Complete positivity of Rieffel's deformation quantization by actions of $\mathbb{R}^d$

Daniel Kaschek, Nikolai Neumaier, and Stefan Waldmann

Abstract. In this paper we consider C*-algebraic deformations by actions of $\mathbb{R}^d$ à la Rieffel and show that every state of the undeformed algebra can be deformed into a state of the deformed algebra in the sense of a continuous field of states. The construction is explicit and involves a convolution operator with a particular Gauß function.

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1. Introduction

In deformation quantization [1] the transition from classical mechanics to quantum mechanics is obtained as an associative deformation of the classical observable algebra, modelled by a certain class of functions on the classical phase space. In *formal* deformation quantization this is accomplished by constructing a new associative product, the star product, as a formal power series with formal parameter $\hbar$. While this theory is by now very well understood, see [3], [13], [14], [16], [18], [24] for existence and classification results, and [29] for a gentle introduction, from a physicist's perspective the formal character of the star products is still not satisfying: $\hbar$ is not a formal parameter after all, whence at the end of the day, some sort of "convergence" in $\hbar$ is needed.

Attacking the convergence problem of the formal series seems to be complicated though in examples this can be done [2]. More successful are approaches that are intrinsically non-formal like the Berezin–Toeplitz inspired quantizations [4], [9], [10], [11], [12] or Rieffel's approach using oscillatory integrals based on group actions of $\mathbb{R}^d$. In this version [25], the starting point is a C*-algebra $\mathfrak{A}$ endowed with an isometric, strongly continuous action by *-automorphisms by some finite-dimensional vector space V. Out of this and the choice of a symplectic form on V, Rieffel constructs a deformation of $\mathfrak{A}$ in the sense of a continuous field of C*-algebras, the field parameter being $\hbar$. While the construction is very general, there are yet many

examples of Poisson manifolds which can be deformation quantized this way. In this framework of strict deformations many results have been obtained, most notably [21], [23].

While the above constructions deal with the observable algebra, for a physically complete description of quantization also the states have to be taken into account. In both approaches, the appropriate notion of states is that of positive linear functionals on the observable algebras. While for C*-algebras this is of course a well-known concept, also in the formal deformation quantization this leads to a physically reasonable definition incorporating a reasonable representation theory; see, e.g., [5], [7], [28] and references therein.

A fundamental question is whether a given classical state arises as the classical limit of a quantum state. In formal deformation quantization there is a general and affirmative answer to this question [6], [8]. In the strict approaches, Landsman discussed this in [20] for a certain class of examples: the appropriate notion of classical limit and deformation of states is that of a continuous field of states with respect to a given continuous field of C*-algebras. His construction is based on particular *-representations and certain coherent states and their Wigner functions. More recently, Landsman uses continuous fields of states in his discussion of the Born rule [22].

In this article we consider Rieffel's deformation by actions of $\mathbb{R}^d$ in general and prove that every state of the undeformed algebra can be deformed into a continuous field of states for the field of deformed algebras. Moreover, we give an explicit construction including a detailed study of the asymptotics of the deformed states for $\hbar \to 0$; see also [17]. It turns out that the asymptotic expansion coincides in a very precise sense with the formal deformations obtained in [6].

The article is organized as follows: in Section 2 we recall Rieffel's deformation in the Fréchet algebraic framework and define an operator $S_\hbar$ being the "convolution" with a Gauß function. The precise form of $S_\hbar$ resembles the Wigner functions Landsman used, however now $S_\hbar$ is defined directly on the algebra. The asymptotics of $S_\hbar$ for $\hbar \to 0^+$ is studied in detail. In Section 3 we show that $S_\hbar$ maps squares $a^* \star_\hbar a$ of the deformed algebra to positive elements of the undeformed algebra. This allows to define a positive functional $\omega_\hbar = \omega \circ S_\hbar$ of the deformed algebra for every positive functional ω of the undeformed algebra. A detailed asymptotic expansion is obtained as well. Section 4 is devoted to the more particular case of a C*-algebra deformation. Here we show that the operator $S_\hbar$ is also continuous in the C*-topology of the deformed algebra whence it extends to the C*-algebraic completion. Finally, in Section 5 we show that the positive functionals $\{\omega_\hbar\}_{\hbar \geq 0}$ indeed form a continuous field of states.

2. The operator S on $\mathscr{A}^\infty$

In this section $\mathscr{A}$ denotes a Fréchet *-algebra endowed with a strongly continuous action α by *-homomorphisms of a finite-dimensional vector space V which we assume without restriction to be even dimensional. Moreover, one requires that there is a system of seminorms $\| \cdot \|_k$ defining the topology of $\mathscr{A}$ such that with respect to these seminorms the action is isometric. By $\mathscr{A}^\infty \subseteq \mathscr{A}$ we denote the subspace of smooth vectors in $\mathscr{A}$ with respect to α. It is well known that $\mathscr{A}^\infty$ is a dense *-subalgebra of $\mathscr{A}$. Moreover, $\mathscr{A}^\infty$ carries a finer topology making it into a Fréchet algebra, too. A system of seminorms defining the topology is explicitly given by

$$\|a\|_{k,\mu} = \sup_{|\beta| \leq \mu} \|\partial^\beta a\|_k,$$

where using multi-index notation $\partial^\beta a$ denotes the derivative of $\alpha_u(a)$ with respect to u at $u = 0$; see, e.g., [27] for more background on smooth vectors.

