: \text{Th}(N) \rightarrow \mathcal{P}_{W_3 \circ \nu_k}(M\text{Spin}^c(k))/M.$$

Notice that the  $\alpha$ -twisted  $\text{Spin}^c$ -structure on  $M$  defines a continuous map

$$\iota_*: \mathcal{P}_{W_3 \circ \nu_k}(M\text{Spin}^c(k))/M \rightarrow \mathcal{P}_\alpha(M\text{Spin}^c(k))/X.$$

The composition  $\iota_* \circ h_* \circ \varepsilon^{-1} \circ c$  is a continuous map of pairs

$$\theta = \theta_{(M, \iota, \nu, \eta)}: (S^{n+k}, \infty) \rightarrow (\mathcal{P}_\alpha(M\text{Spin}^c(k))/X, *),$$

here  $*$  is the base point in  $\mathcal{P}_\alpha(M\text{Spin}^c(k))/X$ , hence represents an element of

$$\lim_{k \rightarrow \infty} \pi_{n+k}(\mathcal{P}_\alpha(M\text{Spin}^c(k))/X).$$

The stable homotopy class of  $\theta$  does not depend on choices of  $i_k, \nu_k, \varepsilon$  in the construction for sufficiently large  $k$ . Thus, we assign an element in

$$\lim_{k \rightarrow \infty} \pi_{n+k}(\mathcal{P}_\alpha(M\text{Spin}^c(k))/X)$$

represented by  $\theta$  to every closed  $\alpha$ -twisted  $\text{Spin}^c$  manifold  $(M, \iota, \nu, \eta)$  over  $X$ .

Now we show that the stable homotopy class of  $\theta$  depends only on the bordism class of  $M$ . Let  $W$  be an  $(n + 1)$ -dimensional  $\alpha$ -twisted  $\text{Spin}^c$  manifold and let  $j: \partial W \rightarrow \mathbb{R}^{n+k}$  be an embedding for some sufficiently large  $k$  with the classifying map  $\nu_k$  for the normal bundle:

$$\begin{array}{ccc} N_{\partial W} & \xrightarrow{\tilde{\nu}_k} & \xi_k \\ \pi \downarrow & & \downarrow \\ \partial W & \xrightarrow{\nu_k} & BSO(k). \end{array}$$

Choose  $i: W \rightarrow \mathbb{R}^{n+k} \times [0, 1]$  to be an embedding agreeing with  $j \times \{1\}$  on  $\partial W$ , embedding a tubular neighborhood of  $\partial W$  orthogonally along  $j(\partial W) \times \{1\}$ , and with the image missing  $\mathbb{R}^{n+k} \times \{0\}$ . The previous construction applied to the embedding  $i$  yields a null-homotopy of the map

$$\theta: (S^{n+k}, \infty) \rightarrow (\mathcal{P}_\alpha(M\text{Spin}^c(k))/X, *).$$

Assigning the stable homotopy class of the map  $\theta$  to each  $\alpha$ -twisted  $\text{Spin}^c$  bordism class, we have defined a map

$$\Theta: \Omega_n^{\text{Spin}^c}(X, \alpha) \rightarrow \pi_n^S(\mathcal{P}_\alpha(M\text{Spin}^c)/X).$$

*Step 2.*  $\Theta$  is a homomorphism.

Let  $(M_1, \iota_1, \nu_1, \eta_1)$  and  $(M_2, \iota_2, \nu_2, \eta_2)$  be a pair of closed  $\alpha$ -twisted  $\text{Spin}^c$  manifolds over  $X$  representing two classes in  $\Omega_n^{\text{Spin}^c}(X, \alpha)$ . Then for  $a = 1$  or  $2$ ,  $\Theta([M_a, \iota_a, \nu_a, \eta_a])$  is represented by a map

$$\theta_a: (S^{n+k}, \infty) \rightarrow (\mathcal{P}_\alpha(M\text{Spin}^c(k))/X, *)$$

constructed as above.

Choose an embedding  $i : M_1 \sqcup M_2 \rightarrow \mathbb{R}^{n+k}$  such that the last coordinate is positive for  $M_1$  and negative for  $M_2$ . Taking small enough  $\varepsilon$ , the previous construction gives us a map

$$(S^{n+k}, \infty) \xrightarrow{d} S^{n+k} \wedge S^{n+k} \xrightarrow{\theta_1 + \theta_2} (\mathcal{P}_\alpha(M\text{Spin}^c(k))/X, *),$$

where  $d$  denotes the collapsing the equator of  $S^{n+k}$ . This map represents the sum of the homotopy classes of  $\theta_1$  and  $\theta_2$ . Hence,

$$\Theta([M_1, \iota_1, \nu_1, \eta_1] + [M_2, \iota_2, \nu_2, \eta_2]) = \Theta([M_1, \iota_1, \nu_1, \eta_1]) + \Theta([M_2, \iota_2, \nu_2, \eta_2]).$$

*Step 3.*  $\Theta$  is a monomorphism.

Let  $(M, \iota, \nu, \eta)$  be an  $\alpha$ -twisted  $\text{Spin}^c$   $n$ -manifold such that  $\Theta([M, \iota, \nu, \eta]) = 0$ . Then for some large  $k$ , the above construction in step 1 defines a continuous map

$$\theta_0 = \iota_* \circ h_* \circ \varepsilon^{-1} \circ c : (S^{n+k}, \infty) \rightarrow \mathcal{P}_\alpha(M\text{Spin}^c(k))/X$$

which is null-homotopic. Since  $\mathcal{P}_\alpha(B\text{Spin}^c(k)) \subset \mathcal{P}_\alpha(M\text{Spin}^c(k))/X$ , since  $M$  is the zero section of  $N$ , and since the map

$$\iota_* \circ h_* : \text{Th}(N) \rightarrow \mathcal{P}_\alpha(M\text{Spin}^c(k))/X$$

sends the zero section of  $N$  to  $\mathcal{P}_\alpha(B\text{Spin}^c(k))$ , we have

$$M = \theta_0^{-1}(\mathcal{P}_\alpha(B\text{Spin}^c(k))).$$

The trivial map, denoted by  $\theta_1$ , maps  $S^{n+k}$  to the base point of  $\mathcal{P}_\alpha(M\text{Spin}^c(k))/X$ , so we know that  $\theta_1^{-1}(\mathcal{P}_\alpha(B\text{Spin}^c(k)))$  is an empty set. Now we can choose a homotopy

$$H : S^{n+k} \times [0, 1] \rightarrow \mathcal{P}_\alpha(M\text{Spin}^c(k))/X$$

between  $\theta_0$  and  $\theta_1$  for some sufficiently large  $k$  such that  $H$  is differentiable near and transversal to

$$\mathcal{P}_\alpha(B\text{Spin}^c(k)) \subset \mathcal{P}_\alpha(M\text{Spin}^c(k))/X.$$

Thus

$$W = H^{-1}(\mathcal{P}_\alpha(B\text{Spin}^c(k)))$$

is a submanifold of  $\mathbb{R}^{n+k} \times [0, 1]$  with  $\partial W = M$  meeting  $\mathbb{R}^{n+k} \times \{0\}$  orthogonally along  $M$ . The map  $H|_W$  sends  $W$  to  $\mathcal{P}_\alpha(B\text{Spin}^c(k))$  because  $\mathcal{P}_\alpha(B\text{Spin}^c(k))$  is a fibration over  $X$ , so we have a continuous map  $\iota_W : W \rightarrow X$ .

Note that the pull-back diagram

$$\begin{array}{ccc} \mathcal{P}_\alpha & \longrightarrow & EK(\mathbb{Z}, 2) \\ \downarrow & & \downarrow \\ X & \xrightarrow{\alpha} & K(\mathbb{Z}, 3) \end{array}$$

induces a homotopy pull-back

$$\begin{array}{ccc}
 \mathcal{P}_\alpha(B\text{Spin}^c(k)) & \longrightarrow & EK(\mathbb{Z}, 2)(B\text{Spin}^c k) \\
 \downarrow & & \downarrow \\
 X & \xrightarrow{\alpha} & K(\mathbb{Z}, 3).
 \end{array}$$

Since  $EK(\mathbb{Z}, 2)$  is contractible, the associated fiber bundle  $EK(\mathbb{Z}, 2)(B\text{Spin}^c(k))$  is homotopy equivalent to  $BSO(k)$ . This implies that the diagram

$$\begin{array}{ccc}
 \mathcal{P}_\alpha(B\text{Spin}^c(k)) & \longrightarrow & BSO(k) \\
 \downarrow & & \downarrow \\
 X & \xrightarrow{\alpha} & K(\mathbb{Z}, 3)
 \end{array}$$

is a homotopy pull-back. We see that the map  $H|_W$  defines a homotopy commutative diagram

$$\begin{array}{ccc}
 W & \xrightarrow{\quad} & BSO(k) \\
 \downarrow H|_W & \searrow & \downarrow W_3 \\
 \mathcal{P}_\alpha(B\text{Spin}^c(k)) & \longrightarrow & BSO(k) \\
 \downarrow & \swarrow \text{dashed} & \downarrow \\
 X & \xrightarrow{\alpha} & K(\mathbb{Z}, 3).
 \end{array}$$

Hence  $W$  admits an  $\alpha$ -twisted  $\text{Spin}^c$  structure such that the boundary inclusion  $M \rightarrow W$  is a morphism in the  $\alpha$ -twisted  $\text{Spin}^c$  bordism category. This implies that  $(M, \iota, \nu, \eta)$  is null-bordant, so  $[M, \iota, \nu, \eta] = 0$  in  $\Omega_n^{\text{Spin}^c}(X, \alpha)$ .

Step 4.  $\Theta$  is an epimorphism.

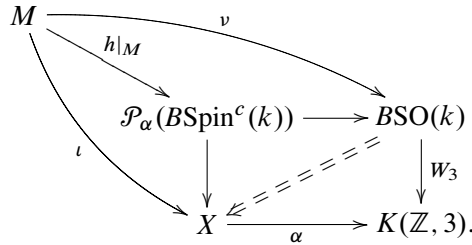
Let  $\theta: (S^{n+k}, \infty) \rightarrow (\mathcal{P}_\alpha(M\text{Spin}^c(k))/X, *)$ , for a large  $k$ , represent an element in

$$\pi_n^S(\mathcal{P}_\alpha(M\text{Spin}^c)/X).$$

As  $S^{n+k}$  is compact, we may find a finite dimensional model for  $B\text{Spin}^c(k)$ , so we may pretend that  $B\text{Spin}^c(k)$  is finite dimensional. We can deform the map  $\theta$  to a map  $h$  such that

- (1)  $h$  agrees with  $\theta$  on an open set containing  $\infty$ ;
- (2)  $h$  is differentiable on the preimage of some open set containing  $\mathcal{P}_\alpha(B\text{Spin}^c(k))$  and is transverse on  $\mathcal{P}_\alpha(B\text{Spin}^c(k))$ ;
- (3) if  $M = h^{-1}(\mathcal{P}_\alpha(B\text{Spin}^c(k)))$ , then  $h$  is a normal bundle map from a tubular neighborhood of  $M$  in  $S^{n+k}$  to  $\mathcal{P}_\alpha(M\text{Spin}^c(k))/X$ .

