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Double constructions of Frobenius algebras, Connes cocycles and their duality

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Abstract. We construct an associative algebra with a decomposition into the direct sum of the underlying vector spaces of another associative algebra and its dual space such that both of them are subalgebras and the natural symmetric bilinear form is invariant or the natural antisymmetric bilinear form is a Connes cocycle. The former is called a double construction of a Frobenius algebra and the latter is called a double construction of the Connes cocycle, which is interpreted in terms of dendriform algebras. Both of them are equivalent to a kind of bialgebras, namely, antisymmetric infinitesimal bialgebras and dendriform D-bialgebras, respectively. In the coboundary cases, our study leads to what we call associative Yang–Baxter equation in an associative algebra and D-equation in a dendriform algebra, respectively, which are analogues of the classical Yang–Baxter equation in a Lie algebra. We show that an antisymmetric solution of the associative Yang–Baxter equation corresponds to the antisymmetric part of a certain operator called $\mathcal{O}$-operator which gives a double construction of a Frobenius algebra, whereas a symmetric solution of the D-equation corresponds to the symmetric part of an $\mathcal{O}$-operator which gives a double construction of the Connes cocycle. By comparing antisymmetric infinitesimal bialgebras and dendriform D-bialgebras, we observe that there is a clear analogy between them. Due to the correspondences between certain symmetries and antisymmetries appearing in this analogy, we regard it as a kind of duality.

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Contents

1. Introduction

Throughout this article, an associative algebra is a non-unital associative algebra. There are two important (non-degenerate) bilinear forms on an associative algebra given as follows.

Definition 1.0.1. A bilinear form $\mathcal{B}(\ ,\ )$ on an associative algebra A is *invariant* if

$$\mathcal{B}(xy,z)=\mathcal{B}(x,yz)\quad \text{for all } x,y,z\in A.$$

Definition 1.0.2. An antisymmetric bilinear form $\omega(\ ,\ )$ on an associative algebra A is a *cyclic* 1*-cocycle in the sense of Connes* if

$$\omega(xy,z)+\omega(yz,x)+\omega(zx,y)=0\quad \text{for all } x,y,z\in A. \tag{1}$$

We also call for abbreviation ω a *Connes cocycle*.

1.1. Frobenius algebras. A Frobenius algebra $(A,\mathcal{B})$ is an associative algebra A with a non-degenerate invariant bilinear form $\mathcal{B}(\ ,\ )$. It was first studied by Frobenius ([Fro]) in 1903 and then named by Brauer and Nesbitt ([BrN]). In fact, Frobenius algebras appear in many fields in mathematics and mathematical physics, such as (modular) representations of finite groups ([Kap]), Hopf algebras ([LS]), statistical models over 2-dimensional graphs ([BFN]), Yang–Baxter equation ([St]), Poisson brackets of hydrodynamic type ([BaN]) and so on. In particular, they play a key role in the study of topological quantum field theory ([Ko], [RFFS], etc.). There are many references concerning the study of Frobenius algebras (for example, see [Kap] or [Y] and the references therein).

A Frobenius algebra $(A,\mathcal{B})$ is symmetric if $\mathcal{B}$ is symmetric. In this article, we mainly consider a class of symmetric Frobenius algebras $(A,\mathcal{B})$ satisfying the conditions

(1) $A=A_1\oplus A_1^*$ as the direct sum of vector spaces;

(2) A_1 and A_1^* are associative subalgebras of A;

(3) $\mathcal{B}$ is the natural symmetric bilinear form on $A_1\oplus A_1^*$ given by

$$\mathcal{B}(x+a^*,y+b^*)=\langle x,b^*\rangle+\langle a^*,y\rangle\quad \text{for all } x,y\in A_1,\ a^*,b^*\in A_1^*, \tag{2}$$

where $\langle\ ,\ \rangle$ is the natural pair between the vector space A_1 and its dual space A_1^*. We call it a double construction of a Frobenius algebra.

Such a double construction of a Frobenius algebra is quite different from the "double extension construction" of a Lie algebra with a non-degenerate invariant bilinear form ([Kac], [MR1]–[MR2], etc.) or the "T^*-extension" of Frobenius algebra given by Bordemann in [Bo].

Moreover, the above double constructions of Frobenius algebras were also considered by Zhelyabin in [Z] and Aguiar in [A3] (under the name of "balanced Drinfeld double $D_b(A)$") with different motivations and approaches respectively. They are closely related to Lie bialgebras. Lie bialgebras were introduced by Drinfeld ([D]) and play a crucial role in symplectic geometry and quantum groups. They are equivalent to Manin triples (see [CP] and the references therein or Section 5.2).

It is easy to show that the commutator of a Frobenius algebra from the above double construction gives a Manin triple (hence a Lie bialgebra). Furthermore, such a double construction has many properties similar to a Lie bialgebra. It is equivalent to an antisymmetric infinitesimal bialgebra (which also goes under the names of "associative D-algebra" in [Z] and "balanced infinitesimal bialgebra" in the sense of the opposite algebra in [A3]), and under a "coboundary" condition it leads to an analogue of the classical Yang–Baxter equation ([Se]) in an associative algebra A_1,

$$r_{12}r_{13} + r_{13}r_{23} - r_{23}r_{12} = 0, \tag{3}$$

where $r = \sum_i x_i \otimes y_i \in A_1 \otimes A_1$ and

$$\begin{aligned} r_{12}r_{13} &= \sum_{i,j} x_i x_j \otimes y_i \otimes y_j, \\ r_{13}r_{23} &= \sum_{i,J} x_i \otimes x_j \otimes y_i y_j, \\ r_{23}r_{12} &= \sum_{i,j} x_j \otimes x_i y_j \otimes y_i. \end{aligned} \tag{4}$$

In particular, an antisymmetric solution of the above equation in A_1 gives a double construction of a Frobenius algebra $(A = A_1 \oplus A_1^*, \mathcal{B})$.

On the other hand, we introduce the new notion of antisymmetric infinitesimal bialgebra in order to express explicitly its relation with the known notion of infinitesimal bialgebra, although there are certain notions for the same or similar structures. An infinitesimal bialgebra is a triple (A, m, Δ), where (A, m) is an associative algebra, (A, Δ) is a coassociative algebra and

$$\Delta(ab) = \sum ab_1 \otimes b_2 + \sum a_1 \otimes a_2 b \quad \text{for all } a, b \in A. \tag{5}$$

It was introduced by Join and Rota ([JR]) in order to provide an algebraic framework for the calculus of divided difference. Furthermore, Aguiar studied the cases of principal derivations and introduced the associative Yang–Baxter equation ([A1])

$$r_{13}r_{12} - r_{12}r_{23} + r_{23}r_{13} = 0. \tag{6}$$

Note that eq. (3) is eq. (6) in the opposite algebra and, when r is antisymmetric, eq. (6) is just eq. (3) under the operation $\sigma_{13}(x \otimes y \otimes z) = z \otimes y \otimes x$.

We would like to point out that although many results on the double constructions of Frobenius algebras have been obtained, a complete and explicit interpretation does not yet exist. In fact, most of these results were given in a scattered way with different motivations. For example, Zhelyabin in [Z] introduced the notion of associative D-algebra as an important step to develop a bialgebra theory of Jordan algebras (an explicit study of coboundary cases for the associative algebras themselves is not performed). In [A3], Aguiar introduced the notion of balanced infinitesimal bialgebra and then studied the antisymmetric solutions of eq. (6) in order to compare them with Lie bialgebras and the classical Yang–Baxter equation in a Lie algebra, respectively, and the balanced Drinfeld double $D_b(A)$ appears as an important consequence. We will formulate the known results by a different and systematic approach (for example, the "invariant" antisymmetry appears naturally). Moreover such an approach is useful and convenient for the whole study in this article.

1.2. $\mathcal{O}$-operators and dendriform algebras. When r is antisymmetric, besides the standard tensor form (3) or (6), the associative Yang–Baxter equation has an equivalent operator form, that is, a special case of a certain operator called $\mathcal{O}$-operator. An $\mathcal{O}$-operator associated to a bimodule (l, r, V) of an associative algebra A is a linear map $T : V \to A$ satisfying

$$T(u) \cdot T(v) = T(l(T(u))v + r(T(v)u)) \quad \text{for all } u, v \in V.$$

In fact, an antisymmetric solution of the associative Yang–Baxter equation is an $\mathcal{O}$-operator associated to the bimodule (R^*, L^*). The notion of $\mathcal{O}$-operator was introduced in [BGN1] (such a structure appeared independently in [U] under the name of generalized Rota–Baxter operator), which is an analogue of the $\mathcal{O}$-operator defined by Kupershmidt as a natural generalization of the operator form of the classical Yang–Baxter equation ([Ku3] and a further study in [Bai1]). Conversely, the antisymmetric part of an $\mathcal{O}$-operator satisfies the associative Yang–Baxter equation in a larger associative algebra.

From an $\mathcal{O}$-operator, one can get a dendriform algebra. Dendriform algebras are equipped with an associative product which can be written as a linear combination of nonassociative compositions. They were introduced by Loday ([Lo1]) with motivation from algebraic K-theory and have been studied quite extensively with connections to several areas in mathematics and physics, including operads ([Lo3]), homology ([Fra1]–[Fra2]), Hopf algebras ([Cha2], [H1]–[H2], [Ron], [LR2]), Lie and Leibniz algebras ([Fra2]), combinatorics ([LR1]), arithmetic ([Lo2]) and quantum field theory ([F1]) and so on (see [EMP] and the references therein).

Furthermore, there is a compatible dendriform algebra structure on an associative algebra A if and only if there exists an invertible $\mathcal{O}$-operator of A, or equivalently, there exists an invertible (usual) 1-cocycle (see eq. (34)) associated to certain suitable bimodule of A ([BGN2]). Thus a close relation between the associative Yang–Baxter equation (hence the antisymmetric infinitesimal bialgebras and the double construction of Frobenius algebras) and dendriform algebras is obviously given (see also [A3], [E1]–[E2]).

1.3. Connes cocycles. Note that a Connes cocycle given by eq. (1) is in fact a Hochschild 2-cocycle which satisfies antisymmetry. It corresponds to the original definition of cyclic cohomology by Connes ([C]). Also note that in cyclic cohomology a cyclic n-cocycle in the sense of Connes is an $(n+1)$-linear form, although a Connes cocycle was called a cyclic 2-cocycle in some references (like [A3]) from some different viewpoints. Moreover, although Connes used it in the unital framework and in the non-unital framework cyclic homology has a very different behavior, we still use the notion of "Connes cocycle" in this article.

We will see that, from a non-degenerate Connes cocycle on an associative algebra A, one can get a compatible dendriform algebra structure on A. Moreover, the dendriform algebra structures play a key role in the following constructions of non-degenerate Connes cocycles, which is one of the main issues of this article. We call (A, ω) a double construction of the Connes cocycle if it satisfies the conditions

(1) $A = A_1 \oplus A_1^*$ as the direct sum of vector spaces;

(2) A is an associative algebra and A_1 and A_1^* are associative subalgebras of A;

(3) ω is the natural antisymmetric bilinear form on $A_1 \oplus A_1^*$ given by

$$\omega(x+a^*, y+b^*) = -\langle x, b^*\rangle + \langle a^*, y\rangle \quad \text{for all } x, y \in A_1,\ a^*, b^* \in A_1^*, \tag{7}$$

and ω is a Connes cocycle on A.

In this article, the double construction of the Connes cocycle is interpreted in terms of dendriform algebras. We find that such a structure is quite similar to a double construction of a Frobenius algebra or a Lie bialgebra. Briefly speaking, a double construction of the Connes cocycle is equivalent to a certain bialgebra structure, namely, a dendriform D-bialgebra structure. Both antisymmetric infinitesimal bialgebras and dendriform D-bialgebras have many similar properties as Lie bialgebras. In particular, there are the so-called coboundary dendriform D-bialgebras which lead to another analogue (D-equation in a dendriform algebra) of the classical Yang–Baxter equation. A symmetric solution of the D-equation corresponds to the symmetric part of an $\mathcal{O}$-operator, which gives a double construction of the Connes cocycle.

1.4. Duality between bialgebras. By comparing antisymmetric infinitesimal bialgebras and dendriform D-bialgebras, we observe that there is a clear analogy between them. Moreover, due to the correspondences between certain symmetries and antisymmetries appearing in the analogy, we regard it as a kind of duality.

There is a similar study in the version of Lie algebras ([CP], [Bai2]). In fact, there is also a double construction of a Lie algebra with a non-degenerate invariant bilinear form (Manin triple or Lie bialgebra) or with a non-degenerate 2-cocycle of Lie algebra (para-Kähler Lie algebra or pre-Lie bialgebra). There are the $\mathcal{O}$-operators and a kind of algebras called pre-Lie algebras (Lie-admissible algebras whose left multiplication operators form a Lie algebra) which play the same role as the $\mathcal{O}$-operators and dendriform algebras. And there is a similar duality between Lie bialgebras and pre-Lie bialgebras.

Moreover, due to Chapoton ([Cha1]), there is a close relationship among the Lie algebras, associative algebras, pre-Lie algebras and dendriform algebras that can be depicted by a commutative diagram of categories:

$$\begin{CD} \text{dendriform algebras} @>>> \text{pre-Lie algebras} \\ @VVV @VVV \\ \text{associative algebras} @>>> \text{Lie algebras.} \end{CD}$$

We will extend the above relationship at the level of bialgebras with the dualities in a commutative diagram. In particular, the relation between antisymmetric infinitesimal bialgebras (the special case of infinitesimal Hopf algebras) and Lie bialgebras was observed in [A3]. Furthermore, these types of bialgebras fit into the general framework of "generalized bialgebras", as introduced by Loday in [Lo4].

The article is organized as follows. In Section 2, we give an explicit and systematic study on the double constructions of Frobenius algebras and then obtain the associative Yang–Baxter equation naturally. In Section 3, we introduce the close relations between $\mathcal{O}$-operators and dendriform algebras. In Section 4, we study the double constructions of Connes cocycles in terms of dendriform algebras. In Section 5, we give a clear analogy between antisymmetric infinitesimal bialgebras and dendriform D-bialgebras, which we regard as a kind of duality. After recalling a similar duality between Lie bialgebras and pre-Lie bialgebras, we express a close relationship among associative algebras, Lie algebras, pre-Lie algebras and dendriform algebras at the level of bialgebras.

Throughout this article, all algebras are finite-dimensional, although many results still hold in the infinite-dimensional case.

2. Double constructions of Frobenius algebras and another approach to associative Yang–Baxter equation

2.1. Bimodules and matched pairs of associative algebras

Definition 2.1.1. Let A be an associative algebra and let V be a vector space. Let $l, r\colon A \to \mathfrak{gl}(V)$ be two linear maps. V (or the pair (l, r), or (l, r, V)) is called a *bimodule* of A if

$$l(xy)v = l(x)l(y)v, \quad r(xy)v = r(y)r(x)v, \quad l(x)r(y)v = r(y)l(x)v$$

for all $x, y \in A$, $v \in V$.

In fact, according to [Sc], (l, r, V) is a bimodule of an associative algebra A if and only if the direct sum $A \oplus V$ of vector spaces is turned into an associative algebra (the semidirect sum) by defining multiplication in $A \oplus V$ by

$$(x_1 + v_1) * (x_2 + v_2) = x_1 \cdot x_2 + (l(x_1)v_2 + r(x_2)v_1)$$

for all $x_1, x_2 \in A$, $v_1, v_2 \in V$. We denote it by $A \ltimes_{l,r} V$ or simply $A \ltimes V$.

The following conclusion is obvious.

Lemma 2.1.2. *Let (l, r, V) be a bimodule of an associative algebra A.*

(1) *Let $l^*, r^* \colon A \to \mathfrak{gl}(V^*)$ be the linear maps given by*

$$\langle l^*(x)u^*, v\rangle = \langle l(x)v, u^*\rangle, \quad \langle r^*(x)u^*, v\rangle = \langle r(x)v, u^*\rangle \tag{8}$$

for all $x \in A$, $u^* \in V^*$, $v \in V$. Then (r^*, l^*, V^*) is a bimodule of A.

(2) *$(l, 0, V)$, $(0, r, V)$, $(r^*, 0, V^*)$ and $(0, l^*, V^*)$ are bimodules of A.*

Example 2.1.3. Let A be an associative algebra. Let $L(x)$ and $R(x)$ denote the left and right multiplication operator, respectively, that is, $L(x)(y) = xy$, $R(x)(y) = yx$ for any $x, y \in A$. Let $L\colon A \to \mathfrak{gl}(A)$ with $x \to L(x)$ and $R\colon A \to \mathfrak{gl}(A)$ with $x \to R(x)$ (for every $x \in A$) be two linear maps. Then $(L, 0)$, $(0, R)$ and (L, R) are bimodules of A. On the other hand, $(R^*, 0)$, $(0, L^*)$ and (R^*, L^*) are bimodules of A, too.

Theorem 2.1.4. *Let $(A, \cdot)$ and $(B, \circ)$ be two associative algebras. Suppose that there are linear maps $l_A, r_A \colon A \to \mathfrak{gl}(B)$ and $l_B, r_B \colon B \to \mathfrak{gl}(A)$ such that (l_A, r_A) is a bimodule of A and (l_B, r_B) is a bimodule of B and they satisfy the following conditions:*

$$l_A(x)(a \circ b) = l_A(r_B(a)x)b + (l_A(x)a) \circ b, \tag{9}$$
$$r_A(x)(a \circ b) = r_A(l_B(b)x)a + a \circ (r_A(x)b), \tag{10}$$
$$l_B(a)(x \cdot y) = l_B(r_A(x)a)y + (l_B(a)x) \cdot y, \tag{11}$$
$$r_B(a)(x \cdot y) = r_B(l_A(y)a)x + x \cdot (r_B(a)y), \tag{12}$$
$$l_A(l_B(a)x)b + (r_A(x)a) \circ b - r_A(r_B(b)x)a - a \circ (l_A(x)b) = 0, \tag{13}$$
$$l_B(l_A(x)a)y + (r_B(a)x) \cdot y - r_B(r_A(y)a)x - x \cdot (l_B(a)y) = 0 \tag{14}$$

for any $x, y \in A$, $a, b \in B$. Then there is an associative algebra structure on the direct sum $A \oplus B$ of the underlying vector spaces of A and B given by

$$(x + a) * (y + b) = (x \cdot y + l_B(a)y + r_B(b)x) + (a \circ b + l_A(x)b + r_A(y)a) \tag{15}$$

for all $x, y \in A$, $a, b \in B$. We denote this associative algebra by $A \bowtie^{l_A, r_A}_{l_B, r_B} B$ or simply $A \bowtie B$. On the other hand, every associative algebra with a decomposition into the direct sum of the underlying vector spaces of two subalgebras can be obtained in this way.

Proof. This is straightforward. □

Definition 2.1.5. Let $(A, \cdot)$ and $(B, \circ)$ be two associative algebras. Suppose that there are linear maps $l_A, r_A \colon A \to \mathfrak{gl}(B)$ and $l_B, r_B \colon B \to \mathfrak{gl}(A)$ such that (l_A, r_A) is a bimodule of A and (l_B, r_B) is a bimodule of B. If eqs. (9)–(14) are satisfied, then $(A, B, l_A, r_A, l_B, r_B)$ is called a *matched pair of associative algebras*.

Remark 2.1.6. Obviously B is an ideal of $A \bowtie B$ if and only if $l_B = r_B = 0$. If B is a trivial (that is, all the products of B are zero) ideal, then $A \bowtie_{0,0}^{l_A,r_A} B \cong A \ltimes_{l_A,r_A} B$. Moreover, some other special cases of Theorem 2.1.4 have already been studied. For example, the case when A is a left B-module and B is a right A-module was considered in [A1], that is, $l_A = 0$ and $r_B = 0$.

2.2. Double constructions of Frobenius algebras and antisymmetric infinitesimal bialgebras. Recall that a (symmetric) Frobenius algebra is an associative algebra A with a non-degenerate (symmetric) invariant bilinear form. Let $(A, \cdot)$ be an associative algebra. Suppose that there is an associative algebra structure "$\circ$" on its dual space A^*. We construct an associative algebra structure on the direct sum $A \oplus A^*$ of the underlying vector spaces of A and A^* such that $(A, \cdot)$ and $(A^*, \circ)$ are subalgebras and the symmetric bilinear form on $A \oplus A^*$ given by eq. (2) is invariant. That is, $(A \oplus A^*, \mathcal{B})$ is a symmetric Frobenius algebra. Such a construction is called a double construction of a Frobenius algebra associated to $(A, \cdot)$ and $(A^*, \circ)$ and we denote it by $(A \bowtie A^*, \mathcal{B})$.

Theorem 2.2.1. *Let $(A, \cdot)$ be an associative algebra. Suppose that there is an associative algebra structure "$\circ$" on its dual space A^*. Then there is a double construction of a Frobenius algebra associated to $(A, \cdot)$ and $(A^*, \circ)$ if and only if $(A, A^*, R_\cdot^*, L_\cdot^*, R_\circ^*, L_\circ^*)$ is a matched pair of associative algebras.*

Proof. If $(A, A^*, R_\cdot^*, L_\cdot^*, R_\circ^*, L_\circ^*)$ is a matched pair of associative algebras, then it is straightforward to show that the bilinear form (2) is invariant on the associative algebra $A \bowtie_{R_\circ^*, L_\circ^*}^{R_\cdot^*, L_\cdot^*} A^*$ given by eq. (15). Conversely, set

$$x * a^* = l_A(x)a^* + r_{A^*}(a^*)x, \quad a^* * x = l_{A^*}(a^*)x + r_A(x)a^*$$

for all $x \in A, a^* \in A^*$. Then $(A, A^*, l_A, r_A, l_{A^*}, r_{A^*})$ is a matched pair of associative algebras. Note that

$$\langle l_A(x)a^*, y\rangle = \langle r_A(y)a^*, x\rangle = \langle y \cdot x, a^*\rangle,$$
$$\langle l_{A^*}(b^*)x, a^*\rangle = \langle r_{A^*}(a^*)x, b^*\rangle = \langle a^* \circ b^*, x\rangle,$$

where $x, y \in A, a^*, b^* \in A^*$. Hence, $l_A = R_\cdot^*$, $r_A = L_\cdot^*$, $l_{A^*} = R_\circ^*$, $r_{A^*} = L_\circ^*$. □

Proposition 2.2.2. *Let $(A, \cdot)$ be an associative algebra. Suppose that there is an associative algebra structure "$\circ$" on its dual space A^*. Then $(A, A^*, R_\cdot^*, L_\cdot^*, R_\circ^*, L_\circ^*)$ is a matched pair of associative algebras if and only if for any $x \in A^*$, $a^*, b^* \in A^*$,*

$$R_\cdot^*(x)(a^* \circ b^*) = R_\cdot^*(L_\circ^*(a^*)x)b^* + (R_\cdot^*(x)a^*) \circ b^*, \quad (16)$$
$$R_\cdot^*(R_\circ^*(a^*)x)b^* + L_\cdot^*(x)a^* \circ b^* = L_\cdot^*(L_\circ^*(b^*)x)a^* + a^* \circ (R_\cdot^*(x)b^*). \quad (17)$$

Proof. Obviously, eq. (16) is just eq. (9) and eq. (17) is just eq. (13) in the case when $l_A = R_{\cdot}^*, r_A = L_{\cdot}^*, l_B = l_{A^*} = R_{\circ}^*, r_B = r_{A^*} = L_{\circ}^*$. By eq. (8), it is easy to show that in this situation

$$\text{eq. (9)} \iff \text{eq. (10)} \iff \text{eq. (11)} \iff \text{eq. (12)},$$
$$\text{eq. (13)} \iff \text{eq. (14)}.$$

Therefore the conclusion holds. □

Before the next study, we give some notations as follows. Let A be an associative algebra. Let $\sigma\colon A \otimes A \to A \otimes A$ be the exchange operator defined as

$$\sigma(x \otimes y) = y \otimes x \quad \text{for all } x, y \in A.$$

There are several ways to make $A \otimes A$ into a bimodule of A. For example, let id be the identity map on A. Then $(\mathrm{id} \otimes L, R \otimes \mathrm{id})$ given by (for any $x, a, b \in A$)

$$(\mathrm{id} \otimes L)(x)(a \otimes b) = (\mathrm{id} \otimes L(x))(a \otimes b) = a \otimes xb,$$
$$(R \otimes \mathrm{id})(x)(a \otimes b) = (R(x) \otimes \mathrm{id})(a \otimes b) = ax \otimes b,$$

is a bimodule of A. Similarly, $(L \otimes \mathrm{id}, \mathrm{id} \otimes R)$ is also a bimodule of A. In fact, eq. (5) given in the introduction can be rewritten as

$$\Delta(ab) = (L(a) \otimes \mathrm{id})\Delta(b) + (\mathrm{id} \otimes R(b))\Delta(a), \tag{18}$$

which gives the notion of infinitesimal bialgebra ([JR]).

