Hopf algebroids and secondary characteristic classes

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Abstract. We study a Hopf algebroid, $\mathcal{H}$, naturally associated to the groupoid $U_n^\delta \ltimes U_n$. We show that classes in the Hopf cyclic cohomology of $\mathcal{H}$ can be used to define secondary characteristic classes of trivialized flat U_n-bundles. For example, there is a cyclic class which corresponds to the universal transgressed Chern character and which gives rise to the continuous part of the ρ-invariant of Atiyah–Patodi–Singer. Moreover, these cyclic classes are shown to extend to pair with the K-theory of the associated C*-algebra. This point of view gives leads to homotopy invariance results for certain characteristic numbers. In particular, we define a subgroup of the cohomology of a group analogous to the Gelfand–Fuchs classes described by Connes [3] and show that the higher signatures associated to them are homotopy invariant.

1. Introduction

We will study some new cyclic classes which forms a subgroup of the cohomology of a discrete group. In common with the Gelfand–Fuchs classes used by Connes in [3], they extend to pair with the K-theory of a certain algebra and hence yield homotopy invariance results for some "higher signatures". The original goal was to study the Connes–Moscovici index theory for hypoelliptic transverse signature operators, but in the much easier case when the foliation is Riemannian. The spectral triple in that case can be constructed analogous to [5] and one needed to find the appropriate adaptation of the Hopf algebra techniques of [9]. As a step in that direction we considered a very special class of Riemannian-foliated flat U_n-bundles. We will use a Hopf algebroid to define characteristic classes and relate them to several earlier constructions. The transverse signature operator in this case comes from the signature operator on the compact Lie group U_n and its Chern character is in the periodic cyclic cohomology of our Hopf algebroid. However, in the present paper we will look at different aspects of these classes.

The Hopf algebroid, which we denote $\mathcal{H}(U_n^\delta \ltimes U_n)$, is one associated to the étale groupoid, $U_n^\delta \ltimes U_n$. We compute its cyclic theory as introduced by Connes and Moscovici [10] and see that it is isomorphic to the cyclic theory of $C_c^\infty(U_n^\delta \ltimes U_n)$. Moreover, it has a convenient description via a double complex.

The double complex has a subcomplex corresponding to invariant forms which we investigate further. These classes have two important properties. The first is that the associated cocycles extend to $K_\bullet(C(U_n) \rtimes U_n^\delta)$, the K-theory of the reduced C*-algebra. This yields homotopy invariance results. The second is that they can be expressed in terms of transgressed classes on the base. A particular case yields the transgressed Chern character, hence yields the homotopy invariance of the (continuous part of) the ρ-invariant of Atiyah–Patodi–Singer.

Using the ideas in Connes [3] we also consider the subgroup of the cohomology of a discrete group which is determined by these classes. All of the classes in this subgroup satisfy the Novikov conjecture. It would be interesting to know how much of the cohomology of a group is spanned by these classes and Gelfand–Fuchs classes. This is analogous to results on low-dimensional cohomology of groups and the Novikov conjecture in the sense that the results do not depend on any special properties of the groups.

2. Cyclic cohomology of Hopf algebroids

In this section we will review Connes and Moscovici's definition, [10], of cyclic cohomology of a Hopf algebroid. We will study a special case related to the classifying space for trivialized flat unitary bundles.

2.1. Hopf algebroids. In [28], Lu introduced the notion of a Hopf algebroid as a generalization of a Hopf algebra. Cyclic cohomology of a Hopf algebroid was developed by Connes and Moscovici, [10], in their study of transverse index theory in the non-flat case.

Let A and B be unital algebras. A bialgebroid structure on A over B consists of the following data:

i) An algebra homomorphism $\alpha\colon B \to A$ called the *source map* and an algebra anti-homomorphism $\beta\colon B \to A$ called the *target map*, satisfying $\alpha(a)\beta(b) = \beta(b)\alpha(a)$ for all $a, b \in A$.

For the next part of the definition let $A \otimes_B A$ be the quotient of $A \otimes A$ by the right $A \otimes A$ ideal generated by $\beta(a) \otimes 1 - 1 \otimes \alpha(a)$ for all $a \in A$.

ii) A B-B bimodule map $\Delta\colon A \to A \otimes_B A$, called the *coproduct*, satisfying

    (a) $\Delta(1) = 1 \otimes 1$,

    (b) $(\Delta \otimes_B \mathrm{id})\Delta = (\mathrm{id} \otimes_B \Delta)\Delta\colon A \to A \otimes_B A \otimes_B A$,

    (c) $\Delta(a)(\beta(b) \otimes 1 - 1 \otimes \alpha(b)) = 0$ for $a \in A$, $b \in B$,

    (d) $\Delta(a_1 a_2) = \Delta(a_1)\Delta(a_2)$ for $a_1, a_2 \in A$.

iii) A B-B bimodule map $\epsilon\colon A \to B$, called the *counit*, satisfying

    (a) $\epsilon(1) = 1$,

    (b) $\ker \epsilon$ is a left A ideal,

    (c) $(\epsilon \otimes_B \mathrm{id})\Delta = (\mathrm{id} \otimes \epsilon)\Delta = \mathrm{id}\colon A \to A$.

A *Hopf algebroid* is a bialgebroid A over B which admits an algebra anti-isomorphism $S\colon A \to A$ satisfying

(1) $S \circ \beta = \alpha$;

(2) $m_A(S \otimes \mathrm{id})\Delta = \beta\epsilon S\colon A \to A$, with $m_A\colon A \otimes A \to A$ the multiplication on A;

(3) there is a linear map $\gamma\colon A \otimes_B A \to A \otimes A$ such that

    (a) If $\pi\colon A \otimes A \to A \otimes_B A$ is the natural projection, $\pi\gamma = \mathrm{id}\colon A \otimes_B A \to A \otimes_B A$;

    (b) $m_A(\mathrm{id} \otimes S)\gamma\Delta = \alpha\epsilon\colon A \to A$.

We note that in the above definition one may allow A and B to be differential graded algebras and require all of the above maps to be compatible with the differentials and to be of degree 0. Thus one would have a differential graded Hopf algebroid. The particular example of a Hopf algebroid which we will study in the present paper is of this type and is inspired by [10], [17].

Example 2.1. We consider a Hausdorff smooth étale groupoid $\mathcal{G} \rightrightarrows G_0$. Define A to be the algebra of differential forms[1] on $\mathcal{G}$, and B to be the algebra of differential forms on G_0. Both A and B are differential graded (commutative) algebras with the de Rham differential.

We define the source and target map $\alpha, \beta\colon A \to B$ as the duals of the source and target maps of the groupoid $\mathcal{G} \rightrightarrows G_0$. Similarly, we define Δ to be the dual of the product on $\mathcal{G} \times_{G_0} \mathcal{G} \to \mathcal{G}$. Note that the algebra of differential forms on $\mathcal{G} \times_{G_0} \mathcal{G}$ is equal to $A \otimes_B A$. The counit map $\epsilon\colon A \to B$ is the embedding $G_0 \to \mathcal{G}$ of the unit space. It is straightforward to check that $(A, B, \alpha, \beta, \Delta, \epsilon)$ is a bialgebroid.

The antipode map $S\colon A \to A$ is defined to be the dual of the inversion map $\iota\colon \mathcal{G} \to \mathcal{G}$. It is easy to check that S satisfies properties (1) and (2) for an antipode of a Hopf algebroid. In the case that $\mathcal{G} \rightrightarrows G_0$ is a transformation groupoid from a discrete

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[1]We remark that we do not require differential forms to be compactly supported.

group Γ action on G_0, we define γ to be the dual of the map $\mathcal{G} \times \mathcal{G} \to \mathcal{G} \times_{G_0} \mathcal{G}$ with $\mathcal{G} \times \mathcal{G} \ni (\alpha_1, m_1, \alpha_2, m_2) \mapsto (\alpha_1, m_1, \alpha_2, \alpha_2^{-1}(m_2)) \in \mathcal{G} \times_{G_0} \mathcal{G}$. It is not difficult to check that $(A, B, \alpha, \beta, \Delta, \epsilon, S, \gamma)$ is a Hopf algebroid. Since A and B are topological algebras, we want to work in that category and so we will use the projective tensor product. This Hopf algebroid will be referred to as $\mathcal{H}(\mathcal{G})$. In general, we do not know a natural construction of the linear map γ required in the above property (3) of an antipode. We refer to [31] for a further discussion about Hopf algebroid associated to a smooth étale groupoid.

2.2. Cyclic cohomology of a Hopf algebroid and Hopf algebroid modules. For completeness, we will review Connes and Moscovici's definition of cyclic cohomology of a Hopf algebroid. The following construction works for a bialgebroid with a *twisted* antipode $S \colon A \to A$. That is, one has an algebra anti-isomorphism such that

$$S^2 = \mathrm{id}, \quad S\beta = \alpha, \quad m_A(S \otimes_B \mathrm{id})\Delta = \beta\epsilon S \colon A \to A,$$

and

$$S(a^{(1)})^{(1)}a^{(2)} \otimes_B S(a^{(1)})^{(2)} = 1 \otimes_B S(a). \tag{2.1}$$

In the above formula we have used Sweedler's notation for the coproduct $\Delta(a) = a^{(1)} \otimes_B a^{(2)}$. We remark that a bialgebroid with this twisted version of an antipode[2] is called a para-Hopf algebroid in [25].

Proposition 2.2. *The bialgebroid $\mathcal{H}(\mathcal{G}) = (A, B, \alpha, \beta, \Delta, \epsilon)$ with the antipode S in Example* 2.1 *for a general étale groupoid forms a para-Hopf algebroid.*

Proof. It is sufficient to check equation (2.1). Let a be an element in A. For $x, y \in \mathcal{G}$ two composable arrows, we notice that

$$\Delta(a)(x, y) = a(xy), \quad (S \otimes \mathrm{id})\Delta(a)(x, y) = a(x^{-1}y).$$

Therefore, one computes

$$(\Delta S \otimes \mathrm{id})\Delta(a)(x, y, z) = a(y^{-1}x^{-1}z),$$
$$S(a^{(1)})^{(1)}a^{(2)} \otimes_B S(a^{(1)})^{(2)}(x, y) = a(y^{-1}x^{-1}x) = a(y^{-1}) = 1 \otimes_B S(a). \quad \square$$

Let Λ be the cyclic category. We construct a cyclic module $A^{\natural}$ for a para-Hopf algebroid $(A, B, \alpha, \beta, \Delta, \epsilon, S)$.