Then  $M$  is a smooth compact  $n$ -dimensional manifold with the following homotopy commutative diagram:



Therefore,  $M$  admits an  $\alpha$ -twisted  $\text{Spin}^c$  structure  $(\iota, \nu, \eta)$ . The above generalized Pontrjagin–Thom construction implies that  $\Theta([M, \iota, \nu, \eta])$  is the class represented by  $\theta$ . □

The index map  $\text{Ind} : M\text{Spin} \rightarrow \mathbb{K}$  (the complex K-theory spectrum) induces a map of bundles of spectra over  $X$

$$\text{Ind} : \mathcal{P}_\alpha(M\text{Spin}^c) \rightarrow \mathcal{P}_\alpha(\mathbb{K}).$$

The stable homotopy group of  $\mathcal{P}_\alpha(\mathbb{K})/X$  by definition is the twisted topological K-homology groups  $K_{\text{ev/odd}}^t(X, \alpha)$ . Due to the periodicity of  $\mathbb{K}$ , we have

$$K_{\text{ev}}^t(X, \alpha) = \varinjlim_{k \rightarrow \infty} \pi_{2k}(\mathcal{P}_\alpha(\mathbb{K})/X)$$

and

$$K_{\text{odd}}^t(X, \alpha) = \varinjlim_{k \rightarrow \infty} \pi_{2k+1}(\mathcal{P}_\alpha(\mathbb{K})/X).$$

Here the direct limits are taken by the double suspension

$$\pi_{n+2k}(\mathcal{P}_\alpha(\mathbb{K})/X) \rightarrow \pi_{n+2k+2}(\mathcal{P}_\alpha(S^2 \wedge \mathbb{K})/X)$$

and then followed by the standard map

$$\pi_{n+2k+2}(\mathcal{P}_\alpha(S^2 \wedge \mathbb{K})/X) \xrightarrow{b \wedge 1} \pi_{n+2k+2}(\mathcal{P}_\alpha(\mathbb{K} \wedge \mathbb{K})/X) \xrightarrow{m} \pi_{n+2k+2}(\mathcal{P}_\alpha(\mathbb{K})/X),$$

where  $b : \mathbb{R}^2 \rightarrow \mathbb{K}$  represents the Bott generator in  $K^0(\mathbb{R}^2)$  and  $m$  is the base point preserving map inducing the ring structure on K-theory.

**Definition 4.5** (Topological index). There is a homomorphism called the topological index,

$$\text{Index}_\iota : \Omega_*^{\text{Spin}^c}(X, \alpha) \rightarrow K_{\text{ev/odd}}^t(X, \alpha), \tag{4.8}$$

defined to be  $\text{Ind}_* \circ \Theta$ , the composition of  $\Theta$  (as in Theorem 4.4),

$$\Theta : \Omega_n^{\text{Spin}^c}(X, \alpha) \xrightarrow{\cong} \pi_n^S(\mathcal{P}_\alpha(M\text{Spin}^c)/X),$$

and the induced index transformation

$$\text{Ind}_* : \lim_{k \rightarrow \infty} \pi_{n+2k}(\mathcal{P}_\alpha(M\text{Spin}^c(2k))/X) \rightarrow \lim_{k \rightarrow \infty} \pi_{n+2k}(\mathcal{P}_\alpha(\mathbb{K})/X).$$

### 5. Topological index = analytical index

In this section we will establish the main result of this paper. It should be thought of as the generalized Atiyah–Singer index theorem for  $\alpha$ -twisted  $\text{Spin}^c$  manifolds over  $X$  with a twisting  $\alpha : X \rightarrow K(\mathbb{Z}, 3)$ . Here we assume throughout that  $X$  is a *closed smooth* manifold.

**Theorem 5.1.** *There is a natural isomorphism  $\Phi : K_{\text{ev/odd}}^t(X, \alpha) \rightarrow K_{\text{ev/odd}}^a(X, \alpha)$  such that the following diagram commutes:*

$$\begin{array}{ccc} & \Omega_{\text{ev/odd}}^{\text{Spin}^c}(X, \alpha) & \\ \text{Index}_t \swarrow & & \searrow \text{Index}_a \\ K_{\text{ev/odd}}^t(X, \alpha) & \xrightarrow[\cong]{\Phi} & K_{\text{ev/odd}}^a(X, \alpha) \end{array}$$

that is, given a closed  $\alpha$ -twisted  $\text{Spin}^c$  manifold  $(M, \nu, \iota, \eta)$  over  $X$ , we have

$$\text{Index}_a(M, \nu, \iota, \eta) = \text{Index}_t(M, \nu, \iota, \eta)$$

under the isomorphism  $\Phi$ .

**Remark 5.2.** If  $\alpha : X \rightarrow K(\mathbb{Z}, 3)$  is the trivial map, then we have following commutative diagram:

$$\begin{array}{ccc} & \Omega_{\text{ev/odd}}^{\text{Spin}^c}(X) & \\ \text{Index}_t \swarrow & & \searrow \text{Index}_a \\ K_{\text{ev/odd}}^t(X) & \xrightarrow[\cong]{} & K_{\text{ev/odd}}^a(X) \end{array} \tag{5.1}$$

where the isomorphism  $K_{\text{ev/odd}}^t(X) \cong K_{\text{ev/odd}}^a(X)$  follows from the work of Atiyah [2], Baum–Douglas [10] and Kasparov [30]. If  $X$  is a point, then the diagram (5.1) is the usual form of Atiyah–Singer index theorem for  $\text{Spin}^c$  manifolds.

*Proof of Theorem 5.1.* Notice that  $K^t_{ev/odd}(X, \alpha)$  and  $K^a_{ev/odd}(X, \alpha)$  are two generalized homology theories dual to the twisted K-theory. The twisted K-cohomology  $K^{ev/odd}(X, \alpha)$  is defined as the homotopy classes of sections of the associated bundle of K-theory spectra,

$$K^{ev}(X, \alpha) = \varinjlim_{k \rightarrow \infty} \pi_0(\Gamma(X, \mathcal{P}_\alpha(\Omega^{2k}\mathbb{K})),$$

$$K^{odd}(X, \alpha) = \varinjlim_{k \rightarrow \infty} \pi_0(\Gamma(X, \mathcal{P}_\alpha(\Omega^{2k+1}\mathbb{K})),$$

and  $\Omega^k\mathbb{K}$  is the iterated loop space of  $\mathbb{K}$ . We will show that there are natural isomorphisms from twisted K-homology (topological and analytical) to twisted K-cohomology with the twisting shifted by

$$\alpha \mapsto \alpha + (W_3 \circ \tau),$$

where  $\tau: X \rightarrow BSO$  is the classifying map of the stable tangent space and  $\alpha + (W_3 \circ \tau)$  denotes the map  $X \rightarrow K(\mathbb{Z}, 3)$  defined in (2.5), representing the class  $[\alpha] + W_3(X)$  in  $H^3(X, \mathbb{Z})$ .

*Step 1.* There exists an isomorphism  $K^t_{ev/odd}(X, \alpha) \cong K^{ev/odd}(X, \alpha + (W_3 \circ \tau))$  with the degree shifted by  $\dim X \pmod{2}$ .

Assume that  $X$  is  $n$  dimensional, choose an embedding  $i_{2k}: X \rightarrow \mathbb{R}^{n+2k}$  for some large  $k$ , with its normal bundle  $\pi: N_{2k} \rightarrow X$  identified as an  $\varepsilon$ -tubular neighborhood of  $X$ . Any two embeddings  $X \rightarrow \mathbb{R}^{n+2k}$  are homotopic through a regular homotopy for a sufficiently large  $k$ . Under the inclusion  $\mathbb{R}^{n+2k} \times 0 \subset \mathbb{R}^{n+2k+2}$ , the Thom spaces of  $N_{2k}$  and  $N_{2k+2}$  are related through the reduced suspension by  $S^2$

$$\text{Th}(N_{2k+2}) = S^2 \wedge \text{Th}(N_{2k}). \tag{5.2}$$

By the Thom isomorphism ([18]), we have an isomorphism

$$K^{ev/odd}(X, \alpha + (W_3 \circ \tau)) \cong \varinjlim_{k \rightarrow \infty} K^{ev/odd}(N_{2k}, \alpha \circ \pi), \tag{5.3}$$

where  $\alpha \circ \pi: N_{2k} \rightarrow K(\mathbb{Z}, 3)$  is the pull-back twisting on  $N_{2k}$ . There is a natural map from  $K^{ev/odd}(N_{2k}, \alpha \circ \pi)$  to  $K^t_{ev/odd}(X, \alpha)$  by considering  $S^{n+2k}$  as  $\mathbb{R}^{n+2k} \cup \{\infty\}$  and the following pull-back diagram:

$$\begin{array}{ccc} \mathcal{P}_{\alpha \circ \pi}(\mathbb{K}) & \longrightarrow & \mathcal{P}_\alpha(\mathbb{K}) \\ \downarrow & & \downarrow \\ N_{2k} & \xrightarrow{\pi} & X. \end{array}$$



