J. Noncommut. Geom. 5 (2011), 331–349
DOI 10.4171/JNCG/78
Journal of Noncommutative Geometry

K-theory for ring C*-algebras attached to polynomial rings over finite fields

Joachim Cuntz[1] and Xin Li[2]

Abstract. We compute the K-theory of ring C*-algebras for polynomial rings over finite fields. The key ingredient is a duality theorem which we had obtained in a previous paper. It allows us to show that the K-theory of these algebras has a ring structure and to determine explicit generators. Our main result also reveals striking similarities between the number field case and the function field case.

Mathematics Subject Classification (2010). Primary 46L05, 46L80; Secondary 14H05.
Keywords. K-theory, ring C*-algebra, function field, polynomial rings over finite fields.

1. Introduction

The theory of ring C*-algebras, initiated in [Cun3], has been developed in [CuLi1], [Li] and [CuLi2]. The present paper continues our work in [CuLi2] where we studied ring C*-algebras associated to rings of integers in number fields. In [CuLi2] we proved a duality theorem which was a key ingredient in the computation of the K-theory of these algebras. It allowed us to pass from the finite adele ring to the infinite one where we could use homotopy arguments to determine the K-theory.

In the present paper, we turn to the case of the function field of the projective line over a finite field. More precisely, our goal is to compute the K-theory of the ring C*-algebra for $\mathbb{F}_q[T]$ where q is a prime power, i.e., $q = p^n$ for some prime number p. Since our duality theorem holds for arbitrary global fields (see [CuLi2]), we can apply it to function fields as well. However, since in that case the infinite adele space is totally disconnected, we cannot hope for homotopy arguments.

Nevertheless, the duality theorem and the passage from the finite to the infinite adele space give us, in a somewhat unexpected way, a different handle on the computation of K-theory. It allows us to find explicit generators for the K-theory which have sufficiently nice properties. These generators are not visible in the representa-

[1]Research supported by the Deutsche Forschungsgemeinschaft (SFB 478).
[2]The second named author is supported by the Deutsche Telekom Stiftung. This work was done in the context of his PhD project at the University of Münster.

tion over the finite adele space. At the same time, this explicit description reveals a ring structure on the K-theory.

As the final result, we obtain that the K-theory for the ring C*-algebra of $\mathbb{F}_q[T]$ can be described as the tensor product over $\mathbb{Z}$ of $\widetilde{K}_0(C^*(\mathbb{F}_q^\times))$ and the exterior $\mathbb{Z}$-algebra over the torsion-free part of the multiplicative group $\mathbb{F}_q(T)^\times$, where $\widetilde{K}_0(C^*(\mathbb{F}_q^\times))$ is the reduced K-theory of $C^*(\mathbb{F}_q^\times)$ (i.e., the cokernel of the canonical map $K_0(\mathbb{C}) \to K_0(C^*(\mathbb{F}_q^\times))$). This formula is compatible with the ring structure.

We proceed as follows: First of all, we recall the notion of ring C*-algebras. We also summarize the results of [CuLi2] (Section 2). Then, we determine the K-theory for the ring C*-algebra of $\mathbb{F}_q[T]$: First, using the duality theorem, we reduce our problem to computing $K_*(C_0(\mathbb{F}_q((T))) \rtimes \mathbb{F}_q(T) \rtimes \mathbb{F}_q(T)^\times)$ (Section 3). Secondly, we start with computing $K_*(C_0(\mathbb{F}_q((T))) \rtimes \mathbb{F}_q(T) \rtimes (\mathbb{F}_q^\times \times \langle T \rangle))$. It turns out that we can find explicit generators, projections and unitaries, for the K-groups (Section 5). The crucial point is that these projections and unitaries commute with all the remaining unitaries one still has to adjoin in order to pass from $C_0(\mathbb{F}_q((T))) \rtimes \mathbb{F}_q(T) \rtimes (\mathbb{F}_q^\times \times \langle T \rangle)$ to $C_0(\mathbb{F}_q((T))) \rtimes \mathbb{F}_q(T) \rtimes \mathbb{F}_q(T)^\times$ (Section 6). Finally, the computation is completed by comparing our situation with commutative tori of suitable dimensions (Section 7).

2. Review

Let K be a global field and R the ring of integers in K. The ring C*-algebra $\mathfrak{A}[R]$ is defined as follows: Consider the Hilbert space $\ell^2(R)$ with canonical orthonormal basis $\{\xi_r : r \in R\}$. Define additive shifts U^a via $U^a(\xi_r) = \xi_{a+r}$ and multiplicative shift operators S_b by $S_b(\xi_r) = \xi_{br}$ (for $b \neq 0$). These unitaries and isometries generate a C*-subalgebra of $\mathcal{L}(\ell^2(R))$, the ring C*-algebra $\mathfrak{A}[R]$. This concrete C*-algebra admits several alternative descriptions. For our purposes, the following one is important (see [CuLi1], Remark 3 and Section 5):

Theorem 2.1. *$\mathfrak{A}[R]$ is Morita equivalent to $C_0(\mathbb{A}_f) \rtimes K \rtimes K^\times$.*

The crossed product is taken with respect to the canonical action of $K \rtimes K^\times$ on the finite adele ring $\mathbb{A}_f$ of K via affine transformations.

Moreover, we proved the following duality result (see [CuLi2], Theorem 4.1 and Corollary 4.2):

Theorem 2.2. *For every global field K, the crossed products $C_0(\mathbb{A}_\infty) \rtimes K \rtimes K^\times$ and $C_0(\mathbb{A}_f) \rtimes K \rtimes K^\times$ are Morita equivalent.*

Here $\mathbb{A}_\infty$ is the infinite adele ring of K. The crossed products arise from the natural actions of $K \rtimes K^\times$ on $\mathbb{A}_\infty$ and $\mathbb{A}_f$ via affine transformations.

This duality theorem allowed us to use homotopy arguments to determine $K_*(\mathfrak{A}[R])$ for the ring of integers R in a number field K. Our final result is (see [CuLi2], Section 6):

Let K be a number field with roots of unity $\mu = \{\pm 1\}$ and ring of integers R. Let $\#\{v_{\mathbb{R}}\}$ be the number of real places of K. There is a decomposition $K^\times = \mu \times \Gamma$, where Γ is a free abelian group on infinitely many generators, such that the K-theory of the ring C*-algebra of R can be described as follows:

Theorem 2.3.

$$K_*(\mathfrak{A}[R]) \cong \begin{cases} K_0(C^*(\mu)) \otimes_{\mathbb{Z}} \Lambda^*(\Gamma) \text{ if } \#\{v_{\mathbb{R}}\} = 0, \\ \Lambda^*(\Gamma) \text{ if } \#\{v_{\mathbb{R}}\} \text{ is odd}, \\ \Lambda^*(\Gamma) \oplus ((\mathbb{Z}/2\mathbb{Z}) \otimes_{\mathbb{Z}} \Lambda^*(\Gamma)) \text{ if } \#\{v_{\mathbb{R}}\} \text{ is even and at least } 2. \end{cases}$$

This isomorphism is meant as an isomorphism between $\mathbb{Z}/2\mathbb{Z}$-graded abelian groups. Here $K_*(\mathfrak{A}[R])$ is the canonically graded group $K_0(\mathfrak{A}[R]) \oplus K_1(\mathfrak{A}[R])$, $K_0(C^*(\mu))$ and $\mathbb{Z}/2\mathbb{Z}$ are trivially graded, the exterior $\mathbb{Z}$-algebra $\Lambda^*(\Gamma)$ is canonically graded and we consider graded tensor products.

3. Applying the duality theorem

Now we turn to function fields. Let us consider the case $K = \mathbb{F}_q(T)$ and $R = \mathbb{F}_q[T]$ for a prime power q. Our goal is to determine the K-theory of $\mathfrak{A}[R]$. By Theorem 2.1, we know that $\mathfrak{A}[R] \sim_M C_0(\mathbb{A}_f) \rtimes K \rtimes K^\times$. Moreover, Theorem 2.2 yields $C_0(\mathbb{A}_\infty) \rtimes K \rtimes K^\times \sim_M C_0(\mathbb{A}_f) \rtimes K \rtimes K^\times$. Thus we have to compute the K-theory of $C_0(\mathbb{A}_\infty) \rtimes K \rtimes K^\times$.

It is our convention that the infinite adele ring over $K = \mathbb{F}_q(T)$ is given by

$$\mathbb{A}_\infty \cong \mathbb{F}_q((T)) = \left\{ \sum_{i=n}^{\infty} a_i T^i : n \in \mathbb{Z},\, a_i \in \mathbb{F}_q \right\}.$$

$\mathbb{F}_q((T))$ is a locally compact field with respect to the valuation

$$\Big| \sum_{i=n}^{\infty} a_i T^i \Big| = q^{-n} \quad \text{if } a_n \neq 0.$$

Moreover, to form the crossed product $C_0(\mathbb{A}_\infty) \rtimes K \rtimes K^\times$, we also need to know how K sits inside $\mathbb{A}_\infty$. The embedding $K \hookrightarrow \mathbb{A}_\infty$ is determined by

$$K \supseteq \mathbb{F}_q[T] \ni a(T) \mapsto a(T^{-1}) \in \mathbb{F}_q((T)) \cong \mathbb{A}_\infty$$

(it is our convention that the infinite place of K is given by the valuation $|a/b|_\infty = q^{\deg(a)-\deg(b)}$ for $a \in \mathbb{F}_q[T]$, $b \in \mathbb{F}_q[T]^\times$).

