On small deformations of paracomplex manifolds

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Abstract. A *paracomplex structure* on a manifold M is an endomorphism K of the tangent bundle TM such that $K^2 = I$, whose ± 1-eigenspaces have the same dimension and are involutive. By using the theory of differential graded Lie algebras, we describe small deformations of paracomplex structures. We also compute the space of invariant small deformations of 4-dimensional nilmanifolds endowed with a fixed paracomplex structure.

Introduction

Let M be a $2n$-dimensional manifold. An *almost paracomplex structure* on M is an endomorphism K of the tangent bundle TM of M such that $K^2 = I_{TM}$ and the two distributions

$$T^{\pm}M = \{X \pm KX \mid X \in \Gamma(M, TM)\}$$

have the same rank (see [11]). An almost paracomplex structure K is said to be a *paracomplex structure* if these two distributions are involutive (this condition is equivalent to the vanishing of the torsion tensor of K). In fact, a paracomplex structure on a manifold can be given by a pair of transverse foliations of the same dimension. If the paracomplex manifold (M, K) admits a pseudo-Riemannian metric g such that $g(KX, KY) = -g(X, Y)$ and the fundamental form $\omega(X, Y) = g(X, KY)$ is closed, then the datum (K, g, ω) is a *para-Kähler structure* on M. It turns out that a para-Kähler structure on M gives rise to a pair of involutive Lagrangian distributions T^-M and T^+M such that $TM = T^+M \oplus T^-M$, called *bi-Lagrangian structure* (see [3]). Conversely, starting from a symplectic manifold (M, ω) whose tangent bundle TM is the direct sum of two Lagrangian involutive distributions $T^{\pm}M$, one can define a para-Kähler structure on M, setting

$$K|_{T^{\pm}M} = \pm I \quad \text{and} \quad g(X, Y) = \omega(X, KY).$$

In this paper we study small deformations of a paracomplex structure on a compact manifold. In order to do this, as in the theory of small deformations of complex structures (see [13]) or CR-structures (see [7]), we introduce a differential graded Lie algebra $(\mathcal{A}, [[\,,\,]], \bar{\partial}_K)$ (see Section 5). Then it turns out that small paracomplex deformations of K are parametrized by degree 1 elements of $\mathcal{A} = \bigoplus_{p \in \mathbb{Z}} \mathcal{A}_p$ satisfying the *Maurer–Cartan equation*. First of all we show that every almost paracomplex structure $\hat{K}$ close to K can be written as

$$\hat{K} = (I + L)K(I + L)^{-1},$$

where $L \in \operatorname{End}(TM)$ anticommutes with K and $\det(I + L) \neq 0$ (see Proposition 3.1). Then we prove the following (see Theorem 4.2).

Theorem. *Let (M, K) be a compact paracomplex manifold. Then the map between*

$$\{L \in \Gamma(M, \operatorname{End}(TM)) \mid LK + KL = 0, \ \bar{\partial}_K L + \tfrac{1}{2}[[L, L]] = 0, \det(I + L) \neq 0\}$$

and

$$\{\hat{K} \in \Gamma\big(M, \operatorname{End}(TM)\big) \mid \hat{K}^2 = I, \ N_{\hat{K}} = 0\},$$

given by

$$L \mapsto \hat{K} = (I + L)K(I + L)^{-1},$$

is a bijection between a neighborhood of $0 \in \Gamma(M, \operatorname{End}(TM))$ and a neighborhood of K.

The paper is organized as follows. In Section 1 we recall some general definitions and results on paracomplex geometry.

Section 2 is devoted to the construction of the differential graded Lie algebra governing the small deformation theory of a paracomplex structure. In particular, we give explicit formulas for the bracket $[[\,,\,]]$ and for the $\bar{\partial}_K$-operator without using local coordinates.

In Section 3 we consider curves of paracomplex structures through a paracomplex structure K, characterizing almost paracomplex structures close to K (see Proposition 3.1) and finding obstructions in order that an endomorphism L anticommuting with K gives rise to a curve of paracomplex structures (see Corollary 3.3).

In Section 4 we give the proof of the main result (Theorem 4.2). A key tool is given by Proposition 4.1.

Finally, in the last section we apply the results previously obtained to describe small deformations on 4-dimensional nilmanifolds endowed with a fixed paracomplex structure. By classification, there are two (non-abelian) nilpotent Lie algebras, whose (compact) nilmanifolds associated are either *primary Kodaira–Thurston surfaces* or have first Betti number $b_1 = 2$. We fix a paracomplex structure on these two nilmanifolds and we determine the invariant small paracomplex deformations of them. We show that such spaces depend on six and four real parameters, respectively.

It has to be remarked that a nilmanifold with $b_1 = 2$ cannot have any complex structure (see Remark 5.2). For small deformations of left invariant complex structures on nilmanifolds see [4] in the case of abelian complex structures and [15] in the general case. We also construct a family of para-Kähler structures on the primary Kodaira–Thurston surface.

