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R-groups and geometric structure in the representation theory of SL(N)

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Abstract. Let F be a nonarchimedean local field of characteristic zero and let $G = \mathrm{SL}(N) = \mathrm{SL}(N, F)$. This article is devoted to studying the influence of the elliptic representations of SL(N) on the K-theory. We provide full arithmetic details. This study reveals an intricate geometric structure. One point of interest is that the R-group is realized as an isotropy group. Our results illustrate, in a special case, part (3) of the recent conjecture in [2].

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1. Introduction

Let F be a nonarchimedean local field of characteristic zero and let $G = \mathrm{SL}(N) = \mathrm{SL}(N, F)$. This article is devoted to studying subspaces of the tempered dual of SL(N) which have an especially intricate geometric structure, and to computing, with full arithmetic details, their K-theory. Our results illustrate, in a special case, part (3) of the recent conjecture in [2].

The subspaces of the tempered dual which are especially interesting for us contain *elliptic* representations. A tempered representation of SL(N) is *elliptic* if its Harish-Chandra character is not identically zero on the elliptic set.

An element in the discrete series of SL(N) is an isolated point in the tempered dual of SL(N) and contributes one generator to K_0 of the reduced C*-algebra of SL(N).

Now SL(N) admits elliptic representations which are not discrete series: we investigate, with full arithmetic details, the contribution of the elliptic representations of SL(N) to the K-theory of the reduced C*-algebra $\mathfrak{A}_N$ of SL(N).

According to [7], $\mathfrak{A}_N$ is a C*-direct sum of fixed C*-algebras. Among these fixed algebras, we will focus on those whose duals contain elliptic representations. Let n be a divisor of N with $1 \le n \le N$ and suppose that the group $\mathcal{U}_F$ of integer units admits a character of order n. Then the relevant fixed algebras are of the form

$$C(\mathbb{T}^n/\mathbb{T}, \mathfrak{K})^{\mathbb{Z}/n\mathbb{Z}} \subset \mathfrak{A}_N.$$

Here, $\mathfrak{K}$ is the C*-algebra of compact operators on standard Hilbert space, $\mathbb{T}^n/\mathbb{T}$ is the quotient of the compact torus $\mathbb{T}^n$ via the diagonal action of $\mathbb{T}$. The compact group $\mathbb{T}^n/\mathbb{T}$ arises as the maximal compact subgroup of the standard maximal torus of the Langlands dual $\mathrm{PGL}(n, \mathbb{C})$. We prove (Theorem 3.1) that this fixed C*-algebra is strongly Morita equivalent to the crossed product

$$C(\mathbb{T}^n/\mathbb{T}) \rtimes \mathbb{Z}/n\mathbb{Z}.$$

The reduced C*-algebra $\mathfrak{A}_N$ is liminal, and its primitive ideal space is in canonical bijection with the tempered dual of $\mathrm{SL}(N)$. Transporting the Jacobson topology on the primitive ideal space, we obtain a locally compact topology on the tempered dual of $\mathrm{SL}(N)$, see [5], 3.1.1, 4.4.1, 18.3.2.

Let $\mathfrak{T}_n$ denote the C*-dual of $C(\mathbb{T}^n/\mathbb{T}, \mathfrak{K})^{\mathbb{Z}/n\mathbb{Z}}$. Then $\mathfrak{T}_n$ is a non-Hausdorff space, and has a very special structure as topological space. When n is a prime number ℓ, then $\mathfrak{T}_\ell$ will contain multiple points. When n is non-prime, $\mathfrak{T}_n$ will contain not only multiple points, but also *multiple subspaces*. This crossed product C*-algebra is a noncommutative unital C*-algebra which fits perfectly into the framework of noncommutative geometry. In the tempered dual of $\mathrm{SL}(N)$, there are connected compact non-Hausdorff spaces, laced with multiple subspaces, and simply described by crossed product C*-algebras.

The K-theory of the fixed C*-algebra is then given by the K-theory of the crossed product C*-algebra. To compute (modulo torsion) the K-theory of this noncommutative C*-algebra, we apply the Chern character for discrete groups [3]. This leads to the cohomology of the *extended quotient* $(\mathbb{T}^n/\mathbb{T})/\!/(\mathbb{Z}/n\mathbb{Z})$. This in turn leads to a problem in classical algebraic topology, namely the determination of the cyclic invariants in the cohomology of the n-torus.

The ordinary quotient will be denoted by $\mathfrak{X}(n)$:

$$\mathfrak{X}(n) := (\mathbb{T}^n/\mathbb{T})/(\mathbb{Z}/n\mathbb{Z}).$$

This is a compact connected orbifold. Note that $\mathfrak{X}(1) = pt$. The orbifold $\mathfrak{X}(n, k, \omega)$ which appears in the following theorem is defined in Section 4. The notation is such that $\mathfrak{X}(n, n, 1)$ is the ordinary quotient $\mathfrak{X}(n)$ and each $\mathfrak{X}(n, 1, \omega)$ is a point. The highest common factor of n and k is denoted (n, k).

Theorem 1.1. *The extended quotient* $(\mathbb{T}^n/\mathbb{T})/\!/(\mathbb{Z}/n\mathbb{Z})$ *is a disjoint union of compact connected orbifolds:*

$$(\mathbb{T}^n/\mathbb{T})/\!/(\mathbb{Z}/n\mathbb{Z}) = \bigsqcup \mathfrak{X}(n, k, \omega)$$

The disjoint union is over all $1 \le k \le n$ *and all* $n/(k, n)$*th roots of unity* ω *in* $\mathbb{C}$.

