

The Heisenberg–Lorentz quantum group

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Abstract. In this article we present a new C^* -algebraic deformation of the Lorentz group. It is obtained by means of the Rieffel deformation applied to $SL(2, \mathbb{C})$. We give a detailed description of the resulting quantum group $\mathbb{G} = (A, \Delta)$ in terms of generators $\hat{\alpha}, \hat{\beta}, \hat{\gamma}, \hat{\delta} \in A^n$ – the quantum counterparts of the matrix coefficients $\alpha, \beta, \gamma, \delta$ of the fundamental representation of $SL(2, \mathbb{C})$. In order to construct $\hat{\beta}$ – the most involved of the four generators – we first define it on the quantum Borel subgroup $\mathbb{G}_0 \subset \mathbb{G}$, then on the quantum complement of the Borel subgroup and finally we perform the gluing procedure. In order to classify representations of the C^* -algebra A and to analyze the action of the comultiplication Δ on the generators $\hat{\alpha}, \hat{\beta}, \hat{\gamma}, \hat{\delta}$ we employ the duality in the theory of locally compact quantum groups.

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1. Introduction

A complete classification of deformations of $SL(2, \mathbb{C})$ on the Hopf $*$ -algebra level was presented in [14]. So far three of the cases contained there have been realized on a deeper, C^* -algebraic level (see [2], [6], [13]). This article is devoted to the C^* -algebraic realization of another case. The method of deformation that we use is the Rieffel deformation which is the same as in the example considered in [2]. Nevertheless, the resulting quantum group $\mathbb{G} = (A, \Delta)$ – the Heisenberg–Lorentz quantum group – is much more complex. One of the difficulties lies in the fact that among the four generators $\hat{\alpha}, \hat{\beta}, \hat{\gamma}, \hat{\delta}$ of the C^* -algebra A only $\hat{\gamma}$ is normal. Also the analysis of the comultiplication Δ is not as straightforward as in the case of [2]. To perform it we use the one-to-one correspondence between representations of the C^* -algebra A and corepresentations of the dual quantum group $\hat{\mathbb{G}}$. This correspondence was also used to describe all representations of A on Hilbert spaces.

Let us briefly describe the contents of the article. In the next section we present the Hopf $*$ -algebraic version of the Heisenberg–Lorentz quantum group. We begin with a description of commutation relations satisfied by generators $\hat{\alpha}, \hat{\beta}, \hat{\gamma}, \hat{\delta}$ – the

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quantum counterparts of the matrix coefficients $\alpha, \beta, \gamma, \delta$ of the fundamental representation of $SL(2, \mathbb{C})$. Formulas for the comultiplication, coinverse and counit on generators are the same as in the classical case. In Section 3 we define Hilbert space representations of the Heisenberg–Lorentz commutation relations. We show that the tensor product of two such representations can be defined. Section 4 is devoted to the construction of the C^* -algebraic version $\mathbb{G} = (A, \Delta)$ of the Heisenberg–Lorentz quantum group. In particular we introduce four affiliated elements $\hat{\alpha}, \hat{\beta}, \hat{\gamma}, \hat{\delta} \in A^\eta$. In order to construct $\hat{\beta}$, we first define it on the quantum Borel subgroup $\mathbb{G}_0 \subset \mathbb{G}$, then on the quantum complement of the Borel subgroup and finally we perform the gluing procedure. Having constructed affiliated elements $\hat{\alpha}, \hat{\beta}, \hat{\gamma}, \hat{\delta}$, we show that they generate A . Moreover, we note that for any representation $\pi \in \text{Rep}(A; \mathcal{H})$ the quadruple $(\pi(\hat{\alpha}), \pi(\hat{\beta}), \pi(\hat{\gamma}), \pi(\hat{\delta}))$ is a Hilbert space representation of the Heisenberg–Lorentz commutation relations. The converse is also true: for any representation $(\tilde{\alpha}, \tilde{\beta}, \tilde{\gamma}, \tilde{\delta})$ of the Heisenberg–Lorentz commutation relations on a Hilbert space \mathcal{H} there exists a unique representation $\pi \in \text{Rep}(A; \mathcal{H})$ such that

$$\pi(\hat{\alpha}) = \tilde{\alpha}, \quad \pi(\hat{\beta}) = \tilde{\beta}, \quad \pi(\hat{\gamma}) = \tilde{\gamma}, \quad \pi(\hat{\delta}) = \tilde{\delta}.$$

At the end of Section 4 we show that the action of Δ on generators has the same form as in the classical case. Appendices collect useful facts concerning the quantization map and the counit in the Rieffel deformation, the complex infinitesimal generator of the Heisenberg group and the product of strongly commuting affiliated elements.

Throughout the article we will freely use the language of C^* -algebras and the theory of locally compact quantum groups. For a locally compact space X , $C_0(X)$ and $C_b(X)$ shall respectively denote the algebra of continuous functions vanishing at infinity and the algebra of continuous bounded functions. If X is also a manifold, then $C^\infty(X)$ denotes the algebra of smooth functions on X and $C_c^\infty(X)$ denotes the algebra of smooth functions of compact supports. For the notion of multipliers, affiliated elements and algebras generated by a family of affiliated elements we refer the reader to [10], [11] and [12]. The set of elements affiliated with a C^* -algebra A will be denoted by A^η and the affiliation relation will be denoted by η , i.e., $T \eta A$ means that $T \in A^\eta$. The z -transform of $T \in A^\eta$ will be denoted by z_T . For the precise definition of z_T we refer to [12]. For the theory of locally compact quantum groups we refer to [3] and [4]. For the theory of quantum groups given by a multiplicative unitary we refer to [1] and [9]. For the notion of Γ -product we refer to [5]. All Hilbert spaces appearing in the article are assumed to be separable. Given a pair of densely defined operators X and Y acting on a Hilbert space \mathcal{H} , the dotted sum $X \dot{+} Y$ is the closure of the usual sum $X + Y$. To define $X \dot{+} Y$ one has to prove that the intersection of domains $D(X) \cap D(Y)$ is dense in \mathcal{H} and $X + Y$ defined on $D(X) \cap D(Y)$ is closable.

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2. Hopf *-algebra level

We fix a deformation parameter $s \in \mathbb{R}$. Let \mathcal{A} be a unital *-algebra generated by four elements $\hat{\alpha}, \hat{\beta}, \hat{\gamma}, \hat{\delta}$ satisfying the following commutation relations:

$$\begin{aligned}
 \hat{\alpha}\hat{\beta} &= \hat{\beta}\hat{\alpha}, & \hat{\alpha}\hat{\gamma} &= \hat{\gamma}\hat{\alpha}, & \hat{\alpha}\hat{\delta} &= \hat{\delta}\hat{\alpha}, \\
 & & \hat{\beta}\hat{\gamma} &= \hat{\gamma}\hat{\beta}, & \hat{\beta}\hat{\delta} &= \hat{\delta}\hat{\beta}, \\
 & & & & \hat{\gamma}\hat{\delta} &= \hat{\delta}\hat{\gamma}, \\
 & & & & \hat{\alpha}\hat{\delta} - \hat{\beta}\hat{\gamma} &= 1; \\
 \hat{\alpha}\hat{\alpha}^* - \hat{\alpha}^*\hat{\alpha} &= -s\hat{\gamma}^*\hat{\gamma}, & \hat{\alpha}\hat{\beta}^* - \hat{\beta}^*\hat{\alpha} &= -s\hat{\gamma}\hat{\delta}^*, & & \\
 \hat{\alpha}\hat{\gamma}^* &= \hat{\gamma}^*\hat{\alpha}, & \hat{\alpha}\hat{\delta}^* &= \hat{\delta}^*\hat{\alpha}, & & \\
 \hat{\beta}\hat{\beta}^* - \hat{\beta}^*\hat{\beta} &= s(\hat{\alpha}^*\hat{\alpha} - \hat{\delta}\hat{\delta}^*), & & & & \\
 \hat{\beta}\hat{\gamma}^* &= \hat{\gamma}^*\hat{\beta}, & \hat{\beta}\hat{\delta}^* - \hat{\delta}^*\hat{\beta} &= s\hat{\gamma}^*\hat{\alpha}, & & \\
 \hat{\gamma}\hat{\gamma}^* &= \hat{\gamma}^*\hat{\gamma}, & \hat{\gamma}\hat{\delta}^* &= \hat{\delta}^*\hat{\gamma}, & & \\
 & & \hat{\delta}\hat{\delta}^* - \hat{\delta}^*\hat{\delta} &= s\hat{\gamma}\hat{\gamma}^*. & &
 \end{aligned} \tag{1}$$

The *-algebra \mathcal{A} was introduced in [14] where it was also proven that it admits the structure of a Hopf *-algebra. The action of the comultiplication $\Delta: \mathcal{A} \rightarrow \mathcal{A} \otimes \mathcal{A}$ on the generators is given by

$$\begin{aligned}
 \Delta(\hat{\alpha}) &= \hat{\alpha} \otimes \hat{\alpha} + \hat{\beta} \otimes \hat{\gamma}, & \Delta(\hat{\beta}) &= \hat{\alpha} \otimes \hat{\beta} + \hat{\beta} \otimes \hat{\delta}, \\
 \Delta(\hat{\gamma}) &= \hat{\gamma} \otimes \hat{\alpha} + \hat{\delta} \otimes \hat{\gamma}, & \Delta(\hat{\delta}) &= \hat{\gamma} \otimes \hat{\beta} + \hat{\delta} \otimes \hat{\delta}.
 \end{aligned}$$

The coinverse $\kappa: \mathcal{A} \rightarrow \mathcal{A}$ is an involutive *-antihomomorphism and its action on the generators is given by

$$\kappa(\hat{\alpha}) = \hat{\delta}, \quad \kappa(\hat{\beta}) = -\hat{\beta}, \quad \kappa(\hat{\gamma}) = -\hat{\gamma}, \quad \kappa(\hat{\delta}) = \hat{\alpha}.$$

Finally, the action of the counit $\varepsilon: \mathcal{A} \rightarrow \mathbb{C}$ on the generators is given by

$$\varepsilon(\hat{\alpha}) = 1, \quad \varepsilon(\hat{\beta}) = 0, \quad \varepsilon(\hat{\gamma}) = 0, \quad \varepsilon(\hat{\delta}) = 1.$$

Note that the formulas defining co-operations on \mathcal{A} coincide with the corresponding formulas for the Hopf *-algebra of polynomial functions on $SL(2, \mathbb{C})$.

3. Hilbert space level

In this section we distinguish a class of representations of commutation relations (1) on a Hilbert space which will be proven to correspond to representations of the C*-algebra A of the Heisenberg–Lorentz quantum group (see Theorem 4.5). Note first

that the pair $(\hat{\alpha}, -s\hat{\gamma}^*\hat{\gamma})$ satisfies the commutation relation defining the Heisenberg Lie algebra (see Appendix C). The same is true of the pair $(\hat{\delta}, s\hat{\gamma}^*\hat{\gamma})$. Furthermore, in the case when $\hat{\gamma}$ is represented by an invertible operator, the equation $\hat{\alpha}\hat{\delta} - \hat{\beta}\hat{\gamma} = 1$ determines $\hat{\beta}$. This gives a motivation for the following definition.

Definition 3.1. Let $\tilde{\alpha}, \tilde{\gamma}, \tilde{\delta}$ be closed operators acting on a Hilbert space \mathcal{H} . We say that the triple $(\tilde{\alpha}, \tilde{\gamma}, \tilde{\delta})$ satisfies the Heisenberg–Lorentz commutation relations if

1. $\tilde{\gamma}$ is normal and $\ker \tilde{\gamma} = \{0\}$;
2. $(\tilde{\alpha}, -s\tilde{\gamma}^*\tilde{\gamma})$ and $(\tilde{\delta}, s\tilde{\gamma}^*\tilde{\gamma})$ are infinitesimal representations of the Heisenberg group \mathbb{H} ;
3. $\tilde{\alpha}, \tilde{\gamma}$ and $\tilde{\delta}$ mutually strongly commute.

For the notion of infinitesimal representation of \mathbb{H} we refer to Definition C.1 and for the notion of strong commutativity we refer to Definition D.1.

Definition 3.1 describes representations of commutation relations (1) in which $\hat{\gamma}$ is represented by an invertible operator $\tilde{\gamma}$. The next definition deals with the representations for which $\tilde{\gamma} = 0$. Note that in this case it is the pair $(\tilde{\beta}, s(\tilde{\alpha}^*\tilde{\alpha} - 1/\tilde{\alpha}^*\tilde{\alpha}))$ that satisfies the Heisenberg Lie algebra relation.

Definition 3.2. Let $\tilde{\alpha}, \tilde{\beta}$ be closed operators acting on a Hilbert space \mathcal{H} . We say that the pair $(\tilde{\alpha}, \tilde{\beta})$ satisfies the Heisenberg–Lorentz commutation relations if

1. $\tilde{\alpha}$ is normal and $\ker \tilde{\alpha} = \{0\}$;
2. $(\tilde{\beta}, s(\tilde{\alpha}^*\tilde{\alpha} - 1/\tilde{\alpha}^*\tilde{\alpha}))$ is an infinitesimal representation of the Heisenberg group \mathbb{H} ;
3. $\tilde{\alpha}$ and $\tilde{\beta}$ strongly commute.

To deal with the general case of representations of Heisenberg–Lorentz commutation relations note that $\hat{\gamma}$ commutes with all of the generators and their adjoints. This fact leads to the idea that any representation of the Heisenberg–Lorentz commutation relations splits into a direct sum of two representations: one with an invertible $\tilde{\gamma}$ and one with $\tilde{\gamma}$ being zero. More precisely we have:

Definition 3.3. Let $\tilde{\alpha}, \tilde{\beta}, \tilde{\gamma}, \tilde{\delta}$ be closed operators acting on a Hilbert space \mathcal{H} , $\tilde{\gamma}$ being normal. By $\mathcal{H}_0, \mathcal{H}_1$ and $\tilde{\gamma}_1$ we denote the kernel of $\tilde{\gamma}$, its orthogonal complement and the restriction of $\tilde{\gamma}$ to \mathcal{H}_1 . We say that the quadruple $(\tilde{\alpha}, \tilde{\beta}, \tilde{\gamma}, \tilde{\delta})$ is a representation of the Heisenberg–Lorentz commutation relations if $\tilde{\alpha}, \tilde{\beta}$ and $\tilde{\delta}$ respect the decomposition $\mathcal{H} = \mathcal{H}_0 \oplus \mathcal{H}_1$, i.e., there exist closed operators $\tilde{\alpha}_0, \tilde{\beta}_0, \tilde{\delta}_0$ acting on \mathcal{H}_0 and $\tilde{\alpha}_1, \tilde{\beta}_1, \tilde{\delta}_1$ acting on \mathcal{H}_1 such that

$$\tilde{\alpha} = \tilde{\alpha}_0 \oplus \tilde{\alpha}_1, \quad \tilde{\beta} = \tilde{\beta}_0 \oplus \tilde{\beta}_1, \quad \tilde{\delta} = \tilde{\delta}_0 \oplus \tilde{\delta}_1,$$

and we have

1. the pair $(\tilde{\alpha}_0, \tilde{\beta}_0)$ satisfies the Heisenberg–Lorentz commutation relations;
2. $\tilde{\alpha}_0$ and $\tilde{\delta}_0$ are mutual inverses: $\tilde{\delta}_0 = \tilde{\alpha}_0^{-1}$;
3. the triple $(\tilde{\alpha}_1, \tilde{\gamma}_1, \tilde{\delta}_1)$ satisfies the Heisenberg–Lorentz commutation relations;
4. $\tilde{\beta}_1 = \tilde{\gamma}_1^{-1}(\tilde{\alpha}_1\tilde{\delta}_1 - 1)$.

Remark 3.4. The product of operators in point 4 above is taken in the sense of Theorem D.2. It is a well-known fact that a representation of the Heisenberg group \mathbb{H} can be decomposed into a direct integral of irreducible representations. In the case of irreducible representations the operator $\tilde{\gamma}$ appearing in Definition 3.1 and $\tilde{\alpha}$ appearing in Definition 3.2 are multiples of identity. This fact will be used in the proof of the next theorem.

As has already been mentioned, the class of representations defined above corresponds to representations of the C^* -algebra A of the Heisenberg–Lorentz quantum group (see Theorem 4.5). Let us recall that for two representations $\pi_1 \in \text{Rep}(A, \mathcal{H})$ and $\pi_2 \in \text{Rep}(A, \mathcal{H}')$ their tensor product is defined by $\pi = (\pi_1 \otimes \pi_2) \circ \Delta \in \text{Rep}(A; \mathcal{H} \otimes \mathcal{H}')$. The next theorem gives a description of the tensor product construction in terms of the Heisenberg–Lorentz commutation relations. This construction will be crucial in the analysis of the comultiplication of Δ on the C^* -algebra level (see Theorem 4.9).