In a next step one chooses a non-degenerate bilinear anti-symmetric form θ on V and $\hbar > 0$. Then Rieffel showed in [25] that

$$a \star_\hbar b = \frac{1}{(\pi\hbar)^{2n}} \int_{V \times V} \alpha_u(a)\alpha_v(b) e^{\frac{2i}{\hbar}\theta(u,v)} \, d(u,v), \tag{2.1}$$

which is defined for $a, b \in \mathscr{A}^\infty$, yields a well-defined associative product such that $\star_\hbar$ is still continuous with respect to the $\mathscr{A}^\infty$-topology. Moreover, the original *-involution of $\mathscr{A}^\infty$ is still a *-involution with respect to $\star_\hbar$. The precise definition of the integral in an oscillatory sense is sophisticated and can be found in Rieffel's booklet [25]. Note that we have to choose a normalization for the Haar measure on V in (2.1). We shall also make use of linear coordinates denoted by $v = u_i e_i$ in the sequel.

Definition 2.1 (The operator S_g). Let $g \colon V \times V \to \mathbb{R}$ be a positive definite inner product on V. Then the linear operator $S_g \colon \mathscr{A} \to \mathscr{A}$ is defined by

$$S_g(a) = \int_V e^{-g(u,u)} \alpha_u(a) \, du. \tag{2.2}$$

Thanks to the fast decay of the Gauß function and the fact that the action α is isometric, the definition of S_g in (2.2) as an improper Riemann integral is possible. More general, we need the following construction: let $B(V, \mathscr{A})$ denote the $\mathscr{A}$-valued functions on V such that $\sup_{v \in V} \|f(v)\|_k < \infty$ for all k, i.e., the bounded functions with respect to the seminorms $\| \cdot \|_k$ of $\mathscr{A}$. Then we define

$$\widetilde{S}_g f = \int_V e^{-g(u,u)} f(u) \, du$$

for $f \in B(V, \mathcal{A})$. Again, a naive definition of the integral is possible. Finally, let $C_u^0(V, \mathcal{A})$ be the uniformly continuous functions in $B(V, \mathcal{A})$ and let $C_u^\infty(V, \mathcal{A})$ be the smooth functions with all partial derivatives in $C_u^0(V, \mathcal{A})$. Clearly, the spaces $C_u^0(V, \mathcal{A})$ as well as $C_u^\infty(V, \mathcal{A})$ are equipped with a natural Fréchet topology by taking the sup-norm over V of seminorms of the values of the (derivatives of the) functions. Then the following Proposition lists some properties of S_g and $\widetilde{S}_g$:

Proposition 2.2 (Continuity of S_g).

(1) *The operator $S_g : \mathcal{A} \to \mathcal{A}$ is continuous.*

(2) *We have $S_g(\mathcal{A}^\infty) \subseteq \mathcal{A}^\infty$ and $S_g : \mathcal{A}^\infty \to \mathcal{A}^\infty$ is continuous, too.*

(3) *The restriction of $\widetilde{S}_g$ to $C_u^0(V, \mathcal{A})$ and $C_u^\infty(V, \mathcal{A})$ is continuous in the respective topologies.*

(4) *The restriction of $\widetilde{S}_g$ to $C_u^0(V, \mathcal{A}^\infty)$ and $C_u^\infty(V, \mathcal{A}^\infty)$ takes values in $\mathcal{A}^\infty$ and is again continuous.*

Proof. The first two statements can be recovered from the third and fourth by considering the function $f(u) = \alpha_u(a)$ for $a \in \mathcal{A}$ or $a \in \mathcal{A}^\infty$, respectively: as the action is isometric we have $f \in C_u^0(V, \mathcal{A})$ and $C_u^\infty(V, \mathcal{A}^\infty)$, respectively. The continuity statements in the third and fourth part are then a straightforward estimate. $\square$

In a next step we want to understand the asymptotics of the operator S_g. To this end we rescale the inner product by $\hbar > 0$ and consider the normalized Gauß function

$$G_\hbar(u) = \frac{\sqrt{\det G}}{(\pi \hbar)^n} e^{-\frac{g(u,u)}{\hbar}}, \tag{2.3}$$

where $\det G > 0$ is the determinant of g with respect to the Haar measure on V and $2n = \dim V$. The normalization constant is chosen such that the integral of $G_\hbar$ is 1. For a fixed choice of g we consider the operator

$$S_\hbar(a) = \int_V G_\hbar(u) \alpha_u(a) \, du. \tag{2.4}$$

Lemma 2.3. *For every $a \in \mathcal{A}$ we have $\lim_{\hbar \searrow 0} S_\hbar(a) = a$ in the topology of $\mathcal{A}$. Moreover, for every $a \in \mathcal{A}^\infty$ we have*

$$\lim_{\hbar \searrow 0} S_\hbar(a) = a \tag{2.5}$$

and

$$\frac{d}{d\hbar}(S_\hbar a) = S_\hbar\left(\tfrac{1}{4} \Delta_g a\right), \tag{2.6}$$

both with respect to the topology of $\mathcal{A}^\infty$ where

$$\Delta_g a = \sum_{i,j}(G^{-1})^{ij}\frac{\partial^2}{\partial u_i\,\partial u_j}\alpha_u(a)\Big|_{u=0}$$

is the Laplacian with respect to the inner product g and the action α viewed as continuous operator on $\mathcal{A}^\infty$. The operator Δ_g does not depend on the choice of linear coordinates.