We apply the Chern character for discrete groups [3], and obtain

Theorem 1.2. *The K-theory groups K_0 and K_1 are given by*

$$K_0(C(\mathbb{T}^n/\mathbb{T}),\mathfrak{K})^{\mathbb{Z}/n\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{C} \simeq \bigoplus H^{\mathrm{ev}}(\mathfrak{X}(n,k,\omega);\mathbb{C}),$$
$$K_1(C(\mathbb{T}^n/\mathbb{T}),\mathfrak{K})^{\mathbb{Z}/n\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{C} \simeq \bigoplus H^{\mathrm{odd}}(\mathfrak{X}(n,k,\omega);\mathbb{C}).$$

The direct sums are over all $1 \le k \le n$ and all $n/(k,n)$th roots of unity ω in $\mathbb{C}$.

For the ordinary quotient $\mathfrak{X}(n)$ we have the following explicit formula (Theorems 6.1 and 6.3). Let $H^\bullet := H^{\mathrm{ev}} \oplus H^{\mathrm{odd}}$ and let ϕ denote the Euler totient.

Theorem 1.3. *Let $\mathfrak{X}(n)$ denote the ordinary quotient $(\mathbb{T}^n/\mathbb{T})/(\mathbb{Z}/n\mathbb{Z})$. Then we have*

$$\dim_{\mathbb{C}} H^\bullet(\mathfrak{X}(n);\mathbb{C}) = \frac{1}{2n} \sum_{d|n,\, d \text{ odd}} \phi(d) 2^{n/d}.$$

Theorem 1.1 lends itself to an interpretation in terms of representation theory. When $n = \ell$ a prime number, the elliptic representations of SL(ℓ) are discussed in Section 2. The extended quotient $(\mathbb{T}^\ell/\mathbb{T})/\!/(\mathbb{Z}/\ell Z)$ is the disjoint union of the ordinary quotient $\mathfrak{X}(\ell)$ and $\ell(\ell-1)$ isolated points. We consider the canonical projection π of the extended quotient onto the ordinary quotient:

$$\pi\colon (\mathbb{T}^\ell/\mathbb{T})/\!/(\mathbb{Z}/\ell\mathbb{Z}) \to \mathfrak{X}(\ell).$$

The points $\tau_1,\dots,\tau_\ell$ constructed in Section 2, are precisely the $\mathbb{Z}/\ell\mathbb{Z}$ fixed points in $\mathbb{T}^\ell/\mathbb{T}$. These are ℓ points of reducibility, each of which admits ℓ elliptic constituents. Note also that, in the canonical projection π, the fibre $\pi^{-1}(\tau_j)$ of each point τ_j contains ℓ points. We may say that the extended quotient encodes, or provides a model of, reducibility. This is a very special case of the recent conjecture in [2].

When n is non-prime, we have points of reducibility, each of which admits elliptic constituents. In addition to the points of reducibility, there is a subspace of reducibility. There are continua of L-packets. Theorem 1.2 describes the contribution, modulo torsion, of all these L-packets to K_0 and K_1.

Let the infinitesimal character of the elliptic representation ϵ be the cuspidal pair (M,σ), where σ is an irreducible cuspidal representation of M with unitary central character. Then ϵ is a constituent of the induced representation $i_{GM}(\sigma)$. Let $\mathfrak{s}$ be the point in the Bernstein spectrum which contains the cuspidal pair (M,σ). To conform to the notation in [2], we will write $E^{\mathfrak{s}} := \mathbb{T}^n/\mathbb{T}$, $W^{\mathfrak{s}} = \mathbb{Z}/n\mathbb{Z}$. The standard projection will be denoted

$$\pi^{\mathfrak{s}}\colon E^{\mathfrak{s}}/\!/W^{\mathfrak{s}} \to E^{\mathfrak{s}}/W^{\mathfrak{s}}.$$

The space of tempered representations of G determined by $\mathfrak{s}$ will be denoted $\mathrm{Irr}^{\mathrm{temp}}(G)^{\mathfrak{s}}$, and the infinitesimal character will be denoted inf.ch.

Theorem 1.4. *There is a continuous bijection*

$$\mu^{\mathfrak{s}}\colon E^{\mathfrak{s}} /\!/ W^{\mathfrak{s}} \to \mathrm{Irr}^{\mathrm{temp}}(G)^{\mathfrak{s}}$$

such that

$$\pi^{\mathfrak{s}} = (\mathrm{inf.ch.}) \circ \mu^{\mathfrak{s}}.$$

This confirms, in a special case, part (3) *of the conjecture in* [2].

In Section 2 of this article, we review elliptic representations of the special linear algebraic group $\mathrm{SL}(N, F)$ over a p-adic field F. Section 3 concerns fixed C*-algebras and crossed products. The extended quotient $(\mathbb{T}^n/\mathbb{T}) /\!/ (\mathbb{Z}/n\mathbb{Z})$ is computed in Section 4. The formation of the R-groups is described in Section 5. In Section 6 we compute the cyclic invariants in the cohomology of the n-torus.

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2. The elliptic representations of SL(N)

Let F be a nonarchimedean local field of characteristic zero. Let $\boldsymbol{G}$ be a connected reductive linear group over F. Let $G = \boldsymbol{G}(F)$ be the F-rational points of $\boldsymbol{G}$. We say that an element x of G is *elliptic* if its centralizer is compact modulo the center of G. We let G^e denote the set of regular elliptic elements of G.