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1. Generalities on paracomplex manifolds

We recall some basic definitions of paracomplex geometry (see e.g [6], [5], [1]). Let V be a $2n$-dimensional real vector space. A *paracomplex structure* on V is an endomorphism $K\colon V \to V$ such that

i) $K^2 = I_V$,

ii) the eigenspaces V^+, V^- of K with eigenvalues 1, -1, respectively, have the same dimension.

The pair (V, K) will be called a *paracomplex vector space*.

Definition 1.1. An *almost paracomplex structure* on a $2n$-dimensional manifold M is the datum of a section K of $\mathrm{End}(TM)$ such that

i) $K^2 = I_{TM}$,

ii) the two eigendistributions $T^{\pm}M := \ker(I \mp K)$ have the same rank.

A *almost paracomplex structure* K is said to be *integrable* if the distributions $T^{\pm}M$ are involutive. In such a case K is called a *paracomplex structure*. A manifold M endowed with an (almost) paracomplex structure K is called an (*almost*) *paracomplex manifold*.

The *Nijenhuis tensor* N_K of an almost paracomplex structure K is the $(1, 1)$-tensor field defined by

$$N_K(X, Y) = [X, Y] + [KX, KY] - K[KX, Y] - K[X, KY]$$

for every pair of vector fields X, Y on M. As in the complex case, an almost paracomplex structure K is integrable if and only if $N_K = 0$ (see e.g. [5]).

A basic example of a paracomplex manifold is given by $\mathbb{R}^{2n}$, with coordinates $(x_1, \ldots, x_n, y_1, \ldots, y_n)$, endowed with the standard paracomplex structure K_n defined as

$$K_n\left(\frac{\partial}{\partial x_j}\right) = \frac{\partial}{\partial y_j}, \quad K_n\left(\frac{\partial}{\partial y_j}\right) = \frac{\partial}{\partial x_j}, \quad j = 1, \ldots, n.$$

We recall that the set of paracomplex numbers

$$\mathcal{C} = \{z = x + ey \mid x, y \in \mathbb{R}, \ e^2 = 1\}$$

is endowed with the natural operations of sum, (distributive) product and multiplication by a real number. Then $\mathcal{C}$ is a ring and a vector space over $\mathbb{R}$. Note that $\mathcal{C}$ is not an integral domain and $z = x + ey \in \mathcal{C}$ is invertible if and only if $x^2 - y^2 \neq 0$. Let (M, K) be an almost para-complex manifold. Set $T^{\mathcal{C}}M = \mathcal{C} \otimes_{\mathbb{R}} TM$ and extend the endomorphism K to a $\mathcal{C}$-linear endomorphism of $T^{\mathcal{C}}M$. Then, for every $p \in M$,

$$T_p^{\mathcal{C}}M = T_p^{1,0}M \oplus T_p^{0,1}M,$$

where

$$T_p^{1,0}M = \{Z \in T_p^{\mathcal{C}}M \mid KZ = eZ\} = \{X + eKX \mid X \in T_pM\},$$
$$T_p^{0,1}M = \{Z \in T_p^{\mathcal{C}}M \mid KZ = -eZ\} = \{X - eKX \mid X \in T_pM\}$$

are the "eigenspaces" of K with "eigenvalues" $\pm e$, being $T_p^{\mathcal{C}}M$ a $\mathcal{C}$-module.

For every $Z = X + eY \in T_p^{\mathcal{C}}M$ denote, as usual, the conjugate of Z by $\bar{Z} = X - eY$. Then $T_p^{0,1}M = \bar{T}_p^{1,0}M$.

The almost para-complex structure K acts naturally on $(T^{\mathcal{C}})^*M$ by

$$K^*\alpha(X) = \alpha(KX).$$

We have a decomposition

$$(T^{\mathcal{C}})^*M = \wedge_K^{1,0}(M) \oplus \wedge_K^{0,1}(M),$$

where

$$\wedge_K^{1,0}(M) := \{\alpha + eK^*\alpha \mid \alpha \in T^*M\},$$
$$\wedge_K^{0,1}(M) := \{\alpha - eK^*\alpha \mid \alpha \in T^*M\}$$

are eigenspaces for K^* with eigenvalues $\pm e$. The last decomposition induces a splitting of the bundle $\wedge^r(T^{\mathcal{C}})^*M$ of para-complex r-forms on (M, K) given by

$$\wedge^r(T^{\mathcal{C}})^*M = \bigoplus_{p+q=r} \wedge_K^{p,q}(M).$$

The sections of $\wedge_K^{p,q}(M)$ are the (p, q)-*forms* on the almost paracomplex manifold (M, K).

A *para-Hermitian metric* on an almost paracomplex manifold (M, K) is the datum of pseudo-Riemannian metric g of signature (n, n) such that

$$g(KX, KY) = -g(X, Y)$$

for every pair of vector fields X, Y in M. The *fundamental form* of the para-Hermitian g is the 2-form ω defined by

$$\omega(X, Y) = g(X, KY).$$

The para-Hermitian metric g is said to be *para-Kähler* if $N_K = 0$ and $d\omega = 0$.