For a linear map $\phi\colon V_1 \to V_2$, we denote the dual (linear) map by $\phi^*\colon V_2^* \to V_1^*$ given by

$$\langle v, \phi^*(u^*)\rangle = \langle \phi(v), u^*\rangle \quad \text{for all } v \in V_1,\ u^* \in V_2.$$

Theorem 2.2.3. *Let $(A, \cdot)$ be an associative algebra. Suppose there is an associative algebra structure "$\circ$" on its dual space A^* given by a linear map $\Delta^*\colon A^* \otimes A^* \to A^*$. Then $(A, A^*, R_{\cdot}^*, L_{\cdot}^*, R_{\circ}^*, L_{\circ}^*)$ is a matched pair of associative algebras if and only if $\Delta\colon A \to A \otimes A$ satisfies the following two conditions:*

$$\Delta(x \cdot y) = (\mathrm{id} \otimes L_{\cdot}(x))\Delta(y) + (R_{\cdot}(y) \otimes \mathrm{id})\Delta(x), \tag{19}$$
$$(L_{\cdot}(y) \otimes \mathrm{id} - \mathrm{id} \otimes R_{\cdot}(y))\Delta(x) + \sigma[(L_{\cdot}(x) \otimes \mathrm{id} - \mathrm{id} \otimes R_{\cdot}(x))\Delta(y)] = 0 \tag{20}$$

for all $x, y \in A$.

Proof. Let $\{e_1, \ldots, e_n\}$ be a basis of A and let $\{e_1^*, \ldots, e_n^*\}$ be its dual basis. Set $e_i \cdot e_j = \sum_{k=1}^n c_{ij}^k e_k$ and $e_i^* \circ e_j^* = \sum_{k=1}^n f_{ij}^k e_k^*$. Therefore, we have $\Delta(e_k) =$

$\sum_{i,j=1}^{n} f_{ij}^{k} e_i \otimes e_j$ and

$$R_{\cdot}^{*}(e_i)e_j^{*} = \sum_{k=1}^{n} c_{ki}^{j} e_k^{*}, \quad L_{\cdot}^{*}(e_i)e_j^{*} = \sum_{k=1}^{n} c_{ik}^{j} e_k^{*},$$
$$R_{\circ}^{*}(e_i^{*})e_j = \sum_{k=1}^{n} f_{ki}^{j} e_k, \quad L_{\circ}^{*}(e_i^{*})e_j = \sum_{k=1}^{n} f_{ik}^{j} e_k.$$

Hence the coefficient of $e_j \otimes e_k$ in

$$\Delta(e_i \cdot e_m) = (\mathrm{id} \otimes L_{\cdot}(e_i))\Delta(e_m) + (R_{\cdot}(e_m) \otimes \mathrm{id})\Delta(e_i)$$

gives the relation (for any i, j, k, m)

$$\sum_{l=1}^{n} c_{mi}^{l} f_{jk}^{l} = \sum_{l=1}^{n} (c_{ml}^{k} f_{jl}^{i} + c_{li}^{j} f_{lk}^{m}),$$

which is just the relation given by the coefficient of e_m^{*} in

$$R_{\cdot}^{*}(e_i)(e_j^{*} \circ e_k^{*}) = R_{\cdot}^{*}(L_{\circ}^{*}(e_j^{*})e_i)e_k^{*} + (R_{\cdot}^{*}(e_i)e_j^{*}) \circ e_k^{*}.$$

Similarly, eq. (20) corresponds to eq. (17). □

Remark 2.2.4. From the symmetry of the associative algebras $(A, \cdot)$ and $(A^*, \circ)$ appearing in the double construction, we also can consider the operation $\beta\colon A^* \to A^* \otimes A^*$ such that $\beta^*\colon A \otimes A \to A$ gives an associative algebra structure on A. It is easy to show that Δ satisfies eqs. (19) and (20) if and only if β satisfies

$$\beta(a^* \circ b^*) = (\mathrm{id} \otimes L_{\circ}(a^*))\beta(b^*) + (R_{\circ}(b^*) \otimes \mathrm{id})\beta(a^*),$$
$$(L_{\circ}(b^*) \otimes \mathrm{id} - \mathrm{id} \otimes R_{\circ}(b^*))\beta(a^*) + \sigma[(L_{\circ}(a^*) \otimes \mathrm{id} - \mathrm{id} \otimes R_{\circ}(a^*))\beta(b^*)] = 0$$

for all $a^*, b^* \in A$.

Definition 2.2.5. Let A be an associative algebra. An *antisymmetric infinitesimal bialgebra* structure on A is a linear map $\Delta\colon A \to A \otimes A$ such that

(a) $\Delta^*\colon A^* \otimes A^* \to A^*$ defines an associative algebra structure on A^*;

(b) Δ satisfies eqs. (19) and (20).

We denote it by (A, Δ) or (A, A^*).

Corollary 2.2.6. *Let $(A, \cdot)$ and $(A^*, \circ)$ be two associative algebras. Then the following conditions are equivalent.*

(1) *There is a double construction of a Frobenius algebra associated to $(A, \cdot)$ and $(A^*, \circ)$;*

(2) $(A, A^*, R_\cdot^*, L_\cdot^*, R_\circ^*, L_\circ^*)$ *is a matched pair of associative algebras;*

(3) (A, A^*) *is an antisymmetric infinitesimal bialgebra.*

Proof. It follows from Theorems 2.2.1 and 2.2.3. □

Remark 2.2.7. As we have pointed out in the introduction, an antisymmetric infinitesimal bialgebra is exactly an associative D-algebra in [Z] where the above equivalence between (1) and (3) was given and a balanced infinitesimal bialgebra in the sense of the opposite algebra in [A3] where the corresponding double construction of a Frobenius algebra was called a balanced Drinfeld double as an important consequence. On the other hand, the notion of antisymmetric infinitesimal bialgebra is due to the fact that eq. (19) (in the sense of the opposite algebra) corresponds to eq. (18) which gives the notion of infinitesimal bialgebra and eq. (20) expresses certain antisymmetry.

Definition 2.2.8. Let (A, Δ_A) and (B, Δ_B) be two antisymmetric infinitesimal bialgebras. A *homomorphism of antisymmetric infinitesimal bialgebras* $\varphi\colon A \to B$ is a homomorphism of associative algebras such that

$$(\varphi \otimes \varphi)\Delta_A(x) = \Delta_B(\varphi(x)) \quad \text{for all } x \in A.$$

An *isomorphism of antisymmetric infinitesimal bialgebras* is an invertible homomorphism of antisymmetric infinitesimal bialgebras.

Definition 2.2.9. Let $(A_1 \bowtie A_1^*, \mathcal{B}_1)$ and $(A_2 \bowtie A_2^*, \mathcal{B}_2)$ be two double constructions of Frobenius algebras. They are *isomorphic* if and only if there exists an isomorphism of associative algebras $\varphi\colon A_1 \bowtie A_1^* \to A_2 \bowtie A_2^*$ such that

$$\varphi(A_1) = A_2, \quad \varphi(A_1^*) = A_2^*, \quad \mathcal{B}_1(x, y) = \varphi^*\mathcal{B}_2(x, y) = \mathcal{B}_2(\varphi(x), \varphi(y))$$

for all $x, y \in A_1 \bowtie A_1^*$.

Proposition 2.2.10. *Two double constructions of Frobenius algebras are isomorphic if and only if their corresponding antisymmetric infinitesimal bialgebras are isomorphic.*

Proof. Let $(A_1 \bowtie A_1^*, \mathcal{B}_1)$ and $(A_2 \bowtie A_2^*, \mathcal{B}_2)$ be two double constructions of Frobenius algebras. Let $\{e_1, \dots, e_n\}$ be a basis of A_1 and $\{e_1^*, \dots, e_n^*\}$ its dual basis. If $\varphi\colon A_1 \bowtie A_1^* \to A_2 \bowtie A_2^*$ is an isomorphism of double constructions of Frobenius algebras, then $\varphi|_{A_1}\colon A_1 \to A_2$ and $\varphi|_{A_1^*}\colon A_1^* \to A_2^*$ are isomorphisms of associative algebras. Moreover, $\varphi|_{A_1^*} = (\varphi|_{A_1})^{*-1}$ since

$$\begin{aligned}
\langle \varphi|_{A_1^*}(e_i^*), \varphi(e_j)\rangle &= \mathcal{B}_2(\varphi|_{A_1^*}(e_i^*), \varphi(e_j)) \\
&= \mathcal{B}_1(e_i^*, e_j) = \delta_{ij} = \langle e_i^*, e_j\rangle \\
&= \langle \varphi^*(\varphi|_{A_1})^{*-1}(e_i^*), e_j\rangle = \langle (\varphi|_{A_1})^{*-1}(e_i^*), \varphi(e_j)\rangle.
\end{aligned}$$

Hence (A_1, A_1^*) and (A_2, A_2^*) are isomorphic as antisymmetric infinitesimal bialgebras. Conversely, let $\varphi'\colon A_1 \to A_2$ be an isomorphism between two antisymmetric infinitesimal bialgebras (A_1, A_1^*) and (A_2, A_2^*). Set $\varphi\colon A_1 \oplus A_1^* \to A_2 \oplus A_2^*$ be a linear map given by

$$\varphi(x) = \varphi'(x), \varphi(a^*) = (\varphi'^*)^{-1}(a^*) \quad \text{for all } x \in A_1,\ a^* \in A_1^*.$$

Then it is easy to show that φ is an isomorphism of double constructions of Frobenius algebras between $(A_1 \bowtie A_1^*, \mathcal{B}_1)$ and $(A_2 \bowtie A_2^*, \mathcal{B}_2)$. □

Example 2.2.11. Let (A, Δ) be an antisymmetric infinitesimal bialgebra. Then its dual (A^*, β) given in Remark 2.2.4 is also an antisymmetric infinitesimal bialgebra.

Example 2.2.12. Let A be an associative algebra. If the associative algebra structure on A^* is trivial, then either $(A, 0)$ or (A, A^*) is an antisymmetric infinitesimal bialgebra. Moreover, its corresponding Frobenius algebra is given by the semidirect sum $A \ltimes_{R^*,L^*} A^*$ with the natural invariant bilinear form $\mathcal{B}$ given by eq. (2). Dually, if A is a trivial associative algebra, then the antisymmetric infinitesimal bialgebra structures on A are in one-to-one correspondence with the associative algebra structures on A^*.

Example 2.2.13. Let (A, A^*) be an antisymmetric infinitesimal bialgebra. In the next subsection, we will prove that there exists a canonical antisymmetric infinitesimal bialgebra structure on the direct sum $A \oplus A^*$ of the underlying vector spaces of A and A^*.

2.3. Coboundary (principal) antisymmetric infinitesimal bialgebras. In fact, for an associative algebra A, $\Delta\colon A \to A \otimes A$ satisfying eq. (19) is a 1-cocycle or a derivation of A associated to the bimodule $(\mathrm{id} \otimes L, R \otimes \mathrm{id})$. So it is natural to consider the special case that Δ is a 1-coboundary or a principal derivation.

Definition 2.3.1. An antisymmetric infinitesimal bialgebra (A, Δ) is called *coboundary* if there exists a $r \in A \otimes A$ such that

$$\Delta(x) = (\mathrm{id} \otimes L(x) - R(x) \otimes \mathrm{id})r \quad \text{for all } x \in A. \tag{21}$$

Let A be an associative algebra and $r \in A \otimes A$. If $\Delta\colon A \to A \otimes A$ is given by eq. (21), then it is obvious that Δ satisfies eq. (19). Therefore, (A, Δ) is an antisymmetric infinitesimal bialgebra if and only if the following two conditions are satisfied:

(1) $\Delta^*\colon A^* \otimes A^* \to A^*$ defines an associative algebra structure on A^*.

(2) Δ satisfies eq. (20).

Lemma 2.3.2 ([A1], Proposition 5.1). *Let A be an associative algebra and $r \in A \otimes A$. Define $\Delta\colon A \to A \otimes A$ by*

$$\Delta(a) = [L(x) \otimes \mathrm{id} - \mathrm{id} \otimes R(x)]r$$

for all $x \in A$. Then $\Delta^*\colon A^* \otimes A^* \to A^*$ defines an associative algebra structure on A^* if and only if

$$(L(x) \otimes \mathrm{id} \otimes \mathrm{id} - \mathrm{id} \otimes \mathrm{id} \otimes R(x))(r_{13}r_{12} + r_{23}r_{13} - r_{12}r_{23}) = 0$$

for all $x \in A$, where the notations $r_{13}r_{12}$, $r_{23}r_{13}$, $r_{12}r_{23}$ are given similarly as in eq. (4).

Therefore for (1), we use a similar discussion to get the following conclusion.

Proposition 2.3.3. *Let A be an associative algebra and $r \in A \otimes A$. Define $\Delta\colon A \to A \otimes A$ by eq. (21). Then $\Delta^*\colon A^* \otimes A^* \to A^*$ defines an associative algebra structure on A^* if and only if*

$$(\mathrm{id} \otimes \mathrm{id} \otimes L(x) - R(x) \otimes \mathrm{id} \otimes \mathrm{id})(r_{12}r_{13} + r_{13}r_{23} - r_{23}r_{12}) = 0 \tag{22}$$

for all $x \in A$.

Proposition 2.3.4. *Let A be an associative algebra and $r \in A \otimes A$. Define $\Delta\colon A \to A \otimes A$ by eq. (21). Then Δ satisfies eq. (20) if and only if r satisfies*

$$[L(x) \otimes \mathrm{id} - \mathrm{id} \otimes R(x)][\mathrm{id} \otimes L(y) - R(y) \otimes \mathrm{id}](r + \sigma(r)) = 0 \tag{23}$$

for all $x, y \in A$.

Proof. This is straightforward. □

Combining Proposition 2.3.3 and Proposition 2.3.4, we have the following conclusion.

Theorem 2.3.5. *Let A be an associative algebra and $r \in A \otimes A$. Then the linear map Δ defined by eq. (21) induces an associative algebra structure on A^* such that (A, A^*) is an antisymmetric infinitesimal bialgebra if and only if eqs. (22) and (23) are satisfied.*

Theorem 2.3.6. *Let (A, Δ_A) be an antisymmetric infinitesimal bialgebra. Then there is a canonical antisymmetric infinitesimal bialgebra structure on the direct sum $A \oplus A^*$ of the underlying vector spaces of A and A^* such that both the inclusions $i_1\colon A \to A \oplus A^*$ and $i_2\colon A^* \to A \oplus A^*$ into the two summands are homomorphisms of antisymmetric infinitesimal bialgebras. Here the antisymmetric infinitesimal bialgebra structure on A^* is $(A^*, -\beta_{A^*})$, where $\beta_{A^*}\colon A^* \to A^* \otimes A^*$ is given in Remark 2.2.4.*

Proof. Let $r \in A \otimes A^* \subset (A \oplus A^*) \otimes (A \oplus A^*)$ correspond to the identity map $\mathrm{id}\colon A \to A$. Let $\{e_1, \ldots, e_n\}$ be a basis of A and $\{e_1^*, \ldots, e_n^*\}$ its dual basis. Then $r = \sum_{i=1}^n e_i \otimes e_i^*$. Suppose that the associative algebra structure "$*$" on $A \oplus A^*$ is given by $\mathcal{AD}(A) = A \bowtie_{R_\circ^*, L_\circ^*}^{R_\cdot^*, L_\cdot^*} A^*$. Then by Theorem 2.1.4, we have (for any $x, y \in A$, $a^*, b^* \in A^*$)

$$x * y = x \cdot y, \qquad a^* * b^* = a^* \circ b^*,$$
$$x * a^* = R_\cdot^*(x)a^* + L_\circ^*(a^*)x, \qquad a^* * x = R_\circ(a^*)x + L_\cdot^*(x)a^*.$$

If r satisfies eqs. (22) and (23), then

$$\Delta_{\mathcal{AD}}(u) = (\mathrm{id} \otimes L(u) - R(u) \otimes \mathrm{id})r$$

for all $u \in \mathcal{AD}(A)$ induces an antisymmetric infinitesimal bialgebra structure on $\mathcal{AD}(A)$.

In fact, for eq. (23), we prove a little stronger conclusion (for any $\mu \in \mathcal{AD}(A)$):

$$\begin{aligned}&(\mathrm{id} \otimes L(\mu) - R(\mu) \otimes \mathrm{id})(r + \sigma(r))\\ &\quad = \sum_i (e_i \otimes \mu * e_i^* + e_i \otimes \mu * e_i - e_i * \mu \otimes e_i^* - e_i * \mu \otimes e_i) = 0.\end{aligned} \tag{24}$$

If $\mu = e_j$, then

$$\begin{aligned}\sum_i e_i \otimes e_j * e_i^* &= \sum_m e_m \cdot e_j \otimes e_m^* + \sum_{i,m} \langle e_i^* \circ e_m^*, e_j \rangle e_i \otimes e_m,\\ \sum_i e_i^* \otimes e_j * e_i &= \sum_i e_i^* \otimes e_j \cdot e_i,\\ \sum_i e_i * e_j \otimes e_i^* &= \sum_i e_i \cdot e_j \otimes e_i^*,\\ e_i^* * e_j \otimes e_i &= \sum_{i,m} \langle e_j, e_m^* \circ e_i^* \rangle e_m \otimes e_i + \sum_m e_m^* \otimes e_j \cdot e_m.\end{aligned}$$

Hence eq. (24) holds for $\mu = e_j$ by exchanging some indices. Similarly, eq. (24) holds for $\mu = e_j^*$. Therefore eq. (23) holds. Furthermore,

$$r_{12}r_{13} + r_{13}r_{23} - r_{23}r_{12} = \sum_{i,j} \{e_j \otimes e_i * e_j^* \otimes e_i^* - e_j \cdot e_i \otimes e_j^* \otimes e_i^* - e_i \otimes e_j \otimes e_i^* \circ e_j^*\}.$$

Since $e_i * e_j^* = \sum_m (\langle e_j^*, e_m \cdot e_i \rangle e_m^* + \langle e_j^* \circ e_m^*, e_i \rangle e_m)$, it follows that $r_{12}r_{13} + r_{13}r_{23} - r_{23}r_{12} = 0$. So $\mathcal{AD}(A)$ is an antisymmetric infinitesimal bialgebra.

For $e_i \in A$, we have

$$\begin{aligned}&\Delta_{\mathcal{AD}}(e_i)\\ &\quad = \sum_{m,k} \{\langle e_m^*, e_k \cdot e_i \rangle e_m \otimes e_k^* + \langle e_m^* \circ e_k^*, e_i \rangle e_m \otimes e_k - \langle e_m^*. e_k \cdot e_i \rangle e_m \otimes e_k^*\}\\ &\quad = \sum_{m,k} \langle e_m^* \circ e_k^*, e_i \rangle e_m \otimes e_k = \Delta_A(e_i).\end{aligned}$$

Therefore the inclusion $i_1\colon A \to A \oplus A^*$ is a homomorphism of antisymmetric infinitesimal bialgebras. Similarly, the inclusion $i_2\colon A^* \to A \oplus A^*$ is also a homomorphism of antisymmetric infinitesimal bialgebras since $\Delta_{\mathcal{A}\mathcal{D}}(e_i^*) = -\beta_{A^*}(e_i^*)$, where β_{A^*} is given in Remark 2.2.4. $\square$

Definition 2.3.7. Let (A, A^*) be an antisymmetric infinitesimal bialgebra. With the antisymmetric infinitesimal bialgebra structure given in Theorem 2.3.6, $A \oplus A^*$ is called an *associative double* of A. We denote it by $\mathcal{A}\mathcal{D}(A)$.

Remark 2.3.8. If we use the opposite algebra, then Theorem 2.3.6 and its proof overlap [A3], Theorem 5.9 and Proposition 5.10 partly. Moreover, the associative double $\mathcal{A}\mathcal{D}(A)$ is a balanced Drinfeld double which was denoted by $D_b(A)$ in [A3].

Corollary 2.3.9. *Let (A, A^*) be an antisymmetric infinitesimal bialgebra. Then the associative double $\mathcal{A}\mathcal{D}(A)$ of A is an antisymmetric infinitesimal bialgebra and it is a symmetric Frobenius algebra with the bilinear form given by eq. (2).*

2.4. The associative Yang–Baxter equation and its properties

Corollary 2.4.1. *Let A be an associative algebra and $r \in A \otimes A$. Suppose that r is antisymmetric. Then the map Δ defined by eq. (21) induces an associative algebra structure on A^* such that (A, A^*) is an antisymmetric infinitesimal bialgebra if*

$$r_{12}r_{13} + r_{13}r_{23} - r_{23}r_{12} = 0. \tag{25}$$

Definition 2.4.2. Let A be an associative algebra and $r \in A \otimes A$. Eq. (25) is called *associative Yang–Baxter equation in A*.

Remark 2.4.3. In [A1] and [A3], the associative Yang–Baxter equation is given as

$$r_{13}r_{12} + r_{23}r_{13} - r_{12}r_{23} = 0. \tag{26}$$

Note that eq. (25) is eq. (26) in the opposite algebra. Moreover, if r satisfies $(L(x) \otimes \mathrm{id} \otimes \mathrm{id} - \mathrm{id} \otimes \mathrm{id} \otimes R(x))(r_{12} + r_{21}) = 0$, then ([A3], Lemma 3.4)

$$\sigma_{13}(r_{12}r_{13} + r_{13}r_{23} - r_{23}r_{12}) = r_{13}r_{12} + r_{23}r_{13} - r_{12}r_{23},$$

where the linear map $\sigma_{13}\colon A \otimes A \otimes A \to A \otimes A \otimes A$ is given by $\sigma_{13}(x \otimes y \otimes z) = z \otimes y \otimes x$ for any $x, y, z \in A$. In particular, when r is antisymmetric, the above two associative Yang–Baxter equations are equivalent.