Let $\tilde{v}^a$, $\tilde{t}_b$ be the unitaries in the multiplier algebra of $C_0(\mathbb{A}_\infty) \rtimes K \rtimes K^\times$ which implement the additive and the multiplicative action, respectively. In other words, we have

$$\tilde{v}^a \tilde{t}_b f \tilde{t}_b^* (\tilde{v}^a)^* = f(\sigma(b)^{-1}(\sqcup - \sigma(a)))$$

for every $f \in C_0(\mathbb{F}_q((T)))$, where σ is the ring isomorphism

$$\mathbb{F}_q(T) \to \mathbb{F}_q(T), \quad a(T) \mapsto a(T^{-1}).$$

We observe that we can equally well consider the crossed product associated to the canonical embedding $\mathbb{F}_q(T) \hookrightarrow \mathbb{F}_q((T))$. We denote this crossed product by $C_0(\mathbb{F}_q((T))) \rtimes \mathbb{F}_q(T) \rtimes \mathbb{F}_q(T)^\times$ and let v^a, t_b be the canonical unitaries in the multiplier algebra of this crossed product corresponding to addition and multiplication, respectively. We can identify $C_0(\mathbb{F}_q((T))) \rtimes \mathbb{F}_q(T) \rtimes \mathbb{F}_q(T)^\times$ and $C_0(\mathbb{A}_\infty) \rtimes K \rtimes K^\times$ via $f v^a t_b \mapsto f \tilde{v}^{\sigma(a)} \tilde{t}_{\sigma(b)}$. To be more precise, this homomorphism identifies the *-algebras $C_c(\mathbb{F}_q(T) \rtimes \mathbb{F}_q(T)^\times, C_0(\mathbb{F}_q((T))))$ and $C_c(K \rtimes K^\times, C_0(\mathbb{A}_\infty))$ viewed as *-subalgebras of $C_0(\mathbb{F}_q((T))) \rtimes \mathbb{F}_q(T) \rtimes \mathbb{F}_q(T)^\times$ or $C_0(\mathbb{A}_\infty) \rtimes K \rtimes K^\times$, respectively. Furthermore, this map is isometric with respect to the ℓ^1-norms, so that it extends to an isomorphism of the crossed products.

Thus our task is to determine the K-theory of $C_0(\mathbb{F}_q((T))) \rtimes \mathbb{F}_q(T) \rtimes \mathbb{F}_q(T)^\times$.

4. Notation

In the following, let $\mathbb{1}_{[X]}$ be the characteristic function of a subset X in $\mathbb{F}_q((T))$. In particular, the ring of power series $\mathbb{F}_q[[T]] = \{\sum_{i=0}^\infty a_i T^i : a_i \in \mathbb{F}_q\}$ sits inside $\mathbb{F}_q((T))$, and we denote by $\mathbb{1}_n$ the characteristic function $\mathbb{1}_{[T^n \cdot \mathbb{F}_q[[T]]]}$. The characteristic function of $\mathbb{F}_q[[T]]$ is denoted by $\mathbb{1}$ (i.e., $\mathbb{1} := \mathbb{1}_0$). Since the subset $T^n \cdot \mathbb{F}_q[[T]]$ is closed and open in $\mathbb{F}_q((T))$, the functions $\mathbb{1}_n$ and $\mathbb{1}$ lie in $C_0(\mathbb{F}_q((T)))$.

Moreover, as we already had above, let v^a, t_b be the canonical unitaries in the multiplier algebra of $C_0(\mathbb{F}_q((T))) \rtimes \mathbb{F}_q(T) \rtimes \mathbb{F}_q(T)^\times$ implementing the additive or the multiplicative action, respectively.

Furthermore, let $f_1, f_2, f_3, \ldots$ be an enumeration of the irreducible polynomials in $\mathbb{F}_q[T]$ with constant term 1, i.e., $f_i \in 1 + T \cdot \mathbb{F}_q[T]$. Let Γ be the subgroup of $\mathbb{F}_q(T)^\times$ generated by the polynomials T and $f_1, f_2, f_3, \ldots$. Γ is a free abelian group, and free generators are precisely given by $T, f_1, f_2, f_3, \ldots$. We have the decomposition $\mathbb{F}_q(T)^\times = \mathbb{F}_q^\times \times \Gamma$. Let $\Gamma_m := \langle T, f_1, \ldots, f_m \rangle$.

We will determine $K_*(C_0(\mathbb{F}_q((T))) \rtimes \mathbb{F}_q(T) \rtimes \mathbb{F}_q(T)^\times)$ step by step, so it will be helpful to choose appropriate C*-subalgebras. Let

$$A_{-1} := C_0(\mathbb{F}_q((T))) \rtimes \mathbb{F}_q(T) \rtimes \mathbb{F}_q^\times$$

and

$$A_m := C_0(\mathbb{F}_q((T))) \rtimes \mathbb{F}_q(T) \rtimes (\mathbb{F}_q^\times \times \Gamma_m) \quad \text{for all } m \in \mathbb{Z}_{\geq 0}.$$

If μ denotes the multiplicative action of Γ on A_{-1}, then we have $A_0 \cong A_{-1} \rtimes_{\mu_T} \mathbb{Z}$ and $A_m \cong A_{m-1} \rtimes_{\mu_{f_m}} \mathbb{Z}$ for all $m \in \mathbb{Z}_{\geq 0}$. Finally, $C_0(\mathbb{F}_q((T))) \rtimes \mathbb{F}_q(T) \rtimes \mathbb{F}_q(T)^\times$ is isomorphic to $\varinjlim A_m$ with respect to the canonical maps $A_{m-1} \to A_m$. Thus we have to determine the K-theory of A_m for each m.

5. Explicit generators for K-theory

The first step is to determine the K-theory of A_{-1}. It turns out that A_{-1} is approximately finite dimensional, so we just have to find a suitable description of A_{-1} as an inductive limit of finite dimensional C*-algebras to compute its K-theory.

5.1. Filtrations. Let μ_T be the endomorphism of $C(\mathbb{F}_q[[T]]) \rtimes (\Gamma \cdot \mathbb{F}_q[T]) \rtimes \mathbb{F}_q^\times$ induced by multiplication with T. μ_T is given by

$$\mu_T(f v^a t_b) = (f(T^{-1}\sqcup) \cdot \mathbb{1}(T^{-1}\sqcup))v^{aT} t_b.$$

Lemma 5.1. *A_{-1} can be identified with the inductive limit of the system*

$$\cdots \xrightarrow{\mu_T} C(\mathbb{F}_q[[T]]) \rtimes (\Gamma \cdot \mathbb{F}_q[T]) \rtimes \mathbb{F}_q^\times \xrightarrow{\mu_T} C(\mathbb{F}_q[[T]]) \rtimes (\Gamma \cdot \mathbb{F}_q[T]) \rtimes \mathbb{F}_q^\times \xrightarrow{\mu_T} \cdots .$$

Proof. The idea is that going over to this inductive limit corresponds to formally inverting μ_T.

To prove the claim, consider for each $n \in \mathbb{Z}_{\geq 0}$ the homomorphism

$$C(\mathbb{F}_q[[T]]) \rtimes (\Gamma \cdot \mathbb{F}_q[T]) \rtimes \mathbb{F}_q^\times \to A_{-1}, \quad f v^a t_b \mapsto f(T^n\sqcup)v^{a/T^n} t_b.$$

This family of homomorphisms is compatible with μ_T and thus gives rise to a homomorphism

$$\varinjlim\{C(\mathbb{F}_q[[T]]) \rtimes (\Gamma \cdot \mathbb{F}_q[T]) \rtimes \mathbb{F}_q^\times; \mu_T\} \to A_{-1}.$$

This homomorphism is clearly surjective. To see injectivity, consider for each $n \in \mathbb{Z}_{\geq 0}$ the commutative square

$$\begin{array}{ccc} C(\mathbb{F}_q[[T]]) \rtimes (\Gamma \cdot \mathbb{F}_q[T]) \rtimes \mathbb{F}_q^\times & \longrightarrow & A_{-1} \\ \downarrow & & \downarrow \\ C(\mathbb{F}_q[[T]]) & \longrightarrow & C_0(\mathbb{F}_q((T))), \end{array}$$

where the upper horizontal arrow is the homomorphism introduced above (for the n we have chosen) and the lower horizontal arrow is given by $f \mapsto f(T^n\sqcup)$. The vertical arrows are the canonical faithful conditional expectations. They exist because we are dealing with discrete amenable groups. As the lower horizontal homomorphism is clearly injective, the upper one has to be so as well. This proves injectivity for each n and thus for the induced homomorphism on the inductive limit. □

Now, for every $d \in \Gamma$ let μ_d be the endomorphism of $C(\mathbb{F}_q[[T]]) \rtimes \mathbb{F}_q[T] \rtimes \mathbb{F}_q^\times$ induced by multiplication with d. It is given by $f v^a t_b \mapsto f(d^{-1}\sqcup)v^{da} t_b$. We have

Lemma 5.2.