Theorem 3.5. *Let $\tilde{\alpha}, \tilde{\beta}, \tilde{\gamma}, \tilde{\delta}$ be closed operators acting on a Hilbert space \mathcal{H} and let $\tilde{\alpha}', \tilde{\beta}', \tilde{\gamma}', \tilde{\delta}'$ be closed operators acting on a Hilbert space \mathcal{H}' . Assume that $(\tilde{\alpha}, \tilde{\beta}, \tilde{\gamma}, \tilde{\delta}), (\tilde{\alpha}', \tilde{\beta}', \tilde{\gamma}', \tilde{\delta}')$ are representations of the Heisenberg–Lorentz commutation relations. Then the quadruple of operators $(\tilde{\alpha}'', \tilde{\beta}'', \tilde{\gamma}'', \tilde{\delta}'')$ acting on $\mathcal{H} \otimes \mathcal{H}'$ and defined by*

$$\begin{aligned} \tilde{\alpha}'' &= \tilde{\alpha} \otimes \tilde{\alpha}' + \tilde{\beta} \otimes \tilde{\gamma}', \\ \tilde{\beta}'' &= \tilde{\alpha} \otimes \tilde{\beta}' + \tilde{\beta} \otimes \tilde{\delta}', \\ \tilde{\gamma}'' &= \tilde{\gamma} \otimes \tilde{\alpha}' + \tilde{\delta} \otimes \tilde{\gamma}', \\ \tilde{\delta}'' &= \tilde{\gamma} \otimes \tilde{\beta}' + \tilde{\delta} \otimes \tilde{\delta}' \end{aligned}$$

is a representation of the Heisenberg–Lorentz commutation relations on $\mathcal{H} \otimes \mathcal{H}'$.

Proof. First, let us introduce some notation. For any $\varepsilon > 0, z \in \mathbb{C}$ we set

$$f_\varepsilon(z) = \frac{1}{\pi\varepsilon} \exp(-\varepsilon^{-1}|z|^2) \in \mathbb{R}_+.$$

Note that for any $\varepsilon \in \mathbb{R}_+, f_\varepsilon \in L^1(\mathbb{C})$ and the family f_ε is a Dirac delta approximation:

$$\lim_{\varepsilon \rightarrow 0} \int d^2z f_\varepsilon(z)g(z) = g(0),$$

where d^2z is a Haar measure on \mathbb{C} . Let V^a be an irreducible unitary representation of the Heisenberg group \mathbb{H} on a Hilbert space \mathcal{H} (for an explanation of the notation V^a we refer to Appendix C). Smearing the family $V_{z,0}^a$ with a function f_ε we get the family

$$I_\varepsilon^a = \int d^2z f_\varepsilon(z) V_{z,0}^a \tag{2}$$

of bounded operators acting on \mathcal{H} . The following properties of I_ε^a will be used in the course of the proof:

$$\begin{aligned} \text{s-lim}_{\varepsilon \rightarrow 0} I_\varepsilon^a &= 1, \\ \text{Ran}(I_\varepsilon) &\subset D(a^n) \quad \text{for any } n \in \mathbb{N}, \\ \lim_{\varepsilon \rightarrow 0} a^n I_\varepsilon h &= a^n h \quad \text{for any } h \in D(a^n), \end{aligned} \tag{3}$$

where s-lim denotes the limit in the strong topology on $B(\mathcal{H})$.

Let us move on to the main part of the proof. Let $c, c' \in \mathbb{C} \setminus \{0\}$. By Remark 3.4 it is enough to prove our theorem in the following three cases:

$$\begin{aligned} \mathcal{H}_0 &= \mathcal{H}'_0 = \{0\}, & \tilde{\gamma} &= c1 & \text{and} & & \tilde{\gamma}' &= c'1, \\ \mathcal{H}_0 &= \mathcal{H}'_1 = \{0\}, & \tilde{\gamma} &= c1 & \text{and} & & \tilde{\alpha}'_0 &= c'1, \\ \mathcal{H}_1 &= \mathcal{H}'_1 = \{0\}, & \tilde{\alpha}_0 &= c1 & \text{and} & & \tilde{\alpha}'_0 &= c'1. \end{aligned}$$

The notation used above coincides with the notation of Definition 3.3. In what follows we shall treat the first case, leaving the second and third cases to the reader.

Note that the pairs $(1 \otimes c\tilde{\alpha}', -s|cc'|^2)$ and $(c'\tilde{\delta} \otimes 1, s|cc'|^2)$ are infinitesimal representations of \mathbb{H} . For any $z \in \mathbb{C}$ we define a unitary operator

$$U_z = U_{z,0}^{1 \otimes c\tilde{\alpha}'} U_{z,0}^{c'\tilde{\delta} \otimes 1} \in B(\mathcal{H}_1 \otimes \mathcal{H}'_1).$$

It is easy to check that the map

$$\mathbb{C} \ni z \mapsto U_z \in B(\mathcal{H}_1 \otimes \mathcal{H}'_1)$$

is a strongly continuous representation of the group $(\mathbb{C}, +)$. Let T be the corresponding infinitesimal generator. By definition T is a normal operator with the domain

$$D(T) = \{h \in \mathcal{H}_1 \otimes \mathcal{H}'_1 \mid \text{the map } \mathbb{C} \ni z \mapsto U_z h \in \mathcal{H}_1 \otimes \mathcal{H}'_1 \text{ is once differentiable}\}$$

and the action of T on $h \in D(T)$ is given by

$$Th = 2 \frac{\partial}{\partial z} U_z h \Big|_{z=0}. \tag{4}$$

With this definition of T , we have $U_z = e^{i\text{Im}(zT)}$, which explains the factor 2 on the right-hand side of (4). Comparing formulas (4) and (75) we see that $\tilde{\gamma}'' \subset T$. In

order to prove the equality $\tilde{\gamma}'' = T$, it is enough to show that $D(1 \otimes \tilde{\alpha}') \cap D(\tilde{\delta} \otimes 1) \subset \mathcal{H}_1 \otimes \mathcal{H}'_1$, which is a core of $\tilde{\gamma}''$, is also a core of T . For this we use the family of operators $I_\varepsilon^{\tilde{\delta}} \otimes I_\varepsilon^{\tilde{\alpha}'}$ in $B(\mathcal{H}_1 \otimes \mathcal{H}'_1)$. It has the following properties:

$$\begin{aligned} \text{s-}\lim_{\varepsilon \rightarrow 0} (I_\varepsilon^{\tilde{\delta}} \otimes I_\varepsilon^{\tilde{\alpha}'}) &= 1, \\ \text{Ran}(I_\varepsilon^{\tilde{\delta}} \otimes I_\varepsilon^{\tilde{\alpha}'}) &\subset D(\tilde{\delta} \otimes 1) \cap D(1 \otimes \tilde{\alpha}'), \\ \lim_{\varepsilon \rightarrow 0} T(I_\varepsilon^{\tilde{\delta}} \otimes I_\varepsilon^{\tilde{\alpha}'})h &= Th \quad \text{for any } h \in D(T). \end{aligned}$$

The first and second properties are direct consequences of (3), while the third property requires a separate proof which is based on formulas (2) and (4). The fact that $D(1 \otimes \tilde{\alpha}') \cap D(\tilde{\delta} \otimes 1)$ is a core of T is now an immediate consequence of all three properties, hence we have $T = \tilde{\gamma}''$.

In the analysis of β'' we shall use the fact that $\tilde{\gamma}''$ defined above is invertible: $\ker \tilde{\gamma}'' = \{0\}$. Assume on the contrary that $\ker \tilde{\gamma}'' \neq \{0\}$. Using the identity

$$(U_{\frac{\tilde{z}}{c}, 0}^{\tilde{\delta}} \otimes U_{-\frac{\tilde{z}}{c}, 0}^{\tilde{\alpha}'})\tilde{\gamma}''(U_{-\frac{\tilde{z}}{c}, 0}^{\tilde{\delta}} \otimes U_{\frac{\tilde{z}}{c}, 0}^{\tilde{\alpha}'}) = \tilde{\gamma}'' + \bar{z},$$

we see that $\tilde{\gamma}''$ has an eigenvector for any complex number. This fact and the normality of $\tilde{\gamma}''$ (eigenvectors of different eigenvalues are perpendicular) contradicts the separability of $\mathcal{H}_1 \otimes \mathcal{H}'_1$, hence $\ker \tilde{\gamma}'' = \{0\}$.

Let us move on to the analysis of the operator $\tilde{\alpha}'' = \tilde{\alpha} \otimes \tilde{\alpha}' \dot{+} \tilde{\beta} \otimes \tilde{\gamma}'$. Our objective is to show that $\tilde{\alpha}''$ is an infinitesimal complex generator of a representation of the Heisenberg group \mathbb{H} (cf. Definition 3.1). In order to do that we define an auxiliary operator

$$T' = \tilde{\gamma}''(c^{-1}\tilde{\alpha} \otimes 1) - c^{-1}c'. \tag{5}$$

Note that $\tilde{\gamma}''$ and $\tilde{\alpha} \otimes 1$ strongly commute, and T' is well defined by Theorem D.2. It is easy to see that $(T', -s\tilde{\gamma}''^* \tilde{\gamma}'')$ is an infinitesimal representation of \mathbb{H} . Hence, to prove that $(\tilde{\alpha}'', -s\tilde{\gamma}''^* \tilde{\gamma}'')$ is also an infinitesimal representation of \mathbb{H} , it is enough to show that $\tilde{\alpha}'' = T'$. For this purpose we use the family of operators

$$I_\varepsilon = I_\varepsilon^{\tilde{\alpha}} I_\varepsilon^{\tilde{\delta}} \otimes I_\varepsilon^{\tilde{\alpha}'} \in B(\mathcal{H}_1 \otimes \mathcal{H}'_1).$$

It has the following properties:

$$\begin{aligned} \text{s-}\lim_{\varepsilon \rightarrow 0} I_\varepsilon &= 1, \\ \text{Ran}(I_\varepsilon) &\subset D(\tilde{\alpha}'') \cap D(T'), \\ T'|_{\text{Ran}(I_\varepsilon)} &= \tilde{\alpha}''|_{\text{Ran}(I_\varepsilon)}, \\ \lim_{\varepsilon \rightarrow 0} T' I_\varepsilon h &= T' h \quad \text{for any } h \in D(T'), \\ \lim_{\varepsilon \rightarrow 0} \tilde{\alpha}'' I_\varepsilon h &= \tilde{\alpha}'' h \quad \text{for any } h \in D(\tilde{\alpha}''). \end{aligned}$$

The third, fourth and fifth properties show that T' and $\tilde{\alpha}''$ coincide on a join core, hence $T' = \tilde{\alpha}''$. Similarly we prove that the operator $\tilde{\delta}'' = \tilde{\gamma} \otimes \tilde{\beta}' + \tilde{\delta} \otimes \tilde{\delta}'$ gives rise to the infinitesimal representation $(\tilde{\delta}'', -s\tilde{\gamma}''^* \tilde{\gamma}'')$ of the Heisenberg group \mathbb{H} .

To complete the proof we have to show that the operator $\tilde{\beta}'' = \tilde{\alpha} \otimes \tilde{\beta}' + \tilde{\beta} \otimes \tilde{\delta}'$ is equal to $\tilde{\gamma}''^{-1}(\tilde{\alpha}''\tilde{\delta}'' - 1)$ (see point 4 of Definition 3.3). In order to do that we use the family of operators

$$J_\varepsilon = I_\varepsilon^{\tilde{\alpha}} I_\varepsilon^{\tilde{\delta}} \otimes I_\varepsilon^{\tilde{\alpha}'} I_\varepsilon^{\tilde{\delta}'} \in B(\mathcal{H}_1 \otimes \mathcal{H}_1').$$

It has the following properties:

$$s - \lim_{\varepsilon \rightarrow 0} J_\varepsilon = 1,$$

$$\text{Ran}(J_\varepsilon) \subset D(\tilde{\gamma}''^{-1}(\tilde{\alpha}''\tilde{\delta}'' - 1)) \cap D(\tilde{\beta}''),$$

$$\tilde{\beta}''|_{\text{Ran}(J_\varepsilon)} = \tilde{\gamma}''^{-1}(\tilde{\alpha}''\tilde{\delta}'' - 1)|_{\text{Ran}(J_\varepsilon)},$$

$$\lim_{\varepsilon \rightarrow 0} \tilde{\beta}'' J_\varepsilon h = \tilde{\beta}'' h \quad \text{for any } h \in D(\tilde{\beta}''),$$

$$\lim_{\varepsilon \rightarrow 0} \tilde{\gamma}''^{-1}(\tilde{\alpha}''\tilde{\delta}'' - 1) J_\varepsilon h = \tilde{\gamma}''^{-1}(\tilde{\alpha}''\tilde{\delta}'' - 1) h \quad \text{for any } h \in D(\tilde{\gamma}''^{-1}(\tilde{\alpha}''\tilde{\delta}'' - 1)).$$

The third, fourth and fifth properties show that $\tilde{\beta}''$ and $\tilde{\gamma}''^{-1}(\tilde{\alpha}''\tilde{\delta}'' - 1)$ coincide on a join core, therefore $\tilde{\beta}'' = \tilde{\gamma}''^{-1}(\tilde{\alpha}''\tilde{\delta}'' - 1)$. □

4. C*-algebra level

In this section we shall describe the Heisenberg–Lorentz quantum group on the C*-algebra level. It is obtained by applying the Rieffel deformation to $SL(2, \mathbb{C})$ (which from now on will be denoted by G). Let us fix an abelian subgroup $\Gamma \subset G$, which will be used in the deformation procedure. Set

$$\Gamma = \left\{ \begin{pmatrix} 1 & z \\ 0 & 1 \end{pmatrix} \mid z \in \mathbb{C} \right\}.$$

Note that Γ is isomorphic to the additive group of complex numbers. In particular Γ and its Pontryagin dual $\hat{\Gamma}$ are isomorphic. The isomorphism that we shall use is given by the non-degenerate pairing on $\mathbb{C} \times \mathbb{C}$:

$$\langle z, z' \rangle = \exp(i \text{Im}(zz')).$$

Let Ψ be the skew bicharacter on $\hat{\Gamma} \simeq \mathbb{C}$ given by

$$\Psi(z, z') = \exp(i \frac{s}{4} \text{Im}(z\bar{z}')). \tag{6}$$

Using the results of Rieffel (see [8]) we know that the abelian subgroup $\Gamma \subset G$ and the bicharacter Ψ on $\hat{\Gamma}$ give rise to a quantum group $\mathbb{G} = (A, \Delta)$. In this article

we shall use a formulation of the Rieffel deformation based on the theory of crossed products which was given in [2]. In this framework the deformation procedure goes as follows. Let $\rho: \Gamma^2 \rightarrow \text{Aut}(C_0(G))$ be the action of Γ^2 given by the left and right shifts of functions on G along Γ ,

$$\rho_{\gamma_1, \gamma_2}(f)(g) = f(\gamma_1^{-1}g\gamma_2),$$

where $\gamma_1, \gamma_2 \in \Gamma$, $f \in C_0(G)$ and $g \in G$. We construct the crossed product C^* -algebra $B = C_0(G) \rtimes_{\rho} \Gamma^2$. Let $\mathbb{B} = (B, \lambda, \hat{\rho})$ be the standard Γ^2 -product structure on B , i.e., $\hat{\rho}$ is the dual action of $\hat{\Gamma}^2$ on B and λ is the representation of Γ^2 on B which implements ρ on $C_0(G) \subset M(B)$:

$$\rho_{\gamma_1, \gamma_2}(f) = \lambda_{\gamma_1, \gamma_2} f \lambda_{\gamma_1, \gamma_2}^*.$$

Let $\hat{\Psi}: \hat{\Gamma} \rightarrow \Gamma$ be the homomorphism given by $\hat{\Psi}(\hat{\gamma}) = \frac{\delta}{4} \bar{\gamma}$ for any $\hat{\gamma} \in \hat{\Gamma}$ (note that we used the identification $\Gamma \cong \hat{\Gamma} \cong \mathbb{C}$). Using $\hat{\Psi}$ we twist $\hat{\rho}$ getting the dual action $\hat{\rho}^{\Psi}: \hat{\Gamma}^2 \rightarrow \text{Aut}(B)$ (in [2] it was denoted by $\hat{\rho}^{\hat{\Psi} \otimes \Psi}$):

$$\hat{\rho}_{\hat{\gamma}, \hat{\gamma}'}^{\Psi}(b) = \lambda_{-\hat{\Psi}(\hat{\gamma}), \hat{\Psi}(\hat{\gamma}')} \hat{\rho}_{\hat{\gamma}, \hat{\gamma}'}(b) \lambda_{-\hat{\Psi}(\hat{\gamma}), \hat{\Psi}(\hat{\gamma}')}^*. \tag{7}$$

As was shown in [2], the triple $\mathbb{B}^{\Psi} = (B, \lambda, \hat{\rho}^{\Psi})$ is also a Γ^2 -product. The C^* -algebra A of the Heisenberg–Lorentz quantum group \mathbb{G} is defined as the Landstad algebra of \mathbb{B}^{Ψ} ; $A \subset M(B)$ is the subalgebra of elements satisfying the Landstad conditions:

$$\begin{aligned} \hat{\rho}_{\hat{\gamma}, \hat{\gamma}'}^{\Psi}(b) &= b \text{ for all } \hat{\gamma}, \hat{\gamma}' \in \hat{\Gamma}, \\ \text{the map } \Gamma^2 \ni (\gamma, \gamma') &\mapsto \lambda_{\gamma, \gamma'} b \lambda_{\gamma, \gamma'}^* \in M(B) \text{ is norm continuous,} \\ x b x' \in B &\text{ for all } x, x' \in C^*(\Gamma) \subset M(B). \end{aligned} \tag{8}$$

The three conditions defining $A \subset M(B)$ will be referred to as the Landstad conditions.

