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Local analytic classification of q-difference equations with $|q| = 1$

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Abstract. In this article, we establish, under convenient diophantine assumptions, a complete analytic classification of q-difference modules over the field of germs of meromorphic functions at zero, proving some analytic analogs of the results by Soibelman–Vologodsky and by Baranovsky–Ginzburg.

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Introduction

For an algebraic complex semisimple group G and for a fixed $q \in \mathbb{C}^* = \mathbb{C} \setminus \{0\}$, $|q| \neq 1$, V. Baranovsky and V. Ginzburg prove the following statement:

Theorem 1 ([3], Theorem 1.2). *There exists a natural bijection between the isomorphism classes of holomorphic principal semistable G-bundles on the elliptic curve $\mathbb{C}^*/q^{\mathbb{Z}}$ and the integral twisted conjugacy classes of the points of G that are rational over $\mathbb{C}((x))$.*

The *twisted conjugation* is an action of $G(\mathbb{C}((x)))$ on itself defined by

$$(g(x), a(x)) \mapsto {}^{g(x)}a(x) = g(qx)a(x)g(x)^{-1}.$$

An equivalence class is call *integral* when it contains a point of G rational over $\mathbb{C}[[x]]$.

As the authors point out, this result is better understood in terms of q-difference equations. If $G = \mathrm{Gl}_\nu$, then the integral twisted conjugacy classes of $G(\mathbb{C}((x)))$ correspond exactly to the isomorphism classes of formal regular singular q-difference systems. In fact, consider a q-difference equation

$$Y(qx) = B(x)Y(x) \quad \text{with } B(x) \in \mathrm{Gl}_\nu(\mathbb{C}((x))).$$

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Then this system is regular singular if there exists $G(x) \in \mathrm{Gl}_\nu(\mathbb{C}((x)))$ such that $B'(x) = G(qx)B(x)G(x)^{-1} \in \mathrm{Gl}_\nu(\mathbb{C}[[x]])$. In this case if $Y(x)$ is a solution of $Y(qx) = B(x)Y(x)$ in some q-difference algebra extending $\mathbb{C}((x))$, then $W(x) = G(x)Y(x)$ is solution of the system $W(qx) = B'(x)W(x)$.

Y. Soibelman and V. Vologodsky in [26] use an analogous approach, via q-difference equations, to understand vector bundles on non-commutative elliptic curves. Their classification, and hence the classification of analytic q-difference systems, with $|q| = 1$, is a step in Y. Manin's *Alterstraum* [14] for understanding real multiplication through non-commutative geometry. On the same topic, we point out the work of Polishchuk and al. (cf. [18], [19], [20]).

In [26], the authors identify the category of coherent modules on the elliptic curve $\mathbb{C}^*/q^{\mathbb{Z}}$, for $q \in \mathbb{C}^*$ not a root of unity, with the category of $\mathcal{O}(\mathbb{C}^*) \rtimes q^{\mathbb{Z}}$-modules of finite presentation over the ring $\mathcal{O}(\mathbb{C}^*)$ of holomorphic functions on $\mathbb{C}^*$ (cf. [26], §2, §3), both in the classic (i.e., $|q| \neq 1$) and in the non-commutative (i.e., $|q| = 1$) case. For $|q| = 1$, they study, under convenient diophantine assumptions, its Picard group and make a list of simple objects. In the second part of the paper, they focus on the classification of formal analogous objects defined over $\mathbb{C}((x))$, namely $\mathbb{C}((x))$-finite vector spaces M equipped with a semilinear invertible operator Σ_q such that $\Sigma_q(f(x)m) = f(qx)\Sigma_q(m)$ for any $f(x) \in \mathbb{C}((x))$ and any $m \in M$.

In this paper we establish, under convenient diophantine assumptions, an analytic classification of q-difference modules over the field $\mathbb{C}(\{x\})$ of germs of meromorphic functions at zero, proving some analytic analogs of the results in [26] and [3].

We fix $q \in \mathbb{C}$, $|q| = 1$, not a root of unity. Let $\mathcal{B}_q$ (resp. $\widehat{\mathcal{B}}_q$) be the category of q-difference module over $\mathbf{K} := \mathbb{C}(\{x\})$ (resp. $\widehat{\mathbf{K}} := \mathbb{C}((x))$). Let us consider a q-difference module over $\mathbf{K}$ and fix a basis $\underline{e}$ such that $\Sigma_q \underline{e} = \underline{e}B(x)$, with $B(x) \in \mathrm{Gl}_\nu(\mathbf{K})$. If it is a regular singular, or equivalently if its Newton polygon has only the zero slope (cf. Section 2.1), then we can choose a basis $\underline{f}$ of $M \otimes_{\mathbf{K}} \mathbb{C}((x))$ such that $\Sigma_q \underline{f} = \underline{f}B'$ and B' is a constant matrix in $\mathrm{Gl}_\nu(\mathbb{C})$. When $|q| \neq 1$ we do not need to extend the scalars to $\mathbb{C}((x))$ and we can find such a basis $\underline{f}$ over $\mathbf{K}$. When $|q| = 1$ this is not possible in general because of some small divisors appearing in the construction of the basis change.

The dichotomy between the "$|q| \neq 1$" and the "$|q| = 1$"case becomes even more evident when the Newton polygons have more than one slope. In fact, let (M, Σ_q) be an object of $\mathcal{B}_q$ with a Newton polygon having slopes $\mu_1 < \cdots < \mu_k$ such that the projection of $\mu_i \in \mathbb{Q}$ on the x-axis has length $r_i \in \mathbb{Z}_{>0}$, and let $(\widehat{M}, \widehat{\Sigma_q})$ be the formal object in $\widehat{\mathcal{B}}_q$ obtained by scalar extension to $\widehat{\mathbf{K}}$. If $|q| \neq 1$, the analytic isomorphism classes in $\mathcal{B}_q$ corresponding to the formal isomorphism class of $(\widehat{M}, \widehat{\Sigma_q})$ in $\widehat{\mathcal{B}}_q$ form a complex affine variety of dimension (cf. [22], [23], [21])

$$\sum_{1 \leq i < j \leq k} r_i r_j (\mu_j - \mu_i).$$

When $|q| = 1$ it may happen that the formal and analytic isomorphism classes are in one-to-one correspondence with each other, or that the situation gets much more complicated than the one described above for $|q| \neq 1$.

The object of this paper is the characterization of the largest full subcategory $\mathcal{B}_q^{\text{iso}}$ of $\mathcal{B}_q$ such that the extension of scalars "$- \otimes_{\mathbf{K}} \mathbb{C}((x))$" induces an equivalence of categories of $\mathcal{B}_q^{\text{iso}}$ onto its image in $\widehat{\mathcal{B}}_q$ (i.e., the formal and analytic isomorphism classes coincide).

The objects of $\mathcal{B}_q^{\text{iso}}$ are q-difference modules over $\mathbf{K}$ satisfying a diophantine condition (cf. Sections 2.2 and 3.4 below). They admit a decomposition associated to their Newton polygon, namely they are *direct sum* of q-difference modules, whose Newton polygon has one single slope. The indecomposable objects, i.e., those objects that cannot be written as direct sum of submodules are obtained by iterated non-trivial extension of a simple objet by itself. The simple objects are all obtained by scalar restriction to $\mathbf{K}$ from rank 1 $q^{1/n}$-difference objects over $\mathbf{K}(t)$, $x = t^n$, associated to equations of the form $y(q^{1/n}t) = \frac{\lambda}{t^\mu} y(t)$, $\lambda \in \mathbb{C}^*$ and $\mu \in \mathbb{Z}$, with $(\mu, n) = 1$.

If we call $\mathcal{B}_q^{\text{iso,f}}$ the subcategory of $\mathcal{B}_q^{\text{iso}}$ of the objects whose Newton polygon has only one slope equal to zero,[1] then we have:

Theorem 2. *The category $\mathcal{B}_q^{\text{iso}}$ is equivalent to the category of $\mathbb{Q}$-graded objects of $\mathcal{B}_q^{\text{iso,f}}$, i.e., each object of $\mathcal{B}_q^{\text{iso}}$ is a direct sum indexed on $\mathbb{Q}$ of objects of $\mathcal{B}_q^{\text{iso,f}}$ and the morphisms of q-difference modules respect the grading.*

Notice that Soibelman and Vologodsky in [26] prove exactly the same statement for the category of formal q-difference module $\widehat{\mathcal{B}}_q$. Moreover we have:

Theorem 3. *The category $\mathcal{B}_q^{\text{iso,f}}$ is equivalent to the category of finite dimensional $\mathbb{C}^*/q^{\mathbb{Z}}$-graded complex vector spaces V, endowed with nilpotent operators which preserve the grading, with the following property:*

Let $\lambda_1, \ldots, \lambda_n \in \mathbb{C}^*$ be a set of representatives of the classes of $\mathbb{C}^*/q^{\mathbb{Z}}$ corresponding to non-zero homogeneous components of V. The series $\Phi_{(q;\underline{\Lambda})}(x)$ (specified in Definition 2.5) is convergent.

Combined with the result proved in [26] that the objects of $\widehat{\mathcal{B}}_q$ of slope zero form a category which is equivalent to the category of $\mathbb{C}/q^{\mathbb{Z}}$-graded complex vector spaces equipped with a nilpotent operator respecting the grading, this gives a characterization of the image of $\mathcal{B}_q^{\text{iso,f}}$ in $\widehat{\mathcal{B}}_q$ via the scalar extension.

To prove the classification described above, one only needs to study the small divisor problem (cf. Section 1). Once this is done, the techniques used are similar to the techniques employed in q-difference equations theory for $|q| \neq 1$ (cf. the

[1]The notation $\widehat{\mathcal{B}}_q^{\text{iso,f}}$ reminds that this is a category of *fuchsian* q-difference modules.

papers of F. Marotte and Ch. Zhang [17], J. Sauloy [25], M. van der Put and M. Reversat [21], which have their roots in the work of G. D. Birkhoff and P. E. Guether [6], and C. R. Adams [1]). The statements we have cited in this introduction are actually consequences of analytic factorization properties of q-difference linear operators (cf. Section 2 below). Finally, we point out a work in progress by C. De Concini, D. Hernandez, and N. Reshetikhin applying the analytic classification of q-difference modules with $|q| \neq 1$ to the study of quantum affine algebras. The study of q-difference equations with $|q| = 1$ should help to complete the theory.

A last remark: the greatest part of the statements proved in this article remain valid also in the ultrametric case, therefore we will mainly work over an algebraically closed normed field $\mathbf{C}$, $|\ |$.

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1. A small divisor problem

Let

$$q = \exp(2i\pi\omega) \quad \text{with } \omega \in (0,1) \setminus \mathbb{Q},$$
$$\lambda = \exp(2i\pi\alpha) \quad \text{with } \alpha \in (0,1] \text{ and } \lambda \notin q^{\mathbb{Z}_{\leq 0}}.$$

We want to study the convergence of the q-hypergeometric series

$$\phi_{(q;\lambda)}(x) = \sum_{n\geq 0} \frac{x^n}{(\lambda;q)_n} \in \mathbb{C}[[x]],$$

where the *q-Pochhammer symbols* appearing in the denominator of the coefficients of $\phi_{(q;\lambda)}(x)$ are defined by

$$\begin{cases} (\lambda;q)_0 = 1, \\ (\lambda;q)_n = (1-\lambda)(1-q\lambda)\dots(1-q^{n-1}\lambda) \quad \text{for } n \geq 1. \end{cases}$$

This is a well-known problem in complex dynamics. Nevertheless we give here some proofs that already contain the problems and the ideas used in the sequel.