2. Paracomplex structures and DGLA

Let (M, K) be an n-dimensional paracomplex manifold. We are going to define the differential graded Lie algebra, shortly DGLA, governing the small deformations of K (for a general theory of deformations see e.g. [9]). Let $\Omega_K^{0,p}(M)$ be the space of sections of the bundle of $(0, p)$-forms on (M, K). Denote by $\Gamma(M, \wedge_K^{0,p}(M) \otimes T^{1,0}M)$ the space of sections of the vector bundle $\wedge_K^{0,p}(M) \otimes T^{1,0}(M)$. Set

$$\mathcal{A}_p = \begin{cases} \Gamma(M, \wedge_K^{0,p}(M) \otimes T^{1,0}M) & \text{if } 0 \le p \le n, \\ 0 & \text{otherwise,} \end{cases}$$

and define

$$\mathcal{A} = \bigoplus_{p \in \mathbb{Z}} \mathcal{A}_p.$$

Note that the real vector space $\mathcal{A}$ is a module over the paracomplex numbers. Now we introduce a bracket $[[\,,\,]]$ and a $\bar{\partial}_K$-operator on the $\mathcal{C}$-module $\mathcal{A}$. Namely,

Definition 2.1. The *bracket*

$$[[\,,\,]] \colon \mathcal{A}_p \times \mathcal{A}_q \to \mathcal{A}_{p+q}$$

is defined in the following way:

- For every pair $Z, W \in \mathcal{A}_0$ set

$$[[Z, W]] = [Z, W],$$

  where $[\,,\,]$ is the usual bracket between complex vector fields.

- For every $\varphi \in \mathcal{A}_1$ and $Z \in \mathcal{A}_0$, define $[[\varphi, Z]] = -[[Z, \varphi]] \in \mathcal{A}_1$ as

$$[[\varphi, Z]](\bar{W}) = [\varphi(\bar{W}), Z] + \frac{1}{2}\varphi([Z, \bar{W}] - eK[Z, \bar{W}]).$$

- For every $\varphi \in \mathcal{A}_1$, define $[[\varphi, \varphi]] \in \mathcal{A}_2$ as

$$[[\varphi, \varphi]](\bar{Z}, \bar{W}) = 2[\varphi(\bar{Z}), \varphi(\bar{W})] - 2\varphi([\varphi(\bar{Z}), \bar{W}] + [\bar{Z}, \varphi(\bar{W})]).$$

- For every $\varphi, \psi \in \mathcal{A}_1$, define $[[\varphi, \psi]] \in \mathcal{A}_2$ as

$$[[\varphi, \psi]] = \frac{1}{2}([[(\varphi + \psi), (\varphi + \psi)]] - [[\varphi, \varphi]] - [[\psi, \psi]]).$$

- For every $\alpha \in \Omega_K^{0,p}(M)$, $\beta \in \Omega_K^{0,q}(M)$, $\varphi, \psi \in \mathcal{A}_1$, set

$$[[\alpha \wedge \varphi, \beta \wedge \psi]] = (-1)^q \alpha \wedge \beta \wedge [[\varphi, \psi]] + (-1)^p (\daleth_\psi \alpha) \wedge \beta \wedge \varphi + \alpha \wedge (\daleth_\varphi \beta) \wedge \psi,$$

where $\daleth_\varphi$ is the skew-symmetric derivation of degree 1 of $\Omega^{0,*}(M)$ such that, for every smooth function f, $(\daleth_\varphi f)(\bar{Z}) = \varphi(\bar{Z})(f)$, and, for every $\alpha \in \Omega_K^{0,1}(M)$,

$$(\daleth_\varphi \alpha)(\bar{Z}, \bar{W}) = \varphi(\bar{Z})(\alpha(\bar{W})) - \varphi(\bar{W})(\alpha(\bar{Z})) - \alpha([\varphi(\bar{Z}), \bar{W}] + [\bar{Z}, \varphi(\bar{W})]).$$

- Finally, extend $[\![\,,\,]\!]$ by bilinearity to any pair of elements of $\mathcal{A}$.

Definition 2.2. The $\bar{\partial}_K$-*operator* is the map

$$\bar{\partial}_K \colon \mathcal{A}_p \to \mathcal{A}_{p+1}$$

defined by:

i) For $Z \in \mathcal{A}_0$, set

$$(\bar{\partial}_K Z)(\bar{W}) = \frac{1}{2}([\bar{W}, Z] + eK[\bar{W}, Z])$$

for vector field $\bar{W}$ of type $(0, 1)$.

ii) For $\varphi \in \mathcal{A}_1$, set

$$(\bar{\partial}_K \varphi)(\bar{Z}, \bar{W}) = (\bar{\partial}_K \varphi(\bar{W}))\bar{Z} - (\bar{\partial}_K \varphi(\bar{Z}))\bar{W} - \varphi([\bar{Z}, \bar{W}])$$

for every pair of vector fields $\bar{Z}, \bar{W}$ of type $(0, 1)$.

iii) Extend $\bar{\partial}_K$ to $\mathcal{A}_p$ by Leibniz rule, i.e.,

$$\bar{\partial}_K(\alpha \wedge \varphi) = \bar{\partial}_K \alpha \wedge \varphi + (-1)^{\deg(\alpha)} \alpha \wedge \bar{\partial}_K \varphi$$

for every $\alpha \in \Omega_K^{0,p}(M)$, $\varphi \in \mathcal{A}_1$.