The C^* -algebra A carries the structure of a quantum group. All structure maps can be described in terms of the Γ^2 -product \mathbb{B} , but in this article we shall rather use the fact that they are related to a multiplicative unitary W of \mathbb{G} , whose construction goes as follows. Let dg be a right invariant Haar measure on G and let $L^2(G)$ be the Hilbert space of square-integrable functions with respect to dg . Let $L_g, R_g \in B(L^2(G))$ be the left and right regular representation of G . Restricting them to $\Gamma \subset G$ we get two representations of Γ on $L^2(G)$. The related representations of $C^*(\Gamma)$ will be denoted by $\pi^L, \pi^R \in \text{Rep}(C^*(\Gamma); L^2(G))$, respectively. Obviously $\Psi \in M(C^*(\Gamma) \otimes C^*(\Gamma))$ is unitary, hence operators $X, Y \in B(L^2(G) \otimes L^2(G))$ given by

$$X = (\pi^R \otimes \pi^R)(\Psi), \quad Y = (\pi^R \otimes \pi^L)(\Psi)$$

are unitary, too. Finally, the multiplicative unitary $W \in B(L^2(G) \otimes L^2(G))$ of \mathbb{G} has the the form

$$W = Y V X,$$

where V is the standard Kac–Takesaki operator of the group G . The C^* -algebra A is isomorphic to the C^* -algebra of slices of the first leg of W :

$$A \simeq \{(\omega \otimes \text{id})W \mid \omega \in B(L^2(G))_*\}^{\|\cdot\| \text{-closure}}. \tag{9}$$

Note that A treated as the algebra of slices of W is naturally represented on $L^2(G)$.

4.1. Affiliated element $\hat{\gamma}$. The idea of the construction of $\hat{\gamma} \in A^\eta$ is based on the observation that the normal operator $\gamma \in B^\eta$ is a central element, which by definition means that $z_\gamma \in M(B)$ is a central element. This together with the invariance of γ under $\hat{\rho}$ implies the invariance of γ under the twisted dual action $\hat{\rho}^\Psi$:

$$\hat{\rho}_{\hat{\gamma}, \hat{\gamma}'}^\Psi(\gamma) = \lambda_{-\hat{\Psi}(\hat{\gamma}), \hat{\Psi}(\hat{\gamma}')} \hat{\rho}_{\hat{\gamma}, \hat{\gamma}'}(\gamma) \lambda_{-\hat{\Psi}(\hat{\gamma}), \hat{\Psi}(\hat{\gamma}')}^* = \lambda_{-\hat{\Psi}(\hat{\gamma}), \hat{\Psi}(\hat{\gamma}')} \gamma \lambda_{-\hat{\Psi}(\hat{\gamma}), \hat{\Psi}(\hat{\gamma}')}^* = \gamma.$$

Hence the first Landstad condition defining elements of A is satisfied for γ . We are not dwelling upon the other Landstad conditions but give the following construction of $\hat{\gamma} \in A^\eta$. Let $\text{id}: \mathbb{C} \rightarrow \mathbb{C}$ be the identity function: $\text{id}(z) = z$ for any $z \in \mathbb{C}$. This function generates $C_0(\mathbb{C})$ in the sense of Woronowicz (see Definition 3.1 of [11]). Let $\pi \in \text{Mor}(C_0(\mathbb{C}); C_0(G))$ be the morphism that sends $\text{id} \in C_0(\mathbb{C})^\eta$ to the coordinate function $\gamma \in C_0(G)^\eta$. From the invariance of γ under the action $\rho: \Gamma^2 \rightarrow \text{Aut}(C_0(G))$ it follows that π satisfies the assumptions of Theorem 3.18 of [2] for the trivial action of Γ^2 on $C_0(\mathbb{C})$. Therefore it gives rise to the twisted morphism $\pi^\Psi \in \text{Mor}(C_0(\mathbb{C}); A)$. We define $\hat{\gamma} \in A^\eta$ as the image of $\text{id} \in C_0(\mathbb{C})^\eta$ under π^Ψ : $\hat{\gamma} = \pi^\Psi(\text{id})$. Obviously, the z -transform $z_{\hat{\gamma}}$ belongs to the center of $M(A)$ and $\hat{\gamma}$ treated as an operator acting on $L^2(G)$ (cf. (9)) coincides with the operator of multiplication by the coordinate γ .

4.2. The affiliated elements $\hat{\alpha}$ and $\hat{\delta}$. As has already been mentioned, the couples $(\hat{\alpha}, -s\hat{\gamma}^*\hat{\gamma})$ and $(\hat{\delta}, s\hat{\gamma}^*\hat{\gamma})$ satisfy the Heisenberg Lie algebra relation (see Appendix C). This observation motivates the idea of prescribing $\hat{\alpha}$ and $\hat{\delta}$ as complex infinitesimal generators of appropriately defined representations $U^{\hat{\alpha}}$ and $U^{\hat{\delta}}$ of the Heisenberg group \mathbb{H} on A . In what follows, we show that this approach is justified, but first we need to introduce some notation.

Let T be a normal element affiliated with a C^* -algebra C and let

$$\lambda(z; T) = \exp(i \text{Im}(zT)) \in M(C).$$

Note that the map

$$\mathbb{C} \ni z \mapsto \lambda(z; T) \in M(C)$$

is a representation of the additive group \mathbb{C} . This representation will be denoted by $\lambda(\cdot; T)$. Let $\mathbb{B} = (B, \lambda, \hat{\rho})$ be the Γ^2 -product introduced at the beginning of Section 4. Let $T_l, T_r \in B^\eta$ be infinitesimal generators of representation $\lambda: \mathbb{C}^2 \rightarrow M(B)$:

$$\lambda_{z_1, z_2} = \lambda(z_1, T_l)\lambda(z_2, T_r). \tag{10}$$

The coordinate function $\gamma \in C_0(G)^\eta \subset B^\eta$ – being central – commutes with λ_{z_1, z_2} :

$$\lambda_{z_1, z_2} \gamma \lambda_{z_1, z_2}^* = \gamma.$$

This implies that T_l, T_r and γ strongly commute in the sense of Definition D.1, hence using Theorem D.2 we may construct a pair of normal elements $\gamma T_l^*, \gamma T_r^* \in B^\eta$. We define the aforementioned representations $U^{\hat{\alpha}}$ and $U^{\hat{\delta}}$ of \mathbb{H} in B by the formulas

$$\begin{aligned} U_{z,t}^{\hat{\alpha}} &= \lambda(z; \alpha) \lambda(z; -\frac{s}{4} \gamma T_l^*) \lambda(t; -\frac{s}{4} \gamma^* \gamma), \\ U_{z,t}^{\hat{\delta}} &= \lambda(z; \delta) \lambda(z; -\frac{s}{4} \gamma T_r^*) \lambda(t; \frac{s}{4} \gamma^* \gamma), \end{aligned} \tag{11}$$

where we used the embedding $C_0(G)^\eta \subset B^\eta$ to treat α, δ and $\gamma^* \gamma$ as normal elements affiliated with B . The analysis of $U^{\hat{\alpha}}$ and $U^{\hat{\delta}}$ will be performed in the proof of the next theorem, but before we formulate it, we have to introduce some auxiliary objects. Let ∂_l and ∂_r denote the vector fields whose actions on a function $f \in C^\infty(G)$ are given by

$$\partial_l f(g) = 2 \frac{\partial}{\partial z} L_z(f)(g) \Big|_{z=0}, \quad \partial_r f(g) = 2 \frac{\partial}{\partial z} R_z(f)(g) \Big|_{z=0}. \tag{12}$$

(Here L_z and R_z denote the operators of the left and right shift by an element $z \in \Gamma$.) Using ∂_l and ∂_r we define differential operators $\text{Op}(\alpha)$ and $\text{Op}(\delta)$ acting on $C_c^\infty(G)$:

$$\text{Op}(\alpha) = \alpha - \frac{s}{4} \gamma \partial_l^*, \quad \text{Op}(\delta) = \delta - \frac{s}{4} \gamma \partial_r^*. \tag{13}$$

The quantization map \mathcal{Q} used in the next theorem is described in Appendix A.

Theorem 4.1. *Let $U_{z,t}^{\hat{\alpha}}, U_{z,t}^{\hat{\delta}} \in M(B)$ be the unitary elements given by (11). Let $\text{Op}(\alpha)$ and $\text{Op}(\delta)$ be the differential operators given by (13) and let A be the Landstad algebra of \mathbb{B}^Ψ (see (8)). Then:*

- (1) $U_{z,t}^{\hat{\alpha}}, U_{z,t}^{\hat{\delta}}$ are elements of $M(A) \subset M(B)$.
- (2) The maps

$$\mathbb{H} \ni (z, t) \mapsto U_{z,t}^{\hat{\alpha}} \in M(A), \quad \mathbb{H} \ni (z, t) \mapsto U_{z,t}^{\hat{\delta}} \in M(A)$$

are strictly continuous, commuting representations of the Heisenberg group.

- (3) Let $\hat{\alpha}, \hat{\delta} \in A^\eta$ be complex generators of $U^{\hat{\alpha}}$ and $U^{\hat{\delta}}$ (see Appendix C). The set $\{\mathcal{Q}(f) \mid f \in C_c^\infty(G)\} \subset A$ is a common core of $\hat{\alpha}, \hat{\delta} \in A^\eta$ and we have

$$\hat{\alpha} \mathcal{Q}(f) = \mathcal{Q}(\text{Op}(\alpha)f), \quad \hat{\delta} \mathcal{Q}(f) = \mathcal{Q}(\text{Op}(\delta)f) \tag{14}$$

for any $f \in C_c^\infty(G)$.

Proof. Let us begin by proving that $U_{z,t}^{\hat{\alpha}}, U_{z,t}^{\hat{\delta}} \in M(A)$. Let $\hat{\rho}^\Psi$ be the twisted dual action given by (7). It is easy to check that

$$\begin{aligned} \hat{\rho}_{z_1, z_2}^\Psi(\lambda(z; \alpha)) &= \lambda(z, \frac{s}{4} \bar{z}_1 \gamma) \lambda(z; \alpha), \\ \hat{\rho}_{z_1, z_2}^\Psi(\lambda(z; -\frac{s}{4} \gamma T_l^*)) &= \lambda(z; -\frac{s}{4} \bar{z}_1 \gamma) \lambda(z; -\frac{s}{4} \gamma T_l^*), \\ \hat{\rho}_{z_1, z_2}^\Psi(\lambda(z; \delta)) &= \lambda(z; \frac{s}{4} \bar{z}_2 \gamma) \lambda(z; \delta), \\ \hat{\rho}_{z_1, z_2}^\Psi(\lambda(z; -\frac{s}{4} \gamma T_r^*)) &= \lambda(z; -\frac{s}{4} \bar{z}_2 \gamma) \lambda(z; -\frac{s}{4} \gamma T_r^*). \end{aligned}$$

Using these equalities and (11) we see that $U_{z,t}^{\hat{\alpha}}$ and $U_{z,t}^{\hat{\delta}}$ satisfy the first Landstad condition (see (8)). Let us move on to the second Landstad condition. One can check that

$$\lambda_{z_1, z_2} U_{z,t}^{\hat{\alpha}} \lambda_{z_1, z_2}^* = \lambda(z, -z_1 \gamma) U_{z,t}^{\hat{\alpha}}, \quad \lambda_{z_1, z_2} U_{z,t}^{\hat{\delta}} \lambda_{z_1, z_2}^* = \lambda(z, z_2 \gamma) U_{z,t}^{\hat{\delta}}. \tag{15}$$

In Section 4.1 we constructed the affiliated element $\hat{\gamma} \in A^\eta$. Its image in B^η coincides with the coordinate function $\gamma \in C_0(G)^\eta \subset B^\eta$, hence using (15) we see that the two maps

$$\begin{aligned} \mathbb{C}^2 \ni (z_1, z_2) &\mapsto \lambda_{z_1, z_2} U_{z,t}^{\hat{\alpha}} \lambda_{z_1, z_2}^* a \in M(B), \\ \mathbb{C}^2 \ni (z_1, z_2) &\mapsto \lambda_{z_1, z_2} U_{z,t}^{\hat{\delta}} \lambda_{z_1, z_2}^* a \in M(B) \end{aligned}$$

are norm continuous for any $a \in A$. This norm-continuity together with the $\hat{\rho}^\Psi$ -invariance of $U_{z,t}^{\hat{\alpha}}, U_{z,t}^{\hat{\delta}} \in M(B)$ implies that $U_{z,t}^{\hat{\alpha}}, U_{z,t}^{\hat{\delta}}$ are indeed elements of $M(A)$. The proof that $U^{\hat{\alpha}}$ and $U^{\hat{\delta}}$ are commuting representations of the Heisenberg group \mathbb{H} is left to the reader.

Let us now prove the strict continuity of these representations. For this purpose we shall treat $U_{z,t}^{\hat{\alpha}}$ and $U_{z,t}^{\hat{\delta}}$ as operators acting on $L^2(G)$. It can be checked that the action of $U^{\hat{\alpha}}$ and $U^{\hat{\delta}}$ on $L^2(G)$ expressed in the coordinates α, δ , and γ is given by

$$\begin{aligned} U_{z,t}^{\hat{\alpha}} f(\alpha, \gamma, \delta) &= \exp(-i \frac{st}{4} \bar{\gamma} \gamma) \exp(i \operatorname{Im}(z\alpha)) f(\alpha - \frac{s}{4} \bar{z} \bar{\gamma} \gamma, \gamma, \delta), \\ U_{z,t}^{\hat{\delta}} f(\alpha, \gamma, \delta) &= \exp(i \frac{st}{4} \bar{\gamma} \gamma) \exp(i \operatorname{Im}(z\delta)) f(\alpha, \gamma, \delta + \frac{s}{4} \bar{z} \bar{\gamma} \gamma) \end{aligned} \tag{16}$$

for any $f \in L^2(G)$. Using Theorem 4.14 of [2] we obtain

$$U_{z,t}^{\hat{\alpha}} \mathcal{Q}(f) = \mathcal{Q}(U_{z,t}^{\hat{\alpha}} f) \tag{17}$$

for any $f \in C_c^\infty(G)$. This together with Theorem A.1 leads to the estimation

$$\|U_{z,t}^{\hat{\alpha}} \mathcal{Q}(f) - \mathcal{Q}(f)\| \leq c \max_{k, k', l, l' \leq 5} \sup_{g \in G} |\partial_l^{k*} \partial_l^{k'} \partial_r^{*m} \partial_r^{m'} (U_{z,t}^{\hat{\alpha}} f - f)|.$$

Using (16) we may see that the right hand side of the above inequality is convergent to zero when $(z, t) \rightarrow (0, 0)$, which shows that

$$\lim_{(z,t) \rightarrow (0,0)} \|U_{z,t}^{\hat{\alpha}} \mathcal{Q}(f) - \mathcal{Q}(f)\| = 0.$$

The density of the set $\{\mathcal{Q}(f) \mid f \in C_c^\infty(G)\}$ in A ensures that $U^{\hat{\alpha}}$ is continuous in the sense of strict topology of $M(A)$. The strict continuity of the representation $U^{\hat{\delta}}$ is shown similarly.