In order to be self-contained, in the following we give some properties of the associative Yang–Baxter equation from the point of view of Frobenius algebras, although some of them have already been given in [A3]. Let A be a vector space. For any $r \in A \otimes A$, r can be regarded as a map from A^* to A in the following way:

$$\langle u^* \otimes v^*, r\rangle = \langle u^*, r(v^*)\rangle \quad \text{for all } u^*, v^* \in A^*.$$

Proposition 2.4.4. *Let $(A,\ \cdot\ )$ be an associative algebra and let $r \in A \otimes A$ be an antisymmetric solution of the associative Yang–Baxter equation in A. Then the associative algebra structure on the associative double $\mathcal{AD}(A)$ is given from the products in A as follows:*

$$a^* * b^* = a^* \circ b^* = R_\cdot^*(r(a^*))b^* + L_\cdot^*(r(b^*))a^* \quad \text{for any } a^*, b^* \in A^*, \tag{27}$$
$$x * a^* = x \cdot r(a^*) - r(R_\cdot^*(x)a^*) + R_\cdot^*(x)a^* \quad \text{for any } x \in A,\ a^* \in A^*, \tag{28}$$
$$a^* * x = r(a^*) \cdot x - r(L_\cdot^*(x)a^*) + L_\cdot^*(x)a^* \quad \text{for any } x \in A,\ a^* \in A^*. \tag{29}$$

Proof. Let $\{e_1, \ldots, e_n\}$ be a basis of A and $\{e_1^*, \ldots, e_n^*\}$ its dual basis. Suppose that $e_i \cdot e_j = \sum_k c_{ij}^k e_k$ and $r = \sum_{i,j} a_{ij} e_i \otimes e_j$, where $a_{ij} = -a_{ji}$. Then for any i, we have

$$\Delta(e_i) = \sum_{\alpha,\beta,l} a_{\alpha\beta}(c_{i\beta}^l e_\alpha \otimes e_l - c_{\alpha i}^l e_l \otimes e_\beta) = \sum_{\alpha,\beta} \sum_l (a_{\alpha l} c_{il}^\beta - a_{l\beta} c_{li}^\alpha) e_\alpha \otimes e_\beta.$$

Therefore we have (for any i, j)

$$\begin{aligned} e_i^* \circ e_j^* &= \sum_{l,t} (a_{il} c_{tl}^j - a_{lj} c_{lt}^i) e_t^* \\ &= \sum_{l,t} (a_{il} \langle e_t \cdot e_l, e_j^* \rangle - a_{lj} \langle e_l \cdot e_t, e_i^* \rangle) e_t^* \\ &= \sum_t (\langle e_t \cdot r(e_i^*), e_j^* \rangle + \langle r(e_j^*) \cdot e_t, e_i^* \rangle) e_t^* \\ &= R_\cdot^*(r(e_i^*)) e_j^* + L_\cdot^*(r(e_j^*)) e_i^*. \end{aligned}$$

Similarly, eqs. (28) and (29) hold. □

Theorem 2.4.5 ([A3], Proposition 2.1). *Let A be an associative algebra and $r \in A \otimes A$. Suppose that r is antisymmetric and non-degenerate. Then r is a solution of the associative Yang–Baxter equation in A if and only if the inverse of the isomorphism $A^* \to A$ induced by r, regarded as a bilinear form ω on A (that is, $\omega(x, y) = \langle r^{-1}x, y \rangle$ for any $x, y \in A$), is a Connes cocycle.*

Corollary 2.4.6. *Let $(A,\ \cdot\ )$ be an associative algebra and let $r \in A \otimes A$ be a non-degenerate antisymmetric solution of the associative Yang–Baxter equation in A. Suppose the associative algebra structure "$\circ$" on A^* is induced by r from eq. (27). Then we have*

$$a^* \circ b^* = r^{-1}(r(a^*) \cdot r(b^*)) \quad \text{for all } a^*, b^* \in A^*. \tag{30}$$

Therefore $r\colon A^* \to A$ is an isomorphism of associative algebras.

Proof. Set $\omega(x,y) = \langle r^{-1}(x), y\rangle$ for any $x, y \in A$. Then ω is a Connes cocycle of A. Hence

$$\begin{aligned}\langle a^* \circ b^*, x\rangle &= \langle r(b^*)\cdot x, a^*\rangle + \langle x \cdot r(a^*), b^*\rangle\\ &= \omega(r(a^*), r(b^*)\cdot x) + \omega(r(b^*), x\cdot r(a^*))\\ &= -\omega(x, r(a^*)\cdot r(b^*))\\ &= \langle r^{-1}(r(a^*)\cdot r(b^*)), x\rangle\end{aligned}$$

for all $a^*, b^* \in A^*$, $x \in A$. So eq. (30) holds. Therefore r is an isomorphism of associative algebras. □

Next we turn to the general antisymmetric solutions of associative Yang–Baxter equation.

Theorem 2.4.7. *Let $(A, \cdot)$ be an associative algebra and $r \in A \otimes A$ antisymmetric. Then r is a solution of the associative Yang–Baxter equation in A if and only if r satisfies*

$$r(a^*)\cdot r(b^*) = r(R^*_\cdot(r(a^*))b^* + L^*_\cdot(r(b^*))a^*)$$

for all $a^*, b^* \in A^*$.

Proof. Let $\{e_1, \ldots, e_n\}$ be a basis of A and $\{e_1^*, \ldots, e_n^*\}$ its dual basis. Suppose that $e_i \cdot e_j = \sum_k c_{ij}^k e_k$ and $r = \sum_{i,j} a_{ij} e_i \otimes e_j$, $a_{ij} = -a_{ji}$. Hence $r(e_i^*) = \sum_k a_{ki} e_k$. Then r is a solution of the associative Yang–Baxter equation in A if and only if (for any i, j, k)

$$\sum_{m,l}\{c_{kl}^m a_{ik} a_{jl} - c_{lk}^i a_{jl} a_{km} - c_{lk}^j a_{lm} a_{ik}\} = 0.$$

The left-hand side of the above equation is just the coefficient of e_m in

$$r(e_i^*)\cdot r(e_j^*) - r(R^*_\cdot(r(e_i^*))e_j^* + L^*_\cdot(r(e_j^*))e_i^*).$$

Therefore the conclusion follows. □

Combining Proposition 2.4.4 and Theorem 2.4.7, we have the following conclusion which extends Corollary 2.4.6.

Corollary 2.4.8. *Let $(A, \cdot)$ be an associative algebra and let $r \in A \otimes A$ be an antisymmetric solution of the associative Yang–Baxter equation in A. Suppose the associative algebra structure "$\circ$" on A^* is induced by r from eq. (27). Then we have*

$$r(a^* \circ b^*) = r(a^*)\cdot r(b^*) \quad \text{for all } a^*, b^* \in A^*.$$

Therefore $r\colon A^* \to A$ is an homomorphism of associative algebras.

Recall that two Frobenius algebras $(A_1, \mathcal{B}_1)$ and $(A_2, \mathcal{B}_2)$ are isomorphic if and only if there exists an isomorphism of associative algebras $\varphi\colon A_1 \to A_2$ such that

$$\mathcal{B}_1(x, y) = \varphi^* \mathcal{B}_2(x, y) = \mathcal{B}_2(\varphi(x), \varphi(y))$$

for all $x, y \in A_1$.

Theorem 2.4.9. *Let $(A, \cdot)$ be an associative algebra. Then, as a Frobenius algebra, the Frobenius algebra $(A \bowtie_{R^*_\circ, L^*_\circ}^{R^*_\cdot, L^*_\cdot} A^*, \mathcal{B})$ given by an antisymmetric solution r of the associative Yang–Baxter equation in A is isomorphic to the Frobenius algebra $(A \ltimes_{R^*_\cdot, L^*_\cdot} A^*, \mathcal{B})$, where $\mathcal{B}$ is given by eq. (2). However, in general, they are not isomorphic as the double constructions of Frobenius algebras (or equivalently, as antisymmetric infinitesimal bialgebras).*

Proof. Let r be an antisymmetric solution of associative the Yang–Baxter equation in A. Define a linear map $\varphi\colon A \ltimes_{R^*_\cdot, L^*_\cdot} A^* \to A \bowtie_{R^*_\circ, L^*_\circ}^{R^*_\cdot, L^*_\cdot} A^*$ satisfying

$$\varphi(x) = x, \quad \varphi(a^*) = -r(a^*) + a^*$$

for all $x \in A$, $a^* \in A^*$. It is straightforward to show that φ is an isomorphism of associative algebras. Moreover,

$$\begin{aligned}\varphi^* \mathcal{B}(x + a^*, y + b^*) &= \langle a^*, -r(b^*) + y\rangle + \langle x - r(a^*), b^*\rangle \\ &= \langle a^*, y\rangle + \langle x, b^*\rangle \\ &= \mathcal{B}(x + a^*, y + b^*).\end{aligned}$$

Therefore φ is an isomorphism of Frobenius algebras. However in general, as antisymmetric infinitesimal bialgebras, they are not isomorphic. In fact, if ψ is an isomorphism of antisymmetric infinitesimal bialgebras between $A \ltimes_{R^*_\cdot, L^*_\cdot} A^*$ and $A \bowtie_{R^*_\circ, L^*_\circ}^{R^*_\cdot, L^*_\cdot} A^*$, then for any $u^*, v^* \in A^*$ there exist $a^*, b^* \in A^*$ such that $\psi(a^*) = u^*$, $\psi(b^*) = v^*$. However, $\psi(a^* \circ b^*) = 0$ and $\psi(a^*) * \psi(b^*) = u^* * v^* = R^*_\cdot(r(a^*))b^* + L^*_\cdot(r(b^*))a^*$ is not zero in general, which is a contradiction. □

Corollary 2.4.10. *Let $(A, \cdot)$ be an associative algebra. Then as Frobenius algebras, the Frobenius algebras $(A \bowtie_{R^*_\circ, L^*_\circ}^{R^*_\cdot, L^*_\cdot} A^*, \mathcal{B})$ given by all antisymmetric solutions of the associative Yang–Baxter equation in A are isomorphic to the Frobenius algebra $(A \ltimes_{R^*_\cdot, L^*_\cdot} A^*, \mathcal{B})$ given by the zero solution.*

2.5. The associative Yang–Baxter equation and $\mathcal{O}$-operators

Definition 2.5.1. Let $(A, \cdot)$ be an associative algebra and (l, r, V) a bimodule. A linear map $T : V \to A$ is called an *$\mathcal{O}$-operator associated to* (l, r, V) if T satisfies

$$T(u) \cdot T(v) = T(l(T(u))v + r(T(v)u)) \quad \text{for all } u, v \in V.$$

Example 2.5.2. Let $(A,\ \cdot)$ be an associative algebra. Then the identity map id is an $\mathcal{O}$-operator associated to the bimodule $(L, 0)$ or $(0, R)$.

Example 2.5.3. Let $(A,\ \cdot)$ be an associative algebra. A linear map $R\colon A \to A$ is called a Rota–Baxter operator on A of weight zero ([Bax], [Rot]) if R satisfies

$$R(x) \cdot R(y) = R(R(x) \cdot y + x \cdot R(y)) \quad \text{for all } x, y \in A.$$

In fact, a Rota–Baxter operator on A is just an $\mathcal{O}$-operator associated to the bimodule (L, R).

Example 2.5.4. Let $(A,\ \cdot)$ be an associative algebra and $r \in A \otimes A$ antisymmetric. Then r is a solution of associative Yang–Baxter equation in A if and only if r is an $\mathcal{O}$-operator associated to the bimodule (R^*, L^*).

Theorem 2.5.5 ([BGN1]). *Let $(A,\ \cdot)$ be an associative algebra and (l, r, V) a bimodule. Let (r^*, l^*, V^*) be the bimodule of A given by Lemma 2.1.2. Let $T : V \to A$ be a linear map which is identified as an element in $(A \ltimes_{r^*,l^*} V^*) \otimes (A \ltimes_{r^*,l^*} V^*)$. Then $r = T - \sigma(T)$ is an antisymmetric solution of the associative Yang–Baxter equation in $A \ltimes_{r^*,l^*} V^*$ if and only if T is an $\mathcal{O}$-operator associated to the bimodule (l, r, V).*

Corollary 2.5.6 (Cf. Corollary 3.1.5). *Let $(A,\ \cdot)$ be an associative algebra. Then*

$$r = \sum_i^n (e_i \otimes e_i^* - e_i^* \otimes e_i) \tag{31}$$

is a solution of the associative Yang–Baxter equation in $A \ltimes_{R^*,0} A^*$ or $A \ltimes_{0,L^*} A^*$, where $\{e_1, \dots, e_n\}$ is a basis of A and $\{e_1^*, \dots, e_n^*\}$ is its dual basis. Moreover there is a natural Connes cocycle ω on $A \ltimes_{R^*,0} A^*$ or $A \ltimes_{0,L^*} A^*$ induced by $r^{-1}\colon A \oplus A^* \to (A \oplus A^*)^*$, which is given by eq. (7).

Proof. Note that id is an $\mathcal{O}$-operator associated to the bimodule $(L, 0, A)$ or $(0, R, A)$. Then the conclusion follows from Theorems 2.5.5 and 2.4.5. □

3. Dendriform algebras

3.1. $\mathcal{O}$-operators and dendriform algebras. There are close relations between $\mathcal{O}$-operators and a class of algebras, namely, dendriform algebras, which are given in [BGN2]. In order to be self-contained, we list them in this subsection.

Definition 3.1.1 ([Lo1]). Let A be a vector space over a field $\mathbb{F}$ with two bilinear products denoted by $\prec$ and $\succ$. Then $(A, \prec, \succ)$ is called a *dendriform algebra* if, for any $x, y, z \in A$,

$$(x \prec y) \prec z = x \prec (y * z),$$
$$(x \succ y) \prec z = x \succ (y \prec z),$$
$$x \succ (y \succ z) = (x * y) \succ z,$$

where $x * y = x \prec y + x \succ y$.

Let $(A, \prec, \succ)$ be a dendriform algebra. For any $x \in A$, let $L_\succ(x)$, $R_\succ(x)$ and $L_\prec(x)$, $R_\prec(x)$ denote the left and right multiplication operators of $(A, \prec)$ and $(A, \succ)$, respectively, that is,

$$L_\succ(x)(y) = x \succ y,\ R_\succ(x)y = y \succ x,\ L_\prec(x)y = x \prec y,\ R_\prec(x)(y) = y \prec x$$

for all $x, y \in A$. Moreover, let $L_\succ, R_\succ, L_\prec, R_\prec : A \to \mathfrak{gl}(A)$ be four linear maps with $x \to L_\succ(x)$, $x \to R_\succ(x)$, $x \to L_\prec(x)$ and $x \to R_\prec(x)$, respectively. It is known that the product given by ([Lo1])

$$x * y = x \prec y + x \succ y, \quad \text{for all } x, y \in A, \tag{32}$$

defines an associative algebra. We call $(A, *)$ the associated associative algebra of $(A, \succ, \prec)$ and $(A, \succ, \prec)$ is called a compatible dendriform algebra structure on the associative algebra $(A, *)$. Moreover, $(L_\succ, R_\prec)$ is a bimodule of the associated associative algebra $(A, *)$.

Theorem 3.1.2 ([BGN2]). *Let A be an associative algebra and (l, r, V) a bimodule. Let $T : V \to A$ be an $\mathcal{O}$-operator associated to (l, r, V). Then there exists a dendriform algebra structure on V given by*

$$u \succ v = l(T(u))v, \quad u \prec v = r(T(v))u$$

for all $u, v \in V$. So there is an associated associative algebra structure on V given by eq. (32) and T is a homomorphism of associative algebras. Moreover, $T(V) = \{T(v) \mid v \in V\} \subset A$ is an associative subalgebra of A and there is an induced dendriform algebra structure on $T(V)$ given by

$$T(u) \succ T(v) = T(u \succ v), \quad T(u) \prec T(v) = T(u \prec v) \tag{33}$$

for all $u, v \in V$. Its corresponding associated associative algebra structure on $T(V)$ given by eq. (32) is just the associative subalgebra structure of A and T is a homomorphism of dendriform algebras.

Corollary 3.1.3 ([BGN2]). *Let $(A, *)$ be an associative algebra. There is a compatible dendriform algebra structure on A if and only if there exists an invertible $\mathcal{O}$-operator of $(A, *)$.*

In fact, if T is an invertible $\mathcal{O}$-operator associated to a bimodule (l, r, V), then the compatible dendriform algebra structure on A is given by

$$x \succ y = T(l(x)T^{-1}(y)), \quad x \prec y = T(r(y)T^{-1}(x))$$

for all $x, y \in A$. Conversely, let $(A, \succ, \prec)$ be a dendriform algebra and $(A, *)$ the associated associative algebra. Then the identity map id is an $\mathcal{O}$-operator associated to the bimodule $(L_{\succ}, R_{\prec})$ of $(A, *)$.

Remark 3.1.4. If T is an invertible $\mathcal{O}$-operator associated to a bimodule (l, r, V), then the linear map $f = T^{-1}\colon A \to V$ satisfies

$$f(x * y) = l(x)f(y) + r(y)f(x) \quad \text{for all } x, y \in A. \tag{34}$$

Such a linear map is a 1-cocycle of $(A, *)$ associated to the bimodule (l, r, V).

Corollary 3.1.5 ([BGN2]). *Let $(A, \succ, \prec)$ be a dendriform algebra. Then*

$$r = \sum_{i}^{n}(e_i \otimes e_i^* - e_i^* \otimes e_i)$$

is a solution of the associative Yang–Baxter equation in $A \ltimes_{R_{\prec}^*, L_{\succ}^*} A^*$, where $\{e_1, \ldots, e_n\}$ is a basis of A and $\{e_1^*, \ldots, e_n^*\}$ is its dual basis. Moreover there is a natural Connes cocycle ω on $A \ltimes_{R_{\prec}^*, L_{\succ}^*} A^*$ induced by $r^{-1}\colon A \oplus A^* \to (A \oplus A^*)^*$, which is given by eq. (7).

Remark 3.1.6. It is easy to see that Corollary 2.5.6 is a special case of the above conclusion, that is, the former corresponds to the trivial dendriform algebra structure on an associative algebra $(A, \cdot)$ given by $\succ = \cdot$, $\prec = 0$ or $\succ = 0$, $\prec = \cdot$.

3.2. Bimodules and matched pairs of dendriform algebras

Definition 3.2.1 ([A4]). Let $(A, \succ, \prec)$ be a dendriform algebra and V a vector space. Let $l_{\succ}, r_{\succ}, l_{\prec}, r_{\prec}\colon A \to \mathfrak{gl}(V)$ be four linear maps. Then V (or $(l_{\succ}, r_{\succ}, l_{\prec}, r_{\prec})$, or $(l_{\succ}, r_{\succ}, l_{\prec}, r_{\prec}, V)$) is called a *bimodule* of A if the following equations hold for any $x, y \in A$:

$$\begin{aligned}
l_{\prec}(x \prec y) &= l_{\prec}(x)l_{*}(y),\ r_{\prec}(x)l_{\prec}(y) = l_{\prec}(y)r_{*}(x),\ r_{\prec}(x)r_{\prec}(y) = r_{\prec}(y * x),\\
l_{\prec}(x \succ y) &= l_{\succ}(x)l_{\prec}(y),\ r_{\prec}(x)l_{\succ}(y) = l_{\succ}(y)r_{\prec}(x),\ r_{\prec}(x)r_{\succ}(y) = r_{\succ}(y \prec x),\\
l_{\succ}(x * y) &= l_{\succ}(x)l_{\succ}(y),\ r_{\succ}(x)l_{*}(y) = l_{\succ}(y)r_{\succ}(x),\ r_{\succ}(x)r_{*}(y) = r_{\succ}(y \succ x),
\end{aligned}$$

where $x * y = x \succ y + x \prec y$, $l_{*} = l_{\succ} + l_{\prec}$, $r_{*} = r_{\succ} + r_{\prec}$.

By a direct computation or according to [Sc], $(l_{\succ}, r_{\succ}, l_{\prec}, r_{\prec}, V)$ is a bimodule of a dendriform algebra $(A, \succ, \prec)$ if and only if there exists a dendriform algebra structure on the direct sum $A \oplus V$ of the underlying vector spaces of A and V given by

$$(x+u) \succ (y+v) = x \succ y + l_{\succ}(x)v + r_{\succ}(y)u,$$
$$(x+u) \prec (y+v) = x \prec y + l_{\prec}(x)v + r_{\prec}(y)u$$

for all $x, y \in A$, $u, v \in V$. We denote it by $A \ltimes_{l_{\succ}, r_{\succ}, l_{\prec}, r_{\prec}} V$.

Proposition 3.2.2. *Let $(l_{\succ}, r_{\succ}, l_{\prec}, r_{\prec}, V)$ be a bimodule of a dendriform algebra $(A, \succ, \prec)$. Let $(A, *)$ be the associated associative algebra. Then we have the following results.*

(1) *Both $(l_{\succ}, r_{\prec}, V)$ and $(l_{\succ} + l_{\prec}, r_{\succ} + r_{\prec}, V)$ are bimodules of $(A, *)$.*

(2) *For any bimodule (l, r, V) of $(A, *)$, $(l, 0, 0, r, V)$ is a bimodule of $(A, \succ, \prec)$.*

(3) *Both $(l_{\succ} + l_{\prec}, 0, 0, r_{\succ} + r_{\prec}, V)$ and $(l_{\succ}, 0, 0, r_{\prec}, V)$ are bimodules of $(A, \succ, \prec)$.*

(4) *The dendriform algebras $A \ltimes_{l_{\succ}, r_{\succ}, l_{\prec}, r_{\prec}} V$ and $A \ltimes_{l_{\succ}+l_{\prec}, 0, 0, r_{\succ}+r_{\prec}} V$ have the same associated associative algebra $A \ltimes_{l_{\succ}+l_{\prec}, r_{\succ}+r_{\prec}} V$.*

(5) *Let $l_{\succ}^*, r_{\succ}^*, l_{\prec}^*, r_{\prec}^* : A \to \mathfrak{gl}(V^*)$ be the linear maps given by*

$$\langle l_{\succ}^*(x)a^*, y\rangle = \langle l_{\succ}(x)y, a^*\rangle, \quad \langle r_{\succ}^*(x)a^*, y\rangle = \langle r_{\succ}(x)y, a^*\rangle,$$
$$\langle l_{\prec}^*(x)a^*, y\rangle = \langle l_{\prec}(x)y, a^*\rangle, \quad \langle r_{\prec}^*(x)a^*, y\rangle = \langle r_{\prec}(x)y, a^*\rangle.$$

Then $(r_{\succ}^* + r_{\prec}^*, -l_{\prec}^*, -r_{\succ}^*, l_{\succ}^* + l_{\prec}^*, V^*)$ is a bimodule of $(A, \succ, \prec)$.

(6) *Both $(r_{\succ}^* + r_{\prec}^*, 0, 0, l_{\succ}^* + l_{\prec}^*, V^*)$ and $(r_{\prec}^*, 0, 0, l_{\succ}^*, V^*)$ are bimodules of $(A, \succ, \prec)$.*

(7) *Both $(r_{\succ}^* + r_{\prec}^*, l_{\succ}^* + l_{\prec}^*, V^*)$ and $(r_{\prec}^*, l_{\succ}^*, V^*)$ are bimodules of $(A, *)$.*

(8) *The dendriform algebras $A \ltimes_{r_{\succ}^*+r_{\prec}^*, -l_{\prec}^*, -r_{\succ}^*, l_{\succ}^*+l_{\prec}^*} V^*$ and $A \ltimes_{r_{\prec}^*, 0, 0, l_{\succ}^*} V^*$ have the same associative algebra $A \ltimes_{r_{\prec}^*, l_{\succ}^*} V^*$.*

Proof. This is straightforward. □

Example 3.2.3. Let $(A, \succ, \prec)$ be a dendriform algebra. Then

$$(L_{\succ}, R_{\succ}, L_{\prec}, R_{\prec}, A), \quad (L_{\succ}, 0, 0, R_{\prec}, A), \quad (L_{\succ} + L_{\prec}, 0, 0, R_{\succ} + R_{\prec}, A)$$

are bimodules of $(A, \prec, \succ)$. On the other hand,

$$(R_{\succ}^* + R_{\prec}^*, -L_{\prec}^*, -R_{\succ}^*, L_{\succ}^* + L_{\prec}^*, A^*), \quad (R_{\prec}^*, 0, 0, L_{\succ}^*, A^*),$$
$$(R_{\succ}^* + R_{\prec}^*, 0, 0, L_{\succ}^* + L_{\prec}^*, A^*)$$

are bimodules of $(A, \succ, \prec)$, too. There are two compatible dendriform algebra structures,

$$A \ltimes_{R_{\succ}^*+R_{\prec}^*, -L_{\prec}^*, -R_{\succ}^*, L_{\succ}^*+L_{\prec}^*} A^* \quad \text{and} \quad A \ltimes_{R_{\prec}^*, 0, 0, L_{\succ}^*} A^*,$$

on the same associative algebra $A \ltimes_{R_{\prec}^*, L_{\succ}^*} A^*$.

