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A universal deformation formula for $\mathcal{H}_1$ without projectivity assumption

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Abstract. We find a universal deformation formula for Connes–Moscovici's Hopf algebra $\mathcal{H}_1$ without any projectivity assumption using Fedosov's quantization of symplectic diffeomorphisms.

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1. Introduction

In the study of index theory of a transverse elliptic differential operator in the case of a codimension one foliation, Connes and Moscovici discovered a Hopf algebra $\mathcal{H}_1$ which governs the local symmetry in computing the Chern character. In this article we study deformation theory for this Hopf algebra. In particular, we prove that the Hopf algebra $\mathcal{H}_1$ has a universal deformation formula.

In [5], inspired from Rankin–Cohen brackets on modular forms, Connes and Moscovici constructed a universal deformation formula for Hopf algebra actions of $\mathcal{H}_1$ with a projective structure. By a universal deformation formula of a Hopf algebra A, we mean an element $R \in A[[\hbar]] \otimes_{\mathbb{C}[[\hbar]]} A[[\hbar]]$ satisfying

$$((\Delta \otimes 1)R)(R \otimes 1) = ((1 \otimes \Delta)R)(1 \otimes R),$$
$$(\epsilon \otimes 1)(R) = 1 \otimes 1 = (1 \otimes \epsilon)(R).$$

In [1], we together with Bieliavsky provided a geometric interpretation of a projective structure in the case of a codimension one foliation. And as a result, we (with Bieliavsky) obtained a geometric way to reconstruct Connes–Moscovici's universal deformation formula. The argument is to construct an associative deformed product on $C_c^\infty(\mathbb{R} \times \mathbb{R}^+) \rtimes \Gamma$ for an arbitrary pseudogroup Γ which acts on the upper-half plane, using Fedosov's quantization procedure. Then we prove the full injectivity of $\mathcal{H}_1$ actions on these algebras, thus enabling us to "pull" the associativity "back" to the Hopf algebra level.

A new and interesting result proved in [1], Prop. 6.1, is that even without any projective structure, the first Rankin–Cohen bracket

$$RC_1 = S(X) \otimes Y + Y \otimes X \in \mathcal{H}_1 \otimes \mathcal{H}_1$$

is a noncommutative Poisson structure, i.e., RC_1 is a Hochschild cocycle and $(1 \otimes \Delta)RC_1(1 \otimes RC_1) - (\Delta \otimes 1)RC_1(RC_1 \otimes 1)$ is a Hochschild coboundary. This inspires the question whether $\mathcal{H}_1$ has a universal deformation formula without any projectivity assumption on its action.

In this article we give a positive answer to the above question and introduce a geometric construction of such a universal deformation formula of $\mathcal{H}_1$. The idea of this construction goes back to Fedosov [6] in his study of deformation quantization of a symplectic diffeomorphism. Fedosov developed in [6] a systematic way to quantize a symplectic diffeomorphism to an endomorphism of the quantum algebra no matter whether it preserves or not the chosen symplectic connection. Fedosov also observed that the homomorphism (functoriality) property of his quantization of symplectic diffeomorphisms fails, i.e., $\hat{\alpha}\hat{\beta} \neq \widehat{\alpha\beta}$. Instead, it satisfies a weaker property that $\hat{\alpha}\hat{\beta}$ and $\widehat{\alpha\beta}$ are related by an inner endomorphism. This picture can be explained using the language of "gerbes and stacks" as [2]. In any case, Fedosov's construction does give rise to a deformation quantization of the groupoid algebra associated with a pseudogroup action on a symplectic manifold.

In this article we apply this idea to the special case where the symplectic manifold is $\mathbb{R} \times \mathbb{R}^+$ and the Poisson structure is $\partial_x \wedge \partial_y$, with x the coordinate on $\mathbb{R}$ and y the coordinate on $\mathbb{R}^+$. We consider symplectic diffeomorphisms on $\mathbb{R} \times \mathbb{R}^+$ of the form

$$\gamma\colon (x, y) \to \Big(\gamma(x), \frac{y}{\gamma'(x)}\Big),$$

where γ is a local diffeomorphism on $\mathbb{R}$. In this case, Fedosov's construction of quantization of symplectic diffeomorphism can be computed explicitly. In particular, we are able to prove that the resulting star product on the groupoid algebra $C_c^\infty(\mathbb{R} \times \mathbb{R}^+) \rtimes \Gamma$ can be expressed by

$$f\alpha \star g\beta = m(R(f\alpha \otimes g\beta)),$$

where m is the multiplication map on $C_c^\infty(\mathbb{R} \times \mathbb{R}^+) \rtimes \Gamma$, and R is an element in $\mathcal{H}_1[[\hbar]] \otimes_{\mathbb{C}[[\hbar]]} \mathcal{H}_1[[\hbar]]$. An important property is that the $\mathcal{H}_1$ action on the collection of all $C_c^\infty(\mathbb{R} \times \mathbb{R}^+) \rtimes \Gamma$ for all pseudogroups Γ is fully injective because this action is equivalent to the action used by Connes and Moscovici to define $\mathcal{H}_1$. With this observation, we can derive all the property of R as a universal deformation formula from the corresponding properties about the star product on $C_c^\infty(\mathbb{R} \times \mathbb{R}^+) \rtimes \Gamma[[\hbar]]$.

The notion of universal deformation formula of a Hopf algebra is closely related to the solution of the quantum Yang–Baxter equation. The results in this article can be used to construct a new Hopf algebra structure on $\mathcal{H}_1[[\hbar]]$. We hope that our construction will shed a light on the study of deformation theory of the Hopf algebra $\mathcal{H}_1$ and also codimension one foliations.

This article is organized as follows. We review in Section 2 Fedosov's theory of deformation quantization of symplectic diffeomorphisms. We provide a detailed proof of the fact that this defines a deformation of the groupoid algebra $C_c^\infty(\mathbb{R}\times\mathbb{R}^+)\rtimes\Gamma$. In Section 3, we prove the main theorem of this paper that $\mathcal{H}_1$ has a universal deformation formula using Fedosov's theory reviewed in Section 2. In Section 4, we compute explicitly our universal deformation formula up to $\hbar^2$. We observe that when the $\mathcal{H}_1$ action is projective, the universal deformation formula obtained in this paper does not agree with the one introduced by Connes and Moscovici in [5]. Instead, our lower order term computation suggests that in the case of a projective action these two universal deformation formulas should be related by an isomorphism expressed by elements in $\mathcal{H}_1[[\hbar]]$ and the projective structure Ω. In the appendix we discuss the associativity of the Eholzer product on modular forms, which was used by Connes and Moscovici in constructing their Rankin–Cohen deformation.

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2. Quantization of symplectic diffeomorphisms

In this section we briefly recall Fedosov's construction of quantization of a symplectic diffeomorphism. Moreover, we use this idea to define a deformation of a groupoid algebra coming from a pseudogroup action on a symplectic manifold. We learned this construction from A. Gorokhovsky, R. Nest, and B. Tsygan.

In Fedosov's approach to deformation quantization of a symplectic manifold (M,ω) a flat connection D (also called Fedosov connection) on the Weyl algebra bundle $\mathcal{W}$ plays an essential role. The elements of each fiber (i.e., the Weyl Algebra) W_x are formal series

$$a(y,\hbar)=\sum_{k,|\alpha|\geq 0}\hbar^k a_{k,\alpha}y^\alpha,$$

where $\hbar$ is the formal parameter, $y=(y^1,\dots,y^{2n})\in T_x^*M$ are coordinate functions of T_xM, and $y^\alpha=(y^1)^{\alpha_1}\dots(y^{2n})^{\alpha_{2n}}$ for $\alpha=(\alpha_1,\dots,\alpha_{2n})\in\mathbb{Z}_{\geq 0}\times\dots\times\mathbb{Z}_{\geq 0}$. Suppose that $\bigwedge^\bullet$ are differential forms on M. A Fedosov connection is a derivation $D\colon\Gamma^\infty(\mathcal{W}\otimes\bigwedge^\bullet)\to\Gamma^\infty(\mathcal{W}\otimes\bigwedge^\bullet)$ and $D^2a=0$ for any $a\in\Gamma^\infty(\mathcal{W}\otimes\bigwedge^\bullet)$.

The quantum algebra (i.e., a noncommutative deformation of the smooth function algebra on M) is identified with the space of flat sections $\mathcal{W}_D:=\{a, Da=0\}$ of $\mathcal{W}$.

The idea is that the elements of $\mathcal{W}_D$ are some global sections of $\mathcal{W}$. The existence of that Fedosov connection guarantees that one can "transport" the Moyal product on one fiber to the whole bundle. Here for $a, b \in W_x$, their Moyal product is

$$a *_M b = \exp\Big(-\frac{i\hbar}{2}\omega^{ij}\frac{\partial}{\partial y^i}\frac{\partial}{\partial z^j}\Big)a(y,\hbar)b(z,\hbar)|_{z=y}$$
$$= \sum_{k=0}^{\infty}\Big(-\frac{i\hbar}{2}\Big)^k\frac{1}{k!}\omega^{i_1 j_1}\dots\omega^{i_k j_k}\frac{\partial^k a}{\partial y^{i_1}\dots\partial y^{i_k}}\frac{\partial^k b}{\partial y^{j_1}\dots\partial y^{j_k}}.$$

The characteristic class Ω_D of the quantum algebra $\mathcal{W}_D$ is defined by $D^2 a = [\Omega_D, a]$. Let ω be the symplectic form on M. Then Ω_D can be written as $-\frac{i\omega}{\hbar} + \omega_0 + \hbar\omega_1 + o(\hbar)$. As $D^2 a = 0$ for any $a \in \mathcal{W}$, one concludes that Ω_D is in the center of $\mathcal{W}$, which implies that $\omega_0 + \hbar\omega_1 + \dots \in \Omega^2(M)[[\hbar]]$. The Bianchi identity of D implies that all ω_i are closed differential 2-forms. In general, the cohomology class of Ω_D in $-i\omega/\hbar + H^2(M)[[\hbar]]$ determines the quantum algebra $\mathcal{W}_D$ up to isomorphisms. In what follows we fix a Fedosov connection D and the corresponding quantum algebra $\mathcal{W}_D$ with characteristic class $\Omega_D = -i\omega/\hbar$.

A question arises when one wants to quantize a symplectic diffeomorphism. Because a symplectic diffeomorphism may not preserve D, the canonical lifting of a symplectic diffeomorphism to the Weyl algebra bundle $\mathcal{W}$ may not act on the quantum algebra $\mathcal{W}_D$. How can we quantize a symplectic diffeomorphism in this case? Fedosov studied this problem in [6]. The answer he came up with fits well the language of "stack of algebras". In the following we briefly review Fedosov's results [6], Section 4.

A symplectic diffeomorphism $\gamma\colon M \to M$ naturally acts on the tangent bundle $\gamma\colon TM \to TM$. Therefore γ lifts to an endomorphism on the Weyl algebra bundle $\gamma\colon \mathcal{W} \to \mathcal{W}$. It is easy to check that if $\gamma(D) := \gamma \circ D \circ \gamma^{-1} = D$, then γ defines an algebra endomorphism on the quantum algebra $\mathcal{W}_D = \ker(D)$, which is called a quantization of the symplectic diffeomorphism γ. We with Bieliavsky in [1] used this idea to construct a universal deformation formula of $\mathcal{H}_1$ with a projective structure.

The quantization of γ when $\gamma(D) \neq D$ is more involved. In [6], Section 4, Fedosov proposed the following construction of quantization. We start with extending the standard Weyl algebra W to W^+:

(1) An element u of W^+ can be written as

$$u = \sum_{2k+l\geq 0}\hbar^k a_{k,i_1,\dots,i_l} y^{i_1}\dots y^{i_l}$$

where $(y^1, \dots, y^{2n})$ are coordinates on the standard symplectic vector space (V, ω). In the above sum, we allow k to be negative.

(2) There are a finite number of terms with a given total degree $2k + l \geq 0$.