Define

$$C^0 = B, \quad C^n = \underbrace{A \otimes_B A \otimes_B \cdots \otimes_B A}_{n}, \quad n \geq 1.$$

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[2]Here in the definition of a twisted antipode, we have fixed a particular modular pair $e = 1 \in A$ and $\delta = \epsilon \circ S \colon A \to B$. And in general we can extend the following definitions to a Hopf algebroid with a modular pair.

Faces and degeneracy operators are defined as follows:

$$\delta_0(a^1 \otimes_B \cdots \otimes_B a^{n-1}) = 1 \otimes_B a^1 \otimes_B \cdots \otimes_B a^{n-1},$$
$$\delta_i(a^1 \otimes_B \cdots \otimes_B a^{n-1}) = a^1 \otimes_B \cdots \otimes_B \Delta a^i \otimes_B \cdots \otimes_B a^{n-1}, \quad 1 \leq i \leq n-1,$$
$$\delta_n(a^1 \otimes_B \cdots \otimes_B a^{n-1}) = a^1 \otimes_B \cdots \otimes_B a^{n-1} \otimes_B 1;$$
$$\sigma_i(a^1 \otimes_B \cdots \otimes_B a^{n+1}) = a^1 \otimes_B \cdots \otimes_B a^i \otimes_B \epsilon(a^{i+1}) \otimes_B a^{i+2} \otimes_B \cdots \otimes_B a^{n+1}.$$

The cyclic operators are given by

$$\tau_n(a^1 \otimes_B \cdots \otimes_B a^n) = (\Delta^{n-1} S(a^1))(a^2 \otimes \ldots a^n \otimes 1).$$

We define the Hopf cyclic cohomology of A to be the cyclic cohomology of $A^{\natural}$. We remark that this definition extends naturally to differential graded Hopf algebroids since all the maps above are maps between differential graded vector spaces and the differential is of degree 1. In the following, we briefly review Gorokhovsky's construction [17] of characteristic class map for a differential graded Hopf algebroid action. We remark that Gorokhovsky only studied characteristic map for a differential graded algebra action. The same construction works for a general para-Hopf algebroid action. In the following we will only state the results and their proofs are actually identical to those in [17].

Let M be a differential graded algebra. Then M is equipped with a Hopf algebroid $(A, B, \alpha, \beta, \Delta, \epsilon, S)$ action if there is a differential graded algebra morphism $\rho \colon B \to M$ and an action $\lambda \colon A \otimes_{\mathbb{C}} M \to M$ satisfying

(1) $\lambda(a_1 a_2, m) = \lambda(a_1, \lambda(a_2, m))$ for $a_1, a_2 \in A, m \in M$;

(2) $\deg(\lambda(a, m)) = \deg(a) + \deg(m)$ and

$$d(\lambda(a, m)) = \lambda(da, m) + (-1)^{\deg(a)} \lambda(a, dm)$$

for $a \in A, m \in M$;

(3) $\lambda(1, m) = m$ for $m \in M$;

(4) $\lambda(a, m_1 m_2) = \sum_i (-1)^{\deg(m_1) \deg(a_2^i)} \lambda(a_1^i, m_1) \lambda(a_2^i, m_2)$ for $a \in A$ and $\Delta(a) = \sum_i a_1^i \otimes a_2^i \in A \otimes_B A$;

and when M has a unit, we require

(5) $\lambda(a, 1) = \lambda(\epsilon(a), 1) = \rho(\epsilon(a))$ for $a \in A$.

We assume that there is a trace τ on M such that

$$\tau(\lambda(a, m)n) = (-1)^{\deg(a) \deg(m)} \tau(m\lambda(S(a), n)) \quad \text{for all } m, n \in M \text{ and all } a \in A.$$

We define a cochain map c from the cyclic cochain complex of A to the cyclic cochain complex of the differential graded algebra M, i.e., for $\Phi = a_1 \otimes \cdots \otimes a_q$ an element in C^q we have

$$c(\Phi)(m_0, \ldots, m_q) = c(\Phi, m_0, \ldots, m_q)\tau(m_0 \lambda(a_1, m_1) \ldots \lambda(a_q, m_q)), \quad m_i \in M,$$

with $c(\Phi, m_0, \ldots, m_1) = (-1)^{\sum_{j=1}^{q} \deg(a_j)(\deg(m_{j-1}) + \cdots + \deg(m_0))}$. We call a cyclic cochain on M in the image of the form $c(\Phi)$ a differentiable cochain, and denote the space of differentiable p-cochains by $C_d^p(M)$. Then we can define differentiable cyclic cohomology of M, $HC_d^*(M)$, to be the cyclic cohomology of $C_d^{\natural}(M)$. One can also define the periodic differentiable cyclic cohomology, $HP_d^*(M)$, by similar means. One has the following natural map

$$c \colon HC^*(A) \to HP_d^*(M) \to HP^*(M).$$

For later applications, we assume that the trace τ is of weight q, i.e., $\tau(m) = 0$ if $\deg(m) \neq q$. Then we easily see that a cochain in $A^{\natural}$ with total degree greater than q is mapped to zero by c. Let $F^l A^{\natural} = \{a_1 \otimes \ldots a_j \mid \deg(a_1) + \cdots + \deg(a_j) \geq l\}$ and $A_q^{\natural} = A^{\natural}/F^{q+1}A^{\natural}$. Then $A_q^{\natural}$ is again a cyclic module with cyclic cohomology $HC^*(A)_q$ and $HP^*(A)_q$. The characteristic map c induces a map $c \colon HC^*(A)_q \to HP^*(M)$.

Let M^0 be the degree 0 part of M. We can map cyclic cochains on M to M^0. And we have the following characteristic map analogous to [17], Theorem 8:

$$c \colon HC^*(A)_q \to HP^{*-q}(M^0).$$

For an étale groupoid $\mathcal{G}$, we consider the space M of compactly supported differential forms on $\mathcal{G}$ with the convolution product. As $\mathcal{G}$ is an étale groupoid, the unit space G_0 is embedded in $\mathcal{G}$ as an open submanifold. This embedding map between spaces defines an algebra homomorphism ρ from $B = \Omega^*(G_0)$ to M. For $a \in A = \Omega^*(\mathcal{G})$ and $m \in M = \Omega_c^*(\mathcal{G})$, define $\lambda(a, m) = am \in \Omega_c^*(\mathcal{G})$ by viewing both a and m as elements of $\mathcal{H}(\mathcal{G})$ and using the product on $\mathcal{H}(\mathcal{G})$. It is a direct check that ρ and λ define an action of $\mathcal{H}(\mathcal{G})$ on M. Furthermore, we assume that there is a $\mathcal{G}$ invariant orientation on G_0. The integration of differential forms on G_0 defines a trace τ_Ω on M of weight $\dim(G_0)$ compatible with the antipode S. Hence as $M^0 = C_c^\infty(\mathcal{G})$, we have a characteristic map $c_\Omega \colon HC^*(\mathcal{H}(\mathcal{G}))_{\dim(G_0)} \to HP^{*-\dim(G_0)}(C_c^\infty(\mathcal{G}))$.

2.3. Hopf cyclic cohomology of the Hopf algebroid $\mathcal{H}(\mathcal{G})$.

In this section we will compute the Hopf cyclic cohomology of the para-Hopf algebroid $\mathcal{H}(\mathcal{G})$ introduced in Proposition 2.2 for a smooth étale groupoid.

To formulate the results, we will use the following constructions. Let $B\mathcal{G}^n = \underbrace{\mathcal{G} \times_{G_0} \cdots \times_{G_0} \mathcal{G}}_{n}$, where $\mathcal{G} \times_{G_0} \mathcal{G}$ is the fiber product with respect to the maps $t \colon \mathcal{G} \to G_0$ and $s \colon \mathcal{G} \to G_0$. This can be given the structure of a simplicial manifold. Let $\Omega^*(B\mathcal{G}^*)$ be the double complex of sheaves of differential forms on the simplicial manifold $B\mathcal{G}^*$, [14]. Then $\Omega^*(B\mathcal{G}^*)$ has a standard cyclic structure (described below). As (Ω^*, d) is a resolution of $\mathbb{C}$, we denote the corresponding cyclic cohomology of $\Omega^*(B\mathcal{G}^*)$ by $HC^*(\mathcal{G}, \mathbb{C})$ and $HP^*(\mathcal{G}, \mathbb{C})$. One can now state the following result.

Theorem 2.3.

$$HC^*(\mathcal{H}(\mathcal{G}))_{\dim(G_0)} = HC^*(\mathcal{G}; \mathbb{C}) \quad and \quad HP^*(\mathcal{H}(\mathcal{G}))_{\dim(G_0)} = HP^*(\mathcal{G}; \mathbb{C}).$$

Proof. As a first step one notes that an m-cochain on $\mathcal{H}(\mathcal{G})$ of total degree less than or equal to $\dim(G_0)$ can be identified with a smooth section of the sheaf of differential forms on $B\mathcal{G}^m$, $m \geq 0$. This identification respects the cyclic simplicial structures on $\Omega^*(B\mathcal{G}^*)$ and $\mathcal{H}(\mathcal{G})^\natural$. Indeed, following the notation in [12], where $(a|g_1, \ldots, g_n)$ with $a \in \Omega^*(G_0)|_{t(g_1)}$, $g_i \in \mathcal{G}$ and $s(g_i) = t(g_{i+1})$, $i = 1, \ldots, n-1$, denotes elements of $\Omega^*(B\mathcal{G}^*)$, the cyclic simplicial structure on $\Omega^*(B\mathcal{G}^*)$ is given by

$$\delta_i(a|g_1, \ldots, g_n), = \begin{cases} (ag_1|g_2, \ldots, g_n), & i = 0, \\ (a|g_1, \ldots, g_i g_{i+1}, \ldots, g_n), & 1 \leq i \leq k-1, \\ (a|g_1, \ldots, g_{n-1}) & i = n, \end{cases}$$

$$\sigma_i(a|g_1, \ldots, g_n), = (a| \ldots, g_{i-1}, 1, g_i, \ldots, g_n),$$

and

$$t(a \mid g_1, \ldots, g_n) = (ag_1 \ldots g_n|(g_1 g_2 \ldots g_n)^{-1}, g_1, \ldots, g_{n-1}).$$

In the above, ag means the right translation of a by $g \in \mathcal{G}$.