Note that, for any $\varphi \in \mathcal{A}_p$, $\psi \in \mathcal{A}_q$, $\rho \in \mathcal{A}_r$, it can be checked that

1) $[\![\varphi, \psi]\!] = -(-1)^{\deg(\varphi)\deg(\psi)}[\![\psi, \varphi]\!]$,
2) $[\![\varphi, [\![\psi, \rho]\!]]\!] = [\![[\![\varphi, \psi]\!], \rho]\!] + (-1)^{\deg(\varphi)\deg(\psi)}[\![\psi, [\![\varphi, \rho]\!]]\!]$,
3) $\bar{\partial}_K^2 = 0$,
4) $\bar{\partial}_K[\![\varphi, \psi]\!] = [\![\bar{\partial}_K \varphi, \psi]\!] + (-1)^{\deg(\varphi)}[\![\varphi, \bar{\partial}_K \psi]\!]$.

Therefore, $(\mathcal{A}, [\![\,,\,]\!], \bar{\partial}_K)$ is a DGLA.

Now we note that the theory can be set in real terms. To this purpose, define

$$\mathcal{A}_p^{\mathbb{R}} = \{\varphi \in \Gamma(M, \textstyle\bigwedge^P(M) \otimes TM) \mid \varphi(X_1, \ldots, KX_j, \ldots, X_p)$$
$$= -K\varphi(X_1, \ldots, X_p) \text{ for all } j = 1, \ldots, p\},$$

e.g., $\mathcal{A}_0^{\mathbb{R}}$ is the space of smooth vector fields and

$$\mathcal{A}_1^{\mathbb{R}} = \{L \in \operatorname{End}(TM) \mid LK + KL = 0\}.$$

Let $m \colon TM \to T^{1,0}M$ be defined as

$$m(X) = \tfrac{1}{2}(X + eKX),$$

with inverse m^{-1} defined by $m^{-1}(R) = R + \bar{R}$.

Then every element $L \in \mathcal{A}_p^{\mathbb{R}}$ can be identified with $m(L) \in \mathcal{A}_p$ by setting

$$m(L) = \tfrac{1}{2}(L + eKL).$$

In particular, for every pair of vector fields X, Y, we have

$$[[X, Y]] = m^{-1}[[m(X), m(Y)]] = \tfrac{1}{2}([X, Y] + [KX, KY]),$$

$$(\bar{\partial}_K X)(Y) = \tfrac{1}{2}([Y, X] - K[KY, X]), \tag{1}$$

and, for every $L \in \mathcal{A}_1^{\mathbb{R}}$,

$$(\bar{\partial}_K L)(X, Y) = (\bar{\partial}_K LY)(X) - (\bar{\partial}_K LX)(Y) - L[[X, Y]],$$

where we used the identification $\bar{\partial}_K = m^{-1} \circ \bar{\partial}_K \circ m$.

By formula (1), it follows immediately that

$$(\bar{\partial}_K KX)(Y) = -(\bar{\partial}_K X)(KY). \tag{2}$$

Finally, in a similar way, for every $L \in \mathcal{A}_1^{\mathbb{R}}$ we have

$$[[L, L]](X, Y) = 2[[LX, LY]] + 2L((\bar{\partial}_K X)(LY) - (\bar{\partial}_K Y)(LX)).$$

Indeed, a direct computation yields

$$\begin{aligned}
[[L, L]]&(X, Y) \\
&= [LX, LY] + [KLX, KLY] \\
&\quad - \tfrac{1}{2}\{L([LX, Y] - [KLX, KY]) + KL([KLX, Y] - [LX, KY])\} \\
&\quad - \tfrac{1}{2}\{L([X, LY] - [KX, KLY]) + KL([X, KLY] - [KX, LY])\}.
\end{aligned}$$

3. Curves of paracomplex structures

We consider the space of linear paracomplex structures on $\mathbb{R}^{2n}$:

$$\mathcal{X}(n) = \{P \in \mathrm{GL}(2n, \mathbb{R}) \mid P^2 = I, \ \mathrm{tr}(P) = 0\}.$$

Let K_n be the standard linear paracomplex structure on $\mathbb{R}^{2n}$, namely,

$$K_n = \begin{pmatrix} 0 & I_n \\ I_n & 0 \end{pmatrix}.$$

Set

$$\mathfrak{h}(n) = \{X \in \mathfrak{gl}(2n, \mathbb{R}) \mid XK_n - K_n X = 0\},$$
$$s(n) = \{X \in \mathfrak{gl}(2n, \mathbb{R}) \mid XK_n + K_n X = 0\}.$$

Then $\mathfrak{gl}(2n, \mathbb{R}) = \mathfrak{h}(n) \oplus s(n)$ and there are defined two projections

$$R \colon \mathfrak{gl}(2n, \mathbb{R}) \to \mathfrak{h}(n)X \mapsto \frac{1}{2}(X + K_n X K_n),$$

$$S \colon \mathfrak{gl}(2n, \mathbb{R}) \to s(n)X \mapsto \frac{1}{2}(X - K_n X K_n).$$

We have the following

Proposition 3.1. *There exists a neighborhood U of K_n in $\mathcal{X}(n)$ such that every $P \in U$ can be written in a unique way as*

$$P = (I + L)K_n(I + L)^{-1},$$

where $LK_n + K_n L = 0$ and $\det(I + L) \neq 0$.

