

The cyclic theory of Hopf algebroids

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Abstract. We give a systematic description of the cyclic cohomology theory of Hopf algebroids in terms of its associated category of modules. Then we introduce a dual cyclic homology theory by applying cyclic duality to the underlying cocyclic object. We derive general structure theorems for these theories in the special cases of commutative and cocommutative Hopf algebroids. Finally, we compute the cyclic theory in examples associated to Lie–Rinehart algebras and étale groupoids.

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Introduction

In geometry, groupoids are a joint generalisation of both spaces and groups. As such they provide a generalised symmetry concept that has found many applications in the theory of foliations, group actions, etc. In particular, the cohomology of the classifying spaces of (Lie) groupoids are the natural domain for the characteristic classes associated to such geometric structures. Symmetries in noncommutative geometry, i.e., the noncommutative analogue of group actions, are encoded by the action or coaction of some Hopf algebra on some algebra or coalgebra, which plays the rôle of a “noncommutative space”.

Hopf algebroids are the noncommutative generalisation of groupoids and as such provide a concept of generalised symmetries in noncommutative geometry: they generalise Hopf algebras to noncommutative base algebras. However, there exists more than one definition. Originally introduced as cogroupoid objects in the category of commutative algebras (see e.g. [Ra]), the main difficulty of defining Hopf algebroids stems from the fact that the involved tensor category of bimodules is not symmetric, so that a straightforward generalisation of the corresponding notion for Hopf algebras does not make sense.

Thinking of a Hopf algebra as a bialgebra equipped with an antipode, the first step, the generalisation to so-called *bialgebroids* (or \times_A -bialgebras), is unambiguous: this is a bialgebra object in the tensor category of bimodules over a (noncommutative) base algebra (cf. [S], [T], [Lu], [Sch1], [X2], [BrzMi]).

Approaches begin to differ when adding the antipode. The first general definition appeared in [Lu], where an auxiliary structure (a section of a certain projection map) was needed. Motivated by cyclic cohomology, as we discuss below, a closely related notion of *para-Hopf algebroid* was introduced in [KR3].

In this paper we will consider the alternative definition of [BSz], [B1], which, roughly speaking, consists of introducing *two* bialgebroid structures on a given algebra, called left and right bialgebroid (cf. [KadSz]), and views the antipode as mapping the left structure to the right one. This setup avoids the somewhat ad hoc choice of a section and makes the definition completely symmetric. Also we will show in §3 that Lie groupoids and Lie algebroids (or rather Lie–Rinehart algebras) lead to natural examples of such structures. However, the immediate generalisation of a Hopf algebra to a noncommutative base ring is, strictly speaking, rather given by a \times_A -Hopf algebra [Sch2], while Hopf algebroids in the sense of [BSz], [B1] generalise Hopf algebras equipped with a character (i.e., with a possibly “twisted” antipode [Cr3], [CM2]). For reasons to be explained in Remark 3.12, we will refer to \times_A -Hopf algebras as *left Hopf algebroids*.

Cyclic cohomology for Hopf algebras, *Hopf-cyclic cohomology*, is the noncommutative analogue of Lie algebra homology (which is recovered when applied to universal enveloping algebras of Lie algebras). It was launched in the work of Connes and Moscovici [CM1] on the transversal index theorem for foliations and defined in general in [Cr3] (cf. also [CM2]). A universal framework suited to describe all

examples of cyclic (co)homology arising from Hopf algebras (up to cyclic duality) was given in [Kay], based on a construction of para-(co)cyclic objects in symmetric monoidal categories in terms of (co)monoids.

The generalisation of Hopf-cyclic cohomology to noncommutative base rings, i.e., to Hopf algebroids, has been less explored. For instance, the general machinery from [Kay] does not apply to this context (because the relevant category of modules is not symmetric and in general not even braided). It appeared for the first time in the particular example of the “extended” Hopf algebra governing the transversal geometry of foliations in [CM3]. In this context, certain bialgebroids (in fact, left Hopf algebroids) carrying a cocyclic structure arise naturally. Extending this construction to general Hopf algebroids is not straightforward: for example, the notion of Hopf algebroid in [Lu] is not well suited to the problem. This led in [KR3] to the definition of para-Hopf algebroids, in which the antipode of [Lu] is replaced by a *para*-antipode. Its axioms are principally designed for the cocyclic structure to be well defined adapting the Hopf algebra case. However, the resulting para-antipode axioms appear quite complicated and do not resemble the original symmetric Hopf algebra axioms. In particular, *guessing* an antipode (and hence the cyclic operator) in concrete examples remains intricate.

In [BŞ] a general cyclic theory for bialgebroids and *left* Hopf algebroids (in terms of so-called (co)monads) is developed that works in an arbitrary category, and hence embraces the construction in [Kay] for symmetric monoidal categories.

In this paper we shall show that the cyclic cohomology theory for Hopf algebroids in the sense of [BSz], [B1] is actually naturally defined and explain how it fits into the monoidal category of modules and the cyclic cohomology of coalgebras, generalising the corresponding Hopf algebra approach from [Cr3], [CM2].

Besides the cyclic cohomology, we develop a dual cyclic homology theory by, roughly speaking, applying cyclic duality to the underlying cocyclic object. This generalises the dual theory for Hopf algebras [Cr2], [KR1] and is more related to a certain category of comodules (over one of the underlying bialgebroid structures). It should be stressed that this homology theory is not simply the Hom-dual of the cohomology theory mentioned above; it can give interesting results even when the cyclic cohomology is trivial, cf. §3.2 for an example. Generally, in each of the classes of examples we consider, one of the two cyclic theories does not furnish new information compared to the respective Hochschild theory, whereas the other one does. However, these examples are in some sense “extremal” with respect to *primitive* and (*weakly*) *grouplike* elements – we do not pursue this any further here.

Outline. This paper is set up as follows: in §1 we review the definition of a Hopf algebroid as in [B1], [BSz] and give a brief description of the associated monoidal categories of modules and comodules. We then give a systematic derivation of the cyclic cohomology complexes using coinvariant localisation in the category of modules over the Hopf algebroid (§2.1 and §2.2). The dual homology is constructed in §2.3 by applying the notion of duality in Connes’ cyclic category, after the cochain

spaces have been mapped isomorphically into the category of certain comodules by means of a *Hopf–Galois map* (cf. [Sch2]) associated to the Hopf algebroid.

The remainder of Section 2 is devoted to some ramifications of the theory. We identify the Hochschild theory as certain derived functors (§2.5) and prove structure theorems which allow to express the cyclic theory of commutative and cocommutative Hopf algebroid in terms of their respective Hochschild theory (§2.6). This generalises a similar approach for Hopf algebras [KR1].

Section 3 is devoted to examples: we discuss Hopf algebroids arising from étale groupoids, Lie–Rinehart algebras (or Lie algebroids), and jet spaces of Lie–Rinehart algebras. In all these examples, the left bialgebroid structure has been described before in the literature, and we add both the right structure and the antipode. For Lie–Rinehart algebras this leads to the following remarkable conclusion: the universal enveloping algebra of a Lie–Rinehart algebra has a canonical left Hopf algebroid structure (in particular it is a left bialgebroid), and a full Hopf algebroid structure depends on the choice of a certain flat right connection (cf. [H2]) on the base algebra. However, its dual jet space does carry a Hopf algebroid structure, free of choices.

Finally, we compute the cyclic homology and cohomology in all these examples and find that it generalises well-known Lie groupoid and Lie algebroid resp. Lie–Rinehart homology and cohomology theories. In particular, it generalises corresponding results in Hopf algebra theory [CM1], [Cr2], [Cr3], [KR1].

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1. Hopf algebroids

1.1. Preliminaries. In this paper, the term “ring” always means “unital and associative ring”, and we fix a commutative ground ring k . Throughout the paper, we work in the symmetric monoidal category of k -modules. For a k -algebra A , its opposite is denoted by A^{op} , the enveloping algebra by $A^e := A \otimes_k A^{\text{op}}$, and the category of left A -modules by $\text{Mod}(A)$. The category of A^e -modules, that is, (A, A) -bimodules with symmetric action of k , is monoidal by means of the tensor product \otimes_A over A . An A -algebra is a monoid in this category, i.e., an (A, A) -bimodule U equipped with (A, A) -bimodule morphisms $\mu: U \otimes_A U \rightarrow U$ and $\eta: A \rightarrow U$ satisfying the usual associativity and unitality axioms. Likewise, the notion of an A -coalgebra is defined as a comonoid in the category of A^e -modules. These notions also appear under the name A -ring and A -coring in the literature; see e.g. [B3], [BrzWi].

1.2. Bialgebroids (cf. [T]). Bialgebroids are a generalisation of bialgebras. An important subtlety is that the algebra and coalgebra structure are defined in different monoidal categories. Let A and \mathcal{H} be (unital) k -algebras, and suppose we have

homomorphisms $s: A \rightarrow \mathcal{H}$ and $t: A^{\text{op}} \rightarrow \mathcal{H}$ whose images commute in \mathcal{H} : this structure is equivalent to the structure of an A^e -algebra on \mathcal{H} . Such objects are also called (s, t) -rings over A , whereas s and t are referred to as *source* and *target* maps. Multiplication in \mathcal{H} from the left equips \mathcal{H} with the following (A, A) -bimodule structure

$$a_1 \cdot h \cdot a_2 := s(a_1)t(a_2)h, \quad a_1, a_2 \in A, h \in \mathcal{H}. \tag{1.1}$$

With respect to this bimodule structure we define the tensor product \otimes_A . Inside $\mathcal{H} \otimes_A \mathcal{H}$, there is a subspace called the *Takeuchi product*:

$$\mathcal{H} \times_A \mathcal{H} := \left\{ \sum_i h_i \otimes_A h'_i \in \mathcal{H} \otimes_A \mathcal{H} \mid \sum_i h_i t_l(a) \otimes h'_i = \sum_i h_i \otimes h'_i s_l(a) \text{ for all } a \in A \right\}.$$

This is a unital algebra via factorwise multiplication and even an (s, t) -ring again.

Definition 1.1. Let A_l be a k -algebra. A *left bialgebroid* over A_l or A_l -*bialgebroid* is an (s_l, t_l) -ring \mathcal{H}_l equipped with the structure of an A_l -coalgebra (Δ_l, ϵ_l) with respect to the (A_l, A_l) -bimodule structure (1.1), subject to the following conditions:

- (i) the (left) coproduct $\Delta_l: \mathcal{H}_l \rightarrow \mathcal{H}_l \otimes_{A_l} \mathcal{H}_l$ maps into $\mathcal{H}_l \times_{A_l} \mathcal{H}_l$ and defines a morphism $\Delta_l: \mathcal{H}_l \rightarrow \mathcal{H}_l \times_{A_l} \mathcal{H}_l$ of unital k -algebras;
- (ii) the (left) counit has the property

$$\epsilon_l(hh') = \epsilon_l(hs_l(\epsilon_l h')) = \epsilon_l(ht_l(\epsilon_l h')) \quad \text{for all } h, h' \in \mathcal{H}_l.$$

We shall indicate such a left bialgebroid by $(\mathcal{H}_l, A_l, s_l, t_l, \Delta_l, \epsilon_l)$, or simply by \mathcal{H}_l .

Given any (s, t) -ring \mathcal{H} , besides the (A, A) -bimodule structure (1.1), one could choose the one coming from the *right* action of \mathcal{H} on itself:

$$a_1 \cdot h \cdot a_2 := ht(a_1)s(a_2), \quad a_1, a_2 \in A, h \in \mathcal{H}. \tag{1.2}$$

Proceeding analogously as above, this leads to the notion of a *right bialgebroid* $(\mathcal{H}_r, A_r, s_r, t_r, \Delta_r, \epsilon_r)$, where the underlying algebra is denoted by A_r . We shall not write out the details but rather refer to [KadSz], [B3]. For example, the corresponding *right* counit $\epsilon_r: \mathcal{H}_r \rightarrow A_r$ satisfies in this case

$$\epsilon_r(hh') = \epsilon_r(s_r(\epsilon_r h)h') = \epsilon_r(t_r(\epsilon_r h)h') \quad \text{for all } h, h' \in \mathcal{H}_r.$$

We will use Sweedler notation with subscripts $\Delta_l(h) = h_{(1)} \otimes h_{(2)}$ for left coproducts, whereas right coproducts are indicated by superscripts: $\Delta_r(h) = h^{(1)} \otimes h^{(2)}$.

1.3. Hopf algebroids. A Hopf algebroid is now, roughly speaking, an algebra equipped with a left and a right bialgebroid structure together with an antipode mapping from the left bialgebroid to the right. This idea leads to the following definition:

Definition 1.2 (cf. [BSz]). A *Hopf algebroid* is given by a triple $(\mathcal{H}_l, \mathcal{H}_r, S)$, where $\mathcal{H}_l = (\mathcal{H}_l, A_l, s_l, t_l, \Delta_l, \epsilon_l)$ is a left A_l -bialgebroid and $\mathcal{H}_r = (\mathcal{H}_r, A_r, s_r, t_r, \Delta_r, \epsilon_r)$ is a right A_r -bialgebroid on the same k -algebra \mathcal{H} , and $S: \mathcal{H} \rightarrow \mathcal{H}$ is a k -module map subject to the conditions:

- (i) the images of s_l and t_r , as well as t_l and s_r , coincide:

$$s_l \epsilon_l t_r = t_r, \quad t_l \epsilon_l s_r r = s_r, \quad s_r \epsilon_r t_l = t_l, \quad t_r \epsilon_r s_l = s_l; \quad (1.3)$$

- (ii) *twisted coassociativity* holds:

$$(\Delta_l \otimes \text{id}_{\mathcal{H}}) \Delta_r = (\text{id}_{\mathcal{H}} \otimes \Delta_r) \Delta_l \quad \text{and} \quad (\Delta_r \otimes \text{id}_{\mathcal{H}}) \Delta_l = (\text{id}_{\mathcal{H}} \otimes \Delta_l) \Delta_r; \quad (1.4)$$

- (iii) for all $a_1 \in A_l, a_2 \in A_r$ and $h \in \mathcal{H}$ we have

$$S(t_l(a_1) h t_r(a_2)) = s_r(a_2) S(h) s_l(a_1);$$

- (iv) the *antipode axioms* are fulfilled:

$$\mu_{\mathcal{H}}(S \otimes \text{id}_{\mathcal{H}}) \Delta_l = s_r \epsilon_r \quad \text{and} \quad \mu_{\mathcal{H}}(\text{id}_{\mathcal{H}} \otimes S) \Delta_r = s_l \epsilon_l, \quad (1.5)$$

where $\mu_{\mathcal{H}}$ denotes multiplication in \mathcal{H} .

Although we do not need this for all constructions in this paper, we shall from now on assume that the antipode S is invertible.

Remark 1.3. The axioms above have the following implications (cf. [BSz], [B3]):

- (i) Applying ϵ_r to the first two and ϵ_l to the second pair of identities in (1.3), one obtains that A_l and A_r are anti-isomorphic as k -algebras, i.e.,

$$\begin{aligned} \phi &:= \epsilon_r \circ s_l: A_l^{\text{op}} \xrightarrow{\cong} A_r, & \phi^{-1} &:= \epsilon_l \circ t_r: A_r \xrightarrow{\cong} A_l^{\text{op}}, \\ \theta &:= \epsilon_r \circ t_l: A_l \xrightarrow{\cong} A_r^{\text{op}}, & \theta^{-1} &:= \epsilon_l \circ s_r: A_r^{\text{op}} \xrightarrow{\cong} A_l. \end{aligned} \quad (1.6)$$

When $S^2 = \text{id}$, i.e., when the antipode is involutive, it follows from (1.8) below that $\theta = \phi$, so there is a canonical way to identify A_l^{op} with A_r .

- (ii) The antipode is an anti-algebra and anti-coalgebra morphism (between different coalgebras) and satisfies

$$\text{tw} \circ (S \otimes S) \Delta_l = \Delta_r S, \quad \text{tw} \circ (S \otimes S) \Delta_r = \Delta_l S, \quad (1.7)$$

where $\text{tw}: \mathcal{H} \otimes_k \mathcal{H} \rightarrow \mathcal{H} \otimes_k \mathcal{H}$ is the tensor flip permuting the two factors (one can check that the maps above do respect the (A_l, A_l) - resp. (A_r, A_r) -bimodule structure). Likewise, one has for the inverse:

$$\text{tw} \circ (S^{-1} \otimes S^{-1}) \Delta_l = \Delta_r S^{-1}, \quad \text{tw} \circ (S^{-1} \otimes S^{-1}) \Delta_r = \Delta_l S^{-1}.$$

(iii) We have the identities

$$\begin{aligned}
 s_r \epsilon_r s_l &= S s_l, & s_l \epsilon_l s_r &= S s_r, & s_r \epsilon_r t_l &= S^{-1} s_l, & s_l \epsilon_l t_r &= S^{-1} s_r, \\
 t_r \epsilon_r s_l &= S t_l, & t_l \epsilon_l s_r &= S t_r, & t_r \epsilon_r t_l &= S^{-1} t_l, & t_l \epsilon_l t_r &= S^{-1} t_r, \\
 \epsilon_r s_l \epsilon_l &= \epsilon_r S, & \epsilon_l s_r \epsilon_r &= \epsilon_l S, & \epsilon_r t_l \epsilon_l &= \epsilon_r S^{-1}, & \epsilon_l t_r \epsilon_r &= \epsilon_l S^{-1}.
 \end{aligned}
 \tag{1.8}$$

We now collect a list of basic identities involving the antipode, the multiplication and the (left or right) comultiplication that we need later in explicit computations. All can be verified directly from the axioms.

Lemma 1.4. *For a Hopf algebroid \mathcal{H} with invertible antipode, the following identities hold true:*

$$\begin{aligned}
 \mu_{\mathcal{H}}(S \otimes s_l \epsilon_l) \Delta_l &= S, & \mu_{\mathcal{H}}(s_r \epsilon_r \otimes S) \Delta_r &= S, \\
 \mu_{\mathcal{H}^{\text{op}}}(S^2 \otimes t_l \epsilon_l S^2) \Delta_l &= S^2, & \mu_{\mathcal{H}^{\text{op}}}(t_r \epsilon_r S^2 \otimes S^2) \Delta_r &= S^2, \\
 \mu_{\mathcal{H}^{\text{op}}}(S^2 \otimes S) \Delta_l &= t_r \epsilon_r S^2, & \mu_{\mathcal{H}^{\text{op}}}(S \otimes S^2) \Delta_r &= t_l \epsilon_l S^2, \\
 \mu_{\mathcal{H}^{\text{op}}}(\text{id}_{\mathcal{H}} \otimes S^{-1}) \Delta_l &= t_r \epsilon_r, & \mu_{\mathcal{H}^{\text{op}}}(S^{-1} \otimes \text{id}_{\mathcal{H}}) \Delta_r &= t_l \epsilon_l, \\
 \mu_{\mathcal{H}^{\text{op}}}(t_l \epsilon_l \otimes S^{-1}) \Delta_l &= S^{-1}, & \mu_{\mathcal{H}^{\text{op}}}(S^{-1} \otimes t_r \epsilon_r) \Delta_r &= S^{-1}, \\
 \mu_{\mathcal{H}}(S^{-1} \otimes S^{-2}) \Delta_l &= s_r \epsilon_r S^{-2}, & \mu_{\mathcal{H}}(S^{-2} \otimes S^{-1}) \Delta_r &= s_l \epsilon_l S^{-2}.
 \end{aligned}$$

Here $\mu_{\mathcal{H}^{\text{op}}}$ is the multiplication in the opposite of \mathcal{H} .

1.4. Modules and comodules. Let $\mathcal{H} = (\mathcal{H}_l, \mathcal{H}_r, S)$ be a Hopf algebroid. In this section we discuss several categories of modules and comodules attached to \mathcal{H} , together with some basic properties.

1.4.1. Left modules (cf. [Sch1]). A left module over \mathcal{H} or left \mathcal{H} -module M is simply a left module over the underlying k -algebra \mathcal{H} . We denote the structure map usually by $(h, m) \mapsto h \cdot m$ and the category of left \mathcal{H} -modules by $\text{Mod}(\mathcal{H})$. The left bialgebroid structure \mathcal{H}_l induces the following structure on this category: first, using the left A_l^e -algebra structure, any module $M \in \text{Mod}(\mathcal{H})$ carries an underlying (A_l, A_l) -bimodule structure by

$$a_1 \cdot m \cdot a_2 := s_l(a_1) \cdot t_l(a_2) \cdot m,
 \tag{1.9}$$

for all $a_1, a_2 \in A_l$ and $m \in M$. This defines a forgetful functor

$$\text{Mod}(\mathcal{H}) \rightarrow \text{Mod}(A_l^e).$$

Secondly, the left coproduct defines a monoidal structure on $\text{Mod}(\mathcal{H})$ by $(M, N) \mapsto M \otimes_{A_l} N$, equipped with the \mathcal{H} -module structure

$$h \cdot (m \otimes n) := h_{(1)} \cdot m \otimes h_{(2)} \cdot n, \quad h \in \mathcal{H}, m \in M, n \in N.
 \tag{1.10}$$

The fundamental theorem of Schauenburg [Sch1], Thm. 5.1, states that conversely such a tensor structure on $\text{Mod}(\mathcal{H})$ is equivalent to a left bialgebroid structure on \mathcal{H} . The unit object in $\text{Mod}(\mathcal{H})$ is given by A_l with left \mathcal{H} -action defined by $\mathcal{H} \rightarrow \text{End}_k(A_l), h \mapsto \{a \mapsto \epsilon_l(hs_l(a))\}$. With this, $\text{Mod}(\mathcal{H})$ is a monoidal tensor category.

1.4.2. Right modules. The category of *right* \mathcal{H} -modules has a similar tensor structure by exploring the *right* bialgebroid structure. Its unit object is given by A_r equipped with a right \mathcal{H} -module structure induced by the right counit: $\mathcal{H} \rightarrow \text{End}_k(A_r), h \mapsto \{a \mapsto \epsilon_r(s_r(a)h)\}$. We write $\text{Mod}(\mathcal{H}^{\text{op}})$ for this tensor category. The antipode defines a functor from $\text{Mod}(\mathcal{H})$ to $\text{Mod}(\mathcal{H}^{\text{op}})$ because it is an anti-homomorphism. When it is involutive, this is obviously an equivalence of categories.

1.4.3. Coinvariant localisation. There is an important functor $(-)\text{coinv} : \text{Mod}(\mathcal{H}) \rightarrow \text{Mod}(k)$ from the category of left \mathcal{H} -modules into the category of k -modules called *coinvariant localisation*, defined by

$$M_{\text{coinv}} := A_r \otimes_{\mathcal{H}} M$$

for $M \in \text{Mod}(\mathcal{H})$. Equivalently, $M_{\text{coinv}} \cong M/I_r$, with I_r the k -module of coinvariants given by

$$I_r := \text{span}_k \{ \epsilon_r(h) \cdot m - h \cdot m, h \in \mathcal{H}, m \in M \},$$

where the (A_r, A_r) -bimodule structure on M is defined by (1.9) via $\theta^{-1} : A_r \rightarrow A_r^{\text{op}}$.