$$C(\mathbb{F}_q[[T]]) \rtimes (\Gamma \cdot \mathbb{F}_q[T]) \rtimes \mathbb{F}_q^\times \cong \varinjlim_{d \in \Gamma} \{C(\mathbb{F}_q[[T]]) \rtimes \mathbb{F}_q[T] \rtimes \mathbb{F}_q^\times; \mu_d\}.$$

Proof. This can be proven analogously to the previous lemma. □

Moreover, let $(\mathbb{F}_q[T])^{(n)}$ be the additive subgroup $\{a_0 + \cdots + a_n T^n : a_i \in \mathbb{F}_q\}$ of $\mathbb{F}_q[T]$. For each n in $\mathbb{Z}_{>0}$, we can identify $(\mathbb{F}_q[T])^{(n-1)}$ and $\mathbb{F}_q[T]/T^n \cdot \mathbb{F}_q[T]$ as additive groups. Thus $(\mathbb{F}_q[T])^{(n-1)}$ acts additively on $C(\mathbb{F}_q[T]/T^n \cdot \mathbb{F}_q[T])$. This additive action and the multiplicative action of $\mathbb{F}_q^\times$ give rise to the crossed product

$$C(\mathbb{F}_q[T]/T^n \cdot \mathbb{F}_q[T]) \rtimes (\mathbb{F}_q[T])^{(n-1)} \rtimes \mathbb{F}_q^\times.$$

Let $\iota_{n,n+1}$ be the homomorphism

$$C(\mathbb{F}_q[T]/T^n \cdot \mathbb{F}_q[T]) \rtimes (\mathbb{F}_q[T])^{(n-1)} \rtimes \mathbb{F}_q^\times \to C(\mathbb{F}_q[T]/T^{n+1} \cdot \mathbb{F}_q[T]) \rtimes (\mathbb{F}_q[T])^{(n)} \rtimes \mathbb{F}_q^\times$$

given by $g v^a t_b \mapsto (g \circ \pi_{n+1,n}) v^a t_b$ with the canonical projection $\pi_{n+1,n}$ from $\mathbb{F}_q[T]/T^{n+1} \cdot \mathbb{F}_q[T]$ onto $\mathbb{F}_q[T]/T^n \cdot \mathbb{F}_q[T]$. We have

Lemma 5.3.

$$C(\mathbb{F}_q[[T]]) \rtimes \mathbb{F}_q[T] \rtimes \mathbb{F}_q^\times \cong \varinjlim_{n} \{C(\mathbb{F}_q[T]/T^n \cdot \mathbb{F}_q[T]) \rtimes (\mathbb{F}_q[T])^{(n-1)} \rtimes \mathbb{F}_q^\times; \iota_{n,n+1}\}.$$

Proof. Again, the proof is analogous to the one of Lemma 5.1. The point is that $\mathbb{F}_q[[T]]$ can be identified with

$$\varprojlim_{n} \{\mathbb{F}_q[T]/T^{n+1} \cdot \mathbb{F}_q[T]; \pi_{n+1,n}\}$$

both algebraically and topologically. □

5.2. Explicit generators for the K-groups. The preceding filtrations allow us to compute the K-theory of A_{-1}. The first step is the following

Lemma 5.4.

$$C(\mathbb{F}_q[T]/T^n \cdot \mathbb{F}_q[T]) \rtimes (\mathbb{F}_q[T])^{(n-1)} \rtimes \mathbb{F}_q^\times \cong M_{q^n}(\mathbb{C}) \otimes C^*(\mathbb{F}_q^\times).$$

Proof. Let $e_n \in C(\mathbb{F}_q[T]/T^n \cdot \mathbb{F}_q[T])$ be the characteristic function of the coset $0 + T^n \cdot \mathbb{F}_q[T]$. It is clear that $\{v^a e_n v^{-a'} : a, a' \in (\mathbb{F}_q[T])^{(n-1)}\}$ are matrix units. Let $e_{a,a'}$ be the canonical rank 1 operator in $\mathcal{L}(\ell^2(\mathbb{F}_q[T]/T^n \cdot \mathbb{F}_q[T]))$ corresponding to the cosets $a + T^n \cdot \mathbb{F}_q[T]$ and $a' + T^n \cdot \mathbb{F}_q[T]$. We can identify $C(\mathbb{F}_q[T]/T^n \cdot \mathbb{F}_q[T]) \rtimes (\mathbb{F}_q[T])^{(n-1)}$ with $\mathcal{L}(\ell^2(\mathbb{F}_q[T]/T^n \cdot \mathbb{F}_q[T])) \cong M_{q^n}(\mathbb{C})$ via

$$v^a e_n v^{-a'} \mapsto e_{a,a'}.$$

Moreover, the action of $\mathbb{F}_q^\times$ on $C(\mathbb{F}_q[T]/T^n \cdot \mathbb{F}_q[T]) \rtimes (\mathbb{F}_q[T])^{(n-1)}$ must be inner as we have seen that $C(\mathbb{F}_q[T]/T^n \cdot \mathbb{F}_q[T]) \rtimes (\mathbb{F}_q[T])^{(n-1)}$ is isomorphic to a matrix algebra. The unitaries implementing the action of $\mathbb{F}_q^\times$ are given by

$$\sum_{a \in (\mathbb{F}_q[T])^{(n-1)}} v^{ba} e_n v^{-a}$$

for $b \in \mathbb{F}_q^\times$.

So on the whole, we obtain the identification

$$C(\mathbb{F}_q[T]/T^n \cdot \mathbb{F}_q[T]) \rtimes (\mathbb{F}_q[T])^{(n-1)} \rtimes \mathbb{F}_q^\times \cong \mathcal{L}(\ell^2(\mathbb{F}_q[T]/T^n \cdot \mathbb{F}_q[T])) \otimes C^*(\mathbb{F}_q^\times)$$

via

$$v^a e_n v^{-a'} t_b \mapsto e_{a,b^{-1}a'} \otimes V_b,$$

where V_b are the canonical unitary generators of $C^*(\mathbb{F}_q^\times)$. □

For every character χ of $\mathbb{F}_q^\times$, let p_χ be the spectral projection

$$\frac{1}{q-1} \sum_{b \in \mathbb{F}_q^\times} \chi(b) t_b$$

in $C(\mathbb{F}_q[T]/T^n \cdot \mathbb{F}_q[T]) \rtimes (\mathbb{F}_q[T])^{(n-1)} \rtimes \mathbb{F}_q^\times$. As an immediate consequence of Lemma 5.4 we get

Corollary 5.5.

$$K_0(C(\mathbb{F}_q[T]/T^n \cdot \mathbb{F}_q[T]) \rtimes (\mathbb{F}_q[T])^{(n-1)} \rtimes \mathbb{F}_q^\times) \cong \bigoplus_{\widehat{\mathbb{F}_q^\times}} \mathbb{Z} \ (\cong \mathbb{Z}^{q-1})$$

and free generators for K_0 *are* $[e_n \cdot p_\chi]$, $\chi \in \widehat{\mathbb{F}_q^\times}$.

$K_1(C(\mathbb{F}_q[T]/T^n \cdot \mathbb{F}_q[T]) \rtimes (\mathbb{F}_q[T])^{(n-1)} \rtimes \mathbb{F}_q^\times)$ *vanishes.*

Recall that $e_n \in C(\mathbb{F}_q[T]/T^n \cdot \mathbb{F}_q[T])$ is the characteristic function of the coset $0 + T^n \cdot \mathbb{F}_q[T]$.

Just a remark on notation: $[\cdot]$ denotes a class in K-theory.

By continuity of K_1 and with the help of Lemmas 5.1, 5.2 and 5.3, we deduce from the previous corollary

Corollary 5.6.

$$K_1(A_{-1}) \cong \{0\}.$$

It remains to determine $K_0(A_{-1})$.

Lemma 5.7. *We can identify* $K_0(C(\mathbb{F}_q[[T]]) \rtimes \mathbb{F}_q[T] \rtimes \mathbb{F}_q^\times)$ *with*

$$\mathbb{Z}[\tfrac{1}{q}] \oplus \bigoplus_{\widehat{\mathbb{F}_q^\times} \setminus \{1\}} \mathbb{Z} \ (\cong \mathbb{Z}[\tfrac{1}{q}] \oplus \mathbb{Z}^{q-2}).$$

Moreover, the identification can be chosen so that the n*-th embedding*

$$\iota_n \colon C(\mathbb{F}_q[T]/T^n \cdot \mathbb{F}_q[T]) \rtimes (\mathbb{F}_q[T])^{(n-1)} \rtimes \mathbb{F}_q^\times \to C(\mathbb{F}_q[[T]]) \rtimes \mathbb{F}_q[T] \rtimes \mathbb{F}_q^\times$$

is given on K_0 *by*

$$(\iota_n)_*([e_n]) = \begin{pmatrix} \frac{1}{q^n} \\ 0 \\ \vdots \\ 0 \end{pmatrix}; \quad (\iota_n)_*([e_n \cdot p_\chi]) = \begin{pmatrix} -\frac{1}{q^n}\frac{q^n-1}{q-1} \\ 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{pmatrix}$$

for every $\chi \in \widehat{\mathbb{F}_q^\times} \setminus \{1\}$. *Here, the "1" in the image of* $[e_n \cdot p_\chi]$ *is the entry corresponding to* χ *in* $\bigoplus_{\widehat{\mathbb{F}_q^\times} \setminus \{1\}} \mathbb{Z}$.