Proof. First of all, we observe that $\mathrm{GL}(2n, \mathbb{R})$ acts transitively on the space $\mathcal{X}(n)$ of paracomplex structures on $\mathbb{R}^{2n}$ by the following:

$$P \mapsto APA^{-1}$$

for every $P \in \mathcal{X}(n)$ and $A \in \mathrm{GL}(2n, \mathbb{R})$. The isotropy subgroup at K_n is given by

$$H(n) = \{A \in \mathrm{GL}(2n, \mathbb{R}) \mid AK_n - K_n A = 0\}.$$

Therefore, $\mathcal{X}(n) = \mathrm{GL}(2n, R)/H(n)$. Consequently, $\mathrm{GL}(2n, \mathbb{R}) \xrightarrow{\pi} \mathcal{X}(n)$ is an $H(n)$-principal bundle and $\pi(A) = AK_n A^{-1}$. By the local triviality, there exists a local section $\sigma \colon U \to \mathrm{GL}(2n, R)$ such that

$$\sigma(K_n) = I, \quad \sigma(P)K_n\sigma(P)^{-1} = P \quad \text{for all } P \in U.$$

By definition of the projection R, we immediately have that $R(\sigma(K_n)) = I$; hence, if U is small enough, $R(\sigma(P)) \in H(n)$, for every $P \in U$. Therefore,

$$\hat{\sigma}(P) = \sigma(P)(R(\sigma(P)))^{-1}$$

is a section over U with $R(\hat{\sigma}(P)) \equiv I$. The section $\hat{\sigma}$ is determined by the two conditions

$$\hat{\sigma}(K_n) = I, \quad R(\hat{\sigma}(P)) \equiv I.$$

Therefore, for every paracomplex structure P, we have

$$P = (R(\hat{\sigma}(P)) + S(\hat{\sigma}(P)))K_n(R(\hat{\sigma}(P)) + S(\hat{\sigma}(P)))^{-1}$$
$$= (I + S(\hat{\sigma}(P)))K_n(I + S(\hat{\sigma}(P)))^{-1}.$$

Setting

$$S(\hat{\sigma}(P)) = L,$$

we obtain that P can be written in a unique way as

$$P = (I + L)K_n(I + L)^{-1},$$

where $LK_n + K_nL = 0$. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ $\square$

Let K_t, for $-\varepsilon < t < \varepsilon$, be a curve of almost paracomplex structures on M such that $K_0 = K$. In view of Proposition 3.1, K_t can be uniquely written as

$$K_t = (I + L_t)K(I + L_t)^{-1},$$

where

$$L_t = tL + o(t) \quad \text{and} \quad L_tK + KL_t = 0.$$

Note that

$$\frac{d}{dt}K_t\Big|_{t=0} = 2LK.$$

Indeed, by definition,

$$\frac{d}{dt}K_t\Big|_{t=0} = \frac{d}{dt}(I + L_t)K(I + L_t)^{-1}\Big|_{t=0} = LK - KL = 2LK.$$

The next proposition allows us to give an obstruction to the integrability of a curve of almost paracomplex structures.

Proposition 3.2. *Let K_t be a curve of almost paracomplex structures, defined for $-\varepsilon < t < \varepsilon$, such that $K_0 = K$. Then*

$$\frac{d}{dt}N_{K_t}(X, Y)\Big|_{t=0} = 4(\bar{\partial}_K L)(X, Y) - N_K(LX, Y) - N_K(X, LY).$$

Proof. Let

$$K_t = (I + L_t)K(I + L_t)^{-1},$$

where

$$L_t = tL + o(t) \quad \text{and} \quad L_tK + KL_t = 0.$$

Then, setting $\dot{K}_0 = \frac{d}{dt}K_t\Big|_{t=0}$, we obtain

$$\begin{aligned}
\frac{d}{dt}N_{K_t}(X, Y)\Big|_{t=0} = {}&[\dot{K}_0X, KY] + [KX, \dot{K}_0Y] - \dot{K}_0[KX, Y]\\
&- K[\dot{K}_0X, Y] - \dot{K}_0[X, KY] - K[X, \dot{K}_0Y].
\end{aligned}$$

Since $\dot{K}_0 = 2LK$, we get

$$\frac{d}{dt} N_{K_t}(X, Y)\Big|_{t=0}$$
$$= 2\{-[KLX, KY] - [KX, KLY] + KL[KX, Y]$$
$$+ K[KLX, Y] + KL[X, KY] + K[X, KLY]\}$$
$$= 2\{KL[KX, Y] + KL[X, KY] + [LX, Y] + [X, LY]$$
$$- K[LX, KY] - K[KX, LY] - N_K(LX, Y) - N_K(X, LY)\}.$$