Let us move on to the proof of the third point of our theorem. Using equation (16) one can check that

$$2 \frac{\partial}{\partial z} U_{z,0}^{\hat{\alpha}} f \Big|_{z=0} = \text{Op}(\alpha) f$$

for any $f \in C_c^\infty(G)$. This together with (17) and (76) proves the first equation of (14). The second formula of (14) is proved similarly. To show that $\mathcal{Q}(C_c^\infty(G))$ is a core of either $\hat{\alpha}$ or $\hat{\delta}$, it is enough to check that the sets

$$\{(1 + \hat{\alpha}^* \hat{\alpha}) \mathcal{Q}(f) \mid f \in C_c^\infty(G)\}, \quad \{(1 + \hat{\delta}^* \hat{\delta}) \mathcal{Q}(f) \mid f \in C_c^\infty(G)\} \quad (18)$$

are dense in A (see Lemma D.3). In what follows we shall sketch the proof of the density for the first of these sets. Let us first note that the set

$$(1 + \hat{\alpha}^* \hat{\alpha})^{-1} \mathcal{Q}(C_c^\infty(G)) = \{(1 + \hat{\alpha}^* \hat{\alpha})^{-1} \mathcal{Q}(f) \mid f \in C_c^\infty(G)\} \subset A$$

is dense in A . This follows from the density of $\mathcal{Q}(C_c^\infty(G))$ in A and the fact that $\hat{\alpha} \in A^\eta$. Let f be an arbitrary element of $C_c^\infty(G)$ and $g \in L^2(G)$ the function given by

$$g = (1 + \hat{\alpha}^* \hat{\alpha})^{-1} f.$$

Using (74) we see that

$$g = \int_{\mathbb{R}_+} dt \exp(-t) \int_{\mathbb{C}} d^2 z h_t(z, -s \frac{1}{2} \hat{\gamma}^* \hat{\gamma}) U_{z,0}^{\hat{\alpha}} f. \quad (19)$$

One can check that g is quantizable in the sense of Theorem A.1, $\mathcal{Q}(g) \in D(\hat{\alpha}^* \hat{\alpha})$ and $\mathcal{Q}(f) = (1 + \hat{\alpha}^* \hat{\alpha}) \mathcal{Q}(g)$. Using formula (19) we can prove the existence of a sequence $f_n \in C_c^\infty(G)$ such that

$$\begin{aligned} \lim_{n \rightarrow \infty} \partial_l^k \partial_l^{*k'} \partial_r^m \partial_r^{*m'} f_n &= \partial_l^k \partial_l^{*k'} \partial_r^m \partial_r^{*m'} g, \\ \lim_{n \rightarrow \infty} \partial_l^k \partial_l^{*k'} \partial_r^m \partial_r^{*m'} (1 + \text{Op}(\alpha)^* \text{Op}(\alpha)) f_n &= \partial_l^k \partial_l^{*k'} \partial_r^m \partial_r^{*m'} (1 + \text{Op}(\alpha)^* \text{Op}(\alpha)) g \end{aligned} \quad (20)$$

for any $k, k', m, m' \leq 5$. By equations (14), (20), Theorem (A.1) and the closedness of $\hat{\alpha}$ we get

$$\mathcal{Q}(f) = \lim_{n \rightarrow \infty} (1 + \hat{\alpha}^* \hat{\alpha}) \mathcal{Q}(f_n).$$

Using the fact that f is an arbitrary smooth function of compact support and that $\mathcal{Q}(C_c^\infty(G))$ is a dense subset of A we get

$$\overline{(1 + \hat{\alpha}^* \hat{\alpha}) \mathcal{Q}(C_c^\infty(G))}^{\|\cdot\|} = A.$$

This ends the proof of (18) for $\hat{\alpha}$. □

4.3. Quantum Borel subgroup \mathbb{G}_0 . This section is a preparation for the construction of the affiliated element $\hat{\beta} \in A^\eta$. For this purpose we have to split A into appropriately defined quantum subspaces. This splitting corresponds to the splitting of the classical group G into its Borel subgroup $G_0 \subset G$ and the set theoretic complement of G_0 , where by $G_0 \subset G$ we understand

$$G_0 = \left\{ \begin{pmatrix} \alpha_0 & \beta_0 \\ 0 & \alpha_0^{-1} \end{pmatrix} \mid \alpha_0 \in \mathbb{C}_*, \beta_0 \in \mathbb{C} \right\}.$$

Let us be more precise. Let $\pi_0 \in \text{Mor}(\mathbb{C}_0(G); \mathbb{C}_0(G_0))$ be the restriction morphism

$$\pi_0(f)(g_0) = f(g_0)$$

for any $f \in \mathbb{C}_0(G)$ and $g_0 \in G_0$. Applying the Rieffel deformation to $(\mathbb{C}_0(G_0), \Delta)$, based on the subgroup $\Gamma \subset G$ we construct a quantum group $\mathbb{G}_0 = (A_0, \Delta)$. Let \mathbb{B}_0 be the respective Γ^2 -product. In this section T_l and T_r denote the infinitesimal generators of the representation $\lambda: \mathbb{C}^2 \rightarrow \text{M}(B_0)$ (see (10)) and ∂_l, ∂_r denote the vector fields on G_0 defined like in (12). By Theorem 3.18 of [2] the restriction morphism $\pi_0 \in \text{Mor}(\mathbb{C}_0(G); \mathbb{C}_0(G_0))$ induces the twisted morphism of C^* -algebras $\pi_0^\Psi: A \rightarrow A_0$ and the surjectivity of π_0 implies the surjectivity of π_0^Ψ . Let $A_{\hat{\gamma}} \subset A$ be the two-sided ideal generated by $z_{\hat{\gamma}}$. Invoking the centrality of $z_{\hat{\gamma}}$ in $\text{M}(A)$ we have $A_{\hat{\gamma}} = \overline{z_{\hat{\gamma}}A}^{\|\cdot\|}$. It is easy to see that $\pi_0^\Psi(z_{\hat{\gamma}}) = 0$, which implies that $A_{\hat{\gamma}} \subset \ker \pi_0^\Psi$. It can also be proven that $\ker \pi_0^\Psi \subset A_{\hat{\gamma}}$, hence we have the exact sequence of C^* -algebras

$$0 \rightarrow A_{\hat{\gamma}} \rightarrow A \xrightarrow{\pi_0^\Psi} A_0 \rightarrow 0. \tag{21}$$

In what follows we shall construct an affiliated element $\hat{\beta}_0 \in A_0^\eta$, which is farther used in the construction of $\hat{\beta} \in A^\eta$. Let us first mention that following the construction of $\hat{\gamma} \in A^\eta$ of Section 4.1, we may introduce an affiliated element $\hat{\alpha}_0 \in A_0^\eta$. It is normal and invertible, and its action on $L^2(G_0)$ is given by the multiplication operator by the coordinate α_0 . Remembering that $\mathbb{C}_0(G_0)^\eta \subset B_0^\eta$ we shall consider α_0 and β_0 affiliated with B_0 . The elements $\alpha_0, T_l, T_r \in B_0^\eta$ strongly commute, hence using Theorem D.2 we construct $\alpha_0 T_r^*, \alpha_0^{-1} T_l^* \in B_0^\eta$. For any $(z, t) \in \mathbb{H}$ we define the unitary element

$$U_{z,t}^{\hat{\beta}_0} = \lambda(z; \beta_0) \lambda(z; -\frac{s}{4} \alpha_0^{-1} T_l^*) \lambda(z; -\frac{s}{4} \alpha_0 T_r^*) \lambda(t; -\frac{s}{4} (|\alpha_0|^{-2} - |\alpha_0|^2)) \in \text{M}(B_0). \tag{22}$$

Let us also define the differential operator

$$\text{Op}(\beta_0) = \beta_0 - \frac{s}{4} \alpha_0^{-1} \partial_l^* - \frac{s}{4} \alpha_0 \partial_r^*. \tag{23}$$

The proof of the next theorem is similar to the proof of Theorem 4.1. The quantization map related to the quantum group \mathbb{G}_0 is denoted by \mathcal{Q}_0 .

Theorem 4.2. *Let $U_{z,t}^{\hat{\beta}_0} \in M(B_0)$ be the unitary element given by formula (22). Then:*

(1) $U_{z,t}^{\hat{\beta}_0}$ is an element of $M(A_0) \subset M(B_0)$ for any $(z, t) \in \mathbb{H}$.

(2) The map

$$\mathbb{H} \ni (z, t) \mapsto U_{z,t}^{\hat{\beta}_0} \in M(A_0)$$

is a strongly continuous representation of the Heisenberg group.

(3) The set $\{\mathcal{Q}_0(f) \mid f \in C_c^\infty(G_0)\} \subset A_0$ is a core of the generator $\hat{\beta}_0 \in A_0^\eta$ of the representation $U^{\hat{\beta}_0}$ and we have

$$\hat{\beta}_0 \mathcal{Q}_0(f) = \mathcal{Q}_0(\text{Op}(\beta_0)f)$$

for any $f \in C_c^\infty(G_0)$. Moreover the set

$$\{\mathcal{Q}_0((1 + \text{Op}(\beta_0)^* \text{Op}(\beta_0))f) \mid f \in C_c^\infty(G_0)\} \tag{24}$$

is dense in A_0 .

4.4. The affiliated element $\hat{\beta}$. After constructing the affiliated element $\hat{\beta}_0 \in A_0$ we shall now move on to the construction of $\hat{\beta}$. In order to do that we first have to introduce $\hat{\beta}_{\hat{\gamma}} \in A_{\hat{\gamma}}^\eta$, which may be treated as a restriction of $\hat{\beta}$ to $A_{\hat{\gamma}}$. The embedding of $A_{\hat{\gamma}}$ into A leads to a morphism $\pi_{\hat{\gamma}} \in \text{Mor}(A, A_{\hat{\gamma}})$, which is defined by the formula $\pi_{\hat{\gamma}}(a)a_{\hat{\gamma}} = aa_{\hat{\gamma}}$ with $a \in A$ and $a_{\hat{\gamma}} \in A_{\hat{\gamma}}$. This morphism is injective, which enables us to treat $\hat{\alpha}, \hat{\gamma}, \hat{\delta} \in A^\eta$ as elements affiliated with $A_{\hat{\gamma}}$. The injectivity of $\pi_{\hat{\gamma}}$ follows from the implication $(az_{\hat{\gamma}} = 0) \implies (a = 0)$, which is true for any $a \in A$. Note that $\hat{\gamma}$ treated as an element of $A_{\hat{\gamma}}$ is invertible, i.e., there exists a unique element $\hat{\gamma}^{-1} \in A_{\hat{\gamma}}^\eta$ strongly commuting with $\hat{\gamma}$ and such that $\hat{\gamma}\hat{\gamma}^{-1} = \hat{\gamma}^{-1}\hat{\gamma} = 1$. Moreover, the elements $\hat{\alpha}, \hat{\delta}, \hat{\gamma}^{-1} \in A_{\hat{\gamma}}^\eta$ mutually strongly commute, so using Theorem D.2 we may define $\hat{\beta}_{\hat{\gamma}}$ by the formula

$$\hat{\beta}_{\hat{\gamma}} = \hat{\gamma}^{-1}(\hat{\alpha}\hat{\delta} - 1) \in A_{\hat{\gamma}}^\eta. \tag{25}$$

In order to give a more direct description of $\hat{\beta}_{\hat{\gamma}}$, let us introduce the differential operator

$$\text{Op}(\beta) = \beta - \frac{s}{4}\delta\partial_t^* - \frac{s}{4}\alpha\partial_r^* + \frac{s^2}{16}\gamma\partial_t^*\partial_r^*. \tag{26}$$

It is easy to check that the determinant relation is satisfied

$$\text{Op}(\alpha)\text{Op}(\delta) - \text{Op}(\gamma)\text{Op}(\beta) = 1,$$

where $\text{Op}(\gamma)$ denotes the operator of multiplication by γ . The following lemma describes $\hat{\beta}_{\hat{\gamma}}$ in terms of $\text{Op}(\beta)$.

Lemma 4.3. Let $\hat{\beta}_{\hat{\gamma}} \in A_{\hat{\gamma}}^{\eta}$ be the affiliated element defined above. The set

$$\{\mathcal{Q}(f)_{z_{\hat{\gamma}}} \mid f \in C_c^{\infty}(G)\}$$

is a core of $\hat{\beta}_{\hat{\gamma}}$ and for any $f \in C_c^{\infty}(G)$ we have

$$\hat{\beta}_{\hat{\gamma}} \mathcal{Q}(f)_{z_{\hat{\gamma}}} = \mathcal{Q}(\text{Op}(\beta)f)_{z_{\hat{\gamma}}}. \quad (27)$$

Moreover, the set

$$\{(1 + \hat{\beta}_{\hat{\gamma}}^* \hat{\beta}_{\hat{\gamma}}) \mathcal{Q}(f)_{z_{\hat{\gamma}}} : f \in C_c^{\infty}(G)\} \quad (28)$$

is dense in $A_{\hat{\gamma}}$.

Proof. Formula (27) follows from equation (25) and point 3 of Theorem 4.1. In order to prove (28) we introduce the affiliated element $T = \hat{\alpha} \hat{\delta} - 1 \in A_{\hat{\gamma}}^{\eta}$. Obviously, we have $\hat{\beta}_{\hat{\gamma}} = \hat{\gamma}^{-1} T$, so we will base the analysis of $\hat{\beta}_{\hat{\gamma}}$ on the analysis of T . Let us check that T and $\hat{\alpha}^* \hat{\alpha} + \hat{\delta}^* \hat{\delta}$ strongly commute:

$$\begin{aligned} & \exp(it(\hat{\alpha}^* \hat{\alpha} + \hat{\delta}^* \hat{\delta})) T \exp(it(\hat{\alpha}^* \hat{\alpha} + \hat{\delta}^* \hat{\delta})) \\ &= \exp(it\hat{\alpha}^* \hat{\alpha}) \hat{\alpha} \exp(-it\hat{\alpha}^* \hat{\alpha}) \exp(it\hat{\delta}^* \hat{\delta}) \hat{\delta} \exp(-it\hat{\delta}^* \hat{\delta}) - 1 \\ &= \exp(it\hat{\gamma}^* \hat{\gamma}) \hat{\alpha} \exp(-it\hat{\gamma}^* \hat{\gamma}) \hat{\delta} - 1 = T. \end{aligned}$$

Using Theorem D.1 we define

$$T' = (1 + T^* T) \exp(-\hat{\alpha}^* \hat{\alpha} - \hat{\delta}^* \hat{\delta}) \in A_{\hat{\gamma}}^{\eta}.$$

The equality $\exp(-\hat{\alpha}^* \hat{\alpha} - \hat{\delta}^* \hat{\delta}) = \exp(-\hat{\alpha} \hat{\alpha}^* - \hat{\delta} \hat{\delta}^*)$ implies that

$$\begin{aligned} T' &= 2 \exp(-\hat{\alpha}^* \hat{\alpha} - \hat{\delta}^* \hat{\delta}) + \hat{\alpha}^* \hat{\alpha} \exp(-\hat{\alpha}^* \hat{\alpha}) \hat{\delta}^* \hat{\delta} \exp(-\hat{\delta}^* \hat{\delta}) \\ &\quad - \hat{\alpha} \exp(-\hat{\alpha}^* \hat{\alpha}) \hat{\delta} \exp(-\hat{\delta}^* \hat{\delta}) - \hat{\alpha}^* \exp(-\hat{\alpha} \hat{\alpha}^*) \hat{\delta}^* \exp(-\hat{\delta} \hat{\delta}^*). \end{aligned}$$

All factors of the above sum belong to $M(A_{\hat{\gamma}})$, hence the resulting operator T' also belongs to $M(A_{\hat{\gamma}})$. Note that

$$T' D(T^* T) = \exp(-\hat{\alpha}^* \hat{\alpha} - \hat{\delta}^* \hat{\delta}) (1 + T^* T) D(T^* T) = \exp(-\hat{\alpha}^* \hat{\alpha} - \hat{\delta}^* \hat{\delta}) A_{\hat{\gamma}}. \quad (29)$$

The right-hand side of (29) is dense in $A_{\hat{\gamma}}$. Using the boundedness of T' and the density of $D(T^* T)$ in $A_{\hat{\gamma}}$ we conclude that the set $T' \mathcal{Q}(C_c^{\infty}(G))_{z_{\hat{\gamma}}}$ is also dense:

$$\overline{T' \mathcal{Q}(C_c^{\infty}(G))_{z_{\hat{\gamma}}}}^{\|\cdot\|} = A_{\hat{\gamma}}. \quad (30)$$