Let $\mathcal{E}_2(G)$ denote the set of equivalence classes of irreducible discrete series representations of G, and denote by $\mathcal{E}_t(G)$ be the set of equivalence classes of irreducible tempered representations of G. Then $\mathcal{E}_2(G) \subset \mathcal{E}_t(G)$. If $\pi \in \mathcal{E}_t(G)$, then we denote its character by Θ_π. Since Θ_π can be viewed as a locally integrable function, we can consider its restriction to G^e, which we denote by Θ_π^e. We say that π is elliptic if $\Theta_\pi^e \neq 0$. The set of elliptic representations includes the discrete series.

Here is a classical example where elliptic representations occur [1]. We consider the group $\mathrm{SL}(\ell, F)$ with ℓ a prime not equal to the residual characteristic of F. Let K/F be a cyclic of order ℓ extension of F. The reciprocity law in local class field theory is an isomorphism

$$F^\times/N_{K/F}\, K^\times \cong \Gamma(K/F) = \mathbb{Z}/\ell\mathbb{Z},$$

where $\Gamma(K/F)$ is the Galois group of K over F. Let now $\mu_\ell(\mathbb{C})$ be the group of ℓth roots of unity in $\mathbb{C}$. A choice of isomorphism $\mathbb{Z}/\ell\mathbb{Z} \cong \mu_\ell(\mathbb{C})$ then produces a character κ of $F^\times$ of order ℓ as follows:

$$\kappa\colon F^\times \to F^\times/N_{K/F}\, K^\times \cong \mathbb{Z}/\ell\mathbb{Z} \cong \mu_\ell(\mathbb{C}).$$

Let B be the standard Borel subgroup of SL(ℓ), let T be the standard maximal torus, and let $B = T \cdot N$ be its Levi decomposition. Let τ be the character of T defined by

$$\tau := 1 \otimes \kappa \otimes \cdots \otimes \kappa^{\ell-1}$$

and let

$$\pi(\tau) := \mathrm{Ind}_B^G(\tau \otimes 1)$$

be the unitarily induced representation of SL(ℓ).

Now $\pi(\tau)$ is a representation in the minimal unitary principal series of SL(ℓ). It has ℓ distinct irreducible elliptic components and the Galois group $\Gamma(K/F)$ acts simply transitively on the set of irreducible components. The set of irreducible components of $\pi(\tau)$ is an L-packet.

Let

$$\pi(\tau) = \pi_1 \oplus \cdots \oplus \pi_\ell$$

be the ℓ components of $\pi(\tau)$. The character Θ of $\pi(\tau)$, as character of a principal series representation, *vanishes on the elliptic set*. The character Θ_1 of π_1 on the elliptic set is therefore *cancelled out* by the sum $\Theta_2 + \cdots + \Theta_\ell$ of the characters of the relatives $\pi_2, \ldots, \pi_\ell$ of π_1.

Let ω denote an ℓth root of unity in $\mathbb{C}$. All the ℓth roots are allowed, including $\omega = 1$. In the definition of τ, we now replace κ by $\kappa \otimes \omega^{\mathrm{val}}$. This will create ℓ characters, which we will denote by $\tau_1, \ldots, \tau_\ell$, where $\tau_1 = \tau$. For each of these characters, the R-group is given as follows:

$$R(\tau_j) = \mathbb{Z}/\ell\mathbb{Z}$$

for all $1 \le j \le \ell$, and the induced representation $\pi(\tau_j)$ admits ℓ elliptic constituents.

If $P = MU$ is a standard parabolic subgroup of G then $i_{GM}(\sigma)$ will denote the induced representation $\mathrm{Ind}_{MU}^G(\sigma \otimes 1)$ (normalized induction). The R-group attached to σ will be denoted $R(\sigma)$.

Let $P = MU$ be the standard parabolic subgroup of $G :=$ SL(N, F) described as follows. Let $N = mn$, let $\widetilde{M}$ be the Levi subgroup $\mathrm{GL}(m)^n \subset \mathrm{GL}(N, F)$ and let $M = \widetilde{M} \cap \mathrm{SL}(N, F)$.

We will use the framework, notation and main result in [6]. Let $\sigma \in \mathcal{E}_2(M)$ and let $\pi_\sigma \in \mathcal{E}_2(\widetilde{M})$ with $\pi_\sigma|M \supset \sigma$. Let $W(M) := N_G(M)/M$ denote the Weyl group of M, so that $W(M)$ is the symmetric group on n letters. Let

$$\bar{L}(\pi_\sigma) := \{\eta \in \widehat{F^\times} \mid \pi_\sigma \otimes \eta \simeq w\pi_\sigma \text{ for some } w \in W\},$$
$$X(\pi_\sigma) := \{\eta \in \widehat{F^\times} \mid \pi_\sigma \otimes \eta \simeq \pi_\sigma\}.$$

By [6], Theorem 2.4, the R-group of σ is given by

$$R(\sigma) \simeq \bar{L}(\pi_\sigma)/X(\pi_\sigma).$$

We follow [6], Theorem 3.4. Let η be a smooth character of $F^\times$ such that $\eta^n \in X(\pi_1)$ and $\eta^j \notin X(\pi_1)$ for $1 \le j \le n-1$. Set

$$\pi_\sigma \simeq \pi_1 \otimes \eta\pi_1 \otimes \eta^2\pi_1 \otimes \cdots \otimes \eta^{n-1}\pi_1, \quad \pi_\sigma|M \supset \sigma, \tag{1}$$

with $\pi_1 \in \mathcal{E}_2(\mathrm{GL}(m))$, $\eta\pi_1 := (\eta \circ \det) \otimes \pi_1$. Then we have

$$\bar{L}(\pi_\sigma)/X(\pi_\sigma) = \langle\eta\rangle$$

and so $R(\sigma) \simeq \mathbb{Z}/n\mathbb{Z}$. The elliptic representations are the constituents of $i_{GM}(\sigma)$ with π_σ as in equation (1).