Lemma 1.5 (Partial integration). *Let \mathcal{H} be a Hopf algebroid as before, and $M, N \in \text{Mod}(\mathcal{H})$. In $M \otimes_{A_l} N$ one has the identity*

$$h \cdot m \otimes n \equiv m \otimes (Sh) \cdot n \pmod{I_r}$$

for all $m \in M, n \in N$, and $h \in \mathcal{H}$.

Proof. First observe that the induced (A_r, A_r) -bimodule structure on $M \otimes_{A_l} N$ reads

$$a_1 \cdot (m \otimes n) \cdot a_2 := S(s_r a_2) \cdot m \otimes s_r(a_1) \cdot n,$$

for $a_1, a_2 \in A_r$ and $m \in M, n \in N$. Then one has

$$\begin{aligned} m \otimes (Sh) \cdot n &= m \otimes (s_r \epsilon_r(h^{(1)}) Sh^{(2)}) \cdot n && \text{(by Lemma 1.4)} \\ &= \epsilon_r(h^{(1)}) \cdot (m \otimes (Sh^{(2)}) \cdot n) \\ &\equiv h^{(1)} \cdot (m \otimes (Sh^{(2)}) \cdot n) \pmod{I_r} \\ &= h_{(1)} \cdot m \otimes h_{(2)}^{(1)} \cdot (Sh_{(2)}^{(2)} \cdot n) && \text{(by twisted coassociativity (1.4))} \\ &= h_{(1)} \cdot m \otimes \epsilon_l(h_{(2)}) \cdot n && \text{(by (1.5))} \\ &= (t_l \epsilon_l(h_{(2)}) h_{(1)}) \cdot m \otimes_A n \\ &= h \cdot m \otimes_A n, \end{aligned}$$

where the last identity is one of the comonoid identities of a left bialgebroid. \square

Considering \mathcal{H} as a module over itself with respect to left multiplication, we get

Proposition 1.6. *For $M \in \text{Mod}(\mathcal{H})$, there is a canonical isomorphism of k -modules*

$$(\mathcal{H} \otimes_{A_l} M)_{\text{coinv}} \xrightarrow{\cong} M,$$

given by

$$h \otimes m \mapsto (Sh) \cdot m. \tag{1.11}$$

Proof. Consider the map $M \rightarrow (\mathcal{H} \otimes_{A_l} M)_{\text{coinv}}$ induced by $m \mapsto 1 \otimes m$. This clearly defines a right inverse to (1.11). By the previous lemma it is also a left inverse. \square

1.4.4. Right and left comodules over \mathcal{H}_r (cf. [Sch1], [B2], [BrzWi]). A right comodule over the underlying right bialgebroid \mathcal{H}_r is a right A_r -module M equipped with a right A_r -module map

$$\Delta_M: M \rightarrow M \otimes_{A_r} \mathcal{H}, \quad m \mapsto m^{(0)} \otimes m^{(1)},$$

satisfying the usual axioms for a coaction, where the involved (A_r, A_r) -bimodule structure on \mathcal{H}_r is given by (1.2). We denote the category of right \mathcal{H}_r -comodules by $\text{Comod}_R(\mathcal{H}_r)$. Any object $M \in \text{Comod}_R(\mathcal{H}_r)$ carries, besides the right A_r -module structure denoted $(m, a) \mapsto m \cdot a$, a commuting left A_r -module structure defined by

$$a \cdot m := m^{(0)} \cdot \epsilon_r(s_r(a)m^{(1)}), \quad a \in A_r. \tag{1.12}$$

This yields a forgetful functor $\text{Comod}_R(\mathcal{H}_r) \rightarrow \text{Mod}(A_r^e)$. The category $\text{Comod}_R(\mathcal{H}_r)$ is monoidal with tensor structure $(M, N) \mapsto M \otimes_{A_r} N$ equipped with the comodule structure

$$m \otimes n \mapsto m^{(0)} \otimes n^{(0)} \otimes m^{(1)}n^{(1)}. \tag{1.13}$$

The unit is given by $A_r \in \text{Comod}_R(\mathcal{H}_r)$ equipped with coaction $a \mapsto 1 \otimes s_r(a)$.

A left comodule N over \mathcal{H}_r is defined similarly as a left A_r -module equipped with a morphism $\Delta_N: N \rightarrow \mathcal{H} \otimes_{A_r} N$, $n \mapsto n^{(-1)} \otimes n^{(0)}$, of left A_r -modules, where as before \mathcal{H}_r is an (A_r, A_r) -bimodule by means of (1.2). Similarly as for right \mathcal{H}_r -comodules, this leads to a monoidal category $\text{Comod}_L(\mathcal{H}_r)$ with unit A_r equipped with the coaction $a \mapsto t_r(a) \otimes 1$.

1.4.5. Comodules over \mathcal{H}_l . Likewise, the underlying left bialgebroid \mathcal{H}_l has associated categories of left and right comodules which we will denote by $\text{Comod}_L(\mathcal{H}_l)$ and $\text{Comod}_R(\mathcal{H}_l)$, respectively. They have analogous structures as the category $\text{Comod}_R(\mathcal{H}_r)$ above. For left and right \mathcal{H}_l -coactions, we shall use an analogous Sweedler notation as above, but with lower indices.

1.4.6. The cotensor product and invariants. The cotensor product (cf. [EMo]) of a right \mathcal{H}_r -comodule M and a left \mathcal{H}_r -comodule M' is defined as

$$M \square_{\mathcal{H}_r} M' := \ker(\Delta_M \otimes \text{id}_{M'} - \text{id}_M \otimes \Delta_{M'}) \subset M \otimes_{A_r} M'.$$

With this, the space of invariants of a, say, right \mathcal{H}_r -comodule M is defined to be

$$M^{\text{inv}} := M \square_{\mathcal{H}_r} A_r.$$

There is a canonical embedding $M^{\text{inv}} \subset M$ as the subspace

$$M^{\text{inv}} \cong \{m \in M \mid \Delta_M(m) = m \otimes 1\}.$$

Likewise, one defines invariants for a, say, left \mathcal{H}_l -comodule N as $N^{\text{inv}} = A_l \square_{\mathcal{H}_l} N \cong \{n \in N \mid \Delta_N(n) = 1 \otimes n\}$. The dual statement to Proposition 1.6 for these two kinds of invariants is now given by

Proposition 1.7. *Let $M \in \text{Comod}_R(\mathcal{H}_r)$, and consider $M \otimes_{A_r} \mathcal{H}$ as a right \mathcal{H}_r -comodule by means of the right coproduct in \mathcal{H} and the coaction (1.13). Then one has a canonical isomorphism of k -modules*

$$M \xrightarrow{\cong} (M \otimes_{A_r} \mathcal{H})^{\text{inv}}, \quad m \mapsto m^{(0)} \otimes S(m^{(1)}).$$

Similarly, for $N \in \text{Comod}_L(\mathcal{H}_l)$,

$$N \xrightarrow{\cong} (\mathcal{H} \otimes_{A_l} N)^{\text{inv}}, \quad n \mapsto S(n_{(-1)}) \otimes n_{(0)},$$

where $\mathcal{H} \otimes_{A_l} N$ is considered as a left \mathcal{H}_l -comodule by means of the left coproduct in \mathcal{H} and the monoidal structure of $\text{Comod}_L(\mathcal{H}_l)$ which is analogous to (1.13).

Proof. It is not difficult to see that both maps indeed map into the space of invariants with respect to the coaction (1.13) and its analogue for $\text{Comod}_L(\mathcal{H}_l)$, respectively. To show that they are isomorphisms, define the two maps

$$\begin{aligned} M \otimes_{A_r} \mathcal{H} &\rightarrow M, & m \otimes h &\mapsto m \cdot \theta(\epsilon_l h), \\ \mathcal{H} \otimes_{A_l} N &\rightarrow N, & h \otimes n &\mapsto \phi^{-1}(\epsilon_r h) \cdot n, \end{aligned} \tag{1.14}$$

for the first and the second case, respectively, where ϕ and θ are as in (1.6). Clearly, these define inverses for the respective maps above. \square

Remark 1.8. For a left A_l -module N , the tensor product $\mathcal{H} \otimes_{A_l} N$ is a left \mathcal{H}_l -comodule by the coaction $\Delta_l \otimes \text{id}_N$. Since the space of invariants of \mathcal{H} as a left \mathcal{H}_l -comodule is precisely given by A_l , we have the standard isomorphism

$$N \cong A_l \square_{\mathcal{H}_l} (\mathcal{H} \otimes_{A_l} N), \quad n \mapsto 1 \otimes n, \tag{1.15}$$

with inverse as in (1.14).

Remark 1.9. In each of the tensor categories discussed in this section, the Hopf algebroid itself defines a canonical object, by either the product or (left or right) coproduct. This defines six – a priori different – bimodule structures on \mathcal{H} :

- (i) \mathcal{H} is a left module over itself. As an object in $\text{Mod}(\mathcal{H})$, this leads to the (A_l, A_l) -bimodule structure (1.1).
- (ii) \mathcal{H} is a right module over itself. This leads to the (A_r, A_r) -bimodule structure given by (1.2).
- (iii) \mathcal{H} is a right \mathcal{H}_r -comodule via the right comultiplication, i.e., an object in $\text{Comod}_R(\mathcal{H}_r)$. In this case, (1.12) leads to the (A_r, A_r) -bimodule structure

$$a_1 \cdot h \cdot a_2 := s_r(a_1)hs_r(a_2). \tag{1.16}$$

- (iv) As a left comodule over \mathcal{H}_r using Δ_r , we get the (A_r, A_r) -bimodule structure $a_1 \cdot h \cdot a_2 := t_r(a_2)ht_r(a_1)$.
- (v) The left comultiplication gives a right comodule structure on \mathcal{H} over \mathcal{H}_l . The associated (A_l, A_l) -bimodule structure reads $a_1 \cdot h \cdot a_2 := t_l(a_2)ht_l(a_1)$.
- (vi) Finally, \mathcal{H} is a left comodule over \mathcal{H}_l using Δ_l . Similar to (iii), this leads to the (A_l, A_l) -bimodule structure $a_1 \cdot h \cdot a_2 := s_l(a_1)hs_l(a_2)$.

2. The cyclic theory

2.1. Hopf-cyclic cohomology: the basic complexes. As before, let $(\mathcal{H}_l, \mathcal{H}_r, S)$ be a Hopf algebroid. We consider \mathcal{H} as a left module over itself, which induces the (A_l, A_l) -bimodule structure (1.1). With this we define

$$C^n(\mathcal{H}) := \underbrace{\mathcal{H} \otimes_{A_l} \cdots \otimes_{A_l} \mathcal{H}}_{n \text{ times}}.$$

For $n \geq 1$, define maps $\delta_i: C^n(\mathcal{H}) \rightarrow C^{n+1}(\mathcal{H})$ by

$$\delta_i(h^1 \otimes \cdots \otimes h^n) := \begin{cases} 1 \otimes h^1 \otimes \cdots \otimes h^n & \text{if } i = 0, \\ h^1 \otimes \cdots \otimes \Delta_l h^i \otimes \cdots \otimes h^n & \text{if } 1 \leq i \leq n, \\ h^1 \otimes \cdots \otimes h^n \otimes 1 & \text{if } i = n + 1. \end{cases} \tag{2.1}$$

For $n = 0$, put $C^0(\mathcal{H}) := A_l$ and define

$$\delta_i(a) := \begin{cases} t_l(a) & \text{if } i = 0, \\ s_l(a) & \text{if } i = 1. \end{cases} \tag{2.2}$$

In the opposite direction we have codegeneracies $\sigma_i: C^n(\mathcal{H}) \rightarrow C^{n-1}(\mathcal{H})$ given by

$$\sigma_i(h^1 \otimes \cdots \otimes h^n) = h^1 \otimes \cdots \otimes \epsilon_l(h^{i+1}) \cdot h^{i+2} \otimes \cdots \otimes h^n, \quad 0 \leq i \leq n - 1. \tag{2.3}$$

One easily verifies that $(\delta_\bullet, \sigma_\bullet)$ equip $C^\bullet(\mathcal{H})$ with the structure of a cosimplicial space which only depends on the underlying left bialgebroid structure of \mathcal{H} . For further use, introduce as usual the Hochschild differential by $b := \sum_{i=0}^{n+1} (-1)^i \delta_i$. Next, define the cyclic operator $\tau_n : C^n(\mathcal{H}) \rightarrow C^n(\mathcal{H})$ by

$$\tau_n(h^1 \otimes \cdots \otimes h^n) = (Sh^1) \cdot (h^2 \otimes \cdots \otimes h^n \otimes 1), \tag{2.4}$$

where the \mathcal{H} -module structure on $C^n(\mathcal{H})$ is given as in (1.10).

Theorem 2.1. *For a Hopf algebroid $(\mathcal{H}_l, \mathcal{H}_r, S)$ the formulae above give $C^\bullet(\mathcal{H})$ the structure of a cocyclic module if and only if $S^2 = \text{id}$. More specifically,*

$$\tau_n^{n+1}(h^1 \otimes \cdots \otimes h^n) = S^2(h^1) \otimes \cdots \otimes S^2(h^n).$$

Remark 2.2. This theorem was first proved by Connes and Moscovici in [CM3] in a special case using a characteristic map associated to a faithful trace. A more general version (for the if -direction) appeared in [KR3] for so-called para-Hopf algebroids.

2.2. The approach via coinvariants. In this section we will prove Theorem 2.1 using coinvariant localisation. This approach is inspired by the analogous procedure for Hopf algebras as in [Cr3].

Let us first define the fundamental cocyclic module associated to a Hopf algebroid arising from its underlying left bialgebroid structure: define the k -module

$$B^n(\mathcal{H}) := A_r \otimes_{A_l^c} C^{n+1}(\mathcal{H}).$$

Here the right A_l^c -module structure on A_r is given using $\theta : A_l \rightarrow A_r^{\text{op}}$, whereas the left A_l^c -module structure on $C^{n+1}(\mathcal{H})$ is defined using s_l and $t'_l := t_l \circ \theta^{-1} \circ \phi$: the k -module $B^n(\mathcal{H})$ is isomorphic to a quotient of $C^{n+1}(\mathcal{H})$ by the k -module $I^n \subset C^{n+1}(\mathcal{H})$ defined by

$$I^n := \text{span}\{s_l(a)h^0 \otimes \cdots \otimes h^n - h^0 \otimes \cdots \otimes t'_l(a)h^n, a \in A_l, h^i \in \mathcal{H}\}.$$

When $S^2 = \text{id}$, and therefore $\theta = \phi$, the right-hand side is the cyclic tensor product (cf. [Q2]) of \mathcal{H} in the category of (A_l, A_l) -bimodules with respect to the bimodule structure induced by the forgetful functor $\text{Mod}(\mathcal{H}) \rightarrow \text{Mod}(A_l^c)$.

Define the coface, codegeneracy, and cocyclic operators on $B^n(\mathcal{H})$ as follows:

$$\begin{aligned} \delta_i(h^0 \otimes h^1 \otimes \cdots \otimes h^n) &= \begin{cases} h^0 \otimes \cdots \otimes \Delta_l h^i \otimes \cdots \otimes h^n & \text{if } 0 \leq i \leq n, \\ h^0_{(2)} \otimes h^1 \otimes \cdots \otimes S^2(h^0_{(1)}) & \text{if } i = n + 1, \end{cases} \\ \sigma_i(h^0 \otimes h^1 \otimes \cdots \otimes h^n) &= h^0 \otimes \cdots \otimes h^i \cdot \epsilon_l(h^{i+1}) \otimes \cdots \otimes h^n, \\ &0 \leq i \leq n - 1, \\ \tau_n(h^0 \otimes h^1 \otimes \cdots \otimes h^n) &= h^1 \otimes h^2 \otimes \cdots \otimes h^n \otimes S^2(h^0). \end{aligned} \tag{2.5}$$

It is easy to verify that with these structure maps $B^*(\mathcal{H})$ is a para-cocyclic module, which is cocyclic if and only if $S^2 = \text{id}$. In this case this is just the canonical cocyclic module associated to the A_l -coalgebra $(\mathcal{H}_l, \Delta_l, \epsilon_l)$ arising from the underlying left bialgebroid structure, and to which we will refer as $\mathcal{H}_{\text{coalg}, \natural}^*$. Likewise, the underlying right bialgebroid gives rise to a similar construction by means of $(\mathcal{H}_r, \Delta_r, \epsilon_r)$.

On the other hand, we have $C^{n+1}(\mathcal{H}) \in \text{Mod}(\mathcal{H})$, and we can apply the functor of coinvariants to get $C^{n+1}(\mathcal{H})_{\text{coinv}} \cong C^n(\mathcal{H})$ by Proposition 1.6. Explicitly, this isomorphism is implemented by the maps

$$C^n(\mathcal{H}) \begin{matrix} \xrightarrow{\Phi_{\text{coinv}}} \\ \xleftarrow{\Psi_{\text{coinv}}} \end{matrix} C^{n+1}(\mathcal{H}),$$

given by

$$\begin{aligned} \Phi_{\text{coinv}}(h^1 \otimes \cdots \otimes h^n) &:= 1 \otimes h^1 \otimes \cdots \otimes h^n, \\ \Psi_{\text{coinv}}(h^0 \otimes \cdots \otimes h^n) &:= S(h^0) \cdot (h^1 \otimes \cdots \otimes h^n). \end{aligned} \tag{2.6}$$

Now observe that $I^n \subseteq \ker(\Psi_{\text{coinv}})$, so that we have a diagram

$$\begin{array}{ccc} C^n(\mathcal{H}) & \xrightarrow{\Phi_{\text{coinv}}} & C^{n+1}(\mathcal{H}) \\ & \searrow \bar{\Psi}_{\text{coinv}} & \swarrow \pi \\ & B^n(\mathcal{H}), & \end{array}$$

where π denotes the canonical projection and $\bar{\Psi}_{\text{coinv}}$ the induced map.

Proposition 2.3. *The morphism $\bar{\Psi}_{\text{coinv}}$ intertwines the maps $\delta_i, \sigma_i,$ and τ_n from (2.1)–(2.4) with the respective ones from (2.5).*

Proof. Consider first the cyclic operator

$$\begin{aligned} \tau_n \bar{\Psi}_{\text{coinv}}(h_0 \otimes h_1 \otimes \cdots \otimes h_n) &= \tau_n(S(h_0) \cdot (h_1 \otimes \cdots \otimes h_n)) \\ &= S((Sh_0)_{(1)}h_1) \cdot ((Sh_0)_{(2)}h_2 \otimes \cdots \otimes (Sh)_{(n)}h_n \otimes 1) \\ &= S(h_1) \cdot S^2(h_0^{(n)}) \cdot (S(h_0^{(n-1)})h_2 \otimes \cdots \otimes S(h_0^{(1)})h_n \otimes 1), \end{aligned}$$

where we used that $C^n(\mathcal{H}) \in \text{Mod}(\mathcal{H})$ with the module structure on tensor products given by (1.10), as well as the fact that the antipode S is an anti-algebra homomorphism. On the other hand,

$$\bar{\Psi}_{\text{coinv}}\tau_n(h_0 \otimes h^1 \otimes \cdots \otimes h^n) = S(h_1) \cdot (h_2 \otimes \cdots \otimes h_n \otimes S^2h_0).$$

The statement therefore follows from the following:

Lemma 2.4. *In $C^n(\mathcal{H})$ the identity*

$$S^2(h_0^{(n)}) \cdot (S(h_0^{(n-1)})h_1 \otimes \cdots \otimes S(h_0^{(1)})h_{n-1} \otimes 1) = h_1 \otimes \cdots \otimes h_{n-1} \otimes S^2h_0.$$

holds.

Proof. This is proved by induction: first, for $n = 2$ we have by the right comonoid identities (1.4), (1.8) and (1.5) for any $h_1, h \in \mathcal{H}$ in $C^2(\mathcal{H})$

$$\begin{aligned} h_1 \otimes Sh &= h_1 \otimes S(h^{(1)}s_r(\epsilon_r h^{(2)})) \\ &= t_l \epsilon_l (Sh^{(2)})h_1 \otimes Sh^{(1)} \\ &= s_r \epsilon_r h^{(2)}h_1 \otimes Sh^{(1)} \\ &= Sh_{(1)}^{(2)}h_{(2)}^{(2)}h_1 \otimes Sh^{(1)} = Sh_{(1)}^{(2)}h_{(2)}h_1 \otimes Sh_{(1)}^{(1)}. \end{aligned} \tag{2.7}$$

Applying this identity to $h := Sh_0$ proves the case $n = 2$. Assume now that the identity holds for $n - 1$. Then we have, using (1.4), (1.8) and (1.5),

$$\begin{aligned} h_1 \otimes \cdots \otimes h_n \otimes S^2h_0 &= h_1 \otimes (S^2(h_0^{(n)}) \cdot (Sh_0^{(n-1)}h_2 \otimes \cdots \otimes Sh_0^{(1)}h_n \otimes 1)) \\ &= h_1 \otimes s_l \epsilon_l (S^2h_{0(1)}^{(n)})S^2h_{0(2)}^{(n)} \cdot (Sh_0^{(n-1)}h_2 \otimes \cdots \otimes Sh_0^{(1)}h_n \otimes 1) \\ &= s_r \epsilon_r (Sh_{0(1)}^{(n)})h_1 \otimes S^2h_{0(2)}^{(n)} \cdot (Sh_0^{(n-1)}h_2 \otimes \cdots \otimes Sh_0^{(1)}h_n \otimes 1) \\ &= S^2(h_{0(1)}^{(n+1)})Sh_0^{(n)}h_2 \otimes S^2h_{0(2)}^{(n+1)} \cdot (Sh_0^{(n-1)}h_2 \otimes \cdots \otimes Sh_0^{(1)}h_n \otimes 1) \\ &= S^2(h_0^{(n+1)}) \cdot (Sh_0^{(n)}h_2 \otimes \cdots \otimes Sh_0^{(1)}h_{n+1} \otimes 1). \end{aligned}$$

This completes the proof of the lemma.

Hence the proposition is proved. □

Remark 2.5. The identity (2.7) for $h_1 = 1$ appears as an axiom in the definition of a para-Hopf algebroid in [KR3].