We remark that the Moyal product extends to a well-defined product on W^+. And we consider the corresponding extension $\mathcal{W}^+$ of the Weyl algebra bundle $\mathcal{W}$ associated to W^+. The Fedosov connection D lifts to an derivation on $\Gamma^\infty(\mathcal{W}^+ \otimes \bigwedge)$ with $D^2 = 0$. We notice that if there are two $\mathcal{W}$-valued 1-forms Δ and Δ' satisfying $i/\hbar[\Delta, \cdot\,] = i/\hbar[\Delta', \cdot\,]$ and $\Delta|_{y^1=\cdots=y^{2n}} = \Delta'|_{y^1=\cdots=y^{2n}} = 0$, then $\Delta = \Delta'$ (the first equation shows that Δ and Δ' are different by a 1-form with value in the center of $\mathcal{W}$, and the second equation shows that $\Delta - \Delta'$ has to be zero as the center of $\mathcal{W}$ is $C^\infty(M)[[\hbar]]$). Now given a symplectic diffeomorphism γ on M, as $\gamma(D)$ is again a connection on $\mathcal{W}$ we can always write $\gamma(D) = D + i/\hbar[\Delta, \cdot\,]$, where Δ is a $\mathcal{W}$-valued 1-form on M. By the above observation, we see that there is a unique choice (if exists) of Δ such that $\Delta|_{y^1=\cdots=y^{2n}=0} = 0$. According to [7], Thm. 5.2.2, one can always find a Fedosov connection D which can be locally written as $D = d + i/\hbar[r, \cdot\,]$ such that $r|_{y^1=\cdots=y^{2n}=0} = 0$ and $\deg(r) \geq 2$. Therefore, we have a canonical choice $\Delta_\gamma = \gamma(r) - r$, a $\mathcal{W}$-valued 1-form on M, with $\Delta_\gamma|_{y^1=\cdots=y^{2n}=0} = 0$ satisfying $\gamma(D) = D + i/\hbar[\Delta_\gamma, \cdot\,]$ and $\deg(\Delta_\gamma) \geq 2$. In the following we will always work with this choice of Δ_γ. We consider the equation

$$DU_\gamma = -\frac{i}{\hbar}\Delta_\gamma \circ U_\gamma, \tag{1}$$

where U_γ is an invertible section of $\mathcal{W}^+$. Fedosov [6], Thm. 4.3, proved that eqn. (1) always has solutions. In general, these solutions are not unique. But the following induction procedure

$$U_{n+1} = 1 + \delta^{-1}\{(D+\delta)U_n + (i/\hbar)\Delta_\gamma \circ U_n\}, \quad U_0 = 1$$

uniquely determines an invertible solution to eqn. (1). Here $\delta\colon \Gamma^\infty(\mathcal{W}^+ \otimes \bigwedge) \to \Gamma^\infty(\mathcal{W}^+ \otimes \bigwedge)$ is defined by

$$\delta a = dx^k \wedge \frac{\partial a}{\partial y^k}.$$

By this induction, we see that U is a solution to the equation

$$U = 1 + \delta^{-1}\{(D+\delta)U + i/\hbar\Delta_\gamma \circ U\}, \tag{2}$$

which actually has a unique solution because $\delta^{-1}\{(D+\delta)U + i/\hbar\Delta_\gamma \circ U\}$ raises the total degree of U by 1. We will always work with this solution in this article. By eqn. (1), for any symplectic diffeomorphism α, U_α^{-1} satisfies the equation

$$DU_\alpha^{-1} = -U_\alpha^{-1} \circ DU_\alpha \circ U_\alpha^{-1} = \frac{i}{\hbar}U_\alpha^{-1} \circ \Delta_\alpha.$$

Using eqn. (2), it is not difficult to check that U_α^{-1} is the unique solution to the equation

$$V = 1 + \delta^{-1}\{(D+\delta)V - i/\hbar V \circ \Delta_\alpha\}, \tag{3}$$

which can be constructed by the same induction as above.

We have the following lemma on Δ_α.

Lemma 2.1. *The mapping $\gamma \mapsto \Delta_\gamma$ defines a cocycle on the group of symplectic diffeomorphisms (more precisely a discrete sub(pseudo)group of the symplectic diffeomorphism group) with value in $\Omega(M, \mathcal{W})$, the space of 1-forms on M with value in $\mathcal{W}$, and*

(1) $\Delta_\alpha + \alpha(\Delta_\beta) = \Delta_{\alpha\beta}$,
(2) $\alpha(\Delta_{\alpha^{-1}}) = -\Delta_\alpha$.

Proof. (1) We have $D + i/\hbar[\Delta_{\alpha\beta}, \cdot] = \alpha\beta(D) = \alpha(\beta(D)) = \alpha(D + i/\hbar[\Delta_\beta, \cdot]) = \alpha(D) + i/\hbar[\alpha(\Delta_\beta), \cdot] = D + i/\hbar[\Delta_\alpha, \cdot] + i/\hbar[\alpha(\Delta_\beta), \cdot] = D + i/\hbar[\Delta_\alpha + \alpha(\Delta_\beta), \cdot]$. Therefore by the defining property of $\Delta_{\alpha\beta}$ and its uniqueness, we conclude that $\Delta_{\alpha\beta} = \Delta_\alpha + \alpha(\Delta_\beta)$.

(2) Corollary of (1) by setting $\beta = \alpha^{-1}$. □

We will need the following properties of U_α later in our construction.

Proposition 2.2. *The assignment $\alpha \mapsto U_\alpha$ satisfies the following properties.*

(1) $D(\alpha(U_\beta)) = -i/\hbar\big(\Delta_{\alpha\beta}\big) \circ \alpha(U_\beta) + i/\hbar\alpha(U_\beta) \circ \Delta_\alpha$;
(2) $\alpha(U_{\alpha^{-1}}) = U_\alpha^{-1}$.

Proof. (1) We can use eqn. (2) to prove a stronger statement. We compute

$$\begin{aligned}\alpha(U_\beta) &= \alpha(1 + \delta^{-1}\{(D + \delta)U_\beta + i/\hbar\Delta_\beta \circ U_\beta\}) \\ &= 1 + \delta^{-1}\{(\alpha(D) + \delta)\alpha(U_\beta) + i/\hbar\alpha(\Delta_\beta) \circ \alpha(U_\beta)\} \\ &= 1 + \delta^{-1}\{(D + \delta)\alpha(U_\beta) + i/\hbar[\Delta_\alpha, \alpha(U_\beta)] + i/\hbar\alpha(\Delta_\beta) \circ \alpha(U_\beta)\} \\ &= 1 + \delta^{-1}\{(D + \delta)\alpha(U_\beta) + i/\hbar(\Delta_\alpha + \alpha(\Delta_\beta)) \circ \alpha(U_\beta) - i/\hbar\alpha(U_\beta) \circ \Delta_\alpha\}.\end{aligned}$$

By applying identity (1) in Lemma 2.1 to the last line, we conclude that $\alpha(U_\beta)$ is the unique solution of the equation

$$\alpha(U_\beta) = 1 + \delta^{-1}\{(D + \delta)\alpha(U_\beta) + i/\hbar\Delta_{\alpha\beta} \circ \alpha(U_\beta) - i/\hbar\alpha(U_\beta) \circ \Delta_\alpha\}. \quad (4)$$

We remark that the solution to eqn. (4) is unique because $\delta^{-1}\{(D + \delta)\alpha(U_\beta) + i/\hbar\Delta_{\alpha\beta} \circ \alpha(U_\beta) - i/\hbar\alpha(U_\beta) \circ \Delta_\alpha\}$ raises the total degree of $\alpha(U_\beta)$ by 1. Taking δ on both sides of the above equation, we obtain the first identity of this proposition.

(2) Setting $\beta = \alpha^{-1}$ in eqn. (4), we have that

$$\begin{aligned}\alpha(U_{\alpha^{-1}}) &= 1 + \delta^{-1}\{(D + \delta)\alpha(U_{\alpha^{-1}}) + i/\hbar(\Delta_{id} \circ \alpha(U_{\alpha^{-1}}) - i/\hbar\alpha(U_{\alpha^{-1}}) \circ \Delta_\alpha\} \\ &= 1 + \delta^{-1}\{(D + \delta)\alpha(U_{\alpha^{-1}}) - i/\hbar\alpha(U_{\alpha^{-1}}) \circ \Delta_\alpha\},\end{aligned}$$

which is same as the equation that defines U_α^{-1}. By the uniqueness of the solution to the above equation, we have $\alpha(U_{\alpha^{-1}}) = U_\alpha^{-1}$. □

Fedosov [6] defined quantization of a symplectic diffeomorphism γ on (M,ω) as

$$\hat{\gamma}(a) = \mathrm{Ad}_{U_\gamma^{-1}}(\gamma(a)) = U_\gamma^{-1} \circ \gamma(a) \circ U_\gamma,\ a \in \Gamma^\infty(\mathcal{W}^+ \otimes \textstyle\bigwedge),$$

which defines an algebra endomorphism of the quantum algebra $\mathcal{W}_D$.

The "defect" of this quantization is that the homomorphism (functoriality) property fails, i.e.,

$$\hat{\alpha}\hat{\beta} \neq \widehat{\alpha\beta}.$$

Instead, Fedosov proved the following property of the associator $v_{\alpha,\beta} := U_\alpha^{-1} \circ \alpha(U_\beta^{-1}) \circ \alpha\beta(U_{(\alpha\beta)^{-1}}^{-1})$.

Proposition 2.3. *The associator $v_{\alpha,\beta}$ is a flat section of $\mathcal{W}$, and*

$$\hat{\alpha}\hat{\beta}(\widehat{\alpha\beta})^{-1} = \mathrm{Ad}_{v_{\alpha,\beta}}.$$

Proof. Using $DU_\alpha^{-1} = i/\hbar U_\alpha^{-1} \circ \Delta_\alpha$, we have

$$\begin{aligned}
Dv_{\alpha,\beta} &= D(U_\alpha^{-1} \circ \alpha(U_\beta^{-1}) \circ \alpha(\beta(U_{(\alpha\beta)^{-1}}^{-1})))\\
&= D(U_\alpha^{-1}) \circ \alpha(U_\beta^{-1}) \circ \alpha(\beta(U_{(\alpha\beta)^{-1}}^{-1})) + U_\alpha^{-1} \circ D(\alpha(U_\beta^{-1})) \circ \alpha\beta(U_{(\alpha\beta)^{-1}}^{-1})\\
&\quad + U_\alpha^{-1} \circ \alpha(U_\beta^{-1}) \circ D(\alpha\beta(U_{(\alpha\beta)^{-1}}^{-1}))\\
&= \frac{i}{\hbar} U_\alpha^{-1} \circ \Delta_\alpha \circ \alpha(U_\beta^{-1}) \circ \alpha\beta(U_{(\alpha\beta)^{-1}}^{-1})\\
&\quad - \frac{i}{\hbar} U_\alpha^{-1} \circ (\Delta_\alpha \circ \alpha(U_\beta^{-1}) - \alpha(U_\beta^{-1}) \circ \Delta_{\alpha\beta}) \circ \alpha\beta(U_{(\alpha\beta)^{-1}}^{-1})\\
&\quad - \frac{i}{\hbar} U_\alpha^{-1} \circ \alpha(U_\beta^{-1}) \circ \Delta_{\alpha\beta} \circ \alpha\beta(U_{(\alpha\beta)^{-1}}^{-1})\\
&= 0.
\end{aligned}$$

We remark that in the above formula we have used the property $D(\alpha(U_\beta^{-1})) = i/\hbar\alpha(U_\beta^{-1}) \circ \Delta_{\alpha\beta} - i/\hbar\Delta_\alpha \circ \alpha(U_\beta)^{-1}$. Therefore, by [6], Lemma 4.2, we conclude that $v_{\alpha,\beta}$ is a flat section of $\mathcal{W}$.

The property of the associator is a straightforward computation. □

We use $f \mapsto \hat{f}$ to represent the bijective map between $C^\infty(M)$ and the quantum algebra $\mathcal{W}_D$. We know that $\alpha(\hat{g})$ is a flat section of the connection $\alpha(D)$. Therefore, $\alpha(\hat{g})$ satisfies the equation

$$D(\alpha(\hat{g})) = -\frac{i}{\hbar}[\Delta_\alpha, \alpha(\hat{g})].$$

Hence $U_\alpha^{-1} \circ \alpha(\hat{g}) \circ U_\alpha$ satisfies the equation

$$\begin{aligned} &D(U_\alpha^{-1} \circ \alpha(\hat{g}) \circ U_\alpha) \\ &\quad = D(U_\alpha^{-1}) \circ \alpha(\hat{g}) \circ U_\alpha + U_\alpha^{-1} \circ D(\alpha(\hat{g})) \circ U_\alpha + U_\alpha^{-1} \circ \alpha(\hat{g}) \circ D(U_\alpha) \\ &\quad = \frac{i}{\hbar} U_\alpha^{-1} \circ \Delta_\alpha \circ \alpha(\hat{g}) \circ U_\alpha - \frac{i}{\hbar} U_\alpha^{-1} \circ \big(\Delta_\alpha \circ \alpha(\hat{g}) - \alpha(\hat{g}) \circ \Delta_\alpha\big) \circ U_\alpha \\ &\qquad - \frac{i}{\hbar} U_\alpha^{-1} \circ \alpha(\hat{g}) \circ \Delta_\alpha \circ U_\alpha \\ &\quad = 0. \end{aligned}$$

In the following, we apply the above idea to quantize the groupoid algebra of a pseudogroup Γ on a symplectic manifold M. We define the following product on $C_c^\infty(M) \rtimes \Gamma[[\hbar]]$:

$$\begin{aligned} f\alpha \star g\beta &:= (\hat{f} \circ U_\alpha^{-1} \circ \alpha(\hat{g}) \circ U_\alpha \circ v_{\alpha,\beta})|_{y=0}\, \alpha\beta \\ &= (\hat{f} \circ U_\alpha^{-1} \circ \alpha(\hat{g}) \circ \alpha(U_\beta^{-1}) \circ \alpha\beta(U^{-1}_{(\alpha\beta)^{-1}}))|_{y=0}\, \alpha\beta, \end{aligned}$$

where the y's are coordinate functions along the fiber direction of T^*M. We remark that because $\hat{f}$, $U_\alpha^{-1} \circ \alpha(\hat{g}) \circ U_\alpha$, and $v_{\alpha,\beta}$ are all flat with respect to the connection D, the product $\hat{f} \circ U_\alpha^{-1} \circ \alpha(\hat{g}) \circ \alpha(U_\beta^{-1}) \circ \alpha\beta(U^{-1}_{(\alpha\beta)^{-1}})$ is also flat with respect to the connection D. Therefore, $\hat{f} \circ U_\alpha^{-1} \circ \alpha(\hat{g}) \circ \alpha(U_\beta^{-1}) \circ \alpha\beta(U^{-1}_{(\alpha\beta)^{-1}})$ is a flat section of $\mathcal{W}$ with respect to D.