Proposition 1.1. *Suppose that $\lambda \notin q^{\mathbb{Z}}$. The series $\phi_{(q;\lambda)}(x)$ converges if and only if both the series $\sum_{n\geq 0} \frac{x^n}{(q;q)_n}$ and the series $\sum_{n\geq 0} \frac{x^n}{1-q^n\lambda}$ converge. Under these assumptions the radius of convergence of $\phi_{(q;\lambda)}(x)$ is at least*

$$R(\omega)\inf(1, r(\alpha)),$$

where $R(\omega)$ and $r(\alpha)$ are the radii of convergence of $\sum_{n\geq 0} \frac{x^n}{(q;q)_n}$ and $\sum_{n\geq 0} \frac{x^n}{1-q^n\lambda}$, respectively.

Remark 1.2. If $\lambda \in q^{\mathbb{Z}_{>0}}$, the series $\phi_{(q;\lambda)}(x)$ is defined and its radius of convergence is equal to $R(\omega)$. Estimates and lower bounds for $R(\omega)$ and $r(\alpha)$ are discussed in the following subsection.

The proof of the Proposition 1.1 obviously follows from the lemma below, which is a q-analogue of a special case of the Kummer transformation formula

$$\sum_{n\geq 0} \frac{x^s}{(1-\alpha)(2-\alpha)\dots(n-\alpha)} = \alpha \exp(x) \sum_{n\geq 0} \frac{(-x)^n}{n!} \frac{1}{\alpha - n},$$

used in some estimates for p-adic Liouville numbers [11], Chapter VI, Lemma 1.1.

Lemma 1.3 ([9], Lemma 20.1). *We have the following formal identity:*

$$\begin{aligned}\phi_{(q;q\lambda)}(x) &= \sum_{n\geq 0} \frac{x^n}{(1-q\lambda)\dots(1-q^n\lambda)} \\ &= (1-\lambda)\Big(\sum_{n\geq 0} \frac{x^n}{(q;q)_n}\Big)\Big(\sum_{n\geq 0} q^{\frac{n(n+1)}{2}} \frac{(-x)^n}{(q;q)_n} \frac{1}{1-q^n\lambda}\Big).\end{aligned}$$

Proof. We set $x = (1-q)t$, $[n]_q = 1 + q + \dots + q^{n-1}$ and $[0]_q = 1$, $[n]_q^! = [n]_q[n-1]_q^!$. Then we have to show the identity

$$\phi_{(q;q\lambda)}((1-q)t) = (1-\lambda)\Big(\sum_{n\geq 0} \frac{t^n}{[n]_q^!}\Big)\Big(\sum_{n\geq 0} q^{\frac{n(n+1)}{2}} \frac{(-t)^n}{[n]_q^!} \frac{1}{1-q^n\lambda}\Big).$$

Consider the q-difference operator $\sigma_q : t \mapsto qt$. One verifies directly that the series $\Phi(t) := \phi_{(q;q\lambda)}((1-q)t)$ is a solution of the q-difference operator

$$\mathcal{L} = [\sigma_q - 1] \circ [\lambda\sigma_q - ((q-1)t + 1)] = \lambda\sigma_q^2 - ((q-1)qt + 1 + \lambda)\sigma_q + (q-1)qt + 1.$$

In fact, we have

$$\mathcal{L}\Phi(t) = [\sigma_q - 1] \circ [\lambda\sigma_q - ((q-1)t + 1)]\Phi(t) = [\sigma_q - 1](\lambda - 1) = 0.$$

Since the roots of the characteristic equation[2] $\lambda T^2 - (\lambda+1)T + 1 = 0$ of $\mathcal{L}$ are exactly $\lambda^{-1} \notin q^{\mathbb{Z}}$ and 1, any solution of $\mathcal{L}y(t) = 0$ of the form $1 + \sum_{n\geq 1} a_n t^n \in \mathbb{C}[[t]]$

[2]The characteristic equation is the one whose coefficients are the constant terms of the coefficients of the q-difference operator. For a complete description of its construction and properties see Section 2.

must coincide with $\Phi(t)$. Therefore, to finish the proof of the lemma, it is enough to verify that

$$\Psi(t) = (1-\lambda)\Big(\sum_{n\geq 0}\frac{t^n}{[n]_q^!}\Big)\Big(\sum_{n\geq 0} q^{\frac{n(n+1)}{2}}\frac{(-t)^n}{[n]_q^!}\frac{1}{1-q^n\lambda}\Big)$$

is a solution of $\mathcal{L}y(t)=0$ and $\Psi(0)=1$.

Let $e_q(t)=\sum_{n\geq 0}\frac{t^n}{[n]_q^!}$. Then $e_q(t)$ satisfies the q-difference equation

$$e_q(qt) = ((q-1)t+1)e_q(t),$$

hence

$$\begin{aligned}\mathcal{L}\circ e_q(t) &= [\sigma_q - 1]\circ e_q(qt)\circ[\lambda\sigma_q - 1]\\ &= e_q(t)((q-1)t+1)[((q-1)qt+1)\sigma_q - 1]\circ[\lambda\sigma_q - 1]\\ &= (*)\big[((q-1)qt+1)\sigma_q - 1\big]\circ[\lambda\sigma_q - 1],\end{aligned}$$

where $(*)$ denotes a coefficient in $\mathbb{C}(t)$ not depending on σ_q.

Consider the series

$$E_q(t) = \sum_{n\geq 0} q^{\frac{n(n+1)}{2}}\frac{t^n}{[n]_q^!},$$

which satisfies $(1-(q-1)t)E_q(qt) = E_q(t)$, and the series

$$g_\lambda(t) = \sum_{n\geq 0} q^{\frac{n(n+1)}{2}}\frac{(-t)^n}{[n]_q^!}\frac{1}{1-q^n\lambda}.$$

Then

$$\begin{aligned}\mathcal{L}\circ e_q(t)g_\lambda(t) &= (*)[((q-1)qt+1)\sigma_q - 1]\circ[\lambda\sigma_q - 1]g_\lambda(t)\\ &= (*)[((q-1)qt+1)\sigma_q - 1]E_q(-qt)\\ &= (*)[((q-1)qt+1)E_q(-q^2t) - E_q(-qt)]\\ &= 0.\end{aligned}$$

It is enough to observe that $e_q(0)g_\lambda(0) = \frac{1}{1-\lambda}$ to conclude that the series $\Psi(t) = (1-\lambda)e_q(t)g_\lambda(t)$ coincides with $\Phi(t)$. □

Remark 1.4. Let $(C,|\ |)$ be a field equipped with an ultrametric norm and let $q\in C$, with $|q|=1$ and q not a root of unity. Then the formal equivalence in Lemma 1.7 is still true. The series $\sum_{n\geq 0}\frac{x^n}{(q;q)_n}$ is convergent for any $q\in C$ such that $|q|=1$ (cf. [2], §2). However the series $\sum_{n\geq 0}\frac{x^n}{q^n-\lambda}$ is not always convergent. If $\left|\frac{\lambda-1}{q-1}\right|<1$ then its radius of convergence coincides with the radius of convergence of the series $\sum_{n\geq 0}\frac{x^n}{n-\alpha}$, where $\alpha=\frac{\log\lambda}{\log q}$ (cf. [9], §19, [11], Ch. VI); otherwise it converges for $|x|<1$.

1.1. Some remarks on Proposition 1.1. Let us make some comments on the convergence of the series $\sum_{n\geq 0} \frac{x^n}{(q;q)_n}$ and $\sum_{n\geq 0} \frac{x^n}{1-q^n\lambda}$. A first contribution to the study of convergence of the series $\sum_{n\geq 0} \frac{x^n}{(q;q)_n}$ can be found in [12]. The subject has been considered in detail in [13].

Definition 1.5 (cf. for instance [16], §4.4). Let $\{\frac{p_n}{q_n}\}_{n\geq 0}$ be the convergents of ω occurring in its continued fraction expansion. Then the *Brjuno function* $\mathcal{B}$ of ω is defined by

$$\mathcal{B}(\omega) = \sum_{n\geq 0} \frac{\log q_{n+1}}{q_n},$$

and ω is a *Brjuno number* if $\mathcal{B}(\omega) < \infty$.

Now we are ready to recall the well-known results by [27], [7], Theorem 2.1, [16], Theorem 5.1.

Theorem 1.6 (Yoccoz's lower bound). *If ω is a Brjuno number then the series $\sum_{n\geq 0} \frac{x^n}{(q;q)_n}$ converges.*

Moreover its radius of convergence is bounded from below by $e^{-B(\omega)-C_0}$, where $C_0 > 0$ is an universal constant (i.e., independent of ω).

Sketch of the proof. Suppose that ω is a Brjuno number. Then our statement is much easier than the results mentioned above, and it is actually an immediate consequence of Davie's lemma (cf. [16], Lemma 5.6 (c), or [7], Lemma B.4, 3)). □

We set $\|x\|_{\mathbb{Z}} = \inf_{k\in\mathbb{Z}} |x + k|$. Then, as far as the series $\sum_{n\geq 0} \frac{x^n}{1-q^n\lambda}$ is concerned, we have:

Lemma 1.7. *The following assertions are equivalent:*

(1) *The series $\sum_{n\geq 0} \frac{x^n}{1-q^n\lambda}$ is convergent.*

(2) $\limsup_{n\to\infty} \frac{\log|1-\lambda q^n|^{-1}}{n} < +\infty$.

(3) $\liminf_{n\to\infty} \|n\omega + \alpha\|_{\mathbb{Z}}^{1/n} > 0$.

Proof. The equivalence between (1) and (2) is straightforward. Let us prove the equivalence "(1) $\Longleftrightarrow$ (3)" (using a really classical argument).

Notice that for any $x \in [0, 1/4]$ we have $f(x) := \sin(\pi x) - x \geq 0$, in fact $f(0) = 0$ and $f'(x) = \pi\cos(\pi x) - 1 \geq 0$. Therefore we conclude that the following inequality holds for any $x \in [0, 1/2]$:

$$\sin(\pi x) > \min(x, 1/4).$$

This implies that

$$\begin{aligned}|q^n\lambda - 1| &= |\exp(2i\pi(n\omega+\alpha)) - 1| \\ &= 2\sin(\pi\|n\omega+\alpha\|_{\mathbb{Z}}) \in [\min(2\|n\omega+\alpha\|_{\mathbb{Z}}, 1/2), 2\pi\|n\omega+\alpha\|_{\mathbb{Z}}[\end{aligned}$$

and ends the proof. □

Remark 1.8. A basic notion in complex dynamics is that a number α is diophantine with respect to another number, say ω. If α is diophantine with respect to ω, then α and ω have the properties of the previous lemma. It is known that for a given $\omega \in [0,1] \setminus \mathbb{Q}$, the complex numbers $\exp(2i\pi\alpha)$ such that α is diophantine with respect to ω form a subset of the unit circle of full Lebesgue measure; cf. [4], §1.3.