It is easy to check from this identification that the $\dim(G_0)$-truncated Hopf cyclic cohomology of $\mathcal{H}(\mathcal{G})$ is equal to the cyclic cohomology of $\Omega^*(B\mathcal{G}^*)$. $\square$

Just as the cohomology of a discrete group is isomorphic to the cohomology of its classifying space, there is an analogous result for étale groupoids, [29]. It states that the groupoid sheaf cohomology of the sheaf $\mathbb{C}$ on $\mathcal{G}$, denoted $H^*(\mathcal{G}; \mathbb{C})$, is isomorphic to the cohomology of its classifying space $B\mathcal{G}$.

Since Ω^* is a projective resolution of the sheaf $\mathbb{C}$ on $\mathcal{G}$, we obtain the following corollary.

Corollary 2.4.

$$HC^*(\mathcal{H}(\mathcal{G}))_{\dim(G_0)} = \bigoplus_{k \in \mathbb{Z}_+} H^{*-2k}(B\mathcal{G}; \mathbb{C}),$$

$$HP^*(\mathcal{H}(\mathcal{G}))_{\dim(G_0)} = \bigoplus_{k \in \mathbb{Z}} H^{*+2k}(B\mathcal{G}; \mathbb{C}).$$

We apply Theorem 2.3 in the following special case.

Let U_n be the compact Lie group of $n \times n$ unitary matrices and let U_n^δ be the same group, but equipped with the discrete topology. Left multiplication of U_n^δ on U_n defines a transformation groupoid $\mathcal{G} = U_n^\delta \ltimes U_n \rightrightarrows U_n$. We will compute the cyclic cohomology of its groupoid algebra, $C_c^\infty(U_n^\delta \ltimes U_n)$. The following result was first proved in [1], and we derive it as an application from [12].

Proposition 2.5.

$$HP^*(C_c^\infty(U_n^\delta \ltimes U_n)) = \bigoplus_{k \in \mathbb{Z}} H^{*+2k+\dim(U_n)}(U_n^\delta \ltimes U_n, \mathbb{C}).$$

Proof. Because U_n^δ acts on U_n freely, the cyclic groupoid $\mathbb{Z}$ introduced in [12] associated to the étale groupoid $U_n^\delta \ltimes U_n$ is equal to itself. Let *or* be the orientation sheaf on $U_n^\delta \ltimes U_n$. Then, in [12], 4.14, it is proved that $HP^*(C_c^\infty(U_n^\delta \ltimes U_n)) = \bigoplus_{k \in \mathbb{Z}} H^{*+2k+\dim(U_n)}(U_n^\delta \ltimes U_n, or)$. Using the left invariant volume form Ω on U_n, one can identify the orientation sheaf *or* and $\mathbb{C}$, and the statement of this proposition follows. $\qquad\square$

Observe that, in the present situation, the groupoid sheaf cohomology $H^*(U_n^\delta \ltimes U_n, \mathbb{C})$ is by definition the cohomology of the total complex of

$$\Gamma((\underbrace{U_n^\delta \times \cdots \times U_n^\delta}_{q+1}) \times_{U_n^\delta} U_n, \Omega^p(U_n)),$$

and the result is equal to the simplicial de Rham cohomology $H^*(EU_n^\delta \times_{U_n^\delta} U_n)$. We record this fact for later use.

Corollary 2.6. *The groupoid sheaf cohomology, $H^*(U_n^\delta \ltimes U_n, \mathbb{C})$ is isomorphic to the de Rham cohomology of the simplicial manifold $H^*(EU_n^\delta \times_{U_n^\delta} U_n)$.*

Let $\overline{BU_n}$ denote the geometric realization of the simplicial manifold $EU_n^\delta \times_{U_n^\delta} U_n$. Recall a map $\phi\colon \oplus_k H^{*+2k}(\overline{BU_n}) \to HP^{*-\dim(U_n)}(C_c^\infty(U_n^\delta \ltimes U_n))$ defined by Connes. This map actually agrees with

$$c\colon \bigoplus_k H^{*+2k}(\overline{BU_n}) = HP^*(\mathcal{H}(U_n^\delta \ltimes U_n))_{\dim(U_n)} \to HP^{*-\dim(U_n)}(C_c^\infty(U_n^\delta \ltimes U_n))$$

by the same argument as in [17]. Note that the composition of the map c with the identification $e\colon HP^*(C_c^\infty(U_n^\delta \ltimes U_n^\delta)) = HP^{*+\dim(U_n)}(U_n^\delta \ltimes U_n, \mathbb{C})$ is the identity map on $H^*(U_n^\delta \ltimes U_n, \mathbb{C})$ since c maps a differential form on $U_n^\delta \ltimes U_n$ to a differential current on $U_n^\delta \ltimes U_n$ to pair with cyclic chains on $C_c^\infty(U_n^\delta \ltimes U_n)$ and e maps vice versa. In summary, we have obtained the following result.

Corollary 2.7. *The map*

$$\phi\colon \bigoplus_k H^{*+2k}(\overline{BU_n}) = HP^*(\mathcal{H}(U_n^\delta \ltimes U_n))_{\dim(U_n)} \to HP^{*-\dim(U_n)}(C_c^\infty(U_n^\delta \ltimes U_n))$$

is an isomorphism.

C. Lazarov and J. Pasternack have introduced secondary characteristic classes for Riemannian foliations with trivial normal bundle. We will relate our constructions to theirs, [26]. Let $\overline{B R \Gamma}_q$ denote the classifying space for Riemannian Haefliger structures of codimension q with trivial normal bundle. We remark that the dimension of U_n is n^2.

Proposition 2.8. *There is a canonical Riemannian Haefliger structure on $EU_n^\delta \times_{U_n^\delta} U_n$ with trivial normal bundle which induces a map $EU_n^\delta \times_{U_n^\delta} U_n \to \overline{B R \Gamma}_{n^2}$.*

Proof. For the existence of a Riemannian Haefliger structure cover EU_n^δ by small open sets V_α which are disjoint from all their translates. Choose an invariant Riemannian metric on U_n and let $\{W_\beta\}$ be a cover by charts. Then $\mathrm{pr}_1 \colon V_\alpha \times W_\beta \to \mathbb{R}^{n^2}$ descends to the quotient by U_n^δ and provides the Haefliger structure. We must show that its normal bundle is trivial. Note first that the tangent bundle TU_n is a trivialized by using left translation. Then $\mathrm{pr}_1^*(TU_n)$ is trivial. Because U_n^δ acts trivially on TU_n, the quotient $\widetilde{TU}_n$ of the bundle $p_1^* TU_n$ on $EU_n^\delta \times_{U_n^\delta} U_n$ is again trivial. But the coordinate transformations for $\widetilde{TU}_n$ are obtained from the differentials of those associated to the Haefliger structure, so the conclusion holds. $\square$

By [26], there is a characteristic class map obtained as the following composition:

$$\kappa \colon H^*(RW_{n^2}) \to H^*(\overline{B R \Gamma}_{n^2}) \to H^*(EU_n^\delta \times_{U_n^\delta} U_n). \qquad (2.2)$$

3. Secondary characteristic classes of trivialized flat U_n-bundles

Let M be a compact smooth manifold and let $\Gamma = \pi_1(M)$ be the fundamental group of M. Let $\widetilde{M}$ be the universal cover of M. Suppose that we are given a finite dimensional unitary representation $\alpha \colon \Gamma \to U_n$. Consider the flat principal U_n-bundle $V = \widetilde{M} \times_\Gamma U_n \to M$. If we assume that this bundle is trivial with a given trivialization $\theta \colon \widetilde{M} \times_\Gamma U_n \to M \times U_n$ then we can relate it to the Lazarov–Pasternack map.

In this section we will construct a map $\chi \colon HC^*(\mathcal{H}(U_n^\delta \ltimes U_n))_{n^2} \to H^*(V)$. It will be a composition of several maps.

First, the homomorphism of algebras $C_c^\infty(\Gamma \ltimes U_n) \to C_c^\infty(U_n^\delta \ltimes U_n)$ induces a homomorphism on cyclic cohomology

$$\nu \colon HP^*(C_c^\infty(U_n^\delta \ltimes U_n)) \to HP^*(C_c^\infty(\Gamma \ltimes U_n)). \qquad (3.1)$$

Next, note that the manifold V is foliated by leaves which are the images of $\widetilde{M} \times \{g\}$. We denote this foliation by $\mathcal{F}$, and the corresponding holonomy groupoid by $\mathcal{G}$. We assume that the representation α is faithful, which will assure that the

holonomy groupoid $\mathcal{G}$ is Hausdorff. Furthermore, $\mathcal{G}$ is Morita equivalent to the transformation groupoid $\Gamma \ltimes U_n$. Therefore the groupoid algebra $C_c^\infty(\mathcal{G})$ is Morita equivalent to $C_c^\infty(\Gamma \ltimes U_n)$. Thus, there is an isomorphism on cyclic cohomology, which provides the second map

$$\iota \colon HP^*(C_c^\infty(\Gamma \ltimes U_n)) \to HP^*(C_c^\infty(\mathcal{G})). \tag{3.2}$$

Recall that in [4], Chapter III, 7.γ, Connes constructs a map $\lambda \colon HP^*(C_c^\infty(\mathcal{G})) \to H^{*+n^2}(B\mathcal{G})$ which is a left inverse to the map Φ which is important in higher index theorems. Finally, the foliated structure on V defines a map $V \to B\mathcal{G}$. The induced map on cohomology will be denoted $\eta \colon H^*(B\mathcal{G}) \to H^*(V)$.