In particular, generators for $K_0(C(\mathbb{F}_q[[T]]) \rtimes \mathbb{F}_q[T] \rtimes \mathbb{F}_q^\times)$ *are given by*

$$[\mathbb{1}_n],\ n \in \mathbb{Z}_{\geq 0}, \quad \text{and} \quad [p_\chi],\ \chi \in \widehat{\mathbb{F}_q^\times} \setminus \{1\}.$$

$\mathbb{1}_n$ *is the characteristic function of* $T^n \cdot \mathbb{F}_q[[T]]$, *and* $1 \in \widehat{\mathbb{F}_q^\times}$ *denotes the trivial character.*

Proof. With Lemma 5.3 in mind, we compute $(\iota_{n,n+1})_*$. By definition,

$$\iota_{n,n+1}(e_n) = \sum_{b \in \mathbb{F}_q} v^{bT^n} e_{n+1} v^{-bT^n}; \qquad \iota_{n,n+1}(t_b) = t_b.$$

Thus, by Corollary 5.5, we have to determine

$$[\iota_{n,n+1}(e_n \cdot p_\chi)] = [(\textstyle\sum_{b \in \mathbb{F}_q} v^{bT^n} e_{n+1} v^{-bT^n}) \cdot p_\chi]$$

in $K_0(C(\mathbb{F}_q[T]/T^{n+1} \cdot \mathbb{F}_q[T]) \rtimes (\mathbb{F}_q[T])^{(n)} \rtimes \mathbb{F}_q^\times)$.

First of all, we have

$$\begin{aligned} p_\chi \cdot \Big(\sum_{b \in \mathbb{F}_q^\times} \psi(b) v^{bT^n} e_{n+1} v^{-bT^n} \Big) &= \frac{1}{q-1} \sum_{b,b' \in \mathbb{F}_q^\times} \chi(b')\psi(b) t_{b'} v^{bT^n} e_{n+1} v^{-bT^n} \\ &= \frac{1}{q-1} \sum_{b,b' \in \mathbb{F}_q^\times} \chi(b')\psi(b) v^{b'bT^n} e_{n+1} v^{-b'bT^n} t_{b'} \\ &= \frac{1}{q-1} \sum_{b' \in \mathbb{F}_q^\times} \Big(\sum_{b \in \mathbb{F}_q^\times} \psi(b')\psi(b) v^{b'bT^n} e_{n+1} v^{-b'bT^n} \Big) \overline{\psi}(b') \chi(b') t_{b'} \\ &= \Big(\sum_{b \in \mathbb{F}_q^\times} \psi(b) v^{bT^n} e_{n+1} v^{-bT^n} \Big) \cdot p_{\overline{\psi} \cdot \chi} \end{aligned} \tag{1}$$

for every ψ, χ in $\widehat{\mathbb{F}_q^\times}$. This result implies that the projections $(e_n - e_{n+1}) \cdot p_{\bar{\psi}\cdot\chi}$ and $(e_n - e_{n+1}) \cdot p_\chi = (\sum_{b\in\mathbb{F}_q^\times} v^{bT^n} e_{n+1} v^{-bT^n}) \cdot p_\chi$ are Murray–von Neumann equivalent via the partial isometry

$$p_\chi \cdot \Big(\sum_{b\in\mathbb{F}_q^\times} \psi(b) v^{bT^n} e_{n+1} v^{-bT^n} \Big).$$

This shows that in $K_0(C(\mathbb{F}_q[T]/T^{n+1}\cdot\mathbb{F}_q[T]) \rtimes (\mathbb{F}_q[T])^{(n)} \rtimes \mathbb{F}_q^\times)$, the following equality holds true:

$$\begin{aligned}(q-1)[e_{n+1}] &= \Big[\sum_{b\in\mathbb{F}_q^\times} v^{bT^n} e_{n+1} v^{-bT^n} \Big] \\ &= [e_n - e_{n+1}] \\ &= \sum_{\psi\in\widehat{\mathbb{F}_q^\times}} [(e_n - e_{n+1}) \cdot p_{\bar{\psi}\cdot\chi}] \\ &= (q-1)[(e_n - e_{n+1}) \cdot p_\chi]\end{aligned}$$

for every $\chi \in \widehat{\mathbb{F}_q^\times}$. Comparing this with

$$(q-1)[e_{n+1}] = (q-1) \sum_{\psi\in\widehat{\mathbb{F}_q^\times}} [e_{n+1}\cdot p_\psi],$$

we deduce

$$[(e_n - e_{n+1}) \cdot p_\chi] = \sum_{\psi\in\widehat{\mathbb{F}_q^\times}} [e_{n+1}\cdot p_\psi]$$

for every $\chi \in \widehat{\mathbb{F}_q^\times}$. Therefore,

$$(\iota_{n,n+1})_*([e_n\cdot p_\chi]) = [(e_{n+1} + (e_n - e_{n+1}))\cdot p_\chi] = [(e_{n+1}\cdot p_\chi)] + \sum_{\psi\in\widehat{\mathbb{F}_q^\times}} [e_{n+1}\cdot p_\psi].$$

Hence, under the identification

$$K_0(C(\mathbb{F}_q[T]/T^n\cdot\mathbb{F}_q[T]) \rtimes (\mathbb{F}_q[T])^{(n-1)} \rtimes \mathbb{F}_q^\times) \cong \mathbb{Z}^{q-1}$$

in Corollary 5.5, we get

$$(\iota_{n,n+1})_* = \begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix} + \begin{pmatrix} 1 & \dots & 1 \\ \vdots & 1 & \vdots \\ 1 & \dots & 1 \end{pmatrix} = \begin{pmatrix} 2 & & 1 \\ & \ddots & \\ 1 & & 2 \end{pmatrix}.$$

Finally, by Lemma 5.3, we compute

$$K_0(C(\mathbb{F}_q[[T]]) \rtimes \mathbb{F}_q[T] \rtimes \mathbb{F}_q^\times) \cong \varinjlim \left\{ \mathbb{Z}^{q-1}; \begin{pmatrix} 2 & & 1 \\ & \ddots & \\ 1 & & 2 \end{pmatrix} \right\} \cong \mathbb{Z}[\tfrac{1}{q}] \oplus \bigoplus_{\widehat{\mathbb{F}_q^\times}\setminus\{1\}} \mathbb{Z}.$$

Moreover, we can choose this identification so that $(\iota_n)_*$ is given by

$$(\iota_n)_*([e_n]) = \begin{pmatrix} \frac{1}{q^n} \\ 0 \\ \vdots \\ 0 \end{pmatrix}; \quad (\iota_n)_*([e_n \cdot p_\chi]) = \begin{pmatrix} -\frac{1}{q^n}\frac{q^n-1}{q-1} \\ 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{pmatrix} \text{ for all } \chi \in \widehat{\mathbb{F}_q^\times} \setminus \{1\}.$$

The last statement about the generators of $K_0(C(\mathbb{F}_q[[T]]) \rtimes \mathbb{F}_q[T] \rtimes \mathbb{F}_q^\times)$ follows from the observation that e_n is sent to $\mathbb{1}_n$ under ι_n. □

From now on, we fix this particular description of $K_0(C(\mathbb{F}_q[[T]]) \rtimes \mathbb{F}_q[T] \rtimes \mathbb{F}_q^\times)$. $\mathbb{1}_n$ is the characteristic function of $T^n \cdot \mathbb{F}_q[[T]] \subseteq \mathbb{F}_q[[T]]$. For all d in Γ,

$$\mu_d(\mathbb{1}_n) = \mathbb{1}_{[T^n \cdot \mathbb{F}_q[[T]]]}(d^{-1} \sqcup) = \mathbb{1}_{[d \cdot (T^n \cdot \mathbb{F}_q[[T]])]} = \mathbb{1}_n$$

because every d in Γ is invertible in $\mathbb{F}_q[[T]]$ (μ and Γ are defined in Section 4). Moreover, μ_d certainly leaves p_χ invariant. Therefore, by Lemma 5.7, $(\mu_d)_* = \text{id}$ on $K_0(C(\mathbb{F}_q[[T]]) \rtimes \mathbb{F}_q[T] \rtimes \mathbb{F}_q^\times)$. This, together with Lemma 5.2, implies

Corollary 5.8.

$$K_0(C(\mathbb{F}_q[[T]]) \rtimes (\Gamma \cdot \mathbb{F}_q[T]) \rtimes \mathbb{F}_q^\times) \cong \mathbb{Z}[\tfrac{1}{q}] \oplus \bigoplus_{\widehat{\mathbb{F}_q^\times} \setminus \{1\}} \mathbb{Z}$$

and the canonical inclusion

$$C(\mathbb{F}_q[[T]]) \rtimes \mathbb{F}_q[T] \rtimes \mathbb{F}_q^\times \to C(\mathbb{F}_q[[T]]) \rtimes (\Gamma \cdot \mathbb{F}_q[T]) \rtimes \mathbb{F}_q^\times$$

is an isomorphism on K_0.

