

## A period map for generalized deformations

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**Abstract.** For every compact Kähler manifold we give a canonical extension of Griffith’s period map to generalized deformations, intended as solutions of Maurer–Cartan equation in the algebra of polyvector fields. Our construction involves the notion of Cartan homotopy and a canonical  $L_\infty$ -structure on mapping cones of morphisms of differential graded Lie algebras.

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### Introduction

Let  $X$  be a compact Kähler manifold and denote by  $H^*(X, \mathbb{C})$  the graded vector space of its de Rham cohomology. The goal of this article is to define a natural transformation

$$\Phi: \text{Def}_X \rightarrow \text{Aut}_{H^*(X, \mathbb{C})}$$

from infinitesimal deformations of  $X$  to automorphisms of  $H^*(X, \mathbb{C})$ . More precisely, for every local Artinian  $\mathbb{C}$ -algebra  $A$  and every deformation of  $X$  over  $\text{Spec}(A)$  we define in a functorial way a canonical morphism of schemes

$$\text{Spec}(A) \rightarrow \text{GL}(H^*(X, \mathbb{C})) = \prod_n \text{GL}(H^n(X, \mathbb{C})).$$

Our construction will be carried out by using the interplay of Cartan homotopies and  $L_\infty$ -morphisms and is compatible with classical constructions of the theory of infinitesimal variations of Hodge structures [11], [21]. In particular:

- (1) Via the natural isomorphism  $H_X \simeq \bigoplus_{p,q} H^q(\Omega_X^p)$  induced by Dolbeault’s theorem and the  $\partial\bar{\partial}$ -lemma, the differential of  $\Phi$ ,

$$d\Phi: H^1(T_X) \rightarrow \text{Hom}^0(H^*(X, \mathbb{C}), H^*(X, \mathbb{C})),$$

is identified with the contraction operator:  $d\Phi(\xi) = \mathbf{i}_\xi$  where  $\mathbf{i}_\xi(\omega) = \xi \lrcorner \omega$ .

(2) The contraction

$$i : H^2(T_X) \rightarrow \text{Hom}^1(H^*(X, \mathbb{C}), H^*(X, \mathbb{C}))$$

is a morphism of obstruction theories. In particular, since  $\text{Aut}_{H^*(X, \mathbb{C})}$  is smooth, every obstruction to deformation of  $X$  is contained in the kernel of  $i$ .

(3) For every  $m$  let  $H^*(F^m) \subseteq H^*(X, \mathbb{C})$  be the subspace of cohomology classes of closed  $(p, q)$ -forms, with  $p \geq m$ . Then the composition of  $\Phi$  with the natural projection

$$\begin{aligned} \text{GL}(H^*(X, \mathbb{C})) &\rightarrow \text{Grass}(H^*(X, \mathbb{C})) = \prod_n \text{Grass}(H^n(X, \mathbb{C})), \\ f &\mapsto f(H^*(F^m)), \end{aligned}$$

is the classical period map.

We will define and study the morphism  $\Phi$  using the framework of  $L_\infty$ -algebras. It is however useful to give also a more geometric definition in the following way.

Denote by  $A_X = \bigoplus_i A_X^i$  the space of complex valued differential forms on  $X$ , by  $d = \partial + \bar{\partial} : A_X^i \rightarrow A_X^{i+1}$  the de Rham differential and by  $\partial A_X \subseteq A_X$  the subspace of  $\partial$ -exact forms. A small variation of the almost complex structure is determined by a form  $\xi \in A_X^{0,1}(T_X)$ : according to Newlander–Nirenberg theorem, the integrability condition of  $\xi$  is equivalent to  $(d + I_\xi)^2 = 0$ , where  $I_\xi : A_X^i \rightarrow A_X^{i+1}$  is the holomorphic Lie derivative, defined by the formula

$$I_\xi(\omega) = \partial(\xi \lrcorner \omega) + \xi \lrcorner \partial\omega.$$

Notice that  $I_\xi(\ker \partial) \subseteq \partial A_X$ .

Assume therefore  $(d + I_\xi)^2 = 0$ . According to  $\partial\bar{\partial}$ -lemma, the complex  $(\partial A_X, d)$  is acyclic so that if  $\xi$  is sufficiently small, the complex  $(\partial A_X, d + I_\xi)$  is still acyclic.

In order to define the automorphism  $\Phi_\xi : H^*(X, \mathbb{C}) \rightarrow H^*(X, \mathbb{C})$ , let  $[\omega] \in H^*(X, \mathbb{C})$  and choose a  $d$ -closed form  $\omega_0 \in A_X$  representing  $[\omega]$  and such that  $\partial\omega_0 = 0$ . Since

$$(d + I_\xi)\omega_0 = \partial(\xi \lrcorner \omega_0) \in \partial A_X \quad \text{and} \quad (d + I_\xi)^2\omega_0 = 0,$$

there exists  $\beta \in A_X$  such that  $(d + I_\xi)\omega_0 = (d + I_\xi)\partial\beta$ . If  $i_\xi$  is the contraction, then it is not difficult to prove that  $d(e^{i_\xi}(\omega_0 - \partial\beta)) = 0$ , and the cohomology class of  $e^{i_\xi}(\omega_0 - \partial\beta)$  does not depend on the choice of  $\beta$  and  $\omega_0$ , allowing to define  $\Phi_\xi([\omega])$  as the cohomology class of  $e^{i_\xi}(\omega_0 - \partial\beta)$ .

Equivalently, for every  $d$ -closed form  $\omega \in A_X$  and every small variation of the complex structure  $\xi$  we have

$$\Phi_\xi([\omega]) = [e^{i_\xi}(\omega - d\gamma - \partial\beta)], \quad \text{where } \partial\omega = \partial d\gamma, \quad I_\xi(\omega - d\gamma) = (d + I_\xi)\partial\beta.$$

Moreover, as direct consequence of the  $L_\infty$ -approach, we will see that  $\Phi_\xi$  is invariant under the gauge action, where two integrable small variation of the almost

complex structure  $\xi_1, \xi_2$  are gauge equivalent if and only if they give isomorphic deformations of  $X$ .

Our construction generalizes in a completely straightforward way to *generalized deformations* of  $X$ , defined as the solutions, up to gauge equivalence, of the Maurer–Cartan equation in the differential graded Lie algebra

$$\text{Poly}_X = \bigoplus_i \text{Poly}_X^i, \quad \text{Poly}_X^i = \bigoplus_{b-a=i-1} A_X^{0,b}(\wedge^a T_X),$$

endowed with the opposite of Dolbeault differential and the Schouten–Nijenhuis bracket.

Putting together all these facts, at the end we get for every  $m$  a commutative diagram of morphism of functors of Artin rings

$$\begin{array}{ccc} \widetilde{\text{Def}}_X & \xrightarrow{\Phi} & \text{Aut}_{H^*(X, \mathbb{C})} \\ \uparrow i & & \downarrow \pi \\ \text{Def}_X & \xrightarrow{p} & \text{Grass}_{H^*(F^m), H^*(X, \mathbb{C})} \end{array}$$

where  $\widetilde{\text{Def}}_X$  is the functor of generalized deformations of  $X$ ,  $i$  is the natural inclusion,  $\text{Grass}_{H^*(F^m), H^*(X, \mathbb{C})}$  is the Grassmann functor with base point  $H^*(F^m)$ ,  $\pi$  is the smooth morphism defined as  $\pi(f) = f(H^*(F^m))$  and  $p$  is the classical  $m^{\text{th}}$  period map.

In view of this result is natural to candidate the composition  $\pi \Phi$  as period map for generalized deformations.

**Example.** Our definition is compatible with yet existing notion of period map for generalized deformations of Calabi–Yau manifolds used in some mirror symmetry constructions [3].