We summarize the above constructions in the following sequence

$$HC^*(\mathcal{H}(U_n^\delta \ltimes U_n))_{n^2} \xrightarrow{c} HP^{*-n^2}(C_c^\infty(U_n^\delta \ltimes U_n)) \xrightarrow{\nu} HP^{*-n^2}(C_c^\infty(\Gamma \ltimes U_n))$$

$$\xrightarrow{\iota} HP^{*-n^2}(C_c^\infty(\mathcal{G})) \xrightarrow{\lambda} \bigoplus_k H^{*+2k}(B\mathcal{G}) \xrightarrow{\eta} \bigoplus_k H^{*+2k}(V).$$

We define

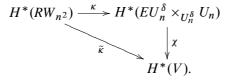$$\chi \colon HC^*(\mathcal{H}(U_n^\delta \ltimes U_n))_{n^2} \to \bigoplus_k H^{*+2k}(V)$$

to be the above composition and

$$\hat{\chi} \colon HC^*(\mathcal{H}(U_n^\delta \ltimes U_n))_{n^2} \to HP^{*-n^2}(C_c^\infty(\Gamma \ltimes U_n))$$

to be the composition of the first two arrows. We will refer to χ as the characteristic map and cohomology classes in the image of χ will be viewed as secondary characteristic classes.

These classes are compatible with those introduced by Lazarov and Pasternack as the next proposition shows.

Proposition 3.1. *Let κ and $\tilde{\kappa}$ be the maps obtained using Lazarov–Pasternack* [26]*, (2.2). Then the following diagram commutes:*

$$
\begin{array}{ccc}
H^*(RW_{n^2}) & \xrightarrow{\ \kappa\ } & H^*(EU_n^\delta \times_{U_n^\delta} U_n) \\
 & \searrow{\scriptstyle \tilde{\kappa}} & \ \ \downarrow{\scriptstyle \chi} \\
 & & H^*(V).
\end{array}
$$

Proof. It is straightforward check that the diagram is commutative. $\square$

In the remainder of this section we provide a more detailed description of the map $\iota \colon HP^*(C_c^\infty(U_n \rtimes \Gamma)) \to HP^*(C_c^\infty(\mathcal{G}))$ following ideas from [4], III.4.α.

Consider the holonomy groupoid $\mathcal{G} \rightrightarrows V$. The space $\mathcal{G}$ consists of holonomy classes of paths connecting two points on a leaf. The set of homotopy classes of paths on a single leaf is equal to $\widetilde{M} \times \widetilde{M}$. Since the leaves are all simply connected, it follows from this that the holonomy groupoid $\mathcal{G}$ can be identified with $(\widetilde{M} \times \widetilde{M}) \times_\Gamma U_n$. The structure maps $s, t \colon \mathcal{G} = (\widetilde{M} \times \widetilde{M}) \times_\Gamma U_n \rightrightarrows V = \widetilde{M} \times_\Gamma U_n$ and groupoid operation are defined as follows:

$$s\langle \tilde{x}, \tilde{y}, g \rangle = \langle \tilde{x}, g \rangle, \quad t\langle \tilde{x}, \tilde{y}, g \rangle = \langle \tilde{y}, g \rangle, \quad \tilde{x}, \tilde{y} \in \widetilde{M}, \ g \in U_n,$$

$$\langle \tilde{x}, \tilde{y}, g \rangle \circ \langle \tilde{y}, \tilde{z}, g \rangle = \langle \tilde{x}, \tilde{z}, g \rangle, \qquad \tilde{x}, \tilde{y}, \tilde{z} \in \widetilde{M}.$$

Using this description of $\mathcal{G}$, we can make the following identifications.

Let $\mathcal{R}$ be the algebra of infinite matrices $(a_{ij})_{i,n \in \mathbb{N}}$ with rapid decay property, i.e.,

$$\sup_{i,j \in \mathbb{N}} i^k j^l |a_{ij}| < \infty \quad \text{for all } k, l \in \mathbb{N}.$$

One defines the map

$$T \colon C_c^\infty(\mathcal{G}) \to M_N(C_c^\infty(\Gamma \ltimes U_n) \otimes \mathcal{R}),$$

where $M_N\big(C_c^\infty(\Gamma \ltimes U_n) \otimes \mathcal{R}\big)$ is the algebra of $(N \times N)$-matrices with entries in $C_c^\infty(\Gamma \ltimes U_n) \otimes \mathcal{R}$.

Since M is a compact manifold, we can choose a finite open cover $(U_i)_{i=1,\dots,N}$, with $\beta_i \colon U_i \to \widetilde{M}$ a local smooth section of the projection $\widetilde{M} \to M$ and $(\varphi_i)_{i=1,\dots,N}$ a smooth partition of unity subordinate to the covering $(U_i)_{i=1,\dots,N}$, with $\varphi_i^{1/2}$ also smooth functions.

Let $\mathcal{R}_M$ denote the algebra of smoothing operators on M. For any $f \in C_c^\infty(\mathcal{G})$, define $T(f) \in M_N\big(C_c^\infty(\Gamma \ltimes U_n) \otimes \mathcal{R}_M\big)$, where $\mathcal{R}_M$ is the algebra of smoothing operators on $C^\infty(M)$ by

$$T(f)_{i,j}(x, y, \gamma, g) = \varphi_i^{\frac{1}{2}}(x)\varphi_j^{\frac{1}{2}}(y) f(\beta_i(x), \gamma^{-1}\beta_j(y), g),$$

for $x, y \in M$, $g \in U_n$, $\gamma \in \Gamma$.

It is straightforward to check that, since f is smooth and compactly supported on $\mathcal{G}$, $T(g)_{ij}(\gamma, g)$ defines a smoothing operator on M with finite support on Γ. Similar to [4], III.4.β, T is an algebra homomorphism and induces a Morita equivalence between $C_c^\infty(\mathcal{G})$ and $C_c^\infty(\Gamma \ltimes U_n)$.

We choose an isomorphism ϱ between $\mathcal{R}_M$ and $\mathcal{R}$, which always exists and is unique up to inner automorphisms. The composition $I = \varrho \circ T$ defines an algebra homomorphism from $C_c^\infty(\mathcal{G})$ to $M_n(C_c^\infty(\Gamma \ltimes U_n) \otimes \mathcal{R})$.

We can use I to define a map $\iota \colon HP^*(C_c^\infty(\Gamma \ltimes U_n)) \to HP^*(C_c^\infty(\mathcal{G}))$ as follows. For $\Psi \in HP^j(C_c^\infty(\Gamma \ltimes U_n))$, let

$$\iota(\Psi)(a_0, \dots, a_j) = \Psi\big(I(a_0)_{i_0 i_1}^{m_0 m_1}, I(a_1)_{i_1 i_2}^{m_1 m_2}, \dots, I(a_n)_{i_n i_0}^{m_n m_0}\big),$$

where $i_0, \ldots, i_n$ run from 1 to ∞, and $m_0, \ldots, m_n$ runs from 1 to N. This construction can be viewed as the sharp product on cyclic cohomology between Ψ with the standard traces on M_n and $\mathcal{R}$.

4. Extendability of cocycles

In this section we will describe a subgroup of the cohomology of $\mathcal{H}(U_n^\delta \ltimes U_n)$, whose image under χ consists of classes extending to pair with the K-theory of the reduced C*-algebra of $\Gamma \ltimes U_n$. The proof is a direct adaptation of Connes' argument in [3].

To describe the subset, consider the double complex $\mathcal{C}^{p,q} := (\Omega^p(U_n) \times_{U_n^\delta} (U_n^\delta)^{\times(q+1)}, (d, b))$ whose total cohomology is $H^*(U_n^\delta \ltimes U_n)$. By [1], the spectral sequence of this double complex degenerates at $E_2 = H^q(BU_n^\delta, H^p(U_n))$. We consider a subcomplex $(\Omega^*(U_n)^{U_n}, d) = \Omega^*(U_n) \times_{U_n^\delta} U_n^\delta = \mathcal{C}^{*,0} \hookrightarrow \mathcal{C}^{p,q}$. It is easy to check that the inclusion is a cochain map and therefore defines a homomorphism $H^*(U_n) \hookrightarrow H^0(BU_n^\delta, H^*(U_n)) \subseteq HC^*(\mathcal{H}(U_n^\delta \ltimes U_n))_{n^2}$.

Theorem 4.1. *The linear functionals on $K_*(C_c^\infty(\Gamma \ltimes U_n))$ defined by elements in $\hat{\chi}(H^*(U_n)) \subset HP^{*-n^2}(C_c^\infty(U_n^\delta \ltimes U_n))$ extend to linear functionals on $K_*(C(U_n) \rtimes \Gamma)$, where $C(U_n) \rtimes \Gamma$ is the reduced crossed product C*-algebra.*

Proof. We consider the following sequence of maps

$$H^*(U_n) \to HC^*(\mathcal{H}(U_n^\delta \ltimes U_n))_{n^2}$$
$$\to HP^{*-n^2}(C_c^\infty(U_n^\delta \ltimes U_n)) \to HP^{*-n^2}(C_c^\infty(\Gamma \ltimes U_n)).$$

Let $\omega \in \Omega^*(U_n))^{U_n}$ be a representative of a cocycle in $H^*(U_n)$ of degree p. Notice that the real dimension of U_n is n^2. Then $\chi(\omega) \in HC^{n^2-p}(C_c^\infty(\Gamma \ltimes U_n))$ is defined as

$$\chi(\omega)(f_0, f_1, \ldots, f_{n^2-p}) = \int_{U_n} \omega \wedge (f_0 df_1 \ldots df_{n^2-k})|_{\mathrm{id}},$$

where $f_0, \ldots, f_{n^2-p}$ are elements in $C_c^\infty(\Gamma \ltimes U_n)$, d is the de Rham differential on $\Omega^*(U_n) \rtimes \Gamma$, and $(\ldots)|_{\mathrm{id}}$ stands for the restriction of an element in $\Omega^*(U_n) \rtimes \Gamma$ to the identity component.