Proof. By substitution $u \to \sqrt{\hbar}u$ we have

$$S_\hbar(a) = \frac{\sqrt{\det G}}{\pi^n}\int_V e^{-g(u,u)}\alpha_{\sqrt{\hbar}u}(a)\,\mathrm{d}u.$$

To exchange the order of integration and $\lim\limits_{\hbar\searrow 0}$ we consider

$$\left\|\int_V e^{-g(u,u)}(\alpha_{\sqrt{\hbar}u}(a)-a)\,\mathrm{d}u\right\|_{k,\mu}$$
$$\leq \int_K e^{-g(u,u)}\|\alpha_{\sqrt{\hbar}u}(a)-a\|_{k,\mu}\,\mathrm{d}u + \int_{V\setminus K} e^{-g(u,u)}\|\alpha_{\sqrt{\hbar}u}(a)-a\|_{k,\mu}\,\mathrm{d}u,$$

where K denotes a compact set in V. For $\hbar \searrow 0$ the function $\alpha_\hbar(a)\colon u \mapsto \alpha_{\sqrt{\hbar}u}(a)$ converges uniformly to the constant function $u \mapsto a$ on every compact set in V. Furthermore, since α is isometric, the estimate $\|\alpha_{\sqrt{\hbar}u}(a)-a\|_{k,\mu} \leq 2\|a\|_{k,\mu}$ holds for all $u \in V$. Thus, choosing K large enough makes the second term small, independently of $\hbar$. Afterwards, choosing $\hbar$ small brings the first term for the fixed K below every positive bound. By the choice of the normalization constant in front of the Gauß function this shows (2.5). The case for $a \in \mathcal{A}$ is analogous. For the last statement we first note that for a fixed a the differentiation in V-directions is a limit in $C_u^\infty(V,\mathcal{A}^\infty)$. By the linearity and continuity of $\widetilde{S}_g$ as in Proposition 2.2 we can thus exchange differentiation and the integral. This gives

$$\frac{\mathrm{d}}{\mathrm{d}\hbar}S_\hbar a = \frac{\mathrm{d}}{\mathrm{d}\hbar}\int_V\frac{\sqrt{\det G}}{\pi^n}e^{-g(u,u)}\alpha_{\sqrt{\hbar}u}(a)\,\mathrm{d}u$$

$$= \frac{\sqrt{\det G}}{\pi^n}\int_V e^{-g(u,u)}\sum_i\frac{u_i}{2\sqrt{\hbar}}\frac{\partial}{\partial u_i}\alpha_u(a)\Big|_{\sqrt{\hbar}u}\,\mathrm{d}u$$

$$= -\frac{1}{4\sqrt{\hbar}}\frac{\sqrt{\det G}}{\pi^n}\int_V\sum_{i,j}(G^{-1})^{ij}\frac{\partial}{\partial u_j}e^{-g(u,u)}\frac{\partial}{\partial u_i}\alpha_u(a)\Big|_{\sqrt{\hbar}u}\,\mathrm{d}u$$

$$= \frac{1}{4}\frac{\sqrt{\det G}}{\pi^n}e^{-g(u,u)}(G^{-1})^{ij}\frac{\partial^2}{\partial u_i\,\partial u_j}\alpha_u(a)\Big|_{\sqrt{\hbar}u}\,\mathrm{d}u$$

$$= \frac{1}{4}\frac{\sqrt{\det G}}{\pi^n}e^{-g(u,u)}\alpha_{\sqrt{\hbar}u}\left((G^{-1})^{ij}\frac{\partial^2}{\partial v_i\,\partial v_j}\alpha_v(a)\Big|_{v=0}\right)\mathrm{d}u,$$

where we have used an integration by parts as well as the fact that α is an action. Note that the operator Δ_g is well defined on $\mathcal{A}^\infty$. This completes the proof. $\qquad\square$

Since with $a \in \mathcal{A}^\infty$ we also have $\Delta_g a \in \mathcal{A}^\infty$, the iteration of (2.6) immediately yields the following statement:

Theorem 2.4 (Asymptotic expansion of $S_\hbar$). *The operator* $S_\hbar \colon \mathcal{A}^\infty \to \mathcal{A}^\infty$ *has the formal asymptotic expansion*

$$S_\hbar \simeq_{\hbar \searrow 0} \mathrm{e}^{\frac{\hbar}{4}\Delta_g}$$

with respect to the topology of $\mathcal{A}^\infty$.

This means that the asymptotic expansion of $S_\hbar$ corresponds to the formal equivalence transformation leading from the Weyl star product to the Wick product; see, e.g., [29], eq. (5.84).