Proof of Theorem 2.1. Since $\bar{\Psi}_{\text{coinv}}$ is surjective with right inverse Φ_{coinv} , this proves Theorem 2.1. □

Definition 2.6. In case $S^2 = \text{id}$, we denote by $\mathcal{H}_1^\bullet := C^\bullet(\mathcal{H})$ the cocyclic module equipped with the operators (2.1)–(2.4) and by $(C^\bullet(\mathcal{H}), b, B)$ its associated mixed complex (cf. [C2], [Kas1]). Its Hochschild and (periodic) cyclic cohomology groups are denoted by $\text{HH}^\bullet(\mathcal{H})$, $\text{HC}^\bullet(\mathcal{H})$ and $\text{HP}^\bullet(\mathcal{H})$, and are referred to as *Hopf-cyclic cohomology groups*.

Remark 2.7. One may think of the forgetful functor $\text{Mod}(\mathcal{H}) \rightarrow \text{Mod}(A_l^c)$ as coming from the morphism of Hopf algebroids $\iota: A_l^c \rightarrow \mathcal{H}$, $a_1 \otimes a_2 \mapsto s_l(a_1)t_l(a_2)$; see

Section 3.1 for the description of the Hopf algebroid structure of A_l^c . From this point of view, $B^n(\mathcal{H})$ is simply the coinvariant localisation of $C^{n+1}(\mathcal{H}) \in \text{Mod}(\mathcal{H})$ with respect to A_l^c . On the other hand, one can also directly show that under the projection $B^n(\mathcal{H}) \rightarrow C^{n+1}(\mathcal{H})_{\text{coinv}}$ induced by ι , the operators (2.5) descend to well-defined maps on $C^{n+1}(\mathcal{H})_{\text{coinv}}$, turning it into a cocyclic module.

2.3. Dual Hopf-cyclic homology

2.3.1. The chain complexes. In this section we consider \mathcal{H} as a right \mathcal{H}_r -comodule with (A_r, A_r) -bimodule structure (1.16) given by left and right multiplication with $s_r(a)$, $a \in A_r$. Using the tensor structure of the category $\text{Comod}_R(\mathcal{H}_r)$, we define

$$C_n(\mathcal{H}) := \underbrace{\mathcal{H} \otimes_{A_r} \cdots \otimes_{A_r} \mathcal{H}}_{n \text{ times}}.$$

Face and degeneracy operators can be introduced by

$$d_i(h^1 \otimes \cdots \otimes h^n) = \begin{cases} \epsilon_r(h^1) \cdot h^2 \otimes \cdots \otimes h^n & \text{if } i = 0, \\ h^1 \otimes \cdots \otimes h^i h^{i+1} \otimes \cdots \otimes h^n & \text{if } 1 \leq i \leq n - 1, \\ h^1 \otimes \cdots \otimes h^{n-1} \cdot \epsilon_r(S^{-1}h^n) & \text{if } i = n, \end{cases}$$

$$s_i(h^1 \otimes \cdots \otimes h^n) = \begin{cases} 1 \otimes h^1 \otimes \cdots \otimes h^n & \text{if } i = 0, \\ h^1 \otimes \cdots \otimes h^i \otimes 1 \otimes h^{i+1} \otimes \cdots \otimes h^n & \text{if } 1 \leq i \leq n. \end{cases} \tag{2.8}$$

Elements of degree zero (i.e., elements of A_r) are mapped to zero, i.e., $d_0(a) = 0$, $a \in A_r$. To define a cyclic structure we assume the antipode S to be invertible and define

$$t_n(h^1 \otimes \cdots \otimes h^n) = S^{-1}(h_{(2)}^1 \cdots h_{(2)}^{n-1} h^n) \otimes h_{(1)}^1 \otimes h_{(1)}^2 \otimes \cdots \otimes h_{(1)}^{n-1}. \tag{2.9}$$

One easily verifies that this operator is well defined. Below we shall prove that these k -modules and maps are canonically isomorphic to the cyclic dual of the cocyclic module $C^*(\mathcal{H})$ of §2.1. This also proves that $C_*(\mathcal{H})$ is indeed a cyclic module.

2.3.2. Cyclic duality (cf. [C1], [Lo]). We recall the notion of cyclic duality. Let Λ denote Connes' cyclic category. A cyclic module is a functor $\Lambda^{\text{op}} \rightarrow \text{Mod}(k)$, i.e., a contravariant functor from Λ to $\text{Mod}(k)$; whereas a cocyclic module is a functor $\Lambda \rightarrow \text{Mod}(k)$. Remarkably, there is a canonical equivalence $\Lambda \cong \Lambda^{\text{op}}$ that allows one to construct a cocyclic module out of a cyclic module and vice versa. Explicitly, in the first direction this is done as follows: let $Y = (Y^*, \delta_*, \sigma_*, \tau_*)$ be a para-cocyclic module with invertible operator τ . Its *cyclic dual* is defined to be $\check{Y} := (\check{Y}_*, d_*, s_*, t_*)$

where $\check{Y}_n := Y^n$ in degree n and

$$\begin{aligned} d_i &:= \sigma_{i-1} : \check{Y}_n \rightarrow \check{Y}_{n-1}, \quad 1 \leq i \leq n, \\ d_0 &:= \sigma_{n-1} \tau_n : \check{Y}_n \rightarrow \check{Y}_{n-1}, \\ s_i &:= \delta_i : \check{Y}_{n-1} \rightarrow \check{Y}_n, \quad 0 \leq i \leq n-1, \\ t_n &:= \tau_n^{-1} : \check{Y}_n \rightarrow \check{Y}_n. \end{aligned}$$

It can be shown that \check{Y} carries the structure of a para-cyclic object in the category of k -modules and is cyclic if Y is cocyclic.

2.3.3. The Hopf–Galois map and cyclic duality. In this section we will prove that the cyclic module dual to $C^\bullet(\mathcal{H})$ is canonically isomorphic to $C_\bullet(\mathcal{H})$. The explicit map implementing this isomorphism is given by generalising the Hopf–Galois map from [Sch2], Thm. 3.5, and its inverse from [BSz] for Hopf algebroids.

Lemma 2.8. *For each $n \geq 0$, the k -modules $C_n(\mathcal{H})$ and $C^n(\mathcal{H})$ are isomorphic by means of the Hopf–Galois map $\varphi_n : C_n(\mathcal{H}) \rightarrow C^n(\mathcal{H})$ defined inductively by $\varphi_1 := \text{id}_{\mathcal{H}}$ and*

$$\varphi_n(h^1 \otimes \cdots \otimes h^n) := h^1 \cdot (1 \otimes \varphi_{n-1}(h^2 \otimes \cdots \otimes h^n)), \quad n \geq 2.$$

For $n = 0$ one defines $\varphi_0 := \phi : A_l^{\text{op}} \rightarrow A_r$.

Proof. The explicit formula for the Hopf–Galois map is given by

$$h^1 \otimes \cdots \otimes h^n \mapsto h^1_{(1)} \otimes h^1_{(2)} h^2_{(1)} \otimes h^1_{(3)} h^2_{(2)} h^3_{(1)} \otimes \cdots \otimes h^1_{(n)} h^2_{(n-1)} \cdots h^{n-1}_{(2)} h^n.$$

Define its inverse $\varphi_n^{-1} : C^n(\mathcal{H}) \rightarrow C_n(\mathcal{H})$ by

$$h_1 \otimes \cdots \otimes h_n \mapsto h_1^{(1)} \otimes S(h_1^{(2)}) h_2^{(1)} \otimes S(h_2^{(2)}) h_3^{(1)} \otimes \cdots \otimes S(h_{n-1}^{(2)}) h_n.$$

To check that this is indeed an inverse, observe that one can decompose

$$\varphi_{n+1} = (\text{id} \otimes \varphi_n) (\varphi_2 \otimes \text{id}^{\otimes n-1}) \quad \text{and} \quad \psi_{n+1} = (\psi_2 \otimes \text{id}^{\otimes n-1}) (\text{id} \otimes \psi_n),$$

and one easily verifies by induction that φ_{n+1} and ψ_{n+1} are mutually inverse. \square

To prove our main theorem about cyclic duality, we also need the inverses of the cyclic operators on $C^\bullet(\mathcal{H})$ and $C_\bullet(\mathcal{H})$. Since we assume S to be invertible, we have:

Lemma 2.9. *Let $n \geq 1$. The inverse of the cocyclic operator τ_n on $C^n(\mathcal{H})$ in (2.4) is given by*

$$\tau_n^{-1}(h^1 \otimes \cdots \otimes h^n) = S^{-1}(h^n) \cdot (1 \otimes h^1 \otimes \cdots \otimes h^{n-1}). \tag{2.10}$$

Likewise, the cyclic operator t_n on $C_n(\mathcal{H})$ has inverse given by

$$t_n^{-1}(h_1 \otimes \cdots \otimes h_n) = h_2^{(1)} \otimes \cdots \otimes h_n^{(1)} \otimes S(h_1 h_2^{(2)} \cdots h_n^{(2)}).$$

Proof. This can be verified directly, but we shall use induction. For $n = 1$, by (2.10) we have $\tau_1^{-1} = S^{-1}$, and the statement is clear. For $n \geq 2$, define the map

$$\tilde{\varphi}(h \otimes h') := h_{(1)}h' \otimes h_{(2)}.$$

This defines a bijection of $\mathcal{H} \otimes_{A_l} \mathcal{H} \in \text{Comod}_L(\mathcal{H}_l)$ to $C^2(\mathcal{H}) \in \text{Mod}(\mathcal{H})$, with inverse

$$\tilde{\varphi}^{-1}(h \otimes h') = h'^{(2)} \otimes S^{-1}(h'^{(1)})h.$$

With these maps, one has

$$\begin{aligned} \tau_{n+1} &= (\text{id}^{\otimes n-1} \otimes \tilde{\varphi})(\tau_n \otimes \text{id}), \\ \tau_{n+1}^{-1} &= (\tau_n^{-1} \otimes \text{id})(\text{id}^{\otimes n-1} \otimes \tilde{\varphi}^{-1}). \end{aligned}$$

This proves the first statement. As for the second part, introduce the maps

$$\begin{aligned} \tilde{\psi}(h \otimes h') &:= h_{(2)}h' \otimes h_{(1)}, \\ \tilde{\psi}^{-1}(h \otimes h') &:= h'^{(1)} \otimes S(h'^{(2)})h. \end{aligned}$$

This time $\tilde{\psi}$ maps the tensor product $\mathcal{H} \otimes_{A_r} \mathcal{H} \in \text{Comod}_R(\mathcal{H}_r)$ to the tensor product

$$\mathcal{H} \otimes_{A_r} \mathcal{H} := \mathcal{H} \otimes_k \mathcal{H} / \text{span}_k \{t^r(a)h \otimes_k h' - h \otimes_k s^r(\phi\theta^{-1}(a))h', a \in A_r\},$$

and one easily checks that $\tilde{\psi}^{-1}$ is its inverse. Then one has

$$\begin{aligned} t_{n+1} &= (t_n \otimes \text{id})(\text{id}^{\otimes n-1} \otimes \tilde{\psi}), \\ t_{n+1}^{-1} &= (\text{id}^{\otimes n-1} \otimes \tilde{\psi}^{-1})(t_n^{-1} \otimes \text{id}), \end{aligned}$$

and with this one proves the second equality. □

Theorem 2.10. *Let \mathcal{H} be a Hopf algebroid with invertible antipode. The Hopf–Galois map $\varphi: C_\bullet(\mathcal{H}) \rightarrow C^\bullet(\mathcal{H})$ identifies $C_\bullet(\mathcal{H})$ as the cyclic dual of the para-cocyclic module $C^\bullet(\mathcal{H})$ of Theorem 2.1.*

Proof. This is now a straightforward verification:

$$\begin{aligned} \tau_n^{-1}\varphi_n(h^1 \otimes \dots \otimes h^n) &= \tau_n^{-1}(h_{(1)}^1 \otimes h_{(2)}^1 h_{(1)}^2 \otimes h_{(3)}^1 h_{(2)}^2 h_{(1)}^3 \otimes \dots \otimes h_{(n)}^1 h_{(n-1)}^2 \dots h_{(2)}^{n-1} h^n) \\ &= S^{-1}(h_{(n)}^1 \dots h_{(2)}^{n-1} h^n) \cdot (1 \otimes h_{(1)}^1 \otimes \dots \otimes h_{(n-1)}^1 h_{(n-2)}^2 \dots h_{(1)}^{n-1}), \end{aligned}$$

and by coassociativity

$$\begin{aligned} \varphi_n t_n(h^1 \otimes \dots \otimes h^n) &= \varphi_n(S^{-1}(h_{(2)}^1 \dots h_{(2)}^{n-1} h^n) \otimes h_{(1)}^1 \otimes h_{(1)}^2 \otimes \dots \otimes h_{(1)}^{n-1}) \\ &= S^{-1}(h_{(2)}^1 \dots h_{(2)}^{n-1} h^n) \cdot (1 \otimes \varphi_{n-1}(h_{(1)}^1 \otimes h_{(1)}^2 \otimes \dots \otimes h_{(1)}^{n-1})) \\ &= S^{-1}(h_{(n)}^1 \dots h_{(2)}^{n-1} h^n) \cdot (1 \otimes h_{(1)}^1 \otimes \dots \otimes h_{(n-1)}^1 h_{(n-2)}^2 \dots h_{(1)}^{n-1}). \end{aligned}$$

Hence, $\varphi_n \circ t_n = \tau_n^{-1} \circ \varphi_n$ or equivalently $t_n = \psi_n \circ \tau_n^{-1} \circ \varphi_n$. In the same fashion,

$$\begin{aligned} & \sigma_{n-1} \tau_n \varphi_n (h^1 \otimes \cdots \otimes h^n) \\ &= \sigma_{n-1} (S(h_{(1)}^1) \cdot (h_{(2)}^1 h_{(1)}^2 \otimes \cdots \otimes h_{(n)}^1 \cdots h_{(2)}^{n-1} h^n \otimes 1)) \\ &= S(h_{(1)}^1) \cdot (h_{(1)}^1 h_{(1)}^2 \otimes \cdots \otimes h_{(n)}^1 \cdots h_{(2)}^{n-1} h^n) \\ &= (S(h_{(1)}^1) h_{(2)}^1 h^2) \cdot (1 \otimes \varphi_{n-2} (h^3 \otimes \cdots \otimes h^n)) \\ &= (s_r \epsilon_r (h^1) h^2) \cdot (1 \otimes \varphi_{n-2} (h^3 \otimes \cdots \otimes h^n)) \\ &= \varphi_{n-1} d_0 (h^1 \otimes \cdots \otimes h^n). \end{aligned}$$

The remaining identities are left to the reader. □

Corollary 2.11. $C_*(\mathcal{H})$ is para-cyclic, and cyclic if and only if $S^2 = \text{id}$.

Definition 2.12. In case $S^2 = \text{id}$, let $\mathcal{H}_\bullet^{\text{cyc}}$:= $C_*(\mathcal{H})$ denote the cyclic module equipped with the operators (2.8)–(2.9). Its respective Hochschild and (periodic) cyclic homology groups are denoted by $\text{HH}_\bullet(\mathcal{H})$, $\text{HC}_\bullet(\mathcal{H})$ and $\text{HP}_\bullet(\mathcal{H})$, and referred to as *dual Hopf-cyclic* homology groups.

2.4. The approach via invariants. As is clear from the explicit formulae, the dual cyclic homology is closely related to the underlying algebra structure of the Hopf algebraoid. To compare this homology to the usual cyclic homology of algebras, we use the following approach, which is dual to that of Section 2.2. Remarkably, it only works in some special cases.

Let $\mathcal{H} = (\mathcal{H}_l, \mathcal{H}_r, S)$ be a Hopf algebraoid. The standard cyclic module of \mathcal{H} as a k -algebra [FeTs] is defined by $\mathcal{H}_\bullet^{\text{alg}, \#} := \mathcal{H}^{\otimes_k (\bullet+1)}$, with face maps

$$d_i (h^0 \otimes \cdots \otimes h^n) = \begin{cases} h^0 \otimes \cdots \otimes h^i h^{i+1} \otimes \cdots \otimes h^n & \text{if } 0 \leq i \leq n-1, \\ h^n h^0 \otimes \cdots \otimes h^{n-1} & \text{if } i = n, \end{cases}$$

and degeneracies

$$s_i (h^0 \otimes \cdots \otimes h^n) = h^0 \otimes \cdots \otimes h^i \otimes 1 \otimes h^{i+1} \otimes \cdots \otimes h^n \quad \text{if } 0 \leq i \leq n.$$

Finally the cyclic structure is given by

$$t_n (h^0 \otimes \cdots \otimes h^n) = h^n \otimes h^0 \otimes \cdots \otimes h^{n-1}.$$

On the other hand, we have $C_{n+1}(\mathcal{H}) \in \text{Comod}_R(\mathcal{H}_r)$. Recall that underlying the comodule structure is an (A_r, A_r) -bimodule, so we can define

$$B_n(\mathcal{H}) := C_{n+1}(\mathcal{H}) \otimes_{A_r^e} A_r.$$

This space is a quotient of $C_{n+1}(\mathcal{H})$ by the k -submodule $I_n \subset C_{n+1}(\mathcal{H})$ given by

$$I_n := \text{span}_k \{ s_r(a) h^0 \otimes \cdots \otimes h^n - h^0 \otimes \cdots \otimes h^n s_r(a), a \in A_r, h^i \in \mathcal{H} \}.$$

One then easily observes that the canonical projection $\mathcal{H}_\bullet^{\text{alg}, \natural} \rightarrow B_\bullet(\mathcal{H})$ equips the latter with the structure of a cyclic module given by the same formulae as above.

By Proposition 1.7, we have $C_n(\mathcal{H}) \cong C_{n+1}(\mathcal{H})^{\text{inv}}$ via the embedding

$$\Psi_{\text{inv}}: h_1 \otimes \cdots \otimes h_n \mapsto h_1^{(1)} \otimes \cdots \otimes h_n^{(1)} \otimes S(h_1^{(2)} \dots h_n^{(2)}). \tag{2.11}$$

Combined with the canonical projection $C_{n+1}(\mathcal{H}) \rightarrow B_n(\mathcal{H})$, this leads to a morphism $\bar{\Psi}_{\text{inv}}: C_\bullet(\mathcal{H}) \rightarrow B_\bullet(\mathcal{H})$. Unfortunately, this is not a map of cyclic objects, let alone simplicial objects in general. However, there are Hopf algebroids for which this is true, cf. §3.2 for an example. Also, the left inverse of the embedding above, given as $\Phi_{\text{inv}}: C_{n+1}(\mathcal{H}) \rightarrow C_n(\mathcal{H})$, $h^1 \otimes \cdots \otimes h^{n+1} \mapsto h^1 \otimes \cdots \otimes h^n \cdot \epsilon_r(S^{-1}h^{n+1})$, does not descend to the quotient $B_n(\mathcal{H})$. We therefore do not have a commutative diagram as for the coinvariant localisation and the cyclic cohomology theory.

2.5. Hochschild theory with coefficients. Both the Hochschild cohomology for bialgebroids as well as the dual homology for Hopf algebroids are part of a more general theory with coefficients that we now describe. First, however, we introduce certain resolutions of the base algebras in the categories discussed in §1.4, defined by the left and right bialgebroid structure on \mathcal{H} .

2.5.1. Cobar resolution. A straightforward generalisation of Theorem A.1.1.3 and Lemma A.1.2.2 in [Ra] to the noncommutative setting (cf. Section 2.4 in [Ko]) shows that if \mathcal{H} is flat as right A_l -module (1.1), the category $\text{Comod}_L(\mathcal{H}_l)$ is abelian and has enough injectives. We call a (say, left) \mathcal{H}_l -comodule N *cofree* if there is a left A_l -module M such that $N \cong \mathcal{H} \otimes_{A_l} M$ as left \mathcal{H}_l -comodules, and it is called *relative injective* if it is a direct summand in a cofree one.

The *cobar* resolution of A_l in the category $\text{Comod}_L(\mathcal{H}_l)$ generalises the well-known construction for bialgebras [D] and for commutative bialgebroids in [Ra]: define the graded space

$$\text{Cobar}^n(\mathcal{H}) := \underbrace{\mathcal{H} \otimes_{A_l} \cdots \otimes_{A_l} \mathcal{H}}_{n+1 \text{ times}},$$

the tensor product being the one in the category $\text{Mod}(\mathcal{H})$. Alternatively, we can view this as the cofree left \mathcal{H}_l -comodule generated by $C^\bullet(\mathcal{H}) \in \text{Mod}(\mathcal{H})$. This allows us to view $\text{Cobar}^\bullet(\mathcal{H}) \in \text{Comod}_L(\mathcal{H}_l)$ by using the left comultiplication on the first component. Introduce the following cosimplicial structure on $\text{Cobar}^\bullet(\mathcal{H})$: first, the coface operators $\delta'_i: \text{Cobar}^n(\mathcal{H}) \rightarrow \text{Cobar}^{n+1}(\mathcal{H})$ are given by

$$\delta'_i(h^0 \otimes h^1 \otimes \cdots \otimes h^n) = \begin{cases} h^0 \otimes \cdots \otimes \Delta_l h^i \otimes \cdots \otimes h^n & \text{if } 0 \leq i \leq n, \\ h^0 \otimes \cdots \otimes h^n \otimes 1 & \text{if } i = n + 1. \end{cases}$$

The codegeneracies $\sigma'_i: \text{Cobar}^n(\mathcal{H}) \rightarrow \text{Cobar}^{n-1}(\mathcal{H})$ are

$$\sigma'_i(h^0 \otimes \cdots \otimes h^n) = h^0 \otimes \cdots \otimes \epsilon_l(h^{i+1}) \otimes \cdots \otimes h^n, \quad 0 \leq i \leq n - 1.$$

These maps are compatible with the left \mathcal{H}_l -comodule structure on $\text{Cobar}^\bullet(\mathcal{H})$ but *not* with the left \mathcal{H}_l -module structure. The left \mathcal{H}_l -coaction on A_l given by the left source map $s_l: A_l \rightarrow \mathcal{H}$ defines a coaugmentation for this cosimplicial object in $\text{Comod}_L(\mathcal{H}_l)$, which yields a cosimplicial resolution of A_l : consider the associated cochain complex

$$A_l \xrightarrow{s_l} \text{Cobar}^0(\mathcal{H}) \xrightarrow{b'} \text{Cobar}^1(\mathcal{H}) \xrightarrow{b'} \dots,$$

with differentials $b' := \sum_{i=0}^{n+1} (-1)^i \delta'_i$. It is easy to check that b' is a morphism of left \mathcal{H}_l -comodules and that the maps $s^{n-1}: \text{Cobar}^n(\mathcal{H}) \rightarrow \text{Cobar}^{n-1}(\mathcal{H})$ given by $h^0 \otimes \dots \otimes h^n \mapsto \epsilon_l(h^0) \cdot h^1 \otimes \dots \otimes h^n$ and $s^{-1}: \mathcal{H} = \text{Cobar}^0(\mathcal{H}) \rightarrow \text{Cobar}^{-1}(\mathcal{H}) := A_l, h \mapsto \epsilon_l h$ define a contracting homotopy for the complex $(\text{Cobar}^\bullet(\mathcal{H}), b')$ over A_l , i.e., $s^n \circ b' + b' \circ s^{n-1} = \text{id}$. In particular, $A_l \xrightarrow{s_l} \text{Cobar}^\bullet(\mathcal{H})$ is a resolution of A_l by cofree (hence relative injective) left \mathcal{H}_l -comodules: from (1.15) follows $\ker b' = \{h \in \mathcal{H}, \Delta_l(h) = h \otimes 1\} \cong A_l$, hence exactness in degree zero.