We shall now prove that the density (30) implies the density (28). Let

$$a = \exp(-\hat{\alpha}^* \hat{\alpha} - \hat{\delta}^* \hat{\delta}) \mathcal{Q}(f)_{z_{\hat{\gamma}}} \quad (31)$$

for some $f \in C_c^\infty(G)$. Using formula (74) one can check that there exists a sequence $f_n \in C_c^\infty(G)$ such that

$$\begin{aligned} \lim_{n \rightarrow \infty} \partial_l^k \partial_l^{*k'} \partial_r^m \partial_r^{*m'} f_n z_\gamma &= \partial_l^k \partial_l^{*k'} \partial_r^m \partial_r^{*m'} \exp(-\hat{\alpha}^* \hat{\alpha} - \hat{\delta}^* \hat{\delta}) f z_\gamma \\ \lim_{n \rightarrow \infty} \partial_l^k \partial_l^{*k'} \partial_r^m \partial_r^{*m'} (1 + T^* T) f_n z_\gamma &= \partial_l^k \partial_l^{*k'} \partial_r^m \partial_r^{*m'} (1 + T^* T) \\ &\quad \cdot \exp(-\hat{\alpha}^* \hat{\alpha} - \hat{\delta}^* \hat{\delta}) f z_\gamma \end{aligned} \tag{32}$$

for any $k, k', m, m' \leq 5$, where in the above formulas we treat $T = \hat{\alpha} \hat{\delta} - 1$ as an operator acting on $L^2(G)$. It may be shown that the convergence in (32) is in the uniform topology on $C_0(G)$. Using Theorem A.1 and the closedness of $1 + T^* T$ we see that

$$(1 + T^* T)a = \lim_{n \rightarrow \infty} (1 + T^* T)\mathcal{Q}(f_n)z_\gamma. \tag{33}$$

Combining (30), (31) and (33) we get

$$\overline{(1 + T^* T)\mathcal{Q}(C_c^\infty(G))z_\gamma}^{\|\cdot\|} = A_{\hat{\gamma}}.$$

Using the above equality, the fact that $\hat{\beta}_{\hat{\gamma}} = \hat{\gamma}^{-1} T$ and Lemma D.5, we get

$$\overline{(1 + \hat{\beta}_{\hat{\gamma}}^* \hat{\beta}_{\hat{\gamma}})(1 + |\hat{\gamma}|^{-2})^{-1} \mathcal{Q}(C_c^\infty(G))z_\gamma}^{\|\cdot\|} = A_{\hat{\gamma}}. \tag{34}$$

The inclusion

$$(1 + |\hat{\gamma}|^{-2})^{-1} \mathcal{Q}(C_c^\infty(G))z_\gamma \subset \mathcal{Q}(C_c^\infty(G))z_\gamma$$

and equation (34) show that

$$\overline{(1 + \hat{\beta}_{\hat{\gamma}}^* \hat{\beta}_{\hat{\gamma}})\mathcal{Q}(C_c^\infty(G))z_\gamma}^{\|\cdot\|} = A_{\hat{\gamma}}.$$

This proves (28). Now from Lemma D.3 it follows that $\mathcal{Q}(C_c^\infty(G))z_\gamma$ is a core of $\hat{\beta}_{\hat{\gamma}}$, which ends the proof of our lemma. \square

Using $\hat{\beta}_{\hat{\gamma}} \in A_\gamma^\eta$ defined above and $\hat{\beta}_0 \in A_0^\eta$ defined in the previous section, we construct the affiliated element $\hat{\beta} \in A^\eta$. Heuristically speaking, it is a gluing of $\hat{\beta}_{\hat{\gamma}}$ and $\hat{\beta}_0$.

Theorem 4.4. *Let $\text{Op}(\beta)$ be the differential operator (26). There exists an affiliated element $\hat{\beta} \in A^\eta$ such that the set $\{\mathcal{Q}(f) \mid f \in C_c^\infty(G)\}$ is a core of $\hat{\beta}$ and*

$$\hat{\beta}\mathcal{Q}(f) = \mathcal{Q}(\text{Op}(\beta)f)$$

for any $f \in C_c^\infty(G)$.

Proof. Let $\text{Graph } \hat{\beta}_{\hat{\gamma}}$ be the graph of the affiliated element $\hat{\beta}_{\hat{\gamma}}$. It is easy to check that the set

$$\left\{ \begin{pmatrix} b \\ b' \end{pmatrix} \mid \begin{pmatrix} b a_{\hat{\gamma}} \\ b' a_{\hat{\gamma}} \end{pmatrix} \in \text{Graph}(\hat{\beta}_{\hat{\gamma}}) \text{ for any } a_{\hat{\gamma}} \in A_{\hat{\gamma}} \right\} \subset A \oplus A$$

is a graph of a closed operator acting on A . This operator will be denoted by $\hat{\beta}$. Let us list some properties of $\text{Graph } \hat{\beta}$:

1. $\text{Graph } \hat{\beta} \subset A \oplus A$ is a submodule of a Hilbert A -module $A \oplus A$.
2. For any $f \in C_c^\infty(G)$ we have $\begin{pmatrix} \mathcal{Q}(f) \\ \mathcal{Q}(\text{Op}(\beta)f) \end{pmatrix} \in \text{Graph } \hat{\beta}$.
3. Let

$$(\text{Graph } \hat{\beta})^\perp = \left\{ \begin{pmatrix} c \\ c' \end{pmatrix} \mid c^*a + c'^*a' = 0 \text{ for any } \begin{pmatrix} a \\ a' \end{pmatrix} \in \text{Graph } \hat{\beta} \right\}$$

be the submodule perpendicular to $\text{Graph } \hat{\beta}$. For any $f \in C_c^\infty(G)$ we have $\begin{pmatrix} -\mathcal{Q}(\text{Op}(\beta)^*f) \\ \mathcal{Q}(f) \end{pmatrix} \in (\text{Graph } \hat{\beta})^\perp$.

4. $\overline{\{\mathcal{Q}((1 + \text{Op}(\beta)^* \text{Op}(\beta))f) \mid f \in C_c^\infty(G)\}}^{\|\cdot\|} = A$.

Properties 1, 2 and 3 are consequences of the definition of $\hat{\beta}$ and Lemma 4.3. In what follows we shall prove property 4:

$$\overline{\{\mathcal{Q}((1 + \text{Op}(\beta)^* \text{Op}(\beta))f) \mid f \in C_c^\infty(G)\}}^{\|\cdot\|} = A.$$

Let $a \in A$ and $\pi_0^\Psi \in \text{Mor}(A, A_0)$ be the morphism entering the exact sequence (21). Using (24) we can see that there exists a sequence $\tilde{f}_n \in C_c^\infty(G_0)$ such that

$$\pi_0^\Psi(a) = \lim_{n \rightarrow \infty} \mathcal{Q}_0((1 + \text{Op}(\beta_0)^* \text{Op}(\beta_0))\tilde{f}_n). \quad (35)$$

Let $f_n \in C_c^\infty(G)$ be an extension of \tilde{f}_n to the whole group G and let $\pi_0 \in \text{Mor}(C_0(G); C_0(G_0))$ be the morphism introduced in Section 4.3. It is not difficult to check that

$$\pi_0^\Psi(\mathcal{Q}(f)) = \mathcal{Q}_0(\pi_0(f)), \quad \pi_0(\text{Op}(\beta)f) = \text{Op}(\beta_0)\pi_0(f).$$

Using these equalities and (35) we see that

$$\lim_{n \rightarrow \infty} \pi_0^\Psi(a - \mathcal{Q}((1 + \text{Op}(\beta)^* \text{Op}(\beta))f_n)) = 0.$$

The exactness of the sequence (21) ensures that for any $\varepsilon > 0$ there exists $n \in \mathbb{N}$ and $a_{\hat{\gamma}} \in A_{\hat{\gamma}}$ such that

$$\|a - \mathcal{Q}((1 + \text{Op}(\beta)^* \text{Op}(\beta))f_n) - a_{\hat{\gamma}}\| \leq \varepsilon. \quad (36)$$

Equality (28) implies that there exists a function $f \in C_c^\infty(G)$ such that

$$\|a_{\hat{\gamma}} - \mathcal{Q}((1 + \text{Op}(\beta)^* \text{Op}(\beta))f)z_{\hat{\gamma}}\| \leq \varepsilon. \quad (37)$$

Combining (36) and (37) we get

$$\|a - \mathcal{Q}((1 + \text{Op}(\beta)^* \text{Op}(\beta))(f_n + fz_\gamma))\| \leq 2\varepsilon.$$

This ends the proof of property 4 above.

Using the properties of Graph $\hat{\beta}$ one can check that it satisfies all assumptions of Proposition 2.2 of [10]. This proposition guaranties that $\hat{\beta} \in A^\eta$. It is easy to check that $\hat{\beta}$ satisfies all the requirements of our theorem. \square

4.5. Representation theory of the C*-algebra A . The results of Appendix B applied to the C*-algebra A of the Heisenberg–Lorentz quantum group \mathbb{G} show that the representation theory of A can be equivalently described by the corepresentation theory of the dual quantum group $\hat{\mathbb{G}}$. As was shown in [2], the C*-algebra of $\hat{\mathbb{G}}$ is the reduced group C*-algebra $C_r^*(G)$. The comultiplication $\Delta_{\hat{\mathbb{G}}}$ is the 2-cocycle twist of the standard comultiplication $\hat{\Delta}$ on $C_r^*(G)$

$$\Delta_{\hat{\mathbb{G}}}(a) = X \hat{\Delta}(a) X^* \tag{38}$$

for any $a \in C_r^*(G)$. The unitary $X \in M(C_r^*(G) \otimes C_r^*(G))$ is the image of $\Psi \in M(C^*(\Gamma) \otimes C^*(\Gamma))$ (see equation (6)) under a morphism which sends the generator $u_\gamma \in M(C^*(\Gamma))$ to the right shift $R_\gamma \in M(C_r^*(G))$.

Let $\pi_U \in \text{Rep}(A; \mathcal{H})$ be a representation of A on a Hilbert space \mathcal{H} . The corresponding corepresentation $U_\pi \in M(\mathcal{K}(H) \otimes \hat{A})$ is given by

$$U_\pi = (\pi_U \otimes \text{id}) \hat{W}, \tag{39}$$

where $\hat{W} \in M(A \otimes \hat{A})$ is the multiplicative unitary of $\hat{\mathbb{G}}$. On the other hand, giving a motivation for Definition 3.3 we claimed that representations of the Heisenberg–Lorentz commutation relations correspond to representations of A on Hilbert spaces. To prove this fact we will show that for any representation $(\tilde{\alpha}, \tilde{\beta}, \tilde{\gamma}, \tilde{\delta})$ on a Hilbert space \mathcal{H} we can construct a corepresentation U of $\hat{\mathbb{G}}$ on \mathcal{H} , which in turn corresponds via (39) to a representation $\pi \in \text{Rep}(A; \mathcal{H})$. This construction of π is performed in the proof of the next theorem, where we also give a more direct characterization of π in terms of $\hat{\alpha}, \hat{\beta}, \hat{\gamma}, \hat{\delta} \in A^\eta$.

Theorem 4.5. *Let $(\tilde{\alpha}, \tilde{\beta}, \tilde{\gamma}, \tilde{\delta})$ be a representation of the Heisenberg–Lorentz commutation relations on a Hilbert space \mathcal{H} (cf. Definition 3.3). There exists a unique representation $\pi \in \text{Rep}(A; \mathcal{H})$ such that*

$$\pi(\hat{\alpha}) = \tilde{\alpha}, \quad \pi(\hat{\beta}) = \tilde{\beta}, \quad \pi(\hat{\gamma}) = \tilde{\gamma}, \quad \pi(\hat{\delta}) = \tilde{\delta}. \tag{40}$$

Moreover, for any $\pi \in \text{Rep}(A; \mathcal{H})$, the quadruple $(\pi(\hat{\alpha}), \pi(\hat{\beta}), \pi(\hat{\gamma}), \pi(\hat{\delta}))$ is a representation of the Heisenberg–Lorentz commutation relations.

Proof. We shall start by fixing some notation. Given any $w \in \mathbb{C} \setminus \{0\}$, g_w denotes an element of G of the form

$$g_w = \begin{pmatrix} 0 & w^{-1} \\ -w & 0 \end{pmatrix}.$$

Let T be a normal, invertible element acting on a Hilbert space \mathcal{H} . We define the unitary operator

$$S(T) = \int dE^T(w) \otimes R_{g_w} \in B(\mathcal{H} \otimes L^2(G)),$$

where E^T is the spectral measure of T and $R_{g_w} \in B(L^2(G))$ is the right shift by g_w . Let E^R be the spectral measure that corresponds to the representation $\Gamma \ni \gamma \mapsto R_\gamma \in B(L^2(G))$ via the Stone–Naimark–Ambrose–Godement theorem. For an infinitesimal representation $(\tilde{a}, \tilde{\lambda})$ of \mathbb{H} on a Hilbert space \mathcal{H} we introduce the unitary operator

$$R(\tilde{a}) = \int U_{-z,0}^{\tilde{a}} \otimes dE^R(z) \in B(\mathcal{H} \otimes L^2(G)). \tag{41}$$

Let us come to the main part of the proof. An immediate consequence of Definition 3.3 is that it is enough to consider two cases of representations:

- (1) $\tilde{\gamma} = 0$;
- (2) $\ker \tilde{\gamma} = \{0\}$.

We give the proof for (2), leaving the first case to the reader. Using Theorem D.2 we define two closed operators $\tilde{\alpha}\tilde{\gamma}^{-1}$ and $\tilde{\delta}\tilde{\gamma}^{-1}$ acting on \mathcal{H} . Note that $(\tilde{\alpha}\tilde{\gamma}^{-1}, -s)$ and $(\tilde{\delta}\tilde{\gamma}^{-1}, s)$ are infinitesimal representations of the Heisenberg group \mathbb{H} . Using the notation introduced above we define the unitary operator

$$U = R(\tilde{\delta}\tilde{\gamma}^{-1})S(\tilde{\gamma})R(\tilde{\alpha}\tilde{\gamma}^{-1}) \in B(\mathcal{H} \otimes L^2(G)).$$

Let us prove that U is a corepresentation of $\hat{\mathbb{G}}$. Let $\hat{\Delta} \in \text{Mor}(C_r^*(G); C_r^*(G) \otimes C_r^*(G))$ be the canonical comultiplication on $C_r^*(G)$. Note that

$$\begin{aligned} (\text{id} \otimes \hat{\Delta})R(\tilde{\delta}\tilde{\gamma}^{-1}) &= \int U_{-(z+z'),0}^{\tilde{\delta}\tilde{\gamma}^{-1}} \otimes dE^R(z) \otimes dE^R(z') \\ &= \int \exp(-i \frac{s}{4} \text{Im}(z\bar{z}')) U_{-z,0}^{\tilde{\delta}\tilde{\gamma}^{-1}} U_{-z',0}^{\tilde{\delta}\tilde{\gamma}^{-1}} \otimes dE^R(z) \otimes dE^R(z') \\ &= X_{23}^* R(\tilde{\delta}\tilde{\gamma}^{-1})_{12} R(\tilde{\delta}\tilde{\gamma}^{-1})_{13}. \end{aligned} \tag{42}$$

The unitary element

$$X = \int \exp(i \frac{s}{4} \text{Im}(z\bar{z}')) dE^R(z) \otimes dE^R(z') \in M(C_r^*(G) \otimes C_r^*(G))$$

used above is the one that twists $\hat{\Delta}$, giving the comultiplication $\Delta_{\hat{\mathbb{G}}}$ (see (38)). Similarly, we check that

$$(\text{id} \otimes \hat{\Delta})R(\tilde{\alpha}\tilde{\gamma}^{-1}) = R(\tilde{\alpha}\tilde{\gamma}^{-1})_{12}R(\tilde{\alpha}\tilde{\gamma}^{-1})_{13}X_{23}. \tag{43}$$

Moreover, the formula $\hat{\Delta}(Z_w) = Z_w \otimes Z_w$ implies that

$$(\text{id} \otimes \hat{\Delta})S(\tilde{\gamma}) = S(\tilde{\gamma})_{12}S(\tilde{\gamma})_{13}. \tag{44}$$

Using equations (42), (43), (44), the fact that the first legs of $R(\tilde{\alpha}\tilde{\gamma}^{-1})$, $R(\tilde{\delta}\tilde{\gamma}^{-1})$ and $S(\tilde{\gamma})$ commute, and formula (38) we get

$$\begin{aligned} (\text{id} \otimes \Delta_{\hat{\mathbb{G}}})U &= X_{23}(\text{id} \otimes \hat{\Delta})(R(\tilde{\delta}\tilde{\gamma}^{-1})S(\tilde{\gamma})R(\tilde{\alpha}\tilde{\gamma}^{-1}))X_{23}^* \\ &= X_{23}X_{23}^*R(\tilde{\delta}\tilde{\gamma}^{-1})_{12}R(\tilde{\delta}\tilde{\gamma}^{-1})_{13}S(\tilde{\gamma})_{12}S(\tilde{\gamma})_{13} \\ &\quad \cdot R(\tilde{\alpha}\tilde{\gamma}^{-1})_{12}R(\tilde{\alpha}\tilde{\gamma}^{-1})_{13}X_{23}X_{23}^* \\ &= R(\tilde{\delta}\tilde{\gamma}^{-1})_{12}S(\tilde{\gamma})_{12}R(\tilde{\alpha}\tilde{\gamma}^{-1})_{12}R(\tilde{\delta}\tilde{\gamma}^{-1})_{13}S(\tilde{\gamma})_{13}R(\tilde{\alpha}\tilde{\gamma}^{-1})_{13} \\ &= U_{12}U_{13}, \end{aligned}$$

which shows that U is a corepresentation of $\hat{\mathbb{G}}$. Let $\pi_U \in \text{Rep}(A; \mathcal{H})$ be the corresponding representation of A . The next step is to prove that π_U is the representation π of our theorem:

$$\pi_U(\hat{\alpha}) = \tilde{\alpha}, \quad \pi_U(\hat{\beta}) = \tilde{\beta}, \quad \pi_U(\hat{\gamma}) = \tilde{\gamma}, \quad \pi_U(\hat{\delta}) = \tilde{\delta}.$$