We check the associativity of $\star$ on $C_c^\infty(M) \rtimes \Gamma[[\hbar]]$:

$$\begin{aligned} &(f\alpha \star g\beta) \star h\gamma \\ &\quad = (\hat{f} \circ U_\alpha^{-1} \circ \alpha(\hat{g}) \circ \alpha(U_\beta^{-1}) \circ \alpha\beta(U^{-1}_{(\alpha\beta)^{-1}}))|_{y=0}\, \alpha\beta \star h\gamma \\ &\quad = (\hat{f} \circ U_\alpha^{-1} \circ \alpha(\hat{g}) \circ \alpha(U_\beta^{-1}) \circ \alpha\beta(U^{-1}_{(\alpha\beta)^{-1}}) \circ U_{\alpha\beta}^{-1} \circ \alpha\beta(\hat{h}) \\ &\qquad \circ \alpha\beta(U_\gamma^{-1}) \circ \alpha\beta\gamma(U^{-1}_{(\alpha\beta\gamma)^{-1}}))|_{y=0} \alpha\beta\gamma \\ &\quad = (\hat{f} \circ U_\alpha^{-1} \circ \alpha(\hat{g}) \circ \alpha(U_\beta^{-1}) \circ \alpha\beta(\hat{h}) \circ \alpha\beta(U_\gamma^{-1}) \circ \alpha\beta\gamma(U^{-1}_{(\alpha\beta\gamma)^{-1}}))|_{y=0}\, \alpha\beta\gamma, \end{aligned}$$

where we have used $\alpha\beta(U^{-1}_{(\alpha\beta)^{-1}}) = U_{\alpha\beta}$,

$$\begin{aligned} &f\alpha \star (g\beta \star h\gamma) \\ &\quad = f\alpha \star (\hat{g} \circ U_\beta^{-1} \circ \beta(\hat{h}) \circ \beta(U_\gamma^{-1}) \circ \beta\gamma(U^{-1}_{(\beta\gamma)^{-1}}))|_{y=0}\, \beta\gamma \\ &\quad = (\hat{f} \circ U_\alpha^{-1} \circ \alpha(\hat{g} \circ U_\beta^{-1} \circ \beta(\hat{h}) \circ \beta(U_\gamma^{-1}) \circ \beta\gamma(U^{-1}_{(\beta\gamma)^{-1}})) \circ \alpha(U^{-1}_{\beta\gamma}) \\ &\qquad \circ \alpha\beta\gamma(U^{-1}_{(\alpha\beta\gamma)^{-1}}))|_{y=0}\, \alpha\beta\gamma \\ &\quad = (\hat{f} \circ U_\alpha^{-1} \circ \alpha(\hat{g}) \circ \alpha(U_\beta^{-1}) \circ \alpha\beta(\hat{h}) \circ \alpha\beta(U_\gamma^{-1}) \circ \alpha\beta\gamma(U^{-1}_{(\alpha\beta\gamma)^{-1}}))|_{y=0} \alpha\, \beta\gamma, \end{aligned}$$

where we have used $\beta\gamma(U^{-1}_{(\beta\gamma)^{-1}}) = U_{\beta\gamma}$.

We conclude that $\star$ defines an associative product on the algebra $C_c^\infty(M) \rtimes \Gamma[[\hbar]]$.

3. A universal deformation formula

In this section we apply the construction described in the previous section to construct a universal deformation formula of Connes–Moscovici's Hopf algebra $\mathcal{H}_1$. We start by recalling briefly the definition of $\mathcal{H}_1$.

Let us consider the defining representation of $\mathcal{H}_1$ on $C_c^\infty(\mathbb{R}\times\mathbb{R}^+) \rtimes \Gamma$. Let (x, y) with $y > 0$ be coordinates on $\mathbb{R}\times\mathbb{R}^+$. Define $\gamma\colon \mathbb{R}\times\mathbb{R}^+ \to \mathbb{R}\times\mathbb{R}^+$ by

$$\gamma(x, y) = \Big(\gamma(x), \frac{y}{\gamma'(x)}\Big).$$

We remark that the above expression of γ action does not agree with the formulas in [4], but the two actions are isomorphic under the transformation $y \mapsto 1/y$.

Consider $X = 1/y\partial_x$, and $Y = -y\partial_y$ acting on $C_c^\infty(\mathbb{R}\times\mathbb{R}^+) \rtimes \Gamma$ as follows

$$X(f\alpha) = \frac{1}{y} f_x\alpha, \quad Y(f\alpha) = -yf_y\alpha.$$

Then

$$\begin{aligned}\alpha(Y(\alpha^{-1}(f))) &= \alpha\big(Y\big(f\big(\alpha(x), \tfrac{y}{\alpha'(x)}\big)\big)\big)\\ &= \alpha\big(-\tfrac{y}{\alpha'(x)} f_y\big(\alpha(x), \tfrac{y}{\alpha'(x)}\big)\big) = -yf_y(x, y) = Y(f)\end{aligned}$$

and

$$\begin{aligned}\alpha(X(\alpha^{-1}(f))) &= \alpha\big(X(f\big(\alpha(x), \tfrac{y}{\alpha'(x)}\big)\big)\big)\\ &= \alpha\big(\tfrac{1}{y}\alpha'(x) f_x\big(\alpha(x), \tfrac{y}{\alpha'(x)}\big) - \tfrac{\alpha''(x)}{(\alpha'(x))^2} f_y\big(\alpha(x), \tfrac{y}{\alpha'(x)}\big)\big)\\ &= \tfrac{1}{y} f_x + \tfrac{\alpha^{-1''}}{\alpha^{-1'}} f_y = Xf - \tfrac{\log(\alpha^{-1'})'}{y} Yf = (X - \delta_1(\alpha)Y)f,\end{aligned}$$

where $\delta_1(f\alpha) = \log(\alpha^{-1'})'/yf$.

We define $\delta_2(\alpha) = X(\delta_1(\alpha)) = \frac{1}{y}\partial_x(\log(\alpha^{-1'})'/y) = \frac{1}{y^2}\frac{\alpha^{-1'''}\alpha^{-1'} - (\alpha^{-1''})^2}{(\alpha^{-1'})^2}$ and $\delta_n(\alpha) = X(\delta_{n-1}(\alpha))$ by induction for $n \geq 2$.

On $\mathbb{R}\times\mathbb{R}^+$ we consider the Poisson structure $\partial_x \wedge \partial_y$, which can be expressed by $-X\otimes Y + Y\otimes X$. Our main goal is to use the method reviewed in the previous section to construct a star product on $C_c^\infty(\mathbb{R}\times\mathbb{R}^+) \rtimes \Gamma[[\hbar]]$. We prove that this star product as a bilinear operator actually can be expressed by an element R of $\mathcal{H}_1[[\hbar]] \otimes_{\mathbb{C}[[\hbar]]} \mathcal{H}_1[[\hbar]]$. The associativity of the star product is equivalent to the property that R is a universal deformation formula. We start by fixing a symplectic connection ∇ on the tangent bundle of $\mathbb{R} \times \mathbb{R}^+$, which was introduced in [1], Section 3:

$$\nabla_{\partial_x}\partial_x = 0, \quad \nabla_{\partial_x}\partial_y = \tfrac{1}{2y}\partial_x, \quad \nabla_{\partial_y}\partial_x = \tfrac{1}{2y}\partial_x, \quad \nabla_{\partial_y}\partial_y = -\tfrac{1}{2y}\partial_y.$$

Using X and Y, we can express the above connection by

$$\nabla_X X = 0, \quad \nabla_X Y = -\tfrac{1}{2}X, \quad \nabla_Y X = \tfrac{1}{2}X, \nabla_Y Y = -\tfrac{1}{2}Y.$$

We compute $\alpha(\nabla)$ by $\alpha\nabla\alpha^{-1}$:

$$\begin{aligned} \alpha(\nabla)_{\partial_x}\partial_x &= -\frac{\alpha^{-1\prime\prime\prime}\alpha^{-1\prime} - \frac{3}{2}(\alpha^{-1\prime\prime})^2}{(\alpha^{-1\prime})^2} y\partial_y, & \alpha(\nabla)_{\partial_x}\partial_y &= \tfrac{1}{2y}\partial_x, \\ \alpha(\nabla)_{\partial_y}\partial_x &= \tfrac{1}{2y}\partial_x, & \alpha(\nabla)_{\partial_y}\partial_y &= -\tfrac{1}{2y}\partial_y, \end{aligned}$$

and

$$\alpha(\nabla)_X X = \delta_2'(\alpha)Y, \quad \alpha(\nabla)_X Y = -\tfrac{1}{2}X, \quad \alpha(\nabla)_Y X = \tfrac{1}{2}X, \quad \alpha(\nabla)_Y Y = -\tfrac{1}{2}Y,$$

where $\delta_2' = \delta_2 - \frac{1}{2}\delta_1^2$. Note that both ∇ and $\alpha(\nabla)$ are flat and torsion-free.

We consider the lifting of ∇ and $\alpha(\nabla)$ onto the Fedosov connection D and $\alpha(D)$ the Weyl algebra bundle. Use u, v to denote the generators along the fiber direction of the Weyl algebra bundle $\mathcal{W}$. We have for any section a of $\mathcal{W}$,

$$\begin{aligned} Da &= da - dx\frac{\partial a}{\partial u} - dy\frac{\partial a}{\partial v} + \frac{i}{\hbar}\Big[\frac{1}{2y}v^2 dx + \frac{1}{2y}2uvdy, a\Big], \\ \alpha(D)a &= da - dx\frac{\partial a}{\partial u} - dy\frac{\partial a}{\partial v} + \frac{i}{\hbar}\Big[\Big(y^3\delta_2'(\alpha)u^2 + \frac{1}{2y}v^2\Big)dx + \frac{1}{2y}2uvdy, a\Big]. \end{aligned}$$

Therefore, using the notation of the previous section, we can fix $\Delta_\alpha = y^3\delta_2'(\alpha)u^2 dx$ satisfying $\deg(\Delta_\alpha) = 3$ and $\Delta_\alpha|_{u=v=0} = 0$.

In the following, we solve the expression for $\hat{f}$, $\alpha(\hat{g})$, U_α^{-1}, $\alpha(U_\beta^{-1})$, and $\alpha\beta(U^{-1}_{(\alpha\beta)^{-1}})$.

3.1. $\hat{f}$**.** The section $\hat{f}$ of $\mathcal{W}$ is a unique solution of

$$D\hat{f} = 0, \quad \hat{f}|_{u=v=0} = f.$$

Set $\hat{f} = \sum_{m,n} f_{m,n}u^m v^n$. Then the above equation can be written as

$$\begin{aligned} &\sum_{m,n} \big(dx\partial_x f_{m,n}u^m v^n + dy\partial_y f_{m,n}u^m v^n - dx f_{m,n} m u^{m-1}v^n - dy f_{m,n}u^m n v^{n-1} \\ &\qquad + dx\tfrac{1}{2y}\tfrac{-1}{2}2v f_{m,n} m u^{m-1}v^n + dy\tfrac{1}{2y}\tfrac{1}{2}(2v f_{m,n}u^m n v^{n-1} - 2u f_{m,n} m u^{m-1}v^n)\big) \\ &= \sum_{m,n} dx\big(\partial_x f_{m,n} - (m+1)f_{m+1,n} - \tfrac{1}{2y}(m+1)f_{m+1,n-1}\big)u^m v^n \\ &\qquad + \sum_{m,n} dy\big(\partial_y f_{m,n} - (n+1)f_{m,n+1} + \tfrac{1}{2y}(n-m)f_{m,n}\big)u^m v^n. \end{aligned}$$

Therefore, we obtain that $\hat{f}$ is the unique solution to the following family of equations

$$\begin{aligned} \partial_x f_{mn} - (m+1) f_{m+1n} - \tfrac{1}{2y}(m+1) f_{m+1n-1} &= 0, \\ \partial_y f_{mn} - (n+1) f_{mn+1} + \tfrac{1}{2y}(n-m) f_{mn} &= 0 \end{aligned} \tag{5}$$

with $f_{00} = f$.

Solving eqn. (5), we get

$$\begin{aligned} f_{mn} &= \frac{1}{m!n!}\Big(\partial_y - \frac{m+1-n}{2y}\Big)\dots\Big(\partial_y - \frac{m}{2y}\Big)\partial_x^m f \\ &= \frac{(-1)^n}{m!} y^{m-n} X^m \Big(Y + \frac{m+n-1}{2}\Big)\dots\Big(Y + \frac{m}{2}\Big)(f). \end{aligned}$$

3.2. $\alpha(\hat{g})$. We know that $\alpha(\hat{g})$ is the unique solution of the equation

$$D(\alpha(\hat{g})) = -\frac{i}{\hbar}[\Delta_\alpha, \alpha(\hat{g})],$$

with $\alpha(\hat{g})|_{u=v=0} = \alpha(g)$. Recall that $\Delta_\alpha = y^3 \delta_2'(\alpha) u^2 dx$.