2.5.2. The bar resolution. Analogous to the standard case (see e.g. [We]), the bar complex gives a resolution of A_l in the category $\text{Mod}(\mathcal{H})$ of left modules over \mathcal{H} . We define

$$\text{Bar}_n(\mathcal{H}) := \underbrace{\mathcal{H} \otimes_{A_r} \dots \otimes_{A_r} \mathcal{H}}_{n+1 \text{ times}},$$

where the tensor product is the one in $\text{Comod}_R(\mathcal{H}_r)$, but we view $\text{Bar}_\bullet(\mathcal{H})$ in $\text{Mod}(\mathcal{H})$, the left \mathcal{H} -action being given by left multiplication on the first factor. The simplicial structure on this graded \mathcal{H} -module is given by the face and degeneracy operators

$$d'_i(h^0 \otimes \dots \otimes h^n) = \begin{cases} h^0 \otimes \dots \otimes h^i h^{i+1} \otimes \dots \otimes h^n & \text{if } 0 \leq i \leq n-1, \\ h^0 \otimes \dots \otimes h^{n-1} \cdot \epsilon_r(S^{-1}h^n) & \text{if } i = n, \end{cases}$$

$$s'_i(h^0 \otimes \dots \otimes h^n) = h^0 \otimes \dots \otimes h^i \otimes 1 \otimes h^{i+1} \otimes \dots \otimes h^n, \quad 0 \leq i \leq n.$$

This time these maps are morphisms of left \mathcal{H} -modules, not of comodules. The augmented bar complex is given by

$$\dots \xrightarrow{d'} \text{Bar}_1(\mathcal{H}) \xrightarrow{d'} \text{Bar}_0(\mathcal{H}) \xrightarrow{\epsilon_l} A_l,$$

with chain operator $d' := \sum_{i=0}^n (-1)^i d'_i$. It is a straightforward check that the bar complex is a contractible resolution of A_l , where the extra degeneracy

$$s_n: \text{Bar}_n(\mathcal{H}) \rightarrow \text{Bar}_{n+1}(\mathcal{H}), \quad s_n(h^0 \otimes \dots \otimes h^n) = 1 \otimes h^0 \otimes \dots \otimes h^n,$$

for $n \geq 0$ and $s_{-1} := t_l$ provide the contracting homotopies. Moreover, $\text{Bar}_\bullet(\mathcal{H})$ is \mathcal{H} -projective if \mathcal{H} is A_r -projective with respect to the left A_r -module structure (1.16), and in this case $\text{Bar}_\bullet(\mathcal{H}) \xrightarrow{\epsilon_l} A_l$ is a projective resolution of A_l in the category of left \mathcal{H} -modules.

2.5.3. The Hochschild theory as derived functors. The main point is now:

Theorem 2.13. *For any Hopf algebroid \mathcal{H} that is flat as right A_l -module (1.1), there are natural isomorphisms*

$$\mathrm{HH}^\bullet(\mathcal{H}) \cong \mathrm{Cotor}_{\mathcal{H}}^\bullet(A_l, A_l),$$

and if \mathcal{H} is projective as left A_r -module (1.16), then

$$\mathrm{HH}_\bullet(\mathcal{H}) \cong \mathrm{Tor}_{\mathcal{H}}^\bullet(A_r, A_l).$$

Proof. Recall (cf. [EMo]) that $\mathrm{Cotor}_{\mathcal{H}}^\bullet(A_l, A_l)$ is the right derived functor of the left cotensor product $A_l \square_{\mathcal{H}_l} - : \mathrm{Comod}_L(\mathcal{H}_l) \rightarrow \mathrm{Mod}(k)$, where A_l is seen as right \mathcal{H}_l -comodule by means of $t_l : A_l \rightarrow \mathcal{H}_l$. That this derived functor can be computed by relative injective resolutions (like the cobar complex) follows from a straightforward generalisation of Lemmas A.1.2.8 and A.1.2.9 in [Ra] to the noncommutative case. A little thought reveals that the space $A_l \square_{\mathcal{H}_l} \mathrm{Cobar}^\bullet(\mathcal{H})$ can be alternatively expressed as $\{h \otimes w \in \mathrm{Cobar}^\bullet(\mathcal{H}) \mid h \otimes w = 1_{\mathcal{H}} \otimes s^l(\epsilon_l h)w\}$, where $w \in C^\bullet(\mathcal{H})$. Using then the isomorphism $f : A_l \square_{\mathcal{H}_l} \mathrm{Cobar}^\bullet(\mathcal{H}) \cong C^\bullet(\mathcal{H})$ given by (1.15), one easily checks that $b \circ f = f \circ (\mathrm{id}_{A_l} \otimes b')$, i.e., the induced differential coincides with that of the Hochschild complex. This proves the first isomorphism.

To prove the second isomorphism, use the bar resolution in $\mathrm{Mod}(\mathcal{H})$ to compute the left derived functor of $A_r \otimes_{\mathcal{H}} - : \mathrm{Mod}(\mathcal{H}) \rightarrow \mathrm{Mod}(k)$. We have $A_r \otimes_{\mathcal{H}} \mathrm{Bar}_\bullet(\mathcal{H}) \cong C_\bullet(\mathcal{H})$, and one easily sees that the differentials coincide. \square

Remark 2.14. The definition of the Hochschild cohomology depends solely on the underlying left bialgebroid structure of \mathcal{H} . This is because for any left bialgebroid \mathcal{H}_l , the base algebra A_l carries canonical left and right \mathcal{H}_l -coactions given by left source and target maps, respectively. By contrast, the definition of dual Hochschild homology *does* depend on the Hopf algebroid structure: although the base algebra A_r of the underlying right bialgebroid is naturally a right \mathcal{H} -module, there is a priori no canonical *left* \mathcal{H} -module structure defined on it without the antipode.

2.5.4. Coefficients. Having identified Hochschild homology and cohomology as derived functors, we can assume a different perspective and put coefficients in: for $M \in \mathrm{Comod}_R(\mathcal{H}_l)$ with coaction Δ_M and \mathcal{H} flat as right A_l -module (1.1), define

$$H^\bullet(\mathcal{H}, M) := \mathrm{Cotor}_{\mathcal{H}}^\bullet(M, A_l).$$

If M is projective as a right A_l -module, one may use the cobar complex to compute these groups: using the isomorphism

$$M \square_{\mathcal{H}_l} \mathrm{Cobar}^\bullet(\mathcal{H}) \cong M \otimes_{A_l} C^\bullet(\mathcal{H}), \quad m \otimes h \otimes w \mapsto m \cdot \epsilon_l(h) \otimes w,$$

with inverse $m \otimes w \mapsto \Delta_M(m) \otimes w$, where $w \in C^\bullet(\mathcal{H})$, as well as the isomorphism

$$M \square_{\mathcal{H}_l} \text{Cobar}^\bullet(\mathcal{H}) \cong \{m \otimes h \otimes w \in M \otimes_{A_l} \text{Cobar}^\bullet(\mathcal{H}_l) \mid m \otimes h \otimes w = \Delta_M(m \cdot \epsilon_l h) \otimes w\},$$

similarly as above, one obtains the explicit corresponding complex with coefficients in M .

Likewise, we put

$$H_\bullet(\mathcal{H}, N) := \text{Tor}_\bullet^{\mathcal{H}}(N, A_l)$$

for $N \in \text{Mod}(\mathcal{H}^{\text{op}})$. If \mathcal{H} is projective as left A_r -module (1.16), one may use the bar resolution to write down the explicit complex computing these groups.

2.6. The case of commutative and cocommutative Hopf algebroids. For commutative and cocommutative Hopf algebroids, one of the respective two cyclic theories is particularly simple to calculate in terms of the associated Hochschild theory. This phenomenon is known for Hopf algebras, cf. [KR1], Thm. 4.1, and originated from Karoubi’s computation of the cyclic homology of $k[G]$ in [Ka], where G is a discrete group.

In a commutative Hopf algebroid, the underlying left bialgebroid may serve to define the right bialgebroid structure by means of the prescriptions $A_r := A_l, s_r := t_l, t_r := s_l, \Delta_r := \Delta_l$, and $\epsilon_r := \epsilon_l$, recovering the notion of Hopf algebroids in [Ra]. On the other hand, cocommutativity for Hopf algebroids is defined as the cocommutativity of the underlying left bialgebroid \mathcal{H}_l (which by (1.7) implies cocommutativity for \mathcal{H}_r as well) and only makes sense for commutative $A_l = A_r$ for which $s_l = t_l$ as well as $s_r = t_r$.

Proposition 2.15. (i) *Let \mathcal{H} be a commutative Hopf algebroid with $A_r = A_l, s_r = t_l, t_r = s_l, \Delta_r = \Delta_l$, and $\epsilon_r = \epsilon_l$. Then $\text{Cobar}^\bullet(\mathcal{H}) \in \text{Comod}_L(\mathcal{H}_l)$ is a para-cocyclic object by means of the cocyclic operator*

$$t'_n(h^0 \otimes \cdots \otimes h^n) = h^0 \cdot (1 \otimes \tau_n(h^1 \otimes \cdots \otimes h^n)),$$

where on the right-hand side the monoidal structure of $\text{Mod}(\mathcal{H})$ is used.

(ii) *Let \mathcal{H} be a cocommutative Hopf algebroid over commutative base algebra A with invertible antipode S . Then $\text{Bar}_\bullet(\mathcal{H})$ is a para-cyclic \mathcal{H} -module with cyclic operator*

$$t'_n(h_0 \otimes \cdots \otimes h_n) = h_0 h_1^{(2)} \dots h_n^{(2)} \otimes t_n(h_1^{(1)} \otimes \cdots \otimes h_n^{(1)}).$$

In both cases one obtains cocyclic resp. cyclic structures if and only if $S^2 = \text{id}$.

Proof. (i) Though we view the cobar complex as cosimplicial object in $\text{Comod}_L(\mathcal{H}_l)$, it has a natural left \mathcal{H} -module structure as in (1.10) from which it is also immediate

that τ' is a morphism of graded left \mathcal{H}_l -comodules. Let us now show that τ' is para-cocyclic: from the explicit formula (2.5) of the cocyclic operator τ , one easily shows by induction that

$$h \cdot \tau_n^j(h^1 \otimes \cdots \otimes h^n) = \tau_n^j(h^1 \otimes \cdots \otimes h^j S^{-1}h \otimes \cdots \otimes h^n),$$

for all $1 \leq j \leq n$. With this equation we can now compute

$$\begin{aligned} \tau_n^{n+1}(h^0 \otimes \cdots \otimes h^n) &= \tau_n^n(h^0 \cdot (1 \otimes \tau_n(h^1 \otimes \cdots \otimes h^n))) \\ &= \tau_n^n(h_{(1)}^0 \otimes h_{(2)}^0 \cdot \tau_n(h^1 \otimes \cdots \otimes h^n)) \\ &= \tau_n^n(h_{(1)}^0 \otimes \tau_n(h^1 S^{-1}h_{(2)}^0 \otimes \cdots \otimes h^n)) \\ &= \tau_n^{n-1}(h_{(1)}^0 \otimes h_{(2)}^0 \cdot \tau_n^2(h^1 S^{-1}h_{(3)}^0 \otimes \cdots \otimes h^n)) \\ &= \tau_n^{n-1}(h_{(1)}^0 \otimes \tau_n^2(h^1 S^{-1}h_{(3)}^0 \otimes h^2 S^{-1}h_{(2)}^0 \otimes \cdots \otimes h^n)) \\ &\vdots \\ &= h_{(1)}^0 \otimes h_{(2)}^0 \cdot \tau_n^{n+1}(h^1 S^{-1}h_{(n+2)}^0 \otimes \cdots \otimes h^n S^{-1}h_{(3)}^0) \\ &= h_{(1)}^0 \otimes h_{(2)}^0 \cdot (S^2(h^1 S^{-1}h_{(n+2)}^0) \otimes \cdots \otimes S^2(h^n S^{-1}h_{(3)}^0)) \\ &= h^0 \otimes S^2 h^1 \otimes \cdots \otimes S^2 h^n. \end{aligned}$$

The last equality is verified by writing out the expression and using the left comonoid identities. This proves that τ' generates an action of the cyclic groups if and only if $S^2 = \text{id}$. The remaining cocyclic identities, compatibility with the δ'_i and σ'_i that is, are easy to verify.

(ii) From $s_r = t_r$ it follows that the space $C_\bullet(\mathcal{H})$ carries a left \mathcal{H}_r -coaction given by $\Delta_r(h_1 \otimes \cdots \otimes h_n) := h_1^{(2)} \dots h_n^{(2)} \otimes h_1^{(1)} \otimes \cdots \otimes h_n^{(1)}$, which appears in the expression of the cyclic operator. One has

$$\begin{aligned} \Delta_r(t_n(h_1 \otimes \cdots \otimes h_n)) &= S^{-1}(h_{1(2)} \dots h_{n-1(2)} h_{n(1)}) h_{1(1)}^{(2)} \dots h_{n-1(1)}^{(2)} \\ &\quad \otimes S^{-1}(h_{1(3)} \dots h_{n-1(3)} h_{n(2)}) \otimes h_{1(1)}^{(1)} \otimes \cdots \otimes h_{n-1(1)}^{(1)} \\ &= S^{-1}h_{n(1)} \otimes t_n(h_1 \otimes \cdots \otimes h_{n-1} \otimes h_{n(2)}). \end{aligned}$$

With this we now compute

$$\begin{aligned} t_n^{n+1}(h_0 \otimes \cdots \otimes h_n) &= t_n^n(h_0 h_1^{(2)} \dots h_n^{(2)} \otimes t_n(h_1^{(1)} \otimes \cdots \otimes h_n^{(1)})) \\ &= t_n^{n-1}(h_0 h_1^{(2)} \dots h_{n-1}^{(2)} h_n^{(2)} S^{-1}h_{n(1)}^{(1)} \otimes t_n^2(h_1^{(1)} \otimes \cdots \otimes h_{n-1}^{(1)} \otimes h_{n(2)}^{(1)})) \\ &= t_n^{n-1}(h_0 h_1^{(2)} \dots h_{n-1}^{(2)} t_r \epsilon_r(h_n^{(1)}) \otimes t_n^2(h_1^{(1)} \otimes \cdots \otimes h_n^{(2)})) \end{aligned}$$

$$\begin{aligned}
 &= t_n^{m-1}(h_0 h_1^{(2)} \cdots h_{n-1}^{(2)} \otimes t_n^2(h_1^{(1)} \otimes \cdots \otimes h_{n-1}^{(1)} \otimes h_n)) \\
 &\vdots \\
 &= h_0 \otimes t_n^{n+1}(h_1 \otimes \cdots \otimes h_n) \\
 &= h_0 \otimes S^{-2}h_1 \otimes \cdots \otimes S^{-2}h_n,
 \end{aligned}$$

where the vertical dots mean the $(n - 1)$ -fold repetition of the previous manipulation. To obtain the fourth line we have used $s_r = t_r$ and

$$a_1 \cdot t_n(h_1 \otimes \cdots \otimes h_n) \cdot a_2 = t_n(h_1 \otimes \cdots \otimes a_2 \cdot h_n \cdot a_1), \quad a_1, a_2 \in A_r,$$

with respect to the respective (A_r, A_r) -bimodule structure (1.16), as follows from (2.9) and by exploiting the Takeuchi condition of the left coproduct on p. 427. \square

Theorem 2.16. *Let \mathcal{H} be a Hopf algebroid with involutive antipode.*

(i) *When \mathcal{H} is commutative with $A_l = A_r$, $s_l = t_r$, $t_l = s_r$, $\Delta_l = \Delta_r$, $\epsilon_l = \epsilon_r$, and flat as right A_l -module (1.1), one has*

$$\mathrm{HC}^\bullet(\mathcal{H}) \cong \bigoplus_{i \geq 0} \mathrm{HH}^{\bullet-2i}(\mathcal{H}).$$

(ii) *When \mathcal{H} is cocommutative and projective as left A_r -module (1.16), there is a natural isomorphism*

$$\mathrm{HC}_\bullet(\mathcal{H}) \cong \bigoplus_{i \geq 0} \mathrm{HH}_{\bullet-2i}(\mathcal{H}).$$

Proof. (i) Consider (cf. [Ts], [Lo]) Tsygan’s double complex $\mathrm{CC}^\bullet(\mathrm{Cobar}^\bullet(\mathcal{H}))$ of the cocyclic left \mathcal{H}_l -comodule $\mathrm{Cobar}^\bullet(\mathcal{H})$. Since $\mathrm{Cobar}^\bullet(\mathcal{H})$ is a resolution of A_l in the category of left \mathcal{H}_l -comodules, the double complex $\mathrm{CC}^\bullet(\mathrm{Cobar}^\bullet(\mathcal{H}))$ is a resolution (in the sense of hypercohomology, see [We], Sec. 5.7) of the cochain complex

$$A_{l\bullet}: \quad 0 \rightarrow A_l \rightarrow 0 \rightarrow A_l \rightarrow 0 \rightarrow \cdots,$$

with the first 0 in degree zero. From the explicit form of the cyclic operator in Proposition 2.15, one easily observes that the natural isomorphism

$$A_l \square_{\mathcal{H}_l} \mathrm{Cobar}^\bullet(\mathcal{H}) \cong C^\bullet(\mathcal{H})$$

of (1.15) is one of cocyclic k -modules. This identifies cyclic cohomology of \mathcal{H} as the hyper-derived \mathbf{Cotor} , written \mathbf{Cotor} , of A_l with values in the chain complex $A_{l\bullet}$:

$$\mathrm{HC}^\bullet(\mathcal{H}) = \mathbf{Cotor}^\bullet(A_l, A_{l\bullet}).$$

Clearly, any resolution for A_l defines a resolution of the complex $A_{l\bullet}$ by putting 0 in the even degree columns, and therefore $\mathbf{Cotor}^\bullet(A_l, A_{l\bullet}) = \bigoplus_{i \geq 0} \mathrm{HH}^{\bullet-2i}(\mathcal{H})$.

(ii) is proved in very much the same fashion, this time identifying $\mathrm{HC}_\bullet(\mathcal{H}) = \mathbf{Tor}_\bullet(A_r, A_{l\bullet})$, the hyper-derived functors of $A_r \otimes_{\mathcal{H}} - : \mathrm{Mod}(\mathcal{H}) \rightarrow \mathrm{Mod}(k)$. \square

3. Examples

In this section we discuss examples of Hopf algebroids and compute their cyclic homology and cohomology groups.

3.1. The enveloping Hopf algebroid of an algebra. A very simple example of a Hopf algebroid is given by the enveloping algebra $A^e = A \otimes_k A^{\text{op}}$ of an arbitrary (unital) k -algebra A . It is a left bialgebroid over A by means of the structure maps $s_l(a) := a \otimes_k 1, t_l(b) := 1 \otimes_k b, \Delta_l(a \otimes b) := (a \otimes_k 1) \otimes_A (1 \otimes_k b), \epsilon_l(a \otimes_k b) := ab$, and a right bialgebroid over A^{op} by means of $s_r(b) := 1 \otimes_k b, t_r(a) := a \otimes_k 1, \Delta_r(a \otimes b) := (1 \otimes_k a) \otimes_A (b \otimes_k 1), \epsilon_r(a \otimes_k b) := ba$. With the antipode $S(a \otimes_k b) := b \otimes_k a$, these data coalesce to a Hopf algebroid.

Proposition 3.1. *Let A be a k -algebra and A^e its enveloping algebra.*

(i) *The Hopf-cyclic cohomology of A^e is trivial, i.e.,*

$$\text{HC}^\bullet(A^e) = \begin{cases} k & \text{if } \bullet = 0, \\ 0 & \text{else.} \end{cases}$$

(ii) *The dual Hopf-cyclic homology of A^e equals the cyclic homology of the k -algebra A*

$$\text{HC}_\bullet(A^e) = \text{HC}_\bullet^{\text{alg}}(A).$$

Proof. (i) was proved in [CM3]. It actually also follows by cyclic duality from (ii). To prove (ii), one just writes out the cyclic object associated to A^e ; it is exactly equal to the cyclic object $A^{\text{alg}, \natural}$ associated to the algebra A . □

Recall that, when passing to the periodic theory, the right-hand side in (ii) yields the noncommutative generalisation of classical de Rham cohomology, cf. [C2].

3.2. Étale groupoids. *Notation.* Let E and F be vector bundles (or more generally, c -soft sheaves of vector spaces) over two manifolds X and Y , respectively. Suppose that $f: X \rightarrow Y$ is an étale map and $\alpha_f: E \cong f^*F$ an isomorphism of vector bundles over X . Then the push-forward (or fibre sum) of f , denoted by $f_*: \Gamma_c(X, E) \rightarrow \Gamma_c(Y, F)$, is defined by

$$(f_*s)(y) = \sum_{f(x)=y} \alpha_f(s(x)),$$

where $x \in X, y \in Y$ and $s \in \Gamma_c(X, E)$. This construction is functorial in the obvious sense.

Another class of examples of Hopf algebroids comes from étale groupoids, as essentially already noted in [Mrč1], [Mrč2] (a different way to obtain a (topological) Hopf algebroid from an étale groupoid is described in [KamTa]). A groupoid G , to

start with, is a small category in which each arrow is invertible. We denote the space of objects by M and the space of arrows by G . The structure maps can be organised in the following diagram:

$$G_2 \xrightarrow{m} G \xrightarrow{i} G \xrightleftharpoons[t]{s} M \xrightarrow{u} G.$$

Here u is the unit map, s and t are the source and target of arrows in G , i is the inversion and m the multiplication defined on the space of composable arrows:

$$G_2 := G \overset{s}{\times}_M^t G = \{(g_1, g_2) \in G \times G, s(g_1) = t(g_2)\}.$$

A Lie groupoid is a groupoid $G \rightrightarrows M$ for which G and M are smooth manifolds and all structure maps listed above are smooth. In an étale groupoid, these are assumed to be local diffeomorphisms. For simplicity of exposition, we will assume that G is Hausdorff.