Given an element of  $K^{\text{ev}}(N_{2k}, \alpha \circ \pi)$  represented by a compactly supported section  $\theta: N_{2k} \rightarrow \mathcal{P}_{\alpha \circ \pi}(\mathbb{K})$ , we have

$$S^{n+2k} \xrightarrow{c} \text{Th}(N_{2k}) \xrightarrow{\theta} \mathcal{P}_{\alpha \circ \pi}(\mathbb{K})/N_{2k} \rightarrow \mathcal{P}_{\alpha}(\mathbb{K})/X,$$

representing an element in  $K^t_{\text{ev/odd}}(X, \alpha)$ . Replacing  $X$  by  $X \times \mathbb{R}$ , this construction gives a map from  $K^{\text{odd}}(N_{2k}, \alpha \circ \pi)$  to  $K^t_{\text{odd}}(X, \alpha)$ . Recall that there is a homotopy equivalence  $\mathbb{K} \sim \Omega^2\mathbb{K}$  induced by the map

$$S^2 \wedge \mathbb{K} \xrightarrow{b \wedge 1} \mathbb{K} \wedge \mathbb{K} \xrightarrow{m} \mathbb{K},$$

where  $b$  represents the Bott generator in  $K^0(\mathbb{R}^2)$  and  $m$  is the base point preserving map inducing the ring structure on K-theory. Together with (5.2), we obtain

$$\begin{array}{ccccc} S^{n+2k} & \xrightarrow{c} & \text{Th}(N_{2k}) & \longrightarrow & \mathcal{P}_{\alpha}(\mathbb{K})/X \\ \downarrow s & & \downarrow s & & \downarrow s \\ S^2 \wedge S^{n+2k} & \xrightarrow{1 \wedge c} & S^2 \wedge \text{Th}(N_{2k}) & \longrightarrow & \mathcal{P}_{\alpha}(S^2 \wedge \mathbb{K})/X \\ \downarrow & & \downarrow & & \downarrow m \circ (b \wedge 1) \\ S^{n+2k+2} & \xrightarrow{c} & \text{Th}(N_{2k+2}) & \longrightarrow & \mathcal{P}_{\alpha}(\mathbb{K})/X, \end{array} \tag{5.4}$$

where  $S$  is the reduced suspension map by  $S^2$ . This implies that the stable homotopy equivalent class of sections defines the same element in  $K^t_*(X, \alpha)$ , with the degree given by  $n \pmod{2}$ . Thus, we have a well-defined homomorphism

$$\Psi_t: K^{\text{ev/odd}}(X, \alpha + (W_3 \circ \tau)) \rightarrow K^t_{\text{ev/odd}}(X, \alpha). \tag{5.5}$$

Conversely, for a sufficiently large  $k$ , let  $\theta: (S^{m+2k}, \infty) \rightarrow (\mathcal{P}_{\alpha}(\mathbb{K})/X, *)$  represent an element in  $K^t_{\text{ev/odd}}(X, \alpha)$  (depending on even or odd  $m$ ). We can lift this map to a map  $\theta_0: S^{m+2k} \rightarrow \mathcal{P}_{\alpha}(M\text{Spin}^c(2k))/X$ . As in step 4 of the proof of Theorem 4.4,  $\theta_0$  can be deformed to a differentiable map  $h$  on the preimage of some open set containing  $\mathcal{P}_{\alpha}(B\text{Spin}^c(2k))$ , is transverse to  $\mathcal{P}_{\alpha}(B\text{Spin}^c(2k))$  and agrees with  $\theta_0$  on an open set containing  $\infty$ . Then

$$M = h^{-1}(\mathcal{P}_{\alpha}(B\text{Spin}^c(2k))) \subset \mathbb{R}^{m+2k} = S^{m+2k} - \{\infty\}$$

is a smooth compact manifold and admits a natural  $\alpha$ -twisted  $\text{Spin}^c$  structure

$$\begin{array}{ccc}
 M & \xrightarrow{v} & BSO \\
 \downarrow h|_M & \searrow & \downarrow W_3 \\
 \mathcal{P}_\alpha(B\text{Spin}^c(2k)) & \longrightarrow & BSO \\
 \downarrow & \swarrow \text{dashed} & \downarrow \\
 X & \xrightarrow{\alpha} & K(\mathbb{Z}, 3)
 \end{array}
 \tag{5.6}$$

Therefore, we can assume that the map  $\theta: (S^{m+2k}, \infty) \rightarrow (\mathcal{P}_\alpha(\mathbb{K})/X, *)$  comes from the following commutative diagram:

$$\begin{array}{ccc}
 & \mathbb{R}^{m+2k} & \xrightarrow{\theta} & \mathcal{P}_\alpha(\mathbb{K}) \\
 & \uparrow j & & \downarrow \\
 N_\varepsilon & & & \\
 & \downarrow \pi & & \\
 M & \xrightarrow{\iota} & X, & 
 \end{array}$$

where  $N_\varepsilon$  is the normal bundle of  $M$  in  $\mathbb{R}^{m+2k}$ , identified as the  $\varepsilon$ -neighborhood of  $M$  in  $S^{m+2k}$ . In particular, the continuous map

$$\theta \circ j: N_\varepsilon \rightarrow S^{m+2k} \rightarrow \mathcal{P}_\alpha(\mathbb{K})/X$$

determines a compactly supported section of  $\mathcal{P}_{\alpha \circ \iota \circ \pi}(\mathbb{K}) = (\iota \circ \pi)^* \mathcal{P}_\alpha(\mathbb{K})$ .

Choose an embedding  $\iota_{2k_0}: M \rightarrow \mathbb{R}^{2k_0}$ , the  $\alpha$ -twisted  $\text{Spin}^c$  structure (5.6) on  $M$  over  $X$  induces a natural  $\alpha \circ \pi_0$ -twisted  $\text{Spin}^c$  structure (5.6) on  $M$  over  $X \times \mathbb{R}^{2k_0}$

$$\begin{array}{ccc}
 M & \xrightarrow{v} & BSO \\
 \downarrow (\iota, \iota_{2k_0}) & \searrow \text{dashed} & \downarrow W_3 \\
 X \times \mathbb{R}^{2k_0} & \xrightarrow{\alpha \circ \pi_0} & K(\mathbb{Z}, 3)
 \end{array}$$

such that  $(\iota, \iota_{2k_0})$  is an embedding. Here  $\pi_0$  is the projection  $X \times \mathbb{R}^{2k_0} \rightarrow X$ . Notice that

$$K_{\text{ev/odd}}^t(X, \alpha) \cong K_{\text{ev/odd}}^t(X \times \mathbb{R}^{2k_0}, \alpha).$$

Therefore, without losing any generality, we may assume that  $\iota: M \rightarrow X$  is an embedding and there is an embedding  $i_{2k}: X \rightarrow \mathbb{R}^{n+2k}$ . Denote by  $N_X$  the normal bundle of the embedding  $i_{2k}$  and by  $N_M$  the normal bundle of  $M$  in  $\mathbb{R}^{n+2k}$ . We

implicitly assume that any normal bundle of an embedding is identified with a tubular neighborhood of the embedding. Then we have the following collapsing map

$$\mathrm{Th}(N_X) \rightarrow \mathrm{Th}(N_M)$$

since  $N_M$  is imbedded in  $N_X$  with appropriate choices of tubular neighborhood.

The map  $\theta: (S^{m+2k}, \infty) \rightarrow (\mathcal{P}_\alpha(\mathbb{K})/X, *)$  is stable homotopic to

$$(S^{n+2k}, \infty) \xrightarrow{c} (\mathrm{Th}(N_M), *) \rightarrow (\mathcal{P}_\alpha(\mathbb{K})/X, *) \tag{5.7}$$

Hence we obtain a map

$$\mathrm{Th}(N_X) \rightarrow \mathrm{Th}(N_M) \rightarrow \mathcal{P}_\alpha(\mathbb{K})/X,$$

which gives a compactly supported section of  $\mathcal{P}_{\alpha \circ \pi}(\mathbb{K})$  where  $\pi$  denotes the projection  $N_X \rightarrow X$ . This section defines an element in  $K^{\mathrm{ev}}(N_X, \alpha \circ \pi)$ , hence an element of  $K^{\mathrm{ev}}(X, \alpha + (W_3 \circ \tau))$  under the isomorphism (5.3) and the diagram (5.4). It is straightforward to show that this map from  $K^t_{\mathrm{ev}/\mathrm{odd}}(X, \alpha)$  to  $K^{\mathrm{ev}/\mathrm{odd}}(X, \alpha + (W_3 \circ \tau))$  is the inverse of  $\Psi_t$  defined before.

Hence we have established the isomorphism

$$\Psi_t: K^{\mathrm{ev}/\mathrm{odd}}(X, \alpha + (W_3 \circ \tau)) \rightarrow K^t_{\mathrm{ev}/\mathrm{odd}}(X, \alpha), \tag{5.8}$$

with the degree shifted by  $\dim X \pmod 2$ . This is the Poincaré duality in topological twisted K-theory.

*Step 2.* There is an isomorphism  $\Psi_\alpha: K^a_{\mathrm{ev}/\mathrm{odd}}(X, \alpha) \cong K^{\mathrm{ev}/\mathrm{odd}}(X, \alpha + (W_3 \circ \tau))$  with the degree shifted by  $\dim X \pmod 2$ .

Recall that for a twisting  $\alpha: X \rightarrow K(\mathbb{Z}, 3)$  there is an associated bundle of  $\mathbb{C}^*$  algebras, denoted by  $\mathcal{P}_\alpha(\mathcal{K})$  where  $\mathcal{K}$  be the  $\mathbb{C}^*$ -algebra of compact operators on an infinite dimensional, complex and separable Hilbert space  $\mathcal{H}$ . Here we identify  $K(\mathbb{Z}, 2)$  as the projective unitary group  $\mathrm{PU}(\mathcal{H})$  with the norm topology (see [5] for details). There is an equivalent definition of  $K^{\mathrm{ev}/\mathrm{odd}}(X, \alpha)$  in [39], using the continuous trace  $\mathbb{C}^*$  algebra  $C_c(X, \mathcal{P}_\alpha(\mathcal{K}))$ , which consists of compactly supported sections of the bundle of  $\mathbb{C}^*$ -algebras,  $\mathcal{P}_\alpha(\mathcal{K})$ . Moreover, Atiyah–Segal established a canonical isomorphism in [5] between  $K^{\mathrm{ev}/\mathrm{odd}}(X, \alpha)$  and the analytical K-theory of  $C_c(X, \mathcal{P}_\alpha(\mathcal{K}))$ . The latter K-theory can be described as the Kasparov KK-theory

$$KK(\mathbb{C}, C_c(X, \mathcal{P}_\alpha(\mathcal{K}))).$$

There is an equivalent definition of  $K^{\mathrm{ev}/\mathrm{odd}}(X, \alpha + (W_3 \circ \tau))$  in [39], using the continuous trace  $\mathbb{C}^*$  algebras, which consists of compactly supported sections of the bundle of  $\mathbb{C}^*$ -algebra,  $\mathcal{P}_{\alpha+(W_3 \circ \tau)}(\mathcal{K})$ .

In [22] (see also [42]), a natural isomorphism, called the Poincaré duality in analytical twisted K-theory,

$$KK(\mathbb{C}, C_c(X, \mathcal{P}_\alpha(\mathcal{K})) \hat{\otimes}_{C_c(X)} C_c(X, \mathrm{Cliff}(TX))) \cong KK(C_c(X, \mathcal{P}_{-\alpha}(\mathcal{K})), \mathbb{C})$$

is constructed using the Kasparov product with the weak dual-Dirac element associated to  $\mathcal{P}_\alpha(\mathcal{K})$ ; see Definition 1.11 and Theorem 1.13 in [22] for details.