Theorem 3.2.4. *Let $(A, \succ_A, \prec_A)$ and $(B, \succ_B, \prec_B)$ be two dendriform algebras. Suppose that there are linear maps $l_{\succ_A}$, $r_{\succ_A}$, $l_{\prec_A}$, $r_{\prec_A} : A \to \mathfrak{gl}(B)$ and $l_{\succ_B}$, $r_{\succ_B}$, $l_{\prec_B}$, $r_{\prec_B} : B \to \mathfrak{gl}(A)$ such that $(l_{\succ_A}, r_{\succ_A}, l_{\prec_A}, r_{\prec_A})$ is a bimodule of A and $(l_{\succ_B}, r_{\succ_B}, l_{\prec_B}, r_{\prec_B})$ is a bimodule of B and they satisfy the following 18 equations:*

$$r_{\prec_A}(x)(a \prec_B b) = a \prec_B (r_A(x)b) + r_{\prec_A}(l_B(b)x)a, \tag{35}$$
$$l_{\prec_A}(l_{\prec_B}(a)x)b + (r_{\prec_A}(x)a) \prec_B b = a \prec_B (l_A(x)b) + r_{\prec_A}(r_B(b)x)a, \tag{36}$$
$$l_{\prec_A}(x)(a *_B b) = (l_{\prec_A}(x)a) \prec_B b + l_{\prec_A}(r_{\prec_B}(a)x)b, \tag{37}$$
$$r_{\prec_A}(x)(a \succ_B b) = r_{\succ_A}(l_{\prec_B}(b)x)a + a \succ_B (r_{\prec_A}(x)b), \tag{38}$$
$$l_{\prec_A}(l_{\succ_B}(a)x)b + (r_{\succ_A}(x)a) \prec_B b = a \succ_B (l_{\prec_A}(x)b) + r_{\succ_A}(r_{\prec_B}(b)x)a, \tag{39}$$
$$l_{\succ_A}(x)(a \prec_B b) = (l_{\succ_A}(x)a) \prec_B b + l_{\prec_A}(r_{\succ_B}(a)x)b, \tag{40}$$
$$r_{\succ_A}(x)(a *_B b) = a \succ_B (r_{\succ_A}(x)b) + r_{\succ_A}(l_{\succ_B}(b)x)a, \tag{41}$$
$$a \succ_B (l_{\succ_A}(x)b) + r_{\succ_A}(r_{\succ_B}(b)x)a = l_{\succ_A}(l_B(a)x)b + (r_A(x)a) \succ_B b, \tag{42}$$
$$l_{\succ_A}(x)(a \succ_B b) = (l_A(x)a) \succ_B b + l_{\succ_A}(r_B(a)x)b, \tag{43}$$
$$r_{\prec_B}(a)(x \prec_A y) = x \prec_A (r_B(a)y) + r_{\prec_B}(l_A(y)a)x, \tag{44}$$
$$l_{\prec_B}(l_{\prec_A}(x)a)y + (r_{\prec_B}(a)x) \prec_A y = x \prec_A (l_B(a)y) + r_{\prec_B}(r_A(y)a)x, \tag{45}$$
$$l_{\prec_B}(a)(x *_A y) = (l_{\prec_B}(a)x) \prec_A y + l_{\prec_B}(r_{\prec_A}(x)a)y, \tag{46}$$
$$r_{\prec_B}(a)(x \succ_A y) = r_{\succ_B}(l_{\prec_A}(y)a)x + x \succ_A (r_{\prec_B}(a)y), \tag{47}$$
$$l_{\prec_B}(l_{\succ_A}(x)a)y + (r_{\succ_B}(a)x) \prec_A y = x \succ_A (l_{\prec_B}(a)y) + r_{\succ_B}(r_{\prec_A}(y)a)x, \tag{48}$$
$$l_{\succ_B}(a)(x \prec_A y) = (l_{\succ_B}(a)x) \prec_A y + l_{\prec_B}(r_{\succ_A}(x)a)y, \tag{49}$$
$$r_{\succ_B}(a)(x *_A y) = x \succ_A (r_{\succ_B}(a)y) + r_{\succ_B}(l_{\succ_A}(y)a)x, \tag{50}$$
$$x \succ_A (l_{\succ_B}(a)y) + r_{\succ_B}(r_{\succ_A}(y)a)x = l_{\succ_B}(l_A(x)a)y + (r_B(a)x) \succ_A y, \tag{51}$$
$$l_{\succ_B}(a)(x \succ_A y) = (l_B(a)x) \succ_A y + l_{\succ_B}(r_A(x)a)y \tag{52}$$

for any $x, y \in A$, $a, b \in B$ and $l_A = l_{\succ_A} + l_{\prec_A}$, $r_A = r_{\succ_A} + r_{\prec_A}$, $l_B = l_{\succ_B} + l_{\prec_B}$, $r_B = r_{\succ_B} + r_{\prec_B}$. Then there is a dendriform algebra structure on the direct sum $A \oplus B$ of the underlying vector spaces of A and B given by

$$\begin{aligned}(x + a) \succ (y + b) &= (x \succ_A y + r_{\succ_B}(b)x + l_{\succ_B}(a)y)\\ &\quad + (l_{\succ_A}(x)b + r_{\succ_A}(y)a + a \succ_B b),\\ (x + a) \prec (y + b) &= (x \prec_A y + r_{\prec_B}(b)x + l_{\prec_B}(a)y)\\ &\quad + (l_{\prec_A} x)b + r_{\prec_A}(y)a + a \prec_B b)\end{aligned}$$

for any $x, y \in A$, $a, b \in B$. Let $A \bowtie^{l_{\succ_A}, r_{\succ_A}, l_{\prec_A}, r_{\prec_A}}_{l_{\succ_B}, r_{\succ_B}, l_{\prec_B}, r_{\prec_B}} B$ or simply $A \bowtie B$ denote this dendriform algebra. On the other hand, every dendriform algebra which is the direct sum of the underlying vector spaces of two subalgebras can be obtained in this way.

Proof. This is straightforward. $\square$

Definition 3.2.5. Let $(A, \succ_A, \prec_A)$ and $(B, \succ_B, \prec_B)$ be two dendriform algebras. Suppose that there are linear maps $l_{\succ_A}$, $r_{\succ_A}$, $l_{\prec_A}$, $r_{\prec_A}: A \to \mathfrak{gl}(B)$ and $l_{\succ_B}$, $r_{\succ_B}$, $l_{\prec_B}$, $r_{\prec_B}: B \to \mathfrak{gl}(A)$ such that $(l_{\succ_A}, r_{\succ_A}, l_{\prec_A}, r_{\prec_A})$ is a bimodule of A and $(l_{\succ_B}, r_{\succ_B}, l_{\prec_B}, r_{\prec_B})$ is a bimodule of B. If eqs. (35)–(52) are satisfied, then $(A, B, l_{\succ_A}, r_{\succ_A}, l_{\prec_A}, r_{\prec_A}, l_{\succ_B}, r_{\succ_B}, l_{\prec_B}, r_{\prec_B})$ is called a *matched pair of dendriform algebras*.

Remark 3.2.6. Obviously B is an ideal of $A \bowtie B$ if and only if $l_{\succ_B} = r_{\succ_B} = l_{\prec_B} = r_{\prec_B} = 0$. If B is a trivial ideal, then $A \bowtie_{0,0,0,0}^{l_{\succ_A}, r_{\succ_A}, l_{\prec_A}, r_{\prec_A}} B \cong A \ltimes_{l_{\succ_A}, r_{\succ_A}, l_{\prec_A}, r_{\prec_A}} B$.

Corollary 3.2.7. *Let $(A, B, l_{\succ_A}, r_{\succ_A}, l_{\prec_A}, r_{\prec_A}, l_{\succ_B}, r_{\succ_B}, l_{\prec_B}, r_{\prec_B})$ be a matched pair of dendriform algebras. Then $(A, B, l_{\succ_A}+l_{\prec_A}, r_{\succ_A}+r_{\prec_A}, l_{\succ_B}+l_{\prec_B}, r_{\succ_B}+r_{\prec_B})$ is a matched pair of the associated associative algebras $(A, *_A)$ and $(B, *_B)$.*

Proof. In fact, the associated associative algebra $(A \bowtie B, *)$ is exactly the associative algebra obtained from the matched pair $(A, B, l_A, r_A, l_B, r_B)$ of associative algebras:

$$(x + a) * (y + b) = x *_A y + l_B(a)y + r_B(b)x + a *_B b + l_A(x)b + r_A(y)a$$

for all $x, y \in A, a, b \in B$, where $l_A = l_{\succ_A} + l_{\prec_A}, r_A = r_{\succ_A} + r_{\prec_A}, l_B = l_{\succ_B} + l_{\prec_B}$, $r_B = r_{\succ_B} + r_{\prec_B}$. □

4. Double constructions of Connes cocycles and an analogue of the classical Yang–Baxter equation

4.1. Connes cocycles and dendriform algebras

Theorem 4.1.1. *Let $(A, *)$ be an associative algebra and let ω be a non-degenerate Connes cocycle. Then there exists a compatible dendriform algebra structure $\succ, \prec$ on A given by*

$$\omega(x \succ y, z) = \omega(y, z * x), \quad \omega(x \prec y, z) = \omega(x, y * z) \quad \textit{for all } x, y, z \in A. \quad (53)$$

Proof. Define a linear map $T: A \to A^*$ by $\langle T(x), y\rangle = \omega(x, y)$ for all $x, y \in A$. Then T is invertible and T^{-1} is an $\mathcal{O}$-operator of the associative algebra $(A, *)$ associated to the bimodule (R_*^*, L_*^*). By Corollary 3.1.3, there is a compatible dendriform algebra structure $\succ, \prec$ on $(A, *)$ given by

$$x \succ y = T^{-1} R_*^*(x) T(y), \quad x \prec y = T^{-1} L_*^*(y) T(x)$$

for all $x, y \in A$, which gives exactly eq. (53). □

Next, we turn to the double construction of the Connes cocycles. Let $(A, *_A)$ be an associative algebra and suppose that there is a associative algebra structure $*_{A^*}$ on its dual space A^*. We construct an associative algebra structure on the direct sum $A \oplus A^*$ of the underlying vector spaces of A and A^* such that both A and A^* are subalgebras and the antisymmetric bilinear form on $A \oplus A^*$ given by eq. (7) is a Connes cocycle on $A \oplus A^*$. Such a construction is called a double construction of the Connes cocycle associated to $(A, *_A)$ and $(A^*, *_{A^*})$ and we denote it by $(T(A) = A \bowtie A^*, \omega)$.

Corollary 4.1.2. *Let $(T(A) = A \bowtie A^*, \omega)$ be a double construction of the Connes cocycle. Then there exists a compatible dendriform algebra structure $\succ$, $\prec$ on $T(A)$ defined by eq. (53). Moreover, A and A^* are dendriform subalgebras with this product.*

Proof. The first half follows from Theorem 4.1.1. Let $x, y \in A$. Set $x \succ y = a + b^*$, where $a \in A$, $b^* \in A^*$. Since A is an associative subalgebra of $T(A)$ and $\omega(A, A) = \omega(A^*, A^*) = 0$, we have

$$\omega(b^*, A^*) = \omega(b^*, A) = \omega(x \succ y, A) = \omega(y, A * x) = 0.$$

Therefore $b^* = 0$ due to the nondependence of ω. Hence $x \succ y = a \in A$. Similarly, $x \prec y \in A$. Thus A is a dendriform subalgebra of $T(A)$ with the product $\succ$, $\prec$. By symmetry of A and A^*, A^* is also a dendriform subalgebra. □

Definition 4.1.3. Let $(T(A_1) = A_1 \bowtie A_1^*, \omega_1)$ and $(T(A_2) = A_2 \bowtie A_2^*, \omega_2)$ be two double constructions of Connes cocycles. They are *isomorphic* if there exists an isomorphism of associative algebras $\varphi\colon T(A_1) \to T(A_2)$ satisfying the conditions

$$\varphi(A_1) = A_2, \quad \varphi(A_1^*) = A_2^*, \quad \omega_1(x, y) = \varphi^*\omega_2(x, y) = \omega_2(\varphi(x), \varphi(y)) \tag{54}$$

for all $x, y \in A_1$.

Proposition 4.1.4. *Two double constructions of Connes cocycles $(T(A_1) = A_1 \bowtie A_1^*, \omega_1)$ and $(T(A_2) = A_2 \bowtie A_2^*, \omega_2)$ are isomorphic if and only if there exists a dendriform algebra isomorphism $\varphi\colon T(A_1) \to T(A_2)$ satisfying eq. (54), where the dendriform algebra structures on $T(A_1)$ and $T(A_2)$ are given by eq. (53), respectively.*

Proof. This is straightforward. □

Theorem 4.1.5. *Let $(A, \succ_A, \prec_A)$ be a dendriform algebra and $(A, *_A)$ the associated associative algebra. Suppose that there is a dendriform algebra structure "$\succ_{A^*}$, $\prec_{A^*}$" on its dual space A^* and $(A^*, *_{A^*})$ is the associated associative algebra. Then there exists a double construction of the Connes cocycle associated to $(A, *_A)$ and $(A, *_{A^*})$ if and only if $(A, A^*, R^*_{\prec_A}, L^*_{\succ_A}, R^*_{\prec_{A^*}}, L^*_{\succ_{A^*}})$ is a matched pair of the associative algebras. Moreover, every double construction of the Connes cocycle can be obtained in this way.*

Proof. The conclusion can be obtained by a similar proof as of Theorem 2.2.1. □

Corollary 4.1.6. *Let $(A, \succ, \prec)$ be a dendriform algebra and $(R^*_\prec, L^*_\succ)$ the bimodule of the associated associative algebra $(A, *)$. Then $(T(A) = A \ltimes_{R^*_\prec, L^*_\succ} A^*, \omega)$ is a double construction of the Connes cocycle. Conversely, let $(T(A) = A \bowtie A^*, \omega)$ be a double construction of the Connes cocycle. If A^* is an ideal of $T(A)$, then A^* is a trivial associative algebra and hence $T(A)$ is isomorphic to the semidirect $A \ltimes_{L_{T(A)}, R_{T(A)}} A^*$. Furthermore, this double construction of the Connes cocycle is isomorphic to the double construction of the Connes cocycle $(T(A) = A \ltimes_{R^*_\prec, L^*_\succ} A^*, \omega)$ and the dendriform algebra structure on A is given by ω from eq. (53).*

Proof. By Remark 2.1.6, $(A, A^*, R^*_\prec, L^*_\succ, 0, 0)$ with the associative algebra structure on A^* being trivial is always a matched pair of associative algebras, the first half follows immediately. Conversely, if A^* is an ideal, then, for any $a^*, b^* \in A^*$, it follows that if $T(A) * a^*,\ b^* * T(A) \in A^*$ then $\omega(a^* * b^*, T(A)) = -\omega(T(A) * a^*, b^*) - \omega(b^* * T(A), a^*) = 0$. Thus $a^* * b^* = 0$. Hence $T(A)$ is isomorphic to $A \ltimes_{L_{T(A)}, R_{T(A)}} A^*$. By Remark 2.1.6, it follows that that $(T(A) = A \bowtie A^*, \omega)$ is isomorphic to the double construction of the Connes cocycle $(T(A) = A \ltimes_{R^*_\prec, L^*_\succ} A^*, \omega)$. □

Theorem 4.1.7. *Let $(A, \succ_A, \prec_A)$ be a dendriform algebra and $(A, *_A)$ the associated associative algebra. Suppose that there is a dendriform algebra structure "$\succ_{A^*}$, $\prec_{A^*}$" on its dual space A^* and $(A^*, *_{A^*})$ is the associated associative algebra. Then $(A, A^*, R^*_{\prec_A}, L^*_{\succ_A}, R^*_{\prec_{A^*}}, L^*_{\succ_{A^*}})$ is a matched pair of associative algebras if and only if*

$$(A, A^*, R^*_{\succ_A} + R^*_{\prec_A}, -L^*_{\prec_A}, -R^*_{\succ_A}, L^*_{\succ_A} + L^*_{\prec_A}, \\ R^*_{\succ_{A^*}} + R^*_{\prec_{A^*}}, -L^*_{\prec_{A^*}}, -R^*_{\succ_{A^*}}, L^*_{\succ_{A^*}} + L^*_{\prec_{A^*}})$$

is a matched pair of dendriform algebras.

Proof. The "if" part follows from Corollary 3.2.7. We need to prove the "only if" part. If $(A, A^*, R^*_{\prec_A}, L^*_{\succ_A}, R^*_{\prec_{A^*}}, L^*_{\succ_{A^*}})$ is a matched pair of associative algebras, then $(A \bowtie^{R^*_{\prec_A}, L^*_{\succ_A}}_{R^*_{\prec_{A^*}}, L^*_{\succ_{A^*}}} A^*, \omega)$ is a double construction of the Connes cocycle. Hence there exists a compatible dendriform algebra structure on $A \bowtie^{R^*_{\prec_A}, L^*_{\succ_A}}_{R^*_{\prec_{A^*}}, L^*_{\succ_{A^*}}} A^*$ given by eq. (53). By a simple and direct computation, we show that A and A^* are its subalgebras and the other products are given by

$$\begin{aligned}
x \succ a^* &= (R^*_{\succ_A} + R^*_{\prec_A})(x)a^* - L^*_{\prec_{A^*}}(a^*)x, \\
x \prec a^* &= -R^*_{\succ_A}(x)a^* + (L^*_{\succ_{A^*}} + L^*_{\prec_{A^*}})(a^*)x, \\
a^* \succ x &= (R^*_{\succ_{A^*}} + R^*_{\prec_{A^*}})(a^*)x - L^*_{\prec_A}(x)a^*, \\
a^* \prec x &= -R^*_{\succ_{A^*}}(a^*)x + (L^*_{\succ_A} + L^*_{\prec_A})(x)a^*,
\end{aligned}$$

for any $x \in A, a^* \in A^*$. Therefore

$$(A, A^*, R^*_{\succ_A} + R^*_{\prec_A}, -L^*_{\prec_A}, -R^*_{\succ_A}, L^*_{\succ_A} + L^*_{\prec_A},\\ R^*_{\succ_{A^*}} + R^*_{\prec_{A^*}}, -L^*_{\prec_{A^*}}, -R^*_{\succ_{A^*}}, L^*_{\succ_{A^*}} + L^*_{\prec_{A^*}})$$

is a matched pair of dendriform algebras. □

4.2. Dendriform D-bialgebras

Theorem 4.2.1. *Let $(A, \succ_A, \prec_A)$ be a dendriform algebra whose products are given by two linear maps $\beta^*_\succ$, $\beta^*_\prec: A \otimes A \to A$. Further suppose that there is a dendriform algebra structure "$\succ_{A^*}$, $\prec_{A^*}$" on its dual space A^* given by two linear maps $\Delta^*_\succ, \Delta^*_\prec: A^* \otimes A^* \to A^*$. Then $(A, A^*, R^*_{\prec_A}, L^*_{\succ_A}, R^*_{\prec_{A^*}}, L^*_{\succ_{A^*}})$ is a matched pair of associative algebras if and only if the following equations hold for any $x, y \in A$ and $a^*, b^* \in A^*$:*

$$\Delta_\prec(x *_A y) = (\mathrm{id} \otimes L_{\prec_A}(x))\Delta_\prec(y) + (R_A(y) \otimes \mathrm{id})\Delta_\prec(x), \tag{55}$$

$$\Delta_\succ(x *_A y) = (\mathrm{id} \otimes L_A(x))\Delta_\succ(y) + (R_{\prec_A}(y) \otimes \mathrm{id})\Delta_\succ(x), \tag{56}$$

$$\beta_\prec(a^* *_{A^*} b^*) = (\mathrm{id} \otimes L_{\prec_{A^*}}(a^*))\beta_\prec(b^*) + (R_{A^*}(b^*) \otimes \mathrm{id})\beta_\prec(a^*), \tag{57}$$

$$\beta_\succ(a^* *_{A^*} b^*) = (\mathrm{id} \otimes L_{A^*}(a^*))\beta_\succ(b^*) + (R_{\prec_{A^*}}(b^*) \otimes \mathrm{id})\beta_\succ(a^*), \tag{58}$$

$$\begin{split}(L_A(x) \otimes \mathrm{id} - \mathrm{id} \otimes R_{\prec_A}(x))\Delta_\prec(y)\\ + \sigma[(L_{\succ_A}(y) \otimes -\mathrm{id} \otimes R_A(y))\Delta_\succ(x)] = 0,\end{split} \tag{59}$$

$$\begin{split}(L_{A^*}(a^*) \otimes \mathrm{id} - \mathrm{id} \otimes R_{\prec_{A^*}}(a^*))\beta_\prec(b^*)\\ + \sigma[(L_{\succ_{A^*}}(b^*) \otimes -\mathrm{id} \otimes R_{A^*}(b^*))\beta_\succ(a^*)] = 0,\end{split} \tag{60}$$

where $L_A = L_{\succ_A} + L_{\prec_A}$, $R_A = R_{\succ_A} + R_{\prec_A}$, $L_{A^*} = L_{\succ_{A^*}} + L_{\prec_{A^*}}$, $R_{A^*} = R_{\succ_{A^*}} + R_{\prec_{A^*}}$.

Proof. Let $\{e_1, \ldots, e_n\}$ be a basis of A and $\{e_1^*, \ldots, e_n^*\}$ its dual basis. Set

$$e_i \succ_A e_j = \sum_{k=1}^n a_{ij}^k e_k, \qquad e_i \prec_A e_j = \sum_{k=1}^n b_{ij}^k e_k,$$

$$e_i^* \succ_{A^*} e_j^* = \sum_{k=1}^n c_{ij}^k e_k^*, \quad e_i^* \prec_{A^*} e_j^* = \sum_{k=1}^n d_{ij}^k e_k^*.$$

Therefore the coefficient of e_l^* in

$$R^*_{\prec_A}(e_i)(e_j^* *_{A^*} e_k^*) = R^*_{\prec_A}(L^*_{\succ_{A^*}}(e_j^*)e_i)e_k^* + R^*_{\prec_A}(e_i)e_j^* *_{A^*} e_k^*$$

gives the following relation (for any i, j, k, l)

$$\sum_{m=1}^n b_{li}^m(c_{jk}^m + d_{jk}^m) = \sum_{m=1}^n [c_{jm}^i b_{lm}^k + b_{mi}^j(c_{mk}^l + d_{mk}^l)],$$

which is precisely the relation given by the coefficient of $e_l^* \otimes e_i^*$ in

$$\beta_{\prec}(e_j^* *_{A^*} e_k^*) = (R_{A^*}(e_k^*) \otimes \mathrm{id})\beta_{\prec}(e_j^*) + (\mathrm{id} \otimes L_{\succ_{A^*}}(e_j^*))\beta_{\prec}(e_k^*).$$

So eq. (9) in the case $l_A = R^*_{\prec_A}, r_A = L^*_{\succ_A}, l_B = l_{A^*} = R^*_{\prec_{A^*}}, r_B = r_{A^*} = L^*_{\succ_{A^*}}$ is eq. (57). Similarly, in this situation, we have the following correspondences:

$$\text{eq. (10)} \iff \text{eq. (58)}, \quad \text{eq. (11)} \iff \text{eq. (55)}, \quad \text{eq. (12)} \iff \text{eq. (56)},$$
$$\text{eq. (13)} \iff \text{eq. (60)}, \quad \text{eq. (14)} \iff \text{eq. (59)}.$$

Therefore the conclusion holds due to Theorem 2.1.4. □

Definition 4.2.2. Let A be a vector space. A *dendriform D-bialgebra* structure on A is a set of linear maps $(\Delta_{\prec}, \Delta_{\succ}, \beta_{\prec}, \beta_{\succ})$ such that $\Delta_{\prec}, \Delta_{\succ} : A \to A \otimes A$, $\beta_{\prec}, \beta_{\succ} : A^* \to A^* \otimes A^*$ and

(a) $(\Delta^*_{\prec}, \Delta^*_{\succ}) : A^* \otimes A^* \to A^*$ defines a dendriform algebra structure $(\succ_{A^*}, \prec_{A^*})$ on A^*;

(b) $(\beta^*_{\prec}, \beta^*_{\succ}) : A \otimes A \to A$ defines a dendriform algebra structure $(\succ_A, \prec_A)$ on A;

(c) eqs. (55)–(60) are satisfied.