In the following we prove that $\chi(\omega)$ defines an $(n^2 - p)$-trace on the reduced C*-algebra $C(U_n) \rtimes \Gamma$ and hence the extension exists. Our proof is essentially the same as Connes' arguments in [3]. However, it simplifies greatly in that we have an isometric (rather than almost isometric) action at hand. In this case the initial step of finding a Banach subalgebra B on which to construct an n-trace is unnecessary, since

the map $\lambda \colon C_c^\infty(\Gamma \ltimes U_n) \to \operatorname{End}_A(\mathcal{E})$ is a *-homomorphism, hence is bounded. Thus one can take $B = C(U_n) \rtimes \Gamma$ with the reduced C*-norm.

What we will show next is that elements in the image $\hat{\chi}(H^*(U_n))$ are n-traces which by [3], Theorem 2.7, will define linear maps from $K_*(A)$ to $\mathbb{C}$.

We recall the definition of an n-trace on a Banach algebra A. An n-trace on A is an $(n+1)$-linear functional τ on a dense subalgebra $\mathcal{A}$ of A such that

(1) τ is a cyclic cocycle on $\mathcal{A}$;

(2) for any $a^i \in \mathcal{A}$, $i = 1, \ldots, n$, there exists $C = C_{a^1,\ldots,a^n} < \infty$ such that

$$|\hat{\tau}((x^1 da^1)(x^2 da^2)\ldots(x^n da^n))| \le C \|x^1\| \ldots \|x^n\|, \quad x^i \in A. \qquad (4.1)$$

For our purpose, we consider the Banach algebra to be $C(U_n) \rtimes \Gamma$. Given any $\omega \in \Omega^*(U_n)^{U_n}$ with $d\omega = 0$, we have that $\chi(\omega)$ defines a cyclic cocycle on $C_c^\infty(\Gamma \ltimes U_n)$. To prove that $\chi(\omega)$ defines an m-trace, it suffices to prove the estimate (4.1) for any $a^i \in \mathcal{A}$ and $x^i \in A$, $i = 1, \ldots, n$.

Define on the collection of spaces $\bigoplus_j \mathcal{E}_j = \bigoplus_j C_c(\Gamma \ltimes U_n, r^*(\wedge^j T_{\mathbb{C}}^* U_n))$ a convolution product $* \colon \mathcal{E}_i \otimes \mathcal{E}_j \to \mathcal{E}_{i+j}$ by

$$\phi * \psi(\alpha) = \sum_{\alpha = \beta\gamma} \phi(\beta) \wedge \beta(\psi(\gamma)), \quad \alpha \in \Gamma \ltimes U_n.$$

Assume that $\omega \in \Omega^p(U_n)^{U_n}$ with $d\omega = 0$. Then $\operatorname{tr}_\omega(\phi) = \int_{U_n} \omega(x)\phi(x, \operatorname{id})$ defines a linear functional on $\mathcal{E}_{n^2 - p}$ with the following property:

$$\operatorname{tr}_\omega(\phi * \psi) = \operatorname{tr}_\omega((-1)^{jk}\psi * \phi), \quad \psi \in \mathcal{E}_j, \ \phi \in \mathcal{E}_k, \ j + k = n^2 - p.$$

The proof of the above trace property goes as follows.

$$\begin{aligned}
\operatorname{tr}_\omega(\phi * \psi) &= \int_{U_n} \omega(x) \sum_{\alpha\beta = \operatorname{id}} \phi(x, \alpha) \wedge \psi(\alpha(x), \beta) \\
&= \int_{U_n} \omega(x)(-1)^{jk} \sum_{\alpha\beta = \operatorname{id}} \psi(\alpha(x), \beta) \wedge \phi(x, \alpha) \\
&= \int_{U_n} \omega(\alpha^{-1}(y))(-1)^{jk} \sum_{\alpha\beta = \operatorname{id}} \psi(y, \beta) \wedge \phi(\beta(y), \alpha) \\
&= \int_{U_n} \omega(y)(-1)^{jk} \sum_{\alpha\beta = \operatorname{id}} \psi(y, \beta) \wedge \phi(\beta(y), \alpha) \\
&= (-1)^{jk} \operatorname{tr}_\omega(\psi * \phi).
\end{aligned}$$

In the above proof, y is $\alpha(x)$ and we have used that $\alpha^{-1} = \beta$, and ω is Γ invariant.

We prove the following properties for tr_ω, which is an analog of [3], Lemma 4.3.

Lemma 4.2. (1) *For any $\phi \in \mathcal{E}_{n^2-p}$, there is a constant $C_\phi < \infty$ such that for any $f \in C(U_n) \rtimes \Gamma$, one has*

$$|\operatorname{tr}_\omega(f\phi)| \leq C_\phi \|f\|_A,$$

where $\|f\|_A$ is the C*-algebra norm of the reduced crossed product C*-algebra $C(U_n) \rtimes \Gamma$.

(2) *For and $\phi_1, \ldots, \phi_{n^2-p} \in \mathcal{E}_1$, there exists a constant $C_{\phi_1,\ldots,\phi_{n^2-p}} < \infty$ such that for any $f_1, \ldots, f_n \in C(U_n) \rtimes \Gamma$,*

$$\operatorname{tr}_\omega(\phi_1 f_1 * \phi_2 f_2 * \cdots * \phi_{n^2-p} f_{n^2-p}) \leq C_{\phi_1,\ldots,\phi_{n^2-p}} \prod_i \|f_i\|.$$

Proof. (1) We recall that for any $f \in C(U_n) \rtimes \Gamma$, the following estimates of the C^*-norm of $\|f\|_A$ holds, i.e., $\sup_{x \in U_n, \gamma \in \Gamma} |f(x, \gamma)| \leq \|f\|_A$. Now for $\operatorname{tr}_\omega(f\phi)$, we have that

$$|\operatorname{tr}_\omega(f\phi)| = \left| \int_{U_n} \sum_{\alpha\beta=\operatorname{id}} \omega(x) f(x, \alpha) \alpha(\phi(\alpha(x), \beta)) \right|$$

$$\leq \int_{u_n} |\omega(x)| \sum_{\alpha\beta=\operatorname{id}} |f(x, \alpha)| |\alpha(\phi(\alpha(x), \beta))|$$

$$\leq \|f\|_A \int_{U_n} |\omega(x)| \sum_\alpha |\alpha(\phi(\alpha(x), \alpha^{-1}))| \leq C_\phi \|f\|_A.$$

We remark that $\sum_\alpha |\alpha(\phi(\alpha(x), \alpha^{-1}))|$ is a finite sum, and therefore the integral is finite.

(2) We apply Connes [3], Theorem 3.7, to the following situation.

Theorem. *Let Ψ be an m-linear function on $\mathcal{E}_1$ satisfying the following conditions:*

(a) $\Psi(\xi_1, \ldots, \xi_j f, \xi_{j+1}, \ldots, \xi_m) = \Psi(\xi_1, \ldots, f\xi_{j+1}, \ldots, \xi_m)$ *for* $j = 1, \ldots, m-1, \xi_k \in \mathcal{E}_1$ *and* $f \in C(U_n) \rtimes \Gamma$.

(b) *For any $\xi_1, \ldots, \xi_m \in \mathcal{E}_1$, there exists $C < \infty$ such that*

$$|\Psi(\xi_1, \ldots, \xi_m f)| \leq C \|f\|_A, \quad f \in C(U_n) \rtimes \Gamma.$$

Then for any $f_0 \in C(U_n) \rtimes \Gamma$ and $\xi_1, \ldots, \xi_m \in \mathcal{E}_1$, there exists $C' < \infty$ with

$$|\Psi(f_0 f_1 \xi_1, f_2 \xi_2, \ldots, f_m \xi_m)| \leq C' \|f_1\| \ldots \|f_m\|, \quad f_i \in C(U_n) \rtimes \Gamma.$$

To apply the above theorem to our situation, we introduce Ψ_ω an $n^2 - p$ linear functional $\mathcal{E}_1$ as follows. For $\xi_1, \ldots, \xi_{n^2-p} \in \mathcal{E}_1$, let

$$\Psi_\omega(\xi_1, \ldots, \xi_{n^2-p}) = \int_{U_n} \omega(x) \wedge (\xi_1 * \cdots * \xi_{n^1-p})|_{\operatorname{id}}.$$

It is easy to see that Ψ_ω satisfies condition (a) of the above theorem since $\xi_i f *$ $\xi_{i+1} = \xi_i * f * \xi_{i+1} = \xi_i * f \xi_{i+1}$ for any $i = 1, \ldots, n^2 - p - 1$. For condition (b), we consider that $\Psi(\xi_1, \ldots, \xi_{n^2-p} f) = \mathrm{tr}_\omega(\xi_1 * \cdots * \xi_{n^2-p} * f) = \mathrm{tr}_\omega(f \xi_1 * \cdots * \xi_{n^2-p})$. Since all ξ_i belong to $\mathcal{E}_1$, $\xi_1 * \cdots * \xi_{n^2-p}$ again belongs to $\mathcal{E}_{n^2-p}$. Hence, by Lemma 3.2 (1), $|\Psi_\omega(\xi_1, \ldots, \xi_{n^2-p} f)| = |\mathrm{tr}_\omega(f \xi_1 * \cdots * \xi_{n^2-p})| \le C_{\xi_1,\ldots,\xi_{n^2-p}} \|f\|_A$. Therefore both condition (a) and (b) are satisfied, and the above theorem implies that

$$|\Psi_\omega(f_0 f_1 \xi_1, f_2 \xi_2, \ldots, f_{n^2-p} \xi_{n^2-p})| \le C' \|f_1\| \ldots \|f_{n^2-p}\|.$$

Now using the tracial property, we have

$$
\begin{aligned}
|\mathrm{tr}_\omega(\phi_1 f_1 * \ldots \phi_{n^2-p} f_{n^2-p})| &= |\mathrm{tr}_\omega(f_{n^2-p} \phi_1 * f_1 \phi_2 * \cdots * f_{n^2-p-1} \phi_{n^2-p})| \\
&= |\Psi_\omega(f_{n^2-p} \phi_1, \ldots, f_{n^2-p-1} \phi_{n^2-p})| \\
&\le C' \|f_1\| \ldots \|f_{n^2-p}\|.
\end{aligned}
$$

This proves Lemma 4.2.