On the other hand, a direct computation gives

$$4(\bar{\partial}_K L)(X, Y) = 2[X, LY] - 2K[KX, LY] + N_K(LY, X)$$
$$- 2[Y, LX] + 2K[KY, LX] - N_K(LX, Y)$$
$$+ 2KL[KX, Y] + 2KL[X, KY].$$

The proof of the proposition follows from the last two equalities.   □

As a direct consequence, we have the following

Corollary 3.3. *If K_t is a curve of paracomplex structures, then $\bar{\partial}_K L = 0$.*

4. Deformations of paracomplex structures

Let M be a compact manifold endowed with a paracomplex structure K. We are going to characterize small deformations of $K \in \Gamma(M, \mathrm{End}(TM))$, using the DGLA $(\mathcal{A}, [\![\,,\,]\!], \bar{\partial}_K)$ introduced in Section 5.

On the space $\Gamma(M, \mathrm{End}(TM))$ of $\mathcal{C}^\infty$-sections of the vector bundle $\mathrm{End}(TM) = TM^* \otimes TM$ we will consider the usual topology.

In view of Proposition 3.1, every almost paracomplex structure $\hat{K}$ close to K can be written in a unique way as

$$\hat{K} = (I + L)K(I + L)^{-1},$$

with $LK + KL = 0$, $\det(I + L) \neq 0$.

The following proposition describes the behavior of the Nijenhuis tensor of the deformed almost paracomplex structure $\hat{K}$.

Proposition 4.1. *Let K be a paracomplex structure on a manifold M and*

$$\hat{K} = (I + L)K(I + L)^{-1},$$

with $LK + KL = 0$, $\det(I + L) \neq 0$. Then

$$(I + L)^{-1} N_{\hat{K}}((I + L)X, (I + L)Y) = 4(I - L^2)^{-1}\left(\bar{\partial}_K L + \frac{1}{2}[\![L, L]\!]\right)(X, Y).$$

Proof. The formula to prove is equivalent to

$$(I - L)N_{\hat{K}}((I + L)X, (I + L)Y) = 4\left(\bar{\partial}_K L + \frac{1}{2}[[L, L]]\right)(X, Y).$$

A straightforward computation, taking into account the integrability condition $N_K = 0$ and $LK + KL = 0$, gives

$$\begin{aligned}
(I - L)&N_{\hat{K}}((I + L)X, (I + L)Y) \\
&= [KX, LKY] + [LKX, KY] + [LKX, LKY] \\
&\quad + [X, LY] + [LX, Y] + [LX, LY] \\
&\quad - K([KX, LY] + [LKX, Y] + [LKX, LY]) \\
&\quad - K([X, LKY] + [LX, KY] + [LX, LKY]) \\
&\quad - L([KX, KY] + [KX, LKY] + [LKX, KY]) \\
&\quad - L([X, Y] + [X, LY] + [LX, Y]) \\
&\quad + KL([KX, Y] + [KX, LY] + [LKX, Y]) \\
&\quad + KL([X, KY] + [X, LKY] + [LX, KY]) \\
&= 4([X, LY] - K[KX, LY] - [Y, LX] + K[KY, LX]) \\
&\quad - 2(L[X, Y] + [KX, KY] - [LX, LY] - [KLX, KLY]) \\
&\quad - L[LX, Y] + L[KLX, KY] - KL[KLX, Y] + KL[LX, KY] \\
&\quad - L[X, LY] + L[KX, KLY] - KL[X, KLY] + KL[KX, LY] \\
&= 4\bar{\partial}_K L(X, Y) + 4[[LX, LY]] + 2L((\bar{\partial}_K X)(LY) - (\bar{\partial}_K Y)(LX)) \\
&\quad + 2L((\bar{\partial}_K KY)(KLX) - (\bar{\partial}_K KX)(KLY)) \\
&= 4\left(\bar{\partial}_K L + \frac{1}{2}[[L, L]]\right)(X, Y),
\end{aligned}$$

where in the last equality we have used formula (2). $\qquad\square$

Therefore, by Propositions 3.1 and 4.1, we obtain the following

Theorem 4.2. *Let (M, K) be a compact paracomplex manifold. Then the map between*

$$\left\{L \in \Gamma(M, \mathrm{End}(TM)) \mid LK + KL = 0, \ \bar{\partial}_K L + \frac{1}{2}[[L, L]] = 0, \det(I + L) \neq 0\right\}$$

and

$$\{\hat{K} \in \Gamma(M, \mathrm{End}(TM)) \mid \hat{K}^2 = I, \ N_{\hat{K}} = 0\},$$

given by

$$L \mapsto \hat{K} = (I + L)K(I + L)^{-1},$$

is a bijection between a neighborhood of $0 \in \Gamma(M, \mathrm{End}(TM))$ and a neighborhood of K.