We also denote it by $(A, A^*, \Delta_{\succ}, \Delta_{\prec}, \beta_{\succ}, \beta_{\prec})$ or simply (A, A^*).

Remark 4.2.3. In fact, the notions of dendriform bialgebra ([LR1]–[LR2], [Ron], [A4]) and bidendriform bialgebra ([F2]), which are the special dendriform bialgebras, were already introduced. We use the terminology "D-bialgebra" in order to express its relation with the double construction. All of these bialgebras are dendriform algebras equipped with coassociative cooperations satisfying some (different) compatibility relations. We would like to point out that the dendriform D-bialgebras are quite different from the other types of bialgebras. For example, one of the differences is that the term $a \otimes b$ appears in both $\Delta_{\prec}(a * b)$ and $\Delta_{\succ}(a * b)$ in a bidendriform bialgebra, whereas it does not appear in a dendriform D-bialgebra.

Theorem 4.2.4. *Let $(A, \prec_A, \succ_A)$ and $(A^*, \prec_{A^*}, \succ_{A^*})$ be two dendriform algebras. Let $(A, *_A)$ and $(A^*, *_{A^*})$ be the associated associative algebras respectively. Then the following conditions are equivalent.*

(1) *There is a double construction of the Connes cocycle associated to $(A, *_A)$ and $(A, *_{A^*})$.*

(2) *$(A, A^*, R^*_{\prec_A}, L^*_{\succ_A}, R^*_{\prec_{A^*}}, L^*_{\succ_{A^*}})$ is a matched pair of the associative algebras.*

(3) *$(A, A^*, R^*_{\succ_A} + R^*_{\prec_A}, -L^*_{\prec_A}, -R^*_{\succ_A}, L^*_{\succ_A} + L^*_{\prec_A}, R^*_{\succ_{A^*}} + R^*_{\prec_{A^*}}, -L^*_{\prec_{A^*}}, -R^*_{\succ_{A^*}}, L^*_{\succ_{A^*}} + L^*_{\prec_{A^*}})$ is a matched pair of dendriform algebras.*

(4) *(A, A^*) is a dendriform D-bialgebra.*

Proof. This follows from Theorems 4.1.5, 4.1.7 and 4.2.1. □

Definition 4.2.5. Let $(A, A^*, \Delta_\succ, \Delta_\prec, \beta_\succ, \beta_\prec)$ and $(B, B^*, \Delta_\succ, \Delta_\prec, \beta_\succ, \beta_\prec)$ be two dendriform D-bialgebras. A *homomorphism of dendriform D-bialgebras* $\varphi\colon A \to B$ is a homomorphism of dendriform algebras such that $\varphi^*\colon B^* \to A^*$ is also a homomorphism of dendriform algebras, that is, φ satisfies

$$(\varphi \otimes \varphi)\Delta_\succ(x) = \Delta_\succ(\varphi(x)), \quad (\varphi \otimes \varphi)\Delta_\prec(x) = \Delta_\prec(\varphi(x)),$$
$$(\varphi^* \otimes \varphi^*)\beta_\succ(a^*) = \beta_\succ(\varphi^*(a^*)), \quad (\varphi^* \otimes \varphi^*)\beta_\prec(a^*) = \beta_\prec(\varphi^*(a^*)),$$

for any $x \in A, a^* \in B^*$. An *isomorphism of dendriform D-bialgebras* is an invertible homomorphism of dendriform D-bialgebras.

Proposition 4.2.6. *Two double constructions of Connes cocycles are isomorphic if and only if their corresponding dendriform D-bialgebras are isomorphic.*

Proof. It follows from a similar proof as of Proposition 2.2.10. □

Example 4.2.7. Let $(A, A^*, \Delta_\succ, \Delta_\prec, \beta_\succ, \beta_\prec)$ be a dendriform D-bialgebra. Then its dual $(A^*, A, \beta_\succ, \beta_\prec, \Delta_\succ, \Delta_\prec)$ is also a dendriform D-bialgebra.

Example 4.2.8. Let $(A, \succ_A, \prec_A)$ be a dendriform algebra. If the dendriform algebra structure on A^* is trivial, then $(A, A^*, 0, 0, \beta_\succ, \beta_\prec)$ is a dendriform D-bialgebra. And its corresponding dendriform algebra is $A \ltimes_{R^*_\succ + R^*_\prec, -L^*_\prec, -R^*_\succ, L^*_\succ + L^*_\prec} A^*$. Moreover, its corresponding double construction of the Connes cocycle is just the semidirect sum $A \ltimes_{R^*_{\prec_A}, L^*_{\succ_A}} A^*$ with the bilinear form ω given by eq. (7). Dually, if A is a trivial dendriform algebra, then the dendriform D-bialgebra structures on A are in one-to-one correspondence with the dendriform algebra structures on A^*.

Example 4.2.9. Let (A, A^*) be a dendriform D-bialgebra. In the next subsection, we will prove that there exists a canonical dendriform D-bialgebra structure on the direct sum $A \oplus A^*$ of the underlying vector spaces of A and A^*.

4.3. Coboundary dendriform D-bialgebras. In Theorem 4.2.1 we showed that both $\Delta_\succ$ and $\Delta_\prec$ ($\beta_\succ$ and $\beta_\prec$, respectively) are the 1-cocycles of the associated associative algebra $(A, *_A)$ (resp. $(A^*, *_{A^*})$). So it is natural to consider the special case that they are 1-coboundaries or principal derivations, as we did in Section 2.3.

Let $(A, \succ, \prec)$ be a dendriform algebra and $r_\succ, r_\prec \in A \otimes A$. Set

$$\Delta_\succ(x) = (\mathrm{id} \otimes L(x) - R_\prec(x) \otimes \mathrm{id})r_\succ, \tag{61}$$
$$\Delta_\prec(x) = (\mathrm{id} \otimes L_\succ(x) - R(x) \otimes \mathrm{id})r_\prec \tag{62}$$

for any $x \in A$. It is obvious that $\Delta_\succ$ satisfies eq. (55) and $\Delta_\prec$ satisfies eq. (56). Moreover, by eq. (59), it follows that

$$(L(x) \otimes \mathrm{id} - \mathrm{id} \otimes R_\prec(x))(\mathrm{id} \otimes L_\succ(y) - R(y) \otimes \mathrm{id})(r_\prec + \sigma(r_\succ)) = 0 \tag{63}$$

for all $x, y \in A$. Therefore $(A, \Delta_\succ, \Delta_\prec, \beta_\succ, \beta_\prec)$ is a dendriform D-bialgebra if and only if the following conditions are satisfied:

(1) $\Delta^*_{\succ}, \Delta^*_{\prec}\colon A^*\otimes A^* \to A^*$ defines a dendriform algebra structure on A^*.

(2) $\beta_{\succ}, \beta_{\prec}$ satisfy eqs. (57), (58) and (60), where the dendriform algebra structure on A^* is given by (1).

Proposition 4.3.1. *Let $(A, \succ, \prec)$ be a dendriform algebra whose products are given by two linear maps $\beta^*_{\succ}, \beta^*_{\prec}\colon A\otimes A \to A$ and $r_{\succ}, r_{\prec} \in A\otimes A$. Suppose there exists a dendriform algebra structure "$\succ_{A^*}, \prec_{A^*}$" on A^* given by $\Delta^*_{\succ}, \Delta^*_{\prec}\colon A^*\otimes A^* \to A^*$, where $\Delta_{\succ}$ and $\Delta_{\prec}$ are two linear maps given by eqs. (61) and (62), respectively. Then*

(1) *Eq. (57) holds if and only if $r_{\succ}, r_{\prec}$ satisfy*

$$[R_{\prec}(x)\otimes L_{\succ}(y) - \mathrm{id}\otimes L_{\succ}(y\prec x) - R_{\prec}(y\succ x)\otimes \mathrm{id}](r_{\succ}+r_{\prec}) = 0 \tag{64}$$

for all $x, y \in A$.

(2) *Eq. (58) holds if and only if $r_{\succ}$, $r_{\prec}$ satisfy eq. (64).*

(3) *Eq. (60) holds if and only if $r_{\succ}$, $r_{\prec}$ satisfy*

$$\begin{aligned}&[L_{\succ}(x)\otimes \mathrm{id} - \mathrm{id}\otimes R_{\prec}(x)][-\mathrm{id}\otimes L_{\succ}(y) + R_{\prec}(y)\otimes \mathrm{id}](r_{\prec}+r_{\succ})\\ &\quad + [L_{\succ}(x)\otimes \mathrm{id} - \mathrm{id}\otimes R_{\prec}(x)][R_{\succ}(y)\otimes \mathrm{id}(r_{\prec}+\sigma(r_{\succ}))\\ &\quad - \mathrm{id}\otimes L_{\prec}(y)(\sigma(r_{\prec})+r_{\succ})] = 0\end{aligned} \tag{65}$$

for any $x, y \in A$.

Proof. Let $\{e_1, \ldots, e_n\}$ be a basis of A and $\{e_1^*, \ldots, e_n^*\}$ its dual basis. Set

$$\begin{aligned} r_{\prec} &= \sum_{i,j} a_{ij} e_i\otimes e_j, & r_{\succ} &= \sum_{i,j} b_{ij} e_i\otimes e_j,\\ e_i\succ e_j &= \sum_{k=1}^n a_{ij}^k e_k, & e_i\prec e_j &= \sum_{k=1}^n b_{ij}^k e_k,\\ e_i^*\succ e_j^* &= \sum_{k=1}^n c_{ij}^k e_k^*, & e_i^*\prec e_j^* &= \sum_{k=1}^n d_{ij}^k e_k^*.\end{aligned}$$

By eqs. (61) and (62), we have (for any i, k, l)

$$\begin{aligned} c_{kl}^i &= \sum_{m=1}^n [b_{km}(a_{im}^l + b_{im}^l) - b_{ml}b_{mi}^k],\\ d_{kl}^i &= \sum_{m=1}^n [a_{km}a_{im}^l - a_{ml}(a_{mi}^k + b_{mi}^k)].\end{aligned} \tag{66}$$

(1) Eq. (57) holds (taking $a^* = e_i^*, b^* = e_j^*$) if and only if (for any i, j, m, t)

$$\sum_{k=1}^n (c_{ij}^k + d_{ij}^k)b_{mt}^k = \sum_{k=1}^n [b_{mk}^j c_{ik}^t + b_{kt}^i(c_{kj}^m + d_{kj}^m)].$$

Substituting eq. (66) into the above equation and after rearranging the terms suitably, we have

$$(\mathrm{F1})+(\mathrm{F2})+(\mathrm{F3})+(\mathrm{F4})+(\mathrm{F5})+(\mathrm{F6})=0,$$

where

$$(\mathrm{F1})=\sum_{k,l}(a_{kl}+b_{kl})(a^j_{ml}b^i_{kt}),\qquad (\mathrm{F2})=\sum_{k,l}(b_{kl}b^i_{kt}b^j_{ml}-b_{lk}b^i_{ln}b^j_{mk});$$

$$(\mathrm{F3})=\sum_{k,l}(a_{il}+b_{il})(-a^j_{kl}b^k_{mt}),\qquad (\mathrm{F4})=\sum_{k,l}b_{il}[b^j_{mk}(a^k_{tl}+b^k_{tl})-b^k_{mt}b^j_{kl}];$$

$$(\mathrm{F5})=\sum_{k,l}(a_{lj}+b_{lj})(b^k_{mt}b^i_{lk}-b^i_{kt}b^k_{lm}),\ (\mathrm{F6})=\sum_{k,l}a_{lj}(a^i_{lk}b^k_{mt}-a^k_{lm}b^i_{kt}).$$

Here (F1) is the coefficient of $e_i\otimes e_j$ in $[R_{\prec}(e_t)\otimes L_{\succ}(e_m)](r_{\succ}+r_{\prec})$;

(F2) $=0$ by interchanging the indices k and l;

(F3) is the coefficient of $e_i\otimes e_j$ in $-[\mathrm{id}\otimes L_{\succ}(e_m\prec e_t)](r_{\succ}+r_{\prec})$;

(F4) $=0$ since the term in the bracket is the coefficient of e_j in

$$e_m\prec(e_t\succ e_l+e_t\prec e_l)-(e_m\prec e_n)\prec e_l=0;$$

(F5) is the coefficient of $e_i\otimes e_j$ in $-[R_{\prec}(e_m\succ e_t)\otimes\mathrm{id}](r_{\succ}+r_{\prec})$.

(F6) $=0$ since the term in the bracket is the coefficient of e_i in

$$e_l\succ(e_m\prec e_t)-(e_l\succ e_m)\prec e_t=0.$$

Therefore we have

$$[R_{\prec}(e_t)\otimes L_{\succ}(e_m)-\mathrm{id}\otimes L_{\succ}(e_m\prec e_t)-R_{\prec}(e_m\succ e_t)\otimes\mathrm{id}](r_{\succ}+r_{\prec})=0.$$

(2) Similarly, eq. (58) holds if and only if $r_{\succ},r_{\prec}$ satisfy eq. (64). In fact, comparing with the proof in (1), the difference appears in (F2)′, (F4)′ and (F6)′, where

(F2)′ $=\sum_{k,l}(a^j_{mk}a^i_{lt}-a^i_{kt}a^j_{ml})=0$ by interchanging the indices k and l;

(F4)′ $=\sum_{k,l}b_{il}(a^k_{mt}b^j_{kl}-a^j_{mk}b^k_{tl})=0$ since the term in the bracket is the coefficient of e_j in

$$(e_m\succ e_t)\prec e_l-e_m\succ(e_t\prec e_l)=0;$$

(F6)′ $=\sum_{k,l}a_{lj}[a^i_{kt}(a^k_{lm}+b^k_{lm})-a^k_{mt}a^i_{lk}]=0$ since the term in the bracket is the coefficient of e_i in $-e_l\succ(e_m\succ e_t)+(e_l\succ e_m+e_l\prec e_m)\succ e_t=0$.

(3) Eq. (60) holds (taking $a^*=e_i^*,b^*=e_j^*$) if and only if (for any i,j,m,t)

$$\sum_{l=1}^n[(c^m_{il}+d^m_{il})b^j_{lt}-b^j_{ml}d^t_{li}+a^i_{lm}c^t_{jl}-a^i_{tl}(c^m_{lj}+d^m_{lj})]=0.$$

Substituting eq. (66) into the above equation and after rearranging the terms suitably, we have

$$(\mathrm{F1})+(\mathrm{F2})+(\mathrm{F3})+(\mathrm{F4})+(\mathrm{F5})+(\mathrm{F6})+(\mathrm{F7})+(\mathrm{F8})+(\mathrm{F9})+(\mathrm{F10})=0,$$

where

$$(\text{F1}) = \sum_{k,l}(a_{kl} + b_{kl})(-b_{lt}^{j} b_{km}^{i}) \implies -R_{\prec}(e_m) \otimes R_{\prec}(e_t)(r_{\succ} + r_{\prec}),$$

$$(\text{F2}) = \sum_{k,l}(a_{lk} + b_{lk})(-a_{mk}^{j} a_{tl}^{i}) \implies -L_{\succ}(e_t) \otimes L_{\succ}(e_m)(r_{\succ} + r_{\prec}),$$

$$(\text{F3}) = \sum_{k,l}(a_{kl} + b_{lk})(-a_{km}^{i} b_{lt}^{j}) \implies -R_{\succ}(e_m) \otimes R_{\prec}(e_t)(\sigma(r_{\succ}) + r_{\prec}),$$

$$(\text{F4}) = \sum_{k,l}(a_{lk} + b_{kl})(-a_{mk}^{j} b_{tl}^{j}) \implies -L_{\succ}(e_t) \otimes L_{\prec}(e_m)(r_{\succ} + \sigma(r_{\prec})),$$

$$(\text{F5}) = \sum_{k,l}(a_{ik} + b_{ik}) a_{mk}^{l} b_{lt}^{j} \implies \mathrm{id} \otimes R_{\prec}(e_t) L_{\succ}(e_m)(r_{\succ} + r_{\prec}),$$

$$(\text{F6}) = \sum_{k,l} a_{ki}(a_{kt}^{l} + b_{kt}^{l}) b_{ml}^{j} \implies \mathrm{id} \otimes R_{\prec}(e_t) L_{\prec}(e_m)(\sigma(r_{\prec})),$$

$$(\text{F7}) = \sum_{k,l} b_{ik} b_{lt}^{j} b_{mk}^{l} \implies \mathrm{id} \otimes R_{\prec}(e_t) L_{\prec}(e_m)(r_{\succ}),$$

$$(\text{F8}) = \sum_{k,l}(a_{kj} + b_{kj}) a_{tl}^{i} b_{km}^{l} \implies L_{\succ}(e_t) R_{\prec}(e_m) \otimes \mathrm{id}(r_{\succ} + r_{\prec}),$$

$$(\text{F9}) = \sum_{k,l} a_{kj} a_{km}^{l} a_{tl}^{i} \implies L_{\succ}(e_t) R_{\succ}(e_m) \otimes \mathrm{id}(r_{\prec}),$$

$$(\text{F10}) = \sum_{k,l} b_{jk} a_{lm}^{i}(a_{tk}^{l} + b_{tk}^{l}) \implies L_{\succ}(e_t) R_{\succ}(e_m) \otimes \mathrm{id}(\sigma(r_{\succ})).$$

Therefore eq. (65) holds. □

By the definition of a dendriform algebra, we have the following conclusion (cf. [F2]).

Lemma 4.3.2. *Let A be a vector space and let $\Delta_{\succ}, \Delta_{\prec} \colon A \otimes A \to A$ be two linear maps. Then $\Delta_{\succ}^{*}, \Delta_{\prec}^{*} \colon A^{*} \otimes A^{*} \to A^{*}$ define a dendriform algebra structure on A^{*} if and only if the following conditions are satisfied:*

$$(\Delta_{\prec} \otimes \mathrm{id})\Delta_{\prec} = (\mathrm{id} \otimes (\Delta_{\succ} + \Delta_{\prec}))\Delta_{\prec}, \tag{67}$$

$$(\mathrm{id} \otimes \Delta_{\prec})\Delta_{\succ} = (\Delta_{\succ} \otimes \mathrm{id})\Delta_{\prec}, \tag{68}$$

$$(\mathrm{id} \otimes \Delta_{\succ})\Delta_{\succ} = ((\Delta_{\succ} + \Delta_{\prec}) \otimes \mathrm{id})\Delta_{\succ}. \tag{69}$$

Proposition 4.3.3. *Let $(A, \succ, \prec)$ be a dendriform algebra and $r_{\succ}, r_{\prec} \in A \otimes A$. Define $\Delta_{\succ}, \Delta_{\prec} \colon A \to A \otimes A$ by eqs. (61) and (62). Then $\Delta_{\succ}^{*}, \Delta_{\prec}^{*} \colon A^{*} \otimes A^{*} \to A^{*}$ define a dendriform algebra structure on A^{*} if and only if the following equations* *are satisfied* (*for any* $x \in A$)

$$\begin{aligned}(R(x) \otimes \mathrm{id} \otimes \mathrm{id})[(r_{\prec,12} * r_{\prec,13} + r_{\prec,13} \prec r_{\succ,23} - r_{\prec,23} \succ r_{\prec,12}) \\ + r_{\prec,13} \succ (r_{\prec,23} + r_{\succ,23}) - (r_{\prec,23} + r_{\succ,23}) \prec r_{\prec,12}] \\ + (r_{\prec,23} + r_{\succ,23}) \prec [(\mathrm{id} \otimes L_{\prec}(x) \otimes \mathrm{id}) r_{\prec,12}] \\ + (\mathrm{id} \otimes \mathrm{id} \otimes L_{\succ}(x))(-r_{\prec,12} * r_{\prec,13} - r_{\prec,13} \prec r_{\succ,23} + r_{\prec,23} \succ r_{\prec,12}) \\ - [(\mathrm{id} \otimes \mathrm{id} \otimes L_{\succ}(x)) r_{13}] \succ (r_{\succ,23} + r_{\prec,23}) = 0;\end{aligned} \tag{70}$$

$$\begin{aligned}(R_{\prec}(x) \otimes \mathrm{id} \otimes \mathrm{id})(r_{\prec,23} * r_{\succ,12} - r_{\succ,12} \prec r_{\prec,13} - r_{\succ,13} \succ r_{\prec,23}) \\ - (\mathrm{id} \otimes \mathrm{id} \otimes L_{\succ}(x))(r_{\prec,23} * r_{\succ,12} - r_{\succ,12} \prec r_{\prec,13} - r_{\succ,13} \succ r_{\prec,23}) = 0;\end{aligned} \tag{71}$$

$$\begin{aligned}(R_{\prec}(x) \otimes \mathrm{id} \otimes \mathrm{id})(-r_{\succ,13} * r_{\succ,23} + r_{\succ,23} \prec r_{\succ,12} - r_{\prec,12} \succ r_{\succ,13}) \\ - (r_{\succ,12} + r_{\prec,12}) \prec [(R_{\prec}(x) \otimes \mathrm{id} \otimes 1) r_{\succ,13}] \\ + [(\mathrm{id} \otimes R_{\prec}(x) \otimes \mathrm{id}) r_{\succ,23}] \succ (r_{\succ,12} + r_{\prec,12}) \\ + (\mathrm{id} \otimes \mathrm{id} \otimes L(x))[r_{\succ,13} * r_{\succ,23} - r_{\succ,23} \prec r_{\succ,12} + r_{\prec,12} \succ r_{\succ,13} \\ + (r_{\succ,12} + r_{\prec,12}) \prec r_{\succ,13} - r_{\succ,23} \succ (r_{\succ,12} + r_{\prec,12})] = 0.\end{aligned} \tag{72}$$

The operation between two r*s is given in an obvious and similar way as eq.* (4).