3. Deformation of positive functionals

Recall that a functional $\omega \colon \mathcal{A} \to \mathbb{C}$ is called positive if for all $a \in \mathcal{A}$ we have

$$\omega(a^*a) \geq 0.$$

While this is a purely algebraic definition, for a topological algebra $\mathcal{A}$ we require furthermore that ω is *continuous*. An algebra element $a \in \mathcal{A}$ is called *positive* if $\omega(a) \geq 0$ for all (continuous) positive functionals ω. The positive algebra elements will be denoted by $\mathcal{A}^+$. Note that for general *-algebras a definition of positivity like $a = b^*b$ will not lead to a reasonable notion of positive elements due to the lack of a functional calculus. Note also that the above definition coincides with the usual definition of positive elements in the case of a C*-algebra. There are even more sophisticated notions of positivity, e.g., for O^*-algebras; see the discussion in [26]. However, for our purposes the above definition will be sufficient as for C*-algebras positive functionals are always continuous.

Now we can use the operator $S_\hbar$ to deform a positive functional of $\mathcal{A}$ into a positive functional with respect to $\star_\hbar$. To this end we observe the following lemma:

Lemma 3.1. *For $a \in \mathcal{A}^\infty$ we have*

$$S_\hbar(a^* \star_\hbar a)$$
$$= \frac{1}{(\pi\hbar)^{2n}} \int_{V \times V} \mathrm{e}^{-\frac{1}{\hbar}g(v,v)} \alpha_v(a^*) \mathrm{e}^{-\frac{1}{\hbar}g(w,w)} \alpha_w(a) \mathrm{e}^{\frac{2}{\hbar}(g(v,w)+\mathrm{i}\theta(v,w))} \, \mathrm{d}v \, \mathrm{d}w.$$

Proof. The proof is a straightforward computation using the fact that α is an action as well as a linear change of coordinates and a Fourier transform of the Gauß function. □

In the particular case that g and θ are *compatible*, i.e., $g(u, v) = \theta(u, Jv)$ with a complex structure J, the combination $h(u, v) = g(u, v) + \mathrm{i}\theta(u, v)$ is known to be a Hermitian metric on the complex vector space (V, J). In this case there exists a symplectic basis $\{e_1, \ldots, e_n, f_1, \ldots, f_n\}$ of V with coordinates q^i and p^i and there exist complex coordinates $z^i = q^i + \mathrm{i}p^i$ and $\bar{z}^i = q^i - \mathrm{i}p^i$ such that

$$g(u, u) = \sum_i z_u^i \bar{z}_u^i = \|z_u\|^2 \quad \text{and} \quad h(v, w) = \sum_i \bar{z}_v^i z_w^i.$$

From now on we assume that g is compatible with θ. Using these coordinates, the above integral can be rewritten as

$$S_\hbar(a^* \star_\hbar a) = \frac{1}{(\pi\hbar)^{2n}} \int_{V \times V} \mathrm{e}^{-\frac{1}{\hbar}\|z_v\|^2} \alpha_v(a^*) \mathrm{e}^{-\frac{1}{\hbar}\|z_w\|^2} \alpha_w(a) \mathrm{e}^{\frac{2}{\hbar}\bar{z}_v \cdot z_w} \, \mathrm{d}v \, \mathrm{d}w. \quad (3.1)$$

Lemma 3.2. *For $a \in \mathcal{A}^\infty$ we have*

$$S_\hbar(a^* \star_\hbar a) = \sum_{L \geq 0} \frac{2^{|L|}}{L!} a_L^* a_L \quad (3.2)$$

with respect to the $\mathcal{A}^\infty$-topology, where for a multi-index $L = (l_1, \ldots, l_n)$ one defines

$$a_L = \frac{1}{\pi^n} \int_V \mathrm{e}^{-\|z_v\|^2} z_v^L \alpha_{\sqrt{\hbar}v}(a) \, \mathrm{d}v.$$

Proof. First note that rescaling the variables in (3.1) by $\sqrt{\hbar}$ allows to get rid of the negative powers of $\hbar$. Then (3.2) is obtained from expanding the exponential function $\mathrm{e}^{2\bar{z}_v \cdot z_w}$ into the Taylor series and exchanging summation and integration. The fact that the latter exchange of limits is allowed follows from a similar argument as in the proof of Lemma 2.3: First we split the integration into two parts, one over a compact subset $K \subseteq V$ and the other over $V \setminus K$. On K the Taylor expansion converges uniformly including all derivatives. Outside K, the Gauß function decays fast enough to over-compensate the exponential increase. Thus first choosing K large enough to make the second integral small then using the uniform convergence gives the result. Note that the convergence is in the sense of $\mathcal{A}^\infty$. □

Theorem 3.3 (Positive deformation of ω). *Let g be a compatible positive definite inner product and $S_\hbar$ the corresponding operator as in* (2.4).

(1) *For every continuous positive linear functional $\omega\colon \mathscr{A} \to \mathbb{C}$ the functional*

$$\omega_\hbar = \omega \circ S_\hbar \colon \mathscr{A}^\infty \to \mathbb{C}$$

is positive and continuous in the $\mathscr{A}^\infty$-topology.