Finally, we have to compute $(\mu_T)_*$ on $K_0(C(\mathbb{F}_q[[T]]) \rtimes (\Gamma \cdot \mathbb{F}_q[T]) \rtimes \mathbb{F}_q^\times)$. We have $\mu_T(\mathbb{1}_n) = \mathbb{1}_{n+1}$ and $\mu_T(p_\chi) = \mathbb{1}_1 \cdot p_\chi$. Thus, under the identifications in Lemma 5.7 and Corollary 5.8, we have

$$(\mu_T)_* = \begin{pmatrix} \frac{1}{q} & -\frac{1}{q} & \cdots & -\frac{1}{q} \\ 0 & 1 & & 0 \\ \vdots & & \ddots & \\ 0 & 0 & & 1 \end{pmatrix}. \tag{2}$$

In particular, $(\mu_T)_*$ is bijective on $K_0(C(\mathbb{F}_q[[T]]) \rtimes (\Gamma \cdot \mathbb{F}_q[T]) \rtimes \mathbb{F}_q^\times)$. Again, combining this result with Lemma 5.1 and Corollary 5.8, we get

Corollary 5.9.

$$K_0(A_{-1}) \cong \mathbb{Z}[\tfrac{1}{q}] \oplus \bigoplus_{\widehat{\mathbb{F}_q^\times} \setminus \{1\}} \mathbb{Z}$$

and the canonical inclusion

$$C(\mathbb{F}_q[[T]]) \rtimes \mathbb{F}_q[T] \rtimes \mathbb{F}_q^\times \to A_{-1}$$

is an isomorphism on K_0.

Generators of $K_0(A_{-1})$ *are*

$$[\mathbb{1}_n] \mathrel{\widehat{=}} \begin{pmatrix} \frac{1}{q^n} \\ 0 \\ \vdots \\ 0 \end{pmatrix} \quad \textit{and} \quad [\mathbb{1} \cdot p_\chi] \mathrel{\widehat{=}} \begin{pmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{pmatrix}.$$

Here "1" is the entry corresponding to χ *in* $\bigoplus_{\widehat{\mathbb{F}_q^\times}\setminus\{1\}} \mathbb{Z}$.

So we have obtained a concrete description of the K-theory of

$$A_{-1} = C_0(\mathbb{F}_q((T))) \rtimes \mathbb{F}_q(T) \rtimes \mathbb{F}_q^\times.$$

We fix this description of $K_0(A_{-1})$ from now on. The next step is to determine the K-theory of

$$A_0 = C_0(\mathbb{F}_q((T))) \rtimes \mathbb{F}_q(T) \rtimes (\mathbb{F}_q^\times \times \Gamma_0) \cong C_0(\mathbb{F}_q((T))) \rtimes \mathbb{F}_q(T) \rtimes \mathbb{F}_q^\times \rtimes_{\mu_T} \mathbb{Z}.$$

For every χ in $\widehat{\mathbb{F}_q^\times}$, let

$$x_{\bar{\chi}} := \sum_{b \in \mathbb{F}_q^\times} \bar{\chi}(b) \mathbb{1}_{[b + T \cdot \mathbb{F}_q[[T]]]} \in A_0. \tag{3}$$

Moreover, we construct

$$w_\chi = t_T(\mathbb{1} \cdot p_1) + x_{\bar{\chi}} p_\chi + (\mathbb{1} \cdot p_\chi) t_T^* + (1 - \mathbb{1} \cdot p_1 - \mathbb{1} \cdot p_\chi) \tag{4}$$

in the unitalization $(A_0)^\sim$ of A_0. 1 denotes the unit in $(A_0)^\sim$. A straightforward computation shows that w_χ is unitary.

Proposition 5.10. *We can identify* $K_0(A_0)$ *with* $\bigoplus_{\widehat{\mathbb{F}_q^\times}\setminus\{1\}} \mathbb{Z}$ *and free generators are* $[\mathbb{1} \cdot p_\chi]$, $1 \neq \chi \in \widehat{\mathbb{F}_q^\times}$.

We also have $K_1(A_0) \cong \bigoplus_{\widehat{\mathbb{F}_q^\times}\setminus\{1\}} \mathbb{Z}$ *and free generators are* $[w_\chi]$, $1 \neq \chi \in \widehat{\mathbb{F}_q^\times}$.

Proof. A_0 can be described as the crossed product $A_{-1} \rtimes_{\mu_T} \mathbb{Z}$. Thus we can apply the Pimsner–Voiculescu sequence. It looks as follows:

$$\{0\} \to K_1(A_0) \xrightarrow{\partial} K_0(A_{-1}) \xrightarrow{\mathrm{id} - (\mu_T)_*} K_0(A_{-1}) \to K_0(A_0) \to \{0\}.$$

If we plug in (2), then we obtain

$$\ker(\mathrm{id} - (\mu_T)_*) = \langle \{ [\mathbb{1} \cdot p_1] - [\mathbb{1} \cdot p_\chi] : \chi \in \widehat{\mathbb{F}_q^\times} \setminus \{1\} \} \rangle;$$
$$\mathrm{im}(\mathrm{id} - (\mu_T)_*) = \langle \{ [\mathbb{1}_n] : n \in \mathbb{Z}_{\geq 0} \} \rangle.$$

As an immediate consequence, we get that $K_0(A_0) \cong \bigoplus_{\widehat{\mathbb{F}_q^\times} \setminus \{1\}} \mathbb{Z}$ with free generators $[\mathbb{1} \cdot p_\chi]$, $1 \neq \chi \in \widehat{\mathbb{F}_q^\times}$, as desired.

To prove our assertion about K_1, we have to show that

$$\partial([w_\chi]) = [\mathbb{1} \cdot p_1] - [\mathbb{1} \cdot p_\chi] \quad \text{(up to sign)}.$$

In order to do so, let us have a closer look at the Pimsner–Voiculescu sequence (compare [PV]). It is derived from the Toeplitz extension associated with the crossed product, where the C*-algebra on which $\mathbb{Z}$ acts is assumed to be unital. As we are in the nonunital case, we have to look at the Toeplitz extension associated to $(A_{-1})^{\sim} \rtimes_{\tilde{\mu}_T} \mathbb{Z}$, i.e.,

$$\{0\} \to \mathcal{K} \otimes (A_{-1})^{\sim} \to \mathcal{T} \to (A_{-1})^{\sim} \rtimes_{\tilde{\mu}_T} \mathbb{Z} \to \{0\}. \tag{5}$$

Here, $\mathcal{K}$ is the C*-algebra of compact operators (on some infinite-dimensional separable Hilbert space) and $\mathcal{T}$ is the C*-subalgebra of $C^*(v) \otimes ((A_{-1})^{\sim} \rtimes_{\tilde{\mu}_T} \mathbb{Z})$ generated by $v \otimes t_T$ and $\{1 \otimes x : x \in (A_{-1})^{\sim}\}$. $C^*(v)$ is the Toeplitz algebra with canonical generator v. The quotient map in (5) maps $v \otimes t_T$ to t_T.

Now, to compute $\partial([w_\chi])$, we consider the partial isometry

$$s_\chi = v \otimes t_T(\mathbb{1} \cdot p_1) + 1 \otimes x_{\bar{\chi}} p_\chi + v^* \otimes (\mathbb{1} \cdot p_\chi) t_T^* + 1 \otimes (1 - \mathbb{1} \cdot p_1 - \mathbb{1} \cdot p_\chi).$$

s_χ is mapped to w_χ under the quotient map in (5). Thus, by definition of ∂, we have (up to sign)

$$\begin{aligned}\partial([w_\chi]) &= [s_\chi^* s_\chi] - [s_\chi s_\chi^*] \\ &= [1 \otimes 1 - (1 - vv^*) \otimes (\mathbb{1}_1 \cdot p_\chi)] - [1 \otimes 1 - (1 - vv^*) \otimes (\mathbb{1}_1 \cdot p_1)] \\ &= [(1 - vv^*) \otimes (\mathbb{1}_1 \cdot p_1)] - [(1 - vv^*) \otimes (\mathbb{1}_1 \cdot p_\chi)]\end{aligned}$$

where we used (1). The last term corresponds to $[\mathbb{1}_1 \cdot p_1] - [\mathbb{1}_1 \cdot p_\chi]$ under the canonical isomorphism $K_0(\mathcal{K} \otimes (A_{-1})^{\sim}) \cong K_0((A_{-1})^{\sim})$. Finally, using Lemma 5.7, we deduce that

$$[\mathbb{1}_1 \cdot p_1] - [\mathbb{1}_1 \cdot p_\chi] = (\iota_1)_*([p_1]) - (\iota_1)_*([p_\chi])$$

corresponds to

$$\begin{pmatrix} \frac{1}{q}(q-1) \\ -1 \\ \vdots \\ -1 \end{pmatrix} - \begin{pmatrix} -\frac{1}{q} \\ 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ \vdots \\ -1 \end{pmatrix} - \begin{pmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{pmatrix}$$

in $\mathbb{Z}[\frac{1}{q}] \oplus \bigoplus_{\widehat{\mathbb{F}_q^\times}\setminus\{1\}} \mathbb{Z}$. But by Lemma 5.7, $[\mathbb{1} \cdot p_1] - [\mathbb{1} \cdot p_\chi]$ also corresponds to

$$\begin{pmatrix} 1 \\ -1 \\ \vdots \\ -1 \end{pmatrix} - \begin{pmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{pmatrix}$$

in $\mathbb{Z}[\frac{1}{q}] \oplus \bigoplus_{\widehat{\mathbb{F}_q^\times}\setminus\{1\}} \mathbb{Z}$, where in the second vector "1" is the entry corresponding to χ in $\bigoplus_{\widehat{\mathbb{F}_q^\times}\setminus\{1\}} \mathbb{Z}$.