3. Fixed algebras and crossed products

Let M denote the Levi subgroup which occurs in Section 2. Denote by $\Psi^1(M)$ the group of unramified unitary characters of M. Now $M \subset \mathrm{SL}(N, F)$ comprises blocks $x_1, \ldots, x_n$ with $x_i \in \mathrm{GL}(m, F)$ and $\prod \det(x_i) = 1$. Each unramified unitary character $\psi \in \Psi^1(M)$ can be expressed as

$$\psi\colon \ \mathrm{diag}(x_1, \ldots, x_n) \to \prod_{j=1}^{n} z_j^{\mathrm{val}(\det x_j)},$$

with $z_1, z_2, \ldots, z_n \in \mathbb{T}$, i.e., $|z_i| = 1$. Such unramified unitary characters ψ correspond to coordinates $(z_1 : z_2 : \cdots : z_n)$ with each $z_i \in \mathbb{T}$. Since

$$\prod_{i=1}^{n} (zz_i)^{\mathrm{val}(\det x_i)} = \prod_{i=1}^{n} z_i^{\mathrm{val}(\det x_i)}$$

we have *homogeneous* coordinates. We have the isomorphism

$$\Psi^1(M) \cong \{(z_1 : z_2 : \cdots : z_n) \mid |z_i| = 1,\ 1 \le i \le n\} = \mathbb{T}^n/\mathbb{T}.$$

If M is the standard maximal torus T of $\mathrm{SL}(N)$ then $\Psi^1(T)$ is the maximal *compact* torus in the dual torus

$$T^\vee \subset G^\vee = \mathrm{PGL}(N, \mathbb{C}),$$

where $G^\vee$ is the Langlands dual group.

Let $\sigma, \pi_\sigma, \pi_1$ be as in equation (1). Let g be the order of the group of unramified characters χ of $F^\times$ such that $(\chi \circ \det) \otimes \pi_1 \simeq \pi_1$. Now let

$$E := \{\psi \otimes \sigma \mid \psi \in \Psi^1(M)\}.$$

The base point $\sigma \in E$ determines a homeomorpism

$$E \simeq \mathbb{T}^n/\mathbb{T}, \quad (z_1^{\mathrm{val}\circ\det} \otimes \cdots \otimes z_n^{\mathrm{val}\circ\det}) \otimes \sigma \mapsto (z_1^g : \cdots : z_n^g).$$

From this point onwards, we will require that the *restriction of* η *to the group* $\mathcal{U}_F$ *of integer units is of order* n. Let $W(M)$ denote the Weyl group of M and let $W(M,E)$ be the subgroup of $W(M)$ which leaves E globally invariant. Then we have $W(M,E) = W(\sigma) = R(\sigma) = \mathbb{Z}/n\mathbb{Z}$.

Let $\mathfrak{K} = \mathfrak{K}(H)$ denote the C*-algebra of compact operators on the standard Hilbert space H. Let $\mathfrak{a}(w,\lambda)$ denote normalized intertwining operators. The fixed C*-algebra $C(E,\mathfrak{K})^{W(M,E)}$ is given by

$$\{f \in C(E,\mathfrak{K}) \mid f(w\lambda) = \mathfrak{a}(w,\lambda\tau) f(\lambda) \mathfrak{a}(w,\lambda\tau)^{-1},\ w \in W(M,E)\}.$$

This fixed C*-algebra is a C*-direct summand of the reduced C*-algebra $\mathfrak{A}_N$ of SL(N), see [7].

Theorem 3.1. *Let* $G = \mathrm{SL}(N,F)$*, and* M *be a Levi subgroup consisting of n blocks of the same size* m*. Let* $\sigma \in \mathcal{E}_2(M)$*. Assume that the induced representation* $i_{GM}(\sigma)$ *has elliptic constituents, then the fixed C*-algebra* $C(E,\mathfrak{K})^{W(M,E)}$ *is strongly Morita equivalent to the crossed product C*-algebra* $C(E) \rtimes \mathbb{Z}/n\mathbb{Z}$.

Proof. For the commuting algebra of $i_{MG}(\sigma)$, we have [12]

$$\mathrm{End}_G((i_{MG}(\sigma)) = \mathbb{C}[R(\sigma)].$$

Let w_0 be a generator of $R(\sigma)$, then the normalized intertwining operator $\mathfrak{a}(w_0,\sigma)$ is a unitary operator of order n. By the spectral theorem for unitary operators, we have

$$\mathfrak{a}(w_0,\sigma) = \sum_{j=0}^{n-1} \omega^j E_j$$

where $\omega = \exp(2\pi i/n)$ and E_j are the projections onto the irreducible subspaces of the induced representation $i_{MG}(\sigma)$. The unitary representation

$$R(\sigma) \to U(H), \quad w \mapsto \mathfrak{a}(w,\sigma)$$

contains each character of $R(\sigma)$ countably many times. Therefore condition (***) in [10], p. 301, is satisfied. The condition (**) in [10], p. 300, is trivially satisfied since $W(\sigma) = R(\sigma)$.