Similar to $\hat{f}$, $\alpha(\hat{g})$ satisfies the equation

$$\begin{aligned} 0 = &\sum_{m,n} dx\big(\partial_x \alpha(g)_{m,n} - (m+1)\alpha(g)_{m+1,n} \\ &- \tfrac{1}{2y}(m+1)\alpha(g)_{m+1,n-1} + y^3\delta_2'(n+1)\alpha(g)_{m-1,n+1}\big)u^m v^n \\ &+ \sum_{m,n} dy\big(\partial_y \alpha(g)_{m,n} - (n+1)\alpha(g)_{m,n+1} + \tfrac{1}{2y}(n-m)\alpha(g)_{m,n}\big)u^m v^n \end{aligned}$$

with $\alpha(g)_{0,0} = \alpha(g)$. Therefore, $\alpha(\hat{g})$ is the unique solution to the following family of equations

$$\begin{cases} 0 = \partial_x \alpha(g)_{m,n} - (m+1)\alpha(g)_{m+1,n} - \frac{1}{2y}(m+1)\alpha(g)_{m+1,n-1} \\ \qquad + y^3\delta_2'(n+1)\alpha(g)_{m-1,n+1}, \\ 0 = \partial_y \alpha(g)_{m,n} - (n+1)\alpha(g)_{m,n+1} + \frac{1}{2y}(n-m)\alpha(g)_{m,n}, \end{cases} \tag{6}$$

with $\alpha(g)_{0,0} = \alpha(g)$.

By the second equation of (6), we have

$$\begin{aligned} \alpha(g)_{m,n} &= \frac{1}{n}\Big(\partial_y + \frac{n-1-m}{2y}\Big)\alpha(g)_{mn-1} \\ &= \dots = \frac{1}{n!}\Big(\partial_y + \frac{n-1-m}{2y}\Big)\dots\Big(\partial_y + \frac{-m}{2y}\Big)\alpha(g)_{m,0}. \end{aligned}$$

Setting $n = 0$ in the first equation of (6), we obtain that

$$\begin{aligned}\alpha(g)_{m+1,0} &= \frac{1}{m+1}(\partial_x \alpha(g)_{m,0} + y^3\delta_2'\alpha(g)_{m-1,1}) \\ &= \frac{1}{m+1}\Big(\partial_x \alpha(g)_{m,0} + y^3\delta_2'\Big(\partial_y + \frac{1-m}{2y}\Big)\alpha(g)_{m-1,0}\Big).\end{aligned}$$

By induction, we can solve the above equation as

$$\alpha(g)_{m,n} = \frac{(-1)^n y^{m-n}}{m!n!} A_m\Big(Y + \frac{n+m-1}{2}\Big)\dots\Big(Y + \frac{m}{2}\Big)(\alpha(g)),$$

where $A_m \in \mathscr{H}_1$ is defined inductively by

$$A_{m+1} = XA_m - m\delta_2'\Big(Y - \frac{m-1}{2}\Big)A_{m-1}, \quad A_0 = 1.$$

3.3. U_α^{-1}. We compute U_α^{-1} using the equation

$$DU_\alpha^{-1} = U_\alpha^{-1} \circ \frac{i}{\hbar}\Delta_\alpha$$

with $\Delta_\alpha = \delta_2'(\alpha)y^3u^2dx$.

Write $U_\alpha^{-1} = \sum_{m,n} u_{m,n}^\alpha u^m v^n$, where $u_{m,n}^\alpha$ takes values in $C_c^\infty(M)[\hbar^{-1}, \hbar]]$. Then $u_{m,n}^\alpha$ satisfies the family of equations

$$\begin{aligned}0 &= \partial_x u_{m,n}^\alpha - (m+1)u_{m+1,n}^\alpha - \frac{1}{2y}(m+1)u_{m+1,n-1}^\alpha - \frac{i}{\hbar}y^3\delta_2' u_{m-2,n}^\alpha \\ &\quad + y^3\delta_2'(n+1)u_{m-1,n+1}^\alpha + \frac{i\hbar}{4}y^3\delta_2'(n+2)(n+1)u_{m,n+2}^\alpha, \\ 0 &= \partial_y u_{m,n}^\alpha - (n+1)u_{m,n+1}^\alpha + \frac{1}{2y}(n-m)u_{m,n}^\alpha\end{aligned} \tag{7}$$

with $u_{0,0} = 1$.

The second equation of (7) implies that

$$\begin{aligned}u_{m,n}^\alpha &= \frac{1}{n}\Big(\partial_y + \frac{n-1-m}{2y}\Big)u_{m,n-1}^\alpha \\ &= \cdots = \frac{1}{n!}\Big(\partial_y + \frac{n-1-m}{2y}\Big)\dots\Big(\partial_y + \frac{-m}{2y}\Big)u_{m,0}^\alpha.\end{aligned}$$

We use the $n = 0$ version of the first equation of eqn. (7) to solve $u_{m,0}$:

$$\begin{aligned}u_{m+1,0}^\alpha = \frac{1}{m+1}\Big(&\partial_x u_{m,0}^\alpha - \frac{i}{\hbar}y^3\delta_2' u_{m-2,0}^\alpha + y^3\delta_2'\Big(\partial_y - \frac{m-1}{2y}\Big)u_{m-1,0}^\alpha \\ &+ \frac{i\hbar}{4}y^3\delta_2'\Big(\partial_y - \frac{m-1}{2y}\Big)\Big(\partial_y - \frac{m}{2y}\Big)u_{m,0}^\alpha\Big).\end{aligned}$$

By induction, we have the following expression of u:

$$u^{\alpha}_{m,n} = \frac{(-1)^n y^{m-n}}{m!n!}\Big(Y + \frac{n-m-1}{2}\Big)\dots\Big(Y - \frac{m}{2}\Big)B_m 1, \tag{8}$$

where B_m is defined by

$$B_{m+1} = \Big(X + \frac{i\hbar}{4}\delta_2'\Big(Y - \frac{m-1}{2}\Big)\Big(Y - \frac{m}{2}\Big)\Big)B_m - \delta_2'\Big(Y - \frac{m-1}{2}\Big)B_{m-1} - \frac{i}{\hbar}\delta_2' B_{m-2}, \tag{9}$$

with $B_0 = 1$.

Remark 3.1. We need to prove that the above obtained solution $\widetilde{U}_\alpha^{-1} = \sum u^{\alpha}_{m,n} u^m v^n$ is the unique solution to the defining eqn. (3) of U_α^{-1}, which implies that $\widetilde{U}_\alpha^{-1} = U_\alpha^{-1}$.

Using (8) and (9) for $u^{\alpha}_{m,n}$, we notice that $u^{\alpha}_{m,n}$ may contain negative power of $\hbar$. From (9) we see that if assume that the negative power of $\hbar$ in B_i is less than or equal to $[i/3]$ for $0 \le i \le m$ ($[\mu]$ means the Gauss integer function), then the negative power of $\hbar$ contained in B_{m+1} is less than or equal to $\max([m/3], [(m-1)/3], [(m-2)/3]+1) = [(m+1)/3]$ for $m \ge 2$. Therefore, by induction and eqn. (8), we can conclude that the negative power of $\hbar$ contained in $u^{\alpha}_{m,n}$ is less than or equal to $[m/3]$.

This shows that once $m + n > 0$, the lowest degree term contained in $u^{\alpha}_{m,n} u^m v^n$ has degree greater than or equal to 1. Therefore the degree 0 term of the solution $\widetilde{U}_\alpha^{-1} = \sum u^{\alpha}_{m,n} u^m v^n$ is equal to 1. Accordingly, using $D\widetilde{U}_\alpha^{-1} - \widetilde{U}_\alpha^{-1} \circ i/\hbar\Delta_\alpha = 0$, we obtain that

$$\begin{aligned}
\widetilde{U}_\alpha^{-1} &= \delta\delta^{-1}\widetilde{U}_\alpha^{-1} + \delta^{-1}\delta\widetilde{U}_\alpha^{-1} + 1 = 1 + \delta^{-1}\delta\widetilde{U}_\alpha^{-1} \\
&= 1 + \delta^{-1}(\delta\widetilde{U}_\alpha^{-1} + D\widetilde{U}_\alpha^{-1} - \widetilde{U}_\alpha^{-1} \circ i/\hbar\Delta_\alpha) \\
&= 1 + \delta^{-1}\{(D+\delta)\widetilde{U}_\alpha^{-1} - \widetilde{U}_\alpha^{-1} \circ i/\hbar\Delta_\alpha\}.
\end{aligned}$$

This remark applies also to the solutions $\alpha(U_\beta^{-1})$ and $\alpha\beta(U_{(\alpha\beta)^{-1}}^{-1})$.

3.4. $\alpha(U_\beta^{-1})$ and $\alpha\beta(U_{(\alpha\beta)^{-1}}^{-1})$. By Proposition 2.2, we know that $\alpha(U_\beta)$ satisfies the equation

$$D(\alpha(U_\beta)) = -\frac{i}{\hbar}(\Delta_{\alpha\beta} \circ \alpha(U_\beta) - \alpha(U_\beta) \circ \Delta_\alpha).$$

Accordingly, $\alpha(U_\beta^{-1}) = (\alpha(U_\beta))^{-1}$ satisfies

$$\begin{aligned}
D(\alpha(U_\beta^{-1})) &= -(\alpha(U_\beta))^{-1} \circ D(\alpha(U_\beta)) \circ (\alpha(U_\beta))^{-1} \\
&= -(\alpha(U_\beta))^{-1} \circ \Big(-\frac{i}{\hbar}\Delta_{\alpha\beta} \circ \alpha(U_\beta) + \frac{i}{\hbar}\alpha(U_\beta) \circ \Delta_\alpha\Big) \circ (\alpha(U_\beta))^{-1} \\
&= \frac{i}{\hbar}(\alpha(U_\beta^{-1}) \circ \Delta_{\alpha\beta} - \Delta_\alpha \circ \alpha(U_\beta^{-1})).
\end{aligned}$$

If we write $\alpha(U_\beta^{-1}) = \sum u_{m,n}^{\alpha,\beta} u^m v^n$, then we have

$$\begin{aligned} 0 &= \partial_x u_{m,n}^{\alpha,\beta} - (m+1)u_{m+1,n}^{\alpha,\beta} - \tfrac{1}{2y}(m+1)u_{m+1,n-1}^{\alpha,\beta} - \tfrac{i}{\hbar} y^3 \alpha(\delta_2'(\beta))u_{m-2,n}^{\alpha,\beta} \\ &\quad + y^3(n+1)(2\delta_2'(\alpha) + \alpha(\delta_2'(\beta)))u_{m-1,n+1}^{\alpha,\beta} + \tfrac{i\hbar}{4} y^3 \alpha(\delta_2'(\beta)) u_{m,n+2}^{\alpha,\beta} \\ 0 &= \partial_y u_{m,n}^{\alpha,\beta} - (n+1)u_{m,n+1}^{\alpha,\beta} + \tfrac{1}{2y}(n-m)u_{m,n}^{\alpha,\beta}. \end{aligned} \tag{10}$$

We can solve eqn. (10) of $u_{m,n}^{\alpha,\beta}$ as follows:

$$u_{m,n}^{\alpha,\beta} = \frac{(-1)^n y^{m-n}}{m!n!}\Big(Y + \frac{n-m-1}{2}\Big)\dots\Big(Y - \frac{m}{2}\Big)C_m 1,$$

where $C_m \in \mathcal{H}_1$ is defined inductively by

$$\begin{aligned} C_{m+1} &= \Big(X + \frac{i\hbar}{4}\alpha(\delta_2'(\beta))\Big(Y - \frac{m-1}{2}\Big)\Big(Y - \frac{m}{2}\Big)\Big)C_m \\ &\quad - (2\delta_2'(\alpha) + \alpha(\delta_2'(\beta)))\Big(Y - \frac{m-1}{2}\Big)C_{m-1} - \frac{i}{\hbar}\alpha(\delta_2'(\beta))C_{m-2}, \quad C_0 = 1. \end{aligned}$$

We know that $\alpha\beta(U_{(\alpha\beta)^{-1}}^{-1})$ is equal to $U_{\alpha\beta}$, which satisfies

$$DU_{\alpha\beta} = -\frac{i}{\hbar}\Delta_{\alpha\beta} \circ U_{\alpha\beta}.$$

We can solve $\alpha\beta(U_{(\alpha\beta)^{-1}}^{-1})$ as $U_{\alpha\beta}$. Write $\alpha\beta(U_{(\alpha\beta)^{-1}}^{-1}) = U_{\alpha\beta} = \sum v_{m,n}^{\alpha\beta} u^m v^n$. Then

$$v_{m,n}^{\alpha\beta} = \frac{(-1)^n y^{m-n}}{m!n!}\Big(Y + \frac{n-m-1}{2}\Big)\dots\Big(Y - \frac{m}{2}\Big)D_m 1,$$

where $D_m \in \mathcal{H}_1$ is defined by

$$\begin{aligned} D_{m+1} &= \Big(X - \frac{i\hbar}{4}\delta_2'(\alpha\beta)\Big(Y - \frac{m-1}{2}\Big)\Big(Y - \frac{m}{2}\Big)\Big)D_m \\ &\quad + \delta_2'(\alpha\beta)\Big(Y - \frac{m-1}{2}\Big)D_{m-1} + \frac{i}{\hbar}\delta_2'(\alpha\beta)D_{m-2}, \end{aligned}$$

with $D_0 = 1$.