Proof. We need to prove that eqs. (67)–(69) are equivalent to eqs. (70)–(72), respectively. Here we only give an explicit proof that eq. (70) holds if and only if eq. (67) holds since the proof of the other two equations is similar. Let $x \in A$. After rearranging the terms suitably, we divide eq. (67) into three parts:

$$(\Delta_{\prec} \otimes \mathrm{id})\Delta_{\prec}(x) - (\mathrm{id} \otimes (\Delta_{\succ} + \Delta_{\prec}))\Delta_{\prec}(x) = (\mathrm{F1}) + (\mathrm{F2}) + (\mathrm{F3}),$$

where

$$\begin{aligned}(\mathrm{F1}) = \sum_{i,j} \{ & (a_i \succ x + a_i \prec x) \otimes [a_j \otimes b_i \succ b_j - (a_j \succ b_i + a_j \prec b_i) \otimes b_j \\ & + c_j \otimes (b_i \succ d_j + b_i \prec d_j) - c_j \prec b_i \otimes d_j] + [a_j \succ (a_i \succ x + a_i \prec x) \\ & + a_j \prec (a_i \succ x + a_i \prec x)] \otimes b_j \otimes b_i \},\end{aligned}$$

$$\begin{aligned}(\mathrm{F2}) = \sum_{i,j} \{ & a_i \otimes [a_j \succ (x \succ b_i) + a_j \prec (x \succ b_i)] \otimes b_j \\ & + a_i \otimes c_j \prec (x \succ b_i) \otimes d_j - a_j \otimes (a_i \succ x + a_i \prec x) \succ b_j \otimes b_i \},\end{aligned}$$

$$\begin{aligned}(\mathrm{F3}) = \sum_{i,j} \{ & [a_i \otimes (a_i \succ b_j) - (a_j \succ a_i + a_j \prec a_i) \otimes b_j] \otimes (x \succ b_i) - a_i \otimes a_j \\ & \otimes [(x \succ b_i) \succ b_j] - a_i \otimes c_j \otimes [(x \succ b_i) \succ d_j + (x \succ b_i) \prec d_j] \}.\end{aligned}$$

On the other hand,

$$\begin{aligned}
(\mathrm{F1a}) &= (R(x)\otimes \mathrm{id}\otimes \mathrm{id})(r_{\prec,12} * r_{\prec,13})\\
&= \sum_{i,j}[(a_i * a_j) * x \otimes b_i \otimes b_j]\\
&= \sum_{i,j}[a_j \succ (a_i \succ x + a_i \prec x) + a_j \prec (a_i \succ x + a_i \prec x)] \otimes b_j \otimes b_i],\\
(\mathrm{F1b}) &= (R(x)\otimes \mathrm{id}\otimes \mathrm{id})(r_{\prec,13} \prec r_{\succ,23})\\
&= \sum_{i,j}[(a_i * x)\otimes c_j \otimes (b_i \prec d_j)]\\
&= \sum_{i,j}[(a_i \succ x + a_i \prec x)\otimes c_j \otimes (b_i \prec d_j)],\\
(\mathrm{F1c}) &= (R(x)\otimes \mathrm{id}\otimes \mathrm{id})(-r_{\prec,23} \succ r_{\prec,12})\\
&= \sum_{i,j}[-(a_i * x)\otimes (a_j \succ b_i)\otimes b_j]\\
&= \sum_{i,j}[-(a_i \succ x + a_i \prec x)\otimes (a_j \succ b_i)\otimes b_j],\\
(\mathrm{F1d}) &= (R(x)\otimes \mathrm{id}\otimes \mathrm{id})[r_{\prec,13} \succ (r_{\prec,23} + r_{\succ,23})]\\
&= \sum_{i,j}\{(a_i \succ x + a_i \prec x)\otimes [a_j \otimes (b_i \succ b_j) + c_j \otimes (b_i \succ d_j)]\},\\
(\mathrm{F1e}) &= (R(x)\otimes \mathrm{id}\otimes \mathrm{id})[-(r_{\prec,23} + r_{\succ,23}) \prec r_{\prec,12}]\\
&= -\sum_{i,j}\{(a_i \succ x + a_i \prec x)\otimes [(a_j \prec b_i)\otimes b_j + (c_j \prec b_i)\otimes d_j]\},\\
(\mathrm{F2}') &= (r_{\prec,23} + r_{\succ,23}) \prec [(\mathrm{id}\otimes L_{\prec}(x)\otimes \mathrm{id})r_{\prec,12}]\\
&= \sum_{i,j} a_i \otimes [a_j \prec (x \succ b_i)\otimes b_j + c_j \prec (x \succ b_i)]\otimes d_j],\\
(\mathrm{F3a}) &= (\mathrm{id}\otimes \mathrm{id}\otimes L_{\succ}(x))(-r_{\prec,12} * r_{\prec,13}) = \sum_{i,j} -a_i * a_j \otimes b_i \otimes (x \succ b_j)\\
&= \sum_{i,j} -[(a_i \succ a_j + a_i \prec a_j)\otimes b_i \otimes (x \succ b_j),\\
(\mathrm{F3b}) &= (\mathrm{id}\otimes \mathrm{id}\otimes L_{\succ}(x))(-r_{\prec,13} \prec r_{\succ,23}) = \sum_{i,j} -[a_i \otimes c_j \otimes x \succ (b_i \prec d_j)],\\
(\mathrm{F3c}) &= (\mathrm{id}\otimes \mathrm{id}\otimes L_{\succ}(x))(r_{\prec,23} \succ r_{\prec,12}) = \sum_{i,j}[a_i \otimes (a_j \succ b_i)\otimes (x \succ b_j)],\\
(\mathrm{F3d}) &= -[(\mathrm{id}\otimes \mathrm{id}\otimes L_{\succ}(x))r_{13}] \succ (r_{\succ,23} + r_{\prec,23})\\
&= -\sum_{i,j} a_i \otimes [a_j \otimes (x \succ b_i) \succ b_j + c_j \otimes (x \succ b_i) \succ d_j].
\end{aligned}$$

It is obvious that

$$\begin{aligned}
(\mathrm{F1}) &= (\mathrm{F1a}) + (\mathrm{F1b}) + (\mathrm{F1c}) + (\mathrm{F1d}) + (\mathrm{F1e}),\\
(\mathrm{F2}) &= (\mathrm{F2})',
\end{aligned}$$

$$\text{(F3)} = \text{(F3a)} + \text{(F3b)} + \text{(F3c)} + \text{(F3d)}.$$

Therefore eq. (70) holds if only if eq. (67) holds. □

Combining Propositions 4.3.1 and 4.3.3, we obtain the following conclusion.

Theorem 4.3.4. *Let $(A, \succ, \prec)$ be a dendriform algebra and $r_\succ, r_\prec \in A \otimes A$. Then the linear maps $\Delta_\succ$, $\Delta_\prec$ defined by eqs. (61) and (62) induce a dendriform algebra structure on A^* such that (A, A^*) is a dendriform D-bialgebra if and only if $r_\succ$ and $r_\prec$ satisfy eqs. (63)–(65) and (70)–(72).*

Definition 4.3.5. A dendriform D-bialgebra (A, A^*) is called *coboundary* if its structure is given by $r_\succ, r_\prec \in A \otimes A$ through Theorem 4.3.4.

Theorem 4.3.6. *Let $(A, A^*, \Delta_\succ, \Delta_\prec, \beta_\succ, \beta_\prec)$ be a dendriform D-bialgebra. Then there is a canonical dendriform bialgebra structure on the direct sum $A \oplus A^*$ of the underlying vector spaces of A and A^* such that both the inclusions $i_1 \colon A \to A \oplus A^*$ and $i_2 \colon A^* \to A \oplus A^*$ into the two summands are homomorphisms of dendriform D-bialgebras, where the dendriform D-bialgebra structure on A^* is given in Example 4.2.7.*

Proof. Let $r = \sum_i e_i \otimes e_i^* \in A \otimes A^* \subset (A \oplus A^*) \otimes (A \oplus A^*)$ which corresponds to the identity map $\mathrm{id} \colon A \to A$, where $\{e_1, \dots, e_n\}$ is a basis of A and $\{e_1^*, \dots, e_n^*\}$ is its dual basis. Suppose that the dendriform D-bialgebra structure "$\succ, \prec$" on $A \oplus A^*$ is given by

$$\mathcal{D}\mathcal{D}(A) = A \bowtie^{R^*_{\succ_A}+R^*_{\prec_A}, -L^*_{\prec_A}, -R^*_{\succ_A}, L^*_{\succ_A}+L^*_{\prec_A}}_{R^*_{\succ_{A^*}}+R^*_{\prec_{A^*}}, -L^*_{\prec_{A^*}}, -R^*_{\succ_{A^*}}, L^*_{\succ_{A^*}}+L^*_{\prec_{A^*}}} A^*.$$

Then we have, for any $x, y \in A$, $a, b \in A^*$,

$$\begin{gathered} x \succ y = x \succ_A y, \quad x \prec y = x \prec_A y, \quad x \succ a = R_A^*(x)a - L^*_{\prec_{A^*}}(a)x, \\ x \prec a = -R^*_{\succ_A}(x)a + L^*_{A^*}(a)x, \quad a \succ x = R^*_{A^*}(a)x - L^*_{\prec_A}(x)a, \\ a \prec x = -R^*_{\succ_{A^*}}(a)x + L^*_A(x)a, \quad a \succ b = a \succ_{A^*} b, \quad a \prec b = a \prec_{A^*} b. \end{gathered}$$

If $r_\succ = r$ and $r_\prec = -r$ satisfies eqs. (63)–(65) and (70)–(72), then

$$\begin{aligned} \Delta_{\mathcal{D}\mathcal{D},\succ}(u) &= (\mathrm{id} \otimes L(u) - R_\prec(u) \otimes \mathrm{id})(r_\succ), \\ \Delta_{\mathcal{D}\mathcal{D},\prec}(u) &= (\mathrm{id} \otimes L_\prec(u) - R(x) \otimes \mathrm{id})(r_\prec), \end{aligned}$$

for all $u \in \mathcal{D}\mathcal{D}(A)$, can induce a dendriform D-bialgebra structure on $\mathcal{D}\mathcal{D}(A)$.

In fact, we have

$$r_\prec + r_\succ = 0, \quad r_\prec + \sigma(r_\succ) = \sum_i (-e_i \otimes e_i^* + e_i^* \otimes e_i).$$

Therefore eq. (64) holds automatically. By a similar proof as of Theorem 2.3.6, it follows that eqs. (63) and (65) hold and

$$\begin{aligned} r_{12} * r_{13} - r_{13} \prec r_{23} - r_{23} \succ r_{12} &= -r_{23} * r_{12} + r_{12} \prec r_{13} + r_{13} \succ r_{23} \\ &= -r_{13} * r_{23} + r_{23} \prec r_{12} + r_{12} \succ r_{13} \\ &= 0. \end{aligned}$$

So eqs. (70)–(72) are satisfied. Hence $\mathcal{DD}(A)$ is a dendriform D-bialgebra. Furthermore, for $e_k \in A$, we have

$$\begin{aligned} \Delta_{\mathcal{DD},\succ}(e_k) &= \sum_i [e_i \otimes e_k * e_i^* - (e_i \prec e_k) \otimes e_i^*] \\ &= \sum_{i,j} \langle e_k, e_i^* \succ e_j^* \rangle e_i \otimes e_j \\ &= \Delta_\succ(e_k), \\ \Delta_{\mathcal{DD},\prec}(e_k) &= \sum_i [-e_i \otimes e_k \prec e_i^* + (e_i * e_k) \otimes e_i^*] \\ &= \sum_{i,j} \langle e_k, e_i^* \prec e_j^* \rangle e_i \otimes e_j \\ &= \Delta_\prec(e_k). \end{aligned}$$

Therefore the inclusion $i_1 : A \to A \oplus A^*$ is a homomorphism of dendriform D-bialgebras. Similarly, the inclusion $i_2 : A^* \to A \oplus A^*$ is also a homomorphism of dendriform D-bialgebras, where the dendriform D-bialgebra structure on A^* is given in Example 4.2.7. □

Definition 4.3.7. Let (A, A^*) be a dendriform D-bialgebra. With the dendriform D-bialgebra structure given in Theorem 4.3.6, $A \oplus A^*$ is called a *dendriform double* of A. We denote it by $\mathcal{DD}(A)$.

Corollary 4.3.8. *Let (A, A^*) be a dendriform D-bialgebra. Then the dendriform double $\mathcal{DD}(A)$ of A is a dendriform D-bialgebra and the bilinear form ω given by eq. (7) is a Connes cocycle.*

We would like to point out here that, unlike the symmetry of 1-cocycles of A and A^* appearing in the definition of a dendriform D-bialgebra (A, A^*), it is not necessary that β is also a 1-coboundary of A^* for a coboundary dendriform D-bialgebra $(A, A^*, \Delta_\succ, \Delta_\prec, \beta_\succ, \beta_\prec)$, where $\Delta_\succ$, $\Delta_\prec$ are given by eqs. (61) and (62).

4.4. The *D*-equation and its properties. In this subsection, we consider some simple and special cases to satisfy the eqs. (63)–(65) and (70)–(72).

At first, due to eq. (63), we consider the condition

$$r_\prec = r, \quad r_\succ = -\sigma(r), \quad r \in A \otimes A. \tag{73}$$

Corollary 4.4.1. *Let $(A, \succ, \prec)$ be a dendriform algebra and $r = \sum_i a_i \otimes b_i \in A \otimes A$. Then the maps $\Delta_\succ$, $\Delta_\prec$ defined by eqs.* (61) *and* (62) *with $r_\succ$, $r_\prec$ satisfying eq.* (73) *induce a dendriform algebra structure on A^* such that (A, A^*) is a dendriform D-bialgebra if and only if r satisfies the following equations.*

$$[P(x \succ y) - (\mathrm{id} \otimes L_\succ(x))P(y)](r - \sigma(r)) = 0, \tag{74}$$

$$\sigma(P(x))P(y)(r - \sigma(r)) = 0, \tag{75}$$

$$\begin{aligned}(R(x) \otimes \mathrm{id} \otimes \mathrm{id} - \mathrm{id} \otimes \mathrm{id} \otimes L_\succ(x))[(r_{12} * r_{13} - r_{13} \prec r_{32} - r_{23} \succ r_{12})\\ + \sum_i (a_i * x) \otimes P(b_i)(r - \sigma(r)) - a_i \otimes [P(x \succ b_i)(r - \sigma(r))] = 0,\end{aligned} \tag{76}$$

$$(R_\prec(x) \otimes \mathrm{id} \otimes \mathrm{id} - \mathrm{id} \otimes \mathrm{id} \otimes L_\prec(x))(-r_{23} * r_{21} + r_{21} \prec r_{13} + r_{31} \succ r_{23}) = 0, \tag{77}$$

$$\begin{aligned}(R_\prec(u) \otimes \mathrm{id} \otimes \mathrm{id} - \mathrm{id} \otimes \mathrm{id} \otimes L(u))(-r_{31} * r_{32} + r_{32} \prec r_{21} + r_{12} \succ r_{31})\\ + \sum_i [P(b_i)(r - \sigma(r)) \otimes x * a_i - P(b_i \prec x)(r - \sigma(r)) \otimes a_i] = 0,\end{aligned} \tag{78}$$

where $x, y \in A$, $P(x) = \mathrm{id} \otimes L_\succ(x) - R_\prec(x) \otimes \mathrm{id}$.

Remark 4.4.2. Let $\sigma_{123}, \sigma_{132} : A \otimes A \otimes A \to A \otimes A \otimes A$ be two linear maps given by

$$\sigma_{123}(x \otimes y \otimes z) = z \otimes x \otimes y, \quad \sigma_{132}(x \otimes y \otimes z) = y \otimes z \otimes x$$

for all $x, y, z \in A$. Then we have

$$\begin{aligned}(r_{23} * r_{21} - r_{21} \prec r_{13} - r_{31} \succ r_{23}) &= \sigma_{123}(r_{12} * r_{13} - r_{13} \prec r_{32} - r_{23} \succ r_{12}),\\ (r_{31} * r_{32} - r_{32} \prec r_{21} - r_{12} \succ r_{31}) &= \sigma_{132}(r_{12} * r_{13} - r_{13} \prec r_{32} - r_{23} \succ r_{12}).\end{aligned}$$

Remark 4.4.3. We can also consider the case $r_\succ + r_\prec = 0$, as we did in the proof of Theorem 4.3.6. Obviously, if in addition, $r_\prec = r$ is symmetric, then we are in the case satisfying eq. (73).

The simplest way to satisfy eqs. (74)–(78) is to assume that r is symmetric and

$$r_{12} * r_{13} = r_{13} \prec r_{23} + r_{23} \succ r_{12}. \tag{79}$$

Corollary 4.4.4. *Let $(A, \succ, \prec)$ be a dendriform algebra and $r \in A \otimes A$. Suppose that r is symmetric and r satisfies eq.* (79). *Then the maps $\Delta_\succ$, $\Delta_\prec$ defined by eqs.* (61) *and* (62) *with $r_\succ = -r$, $r_\prec = r$ induce a dendriform algebra structure on A^* such that (A, A^*) is a dendriform D-bialgebra.*

Definition 4.4.5. Let $(A, \succ, \prec)$ be a dendriform algebra and $r \in A \otimes A$. Eq. (79) is called *D-equation in A*.

By Remark 4.4.2, when r is symmetric, the equivalent forms of the D-equation are given as

$$r_{23} * r_{12} = r_{12} \prec r_{13} + r_{13} \succ r_{23} \quad \text{or} \quad r_{13} * r_{23} = r_{23} \prec r_{12} + r_{12} \succ r_{13}.$$

By a similar proof as of Proposition 2.4.4, we have the following conclusion.

Proposition 4.4.6. *Let $(A, \succ, \prec)$ be a dendriform algebra and let $r \in A \otimes A$ be a symmetric solution of the D-equation in A. Then the dendriform algebra structure and its associated associative algebra structure on the dendriform double $\mathfrak{DD}(A)$ is given from the products in A as follows:*

$$\begin{aligned}
a^* \prec b^* &= -R_{\succ}^*(r(a^*))b^* + L^*(r(b^*))a^*,\\
a^* \succ b^* &= R^*(r(a^*))b^* - L_{\prec}^*(r(b^*))a^*,\\
a^* * b^* &= a^* \succ b^* + a^* \prec b^* = R_{\prec}^*(r(a^*))b^* + L_{\succ}^*(r(b^*))a^*, \qquad (80)\\
x \succ a^* &= x \succ r(a^*) - r(R^*(x)a^*) + R^*(x)a^*,\\
x \prec a^* &= x \prec r(a^*) + r(R_{\succ}^*(x)a^*) - R_{\succ}^*(x)a^*,\\
x * a^* &= x * r(a^*) - r(R_{\prec}^*(x)a^*) + R_{\prec}^*(x)a^*;\\
a^* \succ x &= r(a^*) \succ x + r(L_{\prec}^*(x)a^*) - L_{\prec}^*(x)a^*,\\
a^* \prec x &= r(a^*) \prec x - r(L^*(x)a^*) + L^*(x)a^*,\\
a^* * x &= r(a^*) * x - r(L_{\succ}^*(x)a^*) + L_{\succ}^*(x)a^*
\end{aligned}$$

for any $x \in A$, $a^*, b^* \in A^*$.

Theorem 4.4.7. *Let $(A, \succ, \prec)$ be a dendriform algebra and $r \in A \otimes A$. Suppose that r is symmetric and non-degenerate. Then r is a solution of the D-equation in A if and only if the inverse of the isomorphism $A^* \to A$ induced by r, regarded as a bilinear form $\mathcal{B}$ on A (that is, $\mathcal{B}(x, y) = \langle r^{-1}x, y\rangle$ for any $x, y \in A$) satisfies*

$$\mathcal{B}(x * y, z) = \mathcal{B}(y, z \prec x) + \mathcal{B}(x, y \succ z) \quad \textit{for all } x, y, z \in A. \qquad (81)$$

Proof. Let $r = \sum_i a_i \otimes b_i$. Since r is symmetric, $r(v^*) = \sum_i \langle v^*, a_i\rangle b_i = \sum_i \langle v^*, b_i\rangle a_i$ for any $v^* \in A^*$. Since r is non-degenerate for any $x, y, z \in A$, there exist $u^*, v^*, w^* \in A^*$ such that $x = r(u^*), y = r(v^*), z = r(w^*)$. Therefore

$$\begin{aligned}
\mathcal{B}(x * y, z) &= \langle r(u^*) * r(v^*), w^*\rangle\\
&= \sum_{i,j} \langle u^*, b_i\rangle\langle v^*, b_j\rangle\langle w^*, a_i * a_j\rangle = \langle w^* \otimes u^* \otimes v^*, r_{12} * r_{13}\rangle,\\
\mathcal{B}(y, z \prec x) &= \langle v^*, r(w^*) \prec r(u^*)\rangle\\
&= \sum_{i,j} \langle u^*, b_i\rangle\langle w^*, b_i\rangle\langle v^*, a_i \prec a_j\rangle = \langle w^* \otimes u^* \otimes v^*, r_{13} \prec r_{23}\rangle,\\
\mathcal{B}(x, y \succ z) &= \langle r(v^*) \succ r(w^*), u^*\rangle\\
&= \sum_{i,j} \langle v^*, b_i\rangle\langle w^*, b_j\rangle\langle u^*, a_i \succ a_j\rangle = \langle w^* \otimes u^* \otimes v^*, r_{23} \succ r_{12}\rangle.
\end{aligned}$$

Therefore $\mathcal{B}$ satisfies eq. (81) if and only if r is a solution of the D-equation in A. $\square$

Definition 4.4.8. Let $(A, \succ, \prec)$ be a dendriform algebra. A bilinear form $\mathcal{B}$ on A is called a 2-cocycle if $\mathcal{B}$ satisfies eq. (81).

Remark 4.4.9. Let $\mathcal{B}$ be 2-cocycle on a dendriform algebra $(A, \succ, \prec)$. Then it is easy to show that $\omega(x, y) = \mathcal{B}(x, y) - \mathcal{B}(y, x)$ (for any $x, y \in A$) is a Connes cocycle of the associated associative algebra $(A, *)$. On the other hand, $\mathcal{B}$ satisfies

$$\mathcal{B}(x \cdot y, z) - \mathcal{B}(x, y \cdot z) = \mathcal{B}(y \cdot x, z) - \mathcal{B}(y, x \cdot z) \tag{82}$$

for all $x, y, z \in A$, where $x \cdot y = x \succ y - y \prec x$ for any $x, y \in A$. Furthermore, $(A, \cdot)$ is a pre-Lie algebra (see Sections 5.2 and 5.3) and a bilinear form on a pre-Lie algebra A satisfying eq. (82) is called a 2-cocycle on A ([Ku2]). Moreover, a pre-Lie algebra A over the real number field $\mathbb{R}$ is called Hessian if there exists a symmetric and positive definite 2-cocycle on A. In geometry, a Hessian manifold M is a flat affine manifold provided with a Hessian metric g, that is, g is a Remanning metric such that for any each point $p \in M$ there exists a C^∞-function φ defined on a neighborhood of p such that $g_{ij} = \frac{\partial^2\varphi}{\partial x^i \partial x^j}$. A Hessian pre-Lie algebra corresponds to an affine Lie group G with a G-invariant Hessian metric ([Sh]). Therefore a symmetric and positive definite 2-cocycle on a real dendriform algebra can give a Hessian structure.

Corollary 4.4.10. *Let $(A, \succ_A, \prec_A)$ be a dendriform algebra and let $r \in A \otimes A$ be a non-degenerate symmetric solution of the D-equation in A. Suppose the dendriform algebra structure "$\succ_{A^*}, \prec_{A^*}$" on A^* is induced by r via Proposition 4.4.6. Then we have*

$$a^* \succ_{A^*} b^* = r^{-1}(r(a^*) \succ_A r(b^*)), \quad a^* \prec_{A^*} b^* = r^{-1}(r(a^*) \prec_A r(b^*))$$

for all $a^*, b^* \in A^*$. Therefore, $r\colon A^* \to A$ is an isomorphism of dendriform algebras.

Proof. The conclusion can be obtained by a similar proof as of Corollary 2.4.6. $\square$

Theorem 4.4.11. *Let $(A, \succ, \prec)$ be a dendriform algebra and $r \in A \otimes A$ symmetric. Then r is a solution of the D-equation in A if and only if r satisfies*

$$r(a^*) * r(b^*) = r(R^*_{\prec}(r(a^*))b^* + L^*_{\succ}(r(b^*))a^*) \quad \textit{for all } a^*, b^* \in A^*.$$

Proof. The conclusion can be obtained by a similar proof as of Theorem 2.4.7. $\square$

Combining Theorem 4.4.11 and Theorem 3.1.2, we have the following conclusion.