We have $W(\sigma) = \mathbb{Z}/n\mathbb{Z}$. Then a subgroup $W(\rho)$ of order d is given by $W(\rho) = k\mathbb{Z} \mod n$ with $dk = n$. In that case, we have

$$\mathfrak{a}(w_0,\sigma)|_{W(\rho)} = \sum_{j=0}^{n-1} \omega^{kj} E_j.$$

We compare the two unitary representations

$$\phi_1 : W(\rho) \to U(H), \quad w \mapsto \mathfrak{a}(w,\sigma)|_{W(\rho)},$$

$$\phi_2 \colon W(\rho) \to U(H), \quad w \mapsto \mathfrak{a}(w, \rho).$$

Each representation contains every character of $W(\rho)$. They are *quasi-equivalent* as in [10]. Choose an increasing sequence (e_n) of finite-rank projections in $\mathcal{L}(H)$ which converge strongly to I and commute with each projection E_j. The compressions of ϕ_1, ϕ_2 to $e_n H$ remain quasi-equivalent. Condition (*) in [10], p. 299, is satisfied.

All three conditions of [10], Theorem 2.13, are satisfied. We therefore have a strong Morita equivalence

$$(C(E) \otimes \mathfrak{K})^{W(M,E)} \simeq C(E) \rtimes R(\sigma) = \mathbb{C}(E) \rtimes \mathbb{Z}/n\mathbb{Z}. \qquad \square$$

We will need a special case of the Chern character for discrete groups [3].

Theorem 3.2. *We have an isomorphism*

$$K_i(C(E) \rtimes \mathbb{Z}/n\mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{C} \cong \bigoplus_{j \in \mathbb{N}} H^{2j+i}(E /\!/ (\mathbb{Z}/n\mathbb{Z}); \mathbb{C})$$

with $i = 0, 1$, *where* $E /\!/ (\mathbb{Z}/n\mathbb{Z})$ *denotes the extended quotient of* E *by* $\mathbb{Z}/n\mathbb{Z}$.

When N is a prime number ℓ, this result already appeared in [8], [10].

4. The formation of the fixed sets

Extended quotients were introduced by Baum and Connes [3] in the context of the Chern character for discrete groups. Extended quotients were used in [9], [8] in the context of the reduced group C*-algebras of $\mathrm{GL}(N)$ and $\mathrm{SL}(\ell)$ where ℓ is prime. The results in this section extend results in [8], [10].

Definition 4.1. Let X be a compact Hausdorff topological space. Let Γ be a finite *abelian* group acting on X by a (left) continuous action. Let

$$\widetilde{\mathrm{X}} = \{(x, \gamma) \in \mathrm{X} \times \Gamma \mid \gamma x = x\}$$

with the group action on $\widetilde{\mathrm{X}}$ given by

$$g \cdot (x, \gamma) = (gx, \gamma)$$

for $g \in \Gamma$. Then the *extended quotient* is given by

$$\mathrm{X}/\!/\Gamma := \widetilde{\mathrm{X}}/\Gamma = \bigsqcup_{\gamma \in \Gamma} \mathrm{X}^{\gamma}/\Gamma$$

where X^{γ} is the γ-fixed set.

The extended quotient will always contain the ordinary quotient. The standard projection $\pi\colon X/\!/\Gamma \to X/\Gamma$ is induced by the map $(x,\gamma)\mapsto x$. We note the following elementary fact, which will be useful later (in Lemma 5.2): let $y=\Gamma x$ be a point in X/Γ. Then the cardinality of the pre-image $\pi^{-1}y$ is equal to the order of the isotropy group Γ_x:

$$|\pi^{-1}y| = |\Gamma_x|.$$

We will write X $= E = \mathbb{T}^n/\mathbb{T}$, where $\mathbb{T}$ acts diagonally on $\mathbb{T}^n$, i.e.,

$$t(t_1,t_2,\dots,t_n) = (tt_1,tt_2,\dots,tt_n), \quad t,t_i\in\mathbb{T}.$$

We have the action of the finite group $\Gamma=\mathbb{Z}/n\mathbb{Z}$ on $\mathbb{T}^n/\mathbb{T}$ given by cyclic permutation. The two actions of $\mathbb{T}$ and of $\mathbb{Z}/n\mathbb{Z}$ on $\mathbb{T}^n$ commute. We will write (k,n) for the highest common factor of k and n.

Theorem 4.2. *The extended quotient $(\mathbb{T}^n/\mathbb{T})/\!/(\mathbb{Z}/n\mathbb{Z})$ is a disjoint union of compact connected orbifolds:*

$$(\mathbb{T}^n/\mathbb{T})/\!/(\mathbb{Z}/n\mathbb{Z}) \simeq \bigsqcup_{\substack{1\le k\le n\\ \omega^{n/(k,n)}=1}} \mathfrak{X}(n,k,\omega).$$

Here ω is a $n/(k,n)$th root of unity in $\mathbb{C}$.

Proof. Let γ be the standard n-cycle defined by $\gamma(i)=i+1 \mod n$. Then γ^k is the product of n/d cycles of order $d=n/(n,k)$. Let ω be a dth root of unity in $\mathbb{C}$. All dth roots of unity are allowed, including $\omega=1$. The element $t(\omega)=t(\omega;z_1,\dots,z_n)\in\mathbb{T}^n$ is defined by imposing the relations

$$z_{i+k} = \omega^{-1}z_i,$$

all suffices mod n. This condition allows n/d of the complex numbers $z_1,\dots,z_n$ to vary freely, subject only to the condition that each z_j has modulus 1. The crucial point is that

$$\gamma^k\cdot t(\omega) = \omega t(\omega)$$

Then ω determines a γ^k-fixed set in $\mathbb{T}^n/\mathbb{T}$, namely the set $\mathfrak{Y}(n,k,\omega)$ of all cosets $t(\omega)\cdot\mathbb{T}$. The set $\mathfrak{Y}(n,k,\omega)$ is an $(n/d-1)$-dimensional subspace of fixed points.