We point out that since there is $1/\hbar$ in the induction formula of C_m and D_m, $u_{m,n}^{\alpha,\beta}$ and $v_{m,n}^{\alpha\beta}$ may contain terms with negative powers of $\hbar$. However, as is explained in Remark 3.1, we have the following proposition about negative powers of $\hbar$ contained in $u_{m,n}^{\alpha,\beta}$ and $v_{m,n}^{\alpha\beta}$, the proof of which is explained in Remark 3.1.

Proposition 3.2. *The negative power of $\hbar$ contained in $u_{m,n}^{\alpha}$, $u_{m,n}^{\alpha,\beta}$, and $v_{m,n}^{\alpha\beta}$ is less than or equal to $[m/3]$.*

Terms surviving in the product $\hat{f} \circ U_\alpha^{-1} \circ \alpha(\hat{g}) \circ \alpha(U_\beta^{-1}) \circ \alpha\beta(U_{(\alpha\beta)^{-1}}^{-1})|_{u=v=0}$ are sums of terms of the form

$$C_{m_1,\dots,m_5;n_1,\dots,n_5} = f_{m_1,n_1} u^{\alpha}_{m_2,n_2} \alpha(g)_{m_3,n_3} u^{\alpha,\beta}_{m_4,n_4} v^{\alpha\beta}_{m_5,n_5} u^{m_1} v^{n_1} \circ \dots \circ u^{m_5} v^{n_5}|_{u=v=0},$$

with $m_1 + \dots + m_5 = n_1 + \dots + n_5$.

Theorem 3.3. *There exists an element $R \in \mathcal{H}_1 \otimes \mathcal{H}_1[[\hbar]]$ such that the star product on $C_c^\infty(\mathbb{R} \times \mathbb{R}^+) \rtimes \Gamma$ can be expressed by $f\alpha \star g\beta = m(R(f\alpha \otimes g\beta))$, where $m\colon C_c^\infty(\mathbb{R} \times \mathbb{R}^+) \rtimes \Gamma \otimes C_c^\infty(\mathbb{R} \times \mathbb{R}^+) \rtimes \Gamma \to C_c^\infty(\mathbb{R} \times \mathbb{R}^+) \rtimes \Gamma$ is the multiplication map. Furthermore, R is a universal deformation formula of $\mathcal{H}_1$.*

Proof. We start by rewriting $u^{\alpha}_{m,n}$, $u^{\alpha,\beta}_{m,n}$, $v^{\alpha\beta}_{m,n}$, and $\alpha(\hat{g})$.

(1) As X and Y vanish on 1, $u^{\alpha}_{m,n}$ can be written as y^{m-n} times a sum of terms of powers of X and Y acting on powers of $\delta_2'(\alpha)$. If we rewrite $\delta_2'(\alpha)$ as $\delta_2 - 1/2\delta_1^2$, we can express a term of powers of X and Y acting on powers of $\delta_2'(\alpha)$ as a sum of products $\delta_{i_1}^{j_1}(\alpha)\dots\delta_{i_p}^{j_p}(\alpha)$. According to Proposition 3.2, we know that the negative power of $\hbar$ contained in $u^{\alpha}_{m,n}$ is no more than $[m/3]$. Therefore, we can write $u^{\alpha}_{m,n}$ as $\hbar^{-[\frac{m}{3}]} y^{m-n} \mu_{m,n}(\delta_1(\alpha), \delta_2(\alpha), \dots)$, where $\mu_{m,n}$ is a polynomial of variables $\hbar, \delta_1, \delta_2, \dots$ independent of α.

(2) Analogous to the above analysis, $u^{\alpha,\beta}_{m,n}$ can be written as y^{m-n} times a sum of terms of powers of X and Y acting on products of powers of $\delta_2'(\alpha)$ and $\alpha(\delta_2'(\beta))$. When X and Y act on $\delta_2'(\alpha)$, we can express the resulting terms as polynomials of $\delta_1(\alpha), \dots, \delta_p(\alpha), \dots$. To compute the action by X and Y on $\alpha(\delta_2'(\beta))$, we look at the following properties of X and Y for any function f:

$$\begin{aligned} X(\alpha(f)) &= \alpha(\alpha^{-1}(X(\alpha(f))) \\ &= \alpha(X(f) - \delta_1(\alpha^{-1})Y(f)) \\ &= \alpha(X(f)) - \alpha(\delta_1(\alpha^{-1}))\alpha(Y(f)) \\ &= \alpha(X(f)) + \delta_1(\alpha)\alpha(Y(f)), \\ Y(\alpha(f)) &= \alpha(\alpha^{-1}(Y(\alpha(f)))) = \alpha(Y(f)). \end{aligned}$$

Here we have used the commutation relation between X, Y and α. This implies that powers of X, Y acting on $\alpha(\delta_2'(\beta))$ give a sum of terms $\sigma(\delta_1(\alpha), \dots)\alpha(\xi(\delta_1(\beta), \dots))$ with σ, ξ polynomials in $\hbar, \delta_1, \delta_2, \dots$ independent of α, β. We summarize that $u^{\alpha,\beta}_{m,n}$ can be written as

$$\hbar^{-[\frac{m}{3}]} y^{m-n} \sum_i \nu^i_{m,n}(\delta_1(\alpha), \dots)\alpha(\xi^i_{m,n}(\delta_1(\beta), \dots)),$$

where ν^i, ξ^i are polynomials in $\hbar, \delta_1, \delta_2, \dots$ independent of α, β.

(3) Similar to $u^{\alpha}_{m,n}$, $v^{\alpha\beta}_{m,n}$ can be expressed as a sum of terms of powers of X, Y acting on $\delta_2'(\alpha\beta)$. For our purpose, we need to rewrite $\delta_2'(\alpha\beta)$ as a sum like $\delta_2'(\alpha)+\alpha(\delta_2'(\beta))$. Therefore the situation is similar to $u^{\alpha,\beta}_{m,n}$. We can write $v^{\alpha\beta}_{m,n}$ as

$$\hbar^{-[\frac{m}{3}]}y^{m-n}\sum\eta^j_{m,n}(\delta_1(\alpha),\dots)\alpha(\lambda^j_{m,n}(\delta_1(\beta),\dots)),$$

with $\eta^j_{m,n}$, $\lambda^j_{m,n}$ polynomials independent of α, β.

(4) From the inductive relations, we see that $\alpha(\hat{g})_{m,n}$ can be written as a sum of terms of a product of two parts. One part is powers of X and Y acting on $\delta_2'(\alpha)$, the other is powers of X and Y acting on $\alpha(g)$. We can write the part involving $\delta_2'(\alpha)$ as polynomials of $\delta_1(\alpha)$, $\delta_2(\alpha)$, ..., the part with $\alpha(g)$ like the above $\alpha(\delta_2'(\beta))$ as a sum of terms

$$\varphi(\delta_1(\alpha),\dots)\alpha(\phi(X,Y)(g)).$$

Therefore, we can write $\alpha(\hat{g})_{m,n}$ as

$$\sum y^{m-n}\rho^i_{m,n}(\delta_1(\alpha),\dots)\alpha(\psi^i_{m,n}(X,Y)(g)).$$

Summarizing the above consideration, we can write the term $C_{m_1,\dots,m_5;n_1,\dots,n_5}$ as

$$\begin{aligned}
&c_{m_1,\dots,m_5;n_1,\dots,n_5}\cdot\hbar^{m_1+m_2-[\frac{m_2}{3}]+m_3+m_4-[\frac{m_4}{3}]+m_5-[\frac{m_5}{3}]}\\
&\cdot\sum_{i,j,k}\tau_{m_1,n_1}(X,Y)(f)\mu_{m_2,n_2}(\delta_1(\alpha),\dots)\rho^i_{m_3,n_3}(\delta_1(\alpha),\dots)\alpha(\psi^i_{m_3,n_3}(X,Y)(g))\\
&\quad v^j_{m_4,n_4}(\delta_1(\alpha),\dots)\alpha(\xi^j_{m_4,n_4}(\delta_1(\beta),\dots))\eta^k_{m_5,n_5}(\delta_1(\alpha),\dots)\alpha(\lambda^k_{m_5,n_5}(\delta_1(\beta),\dots)),
\end{aligned}$$

where $c_{m_1,\dots,m_5;n_1,\dots,n_5}$ is a constant.

Now $\hat{f}\circ U_\alpha^{-1}\circ\alpha(\hat{g})\circ\alpha(U_\beta^{-1})\circ\alpha\beta(U^{-1}_{(\alpha\beta)^{-1}})|_{u=v=0}\alpha\beta$ can be written in the form

$$\begin{aligned}
&\sum_{\substack{m_1,\dots,m_5;n_1,\dots,n_5\\ m_1+\cdots+m_5=n_1+\cdots+n_5}} c_{m_1,\dots,m_5;n_1,\dots,n_5}\hbar^{m_1+m_2-[\frac{m_2}{3}]+m_3+m_4-[\frac{m_4}{3}]+m_5-[\frac{m_5}{3}]}\\
&\qquad\cdot\sum_{i,j,k}\tau_{m_1,n_1}(X,Y)(f)\mu_{m_2,n_2}(\delta_1(\alpha),\dots)\rho^i_{m_3,n_3}(\delta_1(\alpha),\dots)\\
&\qquad\quad\cdot\alpha(\psi^i_{m_3,n_3}(X,Y)(g))v^j_{m_4,n_4}(\delta_1(\alpha),\dots)\alpha(\xi^j_{m_4,n_4}(\delta_1(\beta),\dots))\\
&\qquad\quad\cdot\eta^k_{m_5,n_5}(\delta_1(\alpha),\dots)\alpha(\lambda^k_{m_5,n_5}(\delta_1(\beta),\dots))\alpha\beta\\
&=\sum_{m_1,\dots,m_5;n_1,\dots,n_5} c_{m_1,\dots,m_5;n_1,\dots,n_5}\hbar^{m_1+m_2-[\frac{m_2}{3}]+m_3+m_4-[\frac{m_4}{3}]+m_5-[\frac{m_5}{3}]}\\
&\qquad\cdot\sum_{i,j,k}\mu_{m_2,n_2}(\delta_1(\alpha),\dots)\rho^i_{m_3,n_3}(\delta_1(\alpha),\dots)v^j_{m_4,n_4}(\delta_1(\alpha),\dots)\\
&\qquad\quad\cdot\eta^k_{m_5,n_5}(\delta_1(\alpha),\dots)\tau_{m_1,n_1}(X,Y)(f)\alpha(\xi^j_{m_4,n_4}(\delta_1(\beta),\dots)\\
&\qquad\quad\cdot\lambda^k_{m_5,n_5}(\delta_1(\beta),\dots)\psi^i_{m_3,n_3}(X,Y)(g))\alpha\beta
\end{aligned}$$

$$
\begin{aligned}
= \sum_{m_1,\dots,m_5;n_1,\dots,n_5} & c_{m_1,\dots,m_5;n_1,\dots,n_5} \hbar^{m_1+m_2-[\frac{m_2}{3}]+m_3+m_4-[\frac{m_4}{3}]+m_5-[\frac{m_5}{3}]} \\
& \cdot \sum_{i,j,k} \mu_{m_2,n_2}(\delta_1(\alpha),\dots)\rho^i_{m_3,n_3}(\delta_1(\alpha),\dots)\upsilon^j_{m_4,n_4}(\delta_1(\alpha),\dots) \\
& \quad \cdot \eta^k_{m_5,n_5}(\delta_1(\alpha),\dots)\tau_{m_1,n_1}(X,Y)(f)\alpha\xi^j_{m_4,n_4}(\delta_1(\beta),\dots) \\
& \quad \cdot \lambda^k_{m_5,n_5}(\delta_1(\beta),\dots)\psi^i_{m_3,n_3}(X,Y)(g)\beta.
\end{aligned}
$$

Define $R_{m_1,\dots,m_5;n_1,\dots,n_5} \in \mathcal{H}_1[\hbar] \otimes_{\mathbb{C}[\hbar]} \mathcal{H}_1[\hbar]$ as

$$
\begin{aligned}
c_{m_1,\dots,m_5;n_1,\dots,n_5} & \sum_{i,j,k} \mu_{m_2,n_2}(\delta_1,\dots)\rho^i_{m_3,n_3}(\delta_1,\dots)\upsilon^j_{m_4,n_4}(\delta_1,\dots) \\
& \cdot \eta^k_{m_5,n_5}(\delta_1,\dots)\tau_{m_1,n_1}(X,Y) \otimes \xi^j_{m_4,n_4}(\delta_1,\dots)\lambda^k_{m_5,n_5}(\delta_1,\dots)\psi^i_{m_3,n_3}(X,Y).
\end{aligned}
$$

Furthermore, we define

$$
R = \sum_{m_1+\cdots+m_5=n_1+\cdots+n_5} \hbar^{m_1+m_2-[\frac{m_2}{3}]+m_3+m_4-[\frac{m_4}{3}]+m_5-[\frac{m_5}{3}]} R_{m_1,\dots,m_5;n_1,\dots,n_5}.
$$

We conclude that $R \in \mathcal{H}_1[[\hbar]] \otimes_{\mathbb{C}[[\hbar]]} \mathcal{H}_1[[\hbar]]$ satisfies

$$
f\alpha \star g\beta = m(R(f\alpha \otimes g\beta)).
$$

To check that R is a universal deformation formula, we need to make sure that

$$
\begin{aligned}
((\Delta \otimes 1)R)(R \otimes 1) &= ((1 \otimes \Delta)R)(1 \otimes R), \\
(\epsilon \otimes 1)R = 1 \otimes 1 &= (1 \otimes \epsilon)R.
\end{aligned}
$$

The first identity follows from the fact that $\star$ is associative and the $\mathcal{H}_1$-action on the collection of $C^\infty_c(\mathbb{R} \times \mathbb{R}^+) \rtimes \Gamma$ over all pseudogroups Γ is fully injective.