(2) *For every $a \in \mathscr{A}^\infty$ we have*

$$S_\hbar(a^* \star_\hbar a) \in \mathscr{A}^+.$$

Proof. Let $\omega\colon \mathscr{A} \to \mathbb{C}$ be positive and continuous. Since the topology of $\mathscr{A}^\infty$ is finer than the original one, it follows that $\omega\colon \mathscr{A}^\infty \to \mathbb{C}$ is still continuous. Therefore $\omega(S_\hbar(a^* \star_\hbar a)) \geq 0$ follows immediately from (3.2) and the continuity of ω. Moreover, since $S_\hbar$ is continuous the first part follows. Thus the second part is clear. $\qquad\square$

Corollary 3.4. *Let $\omega\colon \mathscr{A} \to \mathbb{C}$ be a positive and continuous linear functional. Then on $\mathscr{A}^\infty$, $\omega_\hbar = \omega \circ S_\hbar$ has the asymptotic expansion*

$$\omega_\hbar \simeq_{\hbar \searrow 0} \sum_{r=0}^{\infty} \frac{1}{r!} \left(\frac{\hbar}{4}\right)^r \omega \circ \Delta_g^r$$

in the $\mathscr{A}^\infty$-topology.

Remark 3.5. This kind of formal positive deformation of a positive functional was discussed in [6] based on the formal equivalence between the Weyl and Wick star product.

4. The operator S in the C*-case

In a next step we want to apply Theorem 3.3 to the more particular case of a C*-algebraic deformation. Let $\mathfrak{A}$ be a unital C*-algebra endowed with an isometric and strongly continuous action of V by *-automorphisms. Then Rieffel has shown how to construct a C*-norm on the Fréchet *-algebra $\mathcal{A}(\hbar) = (\mathfrak{A}^\infty, \star_\hbar, {}^*)$. In general, $\mathcal{A}(\hbar)$ is not complete. The norm completion of $\mathcal{A}(\hbar)$ will then be denoted by $\mathfrak{A}(\hbar)$.

We briefly recall the construction of the C*-norm on $\mathcal{A}(\hbar)$. Let $\mathcal{S}(V, \mathfrak{A}) \subseteq C_u^\infty(V, \mathfrak{A})$ be the subset of functions which are still in $C_u^\infty(V, \mathfrak{A})$ when multiplied by arbitrary polynomials on V. For $f, g \in \mathcal{S}(V, \mathfrak{A})$ one defines the $\mathfrak{A}$-valued inner product

$$\langle f, g \rangle = \int_V f(v)^* g(v) \, \mathrm{d}v,$$

which makes $\mathcal{S}(V, \mathfrak{A})$ into a pre-Hilbert right $\mathfrak{A}$-module; see, e.g., [19] for details on Hilbert modules. In particular, by

$$\|f\|_{\mathcal{S}} = \sqrt{\|\langle f, f \rangle\|}$$

one obtains a norm on $\mathcal{S}(V, \mathfrak{A})$, where the norm on the right-hand side is the C*-norm of $\mathfrak{A}$. Using this norm, Rieffel showed that for every $F \in C_u^\infty(V, \mathfrak{A})$ the operator

$$F \star_\hbar \cdot : \mathcal{S}(V, \mathfrak{A}) \ni f \mapsto F \star_\hbar f \in \mathcal{S}(V, \mathfrak{A})$$

is continuous with respect to $\|\cdot\|_{\mathcal{S}}$ and adjointable with adjoint given by $F^* \star_\hbar \cdot$. Since for $a \in \mathfrak{A}^\infty$ the function $u \mapsto \alpha_u(a)$ is in $C_u^\infty(V, \mathfrak{A})$ we obtain an induced operator on the pre-Hilbert module $\alpha(a) \star_\hbar \cdot$ which is continuous and adjointable. A final computation then shows that $a \mapsto \alpha(a) \star_\hbar \cdot$ is a *-homomorphism with respect to the deformed product $\star_\hbar$ of $\mathfrak{A}^\infty$. This allows to define

$$\|a\|_\hbar = \|\alpha(a) \star_\hbar \cdot\|,$$

where on the right-hand side we use the operator norm. Since it is well known that the continuous and adjointable operators on a (pre-)Hilbert module constitute a C*-algebra, Rieffel arrives at a C*-norm $\|\cdot\|_\hbar$ for $\mathcal{A}(\hbar)$.

   We want to show that the operator $S_\hbar$ being defined only on $\mathcal{A}(\hbar)$ is also continuous in the C*-norm and thus extends to $\mathfrak{A}(\hbar)$. To show the continuity of $S_\hbar$ we will need the following lemma that shows that there is a star root of the Gauß function.

Lemma 4.1. *Let $G_\hbar$ be the normalized Gauß function as in* (2.3) *used to define the operator $S_\hbar$. Then we have*

$$G_\hbar \star G_\hbar = \frac{1}{(2\pi\hbar)^n} \frac{1}{\sqrt{\det G}} G_\hbar.$$

Proof. The proof is a straightforward and well-known computation; see, e.g., [16], Prop. 3.3.1. □

   From equation (3.2) and the trivial fact that $\sqrt{\det G} a_{L=0} = S_\hbar(a)$ we obtain the following statement:

Lemma 4.2 (Leading order of $S_\hbar(a^* \star_\hbar a)$). *For $a \in \mathcal{A}^\infty$ we have*

$$S_\hbar(a^* \star_\hbar a) = \frac{1}{\det G} S_\hbar(a^*) S_\hbar(a) + b,$$

where $b \in \mathcal{A}^+$ is positive.