In fact, if  $X$  is a Calabi–Yau manifold with volume element  $\Omega$ , then by the Tian–Todorov Lemma, every generalized deformation over  $\text{Spec}(A)$  is represented by an element  $\xi \in \text{Poly}_X^1 \otimes \mathfrak{m}_A$  such that

$$D\xi + \frac{1}{2}[\xi, \xi] = 0, \quad \partial(\xi \lrcorner \Omega) = 0.$$

Under these assumptions our recipe gives

$$\Phi_\xi([\Omega]) = [e^{i\xi}(\Omega)]$$

and therefore we recover the construction of [2].

**Keywords and general notation.** We assume that the reader is familiar with the notion and main properties of differential graded Lie algebras and  $L_\infty$ -algebras (we refer to [8], [14], [15], [16], [19] as introduction to such structures); however the basic

definitions are recalled in this article in order to fix notation and terminology. For the whole article,  $\mathbb{K}$  is a field of characteristic 0; every vector space is intended over  $\mathbb{K}$ .  $\mathbf{Art}$  is the category of local Artinian  $\mathbb{K}$ -algebras with residue field  $\mathbb{K}$ . For  $A \in \mathbf{Art}$  we denote by  $\mathfrak{m}_A$  the maximal ideal of  $A$ . By abuse of notation, if  $F: \mathbf{Art} \rightarrow \mathbf{Set}$  is a functor, we write  $\xi \in F$  to mean  $\xi \in F(A)$  for some fixed  $A \in \mathbf{Art}$ .

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### 1. Deformation functors associated with DGLA morphisms

We recall from [20] that to any morphism  $\chi: L \rightarrow M$  of differential graded Lie algebras over a field  $\mathbb{K}$  of characteristic 0 are naturally associated two functors  $\text{MC}_\chi, \text{Def}_\chi: \mathbf{Art} \rightarrow \mathbf{Set}$  of Artin rings in the following way:

$$\begin{aligned} \text{MC}_\chi(A) &= \{(x, e^a) \in (L^1 \otimes \mathfrak{m}_A) \times \exp(M^0 \otimes \mathfrak{m}_A) \mid dx + \frac{1}{2}[x, x] = 0, \\ &\qquad\qquad\qquad e^a * \chi(x) = 0\}, \\ \text{Def}_\chi(A) &= \frac{\text{MC}_\chi(A)}{\text{gauge equivalence}}, \end{aligned}$$

where two solutions of the Maurer–Cartan equation are gauge equivalent if they belong to the same orbit of the gauge action

$$(\exp(L^0 \otimes \mathfrak{m}_A) \times \exp(dM^{-1} \otimes \mathfrak{m}_A)) \times \text{MC}_\chi(A) \xrightarrow{*} \text{MC}_\chi(A)$$

given by the formula

$$(e^l, e^{dm}) * (x, e^a) = (e^l * x, e^{dm} e^a e^{-\chi(l)}) = (e^l * x, e^{dm \bullet a \bullet (-\chi(l))}).$$

The  $\bullet$  in the rightmost term in the above formula is the Baker–Campbell–Hausdorff multiplication; namely  $e^x e^y = e^{x \bullet y}$ . Note that if  $L = 0$  and the differential on  $M$  is trivial, then

$$\text{Def}_\chi(A) = \exp(M^0 \otimes \mathfrak{m}_A).$$

It has been shown in [6] that the suspended cone of  $\chi$ , i.e., the differential complex  $(C_\chi, \mu_1)$  given by the graded vector space

$$C_\chi = \bigoplus_i C_\chi^i, \quad C_\chi^i = L^i \oplus M^{i-1},$$

endowed with the differential

$$\mu_1(l, m) = (dl, \chi(l) - dm), \quad l \in L, m \in M,$$

carries a natural compatible  $L_\infty$ -algebra structure, which we shall denote  $\tilde{C}(\chi)$ , such that the associated deformation functor  $\text{Def}_{\tilde{C}(\chi)}$  is naturally isomorphic to  $\text{Def}_\chi$ . More precisely, the map  $(l, m) \mapsto (l, e^m)$  induces a natural isomorphism  $\text{MC}_{\tilde{C}(\chi)} \xrightarrow{\sim} \text{MC}_\chi$ , and homotopy equivalence on  $\text{MC}_{\tilde{C}(\chi)}$  is identified with gauge equivalence on  $\text{MC}_\chi$ .

The higher brackets

$$\mu_n : \bigwedge^n C_\chi \rightarrow C_\chi[2 - n], \quad n \geq 2,$$

defining the  $L_\infty$ -algebra structure  $\tilde{C}(\chi)$  have been explicitly described in [6]. Namely, one has

$$\mu_2((l_1, m_1) \wedge (l_2, m_2)) = ([l_1, l_2], \frac{1}{2}[m_1, \chi(l_2)] + \frac{(-1)^{\text{deg}(l_1)}}{2}[\chi(l_1), m_2])$$

and for  $n \geq 3$

$$\begin{aligned} \mu_n((l_1, m_1) \wedge \cdots \wedge (l_n, m_n)) \\ = \pm \frac{B_{n-1}}{(n-1)!} \sum_{\sigma \in S_n} \varepsilon(\sigma) [m_{\sigma(1)}, [\dots, [m_{\sigma(n-1)}, \chi(l_{\sigma(n)})] \dots]]. \end{aligned}$$

Here the  $B_n$ 's are the Bernoulli numbers,  $\varepsilon$  is the Koszul sign, and we refer to [6] for the exact determination of the overall  $\pm$  sign in the above formulas (it will not be needed in the present article). Note that the projection on the first factor  $\pi_1 : \tilde{C}(\chi) \rightarrow L$  is a linear  $L_\infty$ -morphism.

By the functoriality of  $\chi \mapsto \tilde{C}(\chi)$ , if

$$\begin{array}{ccc} L_1 & \xrightarrow{f_L} & L_2 \\ \chi_1 \downarrow & & \downarrow \chi_2 \\ M_1 & \xrightarrow{f_M} & M_2 \end{array}$$

is a commutative diagram of morphisms of differential graded Lie algebras, then  $(x, e^a) \mapsto (f_L(x), e^{f_M(a)})$  is a natural transformation of Maurer–Cartan functors inducing a natural transformation

$$\text{Def}_{\chi_1} \rightarrow \text{Def}_{\chi_2}.$$

Moreover, if  $f_L$  and  $f_M$  are quasi-isomorphisms, then  $\text{Def}_{\chi_1} \xrightarrow{\sim} \text{Def}_{\chi_2}$  is an isomorphism.

## 2. An example from Kähler geometry

Let  $X$  be a compact Kähler manifold. Consider the DGLA  $\text{Hom}^*(A_X, A_X)$  of graded endomorphisms of the de Rham complex and their subDGLAs

$$\begin{aligned} L &= \{f \in \text{Hom}^*(A_X, A_X) \mid f(\ker \partial) \subseteq \partial A_X\}, \\ M &= \{f \in \text{Hom}^*(A_X, A_X) \mid f(\ker \partial) \subseteq \ker \partial \text{ and } f(\partial A_X) \subseteq \partial A_X\}. \end{aligned}$$

Then we have a commutative diagram of morphisms of DGLAs, where the vertical arrows are the inclusions

$$\begin{array}{ccccc}
 0 & \longleftarrow & L & \xlongequal{\quad} & L \\
 \downarrow \rho & & \downarrow \eta & & \downarrow \chi \\
 \text{Hom}^* \left( \frac{\ker \partial}{\partial A_X}, \frac{\ker \partial}{\partial A_X} \right) & \longleftarrow & M & \longrightarrow & \text{Hom}^*(A_X, A_X).
 \end{array}$$