1.2. A corollary. Let

$$q = \exp(2i\pi\omega) \text{ with } \omega \in (0,1) \setminus \mathbb{Q},$$
$$m \in \mathbb{Z}_{>0} \text{ and } \lambda_i = \exp(2i\pi\alpha_i) \text{ for } i = 1,\dots,m \text{ with } \alpha_i \in (0,1] \text{ and } \lambda_i \notin q^{\mathbb{Z}}.$$

For further reference we state the corollary below, which is an immediate consequence of Proposition 1.1:

Corollary 1.9. *Let $\Lambda = (\lambda_1,\dots,\lambda_m)$. The series*

$$\phi_{(q;\Lambda)}(x) = \sum_{n\geq 0} \frac{x^n}{(\lambda_1;q)_n \dots (\lambda_m;q)_n} \in \mathbb{C}[[x]] \tag{1}$$

converges if and only if both the series $\sum_{n\geq 0} \frac{x^n}{(q;q)_n}$ and the series $\sum_{n\geq 0} \frac{x^n}{1-q^n\lambda_i}$ converge for $i = 1,\dots,m$. Under these assumptions the radius of convergence of $\phi_{(q;\Lambda)}(x)$ is at least

$$R(\omega)^m \cdot \prod_{i=1}^{m} \inf(1, r(\alpha_i)).$$

2. Analytic factorization of q-difference operators

Notation 2.1. Let $(\mathbf{C}, |\ |)$ be either the field of complex numbers with the usual norm or an algebraically closed field with an ultrametric norm. We fix $q \in \mathbf{C}$ such that $|q| = 1$ and q is not a root of unity, and a set of elements $q^{1/n} \in \mathbf{C}$ such that $(q^{1/n})^n = q$. If $\mathbf{C} = \mathbb{C}$ then let $\omega \in (0,1] \setminus \mathbb{Q}$ be such that $q = \exp(2i\pi\omega)$.

We suppose that the series $\sum_{n\geq 0} \frac{x^n}{(q;q)_n}$ is convergent, which happens for instance if ω is a Brjuno number.

The contents of this section is largely inspired by [25], where the author proves an analytic classification result for q-difference equations with $|q| \neq 1$: the major difference is the small divisor problem that the assumption $|q| = 1$ introduces. Of course, once the small divisor problem is solved, the techniques are the same. For this reason some proofs will be only sketched.

2.1. The Newton polygon. We consider a q-difference operator

$$\mathcal{L} = \sum_{i=0}^{\nu} a_i(x)\sigma_q^i \in \mathbf{C}\{x\}[\sigma_q],$$

i.e., an element of the skew ring $\mathbf{C}\{x\}[\sigma_q]$, where $\mathbf{C}\{x\}$ is the $\mathbf{C}$-algebra of germs of analytic functions at zero and $\sigma_q f(x) = f(qx)\sigma_q$. The associated q-difference equations is

$$\mathcal{L}y(x) = a_\nu(x)y(q^\nu x) + a_{\nu-1}(x)y(q^{\nu-1}x) + \cdots + a_0(x)y(x) = 0\,.$$

We suppose that $\alpha_\nu(x) \neq 0$ and call ν the *order* of $\mathcal{L}$ (or of $\mathcal{L}y = 0$).

Definition 2.2. The *Newton polygon* $\mathrm{NP}(\mathcal{L})$ of the equation $\mathcal{L}y = 0$ (or of the operator $\mathcal{L}$) is the convex envelop in $\mathbb{R}^2$ of the set

$$\{(i,k) \in \mathbb{Z} \times \mathbb{R} \mid i = 0,\dots,\nu;\ a_i(x) \neq 0,\ k \geq \mathrm{ord}_x\, a_i(x)\},$$

where $\mathrm{ord}_x\, a_i(x) \geq 0$ denotes the order of zero of $a_i(x)$ at $x = 0$.

Notice that the polygon $\mathrm{NP}(\mathcal{L})$ has a finite number of finite slopes, which are all rational and possibly negative, and two infinite vertical sides. We will denote by $\mu_1,\dots,\mu_k$ the finite slopes of $\mathrm{NP}(\mathcal{L})$ (or, briefly of $\mathcal{L}$), ordered so that $\mu_1 < \mu_2 < \cdots < \mu_k$ (i.e., from left to right), and by $r_1,\dots,r_k$ the length of their respective projections on the x-axis. Notice that $\mu_i r_i \in \mathbb{Z}$ for any $i = 1,\dots,k$.

We can always – and actually will – assume that the boundary of the Newton polygon of $\mathcal{L}$ and the x-axis intersect only in one point or in a segment by clearing some common powers of x in the coefficients of $\mathcal{L}$. With this convention, the Newton polygon is completely determined by the set $\{(\mu_1,r_1),\dots,(\mu_k,r_k)\} \in \mathbb{Q} \times \mathbb{Z}_{>0}$, therefore we will identify the two data.

Definition 2.3. A q-difference operator whose Newton polygon has only one slope (equal to μ) is called *pure* (*of slope* μ).[3]

[3] Some authors call pure the objects that are the direct sum of objects having only one slope. The objects that we call pure are then called isoclinic.

Remark 2.4. All the properties of Newton polygons of q-difference equations listed in [25], §1.1, are formal and therefore independent of the field $\mathbf{C}$ and of the norm of q: they can be rewritten, with exactly the same proof, in our case. We recall, in particular, two properties of the Newton polygon that we will use in the sequel (cf. [25], §1.1.5):

- Let θ be a solution in some formal extension of $\mathbf{C}(\{x\}) = \mathrm{Frac}(\mathbf{C}\{x\})$ of the q-difference equation $y(qx) = xy(x)$. The twisted conjugate operator $x^C\theta^\mu\mathcal{L}\theta^{-\mu} \in \mathbf{C}\{x\}[\sigma_q]$, where C is a convenient non-negative integer, is associated to the q-difference equation[4]

$$\begin{aligned} &a_\nu(x)q^{-\mu\frac{\nu(\nu+1)}{2}}x^{C-\mu\nu}y(q^\nu x)\\ &\quad + a_{\nu-1}(x)q^{-\mu\frac{\nu(\nu-1)}{2}}x^{C-\mu(\nu-1)}y(q^{\nu-1}x) + \cdots + x^C a_0(x)y(x) = 0, \end{aligned} \tag{2}$$

  and has Newton polygon $\{(\mu_1-\mu, r_1), \dots, (\mu_k-\mu, r_k)\}$.

- If $e_{q,c}(x)$ is a solution of $y(qx) = cy(x)$ with $\in \mathbf{C}^*$, then the twisted operator $e_{q,c}(x)^{-1}\mathcal{L}e_{q,c}(x)$ has the same Newton polygon as $\mathcal{L}$, while all the zeros of the polynomial $\sum_{i=0}^{\nu} a_i(0)T^i$ are multiplied by c.

2.2. Admissible *q*-difference operators. Suppose that 0 is a slope of $\mathrm{NP}(\mathcal{L})$. We call the polynomial

$$a_\nu(0)T^\nu + a_{\nu-1}(0)T^{\nu-1} + \cdots + a_0(0) = 0$$

characteristic polynomial of the zero slope. The characteristic polynomial of a slope $\mu \in \mathbb{Z}$ is the characteristic polynomial of the zero slope of the q-difference operator $x^C\theta^\mu\mathcal{L}\theta^{-\mu}$ (cf. equation (2)). In the general case, when $\mu \in \mathbb{Q}\setminus\mathbb{Z}$, we reduce to the previous assumption by performing a ramification. Namely, for a convenient $n \in \mathbb{Z}_{>0}$ we set $t = x^{1/n}$. With this variable change, the operator $\mathcal{L}$ becomes $\sum a_i(t^n)\sigma^i_{q^{1/n}}$. Notice that the characteristic polynomial does not depend on the choice of n.

Finally, we call the non-zero roots of the characteristic polynomial of the slope μ the *exponents of the slope* μ. The cardinality of the set $\mathrm{Exp}(\mathcal{L}, \mu)$ of the exponents of the slope μ, counted with multiplicities, is equal to the length of the projection of μ on the x-axis.

Definition 2.5. Let $(\lambda_1, \dots, \lambda_r)$ be the exponents of the slope μ of $\mathcal{L}$ and let

$$\underline{\Lambda} = \{\lambda_i\lambda_j^{-1} \mid i, j = 1, \dots, r;\ \lambda_i\lambda_j^{-1} \notin q^{\mathbb{Z}_{\le 0}}\}.$$

We say that a slope $\mu \in \mathbb{Z}$ of $\mathcal{L}$ is *admissible* if the series $\phi_{(q;\underline{\Lambda})}(x)$ (cf. equation (1)) is convergent and that a slope $\mu \in \mathbb{Q}$ is *almost admissible* if it becomes admissible in $\mathbf{C}\{x^{1/n}\}[\sigma_{q^{1/n}}]$ for a convenient $n \in \mathbb{Z}_{>0}$.

[4]Notice that there is no need to determine the function θ.

A q-difference operator is said to be *admissible* (resp. *almost admissible*) if all its slopes are admissible (resp. *almost admissible*).

Remark 2.6. A rank 1 q-difference equation is admissible as long as the series $\sum_{n\geq 0} \frac{x^n}{(q;q)_n}$ is convergent.

2.3. Analytic factorization of admissible q-difference operators. The main result of this subsection is the analytic factorization of admissible q-difference operators. The analogous result in the case $|q| \neq 1$ is well known (cf. [17], [25], §1.2, or, for a more detailed exposition, [24], §1.2). The germs of those works are already in [6], where the authors establish a canonical form for the solution of analytic q-difference systems.

Theorem 2.7. *Suppose that the q-difference operator $\mathcal{L}$ is admissible, with Newton polygon $\{(\mu_1, r_1), \dots, (\mu_k, r_k)\}$. Then for any permutation ϖ of the set $\{1, \dots, k\}$ there exists a factorization of $\mathcal{L}$,*

$$\mathcal{L} = \mathcal{L}_{\varpi,1} \circ \mathcal{L}_{\varpi,2} \circ \cdots \circ \mathcal{L}_{\varpi,k},$$

such that $\mathcal{L}_{\varpi,i} \in \mathbf{C}\{x\}[\sigma_q]$ is admissible and pure of slope $\mu_{\varpi(i)}$ and order $r_{\varpi(i)}$.

Remark 2.8. Given the permutation ϖ, the q-difference operator $\mathcal{L}_{\varpi,i}$ is uniquely determined modulo a factor in $\mathbf{C}\{x\}$.

Exactly the same statement holds for almost admissible q-difference operators (cf. Theorem 3.16 below).

Theorem 2.7 follows from the recursive application of the following statement.