Theorem 4.3. *Let $(\mathfrak{A}, \cdot, \|\cdot\|)$ be a C*-algebra with isometric and strongly continuous action α of V and let $\mathcal{A}(\hbar) = (\mathfrak{A}^\infty, \star_\hbar, \|\cdot\|_\hbar)$ be the Rieffel deformed pre-C*-algebra. Then the operator*

$$S_\hbar \colon \mathcal{A}(\hbar) \to \mathfrak{A}$$

is a continuous operator in the C*-norms of $\mathcal{A}(\hbar)$ and $\mathfrak{A}$.

Proof. Since $\mathfrak{A}$ is a C*-algebra, we have $\|S_\hbar a\|^2 = \|(S_\hbar a)^*(S_\hbar a)\|$. From Lemma 4.2 it follows that $(S_\hbar a)^*(S_\hbar a) \leq \det(G) S_\hbar(a^* \star_\hbar a)$ in the sense of positive elements in $\mathfrak{A}$. From this it follows that the same holds for the norms, i.e., $\|(S_\hbar a)^*(S_\hbar a)\| \leq \det(G)\|S_\hbar(a^* \star_\hbar a)\|$. In order to compute the last norm we need the following fact that

$$\int_V f \star_\hbar g = \int_V fg \tag{4.1}$$

for all $f, g \in \mathcal{S}(V, \mathfrak{A})$; see [25], Lemma 3.8. Moreover, due to the fast decay of functions in $\mathcal{S}(V, \mathfrak{A})$, eq. (4.1) still holds if one of them is in $C_u^\infty(V, \mathfrak{A})$. Using this and Lemma 4.1 we find

$$\|S_\hbar(a^* \star_\hbar a)\| = \det(G)\left\| \int_V (G_\hbar \star_\hbar \alpha(a^* \star_\hbar a))(u)\,\mathrm{d}u \right\|$$

$$= (2\pi\hbar)^n (\det(G))^{\frac{3}{2}} \left\| \int_V (G_\hbar \star_\hbar G_\hbar \star_\hbar \alpha(a^* \star_\hbar a))(u)\,\mathrm{d}u \right\|$$

$$= (2\pi\hbar)^n (\det(G))^{\frac{3}{2}} \left\| \int_V (G_\hbar \star_\hbar \alpha(a)^* \star_\hbar \alpha(a) \star_\hbar G_\hbar)(u)\,\mathrm{d}u \right\|$$

$$= (2\pi\hbar)^n (\det(G))^{\frac{3}{2}} \|\langle \alpha(a) \star_\hbar G_\hbar, \alpha(a) \star_\hbar G_\hbar \rangle\|$$

$$\leq (2\pi\hbar)^n (\det(G))^{\frac{3}{2}} \|G_\hbar\|_{\mathcal{S}}^2 \|a\|_\hbar^2,$$

by observing that $G_\hbar$ is central for the *undeformed* pointwise product of $C_u^\infty(V, \mathfrak{A})$. Thus we have the desired continuity

$$\|S_\hbar a\|^2 \leq (2\pi\hbar)^n (\det(G))^{\frac{3}{2}} \|G_\hbar\|_{\mathcal{S}}^2 \|a\|_\hbar^2. \tag{4.2}$$

$\square$

Corollary 4.4. *Let $\omega \colon \mathfrak{A} \to \mathbb{C}$ be a positive linear functional of the undeformed C*-algebra. Then $\omega_\hbar = \omega \circ S_\hbar \colon \mathcal{A}(\hbar) \to \mathbb{C}$ is continuous with respect to $\|\cdot\|_\hbar$ and extends to a positive linear functional $\omega_\hbar \colon \mathfrak{A}(\hbar) \to \mathbb{C}$.*

Thus we have constructed for every classical state ω a corresponding quantum state using the operator $S_\hbar$. We shall also use the symbol

$$S_\hbar \colon \mathfrak{A}(\hbar) \to \mathfrak{A}$$

for the extension of the operator $S_\hbar$ to the completions in the corresponding C*-topologies.

5. Continuous fields of states

In a last step we want to discuss in which sense $\omega_\hbar$ can be considered as a deformation of ω: clearly we have $\omega(a) = \lim_{\hbar \searrow 0} \omega_\hbar(a)$ pointwise for every $a \in \mathcal{A}^\infty$ but we want to show some continuity properties beyond that trivial observation.