Note that $\mathfrak{Y}(n,k,\omega)$, as a coset of the closed subgroup $\mathfrak{Y}(n,k,1)$ in the compact Lie group E, is homeomorphic (by translation in E) to $\mathfrak{Y}(n,k,1)$. The translation is by the element $t(\omega:1,\dots,1)$. If ω_1,ω_2 are distinct dth roots of unity, then $\mathfrak{Y}(n,k,\omega_1),\mathfrak{Y}(n,k,\omega_2)$ are disjoint.

We define the quotient space

$$\mathfrak{X}(n,k,\omega) := \mathfrak{Y}(n,k,\omega)/(\mathbb{Z}/n\mathbb{Z})$$

and apply Definition 4.1. □

When $k = n$, we must have $\omega = 1$. In that case, the orbifold is the ordinary quotient: $\mathfrak{X}(n, n, 1) = \mathfrak{X}(n)$.

Let $(n, k) = 1$. The number of such k in $1 \le k \le n$ is $\phi(n)$. In this case, ω is an nth root of unity and $\mathfrak{X}(n, k, \omega)$ is a point. There are n such roots of unity in $\mathbb{C}$. Therefore, the extended quotient $(\mathbb{T}^n/\mathbb{T})/\!/(\mathbb{Z}/n\mathbb{Z})$ always contains $\phi(n)n$ isolated points.

Theorem 1.1 is a consequence of Theorems 3.1, 3.2 and 4.2. If, in Theorem 1.1, we take n to be a prime number ℓ, then we recover the following result in [8], p. 30: the extended quotient $(\mathbb{T}^\ell/\mathbb{T})/\!/(\mathbb{Z}/\ell\mathbb{Z})$ is the disjoint union of the ordinary quotient $\mathfrak{X}(\ell)$ and $(\ell - 1)\ell$ points.

5. The formation of the R-groups

We continue with the notation of Section 3. Let σ, π_σ, π_1, η be as in equation (1). The n-tuple $t := (z_1, \dots, z_n) \in \mathbb{T}^n$ determines an element $[t] \in E$. We can interpret $[t]$ as the unramified character

$$\chi_t := (z_1^{\mathrm{val}\circ\det}, \dots, z_n^{\mathrm{val}\circ\det}).$$

Let $\Gamma = \mathbb{Z}/n\mathbb{Z}$, and let $\Gamma_{[t]}$ denote the isotropy subgroup of Γ.

Lemma 5.1. *The isotropy subgroup $\Gamma_{[t]}$ is isomorphic to the R-group of $\chi_t \otimes \sigma$:*

$$\Gamma_{[t]} \simeq R(\chi_t \otimes \sigma).$$

Proof. Let the order of $\Gamma_{[t]}$ be d. Then d is a divisor of n. Let γ be a generator of $\Gamma_{[t]}$. Then γ is a product of n/d disjoint d-cycles, as in Section 4. We must have $t = t(\omega)$ with ω a dth root of unity in $\mathbb{C}$. Note that $\gamma \cdot t(\omega) = \omega t(\omega)$. Then we have

$$\begin{aligned} R(\chi_t \otimes \sigma) &= \bar{L}(\chi_t \otimes \pi_\sigma)/X(\chi_t \otimes \pi_\sigma) \\ &= \{\alpha \in \widehat{F^\times} \mid w\pi_\sigma \simeq \pi_\sigma \otimes \alpha \text{ for some } w \text{ in } W\}/X(\chi_t \otimes \pi_\sigma) \\ &= \langle \omega^{\mathrm{val}\circ\det} \otimes \eta^{n/d} \rangle \\ &= \mathbb{Z}/d\mathbb{Z} \\ &= \Gamma_{[t]} \end{aligned}$$

since, modulo $X(\chi_t \otimes \pi_\sigma)$, the character $\eta^{n/d}$ has order d. □

Lemma 5.2. *In the standard projection $p\colon E/\!/\Gamma \to E/\Gamma$, the cardinality of the fibre of $[t]$ is the order of the R-group of $\chi_t \otimes \sigma$.*

Proof. This follows from Lemma 5.1. □

We will assume that σ is a *cuspidal* representation of M with unitary central character. Let $\mathfrak{s}$ be the point in the Bernstein spectrum of SL(N) which contains the cuspidal pair (M, σ). To conform to the notation in [2], we will write $E^{\mathfrak{s}} := \mathbb{T}^n/\mathbb{T}$, $W^{\mathfrak{s}} = \mathbb{Z}/n\mathbb{Z}$. The standard projection will be denoted

$$\pi^{\mathfrak{s}} \colon E^{\mathfrak{s}} /\!/ W^{\mathfrak{s}} \to E^{\mathfrak{s}}/W^{\mathfrak{s}}.$$

The space of tempered representations of G determined by $\mathfrak{s}$ will be denoted by $\mathrm{Irr}^{\mathrm{temp}}(G)^{\mathfrak{s}}$, and the infinitesimal character will be denoted $inf.ch.$

Theorem 5.3. *We have a commutative diagram*

$$\begin{array}{ccc} E /\!/ W^{\mathfrak{s}} & \xrightarrow{\mu^{\mathfrak{s}}} & \mathrm{Irr}^{\mathrm{temp}}(G)^{\mathfrak{s}} \\ {\scriptstyle \pi^{\mathfrak{s}}}\downarrow & & \downarrow{\scriptstyle \mathrm{inf.ch.}} \\ E/W^{\mathfrak{s}} & \longrightarrow & E/W^{\mathfrak{s}} \end{array}$$

in which the map $\mu^{\mathfrak{s}}$ is a continuous bijection. This confirms, in a special case, part (3) *of the conjecture in* [2].