By the  $\partial\bar{\partial}$ -lemma, we have quasi-isomorphisms

$$(A_X, d) \leftarrow (\ker \partial, d) \rightarrow \left( \frac{\ker \partial}{\partial A_X}, 0 \right) \cong (H^*(X, \mathbb{C}), 0).$$

Hence the horizontal arrows in the above commutative diagram are quasi-isomorphisms and we get isomorphisms of deformation functors

$$\text{Def}_\chi \simeq \text{Def}_\eta \simeq \text{Def}_\rho = \text{Aut}_{H^*(X, \mathbb{C})}.$$

The isomorphism  $\psi: \text{Def}_\chi \simeq \text{Aut}_{H^*(X, \mathbb{C})}$  is explicitly described as follows: Given a Maurer–Cartan element  $(\alpha, e^\alpha) \in \text{MC}_\chi$  and a cohomology class  $[\omega] \in H^*(X, \mathbb{C})$ ,

$$\psi_a([\omega]) = [e^{\tilde{\alpha}} \omega_0 - \partial\beta_0],$$

where  $(\tilde{\alpha}, e^{\tilde{\alpha}}) \in \text{MC}_\eta \subseteq \text{MC}_\chi$  is gauge-equivalent to  $(\alpha, e^\alpha)$ , the differential form  $\omega_0$  is a  $\partial$ -closed representative for the cohomology class  $[\omega]$ , and  $\beta_0 \in A_X$  is such that  $d(e^{\tilde{\alpha}} \omega_0 - \partial\beta_0) = 0$ . The cohomology class  $[e^{\tilde{\alpha}} \omega_0 - \partial\beta_0]$  is independent of the choices of  $(\tilde{\alpha}, e^{\tilde{\alpha}})$ ,  $\omega_0$  and  $\beta_0$ . Since  $e^{\tilde{\alpha}}$  is an automorphism of  $\partial A_X$ , we can write

$$\psi_a([\omega]) = [e^{\tilde{\alpha}}(\omega_0 - \partial\beta_0)]$$

for any  $(\tilde{\alpha}, e^{\tilde{\alpha}})$  and  $\omega_0$  as above and any  $\beta \in A_X$  such that  $d e^{\tilde{\alpha}}(\omega_0 - \partial\beta) = 0$ .

**Remark 2.1.** As  $\chi: L \hookrightarrow \text{Hom}^*(A_X, A_X)$  is injective, the projection on the second factor  $C_\chi \rightarrow \text{Hom}^{*-1}(A_X, A_X)$  induces an identification  $H^*(C_\chi) \simeq H^{*-1}(\text{coker } \chi)$  and so in particular

$$H^1(C_\chi) \xrightarrow{\simeq} H^0(\text{Hom}^*(\ker \partial, A_X/\partial A_X)) = \text{Hom}^0(H^*(\ker \partial), H^*(A_X/\partial A_X)).$$

Hence the differential of  $\psi$  is naturally identified with the linear isomorphism

$$\text{Hom}^0(H^*(\ker \partial), H^*(A_X/\partial A_X)) \simeq \text{Hom}^0(H^*(X, \mathbb{C}); H^*(X, \mathbb{C})),$$

induced by the  $\partial\bar{\partial}$ -lemma and the de Rham isomorphism.

For later use, we give a more explicit description of the map  $\psi$  by writing a map  $\tilde{\psi}: \text{MC}_\chi \rightarrow \text{Aut}_{H^*(X, \mathbb{C})}$  inducing it. To define the map  $\tilde{\psi}$  we need a few preliminary remarks.

**Lemma 2.2.** *If  $(\alpha, e^a) \in \text{MC}_X$ , then in the associative algebra  $\text{Hom}^*(A_X, A_X)$  we have the equality*

$$e^{-a}de^a = d + \alpha.$$

*Proof.* By the definition of the gauge action in the DGLA  $\text{Hom}^*(A_X, A_X)$ , one has for every  $x \in \text{Hom}^0(A_X, A_X)$ ,  $y \in \text{Hom}^1(A_X, A_X)$  the formula

$$e^x * y = e^x(d + y)e^{-x} - d.$$

In particular,  $e^{-a}de^a = d + e^{-a} * 0$ . □

**Corollary 2.3.** *If  $(\alpha, e^a) \in \text{MC}_X$ , then the graded subspaces*

$$e^a(\ker \partial), \quad e^a(\partial A_X)$$

*are subcomplexes of  $(A_X, d)$ . Moreover, the map*

$$e^a : \left( \frac{\ker \partial}{\partial A_X}, 0 \right) \rightarrow \left( \frac{e^a(\ker \partial)}{e^a(\partial A_X)}, 0 \right)$$

*is an isomorphism of complexes and the natural maps*

$$(A_X, d) \leftarrow (e^a(\ker \partial), d) \rightarrow \left( \frac{e^a(\ker \partial)}{e^a(\partial A_X)}, 0 \right)$$

*are quasi-isomorphisms.*

*Proof.* Both  $(\partial A_X, d)$  and  $(\ker \partial, d)$  are subcomplexes of  $(A_X, d)$ . Because  $de^a(v) = e^a(dv + \alpha(v))$  with  $\alpha(\ker \partial) \subseteq \partial A_X$ , we have  $de^a(\partial A_X) \subseteq e^a(\partial A_X)$  and  $de^a(\ker \partial) \subseteq e^a(\ker \partial)$ . The induced differential on the quotient space  $\frac{e^a(\ker \partial)}{e^a(\partial A_X)}$  is trivial since  $d(\ker \partial) \subseteq \partial A_X$  by the  $\partial\bar{\partial}$ -lemma. Again by the  $\partial\bar{\partial}$ -lemma the complex  $(\partial A_X, d)$  is acyclic and therefore the morphisms of complexes

$$(A_X, d) \leftarrow (\ker \partial, d) \rightarrow \left( \frac{\ker \partial}{\partial A_X}, 0 \right)$$

are quasi-isomorphisms. Since every infinitesimal perturbation of an acyclic complex is still acyclic, the complex  $(e^a(\partial A_X), d)$  is acyclic and so the morphisms of complexes

$$(A_X, d) \leftarrow (e^a(\ker \partial), d) \rightarrow \left( \frac{e^a(\ker \partial)}{e^a(\partial A_X)}, 0 \right)$$

are quasi-isomorphisms. □

**Definition 2.4.** The isomorphism

$$\tilde{\psi}_a : H^*(X, \mathbb{C}) \rightarrow H^*(X, \mathbb{C})$$

associated to a Maurer–Cartan element  $(\alpha, e^a)$  via the natural map  $\text{MC}_X \rightarrow \text{Def}_\rho \simeq \text{Aut}^0(H^*(X; \mathbb{C}))$  is obtained by the de Rham isomorphism  $H^*(X, \mathbb{C}) = H^*(A_X, d)$  and the chain of quasi-isomorphisms

$$\begin{array}{ccc} A_X & \longleftarrow \ker \partial & \longrightarrow \frac{\ker \partial}{\partial A_X} \\ & & \downarrow \\ A_X & \longleftarrow e^a(\ker \partial) & \longrightarrow \frac{e^a(\ker \partial)}{e^a(\partial A_X)}. \end{array}$$

More explicitly,

$$\tilde{\psi}_a([\omega]) = [e^a(\omega_0 - \partial\beta)]$$

for any  $\partial$ -closed representative  $\omega_0$  of the cohomology class  $[\omega]$  and any  $\beta \in A_X$  such that  $de^a(\omega_0 - \partial\beta) = 0$ .