Proposition 2.9. *Let $\mu \in \mathbb{Z}$ be an admissible slope of the Newton polygon of $\mathcal{L}$ and let r be the length of its projection on the x-axis. Then the q-difference operator $\mathcal{L}$ admits a factorization $\mathcal{L} = \widetilde{\mathcal{L}} \circ \mathcal{L}_\mu$ such that*

(1) *the operator $\widetilde{\mathcal{L}}$ is in $\mathbf{C}\{x\}[\sigma_q]$ and $\mathrm{NP}(\widetilde{\mathcal{L}}) = \mathrm{NP}(\mathcal{L}) \setminus \{(\mu, r)\}$;*

(2) *the operator $\mathcal{L}_\mu$ has the form*

$$\mathcal{L}_\mu = (x^\mu \sigma_q - \lambda_r) h_r(x) \circ (x^\mu \sigma_q - \lambda_{r-1}) h_{r-1}(x) \circ \cdots \circ (x^\mu \sigma_q - \lambda_1) h_1(x),$$

where $\lambda_1, \dots, \lambda_r \in \mathbf{C}$ are the exponents of the slope μ, ordered so that if $\frac{\lambda_i}{\lambda_j} \in q^{\mathbb{Z}_{>0}}$ then $i < j$, and $h_1(x), \dots, h_r(x) \in 1 + x\mathbf{C}\{x\}$.

Moreover, if $\mathcal{L}$ is admissible (resp. almost admissible), then the operator $\widetilde{\mathcal{L}}$ is also admissible (resp. almost admissible).

Proposition 2.9 follows from an iterated application of the following lemma:

Lemma 2.10. *Let* $(\mu, r) \in \mathrm{NP}(\mathcal{L}) = \{(\mu_1, r_1), \dots, (\mu_k, r_k)\}$ *be an integral slope of* $\mathcal{L}$ *with exponents* $(\lambda_1, \dots, \lambda_r)$. *Fix an exponent* λ *of* μ *such that*

(1) $q^n\lambda$ *is not an exponent of the same slope for any* $n > 0$;

(2) *the series* $\phi_{(q;(\frac{\lambda_1}{\lambda},\dots,\frac{\lambda_r}{\lambda}))}(x)$ *is convergent.*

Then there exists a unique $h(x) \in 1 + x\mathbf{C}\{x\}$ *such that* $\mathcal{L} = \widetilde{\mathcal{L}} \circ (x^\mu \sigma_q - \lambda)h(x)$ *for some* $\widetilde{\mathcal{L}} \in \mathbf{C}\{x\}[\sigma_q]$. *Moreover let* $\iota = 1, \dots, k$ *such that* $\mu_\iota = \mu$.

If $r_\iota = 1$, *then* $\mathrm{NP}(\widetilde{\mathcal{L}}) = \{(\mu_1, r_1), \dots, (\mu_{\iota-1}, r_{\iota-1}), (\mu_{\iota+1}, r_{\iota+1}), \dots, (\mu_k, r_k)\}$.

If $r_\iota > 1$, *then we have* $\mathrm{NP}(\widetilde{\mathcal{L}}) = \{(\mu_1, r_1), \dots, (\mu_\iota, r_\iota - 1), \dots, (\mu_k, r_k)\}$ *and* $\mathrm{Exp}(\widetilde{\mathcal{L}}, \mu_\iota) = \mathrm{Exp}(\mathcal{L}, \mu_\iota) \setminus \{\lambda\}$.

Proof. It is enough to prove the lemma for $\mu = 0$ and $\lambda = 1$ (cf. Remark 2.4). Write $y(x) = \sum_{n\geq 0} y_n x^n$, with $y_0 = 1$, and $a_i(x) = \sum_{n\geq 0} a_{i,n}x^n$. Then we obtain by direct computation that $\mathcal{L}y(x) = 0$ if and only if for any $n \geq 1$ we have

$$F_0(q^n)y_n = -\sum_{l=0}^{n-1} F_{n-l}(q^l)y_l,$$

where $F_l(T) = \sum_{i=0}^{\nu} a_{i,l}T^i$. Observe that assumption (1) is equivalent to the property: $F_0(q^n) \neq 0$ for any $n \in \mathbb{Z}_{>0}$.

The convergence of the coefficients $a_i(x)$ of $\mathcal{L}$ implies the existence of two constants $A, B > 0$ such that $|F_{n-l}(q^l)| \leq AB^{n-l}$ for any $n \geq 0$ and any $l = 0, \dots, n-1$. We set

$$s_n = F_0(1)F_0(q)\dots F_0(q^n)y_n.$$

Then

$$|s_n| \leq \left|\sum_{l=0}^{n-1} s_l F_0(q^{l+1})\dots F_0(q^{n-1})F_{n-l}(q^l)\right| \leq A^nB^n \sum_{l=0}^{n-1} \frac{|s_l|}{(AB)^l},$$

and therefore

$$|t_n| \leq \sum_{l=0}^{n-1} |t_l| \quad \text{with } t_l = \frac{s_l}{(AB)^l}.$$

If $|t_l| < CD^l$ for any $l = 0, \dots, n-1$ with $D > 1$, then $|t_n| \leq C\sum_{l=0}^{n-1} D^l \leq CD^n(D-1)^{-1} \leq CD^n$. Hence $|t_n| \leq CD^n$ for any $n \geq 1$ and so $|s_n| \leq C(ABD)^n$. Hypothesis (2) assures that the series $\sum_{n\geq 1} \frac{x^n}{F_0(1)\dots F_0(q^n)}$ is convergent, which implies that $y(x)$ is convergent. We conclude setting $h(x) = y(x)^{-1}$.

For the assertion on the Newton polygon see [25]. □

For further reference we point out that we have actually proved the following two corollaries:

Corollary 2.11. *Under the hypothesis of Lemma* 2.10*, suppose that* $\mathcal{L}$ *has a right factor of the form* $(\sigma_q^{\mu} - \lambda) \circ h(x)$*, with* $\mu \in \mathbb{Q}$*,* $\lambda \in \mathbf{C}^*$ *and* $h(x) \in \mathbf{C}[[x]]$*. Then* $h(x)$ *is convergent.*

Remark 2.12. Corollary 2.11 above generalizes [5], Theorem 6.1, where J.-P. Bézivin proves that a formal solution of an analytic q-difference operator satisfying some diophantine assumptions is always convergent.

Corollary 2.13. *Any almost admissible* q*-difference operator* $\mathcal{L}$ *admits an analytic factorization in* $\mathbf{C}\{x^{1/n}\}[\sigma_q]$*, with* $\sigma_q x^{1/n} = q^{1/n}x^{1/n}$*, for a convenient* $n \in \mathbb{Z}_{>0}$*.*

The irreducible factors of $\mathcal{L}$ *in* $\mathbf{C}\{x^{1/n}\}[\sigma_q]$ *have the form* $(x^{\mu/n}\sigma_q - \lambda)h(x^{1/n})$*, with* $\mu \in \mathbb{Z}$*,* $\lambda \in \mathbf{C}^*$ *and* $h(x^{1/n}) \in 1 + x^{1/n}\mathbf{C}\{x^{1/n}\}$*.*

The following example shows the importance of considering admissible operators.

Example 2.14. The series $\Phi(x) = \Phi_{(q;q\lambda)}((1-q)x)$ studied in Proposition 1.1 is solution of the q-difference operator $\mathcal{L} = (\sigma_q - 1) \circ [\lambda\sigma_q - ((q-1)x + 1)]$. This operator is already factored.

Suppose that $\lambda \notin q^{\mathbb{Z}_{<0}}$. If the series $\Phi(x)$ is convergent, i.e., if $\mathcal{L}$ is admissible, the operator $(\sigma_q - 1) \circ \Phi(x)^{-1}$ is a right factor of $\mathcal{L}$, as we could have deduced from Lemma 2.10. We conclude that if $\Phi(x)$ is not convergent, the operator $\mathcal{L}$ cannot be factored "starting with the exponents 1".

2.4. A digression on formal factorization of q-difference operators. If we drop the diophantine assumption of admissibility and consider an operator $\mathcal{L} \in \mathbf{C}[[x]][\sigma_q]$, the notions of Newton polygon and exponent still make sense. The following result is well known (cf. [26], [25]) and can be proved reasoning as in the previous section.

Theorem 2.15. *Suppose that the* q*-difference operator* $\mathcal{L} \in \mathbf{C}[[x]][\sigma_q]$ *has Newton polygon* $\{(\mu_1, r_1), \dots, (\mu_k, r_k)\}$ *with integral slopes. Then for any permutation* ϖ *of the set* $\{1, \dots, k\}$ *there exists a factorization of* $\mathcal{L}$*,*

$$\mathcal{L} = \mathcal{L}_{\varpi,1} \circ \mathcal{L}_{\varpi,2} \circ \cdots \circ \mathcal{L}_{\varpi,k},$$

such that $\mathcal{L}_{\varpi,i} \in \mathbf{C}[[x]][\sigma_q]$ *is pure of slope* $\mu_{\varpi(i)}$ *and order* $r_{\varpi(i)}$*. Any* $\mathcal{L}_{\varpi,i}$ *admits a factorization of the form*

$$\begin{aligned}\mathcal{L}_{\varpi,i} = {} & (x^{\mu_{\varpi(i)}}\sigma_q - \lambda_{r_{\varpi(i)}})h_{r_{\varpi(i)}}(x) \circ (x^{\mu_{\varpi(i)}}\sigma_q - \lambda_{r-1})h_{r-1}(x) \circ \cdots \\ & \cdots \circ (x^{\mu_{\varpi(i)}}\sigma_q - \lambda_1)h_1(x),\end{aligned}$$

where $\mathrm{Exp}(\mathcal{L}, \mu_{\varpi(i)}) = (\lambda_1, \dots, \lambda_{r_{\varpi(i)}})$ *are the exponents of the slope* $\mu_{\varpi(i)}$*, ordered so that if* $\frac{\lambda_i}{\lambda_j} \in q^{\mathbb{Z}_{>0}}$ *then* $i < j$*, and* $h_1(x), \dots, h_{r_{\varpi(i)}}(x) \in 1 + x\mathbf{C}[[x]]$*.*

3. Analytic classification of q-difference modules

Let $\mathbf{K} = \mathbf{C}(\{x\})$ be the field of germs of meromorphic function at 0, i.e., the field of fractions of $\mathbf{C}\{x\}$. In the following we will denote by $\widehat{\mathbf{K}} = \mathbf{C}((x))$ the field of Laurent series, and by $\mathbf{K}_n = \mathbf{K}(x^{1/n})$ (resp. $\widehat{\mathbf{K}}_n = \widehat{\mathbf{K}}(x^{1/n})$) the finite extension of $\mathbf{K}$ (resp. $\widehat{\mathbf{K}}$) or degree n, with its natural $q^{1/n}$-difference structure. *We remind that we are assuming throughout the paper that the series $\sum_{n\geq 0} \frac{x^n}{(q;q)_n}$ is convergent.*

3.1. Generalities on q-difference modules.

We recall some generalities on q-difference modules (for a more detailed exposition cf. for instance [8], Part I, [25] or [10]).

Let F be a q-difference field over $\mathbf{C}$, i.e., a field $F/\mathbf{C}$ of functions with an action of σ_q.

Definition 3.1. A *q-difference modules $\mathcal{M} = (M, \Sigma_q)$ over F (of rank ν)* is a finite F-vector space M of dimension ν equipped with a σ_q-linear bijective endomorphism Σ_q, i.e., a $\mathbf{C}$-linear isomorphism such that $\Sigma_q(fm) = \sigma_q(f)\Sigma_q(m)$, for any $f \in F$ and any $m \in M$.