The second identity is equivalent to show that 1 is a unit respect to the $\star$ product on $C^\infty_c(\mathbb{R} \times \mathbb{R}^+) \rtimes \Gamma$. (We adjoin an identity element, the constant function 1, to the algebra $C^\infty_c(\mathbb{R} \times \mathbb{R}^+) \rtimes \Gamma$ and all the quantization constructions in this and previous sections on $C^\infty_c(\mathbb{R} \times \mathbb{R}^+) \rtimes \Gamma$ naturally extend to the unital algebra.)

When f is 1 and α is identity, we have that $\hat{f} = 1$, $U^{-1}_{\mathrm{id}} = 1$, and $U^{-1}_\beta \circ \beta(U^{-1}_{\beta^{-1}}) = 1$. This implies that $1 \star g\beta = g\beta$.

When g is 1 and β is identity, we have that $\alpha(\hat{1}) = 1$, $U^{-1}_\beta = 1$ and $U^{-1}_\alpha \circ \alpha(U^{-1}_{\alpha^{-1}}) = 1$. Therefore, $f\alpha \star 1 = f\alpha$. □

From the computation in the next section, we know that R can be written as $1 \otimes 1 + \hbar R'$, where R' is an element in $\mathcal{H}_1[[\hbar]] \otimes_{\mathbb{C}[[\hbar]]} \mathcal{H}_1[[\hbar]]$. Therefore, R is invertible with $R^{-1} = 1 + \sum_i (-1)^i (\hbar R')^i$. By the property of universal deformation formula, we can introduce a new Hopf algebra structure on $\mathcal{H}_1[[\hbar]]$ by twisting the coproduct Δ by

$$
\tilde{\Delta}(a) = R^{-1}\Delta(a)R,
$$

and the antipode by

$$\tilde{S}(a) = v^{-1} S(a) v,$$

with $v = m(S \otimes 1)(R)$.

4. Formulae of lower order terms

We must say that the formula for R constructed in the previous section (Theorem 3.3) could be very complicated, and we do not know an easy way to write it down explicitly. In this section R will be computed up to the second order of $\hbar$.

It is not difficult to check that if $m_1 + m_2 - [m_2/3] + m_3 + m_4 - [m_4/3] + m_5 - [m_5/3] = 0$ for nonnegative integers m_i, $i = 1, \ldots, 5$, then $m_1 = \cdots = m_5 = 0$. Therefore, the $\hbar^0$ component of the $\star$ product $f\alpha \star g\beta$ is equal to $f\alpha(g)\alpha\beta$. This implies that $R_{0,\ldots,0;0,\ldots,0} = 1$ and $R = 1 \otimes 1 + O(\hbar)$.

Consider $R_{m_1,\ldots,m_5;n_1,\ldots,n_5}$ with $m_1 + m_2 - [m_2/3] + m_3 + m_4 - [m_4/3] + m_5 - [m_5/3] = 1$. It is not difficult to see that one of the m_i, $i = 1, \ldots, 5$, takes value 1, and all others vanish. We also check that $u^{\alpha}_{11} = u^{\alpha}_{01} = u^{\alpha}_{10} = u^{\alpha,\beta}_{11} = u^{\alpha,\beta}_{10} = u^{\alpha,\beta}_{01} = v^{\alpha\beta}_{11} = v^{\alpha\beta}_{10} = v^{\alpha\beta}_{01} = 0$. Therefore, $R_{1,0,\ldots,0;0,0,1,0,0}$ and $R_{0,0,1,0,0;1,0,\ldots,0}$ are the only nonzero terms among all $R_{m_1,\ldots,m_5;n_1,\ldots,n_5}$ with

$$m_1 + m_2 - [m_2/3] + m_3 + m_4 - [m_4/3] + m_5 - [m_5/3] = 1.$$

We compute $R_{1,0,\ldots,0;0,0,1,0,0} = \frac{i\hbar}{2} X \otimes Y$, and $R_{0,0,1,0,0;1,0,\ldots,0} = -\frac{i\hbar}{2}(\delta_1 Y \otimes Y + Y \otimes X)$. Therefore, the $\hbar$ component of R is

$$\frac{-i\hbar}{2}(-X \otimes Y + \delta_1 Y \otimes Y + Y \otimes X) = \frac{-i\hbar}{2}(S(X) \otimes Y + Y \otimes X).$$

Consider $R_{m_1,\ldots,m_5;n_1,\ldots,n_5}$ with $m_1 + m_2 - [m_2/3] + m_3 + m_4 - [m_4/3] + m_5 - [m_5/3] = 2$. There are three classes of possibilities:

i) one of m_2, m_4, m_5 is equal to 3,

ii) one of m_i $(i = 1, \ldots, 5)$ is equal to 2,

iii) two of m_i $(i = 1, \ldots, 5)$ are both equal to 1.

We notice that $u^{\alpha}_{1\cdot} = u^{\alpha}_{2\cdot} = u^{\alpha,\beta}_{1\cdot} = u^{\alpha,\beta}_{2\cdot} = v^{\alpha\beta}_{1\cdot} = v^{\alpha\beta}_{2\cdot} = 0$. This implies that the terms contributing to $m_1 + m_2 - [m_2/3] + m_3 + m_4 - [m_4/3] + m_5 - [m_5/3] = 2$ are from the following three types:

(1) $R_{0,3,0,0,0;1,0,2,0,0}$, $R_{0,3,0,0,0;2,0,1,0,0}$, $R_{0,0,0,3,0;1,0,2,0,0}$, $R_{0,0,0,3,0;2,0,1,0,0}$, $R_{0,0,0,0,3;1,0,2,0,0}$, $R_{0,0,0,0,3;2,0,1,0,0}$;

(2) $R_{0,3,0,0,0;3,0,\ldots,0}$, $R_{0,3,0,0,0;0,0,3,0,0}$, $R_{0,0,0,3,0;3,0,\ldots,0}$, $R_{0,0,0,3,0;0,0,3,0,0}$, $R_{0,0,0,0,3;3,0,\ldots,0}$, $R_{0,0,0,0,3;0,0,3,0,0}$;

(3) $R_{2,0,\ldots,0;0,0,2,0,0}$, $R_{0,0,2,0,0;2,0,0,0,0}$, $R_{1,0,1,0,0;1,0,1,0,0}$.

We compute the above terms separately.

$$R_{0,3,0,0,0;1,0,2,0,0} = (\tfrac{-i\hbar}{2})^3 \tfrac{1}{2} \tfrac{-i}{\hbar} \delta_2' Y \otimes (Y + \tfrac{1}{2})Y = -(\tfrac{-i\hbar}{2})^2 \tfrac{1}{4} \delta_2' Y \otimes (Y + \tfrac{1}{2})Y,$$

$$R_{0,3,0,0,0;2,0,1,0,0} = (\tfrac{-i\hbar}{2})^3 \tfrac{1}{2!} \tfrac{i}{\hbar} \delta_2'(Y + \tfrac{1}{2})Y \otimes Y = (\tfrac{-i\hbar}{2})^2 \tfrac{1}{4} \delta_2'(Y + \tfrac{1}{2})Y \otimes Y,$$

$$R_{0,0,0,3,0;1,0,2,0,0} = -(\tfrac{-i\hbar}{2})^3 \tfrac{1}{2} \tfrac{i}{\hbar} Y \otimes \delta_2'(Y + \tfrac{1}{2})Y = -(\tfrac{-i\hbar}{2})^2 \tfrac{1}{4} Y \otimes \delta_2'(Y + \tfrac{1}{2})Y,$$

$$R_{0,0,0,3,0;2,0,1,0,0} = -(\tfrac{-i\hbar}{2})^3 \tfrac{1}{2} \tfrac{i}{\hbar} (Y + \tfrac{1}{2})Y \otimes \delta_2' Y = -(\tfrac{-i\hbar}{2})^2 \tfrac{1}{4} (Y + \tfrac{1}{2})Y \otimes \delta_2' Y,$$

$$\begin{aligned} R_{0,0,0,0,3;1,0,2,0,0} &= (\tfrac{-i\hbar}{2})^3 \tfrac{1}{2} \tfrac{i}{\hbar} [\delta_2' Y \otimes (Y + \tfrac{1}{2})Y + Y \otimes \delta_2'(Y + \tfrac{1}{2})Y] \\ &= (\tfrac{-i\hbar}{2})^2 \tfrac{1}{4} [\delta_2' Y \otimes (Y + \tfrac{1}{2})Y + Y \otimes \delta_2'(Y + \tfrac{1}{2})Y], \end{aligned}$$

$$\begin{aligned} R_{0,0,0,0,3;2,0,1,0,0} &= (\tfrac{-i\hbar}{2})^3 \tfrac{1}{2} \tfrac{i}{\hbar} [\delta_2'(Y + \tfrac{1}{2})Y \otimes Y + (Y + \tfrac{1}{2})Y \otimes \delta_2' Y] \\ &= (\tfrac{-i\hbar}{2})^2 \tfrac{1}{4} [\delta_2'(Y + \tfrac{1}{2})Y \otimes Y + (Y + \tfrac{1}{2})Y \otimes \delta_2' Y], \end{aligned}$$

$$\begin{aligned} R_{0,3,0,0,0;3,0,0,0,0} &= -(\tfrac{-i\hbar}{2})^3 \tfrac{i}{6\hbar} \delta_2'(Y + 1)(Y + \tfrac{1}{2})Y \otimes 1 \\ &= -(\tfrac{-i\hbar}{2})^2 \tfrac{1}{12} \delta_2'(Y + 1)(Y + \tfrac{1}{2})Y \otimes 1, \end{aligned}$$

$$\begin{aligned} R_{0,3,0,0,0;0,0,3,0,0} &= (\tfrac{-i\hbar}{2})^3 \tfrac{i}{6\hbar} \delta_2' \otimes (Y + 1)(Y + \tfrac{1}{2})Y \\ &= (\tfrac{-i\hbar}{2})^2 \tfrac{1}{12} \delta_2' \otimes (Y + 1)(Y + \tfrac{1}{2})Y, \end{aligned}$$

$$\begin{aligned} R_{0,0,0,3,0;3,0,0,0,0} &= -(\tfrac{-i\hbar}{2})^3 \tfrac{i}{6\hbar} (Y + 1)(Y + \tfrac{1}{2})Y \otimes \delta_2' \\ &= -(\tfrac{-i\hbar}{2})^2 \tfrac{1}{12} (Y + 1)(Y + \tfrac{1}{2})Y \otimes \delta_2', \end{aligned}$$

$$\begin{aligned} R_{0,0,0,3,0;0,0,3,0,0} &= -(\tfrac{-i\hbar}{2})^3 \tfrac{i}{6\hbar} 1 \otimes \delta_2'(Y + 1)(Y + \tfrac{1}{2})Y \\ &= -(\tfrac{-i\hbar}{2})^2 \tfrac{1}{12} 1 \otimes \delta_2'(Y + 1)(Y + \tfrac{1}{2})Y, \end{aligned}$$

$$\begin{aligned} R_{0,0,0,0,3;3,0,0,0,0} &= (\tfrac{-i\hbar}{2})^3 \tfrac{i}{6\hbar} [\delta_2'(Y + 1)(Y + \tfrac{1}{2})Y \otimes 1 \\ &\qquad + (Y + 1)(Y + \tfrac{1}{2})Y \otimes \delta_2'] \\ &= (\tfrac{-i\hbar}{2})^2 \tfrac{1}{12} [\delta_2'(Y + 1)(Y + \tfrac{1}{2})Y \otimes 1 \\ &\qquad + (Y + 1)(Y + \tfrac{1}{2})Y \otimes \delta_2'], \end{aligned}$$

$$\begin{aligned} R_{0,0,0,0,3;0,0,3,0,0} &= (\tfrac{-i\hbar}{2})^3 \tfrac{i}{6\hbar} [\delta_2' \otimes (Y + 1)(Y + \tfrac{1}{2})Y \\ &\qquad + 1 \otimes \delta_2'(Y + 1)(Y + \tfrac{1}{2})Y] \\ &= (\tfrac{-i\hbar}{2})^2 \tfrac{1}{12} [\delta_2' \otimes (Y + 1)(Y + \tfrac{1}{2})Y \\ &\qquad + 1 \otimes \delta_2'(Y + 1)(Y + \tfrac{1}{2})Y], \end{aligned}$$

$$R_{2,0,\ldots,0;0,0,2,0,0} = (\tfrac{-i\hbar}{2})^2 \tfrac{1}{2} X^2 \otimes (Y + \tfrac{1}{2})Y,$$

$$
\begin{aligned}
R_{0,0,2,0,0;2,0,\dots,0,0} = {}& (\tfrac{-i\hbar}{2})^2\tfrac{1}{2}((Y+\tfrac{1}{2})Y\otimes X^2 + \delta_2'(Y+\tfrac{1}{2})Y\otimes Y \\
& + 2\delta_1(Y+\tfrac{1}{2})Y\otimes XY \\
& + \delta_2'(Y+\tfrac{1}{2})Y\otimes X + \delta_1^2(Y+\tfrac{1}{2})Y\otimes Y^2 \\
& + \tfrac{1}{2}\delta_1^2(Y+\tfrac{1}{2})Y\otimes Y - \delta_2' Y(Y+\tfrac{1}{2})\otimes Y),
\end{aligned}
$$

$$
\begin{aligned}
R_{1,0,1,0,0;1,0,1,0,0} = {}& -(\tfrac{-i\hbar}{2})^2(X(Y+\tfrac{1}{2})\otimes X(Y+\tfrac{1}{2}) \\
& + \delta_1 X(Y+\tfrac{1}{2})\otimes Y(Y+\tfrac{1}{2})).
\end{aligned}
$$