Note that there is a natural Morita equivalence

$$C_c(X, \mathcal{P}_\alpha(\mathcal{K})) \hat{\otimes}_{C_c(X)} C_c(X, \text{Cliff}(TX)) \sim C_c(X, \mathcal{P}_{\alpha+(W_3 \circ \tau)}(\mathcal{K}))$$

which induces a canonical isomorphism on their KK-groups. The isomorphism

$$KK(C_c(X, \mathcal{P}_{-\alpha}(\mathcal{K})), \mathbb{C}) \cong KK(C_c(X, \mathcal{P}_\alpha(\mathcal{K})), \mathbb{C})$$

is obvious using the operator conjugation. So in our notation, the Poincaré duality in analytical twisted K-theory can be written in the form

$$\Psi_\alpha : K^{\text{ev/odd}}(X, \alpha + (W_3 \circ \tau)) \rightarrow K^a_{\text{ev/odd}}(X, \alpha), \tag{5.9}$$

with the degree shifted by  $\dim X \pmod 2$  coming from the shift of grading on the ev/odd dimensional complex Clifford algebra.

Applying the Poincaré duality isomorphisms (5.8) and (5.9) in topological twisted K-theory and analytical twisted K-theory, we have a natural isomorphism

$$\Phi : K^t_{\text{ev/odd}}(X, \alpha) \xrightarrow{\Psi_\alpha \circ \Psi_t^{-1}} K^a_{\text{ev/odd}}(X, \alpha),$$

such that the following diagram commutes:

$$\begin{array}{ccc}
 & K^{\text{ev/odd}}(X, \alpha + (W_3 \circ \tau)) & \\
 \Psi_t \swarrow \cong & & \searrow \Psi_\alpha \cong \\
 K^t_{\text{ev/odd}}(X, \alpha) & \xrightarrow{\cong \Phi} & K^a_{\text{ev/odd}}(X, \alpha).
 \end{array} \tag{5.10}$$

*Step 3.* Show that  $\text{Index}_\alpha = \Phi \circ \text{Index}_t$ .

Applying the suspension operation, we only need to prove the even case. Let  $(M, \iota, \nu, \eta)$  be a  $2n$ -dimensional closed  $\alpha$ -twisted  $\text{Spin}^c$  manifold over  $X$ ,

$$\begin{array}{ccc}
 M & \xrightarrow{\nu} & BSO \\
 \downarrow \iota & \eta \swarrow \cong & \downarrow W_3 \\
 X & \xrightarrow{\alpha} & K(\mathbb{Z}, 3),
 \end{array}$$

representing an element in the  $\alpha$ -twisted  $\text{Spin}^c$  bordism group  $\Omega_n^{\text{Spin}^c}(X, \alpha)$ .

The analytical index of  $(M, \iota, \nu, \eta)$ , as defined Definition 3.7, is given by

$$\text{Index}_\alpha(M, \iota, \nu, \eta) = \iota_!([M]) = \iota_! \circ PD([\underline{\mathbb{C}}]), \tag{5.11}$$

where  $PD: K^0(M) \rightarrow K_0^a(M, W_3 \circ \tau)$  is the Poincaré duality isomorphism with  $\tau$  the classifying map for the stable tangent bundle. The push-forward map  $\iota_!$  in (5.11) is obtained from the following sequence of maps

$$K_0^a(M, W_3 \circ \tau) \xrightarrow{I_*} K_0^a(M, W_3 \circ \nu) \xrightarrow{\eta_*} K_0^a(M, \alpha \circ \iota) \xrightarrow{\iota_*} K_0^a(X, \alpha).$$

There is a natural push-forward map

$$\iota_!: \Omega_{\text{ev}}^{\text{Spin}^c}(M, \alpha \circ \iota) \rightarrow \Omega_{\text{ev}}^{\text{Spin}^c}(X, \alpha)$$

such that the following diagrams for the analytical index

$$\begin{array}{ccc} \Omega_{\text{ev}}^{\text{Spin}^c}(M, \alpha \circ \iota) & \xrightarrow{\iota_!} & \Omega_{\text{ev}}^{\text{Spin}^c}(X, \alpha) \\ \text{Index}_a \downarrow & & \downarrow \text{Index}_a \\ K_0^a(M, \alpha \circ \iota) & \xrightarrow{\iota_!} & K_0^a(X, \alpha) \end{array}$$

and for the topological index

$$\begin{array}{ccc} \Omega_{\text{ev}}^{\text{Spin}^c}(M, \alpha \circ \iota) & \xrightarrow{\iota_!} & \Omega_{\text{ev}}^{\text{Spin}^c}(X, \alpha) \\ \text{Index}_t \downarrow & & \downarrow \text{Index}_t \\ K_0^t(M, \alpha \circ \iota) & \xrightarrow{\iota_!} & K_0^t(X, \alpha) \end{array}$$

are commutative.

Since  $(M, \text{Id}, \nu, \text{Id})$  is a natural  $\alpha \circ \iota$ -twisted  $\text{Spin}^c$  manifold over  $M$ , we only need to prove that

$$(I_*)^{-1} \circ \Phi \circ \text{Index}_t(M, \text{Id}, \nu, \text{Id}) = [M] = PD(\mathbb{C}) \tag{5.12}$$

in  $K_0^a(M, W_3 \circ \tau)$ . We will show that the identity (5.12) follows from the Thom isomorphism

$$K^0(M) \cong K^0(N_M, W_3 \circ \nu_k \circ \pi),$$

where we choose an embedding  $i_k: M \rightarrow \mathbb{R}^{2n+k}$  with its normal bundle  $N_M$ ,  $\pi$  is the projection  $N_M \rightarrow M$  and  $\nu_k: M \rightarrow BSO(k)$  is the classifying map of the normal bundle  $N_M$ . The image of  $[\mathbb{C}]$  under the above Thom isomorphism is represented by the map

$$\theta_M: (S^{2n+k}, \infty) \rightarrow (\text{Th}(N_M), *) \rightarrow (\mathcal{P}_{W_3 \circ \nu_k \circ \pi}(\mathbb{k})/N_M, *)$$

arising from the  $W_3 \circ \nu_k$ -twisted  $\text{Spin}^c$  structure on  $M$  as in the following diagram:

$$\begin{array}{ccccc}
 \text{Th}(N_M) & \longrightarrow & \mathcal{P}_{W_3 \circ \nu_k}(M\text{Spin}^c(k))/M & \longrightarrow & \mathcal{P}_{W_3 \circ \nu_k}(\mathbb{K})/M \\
 \uparrow & & \uparrow & & \\
 M & \longrightarrow & \mathcal{P}_{W_3 \circ \nu_k}(B\text{Spin}^c(k)) & \longrightarrow & BSO(k) \\
 \searrow \text{Id} & & \downarrow & & \downarrow W_3 \\
 & & M & \xrightarrow{W_3 \circ \nu_k} & K(\mathbb{Z}, 3).
 \end{array}$$

This same diagram also defines the topological index of  $(M, \text{Id}, \nu, \text{Id})$  under the index map

$$\text{Index}_t : \Omega_{\text{ev}}^{\text{Spin}^c}(M, W_3 \circ \nu) \rightarrow K_0^t(M, W_3 \circ \nu).$$

Hence we establish the following commutative diagram:

$$\begin{array}{ccc}
 & \Omega_{\text{ev/odd}}^{\text{Spin}^c}(X, \alpha) & \\
 \text{Index}_t \swarrow & & \searrow \text{Index}_a \\
 K_{\text{ev/odd}}^t(X, \alpha) & \xrightarrow[\cong]{\Phi} & K_{\text{ev/odd}}^a(X, \alpha).
 \end{array}$$

□

**Remark 5.3.** Let  $\pi_X : TX \rightarrow X$  be the projection. Applying the Thom isomorphism ([18]), we obtain the following isomorphism:

$$K^{\text{ev/odd}}(TX, \alpha \circ \pi_X) \cong K^{\text{ev/odd}}(X, \alpha + (W_3 \circ \tau)).$$

Hence, the above commutative diagram (5.10) becomes

$$\begin{array}{ccc}
 & K^{\text{ev/odd}}(TX, \alpha \circ \pi_X) & \\
 \cong \swarrow \text{PD} & & \searrow \cong \text{PD} \\
 K_{\text{ev/odd}}^t(X, \alpha) & \xrightarrow[\cong]{\Phi} & K_{\text{ev/odd}}^a(X, \alpha),
 \end{array} \tag{5.13}$$

which should be thought of as a generalized Atiyah–Singer index theorem for  $\alpha$ -twisted  $\text{Spin}^c$  manifolds over  $X$  with a twisting  $\alpha : X \rightarrow K(\mathbb{Z}, 3)$ . If  $\alpha : X \rightarrow K(\mathbb{Z}, 3)$  is a trivial map, the commutative diagram (5.13) becomes

$$\begin{array}{ccc}
 & K^{\text{ev/odd}}(TX) & \\
 \text{Index}_t \swarrow & & \searrow \text{Index}_a \\
 K_{\text{ev/odd}}^t(X) & \xrightarrow[\cong]{\Phi} & K_{\text{ev/odd}}^a(X),
 \end{array}$$

which is the basic form for the Atiyah–Singer index theorem. The upper vertex represents the symbols of elliptic pseudo-differential operators on  $X$ . Each of these index maps is essentially just the Poincaré duality isomorphism between the K-cohomology of  $T^*X$  and the two realizations of the K-homology  $K_0(X)$ . See [10] for more details.

If  $\alpha$  is the twisting associated to the classifying map  $W_3 \circ \tau : X \rightarrow K(\mathbb{Z}, 3)$  of the stable tangent bundle, then we have the following twisted index theorem, given by the following commutative diagram:

$$\begin{array}{ccc}
 & K^{\text{ev/odd}}(TX, W_3 \circ \tau \circ \pi) & \\
 \text{Index}_t \swarrow & & \searrow \text{Index}_a \\
 K^t_{\text{ev/odd}}(X, W_3 \circ \tau) & \xrightarrow[\cong]{\Phi} & K^a_{\text{ev/odd}}(X, W_3 \circ \tau)
 \end{array}$$

This is a special case of twisted longitudinal index theorem for foliations. We will return to this issue later.

### 6. Geometric cycles and geometric twisted K-homology

**Definition 6.1.** Let  $X$  be a paracompact Hausdorff space and let  $\alpha : X \rightarrow K(\mathbb{Z}, 3)$  be a twisting over  $X$ . A geometric cycle for  $(X, \alpha)$  is a quintuple  $(M, \iota, \nu, \eta, [E])$  such that

- (1)  $M$  is a smooth closed manifold equipped with an  $\alpha$ -twisted  $\text{Spin}^c$  structure,

$$\begin{array}{ccc}
 M & \xrightarrow{\nu} & BSO \\
 \downarrow \iota & \eta \swarrow & \downarrow W_3 \\
 X & \xrightarrow{\alpha} & K(\mathbb{Z}, 3),
 \end{array}$$

where  $\iota : M \rightarrow X$  is a continuous map,  $\nu$  is a classifying map of the stable normal bundle of  $M$ , and  $\eta$  is a homotopy from  $W_3 \circ \nu$  and  $\alpha \circ \iota$ ;

- (2)  $[E]$  is a K-class in  $K^0(M)$  represented by a  $\mathbb{Z}_2$ -graded vector bundle  $E$  over  $M$ .