**Proposition 2.5.** *The natural transformation  $\tilde{\psi}: \text{MC}_X \rightarrow \text{Aut}_{H^*(X, \mathbb{C})}$  is gauge invariant and therefore factors to  $\psi: \text{Def}_X \rightarrow \text{Aut}_{H^*(X, \mathbb{C})}$ .*

*Proof.* To show that  $\tilde{\psi}_{a \bullet (-l)} = \tilde{\psi}_a$ , note that, since  $l(\ker \partial) \subseteq \partial A_X$ , we have  $e^{-l}(\ker \partial) = \ker \partial$ ,  $e^{-l}(\partial A_X) = \partial A_X$ , and

$$e^{-l}: \left( \frac{\ker \partial}{\partial A_X}, 0 \right) \rightarrow \left( \frac{\ker \partial}{\partial A_X}, 0 \right)$$

is the identity. To prove that  $\tilde{\psi}_{dm \bullet a} = \tilde{\psi}_a$ , notice that we have a commutative diagram of morphisms of complexes

$$\begin{array}{ccc} A_X & \longleftarrow \ker \partial & \longrightarrow \frac{\ker \partial}{\partial A_X} \\ \downarrow e^{dm} & & \downarrow e^{dm} \\ A_X & \longleftarrow e^{dm}(\ker \partial) & \longrightarrow \frac{e^{dm}(\ker \partial)}{e^{dm}(\partial A_X)} \end{array}$$

which, since  $dm$  is homotopy equivalent to zero, induces the commutative diagram of isomorphisms

$$\begin{array}{ccc} H^*(X; \mathbb{C}) & \longleftrightarrow & \frac{\ker \partial}{\partial A_X} \\ \text{id} \downarrow & & \downarrow e^{dm} \\ H^*(X; \mathbb{C}) & \longleftrightarrow & \frac{e^{dm}(\ker \partial)}{e^{dm}(\partial A_X)}. \end{array}$$

Finally, gauge invariance of  $\tilde{\psi}$ , together with the explicit formulae for  $\tilde{\psi}([\omega])$  and  $\psi([\omega])$  written above immediately imply that  $\tilde{\psi}$  induces  $\psi$ . □



### 3. Morphisms of deformation functors associated to Cartan homotopies

In this section we formalize, under the notion of *Cartan homotopy*, a set of standard identities that often arise in algebra and geometry [4], Appendix B, and show how to any Cartan homotopy can be canonically associated a natural transformation of deformation functors.

Let  $L$  and  $M$  be two differential graded Lie algebras. For a given linear map  $i \in \text{Hom}^{-1}(L, M)$ , let  $\mathbf{l} : L \rightarrow M$  be the map defined as

$$a \mapsto \mathbf{l}_a = d\mathbf{i}_a + \mathbf{i}_{da}.$$

**Definition 3.1.** The map  $i$  is called a *Cartan homotopy* for  $\mathbf{l}$  if

$$\mathbf{i}_{[a,b]} = [\mathbf{i}_a, \mathbf{l}_b], \quad [\mathbf{i}_a, \mathbf{i}_b] = 0.$$

for every  $a, b \in L$ .

It is straightforward to show that the condition  $\mathbf{i}_{[a,b]} = [\mathbf{i}_a, \mathbf{l}_b]$  implies that  $\mathbf{l}$  is a morphism of differential graded Lie algebras.

**Example 3.2.** The name Cartan homotopy has a clear origin in differential geometry. Namely, let  $M$  be a differential manifold,  $\mathcal{X}(M)$  be the Lie algebra of vector fields on  $M$ , and  $\text{End}^*(\Omega^*(M))$  be the Lie algebra of endomorphisms of the de Rham algebra of  $M$ . The Lie algebra  $\mathcal{X}(M)$  can be seen as a DGLA concentrated in degree zero, and the graded Lie algebra  $\text{End}^*(\Omega^*(M))$  has a degree one differential given by  $[d_{\text{dR}}, -]$ , where  $d_{\text{dR}}$  is the de Rham differential. Then the contraction

$$\mathbf{i} : \mathcal{X}(M) \rightarrow \text{End}^*(\Omega^*(M))[-1]$$

is a Cartan homotopy and its differential is the Lie derivative

$$[d, \mathbf{i}] = \mathcal{L} : \mathcal{X}(M) \rightarrow \text{End}^*(\Omega^*(M)).$$

In fact, by classical Cartan's homotopy formulas [1], Section 2.4, for any two vector fields  $X$  and  $Y$  on  $M$ , we have

- (1)  $\mathcal{L}_X = d_{\text{dR}}\mathbf{i}_X + \mathbf{i}_X d_{\text{dR}} = [d_{\text{dR}}, \mathbf{i}_X]$ ,
- (2)  $\mathbf{i}_{[X,Y]} = \mathcal{L}_X\mathbf{i}_Y - \mathbf{i}_Y\mathcal{L}_X = [\mathcal{L}_X, \mathbf{i}_Y] = [\mathbf{i}_X, \mathcal{L}_Y]$ ,
- (3)  $[\mathbf{i}_X, \mathbf{i}_Y] = 0$ .

Note that the first Cartan formula above actually states that  $[d, \mathbf{i}] = \mathcal{L}$ . Indeed  $\mathcal{X}(M)$  is concentrated in degree zero and then its differential is trivial.

**Remark 3.3.** The composition of a Cartan homotopy with a morphism of DGLAs is a Cartan homotopy. If  $\mathbf{i} : L \rightarrow M[-1]$  is a Cartan homotopy and  $\Omega$  is a differential graded-commutative algebra, then its natural extension

$$\mathbf{i} \otimes \text{id} : L \otimes \Omega \rightarrow (M \otimes \Omega)[-1], \quad a \otimes \omega \mapsto \mathbf{i}_a \otimes \omega,$$

is a Cartan homotopy.

**Remark 3.4.** By definition,  $\mathbf{l}$  is the differential of  $\mathbf{i}$  in the complex  $\text{Hom}^*(L, M)$  and so  $\mathbf{i}$  is a homotopy between  $\mathbf{l}$  and the trivial map. Then the map  $\mathbf{l} : L \rightarrow M$  is a null-homotopic morphism of DGLAs and

$$\mathbf{i} : L \rightarrow (\text{coker } \mathbf{l})[-1], \quad \mathbf{i} : \ker \mathbf{l} \rightarrow M[-1]$$

are morphisms of differential graded vector spaces.

**Proposition 3.5.** *Let  $\mathbf{l} : L \rightarrow M$  be a DGLA morphism, and let  $\mathbf{i} : L \rightarrow M[-1]$  be a Cartan homotopy for  $\mathbf{l}$ . Then the linear map  $\tilde{\mathbf{i}} : L \rightarrow \tilde{\mathcal{C}}(\mathbf{l})$  given by  $\tilde{\mathbf{i}}(a) = (a, \mathbf{i}_a)$  is an  $L_\infty$ -morphism. In particular, the map  $a \mapsto (a, e^{i_a})$  induces a natural transformation of Maurer–Cartan functors  $\text{MC}_L \rightarrow \text{MC}_{\mathbf{l}}$ , and consequently a natural transformation of deformation functors  $\text{Def}_L \rightarrow \text{Def}_{\mathbf{l}}$ .*

*Proof.* By the explicit expression for the higher brackets

$$\mu_n : \bigwedge^n C_\chi \rightarrow C_\chi[2-n], \quad n \geq 2,$$

defining the  $L_\infty$ -algebra structure  $\tilde{\mathcal{C}}(\chi)$ , it is straightforward to check that  $\tilde{\mathbf{i}}$  is a morphism of complexes commuting with every bracket. Indeed,  $\tilde{\mathbf{i}}(da) = \mu_1(\tilde{\mathbf{i}}(a))$  is the identity  $\mathbf{i}_{da} = (-d)\mathbf{i}_a + \mathbf{l}_a$ ; the identity  $\tilde{\mathbf{i}}([a, b]) = \mu_2(\tilde{\mathbf{i}}(a) \wedge \tilde{\mathbf{i}}(b))$  is

$$\mathbf{i}_{[a,b]} = \frac{1}{2}[\mathbf{i}_a, \mathbf{l}_b] + \frac{(-1)^{\text{deg}(a)}}{2}[\mathbf{l}_a, \mathbf{i}_b] = [\mathbf{i}_a, \mathbf{l}_b],$$

and  $\mu_n(\tilde{\mathbf{i}}(x_1) \wedge \dots \wedge \tilde{\mathbf{i}}(x_n)) = 0$  for any  $x_1, x_2, \dots, x_n$  and any  $n \geq 3$ , since

$$[\mathbf{i}_a, [\mathbf{i}_b, \mathbf{l}_c]] = [\mathbf{i}_a, \mathbf{i}_{[b,c]}] = 0$$

for any  $a, b, c$ .