A *morphism of q-difference modules* $\varphi\colon (M, \Sigma_q^M) \to (N, \Sigma_q^N)$ is a $\mathbf{C}$-linear morphism $M \to N$, commuting with respect to the action of Σ_q^M and Σ_q^N, i.e., $\Sigma_q^N \circ \varphi = \varphi \circ \Sigma_q^M$.

If G is a q-difference field extending F (i.e., G/F is a field extension and the action of σ_q on G extends the one on F), the module $\mathcal{M}_G = (M \otimes_F G, \Sigma_q \otimes \sigma_q)$ is naturally a q-difference module over G.

If $F_n, n \in \mathbb{Z}_{>1}$, is a $q^{1/n}$-difference field containing F and such that ${\sigma_{q^{1/n}}}_{|F} = \sigma_q$ (for instance, think of $\mathbf{K}$ and $\mathbf{K}_n$), to any q-difference modules $\mathcal{M} = (M, \Sigma_q)$ over F we can associate the $q^{1/n}$-difference module $\mathcal{M}_{F_n} = (M \otimes_F F_n, \Sigma_q \otimes \sigma_{q^{1/n}})$ over F_n.

For other algebraic constructions (tensor product, internal Hom, ...) we refer to [8] or [25].

Remark 3.2 (The cyclic vector lemma). The cyclic vector lemma says that a q-difference module $\mathcal{M}$ over F of rank ν contains a cyclic element $m \in M$, i.e., an element such that $m, \Sigma_q m, \dots, \Sigma_q^{\nu-1} m$ is an F-basis of M. This is equivalent to saying that there exists a q-difference operator $\mathcal{L} \in F[\sigma_q]$ of order ν such that we have an isomorphism of q-difference modules

$$M \cong \frac{F[\sigma_q, \sigma_q^{-1}]}{F[\sigma_q, \sigma_q^{-1}]\mathcal{L}}.$$

We will call $\mathcal{L}$ a q-difference operator associated to $\mathcal{M}$, and $\mathcal{M}$ the q-difference module associated to $\mathcal{L}$.

Example 3.3 (Rank 1 q-difference modules[5]). Let $\mu \in \mathbb{Z}$, $\lambda \in \mathbf{C}^*$ and $h(x) \in \mathbf{K}$ (resp. $h(x) \in \widehat{\mathbf{K}}$). Consider the rank 1 q-difference module $\mathcal{M}_{\mu,\lambda} = (M_{\mu,\lambda}, \Sigma_q)$ over $\mathbf{K}$ (resp. $\widehat{\mathbf{K}}$) associated to the operator $(x^\mu \sigma_q - \lambda) \circ h(x) = h(qx)x^\mu \sigma_q - h(x)\lambda$. There exists a basis f of $M_{\mu,\lambda}$ such that $\Sigma_q f = \frac{h(x)}{h(qx)} \frac{\lambda}{x^\mu} f$. If one considers the basis $e = h(x) f$, then $\Sigma_q e = \frac{\lambda}{x^\mu} e$.

A straightforward calculation shows that $\mathcal{M}_{\mu,\lambda}$ is isomorphic, as a q-difference module, to $\mathcal{M}_{\mu',\lambda'}$ if and only if $\mu = \mu'$ and $\frac{\lambda}{\lambda'} \in q^{\mathbb{Z}}$. Moreover, we proved in the previous section that a q-difference operator $\sigma_q - a(x)$ with $a(x) \in \widehat{\mathbf{K}}$ can be always be written in the form $\sigma_q - \frac{\lambda}{x^\mu} \frac{h(x)}{h(qx)}$ for some $h(x) \in \widehat{\mathbf{K}}$. We also know that if q is such that $\sum_{n \geq 0} \frac{x^n}{(q;q)_n}$ converges and if $a(x) \in \mathbf{K}$, then $h(x)$ is a convergent series.

The remark and the example above, together with the results of the previous section, imply that we can attach to a q-difference modules a Newton polygon by choosing a cyclic vector, and that the Newton polygon of a q-difference modules is well defined (cf. [25]). Moreover the classes modulo $q^{\mathbb{Z}}$ of the exponents of each slope are independent of the choice of the cyclic vector (cf. [26], Theorems 3.12 and 3.14, and [25]). Both the Newton polygon and the classes modulo $q^{\mathbb{Z}}$ of the exponents are an invariant of the formal isomorphism class.

3.2. Main result. Let us call $\mathcal{B}_q$ (resp. $\hat{\mathcal{B}}_q$) the category of q-difference module over $\mathbf{K}$ (resp. $\widehat{\mathbf{K}}$). We will use the adjective analytic (resp. formal) to refer to objects, morphisms, isomorphism classes, etc. of $\mathcal{B}_q$ (resp. $\hat{\mathcal{B}}_q$).

We are concerned with the problem of finding the largest full subcategory $\mathcal{B}_q^{\mathrm{iso}}$ of $\mathcal{B}_q$ defined by the following property:

An object $\mathcal{M}$ of $\mathcal{B}_q$ belongs to $\mathcal{B}_q^{\mathrm{iso}}$ if any object $\mathcal{N}$ in $\mathcal{B}_q$ such that $\mathcal{N}_{\widehat{\mathbf{K}}}$ is isomorphic to $\mathcal{M}_{\widehat{\mathbf{K}}}$ in $\widehat{\mathcal{B}}_q$ is already isomorphic to $\mathcal{M}$ in $\mathcal{B}_q$.

This means that restriction of the functor

$$- \otimes_{\mathbf{K}} \widehat{\mathbf{K}} \colon \mathcal{B}_q \to \widehat{\mathcal{B}}_q, \quad \mathcal{M} \mapsto \mathcal{M}_{\widehat{\mathbf{K}}},$$

to $\mathcal{B}_q^{\mathrm{iso}}$ is an equivalence of category between $\mathcal{B}_q^{\mathrm{iso}}$ and its image. We will come back in Section 4 to the characterization of $\mathcal{B}_q^{\mathrm{iso}} \otimes_{\mathbf{K}} \widehat{\mathbf{K}}$ inside $\widehat{\mathcal{B}}_q$. A counterexample of the fact that the functor $- \otimes_{\mathbf{K}} \widehat{\mathbf{K}}$ is not an equivalence of categories in general is considered in Section 3.3.

The category $\mathcal{B}_q^{\mathrm{iso}}$ is related to the notion of admissibility introduced in the previous section.

[5]For more details on the rank 1 case see [26], Proposition 3.6, where the Picard group of q-difference modules over $\mathcal{O}(\mathbb{C}^*)$ satisfying a convenient diophantine assumption is studied.

Definition 3.4. We say that a q-difference module $\mathcal{M}$ over $\mathbf{K}$ is *admissible* (*resp. almost admissible; resp. pure* (*of slope* μ)) if there exists an operator $\mathcal{L} \in \mathbf{C}\{x\}[\sigma_q]$ such that $M \cong \mathbf{K}[\sigma_q]/(\mathcal{L})$ and that $\mathcal{L}$ is admissible (resp. almost admissible; resp. pure (of slope μ)).

Remark 3.5. The considerations in the previous section imply that the notion of (almost) admissible q-difference module is well defined and invariant up to isomorphism.

Our main result is:

Theorem 3.6. *The category $\mathcal{B}_q^{\mathrm{iso}}$ is the full subcategory of $\mathcal{B}_q$ whose objects are almost admissible q-difference modules.*

We introduce some notation that will be useful in the proof of Theorem 3.6. We will denote $q\text{-Diff}_{\mathbf{K}}^{\mathrm{a}}$ (resp. $q\text{-Diff}_{\mathbf{K}}^{\mathrm{aa}}$) the category of admissible (resp. almost admissible) q-difference modules over $\mathbf{K}$, whose objects are the admissible (resp. almost admissible) q-difference modules over $\mathbf{K}$ and whose morphisms are the morphisms of q-difference modules over $\mathbf{K}$.

Remark 3.7. We know that $\mathcal{B}_q$ and $\widehat{\mathcal{B}}_q$ are abelian categories. Therefore, kernel and cokernel of morphisms in $q\text{-Diff}_{\mathbf{K}}^{\mathrm{a}}$ (resp. $q\text{-Diff}_{\mathbf{K}}^{\mathrm{aa}}$) are q-difference modules over $\mathbf{K}$. To prove that they are objects of $q\text{-Diff}_{\mathbf{K}}^{\mathrm{a}}$ (resp. $q\text{-Diff}_{\mathbf{K}}^{\mathrm{aa}}$) we have only to point out that the operator associated to a sub-q-difference module (resp. a quotient module) is a right (resp. left) factor of a convenient operator associated to the module itself. In fact the slopes and the classes modulo $q^{\mathbb{Z}}$ of the exponents associated to each slope are invariants of q-difference modules.

The proof of Theorem 3.6 consists in proving that $\mathcal{B}_q^{\mathrm{iso}} = q\text{-Diff}_{\mathbf{K}}^{\mathrm{aa}}$, which requires the following steps: first we will make a list of simple and indecomposable objects of $q\text{-Diff}_{\mathbf{K}}^{\mathrm{aa}}$; then we prove a structure theorem for almost admissible q-difference modules. We deduce that the formal isomorphism class of an object of $\mathcal{B}_q$ correspond to more than one analytic isomorphism class if and only if the slopes of the Newton polygon are not admissible, which means that $\mathcal{B}_q^{\mathrm{iso}}$ and $q\text{-Diff}_{\mathbf{K}}^{\mathrm{aa}}$ coincide.

3.3. A crucial example. Consider the q-difference operator (cf. Example 2.14)

$$\mathcal{L} = (\sigma_q - 1) \circ [\lambda\sigma_q - ((q-1)x + 1)]$$

and its associated q-difference module $\mathcal{M} = (M = \frac{\mathbf{K}[\sigma_q,\sigma_q^{-1}]}{\mathbf{K}[\sigma_q,\sigma_q^{-1}]\mathcal{L}}, \Sigma_q)$. If $\lambda \in q^{\mathbb{Z}}$, the module is admissible, and we are done. So let us suppose that $\lambda \notin q^{\mathbb{Z}}$.

In $\widehat{\mathcal{B}}_q$, the q-difference module $\widehat{\mathcal{M}} = \mathcal{M}_{\widehat{\mathbf{K}}}$ is isomorphic to the rank 2 module $\widehat{\mathbf{K}}^2$ equipped with the semi-linear operator:

$$\Sigma_q : \widehat{\mathbf{K}}^2 \to \widehat{\mathbf{K}}^2, \quad \begin{pmatrix} f_1(x) \\ f_2(x) \end{pmatrix} \mapsto \begin{pmatrix} 1 & 0 \\ 0 & \lambda^{-1} \end{pmatrix} \begin{pmatrix} f_1(qx) \\ f_2(qx) \end{pmatrix}.$$

In fact, $\mathcal{L}$ has a right factor $\lambda\sigma_q - ((q-1)x+1)$: this corresponds to the existence of an element $f \in M$ such that $\Sigma_q f = \lambda^{-1}((q-1)x+1)f$. Since $e_q(x) = \sum_{n\geq 0} \frac{(1-q)^n x^n}{(q;q)_n}$ is a solution of the equation $y(qx) = ((q-1)x+1)y(x)$, we deduce that $\tilde{f} = e_q(x)f$ satisfies $\Sigma_q \tilde{f} = \lambda^{-1}\tilde{f}$. On the other hand, we saw that there always exists $\Phi \in \mathbf{C}[[x]]$ such that $(\sigma_q - 1)\Phi$ is a right factor of $\mathcal{L}$, which means that there exists $e \in M$ such that $\Sigma_q e = \Phi(x)\Phi(qx)^{-1}e$ and therefore that there is $\tilde{e} \in M_{\widehat{\mathbf{K}}}$ such that $\Sigma_q \tilde{e} = \tilde{e}$. A priori this last base change is only formal: the series Φ converges if and only if the module is admissible; cf. Example 2.14.