Treating $\hat{\alpha}, \hat{\gamma}, \hat{\delta} \in A^n$ as closed operators acting on $L^2(G)$ (in particular $\hat{\gamma}$ is an invertible operator of multiplication by the coordinate γ) one can prove that the multiplicative unitary \hat{W} is given by

$$\hat{W} = R(\hat{\delta}\hat{\gamma}^{-1})S(\hat{\gamma})R(\hat{\alpha}\hat{\gamma}^{-1}). \tag{45}$$

It can also be shown that

$$\begin{aligned} \hat{W}^*(1 \otimes \exp(i \text{Im}(z\hat{\gamma})))\hat{W} &= U_{z,0}^{\hat{\alpha} \otimes \hat{\gamma}} U_{z,0}^{\hat{\gamma} \otimes \hat{\delta}}, \\ U^*(1 \otimes \exp(i \text{Im}(z\hat{\gamma})))U &= U_{z,0}^{\tilde{\alpha} \otimes \tilde{\gamma}} U_{z,0}^{\tilde{\gamma} \otimes \tilde{\delta}}, \end{aligned}$$

which implies that

$$\hat{W}^*(1 \otimes \hat{\gamma})\hat{W} = \hat{\alpha} \otimes \hat{\gamma} \dot{+} \hat{\gamma} \otimes \hat{\delta}, \quad U^*(1 \otimes \hat{\gamma})U = \tilde{\alpha} \otimes \tilde{\gamma} \dot{+} \tilde{\gamma} \otimes \tilde{\delta}.$$

Applying $\pi_U \otimes \text{id}$ to both sides of the left of these equations and using (39) we get

$$\pi_U(\hat{\alpha}) \otimes \hat{\gamma} \dot{+} \pi_U(\hat{\gamma}) \otimes \hat{\delta} = \tilde{\alpha} \otimes \tilde{\gamma} \dot{+} \tilde{\gamma} \otimes \tilde{\delta}. \tag{46}$$

Let $\pi_0^\Psi \in \text{Mor}(A; A_0)$ be the morphism introduced in Section 4.3. It sends $\hat{\gamma}$ to 0 and $\hat{\delta}$ to the normal element $\hat{\delta}_0 = \hat{\alpha}_0^{-1} \in A_0^\eta$. Applying $\text{id} \otimes \pi_0^\Psi$ to both sides of (46) we get

$$\pi_U(\hat{\gamma}) \otimes \hat{\delta}_0 = \tilde{\gamma} \otimes \hat{\delta}_0.$$

This immediately implies that $\pi_U(\hat{\gamma}) = \tilde{\gamma}$. From this equality and (46) we see that $\pi_U(\hat{\alpha}) = \tilde{\alpha}$. Now using (39) and (45) we obtain

$$R(\pi_U(\hat{\delta})\tilde{\gamma}^{-1}) = R(\tilde{\delta}\tilde{\gamma}^{-1}).$$

Equation (41) together with the fact that the support of the measure dE^R is the whole complex plain implies that $\pi_U(\hat{\delta})\tilde{\gamma}^{-1} = \tilde{\delta}\tilde{\gamma}^{-1}$, hence $\pi_U(\hat{\delta}) = \tilde{\delta}$. Finally, $\pi_U(\hat{\beta}) = \tilde{\beta}$, which is a consequence of the related equalities for $\hat{\alpha}$, $\hat{\gamma}$ and $\hat{\delta}$.

That the quadruple $(\pi(\hat{\alpha}), \pi(\hat{\beta}), \pi(\hat{\gamma}), \pi(\hat{\delta}))$ is a representation of the Heisenberg–Lorentz commutation relations for any representation $\pi \in \text{Rep}(A; \mathcal{H})$ follows directly from the definition of the affiliated elements $\hat{\alpha}, \hat{\beta}, \hat{\gamma}, \hat{\delta} \in A^\eta$. □

The above theorem implies the following result.

Corollary 4.6. *Let A be the C^* -algebra of the Heisenberg–Lorentz quantum group. Then the generators $\hat{\alpha}, \hat{\beta}, \hat{\gamma}, \hat{\delta} \in A^\eta$ separate representations of A . That is, if π_1 and $\pi_2 \in \text{Rep}(A; \mathcal{H})$ coincide on $\hat{\alpha}, \hat{\beta}, \hat{\gamma}, \hat{\delta}$, i.e.,*

$$\pi_1(\hat{\alpha}) = \pi_2(\hat{\alpha}) \quad \pi_1(\hat{\beta}) = \pi_2(\hat{\beta}) \quad \pi_1(\hat{\gamma}) = \pi_2(\hat{\gamma}) \quad \pi_1(\hat{\delta}) = \pi_2(\hat{\delta}),$$

then $\pi_1 = \pi_2$.

4.6. $\hat{\alpha}, \hat{\beta}, \hat{\gamma}, \hat{\delta}$ as generators of A . By Corollary 4.6 we know that $\hat{\alpha}, \hat{\beta}, \hat{\gamma}, \hat{\delta} \in A^\eta$ separate representations of A . The aim of this section is to prove that they generate A in the sense of Woronowicz. For this purpose we shall use the following theorem, which is a consequence of Theorem 4.2 of [11].

Theorem 4.7. *Let T_1, T_2, \dots, T_n be elements affiliated with a C^* -algebra A . Let Ω be the subset of $M(A)$ consisting of elements of the form $(1 + T_i^* T_i)^{-1}, (1 + T_i T_i^*)^{-1}, \exp(-T_i^* T_i), \exp(-T_i T_i^*)$. Assume that*

- (1) T_1, T_2, \dots, T_n separate representations;
- (2) there exist elements $r_1, r_2, \dots, r_k \in \Omega$ such that $r_1 r_2 \dots r_k \in A$.

Then T_1, T_2, \dots, T_n generate A .

Theorem 4.8. *The affiliated elements $\hat{\alpha}, \hat{\beta}, \hat{\gamma}, \hat{\delta}$ generate the C^* -algebra A of the Heisenberg–Lorentz quantum group.*

Proof. From Theorem 4.7 and Corollary 4.6 we see that it is enough to prove that

$$(1 + \hat{\beta}^* \hat{\beta})^{-1} \exp(-\hat{\alpha}^* \hat{\alpha}) \exp(-\hat{\delta}^* \hat{\delta}), \tag{47}$$

which is an element of $M(A)$, belongs in fact to $A \subset M(A)$. In order to do that we shall first analyze the element $\exp(-\hat{\alpha}^* \hat{\alpha}) \exp(-\hat{\delta}^* \hat{\delta}) \in M(A)$. For any $g \in G$ we set

$$f(g) = \frac{1}{\cosh^2(\frac{s}{2} \gamma^* \gamma)} \exp\left(-\frac{2(|\alpha|^2 + |\delta|^2) \tanh(\frac{s}{2} \gamma^* \gamma)}{s \gamma^* \gamma}\right).$$

Let h_t be the family of functions defined by (73). A straightforward computation shows that

$$f(g) = \int d^2 z_1 d^2 z_2 h_1(z_1, -\frac{s}{2} |\gamma|^2) h_1(z_2, \frac{s}{2} |\gamma|^2) \exp(i \operatorname{Im}(z_1 \alpha)) \exp(i \operatorname{Im}(z_2 \delta)). \tag{48}$$

One can check that $\exp(-|\gamma|^2) \exp(i \operatorname{Im}(z_1 \alpha))$ and $\exp(-|\gamma|^2) \exp(i \operatorname{Im}(z_2 \delta))$ are quantizable in the sense of Theorem A.3 and that

$$\begin{aligned} \mathcal{Q}(\exp(-|\gamma|^2) \exp(i \operatorname{Im}(z_1 \alpha))) &= \exp(-|\gamma|^2) U_{z,0}^{\hat{\alpha}}, \\ \mathcal{Q}(\exp(-|\gamma|^2) \exp(i \operatorname{Im}(z_2 \delta))) &= \exp(-|\gamma|^2) U_{z,0}^{\hat{\delta}}. \end{aligned} \tag{49}$$

Using (48), (49) and (74) we get

$$\mathcal{Q}(f) = \exp(-\hat{\alpha}^* \hat{\alpha}) \exp(-\hat{\delta}^* \hat{\delta}). \tag{50}$$

Now, for the purpose of analysis of the whole product (47), we define two auxiliary functions $k_1, k_2: G \rightarrow \mathbb{C}$:

$$k_1(g) = \frac{1}{1 + \hat{\beta} \hat{\beta}} f(g), \quad k_2(g) = f - (1 + \operatorname{Op}(\beta)^* \operatorname{Op}(\beta)) k_1. \tag{51}$$

They satisfy the assumptions of Theorem A.1, hence we can quantize them obtaining $\mathcal{Q}(k_1)$ and $\mathcal{Q}(k_2) \in A$. Combining (50) and (51) we see that

$$\begin{aligned} &(1 + \hat{\beta}^* \hat{\beta})^{-1} \exp(-\hat{\alpha}^* \hat{\alpha}) \exp(-\hat{\delta}^* \hat{\delta}) \\ &= (1 + \hat{\beta}^* \hat{\beta})^{-1} (\mathcal{Q}(f)) \\ &= (1 + \hat{\beta}^* \hat{\beta})^{-1} \mathcal{Q}((1 + \operatorname{Op}(\beta)^* \operatorname{Op}(\beta)) k_1 + k_2) \\ &= (1 + \hat{\beta}^* \hat{\beta})^{-1} \mathcal{Q}((1 + \operatorname{Op}(\beta)^* \operatorname{Op}(\beta)) k_1) + (1 + \hat{\beta}^* \hat{\beta})^{-1} \mathcal{Q}(k_2) \\ &= \mathcal{Q}(k_1) + (1 + \hat{\beta}^* \hat{\beta})^{-1} \mathcal{Q}(k_2). \end{aligned}$$

Consequently $(1 + \hat{\beta}^* \hat{\beta})^{-1} \exp(-\hat{\alpha}^* \hat{\alpha}) \exp(-\hat{\delta}^* \hat{\delta}) \in A$ since both factors of the above sum belong to A . □

4.7. Comultiplication

Theorem 4.9. *Let $\mathbb{G} = (A, \Delta)$ be the Heisenberg–Lorentz quantum group. Then the action of Δ on the generators $\hat{\alpha}, \hat{\beta}, \hat{\gamma}, \hat{\delta} \in A^n$ is given by*

$$\begin{aligned}\Delta(\hat{\alpha}) &= \hat{\alpha} \otimes \hat{\alpha} \dot{+} \hat{\beta} \otimes \hat{\gamma}, \\ \Delta(\hat{\beta}) &= \hat{\alpha} \otimes \hat{\beta} \dot{+} \hat{\beta} \otimes \hat{\delta}, \\ \Delta(\hat{\gamma}) &= \hat{\gamma} \otimes \hat{\alpha} \dot{+} \hat{\delta} \otimes \hat{\gamma}, \\ \Delta(\hat{\delta}) &= \hat{\gamma} \otimes \hat{\beta} \dot{+} \hat{\delta} \otimes \hat{\delta}.\end{aligned}\tag{52}$$

Remark 4.10. The elements appearing on the right-hand side of (52) (further denoted by $\tilde{\alpha}, \tilde{\beta}, \tilde{\gamma}, \tilde{\delta}$, respectively) are treated as closed operators acting on $L^2(G) \otimes L^2(G)$. As usual, the sign $\dot{+}$ denotes the closure of the sum of two operators. The idea of the proof of the above theorem goes as follows. Using Theorem 3.5 we see that the quadruple $(\tilde{\alpha}, \tilde{\beta}, \tilde{\gamma}, \tilde{\delta})$ is a representation of the Heisenberg–Lorentz commutation relations. By Theorem 4.5 this quadruple corresponds to a unique corepresentation U of \mathbb{G} on the Hilbert space $L^2(G) \otimes L^2(G)$, which in turn corresponds to a unique representation $\pi \in \text{Rep}(A; L^2(G) \otimes L^2(G))$ such that $\pi(\hat{\alpha}) = \tilde{\alpha}$, $\pi(\hat{\beta}) = \tilde{\beta}$, $\pi(\hat{\gamma}) = \tilde{\gamma}$ and $\pi(\hat{\delta}) = \tilde{\delta}$. On the other hand, $(\Delta \otimes \text{id})\hat{W} = \hat{W}_{23}\hat{W}_{13}$. Using the correspondence $U = (\pi \otimes \text{id})\hat{W}$ and Theorem 4.5 once again we see that to prove (52) it is enough to show that $U = \hat{W}_{23}\hat{W}_{13}$, which will be done in the following proof. Note that in order to prove Theorem 4.9, it is first necessary to show that the quadruple $(\tilde{\alpha}, \tilde{\beta}, \tilde{\gamma}, \tilde{\delta})$ is a representation of the Heisenberg–Lorentz commutation relations. The proof of this fact seems to be as difficult as the proof of the more general Theorem 3.5.

Proof of Theorem 4.9. In this proof we shall use the notation of the proof of Theorem 4.5. Let $\tilde{\alpha}, \tilde{\beta}, \tilde{\gamma}, \tilde{\delta}$ denote the right-hand sides of (52). As was explained in the above remark, to prove our theorem it is enough to show that

$$\hat{W}_{23}\hat{W}_{13} = R(\tilde{\delta}\tilde{\gamma}^{-1})S(\tilde{\gamma})R(\tilde{\alpha}\tilde{\gamma}^{-1}).\tag{53}$$

From equation (45) we can see that

$$\begin{aligned}\hat{W}_{13} &= R(\hat{\delta}\hat{\gamma}^{-1} \otimes 1)S(\hat{\gamma} \otimes 1)R(\hat{\alpha}\hat{\gamma}^{-1} \otimes 1), \\ \hat{W}_{23} &= R(1 \otimes \hat{\delta}\hat{\gamma}^{-1})S(1 \otimes \hat{\gamma})R(1 \otimes \hat{\alpha}\hat{\gamma}^{-1}).\end{aligned}$$

Therefore, the left-hand side of (53) has the form

$$R(1 \otimes \hat{\delta}\hat{\gamma}^{-1})S(1 \otimes \hat{\gamma})R(1 \otimes \hat{\alpha}\hat{\gamma}^{-1})R(\hat{\delta}\hat{\gamma}^{-1} \otimes 1)S(\hat{\gamma} \otimes 1)R(\hat{\alpha}\hat{\gamma}^{-1} \otimes 1).\tag{54}$$

Using the fact that $R(1 \otimes \hat{\alpha}\hat{\gamma}^{-1})$ commutes with $R(\hat{\delta}\hat{\gamma}^{-1} \otimes 1)$ we see that (54) is equal to

$$R(1 \otimes \hat{\delta}\hat{\gamma}^{-1})S(1 \otimes \hat{\gamma})R(\hat{\delta}\hat{\gamma}^{-1} \otimes 1)R(1 \otimes \hat{\alpha}\hat{\gamma}^{-1})S(\hat{\gamma} \otimes 1)R(\hat{\alpha}\hat{\gamma}^{-1} \otimes 1).$$

Formula (5) and the corresponding formula related to $\tilde{\delta}$ imply that

$$\begin{aligned}\hat{\alpha}\hat{\gamma}^{-1} \otimes 1 &= \tilde{\alpha}\tilde{\gamma}^{-1} \dot{+} \tilde{\gamma}^{-1}(\hat{\gamma}^{-1} \otimes \hat{\gamma}), \\ 1 \otimes \hat{\delta}\hat{\gamma}^{-1} &= \tilde{\delta}\tilde{\gamma}^{-1} \dot{+} \tilde{\gamma}^{-1}(\hat{\gamma} \otimes \hat{\gamma}^{-1}).\end{aligned}$$