Two geometric cycles  $(M_1, \iota_1, \nu_1, \eta_1, [E_1])$  and  $(M_2, \iota_2, \nu_2, \eta_2, [E_2])$  are isomorphic if there is an isomorphism  $f : (M_1, \iota_1, \nu_1, \eta_1) \rightarrow (M_2, \iota_2, \nu_2, \eta_2)$ , as  $\alpha$ -twisted  $\text{Spin}^c$  manifolds over  $X$ , such that  $f_!([E_1]) = [E_2]$ .

Let  $\Gamma(X, \alpha)$  be the collection of all geometric cycles for  $(X, \alpha)$ . We now impose an equivalence relation  $\sim$  on  $\Gamma(X, \alpha)$ , generated by the following three elementary relations:

(1) Direct sum – disjoint union.

If  $(M, \iota, \nu, \eta, [E_1])$  and  $(M, \iota, \nu, \eta, [E_2])$  are two geometric cycles with the same  $\alpha$ -twisted  $\text{Spin}^c$  structure, then

$$(M, \iota, \nu, \eta, [E_1]) \cup (M, \iota, \nu, \eta, [E_2]) \sim (M, \iota, \nu, \eta, [E_1] + [E_2]).$$

(2) Bordism.

Let  $(M_1, \iota_1, \nu_1, \eta_1, [E_1])$  and  $(M_2, \iota_2, \nu_2, \eta_2, [E_2])$  be two geometric cycles. Then there exist an  $\alpha$ -twisted  $\text{Spin}^c$  manifold  $(W, \iota, \nu, \eta)$  and  $[E] \in K^0(W)$  such that

$$\partial(W, \iota, \nu, \eta) = -(M_1, \iota_1, \nu_1, \eta_1) \cup (M_2, \iota_2, \nu_2, \eta_2) \text{ and } \partial([E]) = [E_1] \cup [E_2].$$

Here  $-(M_1, \iota_1, \nu_1, \eta_1)$  denotes the manifold  $M_1$  with the opposite  $\alpha$ -twisted  $\text{Spin}^c$  structure.

(3)  $\text{Spin}^c$  vector bundle modification.

Suppose that we are given a geometric cycle  $(M, \iota, \nu, \eta, [E])$  and a  $\text{Spin}^c$  vector bundle  $V$  over  $M$  with even dimensional fibers. Denote by  $\underline{\mathbb{R}}$  the trivial rank one real vector bundle. Choose a Riemannian metric on  $V \oplus \underline{\mathbb{R}}$  and let

$$\widehat{M} = S(V \oplus \underline{\mathbb{R}})$$

be the sphere bundle of  $V \oplus \underline{\mathbb{R}}$ . Then the vertical tangent bundle  $T^v(\widehat{M})$  of  $\widehat{M}$  admits a natural  $\text{Spin}^c$  structure with an associated  $\mathbb{Z}_2$ -graded spinor bundle  $S_V^+ \oplus S_V^-$ . Denote by  $\rho: \widehat{M} \rightarrow M$  the projection which is K-oriented. Then

$$(M, \iota, \nu, \eta, [E]) \sim (\widehat{M}, \iota \circ \rho, \nu \circ \rho, \eta \circ \rho, [\rho^* E \otimes S_V^+]).$$

**Definition 6.2.** Denote by  $K_*^{\text{geo}}(X, \alpha) = \Gamma(X, \alpha)/\sim$  the geometric twisted K-homology. Addition is given by disjoint union/direct sum relation. Note that the equivalence relation  $\sim$  preserves the parity of the dimension of the underlying  $\alpha$ -twisted  $\text{Spin}^c$  manifold. Let  $K_0^{\text{geo}}(X, \alpha)$  (resp.  $K_1^{\text{geo}}(X, \alpha)$ ) the subgroup of  $K_*^{\text{geo}}(X, \alpha)$  determined by all geometric cycles with even (resp. odd) dimensional  $\alpha$ -twisted  $\text{Spin}^c$  manifolds.

**Remark 6.3.** (1) According to Proposition 3.3,  $M$  admits an  $\alpha$ -twisted  $\text{Spin}^c$  structure if and only if

$$\iota^*([\alpha]) + W_3(M) = 0.$$

(If  $\iota$  is an embedding, this is the anomaly cancellation condition obtained by Freed and Witten in [25]. The cycle  $(M, \iota, \nu, \eta, [E])$  is referred to by physicists as a D-brane and appears in type IIB string theory; see [25], [44], [29], [14].)



(2) Different definitions of topological twisted K-homology were proposed in [36] using  $\text{Spin}^c$ -manifolds and twisted bundles. It is not clear to the author if their definition is equivalent to Definition 6.2.

(3) If  $f : X \rightarrow Y$  is a continuous map and  $\alpha : X \rightarrow K(\mathbb{Z}, 3)$  is a twisting, then there is a natural homomorphism of abelian groups

$$f_* : K_{\text{ev/odd}}^{\text{geo}}(X, \alpha) \rightarrow K_{\text{ev/odd}}^{\text{geo}}(Y, \alpha \circ f)$$

sending  $[M, \iota, \nu, \eta, E]$  to  $[M, f \circ \iota, \nu, \eta, E]$ .

Given a geometric cycle  $(M, \iota, \nu, \eta, [E])$ , the analytical index (as in Definition 3.7) determines an element

$$\begin{aligned} \mu(M, \iota, \nu, \eta, [E]) &= \text{Index}_a(M, \iota, \nu, \eta, [E]) \\ &= \iota_* \circ \eta_* \circ I^* \circ PD([E]) \end{aligned}$$

in  $K_{\text{ev/odd}}^a(X, \alpha)$ .

**Theorem 6.4.** *The assignment  $(M, \iota, \nu, \eta, [E]) \rightarrow \mu(M, \iota, \nu, \eta, [E])$ , called the assembly map, defines a natural homomorphism*

$$\mu : K_{\text{ev/odd}}^{\text{geo}}(X, \alpha) \rightarrow K_{\text{ev/odd}}^a(X, \alpha),$$

which is an isomorphism for any closed smooth manifold  $X$  with a twisting  $\alpha : X \rightarrow K(\mathbb{Z}, 3)$ .

*Proof. Step 1.* We need to show that the correspondence is compatible with the three elementary equivalence relations, so the assembly map  $\mu$  is well defined. We only need to discuss the even case.

Proposition 3.8 ensures that  $\text{Index}_a(\widehat{M}, \iota, \nu, \eta, [E])$  is compatible with the bordism relation and disjoint union/direct sum relation. We only need to check that the assembly map is compatible with the relation of  $\text{Spin}^c$  vector bundle modification.

Suppose that  $(M, \iota, \nu, \eta, [E])$  is a geometric cycle of even dimension and  $\text{Spin}^c$  is a vector bundle  $V$  over  $M$  with even dimensional fibers. Then

$$(M, \iota, \nu, \eta, [E]) \sim (\widehat{M}, \iota \circ \rho, \nu \circ \rho, \eta \circ \rho, [\rho^* E \otimes S_V^\pm]),$$

where  $\widehat{M} = S(V \oplus \mathbb{R})$  is the sphere bundle of  $V \oplus \mathbb{R}$  and  $\rho : \widehat{M} \rightarrow M$  is the projection. The vertical tangent bundle  $T^v(\widehat{M})$  of  $\widehat{M}$  admits a natural  $\text{Spin}^c$  structure with an associated  $\mathbb{Z}_2$ -graded spinor bundle  $S_V^+ \oplus S_V^-$ . The K-oriented map  $\rho$  induces a natural homomorphism (see [4])

$$\rho_* : K^0(\widehat{M}) \rightarrow K^0(M)$$



A homotopy equivalence map gives rise to a bordant  $\alpha$ -twisted  $\text{Spin}^c$  manifold. Hence we have a geometric cycle  $(M, \iota, \nu, \eta, [\underline{\mathbb{C}}])$ , whose equivalence class does not depend on various choices in the construction. This defines a map

$$\Psi: K_0^t(X, \alpha) \rightarrow K_0^{\text{geo}}(X, \alpha).$$

It is straightforward to show that  $\Psi$  is a homomorphism. Note that  $\Phi = \mu \circ \Psi$  follows from the definition of  $\Phi$  and Theorem 5.1.

To show that  $\Psi$  is surjective, let  $(M, \iota, \nu, \eta, [E])$  be a geometric cycle. Then the  $\alpha$ -twisted  $\text{Spin}^c$  manifold  $(M, \iota, \nu, \eta)$  defines a bordism class in  $\Omega_{\text{ev}}^{\text{Spin}^c}(X, \alpha)$ . The topological index

$$\text{Index}_t(M, \iota, \nu, \eta) \in K_0^t(X, \alpha)$$

is represented by the canonical map

$$\theta: (S^{m+2k}, \infty) \rightarrow (\text{Th}(N_M), *) \rightarrow (\mathcal{P}_\alpha(M\text{Spin}^c(2k)/X, *) \rightarrow (\mathcal{P}_\alpha(\mathbb{K})/X, *)$$

associated to the normal bundle  $\pi: N_M \rightarrow M$  of an embedding  $i_k: M \rightarrow \mathbb{R}^{m+2k}$  as in step 1 of the proof of Theorem 5.1. This map defines a compactly supported section of  $\mathcal{P}_{\alpha \circ \iota \circ \pi}(\mathbb{K})$  (a bundle of K-theory spectra over  $N_M$ ). We also denote this section by  $\theta$ . Then the homotopy class of the section  $\theta$  defines a twisted K-class in  $K^0(N_M, \alpha \circ \iota \circ \pi)$ , which is mapped to  $[\underline{\mathbb{C}}]$  under the Thom isomorphism

$$K^0(N_M, \alpha \circ \iota \circ \pi) \cong K^0(N_M, W_3 \circ \nu \circ \pi) \cong K^0(M).$$