Now by Lemma 4.2 and the tracial property, we can easily obtain the condition (4.1), and we conclude that $\hat{\chi}(\omega) \in HC^{n^2-p}(C_c^\infty(\Gamma \ltimes U_n))$ defines a linear function from $K_*(C(U_n) \rtimes \Gamma)$ to $\mathbb{C}$ for any $\omega \in \Omega^p(U_n)^{U_n}$ with $d\omega = 0$. $\qquad\square$

Remark 4.3. We can extend U_n in Theorem 4.1 to a general oriented manifold V with a vector bundle E which is equipped with an almost isometric orientation-preserving action of Γ. Then the similar statements hold for the closed Γ invariant differential forms on V. In [3], Connes applies this idea to fundamental cocycles. In the context, one can see that he definitely has this kind of generalization in mind.

Remark 4.4. We remark that Jiang [19] proved that if a discrete group Γ is rapid decay and acts isometrically on a closed oriented Riemannian manifold V preserving the orientation, then cocycles on $E\Gamma \times_\Gamma V$ of polynomial growth can be paired with the K-theory group of the reduced C*-algebra $C_0(V) \rtimes \Gamma$. (See [19] for the precise statement.) Our results in this section show that any Γ-invariant cocycles on V can be paired with the K-theory of the C*-algebra for any orientation-preserving isometrical action of a discrete group Γ on an oriented Riemannian manifold. We do not need to assume the group to be rapid decay.

5. Transgressed classes and cyclic classes

The goal of the present section is to study the transgressed Chern character of a flat trivialized U_n-bundle. By providing a simplicial construction of the universal transgressed Chern character we will be able to show that it is in the image of the

map χ. Indeed, the proof actually shows that any class in $\chi(H^*(U_n))$ is obtained by transgression from an invariant polynomial applied to curvature forms.

We start from the following fibration

$$\overline{BU}_n \to BU_n^\delta \to BU_n,$$

where $\overline{BU}_n$ is the homotopy fiber of the map from $BU_n^\delta \to BU_n$. We can take $\overline{BU}_n$ to be the realization of a simplicial space, $EU_n^\delta \times_{U_n^\delta} U_n$. Now, one knows that BU_n is the classifying space for U_n-bundles, BU_n^δ is the classifying space for flat U_n-bundles, and $\overline{BU}_n$ is the classifying space for trivialized flat U_n-bundles. In particular, we will see that on $EU_n^\delta \times_{U_n^\delta} U_n$ there is a flat U_n-bundle equipped with a canonical trivialization.

Consider the universal flat U_n^δ-bundle over BU_n^δ, which can be identified with EU_n^δ. Therefore, the associated flat principal U_n-bundle over BU_n^δ can be identified with $EU_n^\delta \times_{U_n^\delta} U_n \to BU_n^\delta$. Considering the map of classifying space, we have the following diagram

$$
\begin{array}{ccc}
EU_n^\delta \times_{U_n^\delta} (U_n \times U_n) & \xrightarrow{\iota'} & EU_n^\delta \times_{U_n^\delta} U_n \\
\pi' \downarrow & & \downarrow \pi \\
EU_n^\delta \times_{U_n^\delta} U_n & \xrightarrow{\iota} & BU_n^\delta,
\end{array}
\tag{5.1}
$$

where $EU_n^\delta \times_{U_n^\delta} (U_n \times U_n)$ with U_n^δ acting on $U_n \times U_n$ diagonally is the pullback with respect to the maps ι and π.

We write the maps π, π', ι, ι' in coordinates. We write a point in $EU_n^\delta \times_{U_n^\delta} U_n$ by (z, x) with $z \in EU_n^\delta$ and $x \in U_n$. Then $\pi(z, x) = \iota(z, x) = [z]$, where $[z]$ stands for the U_n^δ orbit of z in EU_n^δ. And (z, x, y) with $z \in EU_n^\delta$, $x, y \in U_n$ form local U_n^δ-equivariant coordinates for $EU_n^\delta \times_{U_n^\delta} (U_n \times U_n)$ such that $\iota'(z, x, y) = (z, y)$ and $\pi'(z, x, y) = (z, x)$.

On the bundle $\pi' \colon EU_n^\delta \times_{U_n^\delta} (U_n \times U_n) \to EU_n^\delta \times_{U_n^\delta} U_n$ there are two natural connections. One is the pull back of the flat connection on $\pi \colon EU_n^\delta \times_{U_n^\delta} U_n \to BU_n^\delta$, the other is from a canonical trivialization $EU_n^\delta \times_{U_n^\delta} (U_n \times U_n) \to (EU_n^\delta \times_{U_n^\delta} U_n) \times U_n$.

(1) The flat connection on $\pi \colon EU_n^\delta \times_{U_n^\delta} U_n \to BU_n^\delta$ is given by Dupont [14] in a simplicial way. We consider the universal U_n^δ-bundle $\pi \colon \overline{NU}_n^\delta \to NU_n^\delta$. $\overline{NU}_n^\delta$ is the simplicial manifold with $\overline{NU}_n^\delta(k) = U_n^\delta \times \cdots \times U_n^\delta$ ($k + 1$ copies), where the face operators are defined by leaving out one of the components. Similarly, NU_n^δ is simplicial manifold with $NU_n^\delta(k) = U_n^\delta \times \cdots \times U_n^\delta$ (k-copies) with the face operator equal to the multiplication of consecutive two components. EU_n^δ and BU_n^δ are the corresponding geometrical realizations of $\overline{NU}_n^\delta$ and NU_n^δ. There is

a canonical map from $\overline{NU}_n^\delta \to NU_n^\delta$ defined by

$$\pi(x_0, \ldots, x_k) = (x_0 x_1^{-1}, x_1 x_2^{-1}, \ldots, x_{k-1} x_k^{-1}).$$

We consider the maps $\mathrm{pr}_i : \overline{NU}_n^\delta \times_{U_n^\delta} U_n \to U_n$ defined by

$$\mathrm{pr}_i(x_0, \ldots, x_k; x) = x_i x.$$

Furthermore, let θ be the Maurer–Cartan form on U_n and $(t^0, \ldots, t^k)$ be the barycentric coordinates of the simplex Δ_k. Then $\Theta = \sum_i t^i \mathrm{pr}_i^*(\theta)$ defines a connection 1-form on $EU_n^\delta \times_{U_n^\delta} U_n \to BU_n^\delta$. The pullback of this connection through ι' defines a connection 1-form on $EU_n^\delta \times_{U_n^\delta} (U_n \times U_n)$, which is denoted by $\Theta_0 = \iota'^*(\Theta)$.

(2) We construct a canonical trivialization of $EU_n^\delta \times_{U_n^\delta} (U_n \times U_n) \to EU_n^\delta \times_{U_n^\delta} U_n$ by using the natural section $\sigma : EU_n^\delta \times_{U_n^\delta} U_n \to EU_n^\delta \times_{U_n^\delta} (U_n \times U_n)$ given by $\sigma(z, x) = (z, x, x)$. Accordingly we have a map $\Sigma : EU_n^\delta \times_{U_n^\delta} (U_n \times U_n) \to U_n$ defined by $\Sigma(z, x, y) = y^{-1} x$. If θ is the Maurer–Cartan form on U_n, then $\Sigma^*(\theta)$ defines a connection 1-form on $EU_n^\delta \times_{U_n^\delta} (U_n \times U_n)$ which is denoted by $\Theta_1 = \Sigma^*(\theta)$ and which is associated to the section σ.

We consider the following connection $D = s\Theta_0 + (1-s)\Theta_1$ on the space $(EU_n^\delta \times_{U_n^\delta} (U_n \times U_n)) \times [0, 1]$, where s is the coordinate on the interval $[0, 1]$. Let R_s be the curvature of D. Let $\mathfrak{U}_n$ be the Lie algebra of U_n. For any $\mathcal{P} \in \mathrm{sym}(\mathfrak{U}_n^*)^{\mathfrak{U}_n}$, we define $T\mathcal{P} \in \Omega^*(EU_n^\delta \times_{U_n^\delta} (U_n \times U_n))$ by

$$T\mathcal{P} = \int_0^1 \mathcal{P}(R_s).$$

Using the expression $D = s\Theta_0 + (1-s)\Theta_1$, we have that

$$R_s = ds \wedge \Theta_0 + s d\Theta_0 - ds \wedge \Theta_1 + (1-s)d\Theta_1 + \tfrac{1}{2}s^2\Theta_0^2 + (1-s)s\Theta_0 \wedge \Theta_1 + \tfrac{1}{2}(1-s)^2\Theta_1^2.$$