One of the main results in Rieffel's work [25] is that the deformed C*-algebras $\{\mathfrak{A}(\hbar)\}_{\hbar \geq 0}$ actually yield a *continuous field* in the sense of Dixmier [15]: Recall that a continuous field structure on a collection $\{\mathfrak{A}(\hbar)\}_{\hbar \geq 0}$ of C*-algebras consists in the choice of *continuous sections* $\Gamma \subseteq \prod_{\hbar \geq 0} \mathfrak{A}(\hbar)$ subject to the following technical conditions: Γ is a *-algebra with respect to the pointwise product of the sections and for each fixed $\hbar$ the set of possible values $\{a(\hbar)\}_{a \in \Gamma} \subseteq \mathfrak{A}(\hbar)$ is dense. For unital C*-algebras, we require that the unit section $\hbar \mapsto \mathbb{1}(\hbar) = \mathbb{1}_{\mathfrak{A}(\hbar)}$ be always in Γ. Moreover, the function $\hbar \mapsto \|a(\hbar)\|_\hbar$ is continuous for all $a \in \Gamma$. Finally, if an arbitrary section $b \in \prod_{\hbar \geq 0} \mathfrak{A}(\hbar)$ can locally be approximated uniformly by continuous sections, it is already continuous itself, i.e., if b is a section such that for all $\varepsilon > 0$ and all $\hbar_0$ there exists an open neighborhood $U \subseteq [0, \infty)$ of $\hbar_0$ and a continuous section $a \in \Gamma$ such that $\|a(\hbar) - b(\hbar)\|_\hbar \leq \varepsilon$ uniformly for all $\hbar \in U$, then $b \in \Gamma$. It follows that Γ necessarily contains $C^0(\mathbb{R}_0^+)$.

In the case of the Rieffel deformation the *-algebra of continuous sections Γ can be obtained from the "constant" sections $a(\hbar) = a \in \mathfrak{A}^\infty$. In detail, one has the following (technical) characterization:

Proposition 5.1. *Let* $\mathcal{A}(\hbar) = (\mathfrak{A}^\infty, \star_\hbar, \|\cdot\|_\hbar)$ *be the Rieffel deformed pre-C*-algebras and let* $\{\mathfrak{A}(\hbar)\}_{\hbar \geq 0}$ *be the corresponding field of C*-algebras. Moreover, let*

$$\Gamma = \{b \in \textstyle\prod_{\hbar \geq 0} \mathfrak{A}(\hbar) \mid \forall \varepsilon > 0 \forall \hbar_0 \geq 0 \exists U(\hbar_0) \exists a \in \Gamma_0 \forall \hbar \in U(\hbar_0):$$
$$\|b(\hbar) - a(\hbar)\|_\hbar \leq \varepsilon\}$$

be the set of sections generated by the set Γ_0 *of sections. Then for all three choices,*

(1) $\Gamma_0 = \mathfrak{A}^\infty$,

(2) $\Gamma_0 = C^0(\mathbb{R}_0^+) \otimes \mathfrak{A}^\infty$,

(3) Γ_0 *is the* *-algebra generated by the vector space* $C^0(\mathbb{R}_0^+) \otimes \mathfrak{A}^\infty$ *with respect to* $\star_\hbar$,

the set Γ *is the same and defines the structure of a continuous field.*

In other words, the *-algebra Γ of continuous sections yields the smallest continuous field built on the collection $\{\mathfrak{A}(\hbar)\}_{\hbar \geq 0}$ which contains the constant sections $a\colon \hbar \mapsto a(\hbar) = a \in \mathfrak{A}^{\infty}$. The second choice of Γ_0 is the smallest $C^0(\mathbb{R}_0^+)$-module, while the last choice corresponds to the smallest *-algebra containing $\mathfrak{A}^{\infty}$ and $C^0(\mathbb{R}_0^+)$. In the following we shall always refer to this continuous field structure Γ.

Turning back to the states we want to show that the set of states $\omega_{\hbar} = \omega \circ S_{\hbar}$, where $\omega\colon \mathfrak{A} \to \mathbb{C}$ is a classical state, form a continuous field of states in the following sense; see, e.g., [21], Def. 1.3.1:

Definition 5.2 (Continuous field of states). A continuous field of states on a continuous field of C*-algebras $(\{\mathfrak{A}(\hbar)\}_{\hbar \geq 0}, \Gamma)$ is a family of states $\omega_{\hbar}$ on $\mathfrak{A}(\hbar)$ such that

$$\hbar \mapsto \omega_{\hbar}(a(\hbar))$$

is continuous for every continuous section $a \in \Gamma$.

Lemma 5.3. *If $a \in \Gamma$ is a continuous section, then the map $\mathbb{R}_0^+ \ni \hbar \mapsto S_{\hbar}a(\hbar) \in \mathfrak{A}$ is continuous in the* (*undeformed*) *C*-norm of $\mathfrak{A}$.*

Proof. Note that here we use the extension of $S_{\hbar}$ to the completion $\mathfrak{A}(\hbar)$. Moreover, by Proposition 5.1 we can approximate a by sections in $\Gamma_0 = C^0(\mathbb{R}_0^+) \otimes \mathfrak{A}^{\infty}$. First, we show the continuity at $\hbar \neq 0$:

$$\begin{aligned}
\|S_{\hbar}a(\hbar) - S_{\hbar'}a(\hbar')\| &\leq \|S_{\hbar}a(\hbar) - S_{\hbar}a(\hbar')\| + \|S_{\hbar}a(\hbar') - S_{\hbar'}a(\hbar')\| \\
&= \|S_{\hbar}(a(\hbar) - a_{\Delta\hbar}(\hbar))\| + \|(S_{\hbar} - S_{\hbar'})(a(\hbar'))\| \\
&\leq c(\hbar)\|a(\hbar) - a_{\Delta\hbar}(\hbar)\|_{\hbar} + \|(S_{\hbar} - S_{\hbar'})(a(\hbar'))\|.
\end{aligned}$$