As an example, we consider the standard $2n$-dimensional torus $\mathbb{T}^{2n} = \mathbb{R}^{2n}/\mathbb{Z}^{2n}$ endowed with the standard paracomplex structure defined by

$$K_n\left(\frac{\partial}{\partial x_j}\right) = \frac{\partial}{\partial y_j}, \quad K_n\left(\frac{\partial}{\partial y_j}\right) = \frac{\partial}{\partial x_j}, \quad j = 1, \dots, n.$$

Let K_t be a curve of paracomplex structures on $\mathbb{T}^{2n}$ such that $K_0 = K_n$, namely

$$K_t = (I + L_t)K_n(I + L_t)^{-1}.$$

Then the tangent vector $V = \dot{K}_0$ is a constant endomorphism of $T(\mathbb{T}^{2n})$ which anticommutes with K_n. Conversely, starting with such a constant $V \in \mathrm{End}(T(\mathbb{T}^{2n}))$ and setting $L = \frac{1}{2}VK_n$, we construct a curve of almost paracomplex structures given by

$$K_t = (I + tL)K_n(I + tL)^{-1},$$

which is in fact integrable since

$$\bar{\partial}_{K_n}(tL) + \frac{1}{2}t^2[[L, L]] = 0.$$

5. Families of paracomplex structures on 4-dimensional nilmanifolds

In this section we consider 4-dimensional nilmanifolds endowed with a fixed paracomplex structure and we compute the invariant small paracomplex deformations of it.

It is well known that there are only two real 4-dimensional nilpotent non-abelian Lie algebras. According to the Malcev Theorem (see [12]) it turns out that the simply-connected Lie groups associated with these two Lie algebras have compact quotients.

5.1. Four-dimensional nilmanifolds with $b_1 = 3$. Let $\mathfrak{h}(3)$ be the 3-dimensional real Heisenberg Lie algebra and $\mathfrak{n}_1 = \mathfrak{h}(3) \oplus \mathbb{R}$. Then we can find a basis $\{f^1, \dots, f^4\}$ of $\mathfrak{n}_1^*$ such that

$$df^1 = 0, \quad df^2 = -f^1 \wedge f^3, \quad df^3 = 0, \quad df^4 = 0.$$

Therefore, denoting by $\{f_1, \dots, f_4\}$ the dual basis of $\{f^1, \dots, f^4\}$, we obtain that the only non-zero bracket is $[f_1, f_3] = f_2$.

Let $N_1 = H(3) \times \mathbb{R}$ be the simply-connected Lie group with Lie algebra $\mathfrak{n}$, where

$$H(3) = \left\{A = \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} \mid x, y, z, \in \mathbb{R}\right\}$$

is the real 3-dimensional Heisenberg group, and let $M = \Gamma \backslash N_1$ be any compact quotient of N_1. As an example, we can take a *primary Kodaira–Thurston surface*

$\Gamma \backslash H(3) \times \mathbb{S}^1$, where $\Gamma \subset H(3)$ is the subgroup of $H(3)$ given by the matrices with integer entries.

Let $K \in \text{End}(TM)$ be the paracomplex structure defined as

$$Kf_1 = f_3, \quad Kf_2 = f_4, \quad Kf_3 = f_1, \quad Kf_4 = f_2.$$

Then, setting $t = (t_1, \ldots, t_8) \in \mathbb{R}^8$, every invariant endomorphism of TM anticommuting with K has the form

$$L_t = \begin{pmatrix} t_1 & t_2 & t_3 & t_4 \\ t_5 & t_6 & t_7 & t_8 \\ -t_3 & -t_4 & -t_1 & -t_2 \\ -t_7 & -t_8 & -t_5 & -t_6 \end{pmatrix}. \tag{3}$$

A straightforward computation yields

$$(\bar{\partial}_K L_t)(f_1, f_2) = \tfrac{1}{2}(-t_4 f_2 + t_2 f_4)$$

and

$$[[L_t, L_t]](f_1, f_2) = \tfrac{1}{2}\{2t_2 t_4 f_1 + (t_2 t_8 + t_4 t_6) f_2 - (t_2^2 + t_4^2) f_3 - (t_2 t_6 + t_4 t_8) f_4\}.$$

Therefore,

$$\bar{\partial}_K L_t + \tfrac{1}{2}[[L_t, L_t]] = 0$$

if and only if

$$t_2 = t_4 = 0.$$

Hence, for such L_t satisfying the above conditions and $|t| < \varepsilon$, it follows that $K_t = (I + L_t)K(I + L_t)^{-1}$ gives rise to a 6-parameter family of paracomplex structures on M, with $K_0 = K$.

Remark 5.1. Consider the pseudo-Riemannian metric on M defined by

$$g = -f^1 \odot f^4 + f^2 \odot f^3.$$

Then g is a para-Hermitian metric, whose fundamental form

$$\omega = -f^1 \wedge f^2 + f^3 \wedge f^4$$

is d-closed, so that g is a para-Kähler metric on (M, K). Let

$$\omega_t = \frac{1 + t_8^2 - t_6^2}{t_8^2 - (t_6 + 1)^2} f^1 \wedge f^2 + \frac{2t_8}{t_8^2 - (t_6 + 1)^2} f^2 \wedge f^3 + f^3 \wedge f^4.$$

A straightforward computation shows that, for

$$t_1 = t_2 = t_3 = t_4 = 0, \quad 1 + t_8^2 - t_6^2 > 0, \quad t_8^2 - (t_6 + 1)^2 < 0,$$