The calculations above say more: the formal isomorphism class of M corresponds to a single analytic isomorphism class if and only if M is admissible, which happens if and only if the series $\sum_{n\geq 0} \frac{x^n}{(q;q)_n}$ and $\sum_{n\geq 0, q^n\neq\lambda} \frac{x^n}{q^n-\lambda}$ converge.

3.4. Simple and indecomposable objects. In differential and difference equation theory simple objects are called *irreducible*. They are those objects $\mathcal{M} = (M, \Sigma_q)$ over $\mathbf{K}$ such that any $m \in M$ is a cyclic vector: this is equivalent to the property of not having a proper q-difference sub-module, or to the fact that any q-difference operator associated to $\mathcal{M}$ cannot be factorized in $\mathbf{K}[\sigma_q]$.

Corollary 3.8. *The only irreducible objects in the category* q-$\mathrm{Diff}^{\mathrm{a}}_{\mathbf{K}}$ *are the rank* 1 *modules described in Example* 3.3.

Proof. This is a consequence of Proposition 2.9. □

Before the irreducible objects of the category q-$\mathrm{Diff}^{\mathrm{aa}}_{\mathbf{K}}$ are described, we need to introduce a functor of restriction of scalars going from q-$\mathrm{Diff}^{\mathrm{a}}_{\mathbf{K}_n}$ to q-$\mathrm{Diff}^{\mathrm{aa}}_{\mathbf{K}}$. In fact, the set $\{1, x^{1/n}, \dots, x^{n-1/n}\}$ is a basis of $\mathbf{K}_n/\mathbf{K}$ such that $\sigma_q x^{i/n} = q^{i/n}x^{i/n}$. Therefore $\mathbf{K}_n$ can be identified with the admissible q-difference module $M_{0,1} \oplus M_{0,q^{1/n}} \oplus \cdots \oplus M_{0,q^{n-1/n}}$ (in the notation of Example 3.3).

In the same way, we can associate to any (almost) admissible $q^{1/n}$-difference module $\mathcal{M}$ of rank ν over $\mathbf{K}_n$ an almost admissible difference module $\mathrm{Res}_n(\mathcal{M})$ of rank $n\nu$ over $\mathbf{K}$ by restriction of scalars. The functor Res_n "stretches" the Newton polygon horizontally, meaning that if the Newton polygon of $\mathcal{M}$ over $\mathbf{K}_n$ is $\{(\mu_1, r_1), \dots, (\mu_k, r_k)\}$, then the Newton polygon of $\mathrm{Res}_n(\mathcal{M})$ over $\mathbf{K}$ is $\{(\mu_1/n, nr_1), \dots, (\mu_k/n, nr_k)\}$.

Example 3.9. Consider, for some $\lambda \in \mathbf{C}^*$, the $q^{1/2}$-module over $\mathbf{K}_2$ associated to the equation $x^{1/2}y(qx) = \lambda y(x)$. This means that we consider a rank 1 module

$\mathbf{K}_2 e$ over $\mathbf{K}_2$ such that $\Sigma_q e = \frac{\lambda}{x^{1/2}} e$. Notice that its Newton polygon over $\mathbf{K}_2$ has only one single slope equal to 1. Since $\mathbf{K}_2 e = \mathbf{K}e + \mathbf{K}x^{1/2}e$, the module $\mathbf{K}_2 e$ is a q-difference module of rank 2 over $\mathbf{K}$ whose q-difference structure is defined by

$$\Sigma_q(e, x^{1/2}e) = (e, x^{1/2}e) \begin{pmatrix} 0 & q^{1/2}\lambda \\ \lambda/x & 0 \end{pmatrix}.$$

Consider the vector $m = e + x^{1/2}e$. We have $\Sigma_q(m) = q^{1/2}\lambda e + \frac{\lambda}{x}(x^{1/2}e)$ and $\Sigma_q^2(m) = \frac{q^{1/2}\lambda^2}{qx} e + \frac{q^{1/2}\lambda^2}{x}(x^{1/2}e)$. Since m and $\Sigma_q(m)$ are linearly independent, m is a cyclic vector for $\mathbf{K}_2 e$ over $\mathbf{K}$. Moreover, for

$$P(x) = -\lambda^2(q^{3/2}x - 1), \quad Q(x) = \lambda(q-1)x, \quad R(x) = -q^{1/2}x(q^{1/2}x - 1)$$

we have $P(x)m + Q(x)\Sigma_q(m) = R(x)\Sigma_q^2(m)$. In other words, the Newton polygon of the rank 2 q-difference module $\mathbf{K}_2 e$ over $\mathbf{K}$ has only one slope equal to $1/2$.

Let $n \in \mathbb{Z}_{>0}$, let μ be an integer prime to n and $\mathcal{M}_{\mu,\lambda,n}$ be the rank one module over $\mathbf{K}_n$ associated to the equation $x^{\mu/n}y(qx) = \lambda y(x)$. In [26], Lemma 3.9, Soibelman and Vologodsky show that $\mathcal{N}_{\mu/n,\lambda} = \mathrm{Res}_n(\mathcal{M}_{\mu,\lambda,n})$ is a simple object over $\mathcal{O}(\mathbb{C}^*)$. We show that all the simple objects of the category q-$\mathrm{Diff}_{\mathbf{K}}^{\mathrm{aa}}$ are of this form (for the case $|q| \neq 1$ see [21]). Observe that $\mathcal{M}_{\mu,\lambda} = \mathcal{M}_{\mu,\lambda,1} = \mathcal{N}_{\mu,\lambda}$ as q-difference modules over $\mathbf{K}$.

We start by proving a lemma.

Lemma 3.10. *Let $\mathcal{M}$ be a q-difference module associated to a q-difference operator $\mathcal{L} \in \mathbf{C}\{x\}[\sigma_q]$. Suppose that the operator $\mathcal{L}$ has a right factor in $\mathbf{C}\{x^{1/n}\}[\sigma_q]$ of the form $(x^{\mu/n}\sigma_q - \lambda) \circ h(x^{1/n})$, with $n \in \mathbb{Z}_{>1}$, $\mu \in \mathbb{Z}$, $(n,\mu) = 1$, $\lambda \in \mathbf{C}^*$ and $h(x) \in \mathbf{C}\{x^{1/n}\}$.*

Then $\mathcal{M}$ has a submodule isomorphic to $\mathcal{N}_{\mu/n,\lambda}$.

Proof. Note that any operator $\mathcal{L} \in \mathbf{C}\{x\}[\sigma_q]$ divisible by $(x^{\mu/n}\sigma_q - \lambda) \circ h(x^{1/n})$ has order $\geq n$. Let $\mathcal{L}_{\mu/n,\lambda} \in \mathbf{C}\{x\}[\sigma_q]$ be a q-difference operator (of order n) associated to $\mathcal{N}_{\mu/n,\lambda}$. Since the ring $\mathbf{C}\{x\}[\sigma_q]$ is euclidean there exist $\mathcal{Q}, \mathcal{R} \in \mathbf{C}\{x\}[\sigma_q]$ such that

$$\mathcal{L} = \mathcal{Q} \circ \mathcal{L}_{\mu/n,\lambda} + \mathcal{R},$$

with $\mathcal{R} = 0$ or $\mathcal{R}$ of order strictly smaller than n and divisible on the right by $(x^{\mu/n}\sigma_q - \lambda) \circ h(x^{1/n})$. Of course, if $\mathcal{R} \neq 0$ we obtain a contradiction. Therefore $\mathcal{L}_{\mu/n,\lambda}$ divides $\mathcal{L}$, and the lemma follows. □

Remark 3.11. The same statement holds for a formal operator $\mathcal{L} \in \mathbf{C}[[x]][\sigma_q]$ having a formal right factor $(x^{\mu/n}\sigma_q - \lambda) \circ h(x^{1/n})$ with $h(x) \in \mathbf{C}[[x^{1/n}]]$.

We have a complete description of the isomorphism classes of almost admissible irreducible q-difference modules over $\mathbf{K}$:

Proposition 3.12. *A system of representatives of the isomorphism classes of the irreducible objects of* $q\text{-Diff}^{\text{aa}}_{\mathbf{K}}$ *(resp.* $\widehat{\mathcal{B}}_q$*) is given by:*

- *rank* 1 *q-difference modules* $\mathcal{M}_{\mu,\lambda}$ *with* $\mu \in \mathbb{Z}$ *and* $c \in \mathbf{C}^*/q^{\mathbb{Z}}$*, i.e., the irreducible objects of* $q\text{-Diff}^{\text{a}}_{\mathbf{K}}$ *up to isomorphism* (*cf. Example* 3.3)*;*
- *q-difference modules* $\mathcal{N}_{\mu/n,\lambda} = \text{Res}_n(\mathcal{M}_{\mu,\lambda,n})$*, where* $n \in \mathbb{Z}_{>0}$*,* $\mu \in \mathbb{Z}$*,* $(n,\mu) = 1$ *and* $\lambda \in \mathbf{C}^*/(q^{1/n})^{\mathbb{Z}}$.

Proof. The corollary is well known for $\widehat{\mathcal{B}}_q$. We prove the statement for the category $q\text{-Diff}^{\text{aa}}_{\mathbf{K}}$. Rank 1 irreducible objects of $q\text{-Diff}^{\text{aa}}_{\mathbf{K}}$ are necessarily admissible, therefore they are of the form $\mathcal{M}_{\mu,\lambda}$ for some $\mu \in \mathbb{Z}$ and $\lambda \in \mathbf{C}^*/q^{\mathbb{Z}}$. Consider an irreducible object $\mathcal{M}$ in $q\text{-Diff}^{\text{aa}}_{\mathbf{K}}$ of higher rank. Because of the previous lemma and Corollary 2.13 it must contain an object of the form $N_{\mu,\lambda,n}$ for convenient μ, λ, n. The irreducibility implies that $\mathcal{M} \cong \mathcal{N}_{\mu,\lambda,n}$. □

Remark 3.13. Consider the rank 1 modules $\mathcal{N}_{\mu,\lambda,n}$ and $\mathcal{N}_{r\mu,\lambda,rn}$ over $\mathbf{K}_n$ and $\mathbf{K}_{rn}$, respectively, for some $\mu, r, n \in \mathbb{Z}$, $r > 1$, $n > 0$, $(\mu, n) = 1$ and $\lambda \in \mathbb{C}^*$. Then $\text{Res}_n(\mathcal{N}_{\mu,\lambda,n})$ is a rank n q-difference module over $\mathbf{K}$, while $\text{Res}_{rn}(\mathcal{N}_{r\mu,\lambda,rn})$ has rank rn, although $\mathcal{N}_{\mu,\lambda,n}$ and $\mathcal{N}_{r\mu,\lambda,rn}$ are associated to the same rank 1 operator.

