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Hopf action and Rankin–Cohen brackets on an Archimedean complex

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Abstract. The Hopf algebra $\mathcal{H}_1$ of "codimension 1 foliations", generated by operators X, Y and δ_n, $n \geq 1$, satisfying certain conditions, was introduced by Connes and Moscovici in [1]. In [2], it was shown that, for any congruence subgroup Γ of $\mathrm{SL}_2(\mathbb{Z})$, the action of $\mathcal{H}_1$ on the "modular Hecke algebra" $\mathcal{A}(\Gamma)$ captures classical operators on modular forms. In this paper, we show that the action of $\mathcal{H}_1$ captures the monodromy and Frobenius actions on a certain module $\mathbb{B}^*(\Gamma)$ that arises from the Archimedean complex of Consani [4]. The object $\mathbb{B}^*(\Gamma)$ replaces the modular Hecke algebra $\mathcal{A}(\Gamma)$ in our theory. We also introduce a "restricted" version $\mathbb{B}^*_r(\Gamma)$ of the module $\mathbb{B}^*(\Gamma)$ on which the operators δ_n, $n \geq 1$, of the Hopf algebra $\mathcal{H}_1$ act as zero. Thereafter, we construct Rankin–Cohen brackets of all orders on $\mathbb{B}^*_r(\Gamma)$.

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1. Introduction

The Hopf algebra $\mathcal{H}_1$ of "codimension 1 foliations" was introduced by Connes and Moscovici in [1]. As an algebra, $\mathcal{H}_1$ is the universal enveloping algebra of the Lie algebra generated by a family $\{X, Y, \{\delta_n\}_{n\geq 1}\}$ satisfying the relations

$$[Y,X] = X, \quad [Y,\delta_n] = n\delta_n, \quad [X,\delta_n] = \delta_{n+1}, \quad [\delta_k,\delta_l] = 0 \quad \text{for all } k,l \in \mathbb{N},$$

while the coproducts on $\mathcal{H}_1$ are given by

$$\begin{aligned}\Delta(X) &= X \otimes 1 + 1 \otimes X + \delta_1 \otimes Y,\\ \Delta(Y) &= Y \otimes 1 + 1 \otimes Y,\\ \Delta(\delta_1) &= \delta_1 \otimes 1 + 1 \otimes \delta_1.\end{aligned}$$

It was shown in [1] that the action of $\mathcal{H}_1$ on a certain crossed product algebra captures several important operators in the theory of foliations. It was discovered in [2] that this has an analogue in the following arithmetic situation.

Let $N \geq 1$ and let $\Gamma = \Gamma(N)$ be a congruence subgroup of $\mathrm{SL}_2(\mathbb{Z})$. Let $\mathcal{M}$ denote the "modular tower" consisting of the direct limit of modular forms (of all weights) over all levels $\Gamma(N)$. In [2], Connes and Moscovici have defined the "modular Hecke algebra" $\mathcal{A}(\Gamma)$ which is an extension of the usual algebra of Hecke operators and shown that there is an action of the same Hopf algebra $\mathcal{H}_1$ on $\mathcal{A}(\Gamma)$. The modular Hecke algebra $\mathcal{A}(\Gamma)$ is defined as the set of functions of finite support from $\Gamma\backslash \mathrm{GL}_2^+(\mathbb{Q})$ to the modular tower $\mathcal{M}$ satisfying a certain covariance condition (see Definition 2.1). Once again, the action of $\mathcal{H}_1$ on $\mathcal{A}(\Gamma)$ captures well-known classical operators on modular forms. Moreover, the action of the Hopf algebra $\mathcal{H}_1$ on $\mathcal{A}(\Gamma)$ is "flat", i.e.,

$$h(a \cdot b) = \sum h_{(1)}(a) \cdot h_{(2)}(b), \quad \Delta(h) = \sum h_{(1)} \otimes h_{(2)},$$

where $h \in \mathcal{H}_1$, $a, b \in \mathcal{A}(\Gamma)$ and $h(a)$ denotes the action of $h \in \mathcal{H}_1$ on $a \in \mathcal{A}(\Gamma)$. Further, it was shown in [2], [3] that the Hopf algebra $\mathcal{H}_1$ may be used to extend Rankin–Cohen brackets of all orders to the modular Hecke algebra $\mathcal{A}(\Gamma)$.