Let  $\sigma: M \rightarrow \mathbb{K}$  be a map representing the K-class  $[E]$ . Then  $\sigma \circ \pi$  is a section of the trivial bundle  $\underline{\mathbb{K}}$  over  $N_M$ . Define a new section of  $\mathcal{P}_{\alpha \circ \iota \circ \pi}(\mathbb{K})$  by applying the fiberwise multiplication  $m: \mathbb{K} \wedge \mathbb{K} \rightarrow \mathbb{K}$  to  $(\theta, \sigma \circ \pi)$ . Then  $m(\theta, \sigma \circ \pi)$  is a compactly supported section of  $\mathcal{P}_{\alpha \circ \iota \circ \pi}(\mathbb{K})$  which determines a map, denoted by  $\theta \cdot \sigma$ ,

$$\theta \cdot \sigma: (S^{m+2k}, \infty) \rightarrow (\text{Th}(N_M), *) \rightarrow (\mathcal{P}_{\alpha \circ \iota \circ \pi}(\mathbb{K})/N_M, *).$$

The homotopy class of  $\theta \cdot \sigma$  as an element in  $K^0(N_M, \alpha \circ \iota \circ \pi)$  is uniquely determined by the stable homotopy class of  $\theta$  and the homotopy class of  $\sigma$ . Under the Thom isomorphism  $K^0(N_M, \alpha \circ \iota \circ \pi) \cong K^0(M)$ ,  $[\theta \cdot \sigma]$  is mapped to  $[E]$ . Hence,

$$\Psi([\theta \cdot \sigma]) = [M, \iota, \nu, \eta, [E]].$$

Therefore,  $\Psi$  is surjective. □

**Corollary 6.5.** *Given a twisting  $\alpha: X \rightarrow K(\mathbb{Z}, 3)$  on a smooth manifold  $X$ , every twisted K-class in  $K^{\text{ev/odd}}(X, \alpha)$  is represented by a geometric cycle supported on an  $(\alpha + (W_3 \circ \tau))$ -twisted closed  $\text{Spin}^c$ -manifold  $M$  and an ordinary K-class  $[E] \in K^0(M)$ .*

*Proof.* We only need to prove the even case, the odd case can be obtained by the suspension operation. Assume that  $X$  is even dimensional and  $\pi : TX \rightarrow X$  is the projection. Then we have the following isomorphisms

$$\begin{aligned} K^0(X, \alpha) &\cong K^0(TX, (\alpha \circ \pi) + (W_3 \circ \tau \circ \pi)) && \text{(Thom isomorphism)} \\ &\cong K_0^a(X, \alpha + (W_3 \circ \tau)) && \text{(Remark 5.3)} \\ &\cong K_0^{\text{geo}}(X, \alpha + (W_3 \circ \tau)) && \text{(Theorem 6.4)}. \end{aligned}$$

From the definition of  $K_0^{\text{geo}}(X, \alpha + (W_3 \circ \tau))$  we know that each element in  $K_0^{\text{geo}}(X, \alpha + (W_3 \circ \tau))$  is represented by a geometric cycle  $(M, \iota, \nu, \eta, [E])$  for  $(X, \alpha + (W_3 \circ \tau))$ , which is a generalized D-brane supported on an  $(\alpha + (W_3 \circ \tau))$ -twisted closed  $\text{Spin}^c$ -manifold  $M$  and an ordinary K-class  $[E] \in K^0(M)$ .  $\square$

**Remark 6.6.** Let  $Y$  be a closed subspace of  $X$ . A relative geometric cycle for  $(X, Y; \alpha)$  is a quintuple  $(M, \iota, \nu, \eta, [E])$  such that

- (1)  $M$  is a smooth manifold (possibly with boundary) equipped with an  $\alpha$ -twisted  $\text{Spin}^c$  structure  $(M, \iota, \nu, \eta)$ ;
- (2) if  $M$  has a non-empty boundary, then  $\iota(\partial M) \subset Y$ ;
- (3)  $[E]$  is a K-class in  $K^0(M)$  represented by a  $\mathbb{Z}_2$ -graded vector bundle  $E$  over  $M$ , or a continuous map  $M \rightarrow \mathbb{K}$ .

The relation  $\sim$  generated by disjoint union/direct sum, bordism and  $\text{Spin}^c$  vector bundle modification is an equivalence relation. The collection of relative geometric cycles modulo the equivalence relation is denoted by

$$K_{\text{ev/odd}}^{\text{geo}}(X, Y; \alpha).$$

Then we have the following commutative diagram whose arrows are all isomorphisms:

$$\begin{array}{ccc} & K_{\text{ev/odd}}^t(X, Y; \alpha) & \\ \Psi \swarrow & & \searrow \Phi \\ K_{\text{ev/odd}}^{\text{geo}}(X, Y; \alpha) & \xrightarrow{\mu} & K_{\text{ev/odd}}^a(X, Y; \alpha). \end{array}$$

### 7. The twisted longitudinal index theorem for foliation

Given a  $C^\infty$  foliated manifold  $(X, F)$ , that is,  $F$  is an integrable sub-bundle of  $TX$ , let  $D$  be an elliptic differentiable operator along the leaves of the foliation. Denote by  $\sigma_D$  the longitudinal symbol of  $D$ , whose class in  $K^0(F^*)$  is denoted by  $[\sigma_D]$ . In [19],

Connes and Skandalis defined the topological index and the analytical index of  $D$  taking values in the K-theory of the foliation  $C^*$ -algebra  $C_r^*(X, F)$  and established the equality between the topological index and the analytical index of  $D$ . See [19] for more details. In this section, we will generalize the Connes–Skandalis longitudinal index theorem to a foliated manifold  $(X, F)$  with a twisting  $\alpha: X \rightarrow K(\mathbb{Z}, 3)$ .

Let  $N_F = TX/F$  be the normal bundle to the leaves whose classifying map is denoted by  $\nu_F: X \rightarrow BSO(k)$ . Here assume that  $F$  is of rank  $k$  and  $X$  is  $n$ -dimensional. We can equip  $X$  with a Riemannian metric such that we have a splitting

$$TX = F \oplus N_F.$$

Then the sphere bundle  $M = S(F^* \oplus \mathbb{R})$  is a  $W_3 \circ \nu_F$ -twisted  $\text{Spin}^c$  manifold over  $X$ . To see this, let  $\pi$  be the projection  $M \rightarrow X$ . We need to calculate the third Stiefel–Whitney class of  $M = S(F^* \oplus \mathbb{R})$  from the following exact sequence of bundles over  $M$ ,

$$0 \rightarrow \pi^*(F \oplus \mathbb{R}) \rightarrow TM \oplus \mathbb{R} \rightarrow \pi^*TX \rightarrow 0, \tag{7.1}$$

from which we have

$$\begin{aligned} W_3(TM) &= \pi^*W_3(F) + \pi^*W_3(TX) \\ &= \pi^*W_3(F) + \pi^*(W_3(F) + W_3(N_F)) \\ &= \pi^*W_3(N_F). \end{aligned}$$

Note that  $\pi^*W_3(N_F) = [W_3 \circ \nu_F \circ \pi] \in H^3(M, \mathbb{Z})$ . So  $S(F^* \oplus \mathbb{R})$  admits a natural  $W_3 \circ \nu_F$ -twisted  $\text{Spin}^c$  structure

$$\begin{array}{ccc} M & \xrightarrow{\nu} & BSO \\ \pi \downarrow & \eta \swarrow \text{dashed} & \downarrow W_3 \\ X & \xrightarrow{W_3 \circ \nu_F} & K(\mathbb{Z}, 3), \end{array}$$

where  $\nu$  is the classifying map of the stable normal of  $M$  and  $\eta$  is a homotopy associated to a splitting of (7.1) as follows. Given a splitting of (7.1), the natural isomorphisms

$$\begin{aligned} TM \oplus \mathbb{R} &\cong \pi^*TX \oplus \pi^*(F \oplus \mathbb{R}) \\ &\cong \pi^*(F \oplus N_F) \oplus \pi^*(F \oplus \mathbb{R}) \\ &\cong \pi^*(F \oplus F) \oplus \pi^*N_F \oplus \mathbb{R} \end{aligned}$$

and the canonical  $\text{Spin}^c$  structure on  $\pi^*(F \oplus F)$  define the homotopy between  $W_3 \circ \tau$  and  $W_3 \circ \nu_F$ . Different choices of splittings of (7.1) gives rise to the same homotopy equivalence class, hence do not change the twisted  $\text{Spin}^c$  bordism class of  $M$ .

Let an elliptic differentiable operator along the leaves of the foliation with longitudinal symbol class  $[\sigma_D] \in K^0(F^*)$  be represented by a map

$$\sigma_D : \pi^* E_1 \rightarrow \pi^* E_2$$

of a pair of vector bundles  $E_1$  and  $E_2$  over  $X$  such that  $\sigma_D$  is an isomorphism away from the zero section of  $F^*$ . Applying the clutching construction as described in [10],  $M = S(F^* \oplus \mathbb{R})$  consists of two copies of the unit ball bundle of  $F^*$  glued together by the identity map of  $S(F^*)$ . We form a vector bundle over  $M$  by gluing  $\pi^* E_1$  and  $\pi^* E_2$  over each copy of the unit ball bundle along  $S(F^*)$  by the symbol map  $\sigma_D$ . Denote the resulting vector bundle by  $\hat{E}$ . The quintuple  $(M, \pi, \nu, \eta, [\hat{E}])$  is a geometric cycle of  $(X, W_3 \circ \nu_F)$ .

We define the topological index of  $[\sigma(D)]$  to be

$$\text{Index}_t([\sigma_D]) = [M, \pi, \nu, \eta, \hat{E}] \in K_{[n+k]}^{\text{geo}}(X, W_3 \circ \nu_F),$$

where  $[n + k]$  denotes the mod 2 sum (even or odd if  $n + k$  is even or odd).

The analytical index of  $[\sigma_D]$  is defined through the following sequence of isomorphisms:

$$\begin{aligned} K^0(F^*) &\cong K^{[k]}(X, W_3(F)) && \text{(Thom isomorphism)} \\ &\cong K_{[n+k]}^a(X, W_3(F \oplus TX)) && \text{(Poincaré duality)} \\ &\cong K_{[n+k]}^a(X, W_3(N_F)) && (F \oplus TX \cong F \oplus F \oplus N_F) \\ &\cong K_{[n+k]}^a(X, W_3 \circ \nu_F). \end{aligned}$$

The resulting element is denoted by

$$\text{Index}_a([\sigma_D]) \in K_{[n+k]}^a(X, W_3 \circ \nu_F).$$

Now we apply Theorems 5.1 and 6.4, and Remark 5.3 to obtain the following version of the longitudinal index theorem for the foliated manifold  $(X, F)$ . This longitudinal index theorem is equivalent to the Connes–Skandalis longitudinal index theorem through the natural homomorphism

$$K_{[n-k]}^a(X, W_3 \circ \nu_F) \rightarrow K_0(C_r^*(X, F)).$$

**Theorem 7.1.** *Given a  $C^\infty$   $n$ -dimensional foliated manifold  $(X, F)$  of rank  $k$ , the longitudinal index theorem for  $(X, F)$  is given by the following commutative diagram*

$$\begin{array}{ccc} & K^0(F^*) & \\ \text{Index}_t \swarrow & & \searrow \text{Index}_a \\ K_{[n-k]}^{\text{geo}}(X, W_3 \circ \nu_F) & \xrightarrow[\cong]{\mu} & K_{[n-k]}^a(X, W_3 \circ \nu_F), \end{array}$$

whose arrows are all isomorphisms.