Using these equalities and the fact that (54) is equal to $\widehat{W}_{23}\widehat{W}_{13}$ we get:

$$\begin{aligned}\widehat{W}_{23}\widehat{W}_{13} &= R(\tilde{\delta}\tilde{\gamma}^{-1}) \exp(-i\text{Im}(\tilde{\gamma}^{-1}(\hat{\gamma} \otimes \hat{\gamma}^{-1}) \otimes T_r))S(1 \otimes \hat{\gamma})R(\hat{\delta}\hat{\gamma}^{-1} \otimes 1) \\ &\cdot R(1 \otimes \hat{\alpha}\hat{\gamma}^{-1})S(\hat{\gamma} \otimes 1) \exp(-i\text{Im}(\tilde{\gamma}^{-1}(\hat{\gamma}^{-1} \otimes \hat{\gamma}) \otimes T_r))R(\tilde{\alpha}\tilde{\gamma}^{-1}).\end{aligned}\tag{55}$$

Noting that

$$R(\hat{\delta}\hat{\gamma}^{-1} \otimes 1)R(1 \otimes \hat{\alpha}\hat{\gamma}^{-1}) = \exp(-i\text{Im}(\tilde{\gamma}(\hat{\gamma}^{-1} \otimes \hat{\gamma}^{-1}) \otimes T_r))$$

and using equation (55) we see that in order to prove equality (53) it is enough to check that

$$\begin{aligned}S(\tilde{\gamma}) &= \exp(-i\text{Im}(\tilde{\gamma}^{-1}(\hat{\gamma} \otimes \hat{\gamma}^{-1}) \otimes T_r))S(1 \otimes \hat{\gamma}) \exp(-i\text{Im}(\tilde{\gamma}(\hat{\gamma}^{-1} \otimes \hat{\gamma}^{-1}) \otimes T_r)) \\ &\cdot S(\hat{\gamma} \otimes 1) \exp(-i\text{Im}(\tilde{\gamma}^{-1}(\hat{\gamma}^{-1} \otimes \hat{\gamma}) \otimes T_r)).\end{aligned}\tag{56}$$

The operators $\tilde{\gamma}$, $1 \otimes \hat{\gamma}$ and $\hat{\gamma} \otimes 1$ which appear in the above expression are normal and strongly commute. Therefore, to prove (56) we have to check that

$$\begin{aligned}S(u) &= \exp(-i\text{Im}(u^{-1}vw^{-1}T_r))S(w) \exp(-i\text{Im}(uv^{-1}w^{-1}T_r)) \\ &\cdot S(v) \exp(-i\text{Im}(u^{-1}v^{-1}wT_r))\end{aligned}\tag{57}$$

for any $u, v, w \in \mathbb{C} \setminus \{0\}$. Noting that

$$S(w) = Z_w, \quad \exp(i\text{Im}(zT_r)) = R_z,$$

where Z_w and R_z are operators defined in the proof of Theorem 4.5, we see that equation (57) is equivalent to the matrix identity

$$\begin{aligned}\begin{pmatrix} 0 & u^{-1} \\ -u & 0 \end{pmatrix} &= \begin{pmatrix} 1 & -vu^{-1}w^{-1} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & w^{-1} \\ -w & 0 \end{pmatrix} \begin{pmatrix} 1 & -uv^{-1}w^{-1} \\ 0 & 1 \end{pmatrix} \\ &\cdot \begin{pmatrix} 0 & v^{-1} \\ -v & 0 \end{pmatrix} \begin{pmatrix} 1 & -wu^{-1}v^{-1} \\ 0 & 1 \end{pmatrix}.\end{aligned}$$

Its verification is a straightforward computation, which is left to the reader. □

Appendices

A. Quantization map

Let \mathcal{Q} be the quantization map introduced in Definition 4.13 of [2]. Recall that \mathcal{Q} was defined on elements of the Fourier algebra

$$\mathcal{F} = \{(\omega \otimes \text{id})V \mid \omega \in B(L^2(G))_*\}$$

and $\mathcal{Q}((\omega \otimes \text{id})V) = (\omega \otimes \text{id})W$, where V is the Kac–Takesaki operator of a locally compact group G and W is the multiplicative unitary related to a Rieffel deformation of G . Given a function $f \in C_0(G)$ it is usually difficult to check if f belongs to \mathcal{F} , which makes \mathcal{Q} not very useful in the practical applications. In the case of the Heisenberg–Lorentz quantum group we shall give a new description of the quantization map, which does not have the aforementioned drawback. \mathcal{Q} is defined on a different class of functions, but when the function happens to be an element of \mathcal{F} then the new definition will coincide with the old one. Consider two representations of $\Gamma \subset G$ on $L^2(G)$:

$$\Gamma \ni \gamma \mapsto R_\gamma \in B(L^2(G)), \quad \Gamma \ni \gamma \mapsto L_\gamma \in B(L^2(G)).$$

Let T_l and T_r be infinitesimal generators of these representations:

$$R_\gamma = \exp(i\text{Im}(\gamma T_r)), \quad L_\gamma = \exp(i\text{Im}(\gamma T_l)) \tag{58}$$

for any $\gamma \in \Gamma \simeq \mathbb{C}$. The related complex vector fields on G are denoted by ∂_l and ∂_r (see eq. (12)). Now consider two differential operators $K_l = (1 + T_l^* T_l)^2$ and $K_r = (1 + T_r^* T_r)^2$ acting on $L^2(G)$. Note that K_l and K_r are positive, invertible, and their inverses K_l^{-1}, K_r^{-1} are bounded.

Let $x, y, v, w \in L^2(G)$ be vectors such that $x, y, w \in D(K_r)$ and $v \in D(K_l)$. Our next objective is to compute the matrix element $\langle x \otimes v \mid W \mid y \otimes w \rangle$. Note that

$$\begin{aligned} \langle x \otimes v \mid W \mid y \otimes w \rangle &= \langle K_r x \otimes K_l v \mid (K_r^{-1} \otimes K_l^{-1}) Y V X (K_r^{-1} \otimes K_r^{-1}) \mid K_r y \otimes K_r w \rangle. \end{aligned} \tag{59}$$

Let $\pi_R, \pi_L \in \text{Rep}(C_0(\mathbb{C}); L^2(G))$ be representations of $C_0(\mathbb{C})$ which send $\text{id} \in C_0(\mathbb{C})^\eta$ to T_r and T_l , respectively. We have the equalities

$$X(K_r^{-1} \otimes K_r^{-1}) = (\pi_R \otimes \pi_R)((K^{-1} \otimes K^{-1})\Psi), \tag{60}$$

$$(K_r^{-1} \otimes K_l^{-1})Y = (\pi_R \otimes \pi_L)((K^{-1} \otimes K^{-1})\Psi), \tag{61}$$

where $K: \mathbb{C} \rightarrow \mathbb{R}$ is the function given by the formula

$$K(z) = (1 + |z|^2)^2$$

and $\Psi \in M(C_0(\mathbb{C}) \otimes C_0(\mathbb{C}))$ is defined by (6). Let $l: \mathbb{C}^2 \rightarrow \mathbb{C}$ be the function given by

$$l(w_1, w_2) = \int d^2 z_1 d^2 z_2 \frac{\exp(-i\text{Im}(z_1 w_1 + z_2 w_2 - \frac{\delta}{4} z_1 \bar{z}_2))}{(1 + |z_1|^2)^2 (1 + |z_2|^2)^2}.$$

Note that $l \in L^1(\mathbb{C}^2)$ and the right-hand side of (60) can be expressed by l :

$$(\pi_R \otimes \pi_R)((K^{-1} \otimes K^{-1})\Psi) = \int d^2w_1 d^2w_2 l(w_1, w_2)(R_{w_1} \otimes R_{w_2}). \quad (62)$$

We have a similar formula for the right-hand side of (61):

$$(\pi_R \otimes \pi_L)((K^{-1} \otimes K^{-1})\Psi) = \int d^2w_1 d^2w_2 l(w_1, w_2)(R_{w_1} \otimes L_{w_2}). \quad (63)$$

Let $f = (\omega_{x,y} \otimes \text{id})V \in C_0(G)$ be the slice of the Kac–Takesaki operator and $\tilde{h} \in C_0(\mathbb{C})$ the function given by

$$\tilde{h}(w) = \int d^2z \frac{\exp(i\text{Im}(wz))}{(1 + s^{-2}|z|^2)^2}. \quad (64)$$

Let $\pi^{\text{can}} \in \text{Rep}(C_0(G) \rtimes \mathbb{C}^2; L^2(G))$ be the representation introduced in Remark 4.5 of [2] and $\lambda^L, \lambda^R \in \text{Mor}(C_0(\mathbb{C}); C_0(G) \rtimes \mathbb{C}^2)$ the morphisms introduced in the paragraph following Proposition 4.2 of [2]. A simple but tedious computation, which starts with inserting (62) and (63) into (59), leads to the equality

$$\begin{aligned} &\langle x \otimes v | W | y \otimes w \rangle \\ &= \int d^2w_1 d^2w_2 \langle v | \pi^{\text{can}}(\hat{\rho}_{w_1, w_2}^\Psi(\lambda^L(\tilde{h})((1 + \partial_r^* \partial_r)^2(1 + \partial_l^* \partial_l)^2 f)\lambda^R(\tilde{h}))) | w \rangle. \end{aligned}$$

Denoting

$$\lambda^L(\tilde{h})((1 + \partial_r^* \partial_r)^2(1 + \partial_l^* \partial_l)^2 f)\lambda^R(\tilde{h}) \in M(C_0(G) \rtimes \mathbb{C}^2)$$

by b_f we get

$$\langle x \otimes v | W | y \otimes w \rangle = \int d^2w_1 d^2w_2 \langle v | \pi^{\text{can}}(\hat{\rho}_{w_1, w_2}^\Psi(b_f)) | w \rangle. \quad (65)$$

If b_f happens to be in the domain of \mathfrak{E}^Ψ – the averaging map with respect to the twisted dual action $\hat{\rho}^\Psi$ – then in formula (65) we can enter the integral under the scalar product to obtain

$$\langle x \otimes v | W | y \otimes w \rangle = \langle v | \pi^{\text{can}}\left(\int d^2w_1 d^2w_2 \hat{\rho}_{w_1, w_2}^\Psi(b_f)\right) | w \rangle. \quad (66)$$

Equation (66) may then be rewritten as

$$\mathcal{Q}(f) = \pi^{\text{can}}(\mathfrak{E}^\Psi(b_f)).$$

Let us show that this last equation holds whenever f is regular enough. In the next theorem we shall keep the same notation T_l and T_r for normal operators acting on $L^2(G)$ (see (58)) and elements affiliated with $C_0(G) \rtimes \mathbb{C}^2$ (see (10)).

Theorem A.1. Let ∂_l, ∂_r be the complex vector fields on G given by (12) and $f \in C_0(G)$ a continuous function such that $\partial_l^{k*} \partial_l^{k'} \partial_r^{*m} \partial_r^{m'} f \in C_0(G)$ whenever $k, k', l, l' \leq 5$. Let $b_f \in M(C_0(G) \rtimes \mathbb{C}^2)$ be the element given by

$$b_f = \lambda^L(\tilde{h})((1 + \partial_r^* \partial_r)^2 (1 + \partial_l^* \partial_l)^2 f) \lambda^R(\tilde{h}).$$

Then b_f is in the domain of the averaging map \mathfrak{E}^Ψ and there exists a positive constant c such that

$$\|\mathfrak{E}^\Psi(b_f)\| \leq c \max_{k,k',l,l' \leq 5} \sup_{g \in G} |\partial_l^{k*} \partial_l^{k'} \partial_r^{*m} \partial_r^{m'} f|.$$

If $f \in C_0(G)$ is quantizable in the sense of Definition 4.13 of [2], then

$$\mathcal{Q}(f) = \pi^{\text{can}}(\mathfrak{E}^\Psi(f)).$$

To prove the above theorem we shall need the following lemma.

Lemma A.2. Let X be a locally compact Hausdorff space, $\rho: \mathbb{C} \rightarrow \text{Aut}(C_0(X))$ a continuous action and $(C_0(X) \rtimes \mathbb{C}, \lambda, \hat{\rho})$ the canonical \mathbb{C} -product associated with ρ . Let ∂, ∂^* be the differential operators acting on the smooth domain $D^\infty(\rho) \subset C_0(X)$ of the action ρ :

$$\partial f = \frac{\partial}{\partial z} \rho_z f \Big|_{z=0}, \quad \partial^* f = \frac{\partial}{\partial \bar{z}} \rho_z f \Big|_{z=0}.$$

Further, let $\tilde{h} \in C_0(\mathbb{C})$ be the function defined by (64) and let $g \in C_0(X)$ be such that $\partial^{*l} \partial^k g \in C_0(\mathbb{C})$ for $k, l \in \{0, 1\}$. Then $\lambda(\tilde{h})g$ is in the domain of the averaging map \mathfrak{E} and there exists a positive constant $c \in \mathbb{R}$ such that

$$\|\mathfrak{E}(\lambda(\tilde{h})g)\| \leq c \max_{l,k \leq 1} \sup_{x \in X} |\partial^{*l} \partial^k g(x)|. \tag{67}$$

Proof. Using the universal properties of the group C^* -algebra $C^*(\mathbb{C})$ we see that the representation $\lambda \in \text{Rep}(\mathbb{C}; C_0(X) \rtimes \mathbb{C})$ corresponds to a unique element of $\text{Mor}(C^*(\mathbb{C}); C_0(X) \rtimes \mathbb{C})$ (which we also denote by λ). Identifying $C^*(\mathbb{C})$ with $C_0(\mathbb{C})$ (note that we use the self-duality of \mathbb{C}) we can apply λ to $\tilde{h} \in C_0(\mathbb{C})$: $\lambda(\tilde{h}) \in M(C_0(X) \rtimes \mathbb{C})$.

In order to show that $\lambda(\tilde{h})g$ is in the domain of the averaging map $D(\mathfrak{E})$ it is enough to express it as a linear combination of elements of the form

$$\lambda(h_1)b\lambda(h_2), \tag{68}$$

where $h_1, h_2 \in C_0(\mathbb{C}) \cap L^2(\mathbb{C})$ and $b \in C_0(X) \rtimes \mathbb{C}$ (see [5]). Let $T \eta C_0(X) \rtimes \mathbb{C}$ be the image of $\text{id} \in C_0(\mathbb{C})$ under $\lambda \in \text{Mor}(C_0(\mathbb{C}); C_0(X) \rtimes \mathbb{C})$: $T = \lambda(\text{id})$. Note that

$$\begin{aligned} \lambda(\tilde{h})g &= \lambda(\tilde{h})g(1 + T^*T)(1 + T^*T)^{-1} \\ &= \lambda(\tilde{h})g(1 + T^*T)^{-1} + \lambda(\tilde{h}\bar{z})gT(1 + T^*T)^{-1} \\ &\quad + \lambda(\tilde{h})\partial^*gT(1 + T^*T)^{-1}, \end{aligned} \tag{69}$$

where we used the relation

$$\partial^* g = [g, T^*]$$

linking ∂^* and T^* . Note also that $(1 + T^*T)^{-1} = \lambda((1 + |z|^2)^{-1})$, hence

$$\lambda(\tilde{h}) = \lambda((1 + |z|^2)\tilde{h})(1 + T^*T)^{-1}.$$

Therefore, the first summand of the right-hand side of (69) is of the form

$$\lambda(\tilde{h})g(1 + T^*T)^{-1} = \lambda((1 + |z|^2)\tilde{h})((1 + T^*T)^{-1}g)\lambda((1 + |z|^2)^{-1}).$$

Using the fact that \tilde{h} is of the Schwartz type and $(1 + |z|^2)^{-1} \in L^2(\mathbb{C})$ we can see that the above element is of the form (68). Now, by inequality (10) of [2], we get

$$\|\mathfrak{E}(\lambda(\tilde{h})g(1 + T^*T)^{-1})\| \leq \|\tilde{h}\|_2 \|g\| \|(1 + |z|^2)^{-1}\|_2$$

and observe that $\|\mathfrak{E}(\lambda(\tilde{h})g(1 + T^*T)^{-1})\|$ may be estimated by the right-hand side of (67) for c' big enough:

$$\|\mathfrak{E}(\lambda(\tilde{h})g(1 + T^*T)^{-1})\| \leq c' \max_{l,k \leq 1} \sup_{x \in X} |\partial^{*l} \partial^k g(x)|. \tag{70}$$

Let us analyze the second summand of the right-hand side of (69). Note that

$$\lambda(\tilde{h}\bar{z})gT(1 + T^*T)^{-1} = \lambda(\tilde{h}|z|^2)g(1 + T^*T)^{-1} + \lambda(\tilde{h}\bar{z})\partial g(1 + T^*T)^{-1}.$$

A reasoning similar to the one above shows that there exists a constant c'' such that

$$\|\mathfrak{E}(\lambda(\tilde{h}\bar{z})gT(1 + T^*T)^{-1})\| \leq c'' \max_{l,k \leq 1} \sup_{x \in X} |\partial^{*l} \partial^k g(x)|. \tag{71}$$

Similarly, we prove that there exists a constant c''' such that

$$\|\mathfrak{E}(\lambda(\tilde{h})\partial^*gT(1 + T^*T)^{-1})\| \leq c''' \max_{l,k \leq 1} \sup_{x \in X} |\partial^{*l} \partial^k g(x)|. \tag{72}$$

Combining (69), (70), (71) and (72) we get (67) for $c = \max\{c', c'', c'''\}$. □

The above lemma is also true if we replace \mathfrak{E} with \mathfrak{E}^Ψ . An extension of this lemma to the case of an action of \mathbb{C}^2 gives a proof of Theorem A.1. Using the same techniques one can also prove the following theorem:

Theorem A.3. *Let $f \in C_b(G)$ be a function such that $\partial_l^{k*} \partial_l^{k'} \partial_r^{*m} \partial_r^{m'} f \in C_b(G)$ whenever $k, k', l, l' \leq 5$. Let $b_f \in M(C_0(G) \rtimes \mathbb{C}^2)$ be given by*

$$b_f = \lambda^L(\tilde{h})((1 + \partial_r^* \partial_r)^2 (1 + \partial_l^* \partial_l)^2 f) \lambda^R(\tilde{h}).$$

Then $b_f \in D(\mathfrak{E}^\Psi)$, $\mathfrak{E}^\Psi(b_f) \in M(A)$ and there exists a positive constant c such that

$$\|\mathfrak{E}^\Psi(b_f)\| \leq c \max_{k,k',l,l' \leq 5} \sup_{g \in G} |\partial_l^{k*} \partial_l^{k'} \partial_r^{*m} \partial_r^{m'} f|.$$

The element $\mathfrak{E}^\Psi(b_f) \in M(A)$ appearing in the above theorem will also be denoted by $\mathcal{Q}(f)$.