In [4], Consani has introduced the Archimedean bi-complex which computes the cohomology of the "fibre at infinity" of an arithmetic variety. For a modular curve $X(\Gamma(N))$, we will define the bi-complex $(\mathcal{K}_N^{**}, d', d'')$ (see (3.1)) whose terms $\mathcal{K}_N^{**}$ are obtained by tensoring the terms in Consani's complex by modular forms of certain weight. We let $(\mathcal{K}^{**}, d', d'')$ denote the direct limit of $(\mathcal{K}_N^{**}, d', d'')$ over all $N \geq 1$. Then, by replacing the modular tower $\mathcal{M}$ by the direct limit $\mathcal{K}^{**}$, we define $\mathbb{B}^{**}(\Gamma)$ to be the set of functions of finite support from $\Gamma\backslash \mathrm{GL}_2^+(\mathbb{Q})$ to the tower $\mathcal{K}^{**}$ satisfying a certain covariance condition. We let $\mathbb{B}^*(\Gamma)$ denote the total complex associated to $\mathbb{B}^{**}(\Gamma)$.

In this paper, our purpose is to show that the Hopf algebra $\mathcal{H}_1$ has an action on $\mathbb{B}^*(\Gamma)$ that captures the Frobenius and monodromy operators on the Archimedean complex $(\mathcal{K}^{**}, d', d'')$. We also show that $\mathbb{B}^*(\Gamma)$ has the structure of a module over an algebra $\mathcal{A}_T(\Gamma)[T]$, which is a slight variant of the modular Hecke algebra $\mathcal{A}(\Gamma)$ that we describe in Section 3. Further, the action of $\mathcal{H}_1$ on the system $(\mathcal{A}_T(\Gamma), \mathbb{B}^*(\Gamma))$ is flat, in the sense that

$$h(a \cdot m) = \sum h_{(1)}(a) \cdot h_{(2)}(m), \quad \Delta(h) = \sum h_{(1)} \otimes h_{(2)}, \tag{1.1}$$

where $h \in \mathcal{H}_1$, $a \in \mathcal{A}_T(\Gamma)$, $m \in \mathbb{B}^*(\Gamma)$.

Finally, in Section 4, we show that the product on $\mathcal{A}_T(\Gamma)$ can be "restricted" (see (4.1)) in a natural manner so that the action of the operators $\delta_n \in \mathcal{H}_1$ becomes zero. With this "restricted" product structure, we refer to the algebra $\mathcal{A}_T(\Gamma)$ as $\mathcal{A}_T^r(\Gamma)$. Then $\mathcal{H}_1$ has a natural flat action on the algebra $\mathcal{A}_T^r(\Gamma)$ such that the action of the elements $\delta_n \in \mathcal{H}_1$ is zero. Further, $\mathbb{B}^*(\Gamma)$ becomes a module over $\mathcal{A}_T^r(\Gamma)$ and with this module structure, we refer to $\mathbb{B}^*(\Gamma)$ as $\mathbb{B}_r^*(\Gamma)$. Since the action of each of the operators $\delta_n \in \mathcal{H}_1$ is zero, the flat action of $\mathcal{H}_1$ on the system $(\mathcal{A}_T^r(\Gamma), \mathbb{B}_r^*(\Gamma))$ reduces to an action of the smaller Hopf algebra $\mathfrak{h}_1 = \mathcal{U}(\mathfrak{l}_1)$, which is the universal enveloping algebra of the unique two dimensional nonabelian Lie algebra $\mathfrak{l}_1$. The action of $\mathfrak{h}_1$ is then used to define Rankin–Cohen brackets of all orders on $\mathbb{B}_r^*(\Gamma)$, which naturally extend the classical Rankin–Cohen brackets on modular forms (see Proposition 4.3).

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2. Modular Hecke algebras and the Archimedean complex

In this section, we shall briefly recall the background theory leading to modular Hecke algebras and to Archimedean cohomology. The theory of modular Hecke algebras and the action of the Hopf algebra $\mathcal{H}_1$ due to Connes and Moscovici [3], [2] is presented in Section 2.1. Thereafter, in Section 2.2, we recall the Archimedean cohomology developed by Consani [4]. We also show that the commutation relations between monodromy and Frobenius operators on the Archimedean complex are identical to relations between certain generators of $\mathcal{H}_1$.

2.1. Modular Hecke algebras and the Hopf algebra $\mathcal{H}_1$.

For the convenience of the reader, we shall briefly recall the construction of the modular Hecke algebra of Connes and Moscovici from [2].

We will start by fixing some notation. Throughout this paper, we will use $\Gamma(N)$ to denote the congruence subgroup of $\mathrm{SL}_2(\mathbb{Z})$ of level $N \geq 1$. The group $\mathrm{SL}_2(\mathbb{Z})$ has a well-known left action on the upper half plane $\mathbb{H} = \{z \in \mathbb{C} \mid \mathrm{im}(z) > 0\}$. For any congruence subgroup $\Gamma(N)$ of $\mathrm{SL}_2(\mathbb{Z})$, we set $Y(\Gamma(N)) = \Gamma(N)\backslash\mathbb{H}$. Then $Y(\Gamma(N))$ can be compactified in a standard manner by adding a finite number of points. The compactification is referred to as the modular curve $X(\Gamma(N))$ of level N. The finitely many points in $X(\Gamma(N))\backslash Y(\Gamma(N))$ are known as cusps. We will often denote $X(\Gamma(N))$ simply by $X(N)$.