Proof. We have

$$\mathbb{C}[R(\sigma)] \simeq \mathrm{End}_G(i_{GM}(\sigma)).$$

This implies that the characters of the cyclic group $R(\sigma)$ parametrize the irreducible constituents of $i_{GM}(\sigma)$. This leads to a labelling of the irreducible constituents of $i_{GM}(\sigma)$, which we will write as $i_{GM}(\sigma : r)$ with $0 \le r < n$.

The map $\mu^{\mathfrak{s}}$ is defined as follows:

$$\mu^{\mathfrak{s}} \colon (t, \gamma^{rd}) \mapsto i_{GM}(\chi_t \otimes \sigma : r).$$

We now apply Lemma 5.2.

Theorem 3.2 in [7] relates the natural topology on the Harish-Chandra parameter space to the Jacobson topology on the tempered dual of a reductive p-adic group. As a consequence, the map $\mu^{\mathfrak{s}}$ is continuous. □

6. Cyclic invariants

We will consider the map

$$\alpha \colon \mathbb{T}^n \to (\mathbb{T}^n/\mathbb{T}) \times \mathbb{T}, \quad (t_1, \dots, t_n) \mapsto ((t_1 : \cdots : t_n), t_1 t_2 \dots t_n),$$

where $(t_1 : \cdots : t_n)$ is the image of $(t_1, \dots, t_n)$ via the map $\mathbb{T}^n \to \mathbb{T}^n/\mathbb{T}$. The map α is a homomorphism of Lie groups. The kernel of this map is

$$\mathcal{G}_n := \{\omega I_n \mid \omega^n = 1\}.$$

We therefore have the isomorphism of compact connected Lie groups:

$$\mathbb{T}^n/\mathcal{G}_n \cong (\mathbb{T}^n/\mathbb{T}) \times \mathbb{T}. \tag{2}$$

This isomorphism is equivariant with respect to the $\mathbb{Z}/n\mathbb{Z}$-action, and we infer that

$$(\mathbb{T}^n/\mathcal{G}_n)/(\mathbb{Z}/n\mathbb{Z}) \cong (\mathbb{T}^n/\mathbb{T})/(\mathbb{Z}/n\mathbb{Z}) \times \mathbb{T}. \tag{3}$$

Theorem 6.1. *Let $H^\bullet(-;\mathbb{C})$ denote the total cohomology group. We have*

$$\dim_{\mathbb{C}} H^\bullet(\mathfrak{X}(n);\mathbb{C}) = \tfrac{1}{2} \cdot \dim_{\mathbb{C}} H^\bullet(\mathbb{T}^n;\mathbb{C})^{\mathbb{Z}/n\mathbb{Z}}.$$

Proof. The cohomology of the orbit space is given by the fixed set of the cohomology of the original space [4], Corollary 2.3, p. 38. We have

$$H^j(\mathbb{T}^n/\mathcal{G}_n;\mathbb{C}) \cong H^j(\mathbb{T}^n;\mathbb{C})^{\mathcal{G}_n} \cong H^j(\mathbb{T}^n;\mathbb{C}) \tag{4}$$

since the action of $\mathcal{G}_n$ on $\mathbb{T}^n$ is homotopic to the identity. We spell this out. Let $z := (z_1,\dots,z_n)$ and define $H(z,t) = \omega^t \cdot z = (\omega^t z_1,\dots,\omega^t z_n)$. Then $H(z,0) = z$, $H(z,1) = \omega \cdot z$. Also, H is equivariant with respect to the permutation action of $\mathbb{Z}/n\mathbb{Z}$. That is to say, if $\epsilon \in \mathbb{Z}/n\mathbb{Z}$ then $H(\epsilon \cdot z, t) = \epsilon \cdot H(z,t)$. This allows us to proceed as follows:

$$\begin{aligned} H^j(\mathbb{T}^n;\mathbb{C})^{\mathbb{Z}/n\mathbb{Z}} &\cong H^j(\mathbb{T}^n/\mathcal{G}_n;\mathbb{C})^{\mathbb{Z}/n\mathbb{Z}} \\ &\cong H^j((\mathbb{T}^n/\mathbb{T}) \times \mathbb{T};\mathbb{C})^{\mathbb{Z}/n\mathbb{Z}} \\ &\cong H^j((\mathbb{T}^n/\mathbb{T})/(\mathbb{Z}/n\mathbb{Z}) \times \mathbb{T};\mathbb{C}). \end{aligned} \tag{5}$$

We apply the Künneth theorem in cohomology (there is no torsion):

$$(H^j(\mathbb{T}^n;\mathbb{C}))^{\mathbb{Z}/n\mathbb{Z}} \cong H^j(\mathfrak{X}(n);\mathbb{C}) \oplus H^{j-1}(\mathfrak{X}(n);\mathbb{C}) \quad \text{with } 0 < j \le n,$$
$$(H^n(\mathbb{T}^n;\mathbb{C}))^{\mathbb{Z}/n\mathbb{Z}} \simeq H^{n-1}(\mathfrak{X}(n);\mathbb{C}), \qquad H^0(\mathbb{T}^n;\mathbb{C})^{\mathbb{Z}/n\mathbb{Z}} \cong H^0(\mathfrak{X}(n);\mathbb{C}) \simeq \mathbb{C},$$
$$H^{\mathrm{ev}}(\mathbb{T}^n;\mathbb{C})^{\mathbb{Z}/n\mathbb{Z}} = H^\bullet(\mathfrak{X}(n);\mathbb{C}), \qquad H^{\mathrm{odd}}(\mathbb{T}^n;\mathbb{C})^{\mathbb{Z}/n\mathbb{Z}} = H^\bullet(\mathfrak{X}(n);\mathbb{C}).$$