Thus, $[\mathbb{1}_1 \cdot p_1] - [\mathbb{1}_1 \cdot p_\chi] = [\mathbb{1} \cdot p_1] - [\mathbb{1} \cdot p_\chi]$ in $K_0(A_{-1})$. This proves our claim. □

At this point, we remark that A_0 can be described as a Cuntz–Krieger algebra. This leads to an alternative way of computing the K-theory for A_0.

First of all, $C(\mathbb{F}_q[[T]]) \rtimes \mathbb{F}_q[T] \rtimes^{\mathrm{e}}_{\mu_T} \mathbb{N}$ is generated by the isometries $v^a t_T, a \in \mathbb{F}_q$, whose range projections sum up to 1. Here t_T is the isometry which implements the endomorphism μ_T. Thus $C(\mathbb{F}_q[[T]]) \rtimes \mathbb{F}_q[T] \rtimes^{\mathrm{e}}_{\mu_T} \mathbb{N}$ is isomorphic to the Cuntz algebra $\mathcal{O}_q$ (by the universal property of $\mathcal{O}_q$ and since the Cuntz algebra is simple, see [Cun1]).

Secondly, consider the crossed product $\mathcal{O}_q \rtimes (\mathbb{Z}/(q-1)\mathbb{Z})$ with respect to the action

$$S_j \longmapsto \begin{cases} S_j & \text{for } j = 1, \\ \zeta^{j-2} S_j & \text{if } j \geq 2 \end{cases}$$

for a primitive $(q-1)$-th root of unity ζ, where the S_j are the canonical generators of $\mathcal{O}_q$. We claim that

$$\mathcal{O}_q \rtimes (\mathbb{Z}/(q-1)\mathbb{Z}) \cong C(\mathbb{F}_q[[T]]) \rtimes \mathbb{F}_q[T] \rtimes^{\mathrm{e}}_{\mu_T} \mathbb{N} \rtimes \mathbb{F}_q^\times.$$

To show this, choose a generator b of $\mathbb{F}_q^\times$ and consider the isometries

$$t_T \quad \text{and} \quad \frac{1}{\sqrt{q-1}} \sum_{n=0}^{q-2} (\zeta^{2-j})^n v^{(b^n)} t_T \quad \text{for } 2 \leq j \leq q.$$

These q isometries generate $C(\mathbb{F}_q[[T]]) \rtimes \mathbb{F}_q[T] \rtimes^{\mathrm{e}}_{\mu_T} \mathbb{N}$, and their range projections sum up to 1. Moreover, we have $t_b t_T t_b^* = t_T$ and

$$t_b \Big(\frac{1}{\sqrt{q-1}} \sum_{n=0}^{q-2} (\zeta^{2-j})^n v^{(b^n)} t_T\Big) t_b^* = \zeta^{j-2} \Big(\frac{1}{\sqrt{q-1}} \sum_{n=0}^{q-2} (\zeta^{2-j})^n v^{(b^n)} t_T\Big)$$

for $2 \leq j \leq q$. Thus we have found a $\mathbb{Z}/(q-1)\mathbb{Z} \cong \mathbb{F}_q^\times$-invariant isomorphism

$$\mathcal{O}_q \cong C(\mathbb{F}_q[[T]]) \rtimes \mathbb{F}_q[T] \rtimes^{\mathrm{e}}_{\mu_T} \mathbb{N}.$$

Here we again used the universal property of $\mathcal{O}_q$ together with the fact that $\mathcal{O}_q$ is simple (see [Cun1]). We conclude that

$$\mathcal{O}_q \rtimes (\mathbb{Z}/(q-1)\mathbb{Z}) \cong C(\mathbb{F}_q[[T]]) \rtimes \mathbb{F}_q[T] \rtimes^{\mathrm{e}}_{\mu_T} \mathbb{N} \rtimes \mathbb{F}_q^{\times},$$

as claimed.

And thirdly, by [CuEv] we know that $\mathcal{O}_q \rtimes (\mathbb{Z}/(q-1)\mathbb{Z})$ is isomorphic to the Cuntz–Krieger algebra $\mathcal{O}_A$ associated with the matrix $A = \begin{pmatrix} 2 & & 1 \\ & \ddots & \\ 1 & & 2 \end{pmatrix}$.

Thus we get

$$C(\mathbb{F}_q[[T]]) \rtimes \mathbb{F}_q[T] \rtimes^{\mathrm{e}}_{\mu_T} \mathbb{N} \rtimes \mathbb{F}_q^{\times} \cong \mathcal{O}_A.$$

Now this isomorphism can be worked out explicitly, and we can compute the K-theory of $\mathcal{O}_A$ in an explicit way (compare [CuKr] and [Cun2]). Therefore we obtain a concrete description for the K-theory of $C(\mathbb{F}_q[[T]]) \rtimes \mathbb{F}_q[T] \rtimes^{\mathrm{e}}_{\mu_T} \mathbb{N} \rtimes \mathbb{F}_q^{\times}$.

Finally, it follows from our computations that the canonical homomorphism

$$C(\mathbb{F}_q[[T]]) \rtimes \mathbb{F}_q[T] \rtimes^{\mathrm{e}}_{\mu_T} \mathbb{N} \rtimes \mathbb{F}_q^{\times} \to A_0$$

is an isomorphism on K-theory. So this is an alternative route of computing the K-theory of A_0.

6. Commuting unitaries

Now we come to the crucial point in our computations. We have invested some effort in describing the K-groups of A_0 as explicitly as possible. The reason is that we are interested in the following observation:

Lemma 6.1. *For every i and $1 \neq \chi \in \widehat{\mathbb{F}_q^{\times}}$, we have*

$$\tilde{\mu}_{f_i}(\mathbb{1} \cdot p_\chi) = \mu_{f_i}(\mathbb{1} \cdot p_\chi) = \mathbb{1} \cdot p_\chi \quad \textit{and} \quad \tilde{\mu}_{f_i}(w_\chi) = w_\chi.$$

Recall that $f_1, f_2, f_3, \ldots$ is an enumeration of the irreducible polynomials in $\mathbb{F}_q[T]$ with constant term 1.

Proof. We have $\mu_{f_i}(\mathbb{1}) = \mathbb{1}_{[f_i \cdot \mathbb{F}_q[[T]]]} = \mathbb{1}$ as f_i is invertible in $\mathbb{F}_q[[T]]$. This shows that $\tilde{\mu}_{f_i}(\mathbb{1} \cdot p_\chi) = \mu_{f_i}(\mathbb{1} \cdot p_\chi) = \mathbb{1} \cdot p_\chi$.

To show that w_χ (defined in (4)) is $\tilde{\mu}_{f_i}$-invariant, it remains to prove that $x_{\bar{\chi}}$ is μ_{f_i}-invariant. By construction of $x_{\bar{\chi}}$ (defined in (3)), it suffices to prove that $\mathbb{1}_{[b+T\cdot\mathbb{F}_q[[T]]]}$ is μ_{f_i}-invariant for all b in $\mathbb{F}_q^{\times}$. Since $\mu_{f_i}(\mathbb{1}_{[b+T\cdot\mathbb{F}_q[[T]]]}) = \mathbb{1}_{[f_i\cdot(b+T\cdot\mathbb{F}_q[[T]])]}$, we have to show that $b + T \cdot \mathbb{F}_q[[T]] = f_i \cdot (b + T \cdot \mathbb{F}_q[[T]])$. As f_i has constant term 1, it is clear that "$\subseteq$" holds. To prove the reverse inclusion, take an arbitrary element $b + Tx$ in $b + T \cdot \mathbb{F}_q[[T]]$. Then

$$b + Tx = f_i \cdot (b + \underbrace{f_i^{-1}}_{\in \mathbb{F}_q[[T]]} \cdot \underbrace{((1 - f_i)b + Tx)}_{\in T\cdot\mathbb{F}_q[[T]]}) \in f_i \cdot (b + T \cdot \mathbb{F}_q[[T]]).$$

This proves our lemma. □

As we will see, this simple observation plays a very important role in our computations. Moreover, note that this observation heavily relies on the fact that we have applied our duality theorem to pass from the finite adele ring to the infinite one. The reason why our lemma holds true basically is that all the f_i are invertible in $\mathbb{F}_q[[T]]$. But this only happens in the canonical subring of the infinite adele ring, whereas in the canonical subring of the finite adele ring, there is for each polynomial f_i a finite place where f_i is not invertible.

Now, the reason why this observation is so important is that it allows us to produce generators for the K-theory of A_m.