Here $a_{\Delta\hbar}(\hbar) = a(\hbar + \Delta\hbar)$ with $\Delta\hbar = \hbar' - \hbar$ and $c(\hbar)$ is the constant from the estimate (4.2). It is now easy to see that the section $a_{\Delta\hbar}$ is approximated by sections of the form $\sum_n \tau_{\Delta\hbar} f_n a_n$, where $(\tau_{\Delta\hbar} f_n)(\hbar) = f_n(\hbar + \Delta\hbar)$. Thus $a_{\Delta\hbar}$ is still in Γ and approximates a for $\Delta\hbar \to 0$. Hence the first term becomes small for $\hbar' \to \hbar$. The second term requires more attention. We can approximate a by sections of the form $\sum_n f_n a_n \in \Gamma_0$ with a finite sum and $f_n \in C^0(\mathbb{R}_0^+)$ and $a_n \in \mathfrak{A}^{\infty}$. Then we have

$$\begin{aligned}
\|(S_{\hbar} - S_{\hbar'})(a(\hbar'))\| &\leq \|S_{\hbar}(a(\hbar') - \sum f_n(\hbar')a_n)\| \\
&\quad + \|(S_{\hbar} - S_{\hbar'})(\sum f_n(\hbar')a_n)\| + \|S_{\hbar'}(a(\hbar') - \sum f_n(\hbar')a_n)\| \\
&\leq c(\hbar)\|a_{\Delta\hbar}(\hbar) - \sum \tau_{\Delta\hbar} f_n(\hbar)a_n\|_{\hbar} \\
&\quad + \|\sum f_n(\hbar')a_n\| \int |G_{\hbar}(u) - G_{\hbar'}(u)|\, \mathrm{d}u \\
&\quad + c(\hbar')\|a(\hbar') - \sum f_n(\hbar')a_n\|_{\hbar'}.
\end{aligned}$$

The constants $c(\hbar)$ and $c(\hbar')$ are bounded in a small neighborhood of $\hbar \neq 0$. Since the functions f_n are continuous, $\|\sum f_n(\hbar')a_n\|$ is bounded on a neighborhood. The other factors become smaller than any $\varepsilon > 0$ for $\hbar' \to \hbar$. This shows the continuity at $\hbar \neq 0$. For the continuity at 0 we have with $S_0 = \mathrm{id}$:

$$\|S_\hbar(a(\hbar)) - S_0(a(0))\| \leq \|S_\hbar(a(\hbar) - S_\hbar(a(0)))\| + \|S_\hbar(a(0)) - a(0)\|.$$

The first term gives

$$\left\| \int G_\hbar(u)\alpha_u(a(\hbar) - a(0))\, du \right\| \leq \|a(\hbar) - a(0)\| \int G_\hbar(u)\, du$$
$$= \|a(\hbar) - a(0)\|,$$

since the Gauß function is normalized and α is isometric. Now $a(\hbar) = a_\hbar(0)$ approximates $a(0)$ in a neighborhood of zero whence this contribution becomes small for $\hbar \searrow 0$. The second term becomes small thanks to the asymptotics from Lemma 2.3 in the topology of $\mathfrak{A}$. This shows the continuity at 0, too. $\qquad\square$

From this lemma we immediately obtain the main result:

Theorem 5.4. *For every classical state $\omega\colon \mathfrak{A} \to \mathbb{C}$ and for every continuous section $a \in \Gamma$ the map*

$$\hbar \mapsto \omega(S_\hbar(a(\hbar))) = \omega_\hbar(a(\hbar))$$

is continuous. Hence $\{\omega_\hbar\}_{\hbar \geq 0}$ is a continuous field of states with $\omega_0 = \omega$.

Remark 5.5 (Completely positive deformation). Since with $\mathfrak{A}$ also the matrices $M_n(\mathfrak{A})$ carry an induced action of V, we can repeat the whole deformation process for $M_n(\mathfrak{A})$. Then it is easy to see that the deformations of $(M_n(\mathfrak{A}))(\hbar)$ are just $M_n(\mathfrak{A}(\hbar))$. Thus the above statement on the deformation of states applies to $M_n(\mathfrak{A})$, too. In [8], such deformations were called *completely positive deformations*. Of course, here we obtain this statement in a strict framework and not for formal power series in $\hbar$.

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D. Kaschek, N. Neumaier, S. Waldmann, Fakultät für Mathematik und Physik,
Albert-Ludwigs-Universität Freiburg, Physikalisches Institut, Hermann Herder Straße 3,
79104 Freiburg, Germany

E-mail: Daniel.Kaschek@physik.uni-freiburg.de;
Nikolai.Neumaier@physik.uni-freiburg.de; Stefan.Waldmann@physik.uni-freiburg.de