□

We now have to find the cyclic invariants in $H^\bullet(\mathbb{T}^n;\mathbb{C})$. The cohomology ring $H^\bullet(\mathbb{T}^n,\mathbb{C})$ is the exterior algebra $\bigwedge V$ of a complex n-dimensional vector space V, as can be seen by considering differential forms $d\theta_1 \wedge \cdots \wedge d\theta_r$. The vector space V admits a basis $\alpha_1 = d\theta_1,\dots,\alpha_n = d\theta_n$. The action of $\mathbb{Z}/n\mathbb{Z}$ on $\bigwedge V$ is induced by permuting the elements $\alpha_1,\dots,\alpha_n$, i.e., by the regular representation ρ of the cyclic group $\mathbb{Z}/n\mathbb{Z}$. This representation of $\mathbb{Z}/n\mathbb{Z}$ on $\bigwedge V$ will be denoted $\bigwedge \rho$. The dimension of the space of cyclic invariants in $H^\bullet(\mathbb{T}^n,\mathbb{C})$ is equal to the multiplicity of the unit representation 1 in $\bigwedge \rho$. To determine this, we use the theory of group characters.

Lemma 6.2. *The dimension of the subspace of cyclic invariants is given by*

$$(\chi_{\bigwedge\rho}, 1) = \tfrac{1}{n}(\chi_{\bigwedge\rho}(0) + \chi_{\bigwedge\rho}(1) + \cdots + \chi_{\bigwedge\rho}(n-1)).$$

Proof. This is a standard result in the theory of group characters [11]. □

Theorem 6.3. *The dimension of the space of cyclic invariants in $H^\bullet(\mathbb{T}^n, \mathbb{C})$ is given by the formula*

$$g(n) := \frac{1}{n} \sum_{d|n,\, d \text{ odd}} \phi(d) 2^{n/d}$$

Proof. We note first that

$$\chi_{\bigwedge\rho}(0) = \text{Trace } 1_{\bigwedge V} = \dim_{\mathbb{C}} \bigwedge V = 2^n.$$

To evaluate the remaining terms, we need to recall the definition of the elementary symmetric functions e_j:

$$\prod_{j=1}^{n} (\lambda - \alpha_j) = \lambda^n - \lambda^{n-1} e_1 + \lambda^{n-2} e_2 - \cdots + (-1)^n e_n.$$

If we need to mark the dependence on $\alpha_1, \ldots, \alpha_n$ we will write $e_j = e_j(\alpha_1, \ldots, \alpha_n)$. Set $\alpha_j = \omega^{j-1}, \omega = \exp(2\pi i/n)$. Then we get

$$\lambda^n - 1 = \prod_{j=1}^{n} (\lambda - \alpha_j) = \lambda^n - \lambda^{n-1} e_1 + \lambda^{n-2} e_2 - \cdots + (-1)^n e_n.$$

Let $d|n$, let ζ be a *primitive* dth root of unity. Let $\alpha_j = \zeta^{j-1}$. We have

$$(\lambda^d - 1)^{n/d} = (\lambda^d - 1) \ldots (\lambda^d - 1) = \prod_{j=1}^{n} (\lambda - \alpha_j). \quad (6)$$

Set $\lambda = -1$. If d is even, we obtain

$$0 = 1 + e_1(1, \zeta, \zeta^2, \ldots) + e_2(1, \zeta, \zeta^2, \ldots) + \cdots + e_n(1, \zeta, \zeta^2, \ldots). \quad (7)$$

If d is odd, we obtain

$$2^{n/d} = 1 + e_1(1, \zeta, \zeta^2, \ldots) + e_2(1, \zeta, \zeta^2, \ldots) + \cdots + e_n(1, \zeta, \zeta^2, \ldots). \quad (8)$$

We observe that the regular representation ρ of the cyclic group $\mathbb{Z}/n\mathbb{Z}$ is a direct sum of the characters $m \mapsto \omega^{rm}$ with $0 \le r \le n$. This direct sum decomposition allows us to choose a basis $v_1, \ldots, v_n$ in V such that the representation $\bigwedge \rho$ is diagonalized by the wedge products $v_{j_1} \wedge \cdots \wedge v_{j_l}$. This in turn allows us to compute the character of $\bigwedge \rho$ in terms of the elementary symmetric functions $e_1, \ldots, e_n$.

With $\zeta = \omega^r$ as above, we have

$$\chi_{\wedge\rho}(r) = 1 + e_1(1,\zeta,\zeta^2,\dots) + e_2(1,\zeta,\zeta^2,\dots) + \cdots + e_n(1,\zeta,\zeta^2,\dots).$$

We now sum the values of the character $\chi_{\wedge\rho}$. Let $d := n/(r,n)$. Then ζ is a primitive dth root of unity. If d is even then $\chi_{\wedge\rho}(r) = 0$. If d is odd, then $\chi_{\wedge\rho}(r) = 2^{n/d}$. There are $\phi(d)$ such terms. So we have

$$\chi_{\wedge\rho}(0) + \chi_{\wedge\rho}(1) + \cdots + \chi_{\wedge\rho}(n-1) = \sum_{d|n,\, d \text{ odd}} \phi(d)2^{n/d}. \tag{9}$$

We now apply Lemma 6.2. □

The sequence $n \mapsto g(n)/2$, $n = 1,2,3,4,\dots$, is

1, 1, 2, 2, 4, 6, 10, 16, 30, 52, 94, 172, 316, 586, 1096, 2048, 3856, 7286, ...

as in http://www.research.att.com/~njas/sequences/A000016. Thanks to Kasper Andersen for alerting us to this web site.

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