We fix the following notation: Let $t(i)$ be the unitary t_{f_i} in the multiplier algebra of A_m for every $1 \leq i \leq m$. We know that $A_m \cong A_{m-1} \rtimes_{\mu_{f_m}} \mathbb{Z}$, and we denote by ∂_m the boundary map in the corresponding Pimsner–Voiculescu sequence. It will become clear from the context whether we mean the index map or the exponential map. Let

$$(\mathbb{1} \cdot p_\chi, t(m)) := t(m)(\mathbb{1} \cdot p_\chi) + (1 - \mathbb{1} \cdot p_\chi) \in (A_m)^{\sim}. \tag{6}$$

Here 1 is the unit in $(A_m)^{\sim}$.

Lemma 6.2. *$(\mathbb{1} \cdot p_\chi, t(m))$ is a unitary in $(A_m)^{\sim}$ with*

$$\partial_m([(\mathbb{1} \cdot p_\chi, t(m))]) = [\mathbb{1} \cdot p_\chi] \in K_0(A_{m-1}) \tag{7}$$

(*up to sign*).

Proof. First of all, $(\mathbb{1} \cdot p_\chi, t(m))$ is a unitary since $t(m)$ commutes with $\mathbb{1} \cdot p_\chi$ as $\mathbb{1} \cdot p_\chi$ is μ_{f_m}-invariant. To prove (7), we have to look at the Toeplitz extension (with generalized Toeplitz algebra $\mathcal{T}$) associated to $(A_{m-1})^{\sim} \rtimes_{\tilde{\mu}_{f_m}} \mathbb{Z}$ as in the proof of Proposition 5.10. Here $\tilde{\mu}_{f_m}$ is the extension of μ_{f_m} to the unitalization.

We find that the partial isometry $\tilde{s}_\chi := v \otimes t(m)(\mathbb{1} \cdot p_\chi) + 1 \otimes (1 - \mathbb{1} \cdot p_\chi)$ is mapped to $(\mathbb{1} \cdot p_\chi, t(m))$ under the quotient map $\mathcal{T} \to (A_{m-1})^{\sim} \rtimes_{\tilde{\mu}_{f_m}} \mathbb{Z}$. Thus,

$$\partial_m([(\mathbb{1} \cdot p_\chi, t(m))]) = [\tilde{s}_\chi^* \tilde{s}_\chi] - [\tilde{s}_\chi \tilde{s}_\chi^*] = [(1 - vv^*) \otimes \mathbb{1} \cdot p_\chi]$$

(up to sign) and the last term corresponds to $[\mathbb{1} \cdot p_\chi]$ under the canonical identification $K_0(\mathcal{K} \otimes (A_{m-1})^{\sim}) \cong K_0((A_{m-1})^{\sim})$. This proves our lemma. □

In the following, we produce generators for $K_*(A_m)$ by comparing our situation with higher-dimensional commutative tori. We denote by $K_*(A_m)$ the $\mathbb{Z}/2\mathbb{Z}$-graded abelian group $K_0(A_m) \oplus K_1(A_m)$.

For each $l \in \mathbb{Z}_{>0}$, let $z_0, \ldots, z_l$ be the canonical unitary generators of $C(\mathbb{T}^{l+1})$. Choose some $1 \neq \chi \in \widehat{\mathbb{F}_q^\times}$. Let Γ'_m be the subgroup of Γ generated by the polynomials $f_1, \ldots, f_m$, i.e.,

$$\Gamma'_m := \langle f_1, \ldots, f_m \rangle. \tag{8}$$

By universal property of $C(\mathbb{T}^{l+1})$, the commuting unitaries $w_\chi, t(i_1), \dots, t(i_l)$ (for some $1 \leq i_1 < \cdots < i_l \leq m$) give rise to a homomorphism

$$C(\mathbb{T}^{l+1}) \to (A_0)^\sim \rtimes_{\tilde{\mu}} \Gamma'_m;\ z_0 \mapsto w_\chi, z_j \mapsto t(i_j).$$

Here $\tilde{\mu}$ is the extension of μ to the unitalization. Note that we can construct such a homomorphism precisely because of Lemma 6.1.

We denote by $[w_\chi, t(i_1), \dots, t(i_l)]$ the image of $[z_0] \times \cdots \times [z_l]$ (see [HiRo], 4.7, for the definition of the product on K-theory) under this homomorphism in K-theory. A priori, $[w_\chi, t(i_1), \dots, t(i_l)]$ lies in $K_*((A_0)^\sim \rtimes_{\tilde{\mu}} \Gamma'_m)$. However, we observe the following:

Lemma 6.3. $[w_\chi, t(i_1), \dots, t(i_l)]$ *lies in*

$$K_*(A_m) = K_*(A_0 \rtimes_\mu \Gamma'_m) \cong \ker(K_*((A_0)^\sim \rtimes_{\tilde{\mu}} \Gamma'_m) \to K_*(C^*(\Gamma'_m))).$$

Proof. The identification

$$K_*(A_0 \rtimes_\mu \Gamma'_m) \cong \ker(K_*((A_0)^\sim \rtimes_{\tilde{\mu}} \Gamma'_m) \to K_*(C^*(\Gamma'_m)))$$

is justified by the split-exact sequence

$$\{0\} \to A_0 \rtimes_\mu \Gamma'_m \to (A_0)^\sim \rtimes_{\tilde{\mu}} \Gamma'_m \to C^*(\Gamma'_m) \to \{0\}.$$

Now, consider the commutative diagram

$$\begin{array}{ccccc} C_0(\mathbb{R}) \otimes C(\mathbb{T}^l) & \longrightarrow & C(\mathbb{T}^{l+1}) & \longrightarrow & C(\mathbb{T}^l) \\ & & \downarrow & & \downarrow \\ & & (A_0)^\sim \rtimes_{\tilde{\mu}} \Gamma'_m & \longrightarrow & C^*(\Gamma'_m). \end{array}$$

The first row is split-exact. Moreover, $[z_0] \times \cdots \times [z_l]$ clearly comes from $K_*(C_0(\mathbb{R}) \otimes C(\mathbb{T}^l))$. Therefore, $[z_0] \times \cdots \times [z_l]$ is mapped to 0 under the homomorphism $K_*(C(\mathbb{T}^{l+1})) \to K_*(C(\mathbb{T}^l))$. Since the diagram above commutes, it follows that $[w_\chi, t(i_1), \dots, t(i_l)]$ must lie in the kernel of $K_*((A_0)^\sim \rtimes_{\tilde{\mu}} \Gamma'_m) \to K_*(C^*(\Gamma'_m))$. □

Lemma 6.4. *If* $i_l = m$, $l \geq 1$, *then*

$$\partial_m([w_\chi, t(i_1), \dots, t(i_l)]) = [w_\chi, t(i_1), \dots, t(i_{l-1})] \quad (\textit{up to sign}). \tag{9}$$

Proof. Under the boundary map $K_*(C(\mathbb{T}^{l+1})) \to K_{*+1}(C(\mathbb{T}^l))$ associated with the Toeplitz extension of $C(\mathbb{T}^{l+1}) \cong C(\mathbb{T}^l) \rtimes_{\mathrm{id}} \mathbb{Z}$, $[z_0] \times \cdots \times [z_l]$ is mapped to $[z_0] \times \cdots \times [z_{l-1}]$ (up to sign). Therefore, by naturality of the Pimsner–Voiculescu sequence, our claim follows. □

Similarly, we can consider $C(\mathbb{T}^l)$ with canonical unitary generators $\tilde{z}_i$, $1 \leq i \leq l$, and the homomorphism $C(\mathbb{T}^l) \to (A_0)^{\sim} \rtimes_{\tilde{\mu}} \Gamma'_m$ (Γ'_m is defined in (8)) given by

$$\tilde{z}_1 \mapsto (\mathbb{1} \cdot p_\chi, t(i_1)); \quad \tilde{z}_j \mapsto t(i_j) \quad \text{for } 2 \leq j \leq l,\ 1 \leq i_1 < \cdots < i_l \leq m.$$

The unitary $(\mathbb{1} \cdot p_\chi, t(i_1))$ is defined as in (6). Again, this homomorphism exists because the unitaries $(\mathbb{1} \cdot p_\chi, t(i_1)), t(i_2), \ldots, t(i_l)$ commute (see Lemma 6.1).

Let $[\mathbb{1} \cdot p_\chi, t(i_1), t(i_2), \ldots, t(i_l)]$ be the image of $[\tilde{z}_1] \times \cdots \times [\tilde{z}_l]$ under the homomorphism above in K-theory. In complete analogy to the preceding two lemmas, we get

Lemma 6.5. $[\mathbb{1} \cdot p_\chi, t(i_1), t(i_2), \ldots, t(i_l)]$ *lies in*

$$K_*(A_m) = K_*(A_0 \rtimes_\mu \Gamma'_m) \cong \ker(K_*((A_0)^{\sim} \rtimes_{\tilde{\mu}} \Gamma'_m) \to K_*(C^*(\Gamma'_m))).$$

and

Lemma 6.6.

$$\partial_m([\mathbb{1} \cdot p_\chi, t(i_1), t(i_2), \ldots, t(i_l)]) = [\mathbb{1} \cdot p_\chi, t(i_1), t(i_2), \ldots, t(i_{l-1})] \qquad (10)$$

(*up to sign*) *for* $i_l = m$ *and* $l > 1$.

7. Final result

Now we are ready to compute the K-theory of A_m (see Section 4 for the definition of A_m). We just have to put everything together.