Since the  $L_\infty$ -morphism  $\tilde{\mathbf{i}}$  is linear, the map  $l \mapsto (l, \mathbf{i}_l)$  is a morphism of Maurer–Cartan functors  $\text{MC}_L \rightarrow \text{MC}_{\tilde{\mathcal{C}}(\mathbf{l})}$ . To conclude the proof, compose this morphism with the isomorphism  $\text{MC}_{\tilde{\mathcal{C}}(\mathbf{l})} \xrightarrow{\sim} \text{MC}_{\mathbf{l}}$  given by  $(l, m) \mapsto (l, e^m)$ .  $\square$

**Corollary 3.6.** *Let  $\mathbf{i} : N \rightarrow M[-1]$  be a Cartan homotopy for  $\mathbf{l} : N \rightarrow M$ , let  $L$  be a subDGLA of  $M$  such that  $\mathbf{l}(N) \subseteq L$ , and let  $\chi : L \hookrightarrow M$  be the inclusion. Then the linear map*

$$\Phi : N \rightarrow \tilde{\mathcal{C}}(\chi), \quad \Phi(a) = (\mathbf{l}_a, \mathbf{i}_a),$$

*is a linear  $L_\infty$ -morphism. In particular, the map  $a \mapsto (\mathbf{l}_a, e^{i_a})$  induces a natural transformation of Maurer–Cartan functors  $\text{MC}_N \rightarrow \text{MC}_\chi$ , and consequently a natural transformation of deformation functors  $\text{Def}_N \rightarrow \text{Def}_\chi$ .*

*Proof.* We have a commutative diagram of differential graded Lie algebras

$$\begin{array}{ccc} N & \xrightarrow{\mathbf{l}} & L \\ \mathbf{i} \downarrow & & \downarrow \chi \\ M & \xlongequal{\quad} & M \end{array}$$

inducing an  $L_\infty$ -morphism  $\tilde{C}(\mathcal{I}) \rightarrow \tilde{C}(\chi)$ . Composing this morphism with the  $L_\infty$ -morphism  $\tilde{i}: N \rightarrow \tilde{C}(\mathcal{I})$  given by Proposition 3.5, one gets the  $L_\infty$ -morphism  $\Phi$ .  $\square$

#### 4. Polyvector fields and generalized periods

The notion of Cartan homotopy generalizes immediately to sheaves of DGLAs. In this section we give another example of Cartan homotopy, which will be used later.

Let  $X$  be a complex manifold and denote by

- $T_{X,\mathbb{C}} = T_X^{1,0} \oplus T_X^{0,1}$  the complexified differential tangent bundle,
- $T_X \simeq T_X^{1,0}$  the holomorphic tangent bundle,
- $\mathcal{A}_X^{p,q}$  the sheaf of differentiable  $(p, q)$ -forms and by  $\mathcal{A}_X^{p,q}(E)$  the sheaf of  $(p, q)$ -forms with values in a holomorphic vector bundle  $E$ ,
- $A_X^{p,q}$  and  $A_X^{p,q}(E)$  the vector spaces of global sections of  $\mathcal{A}_X^{p,q}$  and  $\mathcal{A}_X^{p,q}(E)$ , respectively.

The direct sum

$$\mathcal{A}_X = \bigoplus_i \mathcal{A}_X^i, \quad \text{where } \mathcal{A}_X^i = \bigoplus_{p+q=i} \mathcal{A}_X^{p,q},$$

endowed with the wedge product  $\wedge$ , is a sheaf of graded algebras; we denote by  $\mathcal{H}om^{a,b}(\mathcal{A}_X, \mathcal{A}_X)$  the sheaf of its  $\mathbb{C}$ -linear endomorphisms of  $\mathcal{A}_X$  of bidegree  $(a, b)$ . Notice that  $\partial$  and  $\bar{\partial}$  are global sections of  $\mathcal{H}om^{1,0}(\mathcal{A}_X, \mathcal{A}_X)$  and  $\mathcal{H}om^{0,1}(\mathcal{A}_X, \mathcal{A}_X)$ , respectively. The direct sum  $\mathcal{H}om^*(\mathcal{A}_X, \mathcal{A}_X) = \bigoplus_k \bigoplus_{a+b=k} \mathcal{H}om^{a,b}(\mathcal{A}_X, \mathcal{A}_X)$  is a sheaf of graded associative algebras, and so a sheaf of differential graded Lie algebras with the natural bracket

$$[f, g] = fg - (-1)^{\deg(f)\deg(g)} gf$$

and differential  $[d, -] = [\partial + \bar{\partial}, -]$ .

For any integer  $(a, b)$  with  $a \leq 0$  and  $b \geq 0$ , let  $\mathcal{G}erst_X^{a,b}$  be the sheaf

$$\mathcal{G}erst_X^{a,b} = \mathcal{A}_X^{0,b}(\wedge^{-a} T_X).$$

The direct sum  $\mathcal{G}erst_X^* = \bigoplus_k \bigoplus_{a+b=k} \mathcal{G}erst_X^{a,b}$  is a sheaf of differential Gerstenhaber algebras, with the wedge product

$$\wedge: \mathcal{G}erst_X^{a_1,b_1} \otimes \mathcal{G}erst_X^{a_2,b_2} \rightarrow \mathcal{G}erst_X^{a_1+a_2,b_1+b_2}$$

as graded commutative product, the degree 1 differential

$$\bar{\partial}: \mathcal{G}erst_X^{a,b} \rightarrow \mathcal{G}erst_X^{a,b+1}$$

defined in local coordinates by the formula

$$\bar{\partial}(\phi \frac{\partial}{\partial z_I}) = \bar{\partial}(\phi) \frac{\partial}{\partial z_I}, \quad \phi \in \mathcal{A}_X^{0,*},$$

and the degree 1 bracket

$$[\cdot, \cdot]_G : \mathcal{G}erst_X^{a_1, b_1} \otimes \mathcal{G}erst_X^{a_2, b_2} \rightarrow \mathcal{G}erst_X^{a_1+a_2+1, b_1+b_2}$$

defined in local coordinates by the formula

$$[f d\bar{z}_I \frac{\partial}{\partial z_H}, g d\bar{z}_J \frac{\partial}{\partial z_K}]_G = d\bar{z}_I \wedge d\bar{z}_J [f \frac{\partial}{\partial z_H}, g \frac{\partial}{\partial z_K}]_{SN}.$$

Here  $[\cdot, \cdot]_{SN}$  denotes the Schouten–Nijenhuis bracket on  $\mathcal{A}_X^0(\wedge^* T_X)$ , i.e., the odd graded Lie bracket obtained by extending the usual Lie bracket on  $\mathcal{A}_X^0(T_X)$  by imposing

$$[\xi, f]_{SN} = \xi(f), \quad \xi \in \mathcal{A}^0(T_X), \quad f \in \mathcal{A}_X^0$$

and the odd graded Poisson identity

$$[\xi, \eta_1 \wedge \eta_2]_{SN} = [\xi, \eta_1]_{SN} \wedge \eta_2 + (-1)^{\deg(\xi)(\deg(\eta_1)-1)} \eta_1 \wedge [\xi, \eta_2]_{SN}.$$