Associated to an étale groupoid is its convolution algebra $C_c^\infty(G)$ with product

$$(f_1 * f_2)(g) = \sum_{g_1 g_2 = g} f_1(g_1) f_2(g_2), \tag{3.1}$$

where $f_1, f_2 \in C_c^\infty(G)$ and $g, g_1, g_2 \in G$. We shall equip this noncommutative algebra with the structure of a Hopf algebroid in the following way: the base algebra is given by the commutative algebra $C_c^\infty(M)$ and we put $s_l = t_l = s_r = t_r = u_*$, the push-forward along the inclusion of the units. We are left with two $C_c^\infty(M)$ -actions on $C_c^\infty(G)$ by left and right multiplication with respect to which we define the tensor products \otimes^{ll} , \otimes^{rl} and \otimes^{rr} . The formula

$$\Omega(f_1 \otimes f_2)(g_1, g_2) := f_1(g_1) f_2(g_2),$$

with $f_1, f_2 \in C_c^\infty(G)$ and $g_1, g_2 \in G$, induces isomorphisms

$$\begin{aligned} \Omega_{s,t}: C_c^\infty(G) \otimes^{\text{rl}} C_c^\infty(G) &\xrightarrow{\cong} C_c^\infty(G \overset{s}{\times}_M^t G) = C_c^\infty(G_2), \\ \Omega_{t,t}: C_c^\infty(G) \otimes^{\text{ll}} C_c^\infty(G) &\xrightarrow{\cong} C_c^\infty(G \overset{t}{\times}_M^t G), \\ \Omega_{s,s}: C_c^\infty(G) \otimes^{\text{rr}} C_c^\infty(G) &\xrightarrow{\cong} C_c^\infty(G \overset{s}{\times}_M^s G). \end{aligned} \tag{3.2}$$

That these maps are indeed isomorphisms can be derived from a more general result on sheaves in [Mrč2], p. 271. With this, we define the left coproduct $\Delta_l: C_c^\infty(G) \rightarrow C_c^\infty(G) \otimes^{\text{ll}} C_c^\infty(G) \cong C_c^\infty(G \overset{t}{\times}_M^t G)$ by the formula

$$\Delta_l f(g_1, g_2) := \begin{cases} f(g_1) & \text{if } g_1 = g_2, \\ 0 & \text{else.} \end{cases}$$

Alternatively, this is simply the push-forward along the diagonal inclusion $d^l: G \rightarrow G \overset{t}{\times}_M^t G, g \mapsto (g, g)$. In a similar fashion, the right coproduct is defined as $\Delta_r = d^r_*$,

where $d^r : \mathbb{G} \rightarrow \mathbb{G} \times_M^s \mathbb{G}$ is again the diagonal. Left and right counit are defined as the push-forward along the target resp. source map:

$$(\epsilon_l f)(x) := \sum_{t(g)=x} f(g) \quad \text{and} \quad (\epsilon_r f)(x) := \sum_{s(g)=x} f(g).$$

Finally, the antipode $S : C_c^\infty(\mathbb{G}) \rightarrow C_c^\infty(\mathbb{G})$ is given by the groupoid inversion:

$$(Sf)(g) := f(g^{-1}).$$

Proposition 3.2. *When M is compact, $C_c^\infty(\mathbb{G})$ is a Hopf algebroid over $C^\infty(M)$ by means of the structure maps mentioned above.*

Proof. We remark that compactness of M is needed in order to make both algebras $C_c^\infty(M)$ and $C_c^\infty(\mathbb{G})$ unital. The fact that $(C_c^\infty(\mathbb{G}), C^\infty(M), \Delta_l, \epsilon_l)$ is a left bialgebroid having an antipode S with certain properties was already shown in [Mrč2], Prop. 2.5. The right bialgebroid structure follows at once by replacing \mathbb{G} by its opposite \mathbb{G}^{op} . It remains to verify the Hopf algebroid axioms in which left and right bialgebroid structures are intertwined: for example, twisted coassociativity is obvious. As for the second identity in (1.5), let $f \in C_c^\infty(\mathbb{G})$ and compute

$$\begin{aligned} (f_{(1)} * S(f_{(2)}))(g) &= \sum_{g_1 g_2 = g} f_{(1)}(g_1) f_{(2)}(g_2^{-1}) \\ &= \sum_{g_1 g_1^{-1} = g} f(g_1) \\ &= \begin{cases} \sum_{t(g_1)=x} f(g_1) & \text{if } g = 1_x \text{ for some } x \in M, \\ 0 & \text{else,} \end{cases} \\ &= (s_l \epsilon_l f)(g). \end{aligned}$$

The remaining identities in Definition 1.2 are left to the reader. □

3.2.1. Cyclic cohomology. The Hopf-cyclic cohomology of this example is easily computed:

Proposition 3.3. *The Hopf-Hochschild cohomology of $C_c^\infty(\mathbb{G})$ is trivial except in degree 0, i.e.,*

$$\text{HH}^*(C_c^\infty(\mathbb{G})) \cong \begin{cases} C^\infty(M) & \bullet = 0, \\ 0 & \text{else.} \end{cases}$$

Hence, for the (periodic) Hopf-cyclic cohomology of $C_c^\infty(\mathbb{G})$ one has

$$\text{HP}^0(C_c^\infty(\mathbb{G})) \cong C^\infty(M), \quad \text{HP}^1(C_c^\infty(\mathbb{G})) \cong 0.$$

Proof. Generalising a construction in [Cr3] for group algebras, define the following periodic resolution of $C^\infty(M)$ by cofree (left) $C_c^\infty(\mathbb{G})$ -comodules:

$$I : 0 \rightarrow C^\infty(M) \xrightarrow{u_*} C_c^\infty(\mathbb{G}) \xrightarrow{\alpha} C_c^\infty(\mathbb{G}) \xrightarrow{\beta} C_c^\infty(\mathbb{G}) \xrightarrow{\alpha} \dots,$$

where $\alpha(f) := f - f|_M$ and $\beta(f) := f|_M$. According to Theorem 2.13, the Hochschild cohomology groups are computed by $C^\infty(M) \square_{C_c^\infty(\mathbb{G})} I$, i.e., by means of $0 \rightarrow C^\infty(M) \xrightarrow{0} C^\infty(M) \xrightarrow{\text{id}} C^\infty(M) \xrightarrow{0} C^\infty(M) \xrightarrow{\text{id}} \dots$. Then one has $\text{HH}^n(C_c^\infty(\mathbb{G})) = C^\infty(M)$ for $n = 0$ and zero in all other cases. Applying an *SBI* sequence argument, the second statement follows. \square

3.2.2. Cyclic homology and groupoid homology. For the dual homology theory, consider the nerve $\mathbf{G}_\bullet := \{\mathbf{G}_n\}_{n \geq 0}$ of \mathbb{G} defined as usual

$$\mathbf{G}_0 := M, \quad \mathbf{G}_n := \{(g_1, \dots, g_n) \in \mathbb{G}^{\times n}, s(g_i) = t(g_{i+1}), 1 \leq i \leq n\},$$

equipped with face operators $d_i : \mathbf{G}_n \rightarrow \mathbf{G}_{n-1}$ defined by

$$d_i(g_1, \dots, g_n) = \begin{cases} (g_2, \dots, g_n) & \text{if } i = 0, \\ (g_1, \dots, g_i g_{i+1}, \dots, g_n) & \text{if } 1 \leq i \leq n-1, \\ (g_1, \dots, g_{n-1}) & \text{if } i = n, \end{cases} \quad (3.3a)$$

whereas $d_0, d_1 : \mathbf{G}_1 \rightarrow \mathbf{G}_0$ are given by source and target map, respectively. Equipped with degeneracies $s_i : \mathbf{G}_n \rightarrow \mathbf{G}_{n+1}$ given as

$$s_i(g_1, \dots, g_n) = \begin{cases} (1_{t(g_1)}, g_1, \dots, g_n) & \text{if } i = 0, \\ (g_1, \dots, g_i, 1_{s(g_i)}, g_{i+1}, \dots, g_n) & \text{if } 1 \leq i \leq n, \end{cases} \quad (3.3b)$$

the nerve is a simplicial manifold whose geometric realisation is a model for the classifying space $B\mathbb{G}$. Denote by $\tau_n : \mathbf{G}_n \rightarrow M$ the map $\tau_n(g_1, \dots, g_n) = t(g_1)$. Given a representation E of \mathbb{G} , that is a vector bundle E over M equipped with an action of \mathbb{G} , define

$$C_n^d(\mathbb{G}; E) := \Gamma_c(\mathbf{G}_n, \tau_n^* E).$$

This space of chains carries a differential $\partial : C_n^d(\mathbb{G}; E) \rightarrow C_{n-1}^d(\mathbb{G}; E)$ given by

$$\partial := \sum_{i=0}^n (-1)^i (d_i)_*,$$

where the push-forward is defined with respect to the tautological isomorphisms $\tau_n^* E \cong d_i^* \tau_{n-1}^* E$ for $1 \leq i \leq n$ and $g_1 : E_{s(g_1)} \rightarrow E_{t(g_1)}$ is the isomorphism $\tau_n^* E \cong d_0^* \tau_{n-1}^* E$ at $(g_1, \dots, g_n) \in \mathbf{G}_n$. This defines a differential because of the simplicial identities of the underlying face maps, and its homology is the groupoid homology of E , denoted as $H_\bullet^d(\mathbb{G}, E)$, cf. [CrMoe].

Theorem 3.4. *Let \mathbb{G} be an étale groupoid. There are natural isomorphisms*

$$\begin{aligned} \mathrm{HH}_\bullet(C_c^\infty(\mathbb{G})) &\cong H_\bullet^d(\mathbb{G}, \underline{\mathbb{C}}), \\ \mathrm{HC}_\bullet(C_c^\infty(\mathbb{G})) &\cong \bigoplus_{n \geq 0} H_{\bullet+2n}^d(\mathbb{G}, \underline{\mathbb{C}}). \end{aligned}$$

Proof. The obvious generalisation of the isomorphism (3.2) to higher degrees yields

$$\Omega_{s,t}^n : C_\bullet(C_c^\infty(\mathbb{G})) \xrightarrow{\cong} C_c^\infty(\mathbb{G}_\bullet) = C_\bullet^d(\mathbb{G}, \underline{\mathbb{C}}), \tag{3.4}$$

where $\underline{\mathbb{C}}$ denotes the trivial representation on the line bundle $M \times \mathbb{C}$. To identify the differential, remark that the convolution product (3.1) is simply the push-forward along the multiplication map $m : \mathbb{G}_2 \rightarrow \mathbb{G}$, and right and left counit the push-forwards along source and target maps, i.e., $\epsilon_r = s_*$, $\epsilon_r \circ S^{-1} = \epsilon_l = t_*$. It is then a straightforward check that the isomorphism (3.4) intertwines the simplicial maps (2.8) with the push-forwards along the face operators (3.3a) on \mathbb{G}_\bullet , and this identifies the differential with the groupoid homology differential ∂ . This proves the first assertion. The second follows from Theorem 2.16. \square

Remark 3.5. In particular, the isomorphism (3.4) is an isomorphism of cyclic modules: the operators $\tilde{t}_n : \mathbb{G}_n \rightarrow \mathbb{G}_n$,

$$\tilde{t}_n(g_1, \dots, g_n) := ((g_1 g_2 \dots g_n)^{-1}, g_1, \dots, g_{n-1}) \tag{3.5}$$

for $n \geq 2$, and $t_1(g) := g^{-1}$, $t_0 := \mathrm{id}_{\mathbb{G}_0}$ define a cyclic operation on \mathbb{G}_\bullet such that $C_c^\infty(\mathbb{G})$ together with the push-forwards of (3.3a), (3.3b) and (3.5) becomes a cyclic module. One then has with respect to the dual Hopf-cyclic operator (2.9):

$$\begin{aligned} &(\Omega_{s,t}^n t_n(f^1 \otimes^{\mathrm{rl}} \dots \otimes^{\mathrm{rl}} f^n))(g_1, \dots, g_n) \\ &= \Omega_{s,t}^n(S^{-1}(f_{(2)}^1 * \dots * f_{(2)}^{n-1} * f^n) \otimes^{\mathrm{rl}} f_{(1)}^1 \otimes^{\mathrm{rl}} \dots \otimes^{\mathrm{rl}} f_{(1)}^{n-1})(g_1, \dots, g_n) \\ &= \sum_{g_1^{-1} = g'_1 \dots g'_n} f_{(2)}^1(g'_1) \dots f_{(2)}^{n-1}(g'_{n-1}) f^n(g'_n) f_{(1)}^1(g_2) \dots f_{(1)}^{n-1}(g_n) \\ &= f^1(g_2) \dots f^{n-1}(g_n) f^n((g_1 \dots g_n)^{-1}), \end{aligned}$$

and this is exactly the push-forward of \tilde{t}_n .

Remark 3.6. The first isomorphism of the theorem above readily generalises as follows: let E be a representation of \mathbb{G} . Then $\mathcal{E} := \Gamma_c(M, E)$ is a module over $C_c^\infty(\mathbb{G})$ by the action

$$(f \cdot \varphi)(x) = \sum_{t(g)=x} f(g)\varphi(s(g)),$$

where $f \in C_c^\infty(\mathbb{G})$ and $\varphi \in \Gamma_c(M, E)$. With this module, we have

$$H_\bullet(C_c^\infty(\mathbb{G}); \mathcal{E}) \cong H_\bullet^d(\mathbb{G}, E).$$

Remark 3.7. Analogously as in group theory, a little computation reveals that the Hopf–Galois map from Lemma 2.8 and its inverse are (via the isomorphisms (3.2)) the push-forwards of the following maps on the groupoid level:

$$\tilde{\varphi}_n : \mathbf{G}_n \rightarrow \mathbf{G}^n, \quad (g_1, \dots, g_n) \mapsto (g_1, g_1 g_2, \dots, g_1 g_2 \cdots g_n),$$

where $\mathbf{G}^n := \{(g_1, \dots, g_n) \in \mathbf{G}^{\times n}, t(g_i) = t(g_{i+1}), 1 \leq i \leq n\}$, with inverse

$$\tilde{\psi}_n : \mathbf{G}^n \rightarrow \mathbf{G}_n, \quad (g_1, \dots, g_n) \mapsto (g_1, g_1^{-1} g_2, \dots, g_{n-1}^{-1} g_n).$$

3.2.3. Relation with the computations of Brylinski–Nistor and Crainic. In [BrN], [Cr1] the cyclic homology of $C_c^\infty(\mathbf{G})$ as an algebra, i.e., not as a Hopf algebroid, was computed. Let us show how the present result fits into that computation. A fundamental tool in the papers mentioned above was the “reduction to loops”

$$C_c^\infty(\mathbf{G})_n^{\text{alg}, \natural} \rightarrow \Gamma_c(B_n, \tau_n^{-1} \mathcal{C}_{M^{\times(n+1)}}^\infty), \tag{3.6}$$

where on the left-hand side we have the usual cyclic object associated to an algebra (but using topological tensor products). The space

$$B_n := \{(g_0, \dots, g_n) \in \mathbf{G}^{\times(n+1)} \mid t(g_i) = s(g_{i-1}) \text{ for } 1 \leq i \leq n, t(g_0) = s(g_n)\},$$

above is the so-called higher Burghilea space of closed strings of $n + 1$ composable arrows and $\tau_n : B_n \rightarrow M^{\times(n+1)}$ is here the map $\tau_n(g_0, \dots, g_n) = (t(g_0), \dots, t(g_n))$. This is a simplicial space by defining face operators $d'_i : B_n \rightarrow B_{n-1}$,

$$d'_i(g_0, g_1, \dots, g_n) = \begin{cases} (g_0, \dots, g_i g_{i+1}, \dots, g_n) & \text{if } 0 \leq i \leq n - 1, \\ (g_n g_0, g_1, \dots, g_{n-1}) & \text{if } i = n, \end{cases}$$

and degeneracy operators $s'_i : B_n \rightarrow B_{n+1}$,

$$s'_i(g_0, g_1, \dots, g_n) = \begin{cases} (g_0, \dots, g_i, 1_{t(g_{i+1})}, g_{i+1}, \dots, g_n) & \text{if } 0 \leq i \leq n - 1, \\ (g_0, \dots, g_n, 1_{s(g_n)}) & \text{if } i = n. \end{cases}$$

Furthermore, it has a cyclic operator $t'_n : B_n \rightarrow B_n$ defined by

$$t'_n(g_0, \dots, g_n) = (g_n, g_0, \dots, g_{n-1}),$$

turning B_\bullet into a cyclic object in the category of manifolds. The map (3.6) is a morphism of cyclic objects if we equip the right-hand side with the cyclic structure induced by B_\bullet , together with the (twisted) cyclic structure of the cyclic object $(\mathcal{C}_M^\infty)^\natural$ in the category of sheaves on M . This is the diagonal of a bicyclic complex which is quasi-isomorphic to its total complex. On the level of Hochschild homology, this is the Eilenberg–Zilber theorem (see, for example, [We], Thm. 8.5.1), which – in one

direction – is implemented by the Alexander–Whitney map. Applying the HKR map on the level of sheaves, one eventually finds

$$\mathrm{HH}_\bullet(C_c^\infty(\mathbb{G})) \cong \bigoplus_{p+q=\bullet} H_p(\Lambda(\mathbb{G}), \wedge^q T^*B_0).$$

The groupoid $\Lambda(\mathbb{G}) := B_0 \rtimes \mathbb{G}$ is disconnected in general, which induces a decomposition of the Hochschild and cyclic homology. The component $\mathbb{G} \subseteq \Lambda(\mathbb{G})$ is called the unit component, and for this one finds

$$\mathrm{HH}_\bullet(C_c^\infty(\mathbb{G}))_{[1]} \cong \bigoplus_{p+q=\bullet} H_p(\mathbb{G}, \wedge^q T^*M). \tag{3.7}$$

We compare this with the Hopf-cyclic theory as follows: using the isomorphisms (3.2), one has for the fundamental space from §2.4

$$B_n(C_c^\infty(\mathbb{G})) \cong C_c^\infty(B_n).$$

One easily checks that the induced simplicial and cyclic operators are equal to the push-forwards along the simplicial and cyclic maps on B_\bullet as above. In a similar spirit, the invariant map $C_c^\infty(\mathbb{G}_n) \hookrightarrow C_c^\infty(B_n)$ from (2.11) is induced by the morphism

$$\mathbb{G}_n \rightarrow B_n, \quad (g_1, \dots, g_n) \mapsto ((g_1 \dots g_n)^{-1}, g_1, \dots, g_n).$$

With this, we now see that the map

$$\mathrm{HH}_\bullet^{\mathrm{alg}}(C_c^\infty(\mathbb{G})) \rightarrow \mathrm{HH}_\bullet^{\mathrm{Hopf}}(C_c^\infty(\mathbb{G}))$$

induced by the projection $C_c^\infty(\mathbb{G})^\sharp \rightarrow B_\bullet(C_c^\infty(\mathbb{G}))$ is in turn induced by the projection onto the degree zero component $\bigoplus_{i \geq 0} \wedge^i T^*M \rightarrow \underline{\mathbb{C}}$ of representations of \mathbb{G} .

Remark 3.8. As remarked in [Cr2], the dual cyclic homology of a Hopf algebra captures the full “localisation at units” of the cyclic homology of the underlying algebra. Here we see explicitly that this is not the case for Hopf algebroids: the right-hand side of (3.7) has far more components than those appearing in Theorem 3.4.

3.3. Lie–Rinehart algebras. Important examples of Hopf algebroids also arise from Lie–Rinehart algebras as we shall now explain.

3.3.1. Definitions. Here we briefly recapitulate the basic definitions and properties of Lie–Rinehart algebras, cf. [Ri], [H1]. Let A be a commutative algebra over the ground ring k , containing \mathbb{Q} . A *Lie–Rinehart algebra* over A is a pair (A, L) , where L is a k -Lie algebra equipped with an A -module structure and a morphism of k -Lie algebras $L \rightarrow \mathrm{Der}_k A$, $X \mapsto \{a \mapsto X(a)\}$ such that

$$\begin{aligned} (aX)(b) &= a(X(b)), & X \in L, a, b \in A, \\ [X, aY] &= a[X, Y] + X(a)Y, & X, Y \in L, a \in A. \end{aligned}$$

The morphism $L \rightarrow \text{Der}_k(A)$ is usually referred to as the *anchor* of (A, L) . For convenience we shall also assume that A is unital in what follows.

A Lie–Rinehart algebra is the algebraic analogue of the notion of a Lie algebroid in differential geometry. The algebraic geometric generalisation is given by a sheaf of Lie algebroids, defined over a locally ringed space. In fact, a Lie–Rinehart algebra defines such a sheaf over the affine scheme $\text{Spec}(A)$.

A *left (A, L) -module* over a Lie–Rinehart algebra is a left A -module M which is also a left Lie algebra module over L with action $X \otimes_k m \mapsto X(m)$ satisfying

$$(aX)(m) = a(X(m)), \quad X(am) = X(a)m + aX(m).$$

Alternatively, we can view a left (A, L) -module as an A -module M equipped with a flat left (A, L) -connection: this is a map $\nabla^l : M \rightarrow \text{Hom}_A(L, M)$ satisfying

$$\nabla_X^l(am) = a\nabla_X^l(m) + X(a)m \tag{3.8}$$

for all $a \in A, X \in L$ and $m \in M$. Flatness amounts to the usual condition

$$[\nabla_X^l, \nabla_Y^l] = \nabla_{[X, Y]}^l$$

for all $X, Y \in L$. We write $\text{Mod}(A, L)$ for the category of left (A, L) -modules.