**Remark 7.2.** If  $(X, F)$  comes from a fibration  $\pi_B : X \rightarrow B$  such that the leaves are the fibers of  $\pi_B$ , then  $F$  is given by the vertical tangent bundle  $T(X/B)$  and  $N_F \cong \pi_B^*TB$ . This isomorphism defines a canonical homotopy  $\eta_0$  realizing  $W_3 \circ \nu_F \sim W_3 \circ \tau_B \circ \pi_B$ , where  $\tau_B$  is the classifying map of the stable tangent bundle of  $B$ . The homotopy diagram

$$\begin{array}{ccc}
 S(F^* \oplus \mathbb{R}) & \xrightarrow{\nu} & BSO \\
 \pi \downarrow & \swarrow \eta & \downarrow W_3 \\
 X & \xrightarrow{W_3 \circ \nu_F} & K(\mathbb{Z}, 3) \\
 \pi \downarrow & \swarrow \eta_0 & \nearrow W_3 \circ \tau_B \\
 B & & 
 \end{array}$$

implies that  $(S(F^* \oplus \mathbb{R}), \pi_B \circ \pi, \nu, \eta * \eta_0, [\hat{E}])$ , where  $\eta * \eta_0$  is the obvious homotopy joining  $\eta$  and  $\eta_0$ , is a geometric cycle of  $(B, W_3 \circ \tau_B)$  and

$$(\pi_B)_!(S(F^* \oplus \mathbb{R}), \pi, \nu, \eta, [\hat{E}]) = (S(F^* \oplus \mathbb{R}), \pi_B \circ \pi, \nu, \eta * \eta_0, [\hat{E}]).$$

The commutative diagram

$$\begin{array}{ccccc}
 & & K^0(F^*) & & \\
 & \swarrow \text{Index}_t & & \searrow \text{Index}_a & \\
 K_{[n-k]}^{\text{geo}}(X, W_3 \circ \nu_F) & \xrightarrow{\mu} & & K_{[n-k]}^a(X, W_3 \circ \nu_F) & \\
 \downarrow (\pi_B)_! & & & \downarrow (\pi_B)_! & \\
 K_{[n-k]}^{\text{geo}}(B, W_3 \circ \tau_B) & \xrightarrow{\mu} & & K_{[n-k]}^a(B, W_3 \circ \tau_B) & \\
 \swarrow PD & & & \searrow PD & \\
 & & K^0(B) & & 
 \end{array}$$

becomes the Atiyah–Singer families index theorem in [8].

In the presence of a twisting  $\alpha : X \rightarrow K(\mathbb{Z}, 3)$  on a foliated manifold  $(X, F)$ , Theorems 5.1 and 6.4, and Remark 5.3 give rise to the following twisted longitudinal index theorem.

**Theorem 7.3.** *Given a  $C^\infty$   $n$ -dimensional foliated manifold  $(X, F)$  of rank  $k$  and a twisting  $\alpha : X \rightarrow K(\mathbb{Z}, 3)$ , let  $\pi : F^* \rightarrow X$  be the projection. Then the twisted longitudinal index theorem for the foliated manifold  $(X, F)$  with a twisting  $\alpha$  is given*

by the commutative diagram

$$\begin{array}{ccc}
 & K^0(F^*, \alpha \circ \pi) & \\
 \text{Index}_f \swarrow & & \searrow \text{Index}_a \\
 K_{[n-k]}^{\text{geo}}(X, \alpha + (W_3 \circ \nu_F)) & \xrightarrow[\cong]{\mu} & K_{[n-k]}^a(X, \alpha + (W_3 \circ \nu_F)),
 \end{array}$$

whose arrows are all isomorphisms. In particular, if  $(X, F)$  comes from a fibration  $\pi_B : X \rightarrow B$  and a twisting  $\alpha \circ \pi_B$  on  $X$  comes from a twisting  $\alpha$  on  $B$ , then we have the following twisted version of the Atiyah–Singer families index theorem with notations from Remark 7.2:

$$\begin{array}{ccc}
 & K^0(T^*(X/B), \alpha \circ \pi_B \circ \pi) & \\
 (\pi_B)_! \circ \text{Index}_f \swarrow & & \searrow (\pi_B)_! \circ \text{Index}_a \\
 K_{[n-k]}^{\text{geo}}(B, \alpha + (W_3 \circ \tau_B)) & \xrightarrow[\cong]{\mu} & K_{[n-k]}^a(B, \alpha + (W_3 \circ \tau_B)) \\
 \swarrow PD & & \searrow PD \\
 & K^0(B, \alpha) &
 \end{array}$$

In [34], Mathai–Melrose–Singer established the index theorem for projective families of longitudinally elliptic operators associated to a fibration  $\phi : Z \rightarrow X$  and an Azumaya bundle  $\mathcal{A}_\alpha$  for  $\alpha$  representing a torsion class in  $H^3(X, \mathbb{Z})$ .

Given a local trivialization of  $\mathcal{A}_\alpha$  for an open covering of  $X = \bigcup_i U_i$ , according to [34], a projective family of longitudinally elliptic operators is a collection of longitudinally elliptic pseudo-differential operators acting on finite dimensional vector bundles of fixed rank over each of the open sets  $\{\phi^{-1}(U_i)\}$  such that the compatibility condition over triple overlaps may fail by a scalar factor. The symbol class of such a projective family of elliptic operators determines a class in

$$K^0(T^*(Z/X), \alpha \circ \phi \circ \pi),$$

where  $T^*(Z/X)$  is dual to the vertical tangent bundle of  $Z$  and  $\pi : T^*(Z/X) \rightarrow Z$  is the projection. Let  $n$  be the dimension of  $Z$  and  $k$  be the dimension of the fiber of  $\phi$ . The Thom isomorphism, Theorems 5.1 and 6.4 give rise to the following commutative



diagram

$$\begin{array}{ccc}
 & K^0(T^*(Z/X), \alpha \circ \phi \circ \pi) & \\
 \phi_! \circ \text{Index}_t \swarrow & & \searrow \phi_! \circ \text{Index}_a \\
 K_{[n-k]}^{\text{geo}}(X, \alpha + (W_3 \circ \tau)) & \xrightarrow{\mu} & K_{[n-k]}^a(X, \alpha + (W_3 \circ \tau)) \\
 \searrow PD & & \swarrow \cong \\
 & K^0(X, \alpha) & \\
 & \swarrow PD & \searrow PD
 \end{array}$$

Here  $\alpha + (W_3 \circ \tau)$  represents the class  $[\alpha] + W_3(X) \in H^3(X, \mathbb{Z})$ . Readers familiar with [34] will recognise that the above theorem is another way of writing the Mathai–Melrose–Singer index theorem (cf. Theorem 4 in [34]) for projective families of longitudinally elliptic operators associated a fibration  $\phi: Z \rightarrow X$  and an Azumaya bundle  $\mathcal{A}_\alpha$  for  $\alpha$  representing a torsion class  $[\alpha] \in H^3(X, \mathbb{Z})$ .

### 8. Final remarks

Let  $M$  be an oriented manifold with a map  $\nu: M \rightarrow BSO$  classifying its stable normal bundle. Given any fibration  $\pi: B \rightarrow BSO$ , we can define a  $B$ -structure on  $M$  to be a homotopy class of lifts  $\tilde{\nu}$  of  $\nu$ :

$$\begin{array}{ccc}
 & B & \\
 \tilde{\nu} \nearrow & & \downarrow \pi \\
 M & \xrightarrow{\nu} & BSO.
 \end{array} \tag{8.1}$$

When  $B$  is  $B\text{Spin}^c$ , then a lift  $\tilde{\nu}$  in (8.1) is a  $\text{Spin}^c$  structure on its stable normal bundle. When  $B$  is  $B\text{Spin}$ , then a lift  $\tilde{\nu}$  in (8.1) is a  $\text{Spin}$  structure on its stable normal bundle.

Define

$$\text{String} = \varinjlim_{k \rightarrow \infty} \text{String}(k),$$

where  $\text{String}(k)$  is an infinite dimensional topological group constructed in [40]. There is a map  $\text{String}(k) \rightarrow \text{Spin}(k)$  which induces an isomorphism  $\pi_n(\text{String}(k)) \cong \pi_n(\text{Spin}(k))$  for all  $n$  except  $n = 3$  when  $\pi_3(\text{String}(k)) = 0$  and  $\pi_3(\text{Spin}(k)) \cong \mathbb{Z}$ .

Let  $M$  be a  $\text{Spin}$  manifold with a classifying map  $\nu: M \rightarrow B\text{Spin}$  for the  $\text{Spin}$  structure on its stable normal bundle. A string structure on  $M$  is a lift  $\tilde{\nu}$  of  $\nu$ :

$$\begin{array}{ccc}
 & B\text{String} & \\
 \tilde{\nu} \nearrow & & \downarrow \pi \\
 M & \xrightarrow{\nu} & B\text{Spin}.
 \end{array}$$

We point out that an oriented manifold  $M$  admits a spin structure on its stable normal bundle if and only if its second Stiefel–Whitney class  $w_2(M)$  vanishes, and a spin manifold  $M$  admits a string structure on its stable normal bundle if  $\frac{p_1(M)}{2}$  vanishes, where  $p_1(M)$  denotes the first Pontrjagin class of  $M$  (cf. [37], [40]). If  $M$  is a string manifold, then  $M$  has a canonical orientation with respect to elliptic cohomology.