The contraction of differential forms with vector fields is used to define an injective morphisms of sheaves of bigraded vector spaces: the *contraction map*

$$i : \mathcal{G}erst_X^{a,b} \rightarrow \mathcal{H}om^{a,b}(\mathcal{A}_X, \mathcal{A}_X), \quad \xi \mapsto i_\xi, \quad i_\xi(\omega) = \xi \lrcorner \omega.$$

The contraction map is actually a morphism of sheaves of bigraded associative algebras:

$$i_{\xi \wedge \eta} = i_\xi i_\eta.$$

In particular, since  $(\mathcal{G}erst_X^*, \wedge)$  is a graded commutative algebra, we obtain that

$$[i_\xi, i_\eta] = 0, \quad \text{for all } \xi, \eta \in \mathcal{G}erst_X^*.$$

Note that iterated contractions give a symmetric map

$$i^{(n)} : \odot^n \mathcal{G}erst_X^* \rightarrow \mathcal{H}om^*(\mathcal{A}_X, \mathcal{A}_X), \quad \xi_1 \odot \xi_2 \odot \dots \odot \xi_n \mapsto i_{\xi_1} i_{\xi_2} \dots i_{\xi_n}.$$

Since  $\mathcal{G}erst_X^*$  is a sheaf of differential graded Gerstenhaber algebras, its desuspension

$$\mathcal{P}oly_X^* = \mathcal{G}erst[-1]_X^*$$

is a sheaf of differential graded Lie algebras. Note that, due to the shift, the differential  $D$  in  $\mathcal{P}oly_X^*$  is  $-\bar{\partial}$ , i.e., in local coordinates

$$D : \mathcal{P}oly_X^k \rightarrow \mathcal{P}oly_X^{k+1}$$

is given by the formula

$$D(\phi \frac{\partial}{\partial z_I}) = -\bar{\partial}(\phi) \frac{\partial}{\partial z_I}, \quad \phi \in \mathcal{A}_X^{0,*}.$$

The contraction map  $i : \mathcal{G}erst_X^* \rightarrow \mathcal{H}om^*(\mathcal{A}_X, \mathcal{A}_X)$  can be seen as a linear map

$$i : \mathcal{P}oly_X^* \rightarrow \mathcal{H}om^*(\mathcal{A}_X, \mathcal{A}_X)[-1].$$

More general, via the decalage isomorphism, the iterated contraction is a graded antisymmetric map

$$i^{(n)} : \bigwedge^n \mathcal{P}oly_X^* \rightarrow \mathcal{H}om^*(\mathcal{A}_X, \mathcal{A}_X)[-n].$$

**Lemma 4.1.** *In the notation above, for every  $\xi, \eta \in \mathcal{P}oly_X^*$  we have*

$$i_{D\xi} = -[\bar{\partial}, i_\xi], \quad i_{[\xi, \eta]} = [i_\xi, [\partial, i_\eta]], \quad [i_\xi, i_\eta] = 0.$$

*Proof.* The third equation has been proved above, and the first equation is completely straightforward: it just expresses the Leibniz rule for  $\bar{\partial}$ . To prove the second equation, let

$$\Phi(\xi, \eta) = i_{[\xi, \eta]} - [i_\xi, [\partial, i_\eta]].$$

Then, using  $i_{\xi \wedge \eta} = i_\xi i_\eta$ , the (shifted) odd Poisson identity  $[\xi, \eta_1 \wedge \eta_2] = [\xi, \eta_1] \wedge \eta_2 + (-1)^{(\deg(\xi)-1)\deg(\eta_1)} \eta_1 \wedge [\xi, \eta_2]$  and the third equation  $[i_\xi, i_\eta] = 0$ , one finds

$$\Phi(\xi, \eta_1 \wedge \eta_2) = \Phi(\xi, \eta_1) i_{\eta_2} + (-1)^{(\deg(\xi)-1)\deg(\eta_1)} i_{\eta_1} \Phi(\xi, \eta_2)$$

and

$$\Phi(\xi_1 \wedge \xi_2, \eta) = i_{\xi_1} \Phi(\xi_2, \eta) + (-1)^{(\deg(\xi_2)(\deg(\eta)-1))} \Phi(\xi_1, \eta) i_{\eta_2}.$$

Therefore, to prove  $\Phi(\xi, \eta) = 0$  for any  $\xi, \eta$  one just needs to prove  $\Phi(\xi, \eta) = 0$  for  $\xi, \eta \in \mathcal{A}_X^0 \cup \{d\bar{z}_i\} \cup \{\partial/\partial z_j\}$ , where  $z_1, \dots, z_n$  are local holomorphic coordinates. This is straightforward and is left to the reader; see also Lemma 7 of [18] and Lemma 7.21 of [19]. □

**Corollary 4.2.** *The contraction map  $i : \mathcal{P}oly_X^* \rightarrow \mathcal{H}om^*(\mathcal{A}_X, \mathcal{A}_X)[-1]$  is a Cartan homotopy and the induced morphism  $\mathbf{l}$  of sheaves of differential graded Lie algebras is the holomorphic Lie derivative*

$$\begin{aligned} \mathbf{l} : \mathcal{P}oly_X^* &\rightarrow \mathcal{H}om^*(\mathcal{A}_X, \mathcal{A}_X), \\ \xi &\mapsto \mathbf{l}_\xi = [\partial, i_\xi], \quad \mathbf{l}_\xi(\omega) = \partial(\xi \lrcorner \omega) + (-1)^{\deg(\xi)} \xi \lrcorner \partial\omega. \end{aligned}$$

Moreover,  $\mathbf{l}$  is an injective morphism of sheaves.

*Proof.* As in Section 3, let  $\mathbf{l}_\xi = [d, \mathbf{i}_\xi] + \mathbf{i}_{D\xi}$ . Since  $\mathbf{i}_{D\xi} = -[\bar{\partial}, \mathbf{i}_\xi]$ , we find  $\mathbf{l}_\xi = [\partial, \mathbf{i}_\xi]$ . The identity  $\mathbf{i}_{[\xi, \eta]} = [\mathbf{i}_\xi, [\partial, \mathbf{i}_\eta]]$  then reads  $\mathbf{i}_{[\xi, \eta]} = [\mathbf{i}_\xi, \mathbf{l}_\eta]$  and this together with  $[\mathbf{i}_\xi, \mathbf{i}_\eta] = 0$  tells us that the contraction map  $\mathbf{i}$  is a Cartan homotopy. Consequently, the holomorphic Lie derivative  $\mathbf{l}_\xi = [\partial, \mathbf{i}_\xi]$  is the induced morphism of sheaves of differential graded Lie algebras. Injectivity of  $\mathbf{l}$  is easily checked in local coordinates.  $\square$

Corollary 4.2, applied to global sections, shows that the contraction map

$$\mathbf{i} : \text{Poly}_X = \bigoplus_{i,j} A^{0,i}(\bigwedge^{-j} T_X) \rightarrow \text{Hom}^*(\mathcal{A}_X, \mathcal{A}_X)[-1]$$

is a Cartan homotopy, as well as its restriction to the Kodaira–Spencer DGLA  $\text{KS}_X = \bigoplus_i A^{0,i}(T_X)$ .