It is easy to check that R_s is $\mathfrak{U}_n$ horizontal, and therefore descends to a 2-form on $(EU_n^\delta \times_{U_n^\delta} U_n) \times [0, 1]$. Consider the projection map $\pi' : EU_n^\delta \times_{U_n^\delta} (U_n \times U_n) \times [0, 1] \to EU_n^\delta \times_{U_n^\delta} U_n \times [0, 1]$, which forgets the second U_n-factor. We observe that the restrictions of $\Theta_0 = \sum_i t^i \mathrm{pr}_i^*(\theta)$ and $d\Theta_0 = \sum_i dt^i \wedge \mathrm{pr}_i^*(\theta) + t^i \mathrm{pr}_i^*(d\theta)$ to $U_n^{\delta \times (q+1)}$ (with $q \geq 0$) vanish as U_n^δ is equipped with discrete topology, and any differential form with a positive degree on a discrete set always vanishes. We conclude that restrictions of those terms in R_s that contain the forms Θ_0 or $d\Theta_0$ vanish on the base $EU_n^\delta \times_{U_n^\delta} U_n \times [0, 1]$. Thus R_s simplifies on $EU_n^\delta \times_{U_n^\delta} U_n \times [0, 1]$ to

$$-ds \wedge \Theta_1 + (1-s)d\Theta_1 + \tfrac{1}{2}(1-s)^2\Theta_1^2.$$

For example, when we apply this to the polynomial $\mathcal{P}(X) = \text{Tr}(\exp(-\frac{1}{2\pi}X)) \in \text{sym}(\mathfrak{U}_n^*)^{\mathfrak{U}_n}$, we obtain a formula for the character:

$$\text{Tch} = \int_0^1 \text{Tr}(\exp(R_s)) = \int_0^1 ds \sum_k \frac{1}{k!} \frac{(-1)^{k+1}}{(2\pi)^{k+1}} \text{Tr}(\Theta_1(sd\Theta_1 + \tfrac{1}{2}s^2\Theta_1^2)^k).$$

Consider next Θ_1 and $d\Theta_1$. These are simplicial differential forms on the components of the simplicial manifold $EU_n^\delta \times_{U_n^\delta} (U_n \times U_n)(k) = (\Delta_k \times \overline{NU}_n^\delta(k)) \times_{U_n^\delta} (U_n \times U_n)$ independent of the coordinates on Δ_k.

We now consider the mapping from the space of differential forms $\Omega^l((\Delta_k \times \overline{NU}(k)_n^\delta) \times_{U_n^\delta} U_n)$ on the simplicial manifold $EU_n^\delta \times_{U_n^\delta} U_n$ to the group cochain complex $C^k(U_n^\delta; \Omega^{l-k}(U_n))$ of $\Omega^*(U_n)$-valued U_n^δ-cochains. The map I is realized by integration along the Δ_k component of $(\Delta_k \times \overline{NU}_n^\delta) \times_{U_n^\delta} U_n$. Observing that $T\mathcal{P}$ is independent of the Δ_k coordinates, we conclude that $I(T\mathcal{P})$ gives rise to a zero-dimensional U_n^δ cocycle with values in $\Omega^*(U_n)^{U_n}$.

Finally, we will relate characteristic numbers obtained from these classes to secondary classes for trivialized flat bundles. Note that we can view these constructions in the context of the following diagram.

$$
\begin{array}{ccc}
U_n & \xrightarrow{\quad = \quad} & U_n \\
\downarrow & & \downarrow \\
V = \tilde{M} \times_\Gamma U_n & \xrightarrow{\hat{\alpha}} & EU_n^\delta \times_{U_n^\delta} U_n = \overline{BU}_n \\
\downarrow & \overset{\hat{\theta}}{\nearrow} & \downarrow k \\
M & \xrightarrow{\alpha} & BU_n^\delta
\end{array}
\qquad (5.2)
$$

The bundle that was considered in (5.1) is the pull-back of the left side to its total space. As a trivialized flat bundle it is classified by the map $\hat{\alpha}$. The universal class $T\mathcal{P}$ is in $H^*(\overline{BU}_n)$. If we assume in addition the existence of a trivialization $\theta \colon \tilde{M} \times_\Gamma U_n \to M \times U_n$, then there is a lift of α determined by θ which we will denote by $\hat{\theta} \colon M \to \overline{BU}_n$. The pull back of the universal class by $\hat{\theta}$ we will denote by

$$\hat{\theta}^*(T\mathcal{P}) = T\mathcal{P}(\alpha, \theta). \qquad (5.3)$$

This will play a role in the index theorems of the next section.

Combining this with Theorem 4.1 we obtain the following result.

Proposition 5.1. *The following holds.*

(1) *Let $[c] \in \chi(H^*(U_n))$. Then there exist $i \geq 1$ and polynomials $\mathcal{P}_1, \ldots, \mathcal{P}_i \in \text{sym}(\mathfrak{U}_n^*)^{\mathfrak{U}_n}$ such that $[c] = \hat{\alpha}^*([T\mathcal{P}_1]\ldots[T\mathcal{P}_i])$.*

(2) *Let $\mathcal{P} \in \mathrm{sym}(\mathfrak{U}_n^*)^{\mathfrak{U}_n}$. Then the associated transgressed class $\hat{\alpha}^*(T\mathcal{P})$ is equal to $\chi([c])$ for some cyclic class $[c]$.*

(3) *The cyclic class from* (2) *extends to define a linear map on the K-theory of the reduced crossed-product C*-algebra:*

$$[c] \colon K_*(C(U_n) \rtimes \Gamma) \to \mathbb{C}. \tag{5.4}$$

6. Higher index theorems and secondary classes

In this section we will apply Connes' index theorem [4] to obtain explicit formulas for pairing our cyclic classes with index classes in K-theory. This will lead in the next section to what appears to be a new family of classes in the cohomology of a group which will satisfy the Novikov conjecture. They are the analog of the Gelfand–Fuchs classes which Connes considered in [3].

Recall the geometric setting as described in Section 2. Let M be a closed Spin^c manifold and let $\alpha \colon \Gamma = \pi_1(M) \to U_n$ be an injective homomorphism. Consider the associated flat, foliated, principal U_n-bundle $V = \tilde{M} \times_\alpha U_n \to M$. Let $\eth$ denote the Dirac operator on M and $\tilde{\eth}$ its lift to $\tilde{M}$. Then $\tilde{\eth}$ descends to a leafwise elliptic operator $\eth_\alpha$ on V with respect to the foliation $\mathcal{F}$ with leaves the images of $\tilde{M} \times \{g\}$ in V.

In this context one can apply the higher index theorem of Connes [4]. It provides a topological formula for the pairing of the index of a leafwise elliptic operator, as an element of K-theory, with a cyclic cocycle in $HP^*(C_c^\infty(U_n \rtimes \Gamma))$. We will use the form presented by Connes that involves a localization map, $\lambda \colon HP^*(C_c^\infty(U_n \rtimes \Gamma)) \to H^{*+n^2}(V)$; see [4], p. 274. Since, in the present context, the normal bundles of all foliations, or the tangent bundles of the units in the étale versions, are oriented we will avoid the use of twisted cohomology groups and always assume that we have identified them with an untwisted group. We will make strong use of the fact that ϕ, and hence also λ, are isomorphisms. Suppose also that we are given a vector bundle E on V.

There is an invariant transverse measure for the foliation of V and an associated Ruelle–Sullivan current which we will denote by Λ.

Theorem 6.1 (Connes). *Let $\mathrm{Ch}(\sigma(\eth_\alpha))$ denote the Chern character of the symbol of $\eth_\alpha$, and $\mathrm{Td}(T\mathcal{F} \otimes \mathbb{C})$ the Todd class of the complexified tangent bundle along the leaves. Let Φ be the Thom isomorphism for $T\mathcal{F}$ and let $\mathrm{Ind}(\eth_\alpha \otimes E) \in K_*(C(U_n) \rtimes \Gamma)$ be the K-theoretic index of the operator $\eth_\alpha \otimes E$. Let $[c] \in HP^*(C_c^\infty(U_n \rtimes \Gamma))$ be given. Then one has*

$$\langle \mathrm{Ind}(\eth_\alpha \otimes E), [c] \rangle = \langle \lambda(c) \cup \mathrm{Ch}(E) \cup \Phi_{T^*\mathcal{F}}^{-1}(\mathrm{Ch}(\sigma(\eth_\alpha)) \cup \mathrm{Td}(T\mathcal{F} \otimes \mathbb{C}), [\Lambda] \rangle. \tag{6.1}$$

We next want to obtain a more precise version of the topological pairing in the case that the cyclic class $[c] = \hat{\chi}([u])$, with $[u] \in H^*(U_n)$. According to Proposition 5.1 there are invariant polynomials $\mathcal{P}_1, \ldots, \mathcal{P}_i$ so that $\chi([u]) = \hat{\alpha}^*(T\mathcal{P}_1 \ldots T\mathcal{P}_i)$. Thus, (6.1) can be rewritten as

$$
\begin{aligned}
\langle \mathrm{Ind}(\slashed{\partial}_\alpha \otimes E), [c] \rangle = \langle &\hat{\alpha}^*(T\mathcal{P}_1 \ldots T\mathcal{P}_i) \cup p^* \mathrm{Ch}(E) \\
&\cup \Phi_{T^*\mathcal{F}}^{-1}(\mathrm{Ch}(\sigma(\slashed{\partial}_\alpha)) \cup \mathrm{Td}(T\mathcal{F} \otimes \mathbb{C}), [\Lambda] \rangle.
\end{aligned}
\tag{6.2}
$$

If we further assume the existence of a trivialization $\theta \colon \widetilde{M} \times_\Gamma U_n \to M \times U_n$, then we can simplify to

$$
\begin{aligned}
\langle \mathrm{Ind}(\slashed{\partial}_\alpha \otimes E), [c] \rangle = \langle &\hat{\pi}^* \hat{\theta}^*(T\mathcal{P}_1 \ldots T\mathcal{P}_i) \cup p^* \mathrm{Ch}(E) \\
&\cup \Phi_{T^*\mathcal{F}}^{-1}(\mathrm{Ch}(\sigma(\slashed{\partial}_\alpha)) \cup \mathrm{Td}(T\mathcal{F} \otimes \mathbb{C}), [\Lambda] \rangle.
\end{aligned}
\tag{6.3}
$$

Recall that we defined $T\mathcal{P}(\theta, \alpha) = \hat{\theta}^*(T\mathcal{P}_1 \ldots T\mathcal{P}_i)$. Following the argument in [13], this may be used to obtain the index by a pairing on the base M. Thus, we finally obtain the desired formula.