Corollary 4.4.12. *Let $(A, \succ, \prec)$ be a dendriform algebra and $r \in A \otimes A$ symmetric. Then r is a solution of the D-equation in A if and only if r is an $\mathcal{O}$-operator of the* *associated associative algebra $(A,*)$ associated to $(R_{\prec}^*, L_{\succ}^*)$. Therefore there is a dendriform algebra structure on A^* given by*

$$a^* \succ b^* = R_{\prec}^*(r(a^*))b^*, \quad a^* \prec b^* = L_{\succ}^*(r(b^*))a^*$$

for all $a^*, b^* \in A^*$. It has the same associated associative algebra as the dendriform algebra on A^* given by eq. (80), which is induced by r in the sense of coboundary dendriform D-bialgebras. If r is non-degenerate, then there is a new compatible dendriform algebra structure on A given by

$$x \succ' y = r(R_{\prec}^*(x)r^{-1}y), \quad x \prec' y = r(L_{\succ}^*(y)r^{-1}x) \quad \text{for all } x, y \in A,$$

which is just the dendriform algebra structure given by

$$\mathcal{B}(x \succ' y, z) = \mathcal{B}(y, z * x), \quad \mathcal{B}(x \prec' y, z) = \mathcal{B}(x, y * z) \quad \text{for all } x, y, z \in A,$$

where $\mathcal{B}$ is the symmetric 2-cocycle on A induced by r^{-1}.

Theorem 4.4.13. *Let $(A,*)$ be an associative algebra and (l, r, V) a bimodule. Let (r^*, l^*, V^*) be the bimodule of A given by Lemma 2.1.2. Suppose that $T: V \to A$ is an $\mathcal{O}$-operator associated to (l, r, V). Then $r = T + \sigma(T)$ is a symmetric solution of the D-equation in $T(V) \ltimes_{r^*,0,0,l^*} V^*$, where $T(V) \subset A$ is a dendriform algebra given by eq. (33) and $(r^*, 0, 0, l^*)$ is a bimodule since its associated associative algebra $T(V)$ is an associative subalgebra of A, and T can be identified with an element in $T(V) \otimes V^* \subset (T(V) \ltimes_{r^*,0,0,l^*} V^*) \otimes (T(V) \ltimes_{r^*,0,0,l^*} V^*)$.*

Proof. Let $\{e_1, \ldots, e_n\}$ be a basis of A. Let $\{v_1, \ldots, v_m\}$ be a basis of V and $\{v_1^*, \ldots, v_m^*\}$ its dual basis. Set $T(v_i) = \sum_{k=1}^n a_{ik} e_k, i = 1, \ldots, m$. Then

$$\begin{aligned} T &= \sum_{i=1}^m T(v_i) \otimes v_i^* \\ &= \sum_{i=1}^m \sum_{k=1}^n a_{ik} e_k \otimes v_i^* \in T(V) \otimes V^* \\ &\subset (T(V) \ltimes_{r^*,0,0,l^*} V^*) \otimes (T(V) \ltimes_{r^*,0,0,l^*} V^*). \end{aligned}$$

Therefore we have

$$\begin{aligned} r_{12} * r_{13} &= \sum_{i,j=1}^m \{T(v_i) * T(v_j) \otimes v_i^* \otimes v_j^* + r^*(T(v_i))v_j^* \otimes v_i^* \otimes T(v_j) \\ &\qquad + l^*(T(v_j))v_i^* \otimes T(v_i) \otimes v_j^*\}, \\ r_{13} \prec r_{23} &= \sum_{i,j=1}^m \{v_i^* \otimes v_j^* \otimes T(v_i) \prec T(v_j) + T(v_i) \otimes v_j^* \otimes l^*(T(v_j))v_i^*\}, \\ r_{23} \succ r_{12} &= \sum_{i,j=1}^m \{T(v_j) \otimes r^*(T(v_i))v_j^* \otimes v_i^* + v_j^* \otimes T(v_i) \succ T(v_j) \otimes v_i^*\}. \end{aligned}$$

On the other hand, we have

$$\begin{aligned}
\sum_{i,j=1} r^*(T(v_i))v_j^* \otimes v_i^* \otimes T(v_j) &= \sum_{i,j=1} v_j^* \otimes v_i^* \otimes T(r(T(v_i))v_j),\\
\sum_{i,j=1} l^*(T(v_j))v_i^* \otimes T(v_i) \otimes v_j^* &= \sum_{i,j=1} v_i^* \otimes T(l(T(v_j))v_i) \otimes v_j^*,\\
\sum_{i,j=1} T(v_i) \otimes v_j^* \otimes l^*(T(v_j))v_i^* &= \sum_{i,j=1} T(l(T(v_j))v_i) \otimes v_j^* \otimes v_i^*,\\
\sum_{i,j=1} T(v_j) \otimes r^*(T(v_i))v_j^* \otimes v_i^* &= \sum_{i,j=1} T(r(T(v_i))v_j) \otimes v_j^* \otimes v_i^*.
\end{aligned}$$

Since T is an $\mathcal{O}$-operator of A associated to (l, r, V) and

$$T(u) \succ T(v) = T(l(T(u))v), \quad T(u) \prec T(v) = T(r(T(v))u)$$

for all $u, v \in V$, it follows that r is a symmetric solution of the D-equation in $T(V) \ltimes_{r^*,0,0,l^*} V^*$. □

Remark 4.4.14. Roughly speaking, a symmetric solution of the D-equation corresponds to the symmetric part of an $\mathcal{O}$-operator, whereas an antisymmetric solution of associative Yang–Baxter equation corresponds to the antisymmetric part of an $\mathcal{O}$-operator.

Corollary 4.4.15. *Let $(A, \succ, \prec)$ be a dendriform algebra. Then*

$$r = \sum_{i=1}^{n} (e_i \otimes e_i^* + e_i^* \otimes e_i) \tag{83}$$

is a symmetric solution of the D-equation in $A \ltimes_{R^*_{\prec},0,0,L^*_{\succ}} A^*$, where $\{e_1, \dots, e_n\}$ is a basis of A and $\{e_1^*, \dots, e_n^*\}$ is its dual basis. Moreover, r is non-degenerate and the induced 2-cocycle $\mathcal{B}$ on $A \ltimes_{R^*_{\prec},0,0,L^*_{\succ}} A^*$ is given by eq. (2).

Proof. Let $V = A$, $l = L_{\succ}$, $r = R_{\prec}$ and $T = \mathrm{id}$ in Theorem 4.4.13. Then the conclusion follows immediately. □

Remark 4.4.16. A comparison with Theorem 4.3.6 shows that (the non-symmetric) $T = \sum_{i=1}^{n} e_i \otimes e_i^*$ induces a dendriform D-bialgebra structure on

$$A \ltimes_{R^*,-L^*_{\prec},-R^*_{\succ},L^*} A^*,$$

whereas the above (symmetric) $r = T + \sigma(T)$ induces a dendriform D-bialgebra structure on $A \ltimes_{R^*_{\prec},0,0,L^*_{\succ}} A^*$.

Recall that two Connes cocycles (A_1, ω_1) and (A_2, ω_2) are isomorphic if and only if there exists an isomorphism of associative algebras $\varphi\colon A_1 \to A_2$ such that

$$\omega_1(x, y) = \varphi^*\omega_2(x, y) = \omega_2(\varphi(x), \varphi(y)) \quad \text{for all } x, y \in A_1.$$

By a similar proof as of Theorem 2.4.9, we have the following conclusion.

Theorem 4.4.17. *Let $(A, \succ, \prec)$ be a dendriform algebra. Then, as Connes cocycles of associative algebras, the double construction of Connes cocycle (or the dendriform D-bialgebra) $(T(A) = A \bowtie A^*, \omega)$ given by a symmetric solution r of the D-equation in A and the double construction of Connes cocycle (or the dendriform D-bialgebra) $(T(A) = A \ltimes_{R^*_{\prec}, L^*_{\succ}} A^*, \omega)$ are isomorphic, where ω is given by eq. (7). However, in general, they are not isomorphic as double constructions of Connes cocycles (or dendriform D-bialgebras).*

Corollary 4.4.18. *Let $(A, \succ, \prec)$ be a dendriform algebra. Then as Connes cocycles of associative algebras, the double constructions of Connes cocycles given by all symmetric solutions of the D-equation in A are isomorphic to the double construction of the Connes cocycle $(T(A) = A \ltimes_{R^*_{\prec}, L^*_{\succ}} A^*, \omega)$ given by the zero solution.*

5. Comparison (duality) between bialgebra structures

5.1. Comparison (duality) between antisymmetric infinitesimal bialgebras and dendriform D-bialgebras. The results in the previous sections allow us to compare antisymmetric infinitesimal bialgebras and dendriform D-bialgebras in terms of the following properties: 1-cocycles of associative algebras, matched pairs of associative algebras, associative algebra structures on the direct sum of the associative algebras in the matched pairs, bilinear forms on the direct sum of the associative algebras in the matched pairs, double structures on the direct sum of the associative algebras in the matched pairs, algebraic equations associated to coboundary cases, non-degenerate solutions, $\mathcal{O}$-operators of associative algebras and constructions from dendriform algebras. We list them in Table 1. From this table, we observe that there is a clear analogy between them and in particular, double constructions of Frobenius algebras correspond to double constructions of Connes cocycles in this sense. Moreover, due to the correspondences between certain symmetries and antisymmetries appearing in the Table 1, we regard it as a kind of duality.

Next we consider the case that a dendriform D-bialgebra is also an antisymmetric infinitesimal bialgebra.

Theorem 5.1.1. *Let $(A, A^*, \Delta_{\succ}, \Delta_{\prec}, \beta_{\succ}, \beta_{\prec})$ be a dendriform D-bialgebra. Then (A, A^*) is an antisymmetric infinitesimal bialgebra if and only if the following two equations hold:*

$$\langle L^*_{\prec_{A^*}}(b^*)y, L^*_{\prec_A}(x)a^*\rangle = \langle R^*_{\succ_{A^*}}(a^*)x, R^*_{\succ_A}(y)b^*\rangle, \tag{84}$$

$$\begin{aligned}\langle L^*_{\prec_{A^*}}(b^*)y, R^*_{\succ_A}(x)a^*\rangle + \langle L^*_{\prec_{A^*}}(a^*)x, R^*_{\succ_A}(y)b^*\rangle \\ = \langle R^*_{\succ_{A^*}}(b^*)x, L^*_{\prec_A}(y)a^*\rangle + \langle R^*_{\succ_{A^*}}(a^*)y, L^*_{\prec_A}(x)b^*\rangle\end{aligned} \tag{85}$$

for any $x, y \in A^*$, $a^*, b^* \in A^*$.

Table 1. Comparison between antisymmetric infinitesimal bialgebras and dendriform D-bialgebras.

Algebras	Antisymmetric infinitesimal bialgebras	Dendriform D-bialgebras
1-cocycles of associative algebras	$(\mathrm{id}\otimes L, R\otimes \mathrm{id})$	$(\mathrm{id}\otimes L_{\succ}, R\otimes \mathrm{id})$, $(\mathrm{id}\otimes L, R_{\prec}\otimes \mathrm{id})$
Matched pairs of associative algebras	$(A, A^*, R_A^*, L_A^*, R_{A^*}^*, L_{A^*}^*)$	$(A, A^*, R_{\prec_A}^*, L_{\succ_A}^*, R_{\prec_{A^*}}^*, L_{\succ_{A^*}}^*)$
Associative algebra structures on the direct sum of the associative algebras in the matched pairs	double constructions of Frobenius algebras	double constructions of Connes cocycles
Bilinear forms on the direct sum of the associative algebras in the matched pairs	symmetric	antisymmetric
	$\langle x+a^*, y+b^*\rangle = \langle x, b^*\rangle + \langle a^*, y\rangle$	$\langle x+a^*, y+b^*\rangle = -\langle x, b^*\rangle + \langle a^*, y\rangle$
	invariant	Connes cocycles
Double structures on the direct sum of the associative algebras in the matched pairs	associative doubles	dendriform doubles
Algebraic equations associated to coboundary cases	antisymmetric solutions	symmetric solutions
	associative Yang–Baxter equations	D-equations in dendriform algebras
Non-degenerate solutions	Connes cocycles of associative algebras	2-cocycles of dendriform algebras
$\mathcal{O}$-operators of associative algebras	associated to (R^*, L^*)	associated to $(R_{\prec}^*, L_{\succ}^*)$
	antisymmetric parts	symmetric parts
Constructions from dendriform algebras	$r = \sum_{i=1}^n (e_i\otimes e_i^* - e_i^*\otimes e_i)$	$r = \sum_{i=1}^n (e_i\otimes e_i^* + e_i^*\otimes e_i)$
	induced bilinear forms $\langle x+a^*, y+b^*\rangle = -\langle x, b^*\rangle + \langle a^*, y\rangle$	induced bilinear forms $\langle x+a^*, y+b^*\rangle = \langle x, b^*\rangle + \langle a^*, y\rangle$

Proof. The conclusion can be obtained by a similar proof as of Proposition 2.2.2. □

Corollary 5.1.2. *Let $(A, \succ, \prec)$ be a dendriform algebra and let $r \in A\otimes A$ be a symmetric solution of the D-equation in A. Suppose the dendriform algebra structure on A^* is induced by r from eq. (80). Then (A, A^*) is an antisymmetric infinitesimal bialgebra if and only if the following two equations hold:*

$$\begin{aligned}&\langle y \prec_A (x \succ_A r(a^*)) - y *_A r(R_{\succ_A}^*(x)a^*), b^*\rangle\\ &\quad = \langle r(L_{\prec_A}^*(y)b^*) *_A x - (r(b^*) \prec_A y) \succ x, a^*\rangle;\end{aligned}$$

$$\begin{aligned}&\langle y \prec_A (r(a^*) \prec_A x) - (y \succ_A r(a^*)) \succ_A x \\ &\qquad\qquad + r(R^*_{\succ_A}(y)a^*) *_A x - y *_A r(L^*_{\prec_A}(x)a^*), b^*\rangle \\ &\quad = \langle -x \prec_A (r(b^*) \prec_A y) + (x \succ_A r(b^*)) \succ_A y - r(R^*_{\succ_A}(x)a^*) *_A y \\ &\qquad\qquad + x *_A r(L^*_{\prec_A}(y)a^*), a^*\rangle,\end{aligned}$$

for any $x, y \in A$ and $a^* \in A^*$.

Corollary 5.1.3. *Let $(A, A^*, \Delta_\succ, \Delta_\prec, \beta_\succ, \beta_\prec)$ be a dendriform D-bialgebra. If eqs. (84) and (85) are satisfied, then there are two associative algebra structures $A \bowtie^{R^*_{\prec_A}, L^*_{\succ_A}}_{R^*_{\prec_{A^*}}, L^*_{\succ_{A^*}}} A^*$ and $A \bowtie^{R^*_A, L^*_A}_{R^*_{A^*}, L^*_{A^*}} A^*$ on the direct sum $A \oplus A^*$ of the underlying vector spaces of A and A^* such that both A and A^* are associative subalgebras and the bilinear form given by eq. (7) is a Connes cocycle on $A \bowtie^{R^*_{\prec_A}, L^*_{\succ_A}}_{R^*_{\prec_{A^*}}, L^*_{\succ_{A^*}}} A^*$ and the bilinear form given by eq. (2) is invariant on $A \bowtie^{R^*_A, L^*_A}_{R^*_{A^*}, L^*_{A^*}} A^*$. Moreover, these two associative algebras are not isomorphic in general.*

Example 5.1.4. Let $(A, *_A)$ be an associative algebra and let ω be a Connes cocycle on $(A, *_A)$. Then there is an antisymmetric infinitesimal bialgebra whose associative algebra structure on A^* is given by a non-degenerate solution r of the associative Yang–Baxter equation as follows:

$$\Delta(x) = (\mathrm{id} \otimes L(x) - R(x) \otimes \mathrm{id})r$$

for all $x \in A$, where $r\colon A^* \to A$ is given by $\omega(x, y) = \langle r^{-1}(x), y\rangle$. On the other hand, there exists a compatible dendriform algebra structure "$\succ_A$, $\prec_A$" on A given by eq. (53), that is,

$$\omega(x \succ_A y, z) = \omega(y, z *_A x), \quad \omega(x \prec_A y, z) = \omega(x, y *_A z) \tag{86}$$

for all $x, y, z \in A$. Moreover, there exists a compatible dendriform algebra structure on the associative algebra A^* given by

$$a^* \succ_{A^*} b^* = r^{-1}(r(a^*) \succ_A r(b^*)), \quad a^* \prec_{A^*} b^* = r^{-1}(r(a^*) \prec_A r(b^*)),$$

for all $a^*, b^* \in A$. Furthermore, it is easy to show that

$$\begin{aligned}L^*_{\succ_A}(x)a^* &= r^{-1}(r(a^*) *_A x),\\ R^*_{\succ_A}(x)a^* &= -r^{-1}(x \prec_A r(a^*)),\\ L^*_{\prec_A}(x)a^* &= -r^{-1}(r(a^*) \succ_A x),\\ R^*_{\prec_A}(x)a^* &= r^{-1}(x * r(a^*)),\\ L^*_{\succ_{A^*}}(a^*)x &= x *_A r(a^*),\\ R^*_{\succ_{A^*}}(a^*)x &= -r(a^*) \prec_A x,\\ L^*_{\prec_{A^*}}(a^*)x &= -x \succ_A r(a^*),\\ R^*_{\prec_{A^*}}(a^*)x &= r(a^*) *_A x\end{aligned}$$

for all $x \in A$, $a^* \in A^*$. Therefore, by Theorem 4.2.4, (A, A^*) (as dendriform algebras) is a dendriform D-bialgebra if and only if $(A, A^*, R^*_{\prec_A}, L^*_{\succ_A}, R^*_{\prec_{A^*}}, L^*_{\succ_{A^*}})$ is a matched pair of associative algebras, which is the case if and only if A is 2-step nilpotent, that is, $x *_A y *_A z = 0$ for any $x, y, z \in A$. In this case, by eq. (86), it is equivalent to

$$x \succ_A (y \succ_A z) = x \prec_A (y \prec_A z) = x \succ_A (y \prec_A z) = 0$$

for all $x, y, z \in A$. Therefore, under such conditions, eqs. (84) and (85) hold naturally.

5.2. Duality in the version of Lie algebras: Lie bialgebras and pre-Lie bialgebras. There is a similar duality for Lie algebras, which was presented in [Bai2]. In order to be self-contained, we give a brief introduction in this subsection. We would like to point out that, although we give the Lie bialgebras and pre-Lie bialgebras as structures similar to antisymmetric infinitesimal bialgebras and dendriform D-bialgebras, here, in fact, it is the Manin triples (Lie bialgebras) that have been first studied and then motivate us to study the other structures.

There are two kinds of important (non-degenerate) bilinear forms on Lie algebras. A bilinear form $\mathcal{B}(\ ,\ )$ on a Lie algebra A is invariant if

$$\mathcal{B}([x, y], z) = \mathcal{B}(x, [y, z]) \quad \text{for all } x, y \in A.$$

A 2-cocycle (symplectic form) on a Lie algebra A is an antisymmetric bilinear from ω satisfying

$$\omega([x, y], z) + \omega([y, z], x) + \omega([z, x], y) = 0 \quad \text{for all } x, y, z \in A.$$

Moreover, the algebras that play a role similar to dendriform algebras in the double constructions of Frobenius algebras and Connes cocycles are pre-Lie algebras. In fact, pre-Lie algebras (or under other names like left-symmetric algebras, quasi-associative algebras, Vinery algebras and so on) are a class of natural algebraic systems appearing in many fields in mathematics and mathematical physics (see the survey article [Bu] and the references therein).

Definition 5.2.1. Let A be a vector space over a field $\mathbb{F}$ with a bilinear product $(x, y) \to xy$. A is called a *pre-Lie algebra* if

$$(xy)z - x(yz) = (yx)z - y(xz) \quad \text{for all } x, y, z \in A.$$

Let A be a pre-Lie algebra. For any $x, y \in A$, let $L(x)$ and $R(x)$ denote the left and right multiplication operator, respectively, that is, $L(x)(y) = xy$, $R(x)(y) = yx$. Let $L\colon A \to \mathfrak{gl}(A)$ with $x \to L(x)$ and $R\colon A \to \mathfrak{gl}(A)$ with $x \to R(x)$ (for every $x \in A$) be two linear maps. For a Lie algebra $\mathcal{G}$, we let $\mathrm{ad}(x)$ denote the adjoint operator, that is, $\mathrm{ad}(x)y = [x, y]$, and $\mathrm{ad}\colon \mathcal{G} \to \mathfrak{gl}(\mathcal{G})$ with $x \to \mathrm{ad}(x)$ is a linear map.

Proposition 5.2.2. *Let A be a pre-Lie algebra.*

(1) *The commutator*

$$[x, y] = xy - yx \quad \text{for all } x, y \in A \tag{87}$$

defines a Lie algebra $\mathcal{G}(A)$, which is called the sub-adjacent Lie algebra of A and A is also called a compatible pre-Lie algebra structure on the Lie algebra $\mathcal{G}(A)$.

(2) *The map $L\colon A \to \mathfrak{gl}(A)$ gives a representation of the Lie algebra $\mathcal{G}(A)$.*

Proposition 5.2.3 ([Chu]). *Let $\mathcal{G}$ be a Lie algebra and let ω be a non-degenerate 2-cocycle on $\mathcal{G}$ (such a Lie algebra called a symplectic Lie algebra). Then there exists a compatible pre-Lie algebra structure on $\mathcal{G}$ defined by*

$$\omega(x * y, z) = -\omega(y, [x, z]) \quad \text{for all } x, y, z \in \mathcal{G}.$$

Next we give the "double constructions" of Lie algebras with non-degenerate invariant bilinear forms or non-degenerate 2-cocycles. In fact, both of them have their own (independent) interest in many fields.

At first, recall that $(\mathcal{G}, \mathcal{H}, \rho, \mu)$ is a matched pair of Lie algebras if $\mathcal{G}$ and $\mathcal{H}$ are Lie algebras and $\rho\colon \mathcal{G} \to \mathfrak{gl}(\mathcal{H})$ and $\mu\colon \mathcal{H} \to \mathfrak{gl}(\mathcal{G})$ are representations satisfying

$$\rho(x)[a, b] - [\rho(x)a, b] - [a, \rho(x)b] + \rho(\mu(a)x)b - \rho(\mu(b)x)a = 0,$$
$$\mu(a)[x, y] - [\mu(a)x, y] - [x, \mu(a)y] + \mu(\rho(x)a)y - \mu(\rho(y)a)x = 0,$$

for any $x, y \in \mathcal{G}$ and $a, b \in \mathcal{H}$. In this case, there exists a Lie algebra structure on the direct sum $\mathcal{G} \oplus \mathcal{H}$ of the underlying vector spaces of $\mathcal{G}$ and $\mathcal{H}$ given by

$$[x + a, y + b] = [x, y] + \mu(a)y - \mu(b)x + [a, b] + \rho(x)b - \rho(y)a$$

for all $x, y \in \mathcal{G}$, $a, b \in \mathcal{H}$. We denote it by $\mathcal{G} \bowtie_{\mu}^{\rho} \mathcal{H}$ or simply $\mathcal{G} \bowtie \mathcal{H}$. Moreover, every Lie algebra which is the direct sum of the underlying vector spaces of two subalgebras can be obtained from a matched pair of Lie algebras as above.

Definition 5.2.4. Let $\mathcal{G}$ be a Lie algebra. Suppose that there is a Lie algebra structure on the direct sum of the underlying vector spaces of $\mathcal{G}$ and its dual space $\mathcal{G}^*$ such that $\mathcal{G}$ and $\mathcal{G}^*$ are Lie subalgebras.