Taking the sum of all the above terms, we have that the $\hbar^2$ component of R is equal to

$$
\begin{aligned}
&\Big(\frac{-i\hbar}{2}\Big)^2\Big(\frac{1}{2}\delta_2'\Big(Y+\frac{1}{2}\Big)Y\otimes Y + \frac{1}{6}\delta_2'\otimes(Y+1)\Big(Y+\frac{1}{2}\Big)Y + \frac{1}{2}X^2\otimes\Big(Y+\frac{1}{2}\Big)Y \\
&+\frac{1}{2}\Big(Y+\frac{1}{2}\Big)Y\otimes X^2 - X\Big(Y+\frac{1}{2}\Big)\otimes X\Big(Y+\frac{1}{2}\Big) - \delta_1 X\Big(Y+\frac{1}{2}\Big)\otimes Y\Big(Y+\frac{1}{2}\Big) \\
&+\delta_1\Big(Y+\frac{1}{2}\Big)Y\otimes X\Big(Y+\frac{1}{2}\Big) + \frac{1}{2}\delta_1^2\Big(Y+\frac{1}{2}\Big)Y\otimes Y^2 + \frac{1}{4}\delta_1^2\Big(Y+\frac{1}{2}\Big)Y\otimes Y)\Big) \\
&= \Big(\frac{-i\hbar}{2}\Big)^2\Big(\frac{1}{2}S(X)^2\otimes\Big(Y+\frac{1}{2}\Big)Y + S(X)\Big(Y+\frac{1}{2}\Big)\otimes X\Big(Y+\frac{1}{2}\Big) \\
&\quad + \frac{1}{2}Y\Big(Y+\frac{1}{2}\Big)\otimes X^2 + \frac{1}{2}\delta_2'\Big(Y+\frac{1}{2}\Big)Y\otimes Y + \frac{1}{6}\delta_2'\otimes(Y+1)\Big(Y+\frac{1}{2}\Big)Y \\
&\quad + \frac{1}{2}\delta_2' Y\otimes\Big(Y+\frac{1}{2}\Big)Y\Big).
\end{aligned}
$$

Remark 4.1. The expression of R_2 agrees with the computation in [1], Prop. 6.1, up to a term $\frac{1}{12}(\frac{-i\hbar}{2})^2\delta_2'\otimes Y$. We notice that $\frac{1}{12}(\frac{-i\hbar}{2})^2\delta_2'\otimes Y$ is Hochschild closed. For the purpose of [1], Prop. 6.1, a change of a closed Hochschild 2-cochain on $\hbar^2$ component does not change the answer. This explains the difference.

We remark that when δ_2' is an inner derivation, we can replace δ_2' by $[\Omega,\cdot]$ in the expression of R. But it turns out that the above computed $\hbar^2$-term R_2 does not agree with RC_2 defined by Connes and Moscovici [5], i.e.,

$$
\begin{aligned}
R_2|_{\delta_2'[\Omega,\cdot]} - RC_2 = {}& \Omega Y\otimes Y(2Y+1) + \Omega Y(2Y+1)\otimes Y \\
& + \frac{1}{3}\Omega\otimes(Y+1)(2Y+1)Y - \frac{1}{3}\otimes\Omega(Y+1)(2Y+1)Y.
\end{aligned}
$$

We do not know the explicit relation between these two universal deformation formulas $R_2|_{\delta_2'=[\Omega,\cdot]}$ and RC_2, and have only a heuristic and geometric explanation for their difference. The geometric constructions in [1] and in this article are not exactly the same. In [1], we used a projective structure to redefine a symplectic connection and a Fedosov connection D' on the Weyl algebra bundle $\mathcal{W}$, and therefore a symplectic diffeomorphism preserving this new Fedosov connection naturally lifts to an endomorphism of the quantum algebra $\mathcal{W}_{D'}$. In this article we do not change the symplectic connection because of lack of data, but change the quantization process of a symplectic diffeomorphism by introducing sections such as $U_\alpha, U_\beta, \dots$. Furthermore, we notice that the difference $R_2|_{\delta_2'=[\Omega,\cdot]} - RC_2$ is actually a Hochschild coboundary of a 1-Hochschild cochain $-1/3\Omega(Y+1)(2Y+1)Y$. This suggests that if we define an isomorphism $I = 1 + 1/3\hbar^2\Omega(Y+1)(2Y+1)Y$ on $C_c^\infty(\mathbb{R}\times\mathbb{R}^+)\rtimes\Gamma[[\hbar]]$, then $I^{-1}(m(R(I(a)\otimes I(b)))) = m + \hbar RC_1 + \hbar^2 RC_2 + o(\hbar^2)$. In general, we expect that if $\mathcal{H}_1$ acts on $\mathcal{A}$ with a projective structure Ω, there is an isomorphism I on $\mathcal{A}[[\hbar]]$ that can be expressed using elements in $\mathcal{H}_1[[\hbar]]$ and the projective structure Ω such that

$$I^{-1}(m(R(I(a)\otimes I(b)))) = m(RC(a\otimes b)).$$

5. Appendix: Associativity of the Eholzer product

In this appendix we study associativity of the Eholzer product, which was used in Connes and Moscovici's approach [5] to obtain the general associativity at the Hopf algebra level. This associativity theorem was first proved by Cohen, Manin, and Zagier in [3]. In the first part of this appendix, we give a new proof of the associativity using the method developed by the second author [9]. In the second part, we study an important combinatorial identity used by Cohen, Manin, and Zagier in [3]. This interesting identity was obtained by Zagier [12], but its complete proof is missing in the literature. We prove this identity in the special case corresponding to the Eholzer product.

5.1. Proof of associativity. First we follow the argument developed in [9] according to which the associativity of the product

$$f * g = \sum_{n=0}^{\infty} [f,g]_n \hbar^n$$

is equivalent to prove the identity (with the notation $X_n = \prod_{i=0}^{n-1}(X+i)$)

$$\sum_{r=0} \binom{n-r}{p} \frac{A_{n-r}(2k+2l+2r, 2m) A_r(2k,2l)}{(2k+2l+2r)_{n-p-r}(2m)_p(2k)_r}$$
$$= \sum_{s=0} \binom{n-s}{n-p} \frac{A_{n-s}(2k, 2l+2m+2s) A_s(2l,2m)}{(2k)_{n-p}(2l+2m+2s)_{p-s}(2m)_s},$$

for $p = 0, 1, \dots, n$ and

$$A_n(2k,2l) = \frac{1}{n!}(2k)_n(2l)_n.$$

The above identity is equivalent to

$$\sum_{r=0}\binom{n-r}{p}\frac{\frac{1}{r!}(2k)_r(2l)_r}{(2k)_r}\frac{\frac{1}{(n-r)!}(2k+2l+2r)_{n-r}(2m)_{n-r}}{(2k+2l+2r)_{n-p-r}(2m)_p}$$
$$=\sum_{s=0}\binom{n-s}{n-p}\frac{\frac{1}{s!}(2l)_s(2m)_s}{(2m)_s}\frac{\frac{1}{(n-s)!}(2k)_{n-s}(2l+2m+2s)_{n-s}}{(2l+2m+2s)_{p-s}(2k)_{n-p}}. \tag{11}$$

Our proof is based on manipulation of combinatorial identities. We have, for the left-hand side,

$$\sum_{r=0}\binom{n-r}{p}\frac{\frac{1}{r!}(2k)_r(2l)_r}{(2k)_r}\frac{\frac{1}{(n-r)!}(2k+2l+2r)_{n-r}(2m)_{n-r}}{(2k+2l+2r)_{n-p-r}(2m)_p}$$
$$=\sum_{r=0}\frac{(n-r)!}{p!(n-r-p)!}\frac{1}{r!}\frac{(2k)_r(2l)_r}{(2k)_r}\frac{1}{(n-r)!}\frac{(2k+2l+2r)_{n-r}(2m)_{n-r}}{(2k+2l+2r)_{n-p-r}(2m)_p}$$
$$=\sum_{r=0}\frac{(2l)_r}{r!}\frac{(2k+2l+2r)_{n-r}}{(2k+2l+2r)_{n-p-r}\,p!}\frac{(2m)_{n-r}}{(2m)_p(n-r-p)!} \tag{12}$$
$$=\sum_{r=0}\binom{2l+r-1}{r}\binom{2k+2l+n+r-1}{p}\binom{2m+n-r-1}{n-p-r}.$$

Once n, p are fixed, what needs to be verified is an identity about polynomials in $2k, 2l, 2m$. When $2l$ is a negative integer, we can use the two combinatorial relations

$$\binom{X+n-1}{n}=(-1)^n\binom{-X}{n},\quad n>0,$$

and

$$\sum_i\binom{X}{i}\binom{Y}{n-i}=\binom{X+Y}{n}$$

to get

$$\binom{2l+r-1}{r}=(-1)^r\binom{-2l}{r},$$
$$\binom{2k+2l+n+r-1}{p}=\sum_u\binom{2k+2l+n-1}{p+2l+u}\binom{r}{-2l-u},$$
$$\binom{2m+n-r-1}{n-p-r}=\sum_v\binom{2l+2m+n-1}{n-p+2l+v}\binom{-2l-r}{-2l-r-v}.$$

Then (12) becomes

$$
\begin{aligned}
&= \sum_{r=0} (-1)^r \binom{-2l}{r} \Big[\sum_u \binom{2k+2l+n-1}{p+2l+u} \binom{r}{-2l-u} \Big] \\
&\qquad \Big[\sum_v \binom{2l+2m+n-1}{n-p+2l+v} \binom{-2l-r}{-2l-r-v} \Big] \\
&= \sum_{u,v} \binom{2k+2l+n-1}{p+2l+u} \binom{2l+2m+n-1}{n-p+2l+v} \\
&\qquad \Big[\sum_r (-1)^r \binom{-2l}{r} \binom{r}{-2l-u} \binom{-2l-r}{-2l-r-v} \Big].
\end{aligned}
$$

We then simplify the quantity inside the above brackets by

$$
\begin{aligned}
&\sum_r (-1)^r \binom{-2l}{r} \binom{r}{-2l-u} \binom{-2l-r}{-2l-r-v} \\
&\quad = \sum_r (-1)^r \frac{(-2l)!}{r!(-2l-r)!} \frac{r!}{(-2l-u)!(r+2l+u)!} \frac{(-2l-r)!}{(-2l-r-v)!v!} \\
&\quad = \frac{(-2l)!}{(-2l-u)!v!} \sum_r (-1)^r \frac{1}{(r+2l+u)!(-2l-r-v)!} \\
&\quad = \frac{(-2l)!}{(-2l-u)!v!} \frac{1}{(u-v)!} \sum_r (-1)^r \frac{(u-v)!}{(r+2l+u)!(-2l-r-v)!} \\
&\quad = \frac{(-2l)!}{(-2l-u)!v!} \frac{1}{(u-v)!} \sum_r (-1)^r \binom{u-v}{-2l-r-v} \\
&\quad = \frac{(-2l)!}{(-2l-u)!v!} \frac{1}{(u-v)!} (1-1)^{u-v} (-1)^{-2l-v} = \frac{(-2l)!}{(-2l-u)!v!} (-1)^{-2l-v} \delta_{u,v},
\end{aligned}
$$

where $\delta_{x,y}$ is the Kronecker symbol ($\delta_{x,y} = 1$ if $x = y$, and $\delta_{x,y} = 0$ if $x \neq y$). Finally we get

$$
\begin{aligned}
&\sum_{r=0} \binom{n-r}{p} \frac{\frac{1}{r!}(2k)_r (2l)_r}{(2k)_r} \frac{\frac{1}{(n-r)!}(2k+2l+2r)_{n-r}(2m)_{n-r}}{(2k+2l+2r)_{n-p-r}(2m)_p} \\
&\quad = \sum_{u=v=-2l-t} \binom{2k+2l+n-1}{p+2l+u} \binom{2l+2m+n-1}{n-p+2l+v} \frac{(-2l)!}{(-2l-u)!v!} (-1)^{-2l-v} \\
&\quad = \sum_t \binom{2k+2l+n-1}{p-t} \binom{2l+2m+n-1}{n-p-t} (-1)^t \binom{-2l}{t}.
\end{aligned}
$$

On the right-hand side of (11), we have

$$\sum_{s=0}\binom{n-s}{n-p}\frac{\frac{1}{s!}(2l)_s(2m)_s}{(2m)_s}\frac{\frac{1}{(n-s)!}(2k)_{n-s}(2l+2m+2s)_{n-s}}{(2l+2m+2s)_{p-s}(2k)_{n-p}}$$

$$=\sum_{s=0}\frac{(n-s)!}{(n-p)!(p-s)!}\frac{1}{s!}\frac{(2l)_s(2m)_s}{(2m)_s}\frac{1}{(n-s)!}\frac{(2k)_{n-s}(2l+2m+2s)_{n-s}}{(2k)_{n-p}(2l+2m+2s)_{p-s}}$$

$$=\sum_{s=0}\binom{2l+s-1}{s}\binom{2k+n-s-1}{p-s}\binom{2l+2m+s+n-1}{n-p}.$$

By the same method as before, we compute the above quantity as follows,

$$=\sum_{s=0}(-1)^s\binom{-2l}{s}\Big[\sum_v\binom{2k+2l+n-1}{p+2l+v}\binom{-2l-s}{-2l-s-v}\Big]$$

$$\Big[\sum_u\binom{2l+2m+n-1}{n-p+2l+u}\binom{s}{-2l-u}\Big]$$

$$=\sum_{u,v}\binom{2k+2l+n-1}{p+2l+v}\binom{2l+2m+n-1}{n-p+2l+u}$$

$$\Big[\sum_{s-0}(-1)^s\binom{-2l}{s}\binom{-2l-s}{-2l-s-v}\binom{s}{-2l-u}\Big]$$

$$=\sum_{u=v=-2l-t}\binom{2k+2l+n-1}{p+2l+v}\binom{2l+2m+n-1}{n-p+2l+u}\frac{(-2l)!}{(-2l-u)!v!}(-1)^{-2l-v}$$

$$=\sum_t\binom{2k+2l+n-1}{p-t}\binom{2l+2m+n-1}{n-p-t}(-1)^t\binom{-2l}{t},$$

which gives out the same quantity. We the obtain the following result.