B. Counit in Rieffel deformation

The aim of this section is to show that a quantum group \mathbb{G} obtained by the Rieffel deformation possesses a counit. Our argument is different from the one given by M. Rieffel in [8]. Let G be a locally compact group, $\Gamma \subset G$ its abelian subgroup and Ψ a 2-cocycle on $\hat{\Gamma}$. Let ρ be the action of Γ^2 on $C_0(G)$ given by the left and right shifts and let ρ_Γ be the corresponding action of Γ^2 on $C_0(\Gamma)$. Note that the restriction morphism $\pi_\Gamma: C_0(G) \rightarrow C_0(\Gamma)$ is Γ^2 -covariant. Using Proposition 3.8 of [2] we get the induced morphism $\pi_\Gamma^\Psi \in \text{Mor}(C_0(G)^\Psi; C_0(\Gamma)^\Psi)$. Since Γ is abelian, it follows that the dual quantum group of $(C_0(\Gamma)^\Psi, \Delta)$ coincides with the quantum group $(C^*(\Gamma), \hat{\Delta})$. Therefore $(C_0(\Gamma)^\Psi, \Delta)$ coincides with $(C_0(\Gamma), \Delta)$. This shows that $\pi_\Gamma^\Psi \in \text{Mor}(C_0(G)^\Psi; C_0(\Gamma))$ and enables us to define a counit e for \mathbb{G} by the formula $e(a) = e_\Gamma(\pi_\Gamma^\Psi(a))$ for any $a \in A$, where $e_\Gamma: C_0(\Gamma) \rightarrow \mathbb{C}$ is the counit for $(C_0(\Gamma), \Delta)$.

Let us now draw an important conclusion from the existence of the counit for \mathbb{G} . Using Proposition 5.16 of [9] we can see that the universal dual quantum group of $\hat{\mathbb{G}} = (C_r^*(G), \Delta^\Psi)$ is isomorphic to the reduced dual: $\mathbb{G} = (A, \Delta)$. In particular, representations of C^* -algebra A are in one-to-one correspondence with corepresentations of the quantum group $\hat{\mathbb{G}}$. This follows from Theorem 5.4 of [9].

C. Complex generator of Heisenberg Lie algebra

Let \mathbb{H} be the Heisenberg group, \mathfrak{h} its Lie algebra and \mathcal{E} the enveloping algebra of \mathfrak{h} . \mathcal{E} is generated by an element $a \in \mathcal{E}$ such that the commutator $\lambda = [a^*, a]$ is central in \mathcal{E} . Let A be a C^* -algebra and let $U \in \text{Rep}(\mathbb{H}; A)$ be a representation. As was described in the third chapter of [12], U induces the map

$$dU: \mathcal{E} \rightarrow \{\text{closed maps on } A\}.$$

By $D^\infty(U)$ we shall denote the set of U -smooth elements in A . In the next definition we identify a representation of \mathbb{H} in the C^* -algebra of compact operators $\mathcal{K}(\mathcal{H})$ with the corresponding Hilbert space representation.

Definition C.1. Let \mathcal{H} be a Hilbert space and let $(\tilde{a}, \tilde{\lambda})$ be a pair of closed operators acting on \mathcal{H} . We say that this pair is an infinitesimal representation of \mathbb{H} on \mathcal{H} if there exists a representation $U \in \text{Rep}(\mathbb{H}; \mathcal{K}(\mathcal{H}))$ such that $dU(a) = \tilde{a}$ and $dU(\lambda) = \tilde{\lambda}$.

The representation U in the above definition is determined by \tilde{a} , therefore in this context it will be denoted by $U^{\tilde{a}}$. Let $U \in \text{Rep}(\mathbb{H}; C^*(\mathbb{H}))$ be the canonical representation of \mathbb{H} . The map dU in this case is injective, which enables us to identify $dU(T)$ with $T \in \mathcal{E}$. The aim of this section is to show that $a \in \mathcal{E}$ is affiliated with $C^*(\mathbb{H})$. In fact one can prove that a generates $C^*(\mathbb{H})$ in the sense of Woronowicz, but we shall not use and so will not prove this fact.

Let $M \in \mathcal{E}$. The criterion for a map $dU(M): D(dU(M)) \rightarrow C^*(\mathbb{H})$ to be affiliated with $C^*(\mathbb{H})$ is provided by Theorem 2.1 of [12]. Our proof that $a \eta C^*(\mathbb{H})$ uses a different approach, which is based on the explicit construction of the semigroup $\mathbb{R}_+ \ni t \mapsto \exp(-ta^*a) \in M(C^*(\mathbb{H}))$.

Theorem C.2. *Let a be the complex generator of the algebra \mathcal{E} . Then a is affiliated with $C^*(\mathbb{H})$.*

Proof. For any $z \in \mathbb{C}$, $x \in \mathbb{R}$ and $t \in \mathbb{R}_+$ we set

$$h_t(z, x) = \frac{x \exp tx}{4\pi \sinh tx} \exp\left(-\frac{|z|^2 x \coth tx}{4}\right) \in \mathbb{R}_+. \tag{73}$$

We would like to define an element $H_t \in M(C^*(\mathbb{H}))$ by the integral

$$H_t = \int_{\mathbb{C}} d^2z h_t(z, \frac{1}{2}\lambda) U_{z,0},$$

but there is a problem with its convergence. To circumvent it we observe that for any $b \in C^*(\mathbb{H})$ and $f \in C_c^\infty(\mathbb{C})$ the integral

$$\int d^2z h_t(z, \frac{1}{2}\lambda) U_{z,0} f(\lambda) b$$

converges in the norm sense and the following inequality holds:

$$\left\| \int d^2z h_t(z, \frac{1}{2}\lambda) U_{z,0} f(\lambda) b \right\| \leq \|f(\lambda) b\|.$$

Hence H_t is well defined on the elements of the form $f(\lambda)b$ and by the above inequality it can be extended to the whole $C^*(\mathbb{H})$ giving a self-adjoint element of $M(C^*(\mathbb{H}))$. Let us list some properties of H_t .

- (1) The map $\mathbb{R}_+ \ni t \mapsto H_t \in M(C^*(\mathbb{H}))$ is a norm-continuous semigroup and $\|H_t\| \leq 1$.
- (2) $\lim_{t \rightarrow 0} H_t b = b$ for any $b \in C^*(\mathbb{H})$.
- (3) For any $b \in D^\infty(U)$ the map $\mathbb{R}_+ \ni t \mapsto H_t b$ is differentiable and

$$\frac{d}{dt} H_t b \Big|_{t=0} = -a^* a b.$$

Property (1) enables us to define the element

$$\Xi = \int_{\mathbb{R}_+} dt e^{-t} H_t \in M(C^*(\mathbb{H})).$$

Using properties (2) and (3) we can check that for any $b \in D^\infty(U)$ we have $\Xi b \in D^\infty(U)$ and

$$(1 + a^* a) \Xi b = b.$$

This shows that $\overline{(1 + a^*a)D^\infty(U)}^{\|\cdot\|} = C^*(\mathbb{H})$, which by Proposition 2.2 of [10] is sufficient for a to be affiliated with $C^*(\mathbb{H})$. \square

Remark C.3. Let \mathcal{H} be a Hilbert space. Analyzing the above proof, one can conclude that, given any representation $\pi \in \text{Rep}(C^*(\mathbb{H}); \mathcal{H})$, a compactly supported function $f \in C_0(\mathbb{C})$ and $v \in \mathcal{H}$, we have

$$\exp(-t\pi(a)^*\pi(a))\pi(f(\lambda))v = \int_{\mathbb{C}} d^2z h_t(z, \frac{1}{2}\pi(\lambda))\pi(U_{z,0}f(\lambda))v, \quad (74)$$

where the integral on the right is taken in the sense of norm topology on \mathcal{H} . Let us also note that given any $v \in \mathcal{H}$ such that the differential

$$\frac{\partial}{\partial z}\pi(U_{z,0})v\Big|_{z=0}$$

exists, we have $v \in D(\pi(a))$ and

$$\pi(a)v = 2\frac{\partial}{\partial z}\pi(U_{z,0})v\Big|_{z=0}. \quad (75)$$

Further, let B be a C^* -algebra and $\pi \in \text{Mor}(C^*(\mathbb{H}); B)$. For any $b \in B$, such that the differential

$$\frac{\partial}{\partial z}\pi(U_{z,0})b\Big|_{z=0}$$

exists, we have $b \in D(\pi(a))$ and

$$\pi(a)b = 2\frac{\partial}{\partial z}\pi(U_{z,0})b\Big|_{z=0}. \quad (76)$$

D. Product of affiliated elements

Let A be a C^* -algebra, and let $T_1, T_2 \eta A$. In general, the product of T_1 and T_2 is not well defined, but it can be defined, assuming that T_1 and T_2 commute in a good sense. The construction of the product given here is a generalization of the case when $A = A_1 \otimes A_2$, $T_1 = S_1 \otimes 1$ and $T_2 = 1 \otimes S_2$, where $S_1 \eta A_1$ and $S_2 \eta A_2$. Then the product of T_1 and T_2 is the tensor product $S_1 \otimes S_2 \eta A_1 \otimes A_2$, the construction of which was described in [12].

Definition D.1. Let A be a C^* -algebra and let T_1, T_2 be elements affiliated with A . Let $z_1, z_2 \in M(A)$ be z -transforms of T_1 and T_2 , respectively. We say that T_1 and T_2 strongly commute if

$$z_1z_2 = z_2z_1, \quad z_1^*z_2 = z_2z_1^*.$$

Let T_1 and T_2 be a pair of closed operators acting on a Hilbert space \mathcal{H} . We say that T_1 and T_2 strongly commute if they strongly commute as elements affiliated with the algebra of compact operators $\mathcal{K}(\mathcal{H})$.

Theorem D.2. *Let A be a C^* -algebra and let $T_1, T_2 \in A$ be a strongly commuting pair of affiliated elements. Let us consider the set $D(T_0) = \{a \in D(T_2) \mid T_2 a \in D(T_1)\}$ and define an operator $T_0: D(T_0) \rightarrow A$ by the formula $T_0 a = T_1(T_2 a)$. Then T_0 is a closable operator acting on the Banach space A and its closure T_0^{cl} is affiliated with A . This closure will be denoted by $T_1 T_2$. We also have $T_1 T_2 = T_2 T_1$.*

Proof. We define $T_1 T_2$ using the method described in Theorem 2.3 of [10]. The related matrix $Q \in M(A) \otimes M(\mathbb{C}^2)$ has the form

$$Q = \begin{pmatrix} (1 - z_1^* z_1)^{\frac{1}{2}} (1 - z_2^* z_2)^{\frac{1}{2}} & -z_1^* z_2^* \\ z_1 z_2 & (1 - z_1 z_1^*)^{\frac{1}{2}} (1 - z_2 z_2^*)^{\frac{1}{2}} \end{pmatrix}.$$

(Compare with the matrix Q from the proof of Theorem 6.1 of [12].) Q satisfies all the assumptions of Theorem 2.3, hence it gives rise to an affiliated element. We leave it to the reader to check that this affiliated element is $T_1 T_2 \in A^\eta$ of our theorem. \square

For the needs of this article we shall prove the following lemmas.

Lemma D.3. *Let A be a C^* -algebra, T an element affiliated with A and X a dense subspace of $D(T)$. Then:*

- (1) *If $(1 + T^* T)^{\frac{1}{2}} X$ is dense in A , then X is a core of T .*
- (2) *If $X \subset D(T^* T)$ and $(1 + T^* T) X$ is dense in A , then X is a core of T .*

Proof. It is easy to see that for any dense subspace $X' \subset A$ the set $(1 + T^* T)^{-\frac{1}{2}} X'$ is a core of T . Taking $X' = (1 + T^* T)^{\frac{1}{2}} X$ we get the proof of point (1) of our lemma. To prove point (2) note that $(1 + T^* T)^{-\frac{1}{2}} X'$ is dense in A whenever X' is dense in A . Applying this to the set $(1 + T^* T) X$ of point (2) we see that $(1 + T^* T)^{\frac{1}{2}} X$ is dense in A . Using point (1) we conclude that X is a core of T . \square

Lemma D.4. *Let $T_1, T_2 \in A^\eta$ strongly commute and let $X \subset A$ be a dense subspace. Then the set*

$$(1 + (T_1 T_2)^*(T_1 T_2))(1 + T_1^* T_1)^{-1}(1 + T_2^* T_2)^{-1} X$$

is dense in A . In particular $(1 + T_1^ T_1)^{-1}(1 + T_2^* T_2)^{-1} X$ is a core of $T_1 T_2$.*

Proof. Note that

$$\begin{aligned} & (1 + (T_1 T_2)^*(T_1 T_2))(1 + T_1^* T_1)^{-1}(1 + T_2^* T_2)^{-1} \\ &= (1 + (T_1^* T_1)(T_2^* T_2))(1 + T_1^* T_1)^{-1}(1 + T_2^* T_2)^{-1}. \end{aligned}$$

We express the right-hand side of the above equation using z -transforms of T_1 and T_2 :

$$(1 - z_{|T_1|}^2)(1 - z_{|T_2|}^2) + z_{|T_1|}^2 z_{|T_2|}^2.$$

Let $f: [0, 1] \times [0, 1] \rightarrow \mathbb{R}_+$ be the function defined by

$$f(x_1, x_2) = (1 - x_1^2)(1 - x_2^2) + x_1^2 x_2^2.$$

Note that $f(x_1, x_2) = 0$ if and only if $x_1 = 1$ and $x_2 = 0$, or $x_1 = 0$ and $x_2 = 1$. Let us also define a function $g: [0, 1] \times [0, 1] \rightarrow \mathbb{R}$ by the formula

$$g(x_1, x_2) = (1 - x_1^2)^{\frac{1}{2}}(1 - x_2^2)^{\frac{1}{2}}.$$

We have the implication

$$(f(x_1, x_2) = 0) \implies (g(x_1, x_2) = 0).$$

Using Proposition 6.2 of [12] we get the inclusion

$$\overline{(1 + T_1^* T_1)^{-\frac{1}{2}}(1 + T_2^* T_2)^{-\frac{1}{2}} X}^{\|\cdot\|} \subset \overline{(1 + (T_1^* T_1)(T_2^* T_2))(1 + T_1^* T_1)^{-1}(1 + T_2^* T_2)^{-1} X}^{\|\cdot\|}.$$

We end the proof by noting that $A = \overline{(1 + T_1^* T_1)^{-\frac{1}{2}}(1 + T_2^* T_2)^{-\frac{1}{2}} X}^{\|\cdot\|}$. □

Lemma D.5. *Let $T_1, T_2 \in A^n$ be a strongly commuting pair of operators and let $Y \subset D(T_2^* T_2)$ be such that $(1 + T_2^* T_2)Y$ is dense in A . Then the set*

$$(1 + (T_1 T_2)^*(T_1 T_2))(1 + T_1^* T_1)^{-1} Y$$

is dense in A .

Proof. This follows from the previous proof, with $X = (1 + T_2^* T_2)Y$. □

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