The tower of Eilenberg–MacLane fibrations

$$\begin{array}{ccc}
 B\text{String} & & \\
 \downarrow K(\mathbb{Z},3) & & \\
 B\text{Spin} & \xrightarrow{\frac{p_1}{2}} & K(\mathbb{Z}, 4) \\
 \downarrow K(\mathbb{Z}_2,1) & & \\
 B\text{SO} & \xrightarrow{w_2} & K(\mathbb{Z}_2, 2)
 \end{array}$$

gives rise to Thom spectra

$$M\text{String} \rightarrow M\text{Spin} \rightarrow MSO,$$

with corresponding bordism groups

$$\Omega_*^{\text{String}}(X) \rightarrow \Omega_*^{\text{Spin}}(X) \rightarrow \Omega_*^{\text{SO}}(X).$$

**Remark 8.1.** Given a paracompact space  $X$ , a continuous map  $\alpha: X \rightarrow K(\mathbb{Z}_2, 2)$  is called a KO-twisting, and a continuous map  $\alpha: X \rightarrow K(\mathbb{Z}, 4)$  is called a string twisting. For a principal  $G$ -bundle  $\mathcal{P}$  over  $X$  for a compact Lie group  $G$  equipped with a map  $BG \rightarrow K(\mathbb{Z}, 4)$  representing a degree 4 class in  $H^4(BG, \mathbb{Z})$ , there is a natural string twisting

$$X \rightarrow BG \rightarrow K(\mathbb{Z}, 4).$$

Given a string twisting  $\alpha: X \rightarrow K(\mathbb{Z}, 4)$ , a universal Chern–Simons 2-gerbe was constructed in [16].

For any KO-twisting  $\alpha$ , there is a corresponding notion of an  $\alpha$ -twisted Spin manifold over  $(X, \alpha)$ .

**Definition 8.2.** Let  $(X, \alpha)$  be a paracompact topological space with a twisting  $\alpha: X \rightarrow K(\mathbb{Z}_2, 2)$ . An  $\alpha$ -twisted Spin manifold over  $X$  is a quadruple  $(M, \nu, \iota, \eta)$  where

- (1)  $M$  is a smooth, oriented and compact manifold together with a fixed classifying map of its stable normal bundle

$$\nu: M \rightarrow B\text{SO};$$

- (2)  $\iota: M \rightarrow X$  is a continuous map;
- (3)  $\eta$  is an  $\alpha$ -twisted Spin structure on  $M$ , that is, a homotopy commutative diagram

$$\begin{array}{ccc}
 M & \xrightarrow{\nu} & BSO \\
 \downarrow \iota & \eta \dashv \vDash & \downarrow w_2 \\
 X & \xrightarrow{\alpha} & K(\mathbb{Z}_2, 2),
 \end{array}$$

where  $w_2$  is the classifying map of the principal  $K(\mathbb{Z}_2, 1)$ -bundle  $BSpin \rightarrow BSO$  associated to the second Stiefel–Whitney class and  $\eta$  is a homotopy between  $w_2 \circ \nu$  and  $\alpha \circ \iota$ .

Two  $\alpha$ -twisted Spin structures  $\eta$  and  $\eta'$  on  $M$  are called equivalent if there is a homotopy between  $\eta$  and  $\eta'$ .

**Remark 8.3.** Let  $M$  be a smooth, oriented and compact  $n$ -dimensional manifold and let  $X$  be a paracompact space with a KO-twisting  $\alpha: X \rightarrow K(\mathbb{Z}_2, 2)$ .

(1)  $M$  admits an  $\alpha$ -twisted Spin structure if and only if there exists a continuous map  $\iota: M \rightarrow X$  such that

$$\iota^*([\alpha]) + w_2(M) = 0 \tag{8.2}$$

in  $H^2(M, \mathbb{Z}_2)$ . Here  $w_2(M)$  is the second Stiefel–Whitney class of  $TM$ . (The condition (8.2) is the anomaly cancellation condition for type I D-branes, cf. [44].)

(2) If  $\iota^*([\alpha]) + w_2(M) = 0$ , then the set of equivalence classes of  $\alpha$ -twisted Spin structures on  $M$  is an affine space modelled on  $H^1(M, \mathbb{Z}_2)$ .

Let  $\mathcal{H}_{\mathbb{R}}$  be an infinite dimensional, real and separable Hilbert space. The projective orthogonal group  $PO(\mathcal{H}_{\mathbb{R}})$  with the norm topology (cf. [32]) has the homotopy type of an Eilenberg–MacLane space  $K(\mathbb{Z}_2, 1)$ . The classifying space of  $PO(\mathcal{H}_{\mathbb{R}})$ , as a classifying space of the principal  $PO(\mathcal{H}_{\mathbb{R}})$ -bundle, is a  $K(\mathbb{Z}_2, 2)$ . Thus, the set of isomorphism classes of locally trivial principal  $PO(\mathcal{H}_{\mathbb{R}})$ -bundles over  $X$  is canonically identified with

$$[X, K(\mathbb{Z}_2, 2)] \cong H^2(X, \mathbb{Z}_2).$$

Given a KO-twisting  $\alpha: X \rightarrow K(\mathbb{Z}_2, 2)$ , there is a canonical principal  $K(\mathbb{Z}_2, 1)$ -bundle over  $X$ , or equivalently, a locally trivial principal  $PO(\mathcal{H}_{\mathbb{R}})$ -bundle  $\mathcal{P}_{\alpha}$  over  $X$ . Let  $\mathcal{K}_{\mathbb{R}}$  be the  $C^*$ -algebra of real compact operators on  $\mathcal{H}_{\mathbb{R}}$ . Let  $C_c(X, \mathcal{P}_{\alpha}(\mathcal{K}_{\mathbb{R}}))$  be the  $C^*$ -algebra of compactly supported sections of the associated bundle

$$\mathcal{P}_{\alpha}(\mathcal{K}_{\mathbb{R}}) := \mathcal{P}_{\alpha} \times_{PO(\mathcal{H}_{\mathbb{R}})} \mathcal{K}_{\mathbb{R}}.$$

In [39] (see also [35]), twisted KO-theory is defined for  $X$  with a KO-twisting

$$\alpha: X \rightarrow K(\mathbb{Z}_2, 2)$$

to be

$$KO^i(X, \alpha) := KO_i(C_c(X, \mathcal{P}_\alpha(\mathcal{K}_\mathbb{R}))).$$

Let  $\mathbb{K}_\mathbb{R}$  be the 0-th space of the KO-theory spectrum. Then there is a base-point preserving action of  $K(\mathbb{Z}_2, 1)$  on the real K-theory spectrum

$$K(\mathbb{Z}_2, 1) \times \mathbb{K}\mathbb{O} \rightarrow \mathbb{K}\mathbb{O},$$

which is represented by the action of real line bundles on ordinary KO-groups. This action defines an associated bundle of KO-theory spectrum over  $X$ . Denote

$$\mathcal{P}_\alpha(\mathbb{K}_\mathbb{R}) = \mathcal{P}_\alpha \times_{K(\mathbb{Z}_2, 1)} \mathbb{K}_\mathbb{R}$$

the bundle of based spectra over  $X$  with fiber the KO-theory spectra, and let  $\{\Omega_X^n \mathcal{P}_\alpha(\mathbb{K}_\mathbb{R}) = \mathcal{P}_\alpha \times_{K(\mathbb{Z}_2, 1)} \Omega^n \mathbb{K}_\mathbb{R}\}$  be the fiber-wise iterated loop spaces. Then we have an equivalent definition of twisted KO-groups of  $(X, \alpha)$  (cf. [39]) as the set of homotopy classes of compactly supported sections of the bundle of K-spectra:

$$KO^n(X, \alpha) = \pi_0(C_c(X, \Omega_X^n \mathcal{P}_\alpha(\mathbb{K}_\mathbb{R}))).$$

Due to Bott periodicity, we only have eight different twisted K-groups  $KO^i(X, \alpha)$  ( $i = 0, \dots, 7$ ). Twisted KO-theory is an 8-periodic generalized cohomology theory.

One would expect that results in this paper can be extended to twisted KO-theory. Much of the constructions and arguments in this paper go through in the case of twisted KO-theory. The subtlety is to study the twisted KR-theory for the (co)-tangent bundle with the canonical involution. This may requires additional arguments.

Another interesting generalization is the notion of twisted string structure for a paracompact topological space  $X$  with a string twisting given by

$$\alpha: X \rightarrow K(\mathbb{Z}, 4).$$

**Definition 8.4.** Let  $(X, \alpha)$  be a paracompact topological space with a string twisting  $\alpha: X \rightarrow K(\mathbb{Z}, 4)$ . An  $\alpha$ -twisted *string* manifold over  $X$  is a quadruple  $(M, v, \iota, \eta)$  where

- (1)  $M$  is a smooth compact manifold with a stable spin structure on its normal bundle given by

$$v: M \rightarrow BSpin,$$

with  $BSpin = \lim_{\rightarrow k} BSpin(k)$  the classifying space of the stable spin structure;

- (2)  $\iota: M \rightarrow X$  is a continuous map;
- (3)  $\eta$  is an  $\alpha$ -twisted string structure on  $M$ , that is, a homotopy commutative diagram

$$\begin{array}{ccc} M & \xrightarrow{v} & BSpin \\ \iota \downarrow & \eta \swarrow \doteq & \downarrow \frac{p_1}{2} \\ X & \xrightarrow{\alpha} & K(\mathbb{Z}, 4), \end{array}$$

where  $\frac{p_1}{2}: B\text{Spin} \rightarrow K(\mathbb{Z}, 4)$  is the classifying map of the principal  $K(\mathbb{Z}, 3)$ -bundle  $B\text{String} \rightarrow B\text{Spin}$ , representing the generator of  $H^4(B\text{Spin}, \mathbb{Z})$ , and  $\eta$  is a homotopy between  $\frac{p_1}{2} \circ \nu$  and  $\alpha \circ \iota$ .

Two  $\alpha$ -twisted String structures  $\eta$  and  $\eta'$  on  $M$  are called equivalent if there is a homotopy between  $\eta$  and  $\eta'$ .

**Remark 8.5.** Let  $M$  be a smooth compact spin manifold and let  $X$  be a paracompact space with a string twisting  $\alpha: X \rightarrow K(\mathbb{Z}, 4)$ .

(1)  $M$  admits an  $\alpha$ -twisted string structure if and only if there is a continuous map  $\iota: M \rightarrow X$  such that

$$\iota^*([\alpha]) + \frac{p_1(M)}{2} = 0 \quad (8.3)$$

in  $H^4(M, \mathbb{Z})$ . Here  $p_1(X)$  is the first Pontrjagin class of  $TM$ .

(2) If  $\iota^*([\alpha]) + \frac{p_1(M)}{2} = 0$ , then the set of equivalence classes of  $\alpha$ -twisted string structures on  $M$  is an affine space modelled on  $H^3(M, \mathbb{Z})$ .

Given a manifold  $X$  with a twisting  $\alpha: X \rightarrow K(\mathbb{Z}, 4)$ , one can form a bordism category, called the  $\alpha$ -twisted string bordism over  $(X, \alpha)$ , whose objects are compact smooth spin manifolds over  $X$  with an  $\alpha$ -twisted string structure. The corresponding bordism group  $\Omega_*^{\text{String}}(X, \alpha)$  is called the  $\alpha$ -twisted string bordism group of  $X$ . We will study these  $\alpha$ -twisted string bordism groups and their applications elsewhere.

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