(a) If the natural symmetric bilinear form on $\mathcal{G} \oplus \mathcal{G}^*$ given by eq. (2) is invariant, then $(\mathcal{G} \bowtie \mathcal{G}^*, \mathcal{G}, \mathcal{G}^*)$ is called a (*standard*) *Manin triple*.

(b) If the natural antisymmetric bilinear form on $\mathcal{G} \oplus \mathcal{G}^*$ given by eq. (7) is a 2-cocycle, then it is called a *phase space of the Lie algebra* $\mathcal{G}$ ([Ku1]). $(\mathcal{G} \bowtie \mathcal{G}^*, \mathcal{G}, \mathcal{G}^*)$ is also called a *para-Kähler structure on the Lie algebra* $\mathcal{G} \bowtie \mathcal{G}^*$ ([Kan]).

For a Lie algebra $\mathcal{G}$ and a representation (ρ, V) of $\mathcal{G}$, recall that a 1-cocycle T associated to ρ (and denoted by (ρ, T)) is a linear map from $\mathcal{G}$ to V satisfying

$$T([x, y]) = \rho(x)T(y) - \rho(y)T(x) \quad \text{for all } x, y \in \mathcal{G}.$$

Definition 5.2.5. (a) Let $\mathscr{G}$ be a Lie algebra. A *Lie bialgebra* structure on $\mathscr{G}$ is an antisymmetric linear map $\delta\colon \mathscr{G} \to \mathscr{G} \otimes \mathscr{G}$ such that $\delta^*\colon \mathscr{G}^* \otimes \mathscr{G}^* \to \mathscr{G}^*$ is a Lie bracket on $\mathscr{G}^*$ and δ is a 1-cocycle of $\mathscr{G}$ associated to $\mathrm{ad} \otimes \mathrm{id} + \mathrm{id} \otimes \mathrm{ad}$ with values in $\mathscr{G} \otimes \mathscr{G}$. We denote it by $(\mathscr{G}, \mathscr{G}^*)$ or $(\mathscr{G}, \delta)$.

(b) Let A be a vector space. A *pre-Lie bialgebra* structure on A is a pair of linear maps (Δ, β) such that $\Delta\colon A \to A \otimes A$, $\beta\colon A^* \to A^* \otimes A^*$ and

(1) $\Delta^*\colon A^* \otimes A^* \to A^*$ defines a pre-Lie algebra structure on A^*,

(2) $\beta^*\colon A \otimes A \to A$ defines a pre-Lie algebra structure on A,

(3) Δ is a 1-cocycle of $\mathscr{G}(A)$ associated to $L \otimes \mathrm{id} + \mathrm{id} \otimes \mathrm{ad}$ with values in $A \otimes A$,

(4) β is a 1-cocycle of $\mathscr{G}(A^*)$ associated to $L \otimes \mathrm{id} + \mathrm{id} \otimes \mathrm{ad}$ with values in $A^* \otimes A^*$.

We denote it by (A, A^*, Δ, β) or simply (A, A^*).

Theorem 5.2.6. (a) *Let* $(\mathscr{G}, [\ ,\ ]_{\mathscr{G}})$ *and* $(\mathscr{G}^*, [\ ,\ ]_{\mathscr{G}^*})$ *be two Lie algebras. Then the following conditions are equivalent:*

(1) $(\mathscr{G} \bowtie \mathscr{G}^*, \mathscr{G}, \mathscr{G}^*)$ *is a standard Manin triple with the bilinear form* (2).

(2) $(\mathscr{G}, \mathscr{G}^*, \mathrm{ad}^*_{\mathscr{G}}, \mathrm{ad}^*_{\mathscr{G}^*})$ *is a matched pair of Lie algebras.*

(3) $(\mathscr{G}, \mathscr{G}^*)$ *is a Lie bialgebra.*

(b) *Let* $(A, \cdot)$ *and* $(A^*, \circ)$ *be two pre-Lie algebras. Then the following conditions are equivalent:*

(1) $(\mathscr{G}(A) \bowtie \mathscr{G}(A)^*, \mathscr{G}(A), \mathscr{G}(A^*))$ *is a para-Kähler Lie algebra with the bilinear form* (7).

(2) $(\mathscr{G}(A), \mathscr{G}(A^*), L^*_{\cdot}, L^*_{\circ})$ *is a matched pair of Lie algebras.*

(3) (A, A^*) *is a pre-Lie bialgebra.*

In fact, a Lie bialgebra is the Lie algebra $\mathscr{G}$ of a Poisson-Lie group G equipped with additional structures induced from the Poisson structure on G, and a Poisson-Lie group is a Lie group with a Poisson structure compatible with the group operation in a certain sense. Poisson-Lie groups play an important role in symplectic geometry and quantum group theory (cf. [D] and the references therein). On the other hand, in geometry, a para-Kähler manifold is a symplectic manifold with a pair of transversal Lagrangian foliations ([Li]). A para-Kähler Lie algebra $\mathscr{G}$ is the Lie algebra of a Lie group G with a G-invariant para-Kähler structure ([Kan]).

We have already obtained many properties of Lie bialgebras and pre-Lie algebras which are similar to our study in the previous sections. We collect them in the Appendix and we compare pre-Lie bialgebras and Lie bialgebras in terms of their certain properties in Table 2. From Table 2, we observe that there is also a clear analogy between them and in particular, due to the correspondences between certain symmetries and antisymmetries appearing in the analogy, we can regard it as a kind of duality again which is similar to the duality appearing in the Table 1.

Table 2. Comparison between Lie bialgebras and pre-Lie bialgebras.

Algebras	Lie bialgebras	Pre-Lie bialgebras
Corresponding Lie groups	Poisson-Lie groups	para-Kähler Lie groups
1-cocycles of Lie algebras	$\mathrm{id} \otimes \mathrm{ad} + \mathrm{ad} \otimes \mathrm{id}$	$L \otimes \mathrm{id} + \mathrm{id} \otimes \mathrm{ad}$
Matched pairs of Lie algebras	$(\mathcal{G}, \mathcal{G}^*, \mathrm{ad}^*_{\mathcal{G}}, \mathrm{ad}^*_{\mathcal{G}^*})$	$(\mathcal{G}(A), \mathcal{G}(A^*), L^*_A, L^*_{A^*})$
Lie algebra structures on the direct sum of the Lie algebras in the matched pairs	Manin triples	phase spaces
Bilinear forms on the direct sum of the Lie algebras in the matched pairs	symmetric	antisymmetric
	$\langle x + a^*, y + b^* \rangle = \langle x, b^* \rangle + \langle a^*, y \rangle$	$\langle x + a^*, y + b^* \rangle = -\langle x, b^* \rangle + \langle a^*, y \rangle$
	invariant	2-cocycles
Double structures on the direct sum of the Lie algebras in the matched pairs	Drinfeld doubles	symplectic doubles
Algebraic equations associated to coboundary cases	antisymmetric solutions	symmetric solutions
	classical Yang–Baxter equations in Lie algebras	S-equations in pre-Lie algebras
Non-degenerate solutions	2-cocycles of Lie algebras	2-cocycles of pre-Lie algebras
	symplectic structures	Hessian structures
$\mathcal{O}$-operators of Lie algebras	associated to ad^*	associated to L^*
	antisymmetric parts	symmetric parts
Constructions from pre-Lie algebras	$r = \sum_{i=1}^n (e_i \otimes e_i^* - e_i^* \otimes e_i)$	$r = \sum_{i=1}^n (e_i \otimes e_i^* + e_i^* \otimes e_i)$
	induced bilinear forms $\langle x + a^*, y + b^* \rangle = -\langle x, b^* \rangle + \langle a^*, y \rangle$	induced bilinear forms $\langle x + a^*, y + b^* \rangle = \langle x, b^* \rangle + \langle a^*, y \rangle$

5.3. Relationships among four bialgebras

Proposition 5.3.1 ([Cha1], [A2]). *Let $(A, \succ, \prec)$ be a dendriform algebra. Then there is a pre-Lie algebra structure on $(A, \cdot)$ given by*

$$x \cdot y = x \succ y - y \prec x \quad \textit{for all } x, y \in A. \tag{88}$$

Corollary 5.3.2. *Let $(A, \succ, \prec)$ be a dendriform algebra. Then the sub-adjacent Lie algebra of the pre-Lie algebra $(A, \cdot)$ given by eq. (88) is the same as the commutator Lie algebra of the associated associative algebra $(A, *)$, that is,*

$$[x, y] = x * y - y * x = x \cdot y - y \cdot x = x \succ y + x \prec y - y \succ x - y \prec x$$

for all $x, y \in A$.

Therefore, as Chapoton pointed out in [Cha1] (also see [A2], [A4], [EMP]), there is the following commutative diagram of categories:

$$\begin{array}{ccc} \text{dendriform algebras} & \longrightarrow & \text{pre-Lie algebras} \\ \downarrow & & \downarrow \\ \text{associative algebras} & \longrightarrow & \text{Lie algebras.} \end{array}$$

In this diagram, the left vertical arrow is given by eq. (32), the top horizontal arrow is given by eq. (88), the bottom arrow is given by eq. (87) since an associative algebra is a special pre-Lie algebra, and the right vertical arrow is given by eq. (87).

Obviously, if a symmetric or antisymmetric bilinear form on an associative algebra is invariant or a Connes cocycle respectively, then it is also invariant or a 2-cocycle on the commutator Lie algebra respectively,.

Theorem 5.3.3. (1) *A double construction of a Frobenius algebra gives a standard Manin triple* (*on the commutator Lie algebra*) *naturally.*

(2) *A double construction of Connes cocycles gives a para-Kähler Lie algebra* (*on the commutator Lie algebra*) *naturally.*

Corollary 5.3.4. (1) *Any antisymmetric infinitesimal bialgebra is a Lie bialgebra* (*in the sense of its commutator Lie algebra*).

(2) *Any dendriform D-bialgebra is a pre-Lie bialgebra* (*in the sense of eq.* (88)).

Corollary 5.3.5. *We have the following relationship among the antisymmetric infinitesimal bialgebras, dendriform algebras, Lie bialgebras and pre-Lie bialgebras:*

$$\begin{array}{ccc} \text{dendriform D-bialgebras} & \hookrightarrow & \text{pre-Lie bialgebras} \\ \text{dual}\updownarrow & & \updownarrow\text{dual} \\ \text{antisymmetric infinitesimal bialgebras} & \hookrightarrow & \text{Lie bialgebras.} \end{array}$$

Here $\updownarrow$ *means the duality given in Sections* 5.1 *and* 5.2*, and* $\hookrightarrow$ *means the inclusion in the sense of Corollary* 5.3.4*.*

Remark 5.3.6. Part (1) of Corollary 5.3.4 and the relation given by the bottom $\hookrightarrow$ in the above diagram were also pointed out in [A3].

Corollary 5.3.7. *Let* $(A, A^*, \Delta_\succ, \Delta_\prec, \beta_\succ, \beta_\prec)$ *be a dendriform D-bialgebra. If eqs.* (84) *and* (85) *hold, then* (A, A^*) *is an antisymmetric infinitesimal bialgebra.* (A, A^*) *is also a pre-Lie bialgebra in the sense of eq.* (88)*. Furthermore, as the commutator Lie algebras,* $(\mathcal{G}(A), \mathcal{G}(A)^*)$ *is a Lie bialgebra. Therefore, there is an* *associative algebra structure and a Lie algebra structure on the direct sum $A \oplus A^*$ of the underlying space of A and A^* such that the natural symmetric bilinear form given by eq.* (2) *is invariant on both of them and the natural antisymmetric bilinear form given by eq.* (7) *is a Connes cocycle on the associative algebra and a* 2*-cocycle on the Lie algebra. Moreover, under such a condition, we have the following commutative diagram:*

$$\begin{tikzcd}
\text{dendriform D-bialgebras} \arrow[r, hook] \arrow[d] & \text{pre-Lie bialgebras} \arrow[d] \\
\text{antisymmetric infinitesimal bialgebras} \arrow[r, hook] & \text{Lie bialgebras.}
\end{tikzcd}$$

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Appendix: Some properties of Lie bialgebras and pre-Lie bialgebras

In this appendix, we list some properties of Lie bialgebras and pre-Lie bialgebras. Most of the results can be found in [Bai2] and the references therein.

Proposition A.1. (a) *Let* $(\mathcal{G}, \mathcal{G}^*)$ *be a Lie bialgebra. Then there is a canonical Lie bialgebra structure on* $\mathcal{G} \oplus \mathcal{G}^*$ *such that the inclusions* $i_1 \colon \mathcal{G} \to \mathcal{G} \oplus \mathcal{G}^*$ *and* $i_2 \colon \mathcal{G}^* \to \mathcal{G} \oplus \mathcal{G}^*$ *into the two summands are homomorphisms of Lie bialgebras, where the Lie bialgebra structure on* $\mathcal{G}^*$ *is given by* $-\delta_{\mathcal{G}^*}$. *Such a structure is called a classical* (*Drinfeld*) *double of* $\mathcal{G}$.

(b) *Let* (A, A^*, Δ, β) *be a pre-Lie bialgebra. Then there is a canonical pre-Lie bialgebra structure on* $A \oplus A^*$ *such that both the inclusions* $i_1 \colon A \to A \oplus A^*$ *and* $i_2 \colon A^* \to A \oplus A^*$ *into the two summands are homomorphisms of pre-Lie bialgebras. Such a structure is called a symplectic double of* A.

Definition A.2. (a) A Lie bialgebra $(\mathcal{G}, \delta)$ is called *coboundary* if δ is a 1-coboundary of $\mathcal{G}$ associated to $\mathrm{ad} \otimes \mathrm{id} + \mathrm{id} \otimes \mathrm{ad}$, that is, there exists an $r \in \mathcal{G} \otimes \mathcal{G}$ such that

$$\delta(x) = (\mathrm{ad}(x) \otimes \mathrm{id} + \mathrm{id} \otimes \mathrm{ad}(x))r \quad \text{for all } x \in \mathcal{G}. \tag{A.1}$$

(b) A pre-Lie bialgebra (A, A^*, Δ, β) is called *coboundary* if Δ is a 1-coboundary of $\mathcal{G}(A)$ associated to $L \otimes \mathrm{id} + \mathrm{id} \otimes \mathrm{ad}$, that is, there exists an $r \in A \otimes A$ such that

$$\Delta(x) = (L(x) \otimes \mathrm{id} + \mathrm{id} \otimes \; ad(x))r \quad \text{for all } x \in A. \tag{A.2}$$

Theorem A.3. (a) *Let* $\mathcal{G}$ *be a Lie algebra and let* $r \in \mathcal{G} \otimes \mathcal{G}$. *Then the map* $\delta \colon \mathcal{G} \to \mathcal{G} \otimes \mathcal{G}$ *defined by eq.* (A.1) *induces a Lie bialgebra structure on* $\mathcal{G}$ *if and only if the following two conditions are satisfied for any* $x \in \mathcal{G}$:

(1) $(\mathrm{ad}(x) \otimes \mathrm{id} + \mathrm{id} \otimes \mathrm{ad}(x))(r + \sigma(r)) = 0$,

(2) $(\mathrm{ad}(x) \otimes \mathrm{id} \otimes \mathrm{id} + \mathrm{id} \otimes \mathrm{ad}(x) \otimes \mathrm{id} + \mathrm{id} \otimes \mathrm{id} \otimes \mathrm{ad}(x))([r_{12}, r_{13}] + [r_{12}, r_{23}] + [r_{13}, r_{23}]) = 0$.

(b) *Let A be a pre-Lie algebra and let $r \in A \otimes A$. Then the map Δ defined by eq.* (A.2) *induces a pre-Lie algebra structure on A^* such that (A, A^*) is a pre-Lie bialgebra if and only if the following two conditions are satisfied for any $x, y \in A$:*

(1) $[P(x \cdot y) - P(x)P(y)](r - \sigma(r)) = 0$,

(2) $Q(x)[[r, r]] = 0$,

where $Q(x) = L(x) \otimes \mathrm{id} \otimes \mathrm{id} + \mathrm{id} \otimes L(x) \otimes \mathrm{id} + \mathrm{id} \otimes \mathrm{id} \otimes \mathrm{ad}(x)$, $P(x) = L(x) \otimes \mathrm{id} + \mathrm{id} \otimes L(x)$ *and*

$$[[r, r]] = r_{13} \cdot r_{12} - r_{23} \cdot r_{21} + [r_{23}, r_{12}] - [r_{13}, r_{21}] - [r_{13}, r_{23}].$$

Corollary A.4. (a) *Let $\mathcal{G}$ be a Lie algebra and $r \in \mathcal{G} \otimes \mathcal{G}$. If r is antisymmetric and r satisfies*

$$[r_{12}, r_{13}] + [r_{12}, r_{23}] + [r_{13}, r_{23}] = 0, \tag{A.3}$$

then the map $\delta\colon \mathcal{G} \to \mathcal{G} \otimes \mathcal{G}$ defined by eq. (A.1) *induces a Lie bialgebra structure on $\mathcal{G}$.*

(b) *Let A be a pre-Lie algebra and $r \in A \otimes A$. Suppose that r is symmetric. Then the map Δ defined by eq.* (A.2) *induces a pre-Lie algebra structure on A^* such that (A, A^*) is a pre-Lie bialgebra if*

$$-r_{12} \cdot r_{13} + r_{12} \cdot r_{23} + [r_{13}, r_{23}] = 0. \tag{A.4}$$

Definition A.5. (a) Let $\mathcal{G}$ be a Lie algebra and $r \in \mathcal{G} \otimes \mathcal{G}$. Eq. (A.3) is called the *classical Yang–Baxter equation* in $\mathcal{G}$.

(b) Let A be a pre-Lie algebra and $r \in A \otimes A$. Eq. (A.4) is called *S-equation* in A.

Let $\mathcal{G}$ be a Lie algebra and let $\rho\colon \mathcal{G} \to \mathfrak{gl}(V)$ be a representation. Recall that a linear map $T: V \to \mathcal{G}$ is called an $\mathcal{O}$-operator of $\mathcal{G}$ associated to ρ if T satisfies

$$[T(u), T(v)] = T(\rho(T(u))v - \rho(T(v))u) \quad \text{for all } u, v \in V.$$

Proposition A.6. (a) *Let $\mathcal{G}$ be a Lie algebra and $r \in \mathcal{G} \otimes \mathcal{G}$.*

(1) *Suppose that r is antisymmetric and non-degenerate. Then r is a solution of the classical Yang–Baxter equation in $\mathcal{G}$ if and only if the isomorphism $\mathcal{G}^* \to \mathcal{G}$ induced by r, regarded as a bilinear form on $\mathcal{G}$, is a 2-cocycle on $\mathcal{G}$.*

(2) *Suppose that r is antisymmetric. Then r is a solution of the classical Yang–Baxter equation in $\mathcal{G}$ if and only if r is an $\mathcal{O}$-operator of $\mathcal{G}$ associated to ad^*, that is, r satisfies*

$$[r(a^*), r(b^*)] = r(\mathrm{ad}^*(r(a^*))b^* - \mathrm{ad}^*(r(b^*))a^*) \quad \textit{for all } a^*, b^* \in \mathcal{G}^*.$$

(b) *Let A be a pre-Lie algebra and $r \in A \otimes A$.*

(1) *Suppose that r is symmetric and non-degenerate. Then r is a solution of S-equation in A if and only if the inverse of the isomorphism $A^* \to A$ induced by r, regarded as a bilinear form $\mathcal{B}$ on A, is a 2-cocycle on A (see eq. (82)).*

(2) *Suppose that r is symmetric. Then r is a solution of S-equation in A if and only if r is an $\mathcal{O}$-operator of $\mathcal{G}(A)$ associated to L^*, that is, r satisfies*

$$[r(a^*), r(b^*)] = r(L_\cdot^*(r(a^*))b^* - L_\cdot^*(r(b^*))a^*) \quad \text{for all } a^*, b^* \in A^*.$$

Lemma A.7. *Let $\mathcal{G}$ be a Lie algebra and let $\rho\colon \mathcal{G} \to \mathfrak{gl}(V)$ be a representation. Let $T\colon V \to \mathcal{G}$ be an $\mathcal{O}$-operator associated to ρ. Then the product*

$$u \circ v = \rho(T(u))v \quad \text{for all } u, v \in V$$

defines a pre-Lie algebra structure on V. Therefore V is a Lie algebra as the sub-adjacent Lie algebra of this pre-Lie algebra and T is a homomorphism of Lie algebras. Furthermore, $T(V) = \{T(v) \mid v \in V\} \subset \mathcal{G}$ is a Lie subalgebra of $\mathcal{G}$ and there is an induced pre-Lie algebra structure on $T(V)$ given by

$$T(u) \cdot T(v) = T(u \circ v) = T(\rho(T(u))v) \quad \text{for all } u, v \in V. \tag{A.5}$$

Moreover, its sub-adjacent Lie algebra structure is just the Lie subalgebra structure of $\mathcal{G}$ and T is a homomorphism of pre-Lie algebras.

Proposition A.8. *Let $\mathcal{G}$ be a Lie algebra and let $\rho\colon \mathcal{G} \to \mathfrak{gl}(V)$ be a representation. Let $\rho^*\colon \mathcal{G} \to \mathfrak{gl}(V^*)$ be the dual representation of ρ.*

(a) *A linear map $T\colon V \to \mathcal{G}$ is an $\mathcal{O}$-operator of $\mathcal{G}$ associated to ρ if and only if $r = T - \sigma(T)$ is an antisymmetric solution of the classical Yang–Baxter equation in $\mathcal{G} \ltimes_{\rho^*} V^*$.*

(b) *Let $T\colon V \to \mathcal{G}$ be an $\mathcal{O}$-operator associated to ρ. Then $r = T + \sigma(T)$ is a symmetric solution of the S-equation in $T(V) \ltimes_{\rho^*,0} V^*$, where $T(V) \subset \mathcal{G}$ is a pre-Lie algebra given by eq. (A.5) and $(\rho^*, 0)$ is a bimodule since its sub-adjacent Lie algebra $\mathcal{G}(T(V))$ is a Lie subalgebra of $\mathcal{G}$, and T can be identified with an element in $T(V) \otimes V^* \subset (T(V) \ltimes_{\rho^*,0} V^*) \otimes (T(V) \ltimes_{\rho^*,0} V^*)$.*

Proposition A.9. *Let $(A, \cdot)$ be a pre-Lie algebra. Let $\{e_1, \ldots, e_n\}$ be a basis of A and $\{e_1^*, \ldots, e_n^*\}$ its dual basis.*

(a) *r given by eq. (31) is an antisymmetric solution of the classical Yang–Baxter equation in $\mathcal{G}(A) \ltimes_{L^*} \mathcal{G}(A)^*$. Moreover, r is non-degenerate and the induced 2-cocycle $\mathcal{B}$ of $\mathcal{G}(A) \ltimes_{L^*} \mathcal{G}(A)^*$ is given by eq. (7).*

(b) *r given by eq. (83) is a symmetric solution of the S-equation in $A \ltimes_{L^*,0} A^*$. Moreover, r is non-degenerate and the induced 2-cocycle $\mathcal{B}$ of $A \ltimes_{L^*,0} A^*$ is given by eq. (2).*

Theorem A.10. *Let (A, A^*, Δ, β) be a pre-Lie bialgebra. Then $(\mathcal{G}(A), \mathcal{G}(A^*))$ is a Lie bialgebra if and only if*

$$\begin{aligned}&\langle R_{\cdot}^*(x)a^*, R_{\circ}^*(b^*)y\rangle + \langle R_{\cdot}^*(x)b^*, R_{\circ}^*(a^*)y\rangle\\&\quad = \langle R_{\cdot}^*(y)b^*, R_{\circ}^*(a^*)x\rangle + \langle R_{\cdot}^*(y)a^*, R_{\circ}^*(b^*)x\rangle,\end{aligned}$$

for any $x, y \in A^*$, $a^*, b^* \in A^*$.

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