**Remark 4.3.** The composition of the inclusion  $\text{KS}_X \hookrightarrow \text{Poly}_X$  with the iterated contraction  $\mathbf{i}^{(n)} : \bigwedge^n \text{Poly}_X \rightarrow \text{Hom}^*(\mathcal{A}_X, \mathcal{A}_X)[-n]$  induces in cohomology a graded antisymmetric map  $\bigwedge^n H^*(\text{KS}_X) \rightarrow \text{Hom}^*(H^*(X, \mathbb{C}), H^*(X, \mathbb{C}))[-n]$ . In particular, from the isomorphism of graded vector spaces  $H^1(\text{KS}_X^*) \simeq H^1(T_X)[-1]$  and the decalage isomorphism, the iterated contraction gives a symmetric morphism

$$\mathbf{i}^{(n)} : \bigodot^n H^1(T_X) \rightarrow \text{Hom}^0(H^*(X, \mathbb{C}), H^*(X, \mathbb{C})).$$

It is well known [10] and easy to prove that the image of  $\mathbf{i}^{(n)}$  consists of self-adjoint operators with respect the cup product on  $H^*(X, \mathbb{C})$ .

Under the identification  $H^*(X, \mathbb{C}) = \bigoplus_{p,q} H^q(X, \Omega_X^p)$ , when  $\dim X = n$  the morphism  $\mathbf{i}^{(n)}$  reduces to the Yukawa coupling

$$\mathbf{i}^{(n)} : \bigodot^n H^1(T_X) \rightarrow \bigodot^2 H^n(X, \mathcal{O}_X).$$

Now, as in Section 2, consider the DGLA

$$L = \{f \in \text{Hom}^*(A_X, A_X) \mid f(\ker \partial) \subseteq \partial A_X\},$$

and let  $\chi : L \hookrightarrow \text{Hom}^*(A_X, A_X)$  be the inclusion. Since  $\mathbf{l}(\text{Poly}_X) \subseteq L$  by Corollary 3.6, we have a natural transformation of deformation functors  $\text{Def}_{\text{Poly}_X} \rightarrow \text{Def}_X$  induced, at the Maurer–Cartan level, by the map  $\xi \mapsto (\mathbf{l}_\xi, e^{i\xi})$ .

The functor  $\text{Def}_{\text{Poly}_X}$ , which we denote by  $\widetilde{\text{Def}}_X$ , is called the functor of generalized deformations of  $X$ ; see [2]. We have shown in Section 2 that there exists a natural isomorphism  $\psi : \text{Def}_X \rightarrow \text{Aut}_{H^*(X, \mathbb{C})}$ . By these considerations, we obtain:

**Theorem 4.4.** *The linear map*

$$\text{Poly}_X \rightarrow \widetilde{\mathcal{C}}(\chi), \quad \xi \mapsto (\mathbf{l}_\xi, \mathbf{i}_\xi),$$

is a linear  $L_\infty$ -morphism and induces a natural transformation of functors

$$\Phi: \widetilde{\text{Def}}_X \rightarrow \text{Aut}_{H^*(X, \mathbb{C})},$$

given at the level of Maurer–Cartan functors by the map  $\xi \mapsto \psi_{i_\xi}$ .

**Proposition 4.5.** *Via the natural identifications  $H^1(\text{Poly}_X) = \bigoplus_{i \geq 0} H^i(\wedge^i T_X)$  and  $H^*(X, \mathbb{C}) = \bigoplus_{p,q} H^q(X, \Omega_X^p)$  given by the Dolbeault’s theorem and the  $\partial\bar{\partial}$ -lemma, the differential of  $\Phi$ ,*

$$d\Phi: H^1(\text{Poly}_X) \rightarrow \text{Hom}^0(H^*(X, \mathbb{C}), H^*(X, \mathbb{C})),$$

is identified with the contraction

$$\left(\bigoplus_{i \geq 0} H^i(\wedge^i T_X)\right) \otimes \left(\bigoplus_{p,q} H^q(X, \Omega_X^p)\right) \rightarrow \bigoplus_{i,p,q} H^{q+i}(X, \Omega_X^{p-i}).$$

*Proof.* By Lemma 4.1 we have a commutative diagram of differential complexes

$$\begin{array}{ccc} (\text{Hom}^{*-1}(A_X, A_X), -\text{ad}_{\bar{\partial}}) & \longrightarrow & (\text{Hom}^{*-1}(\ker \partial, A_X/\partial A_X), -\text{ad}_d) \\ \uparrow i & \nearrow i & \\ (\text{Poly}_X, D) & & \end{array}$$

where we have used the fact that on  $\ker \partial$  and on  $A_X/\partial A_X$  the differentials  $\text{ad}_d$  and  $\text{ad}_{\bar{\partial}}$  coincide. Using the identification  $H_{\bar{\partial}}^*(A_X) = H^*(X, \mathbb{C})$  coming from Dolbeault’s theorem and the  $\partial\bar{\partial}$ -lemma, and by Remark 2.1, the above commutative diagram induces the commutative diagram in cohomology

$$\begin{array}{ccc} \text{Hom}^0(H^*(X, \mathbb{C}), H^*(X, \mathbb{C})) & \xleftarrow{d\psi} & \text{Hom}^0(H^*(\ker \partial), H^*(A_X/\partial A_X)) \\ \uparrow i & \nearrow i & \\ H^1(\text{Poly}_X) & & \end{array}$$

Since, by Theorem 4.4, the differential of  $\Phi$  is  $d\Phi = d\psi \circ i$ , which completes the proof. □

As a corollary of Theorem 4.4, the linear map  $\xi \mapsto (\mathbf{I}_\xi, i_\xi)$  induces a morphism of obstruction spaces  $H^2(\text{Poly}) \rightarrow \text{Hom}^1(H^*(X, \mathbb{C}), H^*(X, \mathbb{C}))$  commuting with  $\Phi$  and obstruction maps [5], [17]. The same argument of Proposition 4.5 shows that this morphism is naturally identified with the contraction

$$\left(\bigoplus_{i \geq 0} H^{i+1}(\wedge^i T_X)\right) \otimes \left(\bigoplus_{p,q} H^q(X, \Omega_X^p)\right) \rightarrow \bigoplus_{i,p,q} H^{q+i+1}(X, \Omega_X^{p-i}).$$

Since the deformation functor  $\text{Aut}_{H^*(X, \mathbb{C})}$  is smooth, we obtain the following version of the so-called Kodaira principle (ambient cohomology annihilates obstruction):

**Proposition 4.6.** *The obstructions to extended deformations of a compact Kähler manifold  $X$  are contained in the subspace*

$$\bigoplus_{i \geq 0} \bigcap_{p,q} \ker(H^{i+1}(\bigwedge^i T_X) \xrightarrow{i} \text{Hom}(H^q(X, \Omega_X^p), H^{q+i+1}(X, \Omega_X^{p-i})))$$

of  $H^2(\text{Poly}_X)$ .

As an immediate corollary we recover the fact that extended deformations of compact Calabi–Yau manifolds are unobstructed [2]. Indeed, if  $X$  is an  $n$ -dimensional compact Calabi–Yau manifold, then the contraction pairing

$$H^{i+1}(\bigwedge^i T_X) \otimes H^0(X, \Omega_X^n) \rightarrow H^{i+1}(X, \Omega_X^{n-i})$$

is nondegenerate for any  $i \geq 0$ .