Proposition 7.1. *We have* (*with* Γ'_m *defined in* (8))

$$K_*(A_m) \cong K_*(A_0) \otimes_{\mathbb{Z}} \Lambda^*(\Gamma'_m).$$

Generators for K_0 *are* $[w_\chi, t(i_1), \ldots, t(i_k)]$ *for* $1 \leq i_1 < \cdots < i_k \leq m$, $1 \leq k$ *odd;* $[\mathbb{1} \cdot p_\chi, t(i_1), \ldots, t(i_l)]$ *for* $1 \leq i_1 < \cdots < i_l \leq m$, $0 \leq l$ *even, with* $\chi \in \widehat{\mathbb{F}_q^\times} \setminus \{1\}$.

Generators for K_1 *are* $[w_\chi, t(i_1), \ldots, t(i_k)]$ *for* $1 \leq i_1 < \cdots < i_k \leq m$, $0 \leq k$ *even;* $[\mathbb{1} \cdot p_\chi, t(i_1), \ldots, t(i_l)]$ *for* $1 \leq i_1 < \cdots < i_l \leq m$, $1 \leq l$ *odd, with* $\chi \in \widehat{\mathbb{F}_q^\times} \setminus \{1\}$.

Recall that $K_*(A_m)$ is the $\mathbb{Z}/2\mathbb{Z}$-graded abelian group $K_0(A_m) \oplus K_1(A_m)$. The isomorphism in this proposition is meant as an isomorphism of $\mathbb{Z}/2\mathbb{Z}$-graded abelian groups, where $\Lambda^*(\Gamma'_m)$ is canonically graded and we consider graded tensor products.

Proof. First of all, the statement makes sense because of Lemma 6.3 and Lemma 6.5. Now, to prove our claim, we proceed inductively. For $m = 0$ the claim about the generators has been proven in Proposition 5.10.

Assume that $m \geq 1$ and that our assertion holds true for $m-1$. We consider the Pimsner–Voiculescu sequence associated to $A_m \cong A_{m-1} \rtimes_{\mu_{f^m}} \mathbb{Z}$. The boundary map ∂_m is surjective because we have (up to sign)

$$\partial_m([w_\chi, t(i_1), \ldots, t(i_k), t(m)]) = [w_\chi, t(i_1), \ldots, t(i_k)]$$

for every $i_1 < \cdots < i_k < m$, $0 \leq k$, by (9) and

$$\partial_m([\mathbb{1} \cdot p_\chi, t(i_1), \ldots, t(i_l), t(m)]) = [\mathbb{1} \cdot p_\chi, t(i_1), \ldots, t(i_l)]$$

for every $i_1 < \cdots < i_l < m$, $0 \leq l$, by (7) and (10).

Therefore, our claim follows from the exactness of the Pimsner–Voiculescu sequence for $A_m \cong A_{m-1} \rtimes_{\mu_{f^m}} \mathbb{Z}$ and by the induction hypothesis. In particular, we have $(\mu_{f^m})_* = \mathrm{id}$ on $K_*(A_{m-1})$ for every $m \in \mathbb{Z}_{>0}$. □

Finally, this result allows us to compute the K-theory of the ring C*-algebra associated to $\mathbb{F}_q[T]$. By our considerations in Section 3, we know that the K-theory of the ring C*-algebra $\mathfrak{A}[\mathbb{F}_q[T]]$ can be identified with the K-theory of $C_0(\mathbb{F}_q((T))) \rtimes \mathbb{F}_q(T) \rtimes \mathbb{F}_q(T)^\times$. Moreover, we have

$$C_0(\mathbb{F}_q((T))) \rtimes \mathbb{F}_q(T) \rtimes \mathbb{F}_q(T)^\times \cong \varinjlim A_m.$$

Thus, using continuity of K-theory together with Propositions 5.10 and 7.1, we arrive at the following final result:

Theorem 7.2. $K_*(\mathfrak{A}[\mathbb{F}_q[T]]) \cong \widetilde{K}_0(C^*(\mathbb{F}_q^\times)) \otimes_{\mathbb{Z}} \Lambda^*(\Gamma)$.

$\widetilde{K}_0(C^*(\mathbb{F}_q^\times))$ denotes the reduced K-theory of $C^*(\mathbb{F}_q^\times)$, i.e., the cokernel of the canonical map $K_0(\mathbb{C}) \to K_0(C^*(\mathbb{F}_q^\times))$. Γ is defined in Section 4. Moreover, $K_*(\mathfrak{A}[\mathbb{F}_q[T]])$ is the $\mathbb{Z}/2\mathbb{Z}$-graded abelian group $K_0(\mathfrak{A}[\mathbb{F}_q[T]]) \oplus K_1(\mathfrak{A}[\mathbb{F}_q[T]])$, and the isomorphism in the theorem above is meant as an isomorphism of $\mathbb{Z}/2\mathbb{Z}$-graded abelian groups. Here $\widetilde{K}_0(C^*(\mathbb{F}_q^\times))$ is trivially graded, $\Lambda^*(\Gamma)$ is canonically graded and we consider graded tensor products.

Our computations show how to define a product structure on $K_*(\mathfrak{A}[\mathbb{F}_q[T]])$ which corresponds to the canonical product structure on $\widetilde{K}_0(C^*(\mathbb{F}_q^\times)) \otimes_{\mathbb{Z}} \Lambda^*(\Gamma)$ under the isomorphism above. Actually, it follows from Lemma 6.1 that for every χ in $\widehat{\mathbb{F}_q^\times} \setminus \{1\}$, the elements $t(1)(\mathbb{1} \cdot p_\chi), t(2)(\mathbb{1} \cdot p_\chi), t(3)(\mathbb{1} \cdot p_\chi), \ldots$ are commuting unitaries in $(\mathbb{1} \cdot p_\chi)(C_0(\mathbb{F}_q((T))) \rtimes \mathbb{F}_q(T) \rtimes \mathbb{F}_q(T)^\times)(\mathbb{1} \cdot p_\chi)$. So they give rise to a homomorphism of the algebra of continuous functions on the infinite dimensional torus to $C_0(\mathbb{F}_q((T))) \rtimes \mathbb{F}_q(T) \rtimes \mathbb{F}_q(T)^\times$. It follows from Proposition 7.1 that this homomorphism induces an embedding on K-theory. Thus we just have to carry over the product structure on the K-theory of the infinite dimensional torus to $K_*(\mathfrak{A}[\mathbb{F}_q[T]]) \cong K_*(C_0(\mathbb{F}_q((T))) \rtimes \mathbb{F}_q(T) \rtimes \mathbb{F}_q(T))$.

As the last comment, we point out that there are striking similarities between the number field case and the function field case (compare Theorem 2.3 and Theorem 7.2). So from this point of view, our results fit nicely into the general picture concerning analogies between number fields and function fields.

References

[Cun1] J. Cuntz, Simple C^*-algebras generated by isometries. *Comm. Math. Phys.* **57** (1977), 173–185. Zbl 0399.46045 MR 0467330

[Cun2] J. Cuntz, A class of C^*-algebras and topological Markov chains II: Reducible chains and the Ext-functor for C^*-algebras. *Invent. Math.* **63** (1981), 25–40. Zbl 0461.46047 MR 608527

[Cun3] J.Cuntz, C*-algebras associated with the $ax + b$-semigroup over $\mathbb{N}$. In *K-theory and noncommutative geometry*, EMS Ser. Congr. Rep., 201–215, European Math. Soc., Zurich 2008. Zbl 1162.46036 MR 2513338

[CuEv] J. Cuntz and D. E. Evans, Some remarks on the C^*-algebras associated with certain topological Markov chains. *Math. Scand.* **48** (1981), 235–240. Zbl 0467.46043 MR 631338

[CuKr] J. Cuntz and W. Krieger, A class of C^*-algebras and topological Markov chains. *Invent. Math.* **56** (1980), 251–268. Zbl 0434.46045 MR 561974

[CuLi1] J. Cuntz and X. Li, The regular C^*-algebra of an integral domain. In *Quanta of maths*, Clay Math. Proc. 11, Amer. Math. Soc., Providence, RI, 2010, 149–170. MR 2732050

[CuLi2] J. Cuntz and X. Li, C^*-algebras associated with integral domains and crossed products by actions on adele spaces. *J. Noncommut. Geom.* **5** (2011), 1–37. Zbl 05843242 MR 2746649

[HiRo] N. Higson and J. Roe, *Analytic K-homology*. Oxford Math. Monogr., Oxford University Press, Oxford 2000. Zbl 0968.46058 MR 1817560

[Li] X. Li, Ring C^*-algebras. *Math. Ann.* **348** (2010), 859–898. Zbl 1207.46050 MR 2721644

[PV] M. Pimsner and D. Voiculescu, Exact sequences for K-groups and Ext-groups of certain cross-product C^*-algebras. *J. Operator Theory* **4** (1980), 93–118. Zbl 0474.46059 MR 587369

Received November 26, 2009

J. Cuntz, Mathematisches Institut, Westfälische Wilhelms-Universität Münster, Einsteinstraße 62, 48149 Münster, Germany

E-mail: cuntz@math.uni-muenster.de

X. Li, Mathematisches Institut, Westfälische Wilhelms-Universität Münster, Einsteinstraße 62, 48149 Münster, Germany

E-mail: xinli.math@uni-muenster.de