The *universal enveloping algebra* of a Lie–Rinehart algebra (A, L) is constructed as follows [Ri]: the direct A -module sum $A \oplus L$ can be made into a k -Lie algebra by means of the Lie bracket

$$[(a_1, X_1), (a_2, X_2)] := (X_1(a_2) - X_2(a_1), [X_1, X_2]).$$

Let $\mathcal{U}(A \oplus L)$ denote its universal enveloping algebra and $\mathcal{U}^+(A \oplus L)$ the subalgebra generated by the canonical image of $A \oplus L$ in $\mathcal{U}(A \oplus L)$. For $z \in A \oplus L$, denote by z' its canonical image in $\mathcal{U}^+(A \oplus L)$. The quotient $\mathcal{V}L := \mathcal{U}^+(A \oplus L)/I$, where I is the two-sided ideal in $\mathcal{U}^+(A \oplus L)$ generated by the elements $(az)' - a'z', a \in A$, is called the universal enveloping algebra of the Lie–Rinehart algebra (A, L) . It comes equipped with a k -algebra morphism $i_A : A \rightarrow \mathcal{V}L$, as well as a morphism $i_L : L \rightarrow \text{Lie}(\mathcal{V}L)$ of k -Lie algebras, subject to the conditions

$$i_A(a)i_L(X) = i_L(aX), \quad i_L(X)i_A(a) - i_A(a)i_L(X) = i_A(X(a)), \quad a \in A, X \in L.$$

It is universal in the following sense: for any other triple $(\mathcal{W}, \phi_L, \phi_A)$ of a k -algebra \mathcal{W} and two morphisms $\phi_A : A \rightarrow \mathcal{W}, \phi_L : L \rightarrow \text{Lie}(\mathcal{W})$ of k -algebras and k -Lie algebras, respectively, that obey

$$\phi_A(a)\phi_L(X) = \phi_L(aX), \quad \phi_L(X)\phi_A(a) - \phi_A(a)\phi_L(X) = \phi_A(X(a))$$

for all $a \in A, X \in L$, there is a unique morphism $\Phi : \mathcal{V}L \rightarrow \mathcal{W}$ of k -algebras such that $\Phi \circ i_A = \phi_A$ and $\Phi \circ i_L = \phi_L$. This property shows that the natural functor $\text{Mod}(\mathcal{V}L) \rightarrow \text{Mod}(A, L)$ is an equivalence of categories. With this, the Lie–Rinehart cohomology of (A, L) with values in a left (A, L) -module M is defined as

$$H^\bullet(L, M) := \text{Ext}_{\mathcal{V}L}^\bullet(A, M). \tag{3.9}$$

The Poincaré–Birkhoff–Witt theorem. The algebra $\mathcal{V}L$ carries a canonical filtration

$$\mathcal{V}L_{(0)} \subset \mathcal{V}L_{(1)} \subset \mathcal{V}L_{(2)} \subset \dots \tag{3.10}$$

by defining $\mathcal{V}L_{(-1)} := 0$, $\mathcal{V}L_{(0)} := A$ and $\mathcal{V}L_{(p)}$ to be the left A -submodule of $\mathcal{V}L$ generated by $i_L(L)^p$, i.e., products of the image of L in $\mathcal{V}L$ of length at most p . Since $aD - Da \in \mathcal{V}L_{(p-1)}$ for all $a \in A$ and $D \in \mathcal{V}L_{(p)}$, left and right A -module structures coincide on $\mathcal{V}L_{(p)}/\mathcal{V}L_{(p-1)}$. It follows that the associated graded object $\text{gr}(\mathcal{V}L)$ inherits the structure of a graded commutative A -algebra.

Let $S_A^\bullet L$ be the graded symmetric A -algebra of L and $S_A^p L$ its degree p part. When L is projective over A , the Poincaré–Birkhoff–Witt theorem (cf. [Ri], and [NWX] in the context of Lie algebroids) states that the canonical A -linear epimorphism $S_A L \rightarrow \text{gr}(\mathcal{V}L)$ is an isomorphism of A -algebras. While i_A is always injective, in this case even i_L is injective and we may identify elements $a \in A$ and $X \in L$ with their images in $\mathcal{V}L$. Hence, the symmetrisation

$$\pi : S_A^p L \rightarrow \mathcal{V}L_{(p)}, \quad X_1 \otimes \dots \otimes X_p \mapsto \frac{1}{p!} \sum_{\sigma \in S_p} X_{\sigma(1)} \dots X_{\sigma(p)},$$

where $X_i \in L$ or $X_i \in A$ induces an isomorphism of left A -modules $S_A L \rightarrow \mathcal{V}L$.

3.3.2. The associated Hopf algebroid. The fact that Lie–Rinehart algebras give rise to left bialgebroids in the sense of Definition 1.1 by means of their enveloping algebras has been observed before in the literature, cf. [X2], [KR2], [MoeMrč]. In this section we shall determine the extra datum needed to define a Hopf algebroid structure.

In the previous section, we have discussed the category of left $\mathcal{V}L$ -modules, and its interpretation on the level of the Lie–Rinehart algebra as flat connections (3.8). Let us now consider *right* $\mathcal{V}L$ -modules. A *right* (A, L) -connection (cf. [H2]) on an A -module N is a map $\nabla^r : N \rightarrow \text{Hom}_k(L, N)$ which fulfills

$$\nabla_X^r(an) = a\nabla_X^r n - X(a)n, \tag{3.11}$$

$$\nabla_{aX}^r n = a\nabla_X^r n - X(a)n, \quad a \in A, X \in L, n \in N. \tag{3.12}$$

Again, the connection is called *flat* if one has $[\nabla_X^r, \nabla_Y^r] = \nabla_{[Y,X]}^r$ for all $X, Y \in L$, in which case they integrate to a right $\mathcal{V}L$ -module. If L is A -projective of finite constant rank, then by [H2], Thm. 3, flat right (A, L) -connections on A correspond to flat *left* (A, L) -connections on $\bigwedge_A^{\max} L$, the maximal exterior power of L . As such they were introduced in [X1] in the context of Lie algebroids to define Lie algebroid homology. In the more general context of Lie–Rinehart algebras such flat right (A, L) -connections on A need not exist at all, cf. Remark 3.12.

For M a right (A, L) -module – or, equivalently, a right $\mathcal{V}L$ -module – we define Lie–Rinehart homology with coefficients in M by

$$H_\bullet(L, M) := \text{Tor}_\bullet^{\mathcal{V}L}(M, A). \tag{3.13}$$

We will now describe left and right bialgebroid structures on $\mathcal{V}L$: to start with, set

$$s_r \equiv t_r \equiv s_l \equiv t_l \equiv i_A: A \hookrightarrow \mathcal{V}L.$$

With this identification at hand, the various A -module structures on $\mathcal{V}L$ reduce to left and right multiplication in $\mathcal{V}L$. With this, we write \otimes^{\parallel} for the tensor product in $\text{Mod}(\mathcal{V}L)$ and \otimes^{tr} for the one in $\text{Mod}(\mathcal{V}L^{\text{op}})$.

Proposition 3.9. *Flat left and right (A, L) -connections on A correspond to respectively left and right bialgebroid structures on $\mathcal{V}L$ over A .*

Proof. Flat left and right (A, L) -connections ∇^l and ∇^r on A give rise to resp. left and right $\mathcal{V}L$ -actions on A which will be denoted, only in this proof, by $(D, a) \mapsto D \cdot a$ and $(a, D) \mapsto a \cdot D$ for $a \in A$ and $D \in \mathcal{V}L$. Define left and right counit by

$$\epsilon_l(D) := D \cdot 1_A, \quad \epsilon_r(D) := 1_A \cdot D$$

for $D \in \mathcal{V}L$. In particular, we have of course $\epsilon_l(a) = a = \epsilon_r(a)$ for $a \in A$. Seen as maps $\mathcal{V}L \rightarrow A$, one has by the properties of a left connection

$$\epsilon_l(D\epsilon_l(E)) = (D\epsilon_l(E)) \cdot 1_A = D \cdot \epsilon_l(E) = D \cdot (E \cdot 1_A) = (DE) \cdot 1_A = \epsilon_l(DE)$$

with $D, E \in \mathcal{V}L$, and also, by (3.11) and (3.12),

$$\epsilon_r(\epsilon_r(D)E) = 1_A \cdot (\epsilon_r(D)E) = \epsilon_r(D) \cdot E = (1_A \cdot D) \cdot E = 1_A \cdot (DE) = \epsilon_r(DE).$$

Define left and right coproduct by setting on generators $X \in L, a \in A$,

$$\begin{aligned} \Delta_l X &= 1 \otimes^{\parallel} X + X \otimes^{\parallel} 1 - \epsilon_l(X) \otimes^{\parallel} 1, & \Delta_l a &= a \otimes^{\parallel} 1, \\ \Delta_r X &= 1 \otimes^{\text{tr}} X + X \otimes^{\text{tr}} 1 - \epsilon_r(X) \otimes^{\text{tr}} 1, & \Delta_r a &= a \otimes^{\text{tr}} 1. \end{aligned}$$

Extending these maps to the whole of $\mathcal{V}L$ by requiring them to corestrict to k -algebra morphisms $\Delta_l: \mathcal{V}L \rightarrow \mathcal{V}L \times_A \mathcal{V}L$ and $\Delta_r: \mathcal{V}L \rightarrow \mathcal{V}L \times^A \mathcal{V}L$ into the respective Takeuchi products (cf. p. 427) associated to the (A, A) -bimodule structures (1.1) and (1.2), respectively, one easily checks that $(\mathcal{V}L, A, i_A, \Delta_l, \epsilon_l)$ is a left and $(\mathcal{V}L, A, i_A, \Delta_r, \epsilon_r)$ is a right bialgebroid, respectively. \square

Remark 3.10. The anchor of a Lie–Rinehart algebra yields a canonical flat left (A, L) -connection and therefore defines a left bialgebroid structure. The associated left counit ϵ_l is simply the projection $\mathcal{V}L \rightarrow A$, and one has $\epsilon_l(X) = 0$ and $\Delta_l X = X \otimes^{\parallel} 1 + 1 \otimes^{\parallel} X$ for $X \in L$. This is the left bialgebroid structure on $\mathcal{V}L$ of [X2], [KR2], [MoeMrč], which we from now on will fix as *the* (canonical) left bialgebroid structure on $\mathcal{V}L$. Observe however that for the *right* bialgebroid structure there is *no* canonical choice, and in general $\epsilon_r(X) \neq 0$ for $X \in L$.

Next we will define an antipode: let (A, L) be a Lie–Rinehart algebra and ∇^r a right (A, L) -connection on A , and define the operator $\epsilon_r^L = \epsilon_r : L \rightarrow A, X \mapsto \nabla_X^r 1_A$. Define a pair of maps $S_L : L \rightarrow \mathcal{V}L$ and $S_A : A \rightarrow \mathcal{V}L$ by

$$S_L(X) = -X + \epsilon_r(X), \quad S_A(a) = a, \quad a \in A, X \in L.$$

Combining (3.11) with (3.12), this implies that $S_L(aX) = -aX + \nabla_X^r a$.

Proposition 3.11 (Antipodes for Lie–Rinehart algebras). *The pair (S_A, S_L) extends to a k -algebra antihomomorphism $S : \mathcal{V}L \rightarrow \mathcal{V}L$ if and only if the underlying right (A, L) -connection on A is flat. In such a case S is an involutive antipode with respect to the canonical left bialgebroid structure and the right bialgebroid structure from Proposition 3.9.*

Conversely, given a k -module isomorphism $S : \mathcal{V}L \rightarrow \mathcal{V}L$ satisfying $S(a_1 D a_2) = a_2 S(D) a_1$ for all $D \in \mathcal{V}L, a_1, a_2 \in A$, and $S(1) = 1$, the assignment

$$\nabla^r : A \rightarrow \text{Hom}_k(L, A), \quad a \mapsto \{X \mapsto \epsilon_L(S(X)a)\},$$

defines a right (A, L) -connection on A which is flat if and only if S is a k -algebra antihomomorphism.

Proof. We use the universal property of $\mathcal{V}L$: clearly $S_A : A \rightarrow \mathcal{V}L$ is a morphism of k -algebras. Next, compute

$$\begin{aligned} [S_L X, S_L Y] &= [X, Y] + [Y, \epsilon_r(X)] - [X, \epsilon_r(Y)] + \epsilon_r(X)\epsilon_r(Y) - \epsilon_r(Y)\epsilon_r(X) \\ &= [X, Y] + Y\epsilon_r(X) - X\epsilon_r(Y) \\ &= S_L([Y, X]) - \epsilon_r([Y, X]) + Y\epsilon_r(X) - X\epsilon_r(Y) \\ &\quad + \epsilon_r(X)\epsilon_r(Y) - \epsilon_r(Y)\epsilon_r(X) \\ &= S_L([Y, X]) - \nabla_{[Y, X]}^r 1_A + \nabla_X^r \epsilon_r(Y) - \nabla_Y^r \epsilon_r(X) \\ &= S_L([Y, X]) - \nabla_{[Y, X]}^r 1_A + \nabla_X^r \nabla_Y^r 1_A - \nabla_Y^r \nabla_X^r 1_A \\ &= S_L([Y, X]) + ([\nabla_X^r, \nabla_Y^r] - \nabla_{[Y, X]}^r)(1_A). \end{aligned}$$

The term between brackets is the curvature of ∇^r , so $S_L : L \rightarrow \text{Lie}(\mathcal{V}L^{\text{op}})$ is a morphism of k -Lie algebras if and only if ∇^r is flat. We now check

$$S_A(a)S_L(X) = -Xa + a\epsilon_r(X) = -aX + \nabla_X^r a = S_L(aX)$$

and also

$$S_L(X)S_A(a) - S_A(a)S_L(X) = -aX + a\epsilon_r(X) + Xa - a\epsilon_r(X) = S_A(X(a)).$$

Thus by the universal property of $\mathcal{V}L$, there exists a unique homomorphism $S : \mathcal{V}L \rightarrow \mathcal{V}L^{\text{op}}$ which fulfills $S \circ i_A = S_A$ and $S \circ i_L = S_L$. If the connection is flat, the antipode axioms including $S^2 = \text{id}$ are straightforward to check by considering a PBW basis of $\mathcal{V}L$ and making use of the antihomomorphism property.

For the converse statement, we need to check the properties (3.11) and (3.12) in order to be a right connection. As for (3.11), we compute

$$\begin{aligned} \nabla_X^r(ab) &= \epsilon_l(S(X)ab) \\ &= \epsilon_l((-Xa + a\epsilon_r(X))b) \\ &= \epsilon_l((-aX - X(a) + a\epsilon_r(X))b) \\ &= a\epsilon_l(S(X)b) - X(a)b \\ &= a\nabla_X^r b - X(a)b, \end{aligned}$$

and (3.12) is left to the reader. To show flatness if and only if S is a k -algebra antihomomorphism, use again the universal property of $\mathcal{V}L$ to compare

$$\begin{aligned} [\nabla_Y, \nabla_X](a) &= \epsilon_l(S(Y)\epsilon_l(S(X)a)) - \epsilon_l(S(X)\epsilon_l(S(Y)a)) \\ &= \epsilon_l(S(Y)S(X)a) - \epsilon_l(S(X)S(Y)a). \end{aligned}$$

with $\nabla_{[X,Y]}a = \epsilon_l(S([X, Y])a)$. This completes the proof. □

Remark 3.12 (The left Hopf algebroid structure). Let $\mathcal{V}L \otimes^{\text{rl}} \mathcal{V}L$ denote the tensor product defined with respect to the ideal generated by $\{Da \otimes E - D \otimes aE, D, E \in \mathcal{V}L, a \in A\}$. Although the antipode S depends on the right connection ∇^r , the *translation map* $\mathcal{V}L \rightarrow \mathcal{V}L \otimes^{\text{rl}} \mathcal{V}L, D \mapsto D_+ \otimes D_- := D^{(1)} \otimes S(D^{(2)})$ is independent of ∇^r . Indeed, evaluated on a PBW basis $D = aX_{i_1} \dots X_{i_n}$, one finds

$$D_+ \otimes D_- = \sum_{j=0}^n \sum_{\substack{i_1 < \dots < i_j \\ i_{j+1} < \dots < i_n}} (-1)^{n-j} aX_{i_1} \dots X_{i_j} \otimes X_{i_{j+1}} \dots X_{i_n}.$$

In [KoKr] it was proved that this defines a *left Hopf algebroid* (\times_A -Hopf algebra [Sch2]) structure on $\mathcal{V}L$, i.e., the Hopf–Galois map $\beta := \varphi_2: \mathcal{V}L \otimes^{\text{rl}} \mathcal{V}L \rightarrow \mathcal{V}L \otimes^{\text{ll}} \mathcal{V}L, D \otimes E \mapsto D_{(1)} \otimes D_{(2)}E$ (see §2.3.3) is bijective with inverse $D \otimes E \mapsto D_+ \otimes D_-E$. Among others, this implies that the map above satisfies several identities of which we will only list the three needed later on when dealing with jet spaces:

$$D_{+(1)} \otimes D_{+(2)}D_- = D \otimes 1 \in \mathcal{V}L \otimes^{\text{ll}} \mathcal{V}L, \tag{3.14}$$

$$D_{+(1)} \otimes D_{+(2)} \otimes D_- = D_{(1)} \otimes D_{(2)+} \otimes D_{(2)-} \in \mathcal{V}L \otimes^{\text{ll}} \mathcal{V}L \otimes^{\text{rl}} \mathcal{V}L, \tag{3.15}$$

$$D_+D_- = \epsilon_l(D). \tag{3.16}$$

Hence, if A does not admit a flat right (A, L) -connection (see [KoKr] for a counterexample), $\mathcal{V}L$ is merely a left Hopf algebroid, but not a Hopf algebroid. Since every Hopf algebroid (with bijective antipode) can be described by *two* different kinds of bijective Hopf–Galois maps (see Prop. 4.2 in [BSz] for details), we thus propose the name *left Hopf algebroid* rather than \times_A -Hopf algebra (see also [Ko], §2.6.14, why this is a reasonable terminology, apart from solving a pronunciation problem).

3.3.3. The cyclic theory of $\mathcal{V}L$. In this section we present the computations of the Hopf-cyclic cohomology and dual Hopf-cyclic homology of the universal enveloping algebra $\mathcal{V}L$ of a Lie–Rinehart algebra (A, L) . Let for the rest of this section L be projective as a left A -module. Furthermore, let $\nabla = \nabla^r$ be a flat right (A, L) -connection on A with associated right counit ϵ_r , and denote A_∇ for A equipped with this right (A, L) -module structure. By Proposition 3.11, the connection ∇ determines an antipode, and therefore the cyclic cohomology and homology are defined. In the following we shall write B_∇ for the corresponding cyclic cohomology operator to stress this dependence; remark that the Hochschild operator b is independent of ∇ . Consider the exterior algebra $\bigwedge_A^\bullet L$ over A equipped with the differential $\partial: \bigwedge_A^n L \rightarrow \bigwedge_A^{n-1} L$ defined by

$$\begin{aligned} \partial(aX_1 \wedge \cdots \wedge X_n) &:= \sum_{i=1}^n (-1)^{i+1} \epsilon_r(aX_i) X_1 \wedge \cdots \wedge \widehat{X}_i \wedge \cdots \wedge X_n \\ &\quad + \sum_{i < j} (-1)^{i+j} a[X_i, X_j] \wedge X_1 \wedge \cdots \wedge \widehat{X}_i \wedge \cdots \wedge \widehat{X}_j \wedge \cdots \wedge X_n. \end{aligned}$$

Theorem 3.13. *Let (A, L) be a Lie–Rinehart algebra with L projective over A and equipped with a flat right (A, L) -connection ∇ on A . The antisymmetrisation map*

$$X_1 \wedge \cdots \wedge X_n \mapsto \frac{1}{n!} \sum_{\sigma \in S_n} (-1)^\sigma X_{\sigma(1)} \otimes^{\text{ll}} \cdots \otimes^{\text{ll}} X_{\sigma(n)}$$

defines a quasi-isomorphism of mixed complexes

$$\text{Alt}: (\bigwedge_A^\bullet L, 0, \partial) \rightarrow (C^\bullet(\mathcal{V}L), b, B_\nabla),$$

which induces natural isomorphisms

$$\text{HH}^\bullet(\mathcal{V}L) \cong \bigwedge_A^\bullet L, \quad \text{HP}^\bullet(\mathcal{V}L) \cong \bigoplus_{n \equiv \bullet \pmod 2} H_n(L, A_\nabla).$$

Proof. The isomorphism for the Hochschild groups relies on a similar consideration for k -modules [Car], [Kas2] and is also known in the Lie algebroid case [Cal], Thm. 1.2. The proof of the algebraic case proceeds analogously: first one checks that the morphism $\text{Alt}: \bigwedge_A^\bullet L \rightarrow C^\bullet(\mathcal{V}L)$ indeed commutes with the differentials, $b \circ \text{Alt} = 0$. Since the Hochschild cohomology only depends on the A -coalgebra structure, it suffices to prove that the morphism $\text{gr}(\text{Alt}): \bigwedge_A^\bullet L \rightarrow C^\bullet(\text{gr}(\mathcal{V}L)) \cong C^\bullet(S_A L)$ is a quasi-isomorphism: observe that $S_A L$ can be seen as the universal enveloping algebra of the Lie–Rinehart algebra defined by the A -module L equipped with zero bracket and zero anchor. With this, the PBW map $S_A L \rightarrow \mathcal{V}L$ is an isomorphism of A -coalgebras.

Assume first L to be finitely generated projective over A . Localising with respect to a maximal ideal $\mathfrak{m} \subset A$, the module $L_\mathfrak{m}$ is free over $A_\mathfrak{m}$ of rank (say) r , and the

morphism descends to a cochain morphism

$$\mathrm{gr}(\mathrm{Alt})_{\mathfrak{m}}: \bigwedge_{A_{\mathfrak{m}}}^{\bullet} L_{\mathfrak{m}} \rightarrow C^{\bullet}(S_{A_{\mathfrak{m}}} L_{\mathfrak{m}}).$$

We shall prove that this map is a quasi-isomorphism for all \mathfrak{m} . Fix a basis $e_i \in L_{\mathfrak{m}}$, $i = 1, \dots, r$, over $A_{\mathfrak{m}}$ as well as a dual basis $e^i \in L_{\mathfrak{m}}^*$. We then have $S_{A_{\mathfrak{m}}} L_{\mathfrak{m}} \cong A_{\mathfrak{m}}[e_1, \dots, e_r]$. The dual Koszul resolution of $A_{\mathfrak{m}}$ by left $S_{A_{\mathfrak{m}}} L_{\mathfrak{m}}$ -comodules has the form

$$K': A_{\mathfrak{m}} \xrightarrow{S} S_{A_{\mathfrak{m}}} L_{\mathfrak{m}} \xrightarrow{d} S_{A_{\mathfrak{m}}} L_{\mathfrak{m}} \otimes_{A_{\mathfrak{m}}} L_{\mathfrak{m}} \xrightarrow{d} S_{A_{\mathfrak{m}}} L_{\mathfrak{m}} \otimes_{A_{\mathfrak{m}}} \bigwedge_{A_{\mathfrak{m}}}^2 L_{\mathfrak{m}} \xrightarrow{d} \dots$$

with $d = \sum_{i=1}^r \iota_{e^i} \otimes e_i$, that is,

$$d(D \otimes X_1 \wedge \dots \wedge X_n) := \sum_{i=1}^r \iota_{e^i} D \otimes e_i \wedge X_1 \wedge \dots \wedge X_n.$$

Here ι_{α} denotes the action of $\alpha \in L^* := \mathrm{Hom}_A(L, A)$ by derivations: $\iota_{\alpha} D := \alpha(D_{(1)})D_{(2)}$, $D \in S_A L$, with respect to the coproduct Δ_{S_L} on $S_A L$. Defining a contracting homotopy by

$$s(D \otimes X_1 \wedge \dots \wedge X_n) := \sum_{i=1}^r (-1)^{i+1} D X_i \otimes X_1 \wedge \dots \wedge \widehat{X}_i \wedge \dots \wedge X_n,$$

it can be checked that K' yields a cofree resolution in the category $\mathrm{Comod}_L(S_{A_{\mathfrak{m}}} L_{\mathfrak{m}})$, hence the resolution is also relative injective. To compare this with the cobar resolution, one shows that the natural map

$$D_0 \otimes D_1 \otimes \dots \otimes D_n \mapsto D_0 \otimes \mathrm{pr}(D_1) \wedge \dots \wedge \mathrm{pr}(D_n),$$

where $\mathrm{pr}: S_{A_{\mathfrak{m}}} L_{\mathfrak{m}} \rightarrow L_{\mathfrak{m}}$ denotes the canonical projection, defines a cochain equivalence. Indeed this amounts to the identity

$$(\mathrm{id} \otimes \mathrm{pr})\Delta_{S_L} D = \sum_{i=1}^r \iota_{e^i} D \otimes e_i$$

for all $D \in S_{A_{\mathfrak{m}}} L_{\mathfrak{m}}$, an identity that is easily checked on generators. To compute the Cotor groups, we use the natural isomorphism

$$A_{\mathfrak{m}} \square_{S_{A_{\mathfrak{m}}} L_{\mathfrak{m}}} (S_{A_{\mathfrak{m}}} L_{\mathfrak{m}} \otimes_{A_{\mathfrak{m}}} \bigwedge_{A_{\mathfrak{m}}}^{\bullet} L_{\mathfrak{m}}) \cong \bigwedge_{A_{\mathfrak{m}}}^{\bullet} L_{\mathfrak{m}},$$

which induces the zero differential on the right-hand side. By the fact that the projection $(S_{A_{\mathfrak{m}}} L_{\mathfrak{m}})^{\otimes n} \rightarrow \bigwedge_{A_{\mathfrak{m}}}^n L_{\mathfrak{m}}$ is a left inverse to Alt , the claim now follows.