(K_t, ω_t) is a family of para-Kähler structures on M such that $(K_0, \omega_0) = (K, \omega)$, where

$$
K_t = \begin{pmatrix}
0 & 0 & 1 & 0 \\
-2\frac{t_5 t_8 - t_7 t_6 - t_7}{-t_6^2 + 1 + t_8^2} & 2\frac{t_8}{-t_6^2 + 1 + t_8^2} & -2\frac{-t_5 t_6 - t_5 + t_7 t_8}{-t_6^2 + 1 + t_8^2} & -\frac{t_8^2 - t_6^2 - 2t_6 - 1}{-t_6^2 + 1 + t_8^2} \\
1 & 0 & 0 & 0 \\
-2\frac{-t_5 t_6 + t_5 + t_7 t_8}{-t_6^2 + 1 + t_8^2} & -\frac{-t_6^2 + 2t_6 - 1 + t_8^2}{-t_6^2 + 1 + t_8^2} & -2\frac{-t_7 t_6 + t_7 + t_5 t_8}{-t_6^2 + 1 + t_8^2} & -2\frac{t_8}{-t_6^2 + 1 + t_8^2}
\end{pmatrix}.
$$

5.2. Four-dimensional nilmanifolds with $b_1 = 2$.

Let $\mathfrak{n}_2$ be the 4-dimensional real Lie algebra whose dual space is spanned by $\{e^1, \dots, e^4\}$ satisfying the following Cartan structure equations:

$$
de^1 = 0, \quad de^2 = 0, \quad de^3 = e^1 \wedge e^2, \quad de^4 = e^1 \wedge e^3.
$$

Denoting by $\{e_1, \dots, e_4\}$ the dual basis of $\{e^1, \dots, e^4\}$, we obtain

$$
[e_1, e_2] = -e_3, \quad [e_1, e_3] = -e_4.
$$

Let N_2 be the simply-connected Lie group having Lie algebra $\mathfrak{g}$. Let $M = \Gamma \backslash N_2$ be any compact quotient of N_2, i.e., M is a compact 4-dimensional nilmanifold. Define an N_2-invariant almost paracomplex structure K on M, by setting

$$
Ke_1 = e_1, \quad Ke_2 = -e_2, \quad Ke_3 = -e_3, \quad Ke_4 = e_4.
$$

Then, a direct computation shows that $N_K = 0$, i.e., K gives rise to a paracomplex structure on M. Now, using Theorem 4.2, we are going to compute the small deformations of K corresponding to invariant endomorphisms L of TM, which anticommute with K.

In order to perform the computation, it will be useful to make a change of basis of forms, namely, we set

$$
f^1 = e^1 + e^2, \quad f^2 = e^3 + e^4, \quad f^3 = e^1 - e^2, \quad f^4 = -e^3 + e^4,
$$

with dual basis

$$
f_1 = \tfrac{1}{2}(e_1 + e_2), \quad f_2 = \tfrac{1}{2}(e_3 + e_4), \quad f_3 = \tfrac{1}{2}(e_1 - e_2), \quad f_4 = \tfrac{1}{2}(-e_3 + e_4).
$$

Then, we have

$$
Kf_1 = f_3, \quad Kf_2 = f_4, \quad Kf_3 = f_1, \quad Kf_4 = f_2.
$$

Setting $t = (t_1, \dots, t_8) \in \mathbb{R}^8$, any invariant endomorphism of TM anticommuting with K is represented with respect to the basis $\{f_1, \dots, f_4\}$ by the matrix

$$
L_t = \begin{pmatrix}
t_1 & t_2 & t_3 & t_4 \\
t_5 & t_6 & t_7 & t_8 \\
-t_3 & -t_4 & -t_1 & -t_2 \\
-t_7 & -t_8 & -t_5 & -t_6
\end{pmatrix}.
$$

A tedious computation shows that the Maurer–Cartan equation

$$\bar{\partial}_K L_t + \frac{1}{2}[[L_t, L_t]] = 0$$

holds if and only if

$$t_1 = t_3, \quad t_2 = t_4 = 0, \quad t_6 = -t_8.$$

In such a case,

$$K_t = (I + L_t)K(I + L_t)^{-1}$$

is a 4-parameter family of paracomplex structures on M, with $K_0 = K$.

Remark 5.2. It has to be noted that any compact quotient M of N_2 has no complex structures for cohomological reasons (see e.g. [8]). Indeed, by Nomizu's Theorem (see [14]), we have that $b_1(M) = 2$. Furthermore, by a result of Kodaira (see [10], Theorem 25), a complex surface is a deformation of an algebraic surface if and only if its first Betti number is even. Therefore, if J were a complex structure on M, then (M, J) should have a Kähler metric. But this is not possible since, by the theorem of Benson and Gordon (see [2]), if M is a compact nilmanifold carrying a Kähler structure, then M is diffeomorphic to a torus.

5.3. Four-dimensional tori. According to §4, any matrix L_t as in (3), with $|t| < \varepsilon$, gives rise to a paracomplex structure on $\mathbb{T}^4$.

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