### 5. Restriction to classical deformations

Let  $(X, \mathcal{O}_X)$  be a complex manifold. It is well known that the infinitesimal deformations of the complex structure of  $X$  are governed by the Kodaira–Spencer DGLA of  $X$ . More precisely, there is a natural isomorphism of deformation functors

$$\text{Def}_{\text{KS}_X} \xrightarrow{\sim} \text{Def}_X,$$

which map a Maurer–Cartan element  $\xi \in A_X^{0,1}(T_X)$  to the complex manifold  $(X, \mathcal{O}_\xi)$ , where the structure sheaf  $\mathcal{O}_\xi$  is defined by

$$\mathcal{O}_\xi = \ker\{\bar{\partial}_\xi : \mathcal{A}_X^{0,0} \rightarrow \mathcal{A}_X^{0,1}\} = \{f \in \mathcal{A}_X^0 \mid (\bar{\partial} + I_\xi)f = 0\};$$

see [4], [9], [14], Ex. 3.4.1, or [13]. The above equations must be interpreted as identities among functors of Artin rings; namely, they mean that for any local Artin algebra  $(B, \mathfrak{m}_B)$  the Kuranishi data  $\xi \in \text{MC}_{\text{KS}_X}(B) \subseteq A_X^{0,1}(T_X) \otimes \mathfrak{m}_B$  are mapped to the family  $(X, \mathcal{O}_\xi)$  of complex manifolds over  $\text{Spec}(B)$ , whose structure sheaf  $\mathcal{O}_\xi$  is defined by

$$\mathcal{O}_\xi = \ker\{\bar{\partial}_\xi : \mathcal{A}_X^{0,0} \otimes B \rightarrow \mathcal{A}_X^{0,1} \otimes B\} = \{f \in \mathcal{A}_X^0 \otimes B \mid (\bar{\partial} + I_\xi)f = 0\}.$$

Let now

$$A_X = F_\xi^0 \supseteq F_\xi^1 \supseteq \dots$$

be the Hodge filtration of differential forms on the complex manifold  $(X, \mathcal{O}_\xi)$ , i.e., for every  $m \geq 0$ ,  $F_\xi^m$  is the complex of global sections of the differential ideal sheaf  $\mathcal{F}_\xi^m \subseteq \mathcal{A}_X$  generated by  $(d\mathcal{O}_\xi)^m$ . Again, here we write  $\mathcal{A}_X$  for the functor of Artin



rings defined by  $B \mapsto \mathcal{A}_X \otimes B$ . If  $X$  is a compact Kähler manifold, the cohomology of  $(F_\xi^m, d)$  naturally embeds into the cohomology of  $(A_X, d)$ . Since the dimension of  $H^*(F_\xi^m, d)$  is independent of  $\xi$ , one can look at  $H^*(F_\xi^m, d)$  as a different linear embedding of  $H^*(F^m, d)$  into  $H^*(X; \mathbb{C})$ . Hence  $\xi \mapsto H^*(F_\xi^m, d)$  is a map

$$\text{Def}_X \rightarrow \text{Grass}_{H^*(F^m), H^*(X; \mathbb{C})},$$

called the  $m$ -th period map.

The inclusion of DGLAs  $\text{KS}_X \hookrightarrow \text{Poly}_X$  induces an embedding of deformation functors  $\text{Def}_X \rightarrow \widetilde{\text{Def}}_X$ . Hence, the restriction of  $\Phi$  to  $\text{Def}_X$  is a natural transformation

$$\Phi: \text{Def}_X \rightarrow \text{Aut}_{H^*(X, \mathbb{C})}.$$

**Theorem 5.1.** *For any  $m \geq 0$ , the map  $\Phi: \text{Def}_X \rightarrow \text{Aut}_{H^*(X, \mathbb{C})}$  lifts the  $m$ -th period map  $\text{Def}_X \rightarrow \text{Grass}_{H^*(F^m), H^*(X; \mathbb{C})}$ .*

*Proof.* Let  $\xi$  be a Maurer–Cartan element in  $\text{KS}_X$ . Then  $(I_\xi, e^{i\xi}) \in \text{MC}_X$  is a Maurer–Cartan element with  $I_\xi$  of bidegree  $(0, 1)$ . Let  $[\omega]$  be an element in  $H^*(F^m)$ . To compute  $\psi_\xi[\omega]$  we pick a  $\partial$ -closed representative for the class  $[\omega]$ , which we can assume to be  $\omega$ , and then we take the cohomology class of a  $d$ -closed representative of  $e^{i\xi}\omega$  in  $e^{i\xi}(\ker \partial)/e^{i\xi}(\partial A_X)$ , i.e., we have  $\psi_\xi[\omega] = [e^{i\xi}(\omega - \partial\beta)]$  for any  $\beta \in A_X$  such that  $de^{i\xi}(\omega - \partial\beta) = 0$ . For such a  $\beta$  we have

$$0 = e^{-i\xi} de^{i\xi}(\omega - \partial\beta) = -\bar{\partial}\partial\beta + I_\xi(\omega) - I_\xi(\partial\beta)$$

and so

$$(\bar{\partial} + I_\xi)\partial\beta = I_\xi(\omega).$$

Write  $\eta_{< m}$  and  $\eta_{\geq m}$  for the components of a differential form  $\eta$  in  $\bigoplus_{i < m} A_X^{i,*}$  and in  $\bigoplus_{i \geq m} A_X^{i,*}$ , respectively. Since both  $I_\xi$  and  $(\bar{\partial} + I_\xi)$  are homogeneous of bidegree  $(0, 1)$ , we have

$$I_\xi(\omega) = (I_\xi(\omega))_{\geq m} = ((\bar{\partial} + I_\xi)\partial\beta)_{\geq m} = (\bar{\partial} + I_\xi)\partial(\beta_{\geq m-1}).$$

Hence  $\psi_{i_\xi}[\omega] = [e^{i\xi}(\omega - \partial(\beta_{\geq m-1}))] \in H^*(e^{i\xi} F^m)$ , and so

$$\Phi_\xi(H^*(F^m)) = H^*(e^{i\xi} F^m).$$

On the other hand, the period of the infinitesimal deformation  $\mathcal{O}_\xi = \ker(\bar{\partial} + I_\xi)$  is  $H^*(F_\xi^m) \subseteq H^*(A_X)$ , where  $F_\xi^m$  is the complex of global sections of the differential ideal sheaf  $\mathcal{F}_\xi^m \subseteq \mathcal{A}_X$  generated by  $(d\mathcal{O}_\xi)^m$ . Since  $e^{i\xi}$  is the identity on  $\mathcal{A}_X^{0,0}$ , by Lemma 2.2 we can write

$$e^{-i\xi}(d\mathcal{O}_\xi) = e^{-i\xi} de^{i\xi} \mathcal{O}_\xi = (\partial + \bar{\partial} + I_\xi)\mathcal{O}_\xi = \partial\mathcal{O}_\xi \subseteq \partial\mathcal{A}_X^{0,0} \subseteq \mathcal{A}_X^{1,0}.$$

Since  $e^{i\xi}: \mathcal{A}_X \rightarrow \mathcal{A}_X$  is a morphism of sheaves of differential graded commutative algebras, we get  $e^{-i\xi}(\mathcal{F}_\xi^m) \subseteq \mathcal{F}^m$  and then, by rank considerations,  $e^{i\xi}(\mathcal{F}^m) = \mathcal{F}_\xi^m$ . Hence

$$\Phi_\xi(H^*(F^m)) = H^*(\mathcal{F}_\xi^m). \quad \square$$

**Remark 5.2.** That the  $m$ -th period map  $\text{Def}_X \rightarrow \text{Grass}_{H^*(F^m), H^*(X; \mathbb{C})}$  is induced by an  $L_\infty$ -morphism was shown in [7]. Namely, let  $\chi_m: L_m \hookrightarrow \text{Hom}^*(A_X, A_X)$  the inclusion of the subalgebra

$$L_m = \{f \in \text{Hom}^*(A_X, A_X) \mid f(F^m) \subseteq F^m\}$$

in the DGLA of endomorphisms of  $A_X$ . Then  $\text{Def}_{\chi_m} \simeq \text{Grass}_{H^*(F^m), H^*(X)}$ , and the map  $\xi \mapsto (I_\xi, i_\xi)$  is an  $L_\infty$ -morphism between  $\text{KS}_X$  and  $C_{\chi_m}$  inducing the  $m$ -th period map.

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