Writing explicitly the basis of $\mathbf{K}_{rn}$ over $\mathbf{K}_n$ and $\mathbf{K}$, it follows that $\text{Res}_{rn}(\mathcal{N}_{r\mu,\lambda,rn})$ is a direct sum of r copies of $\text{Res}_n(\mathcal{N}_{\mu,\lambda,n})$.

3.5. Structure theorem for almost admissible *q*-difference modules. Now we are ready to state a structure theorem for almost admissible q-difference modules.

Theorem 3.14. *Suppose that the q-difference module* $\mathcal{M} = (M, \Sigma_q)$ *over* $\mathbf{K}$ *is almost admissible, with Newton polygon* $\{(\mu_1, r_1), \dots, (\mu_k, r_k)\}$*. Then*

$$M = M_1 \oplus M_2 \oplus \cdots \oplus M_k,$$

where the q-difference modules $\mathcal{M}_i = (M_i, \Sigma_{q|M_i})$ *are defined over* $\mathbf{K}$*, and are almost admissible and pure of slope* μ_i *and rank* r_i*.*

Each $\mathcal{M}_i$ *is a direct sum of almost admissible indecomposable q-difference modules, i.e., an iterated non-trivial extension of a simple almost admissible q-difference module by itself.*

Remark 3.15. More precisely, consider the rank ν unipotent q-difference module $\mathcal{U}_\nu = (U_\nu, \Sigma_q)$ defined by the property of having a basis $\underline{e}$ such that the action of Σ_q on $\underline{e}$ is described by a matrix composed by a single Jordan block with eigenvalue 1. Then the indecomposable modules $\mathcal{N}$ in the previous theorem are isomorphic to $\mathcal{N} \otimes_{\mathbf{K}} \mathcal{U}_\nu$ for some irreducible module $\mathcal{N}$ of $q\text{-Diff}^{\text{aa}}_{\mathbf{K}}$ and some ν.

The theorem above is equivalent to a stronger version of Theorem 2.7 for almost admissible q-difference operators:

Theorem 3.16. *Suppose that the q-difference operator $\mathcal{L}$ is almost admissible, with Newton polygon $\{(\mu_1, r_1), \dots, (\mu_k, r_k)\}$. Then for any permutation ϖ on the set $\{1, \dots, k\}$ there exists a factorization of $\mathcal{L}$,*

$$\mathcal{L} = \mathcal{L}_{\varpi,1} \circ \mathcal{L}_{\varpi,2} \circ \dots \circ \mathcal{L}_{\varpi,k},$$

such that $\mathcal{L}_{\varpi,i} \in \mathbf{C}\{x\}[\sigma_q]$ is almost admissible and pure of slope $\mu_{\varpi(i)}$ and order $r_{\varpi(i)}$.

Moreover, for any $i = 1, \dots, k$, write $\mu_i = d_i/s_i$ with $d_i, s_i \in \mathbb{Z}$, $s_i > 0$ and $(d_i, s_i) = 1$. We have

$$\mathcal{L}_{\varpi,i} = \mathcal{L}_{d_{\varpi(i)},\lambda_l^{\varpi(i)},s_{\varpi(i)}} \circ \dots \circ \mathcal{L}_{d_{\varpi(i)},\lambda_1^{\varpi(i)},s_{\varpi(i)}},$$

where $\lambda_1^{\varpi(i)}, \dots, \lambda_l^{\varpi(i)}$ are exponents of the slope $\mu_{\varpi(i)}$, ordered so that if $\lambda_j^{\varpi(i)}(\lambda_{j'}^{\varpi(i)})^{-1} \in q^{\mathbb{Z}_{>0}}$ then $j < j'$, and the operator $\mathcal{L}_{d_{\varpi(i)},\lambda_j^{\varpi(i)},s_{\varpi(i)}}$ is associated to the module $\mathcal{N}_{d_{\varpi(i)},\lambda_j^{\varpi(i)},s_{\varpi(i)}}$.

Proof. Suppose that the operator has at least one non-integral slope. A priori the operators $\mathcal{L}_{\varpi,i}$ are defined over $\mathbf{C}\{x^{1/n}\}$ for some $n > 1$. But it follows from Lemma 3.10 that they are the product of operators associated to q-difference modules defined over $\mathbf{K}$ of the form $\mathcal{N}_{\mu,\lambda,n}$ for some $\mu, n \in \mathbb{Z}$, $n > 0$ and $\lambda \in \mathbf{C}^*$. □

3.6. Analytic versus formal classification. The formal classification of q-difference modules with $|q| = 1$ is studied in [26] by different techniques. It can also be deduced by the results of the previous section, dropping the diophantine assumptions and establishing a formal factorization theorem for q-difference operators:

Theorem 3.17. *Consider a q-difference module $\mathcal{M} = (M, \Sigma_q)$ over $\widehat{\mathbf{K}}$, with Newton polygon $\{(\mu_1, r_1), \dots, (\mu_k, r_k)\}$. Then*

$$M = M_1 \oplus M_2 \oplus \dots \oplus M_k\,,$$

where the q-difference modules $\mathcal{M}_i = (M_i, \Sigma_{q|M_i})$ are defined over $\widehat{\mathbf{K}}$ and are pure of slope μ_i and rank r_i.

Each $\mathcal{M}_i$ is a direct sum of almost admissible indecomposable q-difference modules, i.e., an iterated non-trivial extension of a simple almost admissible q-difference module by itself.

Remark 3.18. Irreducible objects are q-difference modules over $\widehat{\mathbf{K}}$ obtained by rank 1 modules associated to q-difference equations of the form $x^{\mu} y(qx) = \lambda y(x)$, with $\mu \in \mathbb{Q}$ and $\lambda \in \mathbf{C}^*$, by restriction of scalars.

Hence the first part of Theorem 3.6 can now be proved.

Proposition 3.19. *Let $\mathcal{M} = (M, \Sigma_q^M)$ and $\mathcal{N} = (N, \Sigma_q^N)$ be two almost admissible q-difference modules over* $\mathbf{K}$. *Then $\mathcal{M}$ is isomorphic to $\mathcal{N}$ over $\mathbf{K}$ if and only if $\mathcal{M}_{\widehat{\mathbf{K}}}$ is isomorphic to $\mathcal{N}_{\widehat{\mathbf{K}}}$ over $\widehat{\mathbf{K}}$.*

Proof. It follows from the analytic (resp. formal) factorizations of q-difference modules over $\mathbf{K}$ (resp. $\widehat{\mathbf{K}}$) that

$$\mathcal{M} \cong \mathcal{N} \iff \mathcal{M}_{\mathbf{K}_n} \cong \mathcal{N}_{\mathbf{K}_n} \quad \text{and} \quad \mathcal{M}_{\widehat{\mathbf{K}}} \cong \mathcal{N}_{\widehat{\mathbf{K}}} \iff \mathcal{M}_{\widehat{\mathbf{K}}_n} \cong \mathcal{N}_{\widehat{\mathbf{K}}_n}$$

for an integer $n \geq 1$ such that the slopes of the two modules become integral over $\mathbf{K}_n$. So we can suppose that the two modules are actually admissible.

If $\mathcal{M}$ and $\mathcal{N}$ are isomorphic over $\mathbf{K}$, then they are necessarily isomorphic over $\widehat{\mathbf{K}}$. On the other hand suppose that $\mathcal{M}_{\widehat{\mathbf{K}}} \cong \mathcal{N}_{\widehat{\mathbf{K}}}$. Then the claims follow from the fact that any formal factorization must actually be analytic (cf. Corollary 2.11). □

For further reference we point out that we have proved the following statement:

Corollary 3.20. *Let $\mathcal{M} = (M, \Sigma_q)$ be a pure q-difference module over $\widehat{\mathbf{K}}$ (resp. a pure almost admissible q-difference module over $\mathbf{K}$) of slope μ and rank ν. Then for any $n \in \mathbb{Z}_{\geq 1}$ such that $n\mu \in \mathbb{Z}$, there exists a $\mathbf{C}$-vector space V contained in $M_{\widehat{\mathbf{K}}_n}$ (resp. $M_{\mathbf{K}_n}$) of dimension ν such that $x^{\mu}\Sigma_q(V) \subset V$.*

3.7. End of the proof of Theorem 3.6. Theorem 3.6 states that $\mathcal{B}_q^{\mathrm{iso}} = q\text{-Diff}_{\mathbf{K}}^{\mathrm{aa}}$. Proposition 3.19 implies that $q\text{-Diff}_{\mathbf{K}}^{\mathrm{aa}}$ is a subcategory of $\mathcal{B}_q^{\mathrm{iso}}$. To conclude it is enough to prove the following lemma.

Lemma 3.21. *Let $\mathcal{M} \in \mathcal{B}_q$. Suppose that any $\mathcal{N} \in \mathcal{B}_q$ with $\mathcal{M}_{\widehat{\mathbf{K}}} \cong \mathcal{N}_{\widehat{\mathbf{K}}}$ in $\widehat{\mathcal{B}}_q$ is already isomorphic to $\mathcal{M}$ in $\mathcal{B}_q$. Then $\mathcal{M}$ is almost admissible.*

Proof. With no loss of generality, we may suppose that the slope of the Newton polygon of $\mathcal{M}$ are integral. We know that the lemma is true for rank one modules. In the general case we prove the lemma by steps:

Step 1. Pure rank 2 modules of slope zero.

Let us suppose that $\mathcal{M}$ is pure with Newton polygon $\{(0, 2)\}$. Then there exists a basis $\underline{e}$ of $\mathcal{M}_{\widehat{\mathbf{K}}}$ such that $\Sigma_q \underline{e} = \underline{e}A$ with $A \in \mathrm{Gl}_2(\mathbf{C})$ in the Jordan normal form. The assumptions of the lemma actually say that the basis $\underline{e}$ can chosen to be a basis of $\mathcal{M}$ over $\mathbf{K}$. If $\mathcal{M}$ has only one exponent modulo $q^{\mathbb{Z}}$, then $\mathcal{M}$ is admissible. So let us suppose that $\mathcal{M}$ has at least two different exponents modulo $q^{\mathbb{Z}}$: $\alpha, \beta \in \mathbf{C}$. An elementary manipulation on the exponents (cf. Remark 2.4) allows to assume that $\beta = 1$. This means that A is a diagonal matrix of eigenvalues $1, \alpha$. We are in the case of Section 3.3, so we already know that there exists only one isoformal analytic isomorphism class if and only if the module is admissible.

Step 2. Proof of the lemma in the case of a pure module of slope zero.