In the general case where L is projective over A , but not finitely generated, there exists as in [Lo], Thm. 3.2.2, a filtered ordered set J as well as an inductive system of finitely generated projective (or even free) A -modules L_j such that

$$L \simeq \lim_{\longrightarrow_{j \in J}} L_j.$$

Since both HH (which is the derived functor Cotor here) as well as S commute with inductive limits over a filtered ordered set, the projective case follows from the finitely generated projective case.

To prove the second isomorphism, we need to show that Alt intertwines the cyclic cohomology differential with ∂ . The best way to do this is to use localisation onto coinvariants. Let $B_{\nabla} : C^{\bullet}(\mathcal{V}L) \rightarrow C^{\bullet-1}(\mathcal{V}L)$ and $B : B^{\bullet}(\mathcal{V}L) \rightarrow B^{\bullet-1}(\mathcal{V}L)$ denote the cyclic cohomology differentials of the mixed complexes associated to the Hopf-cocyclic module $\mathcal{V}L_{\natural}$ and the fundamental A -coalgebra cocyclic module $\mathcal{V}L_{\text{coalg}, \natural}$, respectively. As usual, $B = N\sigma_{-1}(1 - \lambda)$, where $\lambda := (-1)^n \tau_n$, $N := \sum_{i=0}^n \lambda^i$, and $\sigma_{-1} := \sigma_{n-1} \tau_n$. Hence, $B : B^n(\mathcal{V}L) \rightarrow B^{n-1}(\mathcal{V}L)$ is given explicitly by

$$\begin{aligned} & B(D_0 \otimes \cdots \otimes D_n) \\ &= \sum_{i=0}^n ((-1)^{ni} \epsilon_l(D_0) D_{i+1} \otimes \cdots \otimes D_n \otimes D_1 \otimes \cdots \otimes D_{i-1}) \\ &\quad - (-1)^{n(i-1)} \epsilon_l(D_n) D_{i+1} \otimes \cdots \otimes D_{n-1} \otimes D_0 \otimes \cdots \otimes D_{i-1}. \end{aligned}$$

Note that $B^n(\mathcal{V}L) \cong C^{n+1}(\mathcal{V}L)$ as (A, A) -bimodules in this example. From our general considerations in §2.2, we have $B_{\nabla} \circ \bar{\Psi}_{\text{coinv}} = \bar{\Psi}_{\text{coinv}} \circ B$ for the morphism $\bar{\Psi}_{\text{coinv}} : B^n(\mathcal{V}L) \rightarrow C^n(\mathcal{V}L)$. Using its right inverse (2.6), it is seen that

$$\text{Alt}(aX_1 \wedge \cdots \wedge X_n) = \bar{\Psi}_{\text{coinv}} \left(\frac{1}{n!} \sum_{\sigma \in S_n} (-1)^{\sigma} a \otimes X_{\sigma(1)} \otimes \cdots \otimes X_{\sigma(n)} \right).$$

Since $L \subset \ker \epsilon_l$ and because ϵ_l is a left A -module map, we can compute

$$\begin{aligned} B_{\nabla}(\text{Alt}(aX_1 \wedge \cdots \wedge X_n)) &= B_{\nabla} \bar{\Psi}_{\text{coinv}} \left(\frac{1}{n!} \sum_{\sigma \in S_n} (-1)^{\sigma} a \otimes X_{\sigma(1)} \otimes \cdots \otimes X_{\sigma(n)} \right) \\ &= \bar{\Psi}_{\text{coinv}} B \left(\frac{1}{n!} \sum_{\sigma \in S_n} (-1)^{\sigma} a \otimes X_{\sigma(1)} \otimes \cdots \otimes X_{\sigma(n)} \right) \\ &= \bar{\Psi}_{\text{coinv}} \left(\frac{1}{(n-1)!} \sum_{\sigma \in S_n} (-1)^{\sigma} a X_{\sigma(1)} \otimes \cdots \otimes X_{\sigma(n)} \right) \\ &= \frac{1}{(n-1)!} \sum_{\sigma \in S_n} (-1)^{\sigma} S(aX_{\sigma(1)}) \cdot (X_{\sigma(2)} \otimes \cdots \otimes X_{\sigma(n)}). \end{aligned}$$

Now as an element in $\mathcal{V}L^{\otimes n}$ it is easy to see that

$$\Delta_l^{n-1} S(aX) = - \sum_{i=1}^n \underbrace{1 \otimes \cdots \otimes 1}_{i-1 \text{ times}} \otimes aX \otimes \underbrace{1 \otimes \cdots \otimes 1}_{n-i \text{ times}} + \epsilon_r(aX) \otimes \underbrace{1 \otimes \cdots \otimes 1}_{n-1 \text{ times}}$$

for $a \in A, X \in L$. With this one then obtains

$$\begin{aligned}
 & \frac{1}{(n-1)!} \sum_{\sigma \in S_n} (-1)^\sigma S(aX_{\sigma(1)}) \cdot (X_{\sigma(2)} \otimes \cdots \otimes X_{\sigma(n)}) \\
 &= \frac{1}{(n-1)!} \sum_{\sigma \in S_n} (-1)^\sigma \epsilon_r(aX_{\sigma(1)}) X_{\sigma(2)} \otimes \cdots \otimes X_{\sigma(n)} \\
 & \quad - \frac{1}{(n-1)!} \sum_{i=1}^n \sum_{\sigma \in S_n} (-1)^\sigma a X_{\sigma(2)} \otimes \cdots \otimes X_{\sigma(1)} X_{\sigma(i)} \otimes \cdots \otimes X_{\sigma(n)} \quad (3.17) \\
 &= \text{Alt}(\sum_{i=1}^n (-1)^{i+1} \epsilon_r(aX_i) X_1 \wedge \cdots \wedge \widehat{X}_i \wedge \cdots \wedge X_n \\
 & \quad + \sum_{i < j} (-1)^{i+j} a[X_i, X_j] \wedge X_1 \wedge \cdots \wedge \widehat{X}_i \wedge \cdots \wedge \widehat{X}_j \wedge \cdots \wedge X_n) \\
 &= \text{Alt}(\partial(a \otimes X_1 \wedge \cdots \wedge X_n)).
 \end{aligned}$$

This completes the proof. □

Theorem 3.14. *Let (A, L) be a Lie–Rinehart algebra. Under the same assumptions as in Theorem 3.13, there are natural isomorphisms*

$$\text{HH}_*(\mathcal{V}L) \cong H_*(L, A_\nabla), \quad \text{HC}_*(\mathcal{V}L) \cong \bigoplus_{i \geq 0} H_{*-2i}(L, A_\nabla).$$

Proof. The first isomorphism follows from Theorem 2.13, together with the definition (3.13) of Lie–Rinehart homology as a Tor functor. The second isomorphism follows from Theorem 2.16 (ii). □

Proposition 3.15. *The isomorphism of Hochschild homology above is induced by the chain morphism*

$$\varphi_n^{-1} \circ n! \text{Alt}: (\wedge_A^n L, \partial) \rightarrow (C_n(\mathcal{V}L), b),$$

where φ is the Hopf–Galois map of Lemma 2.8.

Proof. In view of Theorem 2.10 it is equivalent to prove that the map $\text{Alt}: \wedge_A^\bullet L \rightarrow C^\bullet(\mathcal{V}L)$ maps the differential $\partial: \wedge_A^n L \rightarrow \wedge_A^{n-1} L$ to

$$\check{b} := \sigma_{n-1} \circ \tau_n + \sum_{i=0}^{n-1} (-1)^{i+1} \sigma_i,$$

i.e., $\text{Alt} \circ \partial = \check{b} \circ \text{Alt}$ on $C^\bullet(\mathcal{V}L)$. Since the maps σ_i are just given by the left counit acting on the i^{th} slot of the tensor product, the second sum is zero when evaluated on the image of Alt , and we are left with the term $\sigma_{n-1} \circ \tau_n$, which gives

$$\sigma_{n-1} \tau_n \text{Alt}(aX_1 \wedge \cdots \wedge X_n) = \frac{1}{n!} \sum_{\sigma \in S_n} (-1)^\sigma S(aX_{\sigma(1)}) \cdot (X_{\sigma(2)} \otimes^{\parallel} \cdots \otimes^{\parallel} X_{\sigma(n)}).$$

Inspection of the calculation (3.17) shows that this is exactly $\frac{1}{n} \text{Alt}(\partial(aX_1 \wedge \cdots \wedge X_n))$. Hence $\varphi_n^{-1} \circ n! \text{Alt}$ is a morphism of complexes. To prove that it is a quasi-isomorphism, consider the so-called Koszul–Rinehart resolution

$$A \xrightarrow{i_A} \mathcal{V}L \xrightarrow{b'} \mathcal{V}L \otimes_A L \xrightarrow{b'} \mathcal{V}L \otimes_A \wedge_A^2 L \xrightarrow{b'} \cdots,$$

with $b': \mathcal{V}L \otimes_A \wedge^n L \rightarrow \mathcal{V}L \otimes_A \wedge^{n-1} L$ given by

$$\begin{aligned} & b'(D \otimes X_1 \wedge \cdots \wedge X_n) \\ &= \sum_{i=1}^n (-1)^{i-1} DX_i \otimes X_1 \wedge \cdots \widehat{X}_i \cdots \wedge X_n \\ & \quad + \sum_{1 \leq i < j \leq n} (-1)^{i+j} D \otimes [X_i, X_j] \wedge X_1 \wedge \cdots \widehat{X}_i \cdots \widehat{X}_j \cdots \wedge X_n, \end{aligned}$$

where $D \in \mathcal{V}L$ and $X_1, \dots, X_n \in L$. This is a projective resolution of A in the category $\text{Mod}(\mathcal{V}L)$. By the same computation as above, one shows that the map

$$n!(\text{id} \otimes \varphi_n^{-1} \circ \text{Alt}): \mathcal{V}L \otimes_A \wedge_A^n L \rightarrow \text{Bar}_n(\mathcal{V}L)$$

is a homotopy equivalence. Taking $A \otimes_{\mathcal{V}L} -$ on both sides, one finds the map of the proposition. This proves that it is a quasi-isomorphism. \square

3.4. Jet spaces

3.4.1. The dual jet space of a Lie–Rinehart algebra. In this section we describe another Hopf algebroid associated to a Lie–Rinehart algebra (A, L) , the Hopf algebroid of L -jets. Some of its structure maps have been used before in the literature, cf. [NeTs], [CalVdB], here we give a complete description: it is in a certain sense the dual of $\mathcal{V}L$. (*Note added in proof:* this Hopf algebroid was later independently reobtained in [CalRoVdB], App. A.) In general, duality in the category of bialgebroids has been described in [KadSz] (see [BSz] for an extension to Hopf algebroids) assuming that the bialgebroid is finitely generated projective over the base algebra. This is clearly not the case for $\mathcal{V}L$, but each successive quotient $\mathcal{V}L_{(p)}/\mathcal{V}L_{(p-1)}$ in the Poincaré–Birkhoff–Witt filtration (3.10) is projective, provided L is projective over A .

For the remainder of this section, let L be finitely generated projective of constant rank as an A -module. The space of p -jets of (A, L) is then defined as

$$\mathcal{J}^p L := \text{Hom}_A(\mathcal{V}L_{\leq p}, A),$$

where $\mathcal{V}L_{\leq p}$ denotes the elements in $\mathcal{V}L$ of degree p or less. The infinite jet space is defined as the projective limit

$$\mathcal{J}^\infty L := \varprojlim \mathcal{J}^p L.$$

By definition, $\mathcal{J}^\infty L$ is complete with respect to the canonical PBW filtration (3.10). In this section we will therefore always complete tensor products using this filtration (cf. [Q1]).

We are now going to show that this space carries the structure of a Hopf algebroid over A : first of all, there is a commutative algebra structure that can be described using the (left) comultiplication on $\mathcal{V}L$:

$$\phi_1 \phi_2(D) = \phi_1(D_{(1)}) \phi_2(D_{(2)}), \quad \phi_1, \phi_2 \in \mathcal{J}^\infty L, \quad D \in \mathcal{V}L.$$

The unit for this multiplication is given by the left counit $\epsilon_l: \mathcal{V}L \rightarrow A$, since

$$\epsilon_l \phi(D) = \epsilon_l(D_{(1)}) \phi(D_{(2)}) = \phi(\epsilon_l(D_{(1)}) D_{(2)}) = \phi(D).$$

There are two homomorphisms $s, t: A \rightarrow \mathcal{J}^\infty L$ given by

$$s(a)(D) := \epsilon_l(aD) = a \epsilon_l(D), \quad t(a)(D) := \epsilon_l(Da) = D(a), \quad a \in A, \quad D \in \mathcal{V}L,$$

where we recall that here and in the rest of this section $D(a) := \epsilon_l(Da)$, $D \in \mathcal{V}L$, is the canonical $\mathcal{V}L$ -action on A given by extension of the anchor of (A, L) . A small computation shows that the images commute, and therefore $(\mathcal{J}^\infty L, A, s, t)$ is an (s, t) -ring. Next we consider the coproduct. For this we need the following:

Lemma 3.16. *There is a canonical isomorphism*

$$\mathcal{J}^\infty L \otimes_A \mathcal{J}^\infty L \cong \varprojlim_p \text{Hom}_A((\mathcal{V}L \otimes_A^{\text{rl}} \mathcal{V}L)_{\leq p}, A).$$

Proof. By definition, $\mathcal{J}^\infty L \otimes_A \mathcal{J}^\infty L$ is the quotient of $\mathcal{J}^\infty L \otimes_k \mathcal{J}^\infty L$ by the ideal generated by $\{t(a)\phi_1 \otimes \phi_2 - \phi_1 \otimes s(a)\phi_2, \phi_1, \phi_2 \in \mathcal{J}^\infty L, a \in A\}$. The first term in this ideal, evaluated on $D \otimes E \in \mathcal{V}L \otimes_k \mathcal{V}L$, we write as

$$\begin{aligned} (t(a)\phi_1 \otimes \phi_2)(D \otimes E) &= t(a)\phi_1(D) \otimes \phi_2(E) \\ &= D_{(1)}(a)\phi_1(D_{(2)}) \otimes \phi_2(E) \\ &= \phi_1(\epsilon_l(D_{(1)}a)D_{(2)}) \otimes \phi_2(E) \\ &= \phi_1(Da) \otimes \phi_2(E). \end{aligned}$$

The second term gives

$$(\phi_1 \otimes s(a)\phi_2)(D \otimes E) = \phi_1(D) \otimes a\epsilon_l(E_{(1)})\phi_2(E_{(2)}) = \phi_1(D) \otimes \phi_2(aE).$$

Note that these two expressions use exactly the (A, A) -bimodule structure on $\mathcal{V}L$ used in the \otimes^{rl} -tensor product. It therefore follows that the map

$$\phi_1 \otimes \phi_2 \mapsto \{D \otimes E \mapsto \phi_1(D\phi_2(E))\}$$

induces the desired isomorphism. \square

Observe now that the product on $\mathcal{V}L$ descends to a map $m: \mathcal{V}L \otimes^{\text{rl}} \mathcal{V}L \rightarrow \mathcal{V}L$. We can therefore dualise the product to obtain a coproduct $\Delta: \mathcal{J}^\infty L \rightarrow \mathcal{J}^\infty L \otimes_A \mathcal{J}^\infty L$,

$$\phi(DE) =: \Delta(\phi)(D \otimes^{\text{rl}} E) = \phi_{(1)}(D\phi_{(2)}(E)).$$

Associativity of the multiplication implies that Δ is coassociative. The counit for this coproduct is given by $\epsilon: \phi \mapsto \phi(1_{\mathcal{V}L})$. It is now easy to verify that $(\mathcal{J}^\infty L, A, s, t, \epsilon, \Delta)$ is a left bialgebroid, and since $\mathcal{J}^\infty L$ is commutative, it is also a right bialgebroid. Hence, to obtain a Hopf algebroid all we need is an antipode.

As observed in [NeTs], there are two left $\mathcal{V}L$ -module structures on $\mathcal{J}^\infty L$. First there is the “obvious” module structure given by

$$(D \cdot_1 \phi)(E) := \phi(ED), \quad \phi \in \mathcal{J}^\infty L, \quad D, E \in \mathcal{V}L.$$

Second, there is another left $\mathcal{V}L$ -module structure constructed as follows: consider the A -module structure defined by left multiplication by the source map, i.e., $(a \cdot \phi)(D) := (s(a)\phi)(D) = \phi(aD)$. On this A -module, there is a canonical left connection, also called the *Grothendieck connection*, given by

$$\nabla_X^l(\phi)(D) := X(\phi(D)) - \phi(XD), \quad \phi \in \mathcal{J}^\infty L, \quad D \in \mathcal{V}L, \quad X \in L. \quad (3.18)$$

One easily checks that this connection is flat, and we can write the induced $\mathcal{V}L$ -module structure as

$$(D \cdot_2 \phi)(E) = D_+(\phi(D_-E)), \quad D, E \in \mathcal{V}L,$$

where we used the canonical left Hopf algebroid structure on $\mathcal{V}L$, cf. Remark 3.12. With respect to the coproduct, these two module structures satisfy

$$\Delta(D \cdot_1 \phi) = D \cdot_1 \phi_{(1)} \otimes \phi_{(2)}, \quad \Delta(D \cdot_2 \phi) = \phi_{(1)} \otimes D \cdot_2 \phi_{(2)}. \quad (3.19)$$

We now define the antipode on $\mathcal{J}^\infty L$ to be

$$(S\phi)(D) := \epsilon(D \cdot_2 \phi) = D_+(\phi(D_-)).$$

By construction, this is the map that intertwines the two module structures.

Theorem 3.17. *Equipped with this antipode, $\mathcal{J}^\infty L$ is a Hopf algebroid with involutive antipode in the sense of Definition 1.2.*

Proof. Since L acts on $\mathcal{V}L$ via (3.18) by derivations, $L \rightarrow \text{Der}_k(\mathcal{J}^\infty L)$ is a morphism of Lie algebras. It therefore follows from the Poincaré–Birkhoff–Witt theorem that

$$D \cdot_2 (\phi_1\phi_2) = (D_{(1)} \cdot_2 \phi_1)(D_{(2)} \cdot_2 \phi_2).$$

Using this property, one finds that S is a homomorphism of commutative algebras:

$$\begin{aligned} S(\phi_1\phi_2)(D) &= (D \cdot_2 (\phi_1\phi_2))(1_{\mathcal{V}L}) \\ &= ((D_{(1)} \cdot_2 \phi_1)(D_{(2)} \cdot_2 \phi_2))(1_{\mathcal{V}L}) \\ &= ((S\phi_1)(S\phi_2))(D). \end{aligned}$$

To prove that $S^2 = \text{id}$, one first computes $(S^2\phi)(D) = \epsilon_l(D_+D_{-+}\phi(D_{--}))$, using the properties of a left counit. Next, to find a simpler expression for $D_+D_{-+} \otimes D_{--} \in \mathcal{V}L \otimes^{\text{rl}} \mathcal{V}L$, apply the Hopf–Galois map β from Remark 3.12 to it:

$$\begin{aligned} \beta(D_+D_{-+} \otimes D_{--}) &= D_{+(1)}D_{-(1)} \otimes D_{+(2)}D_{-(2)}D_{--} \\ &= D_{+(1)}D_- \otimes D_{+(2)} \\ &= 1 \otimes D \in \mathcal{V}L \otimes^{\text{ll}} \mathcal{V}L. \end{aligned}$$

Here (3.14) and the fact that $\mathcal{V}L$ is cocommutative were used. Hence

$$D_+D_{-+} \otimes D_{--} = \beta^{-1}(1 \otimes D) = 1_+ \otimes 1_-D = 1 \otimes D \in \mathcal{V}L \otimes^{\text{rl}} \mathcal{V}L,$$

and therefore $(S^2\phi)(D) = \phi(D)$.

We now verify the axioms in Definition 1.2: since $s = s_l = t_r$, $t = t_l = s_r$, the first one is trivially satisfied, whereas the second is equivalent to the coassociativity of Δ , because $\Delta = \Delta_l = \Delta_r$. For the third one, with (3.14), (3.16), and the Leibniz rule for the canonical left $\mathcal{V}L$ -action on A we compute

$$S(s(a))(D) = D_+(a\epsilon_l(D_-)) = \epsilon_l(D_{+(1)}a)\epsilon_l(D_{+(2)}D_-) = D(a) = t(a)(D),$$

for $a \in A$, $D \in \mathcal{V}L$, and $S \circ t = s$ then follows using $S^2 = \text{id}$. Finally, since S is an algebra homomorphism and an involution, it suffices to prove one of the two identities in (1.5). For example, with (3.15) and (3.16) we obtain

$$\begin{aligned} \phi_{(1)}S(\phi_{(2)})(D) &= \phi_{(1)}(D_{(1)})D_{(2)+}(\phi_{(2)}(D_{(2)-})) \\ &= \phi_{(1)}(D_{+(1)})D_{+(2)}(\phi_{(2)}(D_-)) \\ &= \phi_{(1)}(D_+\phi_{(2)}(D_-)) \\ &= \phi(D_+D_-) \\ &= \epsilon_l(D)\phi(1) = s(\epsilon(\phi))(D), \end{aligned}$$

and this is precisely the second identity in (1.5). This completes the proof that $\mathcal{J}^\infty L$ has the structure of a Hopf algebroid with involutive antipode. \square

Remark 3.18. Theorem 3.17 is remarkable in the sense that whereas the universal enveloping algebra $\mathcal{V}L$ of a Lie–Rinehart algebra carries no canonical Hopf algebroid structure, its dual $\mathcal{J}^\infty L$ is a Hopf algebroid without making further choices. Close inspection of the preceding proof shows that the Hopf algebroid structure – more precisely the antipode – depends solely on the *left* Hopf algebroid structure on $\mathcal{V}L$ which is canonical, i.e., does not depend on the choice of a flat right connection.