Proposition 5.1. *The Eholzer product is associative.*

Remark 5.2. We have proved that the following combinatorial identity holds for every triple of indices (l_1, l_2, l_3):

$$\sum_{r=0}^{l_1}\sum_{s=0}^{l_2}\sum_{t=0}^{l_3}(-1)^{l_1+l_2-s}\binom{E+l_1+s+l_3-t-1}{l_3-t}\binom{E+r+s-1}{s}f$$

$$\binom{E+l_2+r+t-1}{t}\binom{E+r+s-1}{r}g$$

$$\binom{E+l_1-r+l_2-s+l_3-1}{l_1-r}\binom{E+l_3+l_2-s-1}{l_2-s}h$$

$$
\begin{aligned}
= \sum_{r=0}^{l_1}\sum_{s=0}^{l_2}\sum_{t=0}^{l_3}(-1)^{l_1+l_2-s}&\binom{E+l_1+s+l_3-t-1}{l_3-t}\binom{E+l_1+s-1}{s}f\\
&\binom{E+l_2-s+t-1}{t}\binom{E+l_2+t+r-1}{r}g\\
&\binom{E+l_1-r+l_2-s+l_3-1}{l_1-r}\binom{E+t+l_2-s-1}{l_2-s}h,
\end{aligned}
$$

where E is the Euler operator. This equality identifies the coefficients of $d^u f d^v g d^h$ in $(f * g) * h$ and $f * (g * h)$.

5.2. Zagier's identity. In this part we turn to the original proof by Cohen, Manin and Zagier [3] of the Eholzer product. Their proof relies on the following combinatorial identity:

$$
\frac{(-4)^n}{\binom{2x}{n}}\sum_{r+s=n}\frac{\binom{y}{r}\binom{y-a}{r}}{\binom{2y}{r}}\frac{\binom{z}{s}\binom{z+a}{s}}{\binom{2z}{s}} = \sum_{j\geq 0}\binom{n}{2j}\frac{\binom{-\frac{1}{2}}{j}\binom{a-\frac{1}{2}}{j}\binom{-a-\frac{1}{2}}{j}}{\binom{x-\frac{1}{2}}{j}\binom{y-\frac{1}{2}}{j}\binom{z-\frac{1}{2}}{j}}, \tag{13}
$$

where $n \geq 0$ and the variables a, x, y, z satisfy $x + y + z = n - 1$.

Here we give a proof of this identity when $a = \frac{1}{2}$. We start with some transformations. Our aim is to eliminate the binomial coefficients in the denominator of both sides. Using the identity

$$
\frac{1}{\binom{X}{r}} = \frac{\binom{X-r}{n-r}}{\binom{X}{n}}\frac{r!(n-r)!}{n!},
$$

we rewrite the left-hand side of (13) as

$$
\frac{(-4)^n}{\binom{2x}{n}}\sum_{r=0}^{n}\frac{\binom{y}{r}\binom{y-a}{r}\binom{2y-r}{n-r}}{\binom{2y}{n}}\frac{\binom{z}{n-r}\binom{z+a}{n-r}\binom{2z-n+r}{r}}{\binom{2z}{n}}\frac{(r!(n-r)!)^2}{(n!)^2}. \tag{14}
$$

Using the identity

$$
\binom{2Y}{2j} = \binom{Y-\frac{1}{2}}{j}\binom{Y}{j}\frac{(j!)^2 4^j}{(2j)!},
$$

we rewrite the right-hand side of (13) in the form

$$
\begin{aligned}
&= \sum_{j=0}^{[\frac{n}{2}]}\binom{n}{2j}\binom{-\frac{1}{2}}{j}\binom{a-\frac{1}{2}}{j}\binom{-a-\frac{1}{2}}{j}\frac{\binom{x}{j}4^j(j!)^2}{\binom{2x}{2j}(2j)!}\frac{\binom{y}{j}4^j(j!)^2}{\binom{2y}{2j}(2j)!}\frac{\binom{z}{j}4^j(j!)^2}{\binom{2z}{2j}(2j)!}\\
&= \sum_{j=0}^{[\frac{n}{2}]}\binom{n}{2j}\binom{-\frac{1}{2}}{j}\binom{a-\frac{1}{2}}{j}\binom{-a-\frac{1}{2}}{j}\left[\frac{4^j(j!)^2(n-2j)!}{n!}\right]^3 \\
&\qquad\qquad \frac{\binom{x}{j}\binom{2x-2j}{n-2j}}{\binom{2x}{n}}\frac{\binom{y}{j}\binom{2y-2j}{n-2j}}{\binom{2y}{n}}\frac{\binom{z}{j}\binom{2z-2j}{n-2j}}{\binom{2z}{n}}.
\end{aligned} \tag{15}
$$

By combining (14) and (15) we can then multiply both sides of (13) by the common denominator $\binom{2x}{n}\binom{2y}{n}\binom{2z}{n}$. We obtain a polynomial $P_n(y,z,a)$ on the left-hand side of (13),

$$P_n(y,z,a) := (-4)^n \sum_{r=0}^{n} \binom{y}{r}\binom{y-a}{r}\binom{2y-r}{n-r} \binom{z}{n-r}\binom{z+a}{n-r}\binom{2z-n+r}{r}(r!(n-r)!)^2,$$

and a polynomial $Q_n(y,z,a)$ on the right-hand side of (13), (we replace x by $n-y-z-1$)

$$Q_n(y,z,a) := \sum_{j=0}^{[\frac{n}{2}]} \frac{((n-2j)!)^2(j!)^6 2^{6j}}{(2j)!}\binom{-\frac{1}{2}}{j}\binom{a-\frac{1}{2}}{j}\binom{-a-\frac{1}{2}}{j} \binom{n-y-z-1}{j}\binom{2n-2y-2z-2-2j}{n-2j} \binom{y}{j}\binom{2y-2j}{n-2j}\binom{z}{j}\binom{2z-2j}{n-2j}.$$

In summary, identity (13) is equivalent to the identity $P_n(y,z,a) = Q_n(y,z,a)$ for $n \geq 0$. In order to have the associativity of the Eholzer product, we shall prove this identity when $a = \frac{1}{2}$. Explicitly, we want to prove the following:

$$(-1)^n \sum_{r=0}^{n} \binom{2y}{2r}\binom{2y-r}{n-r}\binom{2z+1}{2(n-r)}\binom{2z-n+r}{r}\frac{(2r)!(2(n-r))!}{(n!)^2} = \binom{2n-2y-2z-2}{n}\binom{2y}{n}\binom{2z}{n}. \tag{16}$$

5.2.1. Simplification. We apply the following identities to the left-hand side of (16):

$$\frac{\binom{2y}{2r}\binom{2y-r}{n-r}(2r)!}{n!} = \binom{2y}{n}\binom{2y-r}{r}\frac{r!}{(n-r)!},$$

$$\frac{\binom{2z+1}{2(n-r)}\binom{2z-n+r}{r}(2(n-r))!}{n!} = \binom{2z}{n}\left[\binom{2z-n+r}{n-r} + 2\binom{2z-n+r}{n-r-1}\right]\frac{(n-r)!}{r!}.$$

Taking the quotients on both sides of (16) by $\binom{2y}{n}\binom{2z}{n}$, we have the equivalent identity

$$(-1)^n \sum_{r=0}^{n} \binom{2y-r}{r}\left[\binom{2z-n+r}{n-r} + 2\binom{2z-n+r}{n-r-1}\right] = \binom{2n-2y-2z-2}{n}.$$

We can rewrite the above identity as

$$\sum_{r=0}^{n}\binom{2y-r}{r}\Big[\binom{2z-n+r}{n-r}+2\binom{2z-n+r}{n-r-1}\Big]=\binom{2y+2z-n+1}{n}, \quad (17)$$

using $\binom{2n-2y-2z-2}{n}=(-1)^n\binom{2y+2z-n+1}{n}$.

5.2.2. The sums $S_0(n;A,B)$ and $S(n;X)$. Consider the sum

$$S_0(n;A,B)=\sum_{k=0}^{n}\binom{k+A}{n-k}\binom{n-k+B}{k}.$$

We have $S_0(0;A,B)=1$, and by $\binom{X+1}{n}=\binom{X}{n}+\binom{X}{n-1}$,

$$\begin{aligned}S_0(n+1;A,B)&=\sum_{k=0}^{n+1}\binom{k+A}{n+1-k}\binom{n+1-k+B}{k}\\&=S_0(n;A-1,B+1)+S_0(n+1;A-1,B).\end{aligned}$$

In the same way, $S_0(n+1;A,B)=S_0(n;A+1,B-1)+S_0(n+1;A,B-1)$.

Moreover, we define

$$S(n;X)=\sum_{p=0}^{[\frac{n}{2}]}\binom{X+n-1-2p}{n-2p}.$$

It is clear that $S_0(0;X)=1$ and $S(n+1;X)=S(n+1;X-1)+S(n;X)$ for the same reason as above.

Another useful relation is that

$$S(n,X-1)+2S(n-1,X)=\binom{X+n}{n}, \quad (18)$$

because

$$\begin{aligned}&S(n,X-1)+2S(n-1,X)\\&\quad=S(n;X)+S(n-1;X)\\&\quad=\sum_{p=0}^{[\frac{n}{2}]}\binom{X+n-1-2p}{n-2p}+\sum_{p=0}^{[\frac{n-1}{2}]}\binom{X+n-2-2p}{n-1-2p}\\&\quad=\sum_{i=0}^{n}(-1)^i\binom{-X}{i}=(-1)^n\binom{-X-1}{n}=\binom{X+n}{n}.\end{aligned}$$

The following lemma holds:

Lemma 5.3. $S_0(n; A, B) = S(n; A + B)$.

Proof. We use induction on n. For $n = 0$ the assertion obviously holds. If the claim is valid for $0, 1, \ldots, n$, then by the above two induction relations we have

$$\begin{aligned}
&S_0(n+1; A, B) - S(n+1; A+B) \\
&\qquad = (S_0(n+1; A, B-1) + S_0(n; A+1, B-1)) \\
&\qquad\quad - (S(n+1; A+B-1) + S(n; A+B)) \\
&\qquad = (S_0(n+1; A, B-1) - S(n+1; A+B-1)) \\
&\qquad\quad + (S_0(n; A+1, B-1) - S(n; A+B)) \\
&\qquad = S_0(n+1; A, B-1) - S(n+1; A+B-1).
\end{aligned}$$

Using the same method, we can prove that this difference is also equal to $S_0(n+1; A-1, B) - S(n+1; A+B-1)$. Hence it follows that the difference $S_0(n+1; A, B) - S(n+1; A+B)$ has the same value for all pairs $(A, B) \in \mathbb{N}^2$. But by definition we know that

$$S(n+1; 0, 0) - S(n+1; 0) = \begin{cases} 0 - 0 = 0 & \text{if } n \text{ is odd,} \\ 1 - 1 = 0 & \text{if } n \text{ is even.} \end{cases}$$

We conclude that the identity holds for all n and all pairs $(A, B) \in \mathbb{N}^2$. But since what we want to prove is a polynomial identity in A and B (for fixed n), the identity for all natural numbers imply its correctness for arbitrary (A, B). □

5.2.3. Resummation

Theorem 5.4. *The identity* (17) *is valid.*

Proof. In fact, by using (18), we have

$$\begin{aligned}
&\sum_{r=0}^{n} \binom{2y-r}{r} \left[\binom{2z-n+r}{n-r} + 2\binom{2z-n+r}{n-r-1} \right] \\
&\qquad = [S_0(n; 2z-n, 2y-n) + 2S_0(n-1; 2z-n, 2y-n+1)] \\
&\qquad = [S(n; 2y+2z-2n) + 2S(n-1; 2y+2z-2n+1)] \\
&\qquad = \binom{2y+2z-n+1}{n}.
\end{aligned}$$

□

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