Proposition 6.2. *Let $V = \widetilde{M} \times_\alpha U_n$ be the foliated, flat, bundle obtained from the representation $\alpha \colon \pi_1(M) \to U_n$. Let $\theta \colon \widetilde{M} \times_\alpha U_n \to M \times U_n$ be a trivialization. Let $[c] \in \chi(H^*(U_n))$ correspond to $T\mathcal{P}_1 \ldots T\mathcal{P}_i$. Then we have*

$$
\begin{aligned}
\langle \mathrm{Ind}(\slashed{\partial}_\alpha \otimes E), [c] \rangle = \langle &T\mathcal{P}(\theta, \alpha) \cup \mathrm{Ch}(E) \\
&\cup \Phi_{T^*M}^{-1}(\mathrm{Ch}(\sigma(\slashed{\partial})) \cup \mathrm{Td}(TM \otimes \mathbb{C}), [M] \rangle.
\end{aligned}
\tag{6.4}
$$

If we now specialize to the case where the bundle E is the spinors, so that $\slashed{\partial} \otimes E$ is the signature operator on M, which we denote D_M^{sign}, then the operator on $\widetilde{M} \times_\Gamma U_n$ is the leafwise signature operator, and we obtain the following formula, which we will apply in the next section.

Proposition 6.3.

$$
\langle \mathrm{Ind}(D_{M,\alpha}^{\mathrm{sign}}), [c] \rangle = \langle T\mathcal{P}(\theta, \alpha) \cup L(M), [M] \rangle.
\tag{6.5}
$$

7. Homotopy invariance of characteristic numbers

In this section we will obtain homotopy invariance results for characteristic numbers obtained from the cyclic classes considered above. We will describe two methods for doing this. The first is based on Connes' method of extending cocycles, cf. [4], [3]. The second was communicated to us by Guoliang Yu, and we will present his argument as an alternative approach.

7.1. First method. Let $h\colon M_1 \to M_2$ be an orientation-preserving homotopy equivalence of closed, oriented manifolds. Let $\alpha_i\colon \Gamma_i = \pi_1(M_i) \to U_n$ be unitary representations such that $\alpha_2 h_* = \alpha_1$.

The first step is to do a suspension operation. For this we lift h to a homotopy equivalence between universal covers, $\tilde{h}\colon \tilde{M}_1 \to \tilde{M}_2$, and this descends to a leafwise homotopy equivalence between the foliated, flat, principal bundles:

$$\hat{h}\colon V_1 = \tilde{M}_1 \times_{\alpha_1} U_n \to \tilde{M}_2 \times_{\alpha_2} U_n = V_2. \qquad (7.1)$$

The signature operators on M_i induce operators $D^{\text{sign}}_{M_i,\alpha_i}$ on V_i, which are the leafwise signature operators. The map $\hat{h}$ yields a Morita equivalence which induces a map on the K-theory of the associated (étale) foliation algebras,

$$\hat{h}_*\colon K_*(C(U_n) \rtimes_{\alpha_1} \Gamma_1)) \to K_*(C(U_n) \rtimes_{\alpha_2} \Gamma_2)), \qquad (7.2)$$

and it follows from [22], [18] that

$$\hat{h}_*(\text{Ind}(D^{\text{sign}}_{M_1,\alpha_1})) = \text{Ind}(D^{\text{sign}}_{M_2,\alpha_2}). \qquad (7.3)$$

Recall that one can always represent the K-theory class of the index in the image of $K_*(C_c^\infty(\Gamma \ltimes U_n))$ in such a way that any of our cocycles can pair with it. However, the equivalence in (7.3) is in $K_*(C(U_n) \rtimes \Gamma)$, so there is no guarantee that $\langle \hat{h}_*(\text{Ind}(D^{\text{sign}}_{M_1,\alpha_1})), [c] \rangle$ will agree with $\langle \text{Ind}(D^{\text{sign}}_{M_1,\alpha_1}), \hat{h}^*([c]) \rangle$. If the cocycle $[c]$ extends to the crossed-product C*-algebra, then we do obtain equality. Thus, we have the following version of the homotopy invariance property expressed in the Novikov conjecture.

Proposition 7.1.

$$\langle \text{Ind}(D^{\text{sign}}_{M_2,\alpha_2}), [c] \rangle = \langle \hat{h}_*(\text{Ind}(D^{\text{sign}}_{M_1,\alpha_1})), [c] \rangle = \langle \text{Ind}(D^{\text{sign}}_{M_1,\alpha_1}), \hat{h}^*([c]) \rangle, \qquad (7.4)$$

where $\hat{h}^*$ is the map induced on cyclic theory of the crossed-products.

Now if we interpret the pairings using the results from the previous sections, these equalities can be expressed in terms of characteristic numbers. Suppose that the cyclic class $[c]$ corresponds to a cohomology class in $\chi(H^*(U_n))$. There is an integer N so that the bundle defined by α_1^N is trivial, and we will choose an explicit trivialization θ. With this data, we can apply the index formulas (6.3) to (7.4) and obtain homotopy invariance results for certain characteristic numbers.

Theorem 7.2. $\langle h^*(L(M_2)) \cup T\mathcal{P}(\alpha_1^N, \theta), [M_1] \rangle = \langle L(M_2) \cup T\mathcal{P}(\alpha_2^N, \theta), [M_2] \rangle$.

Proof. By the index theorem, (6.1), the first and third terms in (7.4) can be rewritten in terms of topological pairings with the Ruelle–Sullivan currents associated to the Haar measure, and then, because of the form of $\chi([c])$, the pairing can be done on the base. The resulting formula is as stated.   □

This should be viewed as a homotopy invariance property for the "local part" of an invariant which is defined in special cases. For the case of Tch, the transgressed Chern character, the invariant is the ρ-invariant of Atiyah–Patodi–Singer and we are obtaining $\int_0^1 \dot{\eta}(D_t)dt$, where D_t is a path of operators determined by the two flat connections. Homotopy invariance for the ρ-invariant itself has been proved under the condition that the Baum–Connes conjecture holds for $C_{\max}^*(\Gamma)$. However, it has been shown to hold in $\mathbb{R}/\mathbb{Q}$ in general; see [16], [24], [32].

7.2. Second method. The second approach, communicated to us by Guoliang Yu, does not require the extending of cocycles. We will sketch the argument, which is an adaptation of the usual proof that injectivity of the assembly map implies the Novikov conjecture.

Proposition 7.3. *The assembly map*

$$\mu \colon K^{\mathrm{top}}(U_n^\delta; C(U_n)) \to K_*(C(U_n) \rtimes_{\max} U_n^\delta) \tag{7.5}$$

is injective. Moreover, there is an injective map

$$\xi \colon K_*(\underline{EU_n^\delta \times_{U_n^\delta} U_n}) \otimes \mathbb{Q} = K_*(\overline{BU}_n) \otimes \mathbb{Q} \to K^{\mathrm{top}}(U_n^\delta; C(U_n)) \otimes \mathbb{Q}. \tag{7.6}$$

Assuming for $(M_i, \alpha_i, \theta_i)$ the same data as in Section 7.1, we get leafwise signature operators and a map $\hat{\theta}_i \colon M_i \to \overline{BU}_n$ with the property that $\mu(\xi[(M_i, \theta_i)]) = \mathrm{Ind}(D_{M_i, \alpha_i}^{\mathrm{sign}})$. According to [18], one has $\mathrm{Ind}(D_{M_2, \alpha_2}^{\mathrm{sign}}) = \hat{h}_*(\mathrm{Ind}(D_{M_1, \alpha_1}^{\mathrm{sign}}))$ and since μ and ξ are injective we have that $[(M_2, \hat{\theta}_2 h)] = [(M_1, \hat{\theta}_1)]$. Our goal is to show that for any class $[u] \in HC^*(\mathcal{H}(U_n^\delta \ltimes U_n))_{n^2}$ we have

$$\langle \mathrm{Ind}(D_{M_2, \alpha_2}^{\mathrm{sign}}), \chi([u])\rangle = \langle \hat{h}_*(\mathrm{Ind}(D_{M_1, \alpha_1}^{\mathrm{sign}})), \chi([u])\rangle. \tag{7.7}$$

Since the map $\phi \colon \bigoplus_k H^{*+2k}(\overline{BU}_n) \to HP^{*-n^2}(C_c^\infty(U_n^\delta \ltimes U_n))$ is an isomorphism, there is a class $x \in H^*(\overline{BU}_n)$ satisfying

$$\langle \mathrm{Ch}([M_i, \theta_i]), x\rangle = \langle \mathrm{Ind}(D_{M_i, \alpha_i}^{\mathrm{sign}}), \chi([u])\rangle. \tag{7.8}$$

Since the left side is independent of i so is the right, and we obtain

Proposition 7.4.

$$\langle \mathrm{Ind}(D_{M_2, \alpha_2}^{\mathrm{sign}}), \chi([u])\rangle = \langle (\mathrm{Ind}(D_{M_1, \alpha_1}^{\mathrm{sign}})), \hat{h}^*\chi([u])\rangle. \tag{7.9}$$

Note that this last step is the reverse of the usual procedure. Here we have a cyclic class and we must find a cohomology class corresponding to it, while in Connes' argument one has a cohomology class and one must find a corresponding cyclic class which also extends to K-theory of the crossed-product.

8. Concluding remarks

For a finitely presented group Γ, let $\mathrm{LP}(\Gamma) \subseteq H^*(\Gamma, \mathbb{Q})$ denote the elements obtained by the above process. That is, given a finite dimensional unitary representation of Γ, we consider $\chi(HC^*(\mathcal{H}(U_n^{\delta} \ltimes U_n)_{n^2}) \subseteq H^{*-n^2}(\widetilde{M} \times_{\Gamma} U_n)$ and we integrate these classes along the fiber to obtain a subgroup of $H^*(B\Gamma)$. By the above arguments, pairing these classes with the index of signature operators gives homotopy invariant characteristic numbers. This is the unitary analog of Connes' work on Gelfand–Fuchs classes (LP stands for Lazarov–Pasternack classes).

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