Remark 3.19. In the construction of the jet space – now written as $\mathcal{J}_l^\infty L$ – we considered $\mathcal{V}L$ as an A -module by left multiplication. *Right* multiplication leads to a space $\mathcal{J}_r^\infty L$, a priori without much structure. Only after introducing a flat right

(A, L) -connection on A we can introduce a ring structure using the right coproduct Δ_r on $\mathcal{V}L$, as well as source and target maps using the right counit ϵ_r . This does again lead to a Hopf algebroid, but one easily proves that the map $\phi \mapsto \phi \circ S$ defines an isomorphism $\mathcal{J}_l^\infty L \rightarrow \mathcal{J}_r^\infty L$ of Hopf algebroids, where S is the antipode on $\mathcal{V}L$ constructed from the same flat right connection as in Proposition 3.11.

3.4.2. The cyclic theory of $\mathcal{J}^\infty L$. Let (A, L) be a Lie–Rinehart algebra. If L is A -projective, Lie–Rinehart cohomology with values in A (cf. (3.9)) can be computed by the complex $(\text{Hom}_A(\wedge_A^\bullet L, A), d)$ with differential $d : \wedge_A^n L \rightarrow \wedge_A^{n+1} L$ defined by

$$d\omega(X_0 \wedge \cdots \wedge X_n) = \sum_{i=0}^n (-1)^i X_i(\omega(X_0, \dots, \widehat{X}_i, \dots, X_n)) + \sum_{i < j} (-1)^{i+j} \omega([X_i, X_j], X_0, \dots, \widehat{X}_i, \dots, \widehat{X}_j, \dots, X_n).$$

Theorem 3.20. *Let (A, L) be a Lie–Rinehart algebra, where L is finitely generated A -projective of constant rank. There are canonical isomorphisms*

$$\text{HH}^\bullet(\mathcal{J}^\infty L) \cong H^\bullet(L, A), \quad \text{HC}^\bullet(\mathcal{J}^\infty L) \cong \bigoplus_{i \geq 0} H^{\bullet+2i}(L, A).$$

Proof. Write $L^* := \text{Hom}_A(L, A)$. By the given conditions we have $\wedge_A^\bullet L^* \cong \text{Hom}_A(\wedge_A^\bullet L, A)$. To compute Hochschild cohomology, instead of the cobar resolution one can use the dual of the Koszul–Rinehart resolution given by (cf. [NeTs])

$$0 \rightarrow A \xrightarrow{s} \mathcal{J}^\infty L \xrightarrow{\nabla} \mathcal{J}^\infty L \otimes_A \wedge_A^1 L^* \xrightarrow{\nabla} \mathcal{J}^\infty L \otimes_A \wedge_A^2 L^* \xrightarrow{\nabla} \cdots,$$

where ∇ is the continuation of the Grothendieck connection, cf. (3.18):

$$\begin{aligned} &\nabla(\phi \otimes \omega)(X_1, \dots, X_{n+1}) \\ &= \sum_{i=1}^{n+1} (-1)^{i-1} \nabla_{X_i}^l \phi \otimes \omega(X_1, \dots, \widehat{X}_i, \dots, X_{n+1}) \\ &\quad + \sum_{i < j} (-1)^{i+j} \phi \otimes \omega([X_i, X_j], X_1, \dots, \widehat{X}_i, \dots, \widehat{X}_j, \dots, X_{n+1}) \end{aligned}$$

for $\phi \in \mathcal{J}^\infty L$, $\omega \in \wedge_A^n L^*$ and $X_1, \dots, X_{n+1} \in L$. It follows from (3.19) that this is indeed a cofree resolution of A in the category of left $\mathcal{J}^\infty L$ -comodules (observe that $s : A \rightarrow \mathcal{J}^\infty L$ is a morphism of left $\mathcal{J}^\infty L$ -comodules). To compute the Cotor groups, we take invariants and apply the isomorphism (1.15),

$$\wedge_A^\bullet L^* \xrightarrow{\cong} A \square_{\mathcal{J}^\infty L} (\mathcal{J}^\infty L \otimes_A \wedge_A^\bullet L^*),$$

given by $X_1 \wedge \cdots \wedge X_n \mapsto 1_A \otimes 1_{\mathcal{J}^\infty L} \otimes X_1 \wedge \cdots \wedge X_n$. Since the unit in $\mathcal{J}^\infty L$ is given by the left counit $\epsilon_l : \mathcal{V}L \rightarrow A$, the induced differential is exactly the differential for Lie–Rinehart cohomology. This proves the isomorphism for Hochschild cohomology. The second isomorphism on cyclic cohomology follows from Theorem 2.16 (i). \square

Theorem 3.21. *Let (A, L) be a Lie–Rinehart algebra, where L is finitely generated A -projective of constant rank. There is a natural morphism of mixed complexes*

$$F: (C_\bullet(\mathcal{J}^\infty L), b, B) \rightarrow (\bigwedge_A^\bullet L^*, 0, d)$$

defined in degree n by

$$F(\phi^1 \otimes \cdots \otimes \phi^n)(X_1 \wedge \cdots \wedge X_n) := (-1)^n (S(\phi^1) \wedge \cdots \wedge S(\phi^n))(X_1, \dots, X_n),$$

which induces isomorphisms

$$\mathrm{HH}_\bullet(\mathcal{J}^\infty L) \cong \bigwedge_A^\bullet L^*, \quad \mathrm{HP}_\bullet(\mathcal{J}^\infty L) \cong \prod_{i \geq 0} H^{\bullet+2i}(L, A).$$

Proof. This statement is very much the dual of Theorem 3.13. The dual of the PBW isomorphism gives $\mathcal{J}^\infty L \cong \widehat{S}_A L^*$ as commutative algebras. Similar to Lemma 3.16 there is a canonical isomorphism

$$C_n(\mathcal{J}^\infty L) \cong \varprojlim_p \mathrm{Hom}_A((\mathcal{V}L^{\otimes n})_{(p)}, A),$$

induced by the map

$$(\phi^1 \otimes \cdots \otimes \phi^n)(D_1 \otimes \cdots \otimes D_n) = S(\phi^1)(D_1) \cdots S(\phi^n)(D_n).$$

Observe that $C_n(\mathcal{J}^\infty L)$ is defined here with respect to the tensor product in the category $\mathrm{Comod}_R(\mathcal{J}^\infty L)$, the dual of \otimes^r , and the antipode is needed to go from (the duals of) $\mathcal{V}L^{\otimes n}$ to $\mathcal{V}L^{\otimes n}$ to make the map F well defined. Since $\mathcal{J}^\infty L$ is a commutative algebra, it maps the Hochschild differential b to zero.

Clearly, F is a morphism of A -modules, where A acts on $C_\bullet(\mathcal{J}^\infty L)$ by multiplication by $t(a)$, $a \in A$, on the first component. We can therefore localise with respect to a maximal ideal $\mathfrak{m} \subset A$ to prove that F is a quasi-isomorphism. As in the proof of Theorem 3.13, $L_\mathfrak{m}$ is free of rank r over $A_\mathfrak{m}$, and we choose a basis $e_i \in L_\mathfrak{m}$, $e^i \in L_\mathfrak{m}^*$, $i = 1, \dots, r$. The Koszul resolution

$$0 \leftarrow A_\mathfrak{m} \xleftarrow{\epsilon} \mathcal{J}^\infty L_\mathfrak{m} \xleftarrow{\partial'} \mathcal{J}^\infty L_\mathfrak{m} \otimes_{A_\mathfrak{m}} L_\mathfrak{m}^* \xleftarrow{\partial'} \mathcal{J}^\infty L_\mathfrak{m} \otimes_{A_\mathfrak{m}} \bigwedge_{A_\mathfrak{m}}^2 L_\mathfrak{m}^* \xleftarrow{\partial'} \cdots$$

is a free resolution of $A_\mathfrak{m}$ in the category $\mathrm{Mod}(\mathcal{J}^\infty L_\mathfrak{m})$ with differential

$$\partial'(\phi \otimes \omega) = \sum_{i=1}^r e^i \phi \otimes \iota_{e_i} \omega.$$

The natural map $\mathcal{J}^\infty L_\mathfrak{m} \otimes_{A_\mathfrak{m}} \bigwedge_{A_\mathfrak{m}}^\bullet L_\mathfrak{m}^* \rightarrow \mathrm{Bar}_\bullet(\mathcal{J}^\infty L_\mathfrak{m})$ given by

$$\phi \otimes \alpha_1 \wedge \cdots \wedge \alpha_n := \phi_0 \otimes (\alpha_1 \circ \mathrm{pr}) \wedge \cdots \wedge (\alpha_n \circ \mathrm{pr}),$$

is a morphism of complexes as one easily checks. Since $S(\alpha \circ \mathrm{pr}) = -\alpha \circ \mathrm{pr}$ for $\alpha \in L^*$, the map $\mathrm{id} \otimes F_\mathfrak{m}: \mathrm{Bar}_\bullet(\mathcal{J}^\infty L_\mathfrak{m}) \rightarrow \mathcal{J}^\infty L_\mathfrak{m} \otimes_{A_\mathfrak{m}} \bigwedge_{A_\mathfrak{m}}^\bullet L_\mathfrak{m}^*$ is a right inverse

and induces the morphism F when taking the tensor product $A_m \otimes_{\mathcal{J}^\infty L_m} -$ on both sides. This proves the first claim.

As for the second, notice that one has $B_n(\mathcal{J}^\infty L) \cong C_{n+1}(\mathcal{J}^\infty L)$ since $\mathcal{J}^\infty L$ is commutative, and the map to invariants $\Psi_{\text{inv}} : C_n(\mathcal{J}^\infty L) \rightarrow C_{n+1}(\mathcal{J}^\infty L)$ of §2.4 is a morphism of cyclic modules. Explicitly, this map, when restricted to L^* , is given by

$$\begin{aligned} & \Psi_{\text{inv}}(\phi^1 \otimes \cdots \otimes \phi^n)(X_1 \otimes \cdots \otimes X_{n+1}) \\ &= S\phi_{(1)}^1(X_1) \dots S\phi_{(1)}^n(X_n) \nabla_{X_{n+1}}^l(\phi_{(2)}^1 \cdots \phi_{(2)}^n)(1) \\ &= \sum_{i=1}^n (S\phi^1(X_1) \dots \widehat{S\phi^i(X_i)} \dots S\phi^n(X_n)) S\phi_{(1)}^i(X_i)(X_{n+1}(\phi_{(2)}^i(1)) - \phi_{(2)}^i(X_{n+1})). \end{aligned}$$

Since the cyclic structure on $C_{\bullet+1}(\mathcal{J}^\infty L)$ depends only on the structure of $\mathcal{J}^\infty L$ as a commutative algebra, it is well known (see, for example, [Lo]) that the morphism

$$\phi^1 \otimes \cdots \otimes \phi^{n+1} \mapsto \phi^{n+1} d\phi^1 \wedge \cdots \wedge d\phi^n$$

induces a morphism of mixed complexes $(C_\bullet(\mathcal{J}^\infty L)[1], b, B) \rightarrow (\wedge_A^\bullet L^*, 0, d)$. Composing this morphism with Ψ_{inv} as above, one finds exactly the map stated in the theorem. This proves that it intertwines the B -operator with the coboundary operator for Lie–Rinehart cohomology. Since we already know that this map is a quasi-isomorphism on the level of Hochschild homology, the *SBI* sequence implies that it is a quasi-isomorphism for cyclic homology. This proves the theorem. \square

3.4.3. Lie groupoids. Here we explain the relationship between the previous constructions and so-called *formal Lie groupoids* [Kar], justifying the name jet spaces. Among others, it gives a natural explanation of the Hopf algebroid structure.

Let $X \subset Y$ be a closed subset of a smooth manifold Y . Its *formal neighbourhood* is the commutative ring

$$\mathcal{J}_Y^\infty(X) := C^\infty(Y)/I_X^\infty,$$

where I_X denotes the ideal of functions vanishing on X and $I_X^\infty = \bigcap I_X^{n+1}$. It has the following functorial property: let $f : (X_1, Y_1) \rightarrow (X_2, Y_2)$ be a smooth map from Y_1 to Y_2 with the property that $f(X_1) \subset X_2$. This induces a canonical morphism of rings $f^* : \mathcal{J}_{Y_2}^\infty(X_2) \rightarrow \mathcal{J}_{Y_1}^\infty(X_1)$ by pull-back.

Consider the Lie–Rinehart algebra arising from a Lie algebroid $E(\mathbb{G})$ of a Lie groupoid $s, t : \mathbb{G} \rightrightarrows M$: this is the vector bundle over M defined by the kernel of the derivative of the source map: $E(\mathbb{G}) := \ker(ds)|_M$. The derivative of the target map, restricted to M , provides the anchor, so that the space of sections of $E(\mathbb{G})$ defines a Lie–Rinehart algebra over $A = C^\infty(M)$. Let \mathcal{C}_M^∞ denote the structure sheaf of smooth functions on M , and define the sheaf

$$\mathcal{J}_\mathbb{G}^n := s_*(\mathcal{C}_\mathbb{G}^\infty / \mathcal{I}_M^{n+1}),$$

on M , where \mathcal{I}_M denotes the sheaf of smooth functions on \mathbb{G} vanishing on M . This defines a sheaf of commutative algebras on M which has two natural inclusions

$\mathcal{C}_M^\infty \hookrightarrow \mathcal{J}_G^n$ given by pull-back via s or t . As above, \mathcal{J}_G^∞ denotes the projective limit of these sheaves. The pair $(M, \mathcal{J}_G^\infty)$ is a locally ringed space, and the ring of global sections $\mathcal{J}_G^\infty(M)$ is the formal neighbourhood of M in G as defined above.

Remark 3.22. For the so-called *pair groupoid* $M \times M \rightrightarrows M$, source and target map are given by the projection onto the first resp. second component. The associated Lie algebroid over M is then nothing but the tangent bundle TM . Since the unit inclusion is just the diagonal map, the definition above is the standard definition of the sheaf of jets of smooth functions on M , cf., e.g., [KuSp], Ch. 1.

Proposition 3.23. *There is a canonical isomorphism $\mathcal{J}_G^p(M) \cong \mathcal{J}^p(E(G))$.*

Proof. On M there is a short exact sequence of vector bundles

$$0 \rightarrow E(G) \rightarrow TG|_M \xrightarrow{ds} TM \rightarrow 0.$$

There is therefore a canonical map

$$\mathcal{J}_G^p(M) \rightarrow \mathcal{J}^p(E(G)), \quad f \mapsto \{D \mapsto D(f)\},$$

where we view $D \in \mathcal{V}E(G)_{\leq p}$ as a germ of a differential operator on G of order $\leq p$. This map is clearly left \mathcal{C}_M^∞ -linear, so it indeed defines an element in $\mathcal{J}^p(E(G))$. Let $(x_1, \dots, x_s, y_1, \dots, y_r) : U \rightarrow \mathbb{R}^{s+r}$ be local coordinates on $U \subset G$, where (x_1, \dots, x_s) are defined on $U \cap M$. For some $f \in \mathcal{C}_G^\infty(U)$ we have by Taylor's expansion

$$f(x, y) = \sum_{|\alpha| \leq p} D_y^\alpha f(x, 0) \frac{y^\alpha}{\alpha!} \pmod{\mathcal{I}_M^{p+1}}, \tag{3.20}$$

where $\alpha = (\alpha_1, \dots, \alpha_r)$ denotes a multiindex, $|\alpha| = \sum_i \alpha_i$, $\alpha! = \alpha_1! \dots \alpha_r!$, and $D_y^\alpha = \partial^{|\alpha|} / \partial y_1^{\alpha_1} \dots \partial y_r^{\alpha_r}$. This gives locally a representative of each local section of \mathcal{J}_G^p as a polynomial of degree $\leq p$ in the y -coordinates. A general element $D \in \mathcal{V}L_{\leq p}$ can locally be written as

$$D = \sum_{|\alpha| \leq p} c_\alpha(x) D_y^\alpha,$$

with $c_\alpha \in \mathcal{C}_M^\infty(U)$, and this shows that the map defined above is an isomorphism in each degree. Taking the projective limit proves the proposition. \square

As remarked, the formal neighbourhood $\mathcal{J}_G^\infty(M)$ comes equipped with two homomorphisms $s, t : C^\infty(M) \rightarrow \mathcal{J}_G^\infty(M)$ given by pull-back along the groupoid source resp. target map. As a commutative algebra it therefore inherits the structure of an (s, t) -ring.

Consider now the inclusion $(u, u) : M \hookrightarrow G_2$ and define the sheaf

$$\mathcal{J}_{G_2}^\infty := \varprojlim_p s_*(\mathcal{C}_{G_2}^\infty / \mathcal{I}_M^{p+1}),$$

on M , where $s : G_2 \rightarrow M$ is given by $s(g_1, g_2) = s(g_1)$.

Proposition 3.24. *There is a canonical isomorphism of sheaves*

$$\mathcal{J}_G^\infty \otimes_{\mathcal{C}_M^\infty} \mathcal{J}_G^\infty \xrightarrow{\cong} \mathcal{J}_{G_2}^\infty$$

by which the coproduct Δ is identified with the pull-back of the multiplication.

Proof. Define the morphism of sheaves as follows: let f_1 and f_2 be local sections of \mathcal{J}_G^∞ . Define the local section f of $\mathcal{J}_{G_2}^\infty$ stalkwise by

$$[f]_{(g_1, g_2)} := [f_1]_{g_1} [f_2]_{g_2}.$$

Clearly, this morphism factors over the ideal generated by $(s^* \mathcal{C}_M^\infty \otimes 1 - 1 \otimes t^* \mathcal{C}_M^\infty)$ in $\mathcal{J}_G^\infty \otimes \mathcal{J}_G^\infty$ defining the tensor product and is therefore well defined. With respect to G_2 , there is a short exact sequence

$$0 \rightarrow E(G_2) \rightarrow TG_2 \xrightarrow{ds} TM \rightarrow 0$$

of vector bundles over M , where $E(G_2)$ is the vector bundle with fiber at $x \in M$ given by

$$E(G_2)_x = \{(X, Y) \in E(G)_x \oplus E(G)_x \mid dt(X) = ds(Y)\}.$$

It then follows from (3.20) that the map defined above is an isomorphism. □

Next we turn to the antipode, given by the dual of the groupoid inversion map, $S := i^*$. Notice that on the level of sheaves $i : G \rightarrow G$ induces a morphism

$$i^* : \mathcal{J}_G^\infty \rightarrow t_*(\mathcal{C}_G^\infty / \mathcal{I}_M^\infty),$$

but on the level of global sections it defines a homomorphism $S : \mathcal{J}_G^\infty(M) \rightarrow \mathcal{J}_G^\infty(M)$ satisfying $S(s^* f_1 \phi t^* f_2) = s^* f_2 S(\phi) t^* f_1$ for all $\phi \in \mathcal{J}_G^\infty(M)$ and $f_1, f_2 \in C^\infty(M)$. With this antipode, it is easy to check that all Hopf algebroid axioms in Definition 1.2 are satisfied by the fact that $G \rightrightarrows M$ is a Lie groupoid.

Remark 3.25. It is clear from the construction above that not the full groupoid $G \rightrightarrows M$ is needed, but rather its structure in a neighbourhood of M in G . Such an object is called a *local groupoid*. Although for a general Lie algebroid there may be obstructions to integrate to a Lie groupoid [CrF], one can always find an integrating local Lie groupoid, see Cor. 5.1 of [CrF]. The previous construction gives therefore an alternative proof of Theorem 3.17 for Lie algebroids.

Remark 3.26 (The van Est isomorphism). Let G be a compact Lie group with Lie algebra \mathfrak{g} . One may consider G as a Lie groupoid with only one object, the unit, and the previous construction defines a Hopf algebra of jets of functions on G at the unit.

In this case, $\mathcal{V}\mathfrak{g} = \mathcal{U}\mathfrak{g}$, the universal enveloping algebra of \mathfrak{g} . Therefore $\mathcal{J}^\infty \mathfrak{g} = \hat{S}\mathfrak{g}^*$, and the preceding theorem gives

$$\mathrm{HC}_\bullet(\mathcal{J}^\infty(\mathfrak{g})) \cong H_{\mathrm{Lie}}^\bullet(\mathfrak{g}, \mathbb{R}).$$

On the other hand, $C^\infty(G)$ has a Hopf algebra structure by dualising the structure maps of G , provided one uses the projective tensor product $\hat{\otimes}$ and its property $C^\infty(G) \hat{\otimes} C^\infty(G) \cong C^\infty(G \times G)$, cf. [G]. For this Hopf algebra one has

$$\mathrm{HC}_\bullet(C^\infty(G)) \cong \bigoplus_{i \geq 0} H_{\mathrm{diff}}^{\bullet-2i}(G, \mathbb{R}).$$

There is an obvious morphism $C^\infty(G) \rightarrow \mathcal{J}^\infty(\mathfrak{g})$ of Hopf algebras by taking the jet of a function at the unit. On the level of cyclic homology, this induces a map $H_{\mathrm{diff}}^\bullet(G, \mathbb{R}) \rightarrow H_{\mathrm{Lie}}^\bullet(\mathfrak{g}, \mathbb{R})$, which is the van Est map.

Example 3.27 (The coordinate ring of an affine variety). Let A be the coordinate ring of an affine variety X . For the Lie–Rinehart algebra $(A, \mathrm{Der}_k A)$ we have

$$\mathcal{J}^\infty(\mathrm{Der}_k(A)) = \varprojlim_p (A \otimes A/\mathfrak{m}^{p+1}),$$

where $\mathfrak{m} \subset A \otimes A$ is the kernel ideal of the multiplication. We can consider this Hopf algebroid to be the localisation of the enveloping algebra A^e – viz. the pair groupoid – of §3.1 to the diagonal $X \subset X \times X$. By Theorem 3.21 we have

$$\mathrm{HH}_\bullet(\mathcal{J}^\infty(X)) \cong \Omega^\bullet X, \quad \mathrm{HP}_\bullet(\mathcal{J}^\infty(X)) \cong \prod_{i \geq 0} H_{\mathrm{alg}}^{\bullet+2i}(X).$$

Since A is commutative, we also have $\mathrm{HC}_\bullet(A) \cong H_{\mathrm{alg}}^\bullet(X)$. In view of Proposition 3.1, compare this with the van Est isomorphism of the previous remark.

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