Let us suppose that $\mathcal{M}$ is pure with Newton polygon $\{(0,r)\}$. Then there exists a basis $\underline{e}$ of $\mathcal{M}$ over $\mathbf{K}$ such that $\Sigma_q\underline{e} = \underline{e}A$ with $A \in \mathrm{Gl}_r(\mathbf{C})$ in the Jordan normal form. If $\mathcal{M}$ has only one exponent modulo $q^{\mathbb{Z}}$, then $\mathcal{M}$ is admissible. So let us suppose that $\mathcal{M}$ has at least two different exponents modulo $q^{\mathbb{Z}}$. For any pair of exponents α, β distinct modulo $q^{\mathbb{Z}}$, the module $\mathcal{M}$ has a rank two submodule isomorphic to the module considered in step 1. This implies that $\phi_{q,\alpha\beta^{-1}}$ is convergent, and hence that $\mathcal{M}$ is admissible.

Step 3. General case.

Let $\{(r_i,\mu_i) : i = 1,\dots,k\}$ be the Newton polygon of $\mathcal{M}$. The formal module $\mathcal{M}_{\widehat{\mathbf{K}}}$ admits a basis $\underline{e}$ such that the matrix of Σ_q with respect to $\underline{e}$ is a block diagonal matrix of the form (cf. Corollary 3.20)

$$\Sigma_q\underline{e} = \underline{e}\,\mathrm{diag}\left(\frac{A_1}{x^{\mu_1}} \quad \dots \quad \frac{A_k}{x^{\mu_k}}\right),$$

where $A_1,\dots,A_k$ are constant square matrices that we can suppose to be in Jordan normal form. The assumption says that $\mathcal{M}$ is isomorphic in $\mathcal{B}_q$ to the q-difference module $\mathcal{N}$ over $\mathbf{K}$ generated by the basis $\underline{e}$. Since the slopes and the classes modulo $q^{\mathbb{Z}}$ of the exponents are both analytic and formal invariants, it is enough to prove the statement for pure modules. If $\mathcal{M}$ is pure, this follows from step 2 by elementary manipulation of the slopes (cf. Remark 2.4).

This finishes the proof of the lemma and therefore the proof of Theorem 3.6. □

4. Structure of the category $\mathcal{B}_q^{\mathrm{iso}}$. Comparison with the results in [3] and [26]

The formal results above give another proof of the following:

Theorem 4.1 ([26], Theorems 3.12 and 3.14). *The subcategory $\widehat{\mathcal{B}}_q^f$ of $\widehat{\mathcal{B}}_q$ of pure q-difference modules of slope zero is equivalent to the category of $\mathbf{C}^*/q^{\mathbb{Z}}$-graded finite dimensional $\mathbf{C}$-vector spaces equipped with a nilpotent operator that preserves the grading.*

The category $\widehat{\mathcal{B}}_q$ is equivalent to the category of $\mathbb{Q}$-graded objects of $\widehat{\mathcal{B}}_q^f$.

Let $\mathcal{B}_q^{\mathrm{iso,f}}$ be the full subcategory of $\mathcal{B}_q^{\mathrm{iso}}$ of pure q-difference modules of slope zero. We have an analytic version of the result above:

Theorem 4.2. *The category $\mathcal{B}_q^{\mathrm{iso}}$ is equivalent to the category of $\mathbb{Q}$-graded objects of $\mathcal{B}_q^{\mathrm{iso,f}}$, i.e., each object of $\mathcal{B}_q^{\mathrm{iso}}$ is a direct sum indexed on $\mathbb{Q}$ of objects of $\mathcal{B}_q^{\mathrm{iso,f}}$ and the morphisms of q-difference modules respect the grading.*

Proof. For any $\mu \in \mathbb{Q}$, the component of degree μ of an object of $\mathcal{B}_q^{\mathrm{iso}}$ is its maximal pure submodule of slope μ. The theorem follows from the remark that there are no non-trivial morphisms between two pure modules of different slope. □

As far the structure of the category $\mathcal{B}_q^{\mathrm{iso,f}}$ is concerned, we have an analytic analog of [26], Theorem 3.14, and [3], Theorem 1.6′:

Theorem 4.3. *The category $\mathcal{B}_q^{\mathrm{iso,f}}$ is equivalent to the category of finite dimensional $\mathbb{C}^*/q^{\mathbb{Z}}$-graded complex vector spaces V, endowed with nilpotent operators which preserve the grading, with the following property:*

> *Let $\lambda_1, \dots, \lambda_n \in \mathbb{C}^*$ be a set of representatives of the classes of $\mathbb{C}^*/q^{\mathbb{Z}}$ corresponding to non-zero homogeneous components of V. The series $\Phi_{(q;\underline{\Lambda})}(x)$, where $\underline{\Lambda} = \{\lambda_i\lambda_j^{-1} \mid i, j = 1, \dots, r;\ \lambda_i\lambda_j^{-1} \notin q^{\mathbb{Z}_{\leq 0}}\}$, is convergent.*

Proof. We saw that a module $\mathcal{M} = (M, \Sigma_q)$ in $\mathcal{B}_q^{\mathrm{iso,f}}$ contains a $\mathbb{C}$-vector space V invariant under Σ_q such that $M \cong V \otimes \mathbf{K}$. Hence there exists a basis $\underline{e}$ such that $\Sigma_q \underline{e} = \underline{e}B$, with $B \in \mathrm{Gl}_\nu(\mathbb{C})$ in the Jordan normal form. This means that $B = D + N$, where D is a diagonal constant matrix and N a nilpotent one. The operator $\Sigma_q - D$ is nilpotent on V.

Since any eigenvalue λ of D is uniquely determined modulo $q^{\mathbb{Z}}$, we obtain the $\mathbb{C}^*/q^{\mathbb{Z}}$-grading by considering the kernel of the operators $(\Sigma_q - \lambda)^n$ for $n \in \mathbb{Z}$ large enough. □

References

[1] C. R. Adams, On the linear ordinary q-difference equation. *Ann. of Math.* (2) **30** (1929), 195–205. JFM 55.0263.01 MR 1502876

[2] Y. André and L. Di Vizio, q-difference equations and p-adic local monodromy. *Astérisque* **296** (2004), 55–111. Zbl 1075.39017 MR 2135685

[3] V. Baranovsky and V. Ginzburg, Conjugacy classes in loop groups and G-bundles on elliptic curves. *Internat. Math. Res. Notices* **1996** (1996), 733–751. Zbl 0992.20034 MR 1413870

[4] V. I. Bernik and M. M. Dodson, *Metric Diophantine approximation on manifolds*. Cambridge Tracts in Math. 137, Cambridge University Press, Cambridge 1999. Zbl 0933.11040 MR 1727177

[5] J.-P. Bézivin, Sur les équations fonctionnelles aux q-différences. *Aequationes Math.* **43** (1992), 159–176. Zbl 0757.39002 MR 1158724

[6] G. D. Birkhoff and P. E. Guenther, Note on a canonical form for the linear q-difference system. *Proc. Nat. Acad. Sci. U. S. A.* **27** (1941), 218–222. JFM 67.0305.02 Zbl 0061.20002 MR 0004047

[7] T. Carletti and S. Marmi, Linearization of analytic and non-analytic germs of diffeomorphisms of $(\mathbb{C}, 0)$. *Bull. Soc. Math. France* **128** (2000), 69–85. Zbl 0997.37017 MR 1765828

[8] L. Di Vizio, Arithmetic theory of q-difference equations: the q-analogue of Grothendieck-Katz's conjecture on p-curvatures. *Invent. Math.* **150** (2002), 517–578. Zbl 1023.12004 MR 1946552

[9] L. Di Vizio, Introduction to p-adic q-difference equations (weak Frobenius structure and transfer theorems). In *Geometric aspects of Dwork theory*, Vol. II, Walter de Gruyter GmbH & Co, Berlin 2004, 615–675. Zbl 1070.12004 MR 2099081

[10] L. Di Vizio, J.-P. Ramis, J. Sauloy, and C. Zhang, Équations aux q-différences. *Gaz. Math.* **96** (2003), 20–49. Zbl 1063.39015 MR 1988639

[11] B. Dwork, G. Gerotto, and F. J. Sullivan, *An introduction to G-functions*. Ann. of Math. Stud. 133, Princeton University Press, Princeton, NJ, 1994. Zbl 0830.12004 MR 1274045

[12] G. H. Hardy and E. M. Wright, *An introduction to the theory of numbers*. 5th ed., Oxford Science Publications, Oxford 1979. Zbl 0423.10001 MR 0568909

[13] D. S. Lubinsky, On q-exponential functions for $|q| = 1$. *Canad. Math. Bull.* **41** (1998), 86–97. Zbl 0901.33006 MR 1618963

[14] Yu. I. Manin, Real multiplication and noncommutative geometry (ein Alterstraum). In *The legacy of Niels Henrik Abel*, Springer-Verlag, Berlin 2004, 685–727. Zbl 1091.11022 MR 2077591

[15] S. Mahanta and W.D. van Suijlekom, Noncommutative tori and the Riemann–Hilbert correspondence. Preprint 2007. arXiv:0705.1076

[16] S. Marmi, An introduction to small divisors. Preprint 2000. arXiv:math/0009232

[17] F. Marotte and C. Zhang, Multisommabilité des séries entières solutions formelles d'une équation aux q-différences linéaire analytique. *Ann. Inst. Fourier* (*Grenoble*) **50** (2000), 1859–1890. Zbl 1063.39001 MR 1817385

[18] A. Polishchuk and A. Schwarz, Categories of holomorphic vector bundles on noncommutative two-tori. *Comm. Math. Phys.* **236** (2003), 135–159. Zbl 1033.58009 MR 1977884

[19] A. Polishchuk, Classification of holomorphic vector bundles on noncommutative two-tori. *Doc. Math.* **9** (2004), 163–181. Zbl 1048.32012 MR 2054986

[20] A. Polishchuk, Quasicoherent sheaves on complex noncommutative two-tori. *Selecta Math.* (*N.S.*) **13** (2007), 137–173. Zbl 1135.32011 MR 2330589

[21] M. van der Put and M. Reversat. Galois theory of q-difference equations. *Ann. Fac. Sci. Toulouse Math.* (6) **16** (2007), 665–718. MR 2379057

[22] J.-P. Ramis, J. Sauloy, and C. Zhang, La variété des classes analytiques d'équations aux q-différences dans une classe formelle. *C. R. Math. Acad. Sci. Paris* **338** (2004), 277–280. Zbl 1038.39011 MR 2076495

[23] J. Sauloy, Classification analytique locale des équations aux q-différences irrégulière. http://picard.ups-tlse.fr/ sauloy/

[24] J. Sauloy, La filtration canonique par les pentes d'un module aux q-différences et le gradué associé. arXiv:math.QA/0210430

[25] J. Sauloy, La filtration canonique par les pentes d'un module aux q-différences et le gradué associé. *Ann. Inst. Fourier* (*Grenoble*) **54** (2004), 181–210. Zbl 1061.39013 MR 2069126

[26] Y. Soibelman and V. Vologodsky, Noncommutative compactifications and elliptic curves. *Int. Math. Res. Not.* (2003), 1549–1569. Zbl 1052.14004 MR 1976601

[27] J.-C. Yoccoz, Théorème de Siegel, nombres de Bruno et polynômes quadratiques. *Astérisque* **231** (1995), 3–88. Zbl 0836.30001 MR 1367353

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