For any holomorphic function $f\colon \mathbb{H} \to \mathbb{C}$ and any $\gamma \in \mathrm{GL}_2^+(\mathbb{Q})$, we set (for $k \geq 0$)

$$f|_{2k}\gamma = f(z)(cz+d)^{-2k} \quad \text{if } \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

For given $N \geq 1$, set $q = e^{2\pi i z/N}$ and suppose that $f\colon \mathbb{H} \to \mathbb{C}$ is such that $f|_{2k}\gamma = f$ for each $\gamma \in \Gamma(N)$. Then, for any choice of $\log q$, there exists a well-defined function $f_\infty\colon \{q \in \mathbb{C} \mid 0 < |q| < 1\} \to \mathbb{C}$ such that

$$f_\infty(q) = f\big(N\tfrac{\log q}{2\pi i}\big).$$

Then if f_∞ can be continued holomorphically at $q = 0$ (for each choice of $\log q$), we say that f is a modular form of weight $2k$ and level N. Further, if $f_\infty(0) = 0$, we say that f is a cuspidal form of weight $2k$ and level $\Gamma(N)$.

Given any congruence subgroup $\Gamma = \Gamma(N)$, the space of modular (resp. cuspidal) forms of level Γ and weight $2k$ will be denoted by $\mathcal{M}_{2k}(\Gamma)$ (resp. $\mathcal{M}^0_{2k}(\Gamma)$) and we set

$$\mathcal{M}(\Gamma) = \bigoplus_{k\geq 0} \mathcal{M}_{2k}(\Gamma), \quad \mathcal{M}^0(\Gamma) = \bigoplus_{k\geq 0} \mathcal{M}^0_{2k}(\Gamma).$$

If N' is a multiple of N, we have an inclusion morphism $\mathcal{M}(\Gamma(N)) \to \mathcal{M}(\Gamma(N'))$ (resp. $\mathcal{M}^0(\Gamma(N)) \to \mathcal{M}^0(\Gamma(N')))$. We define $\mathcal{M}$ (resp. $\mathcal{M}^0$) to be the direct limit

$$\mathcal{M} = \varinjlim \mathcal{M}(\Gamma(N)), \quad \mathcal{M}^0 = \varinjlim \mathcal{M}^0(\Gamma(N)),$$

and we refer to $\mathcal{M}$ as the "modular tower".

Definition 2.1 (see [2]). Let $\Gamma = \Gamma(N)$ be a congruence subgroup of $\mathrm{SL}_2(\mathbb{Z})$. A Hecke operator form of level Γ is a function

$$F : \Gamma\backslash \mathrm{GL}_2^+(\mathbb{Q}) \to \mathcal{M}, \quad \Gamma\alpha \mapsto F_\alpha \in \mathcal{M},$$

of finite support satisfying the covariance condition

$$F_\alpha|\gamma = F_{\alpha\gamma} \quad \text{for all } \alpha \in \mathrm{GL}_2^+(\mathbb{Q}),\ \gamma \in \Gamma.$$

The Hecke operator form is said to be cuspidal if

$$F_\alpha \in \mathcal{M}^0 \quad \text{for all } \alpha \in \mathrm{GL}_2^+(\mathbb{Q}).$$

The Hecke operator forms of level Γ form an associative algebra $\mathcal{A}(\Gamma)$ (see [2], 1.9) under the product

$$(F^1 * F^2)_\alpha = \sum_{\beta \in \Gamma\backslash \mathrm{GL}_2^+(\mathbb{Q})} F^1_\beta \cdot F^2_{\alpha\beta^{-1}}|\beta \quad \text{for all } F^1, F^2 \in \mathcal{A}(\Gamma), \tag{2.1}$$

where the summation ranges over all the cosets of Γ in $\mathrm{GL}_2^+(\mathbb{Q})$. The usual Hecke algebra $\mathcal{H}(\Gamma)$ is the algebra of functions from the set of double cosets of Γ in $\mathrm{GL}_2^+(\mathbb{Q})$ to $\mathbb{C}$ having finite support. Then $\mathcal{H}(\Gamma)$ embeds into $\mathcal{A}(\Gamma)$ as

$$j : \mathcal{H}(\Gamma) \hookrightarrow \mathcal{A}(\Gamma), \quad j(h)_\alpha = h(\Gamma\alpha\Gamma), \quad \alpha \in \mathrm{GL}_2^+(\mathbb{Q}).$$

The cuspidal Hecke operators form an ideal in $\mathcal{A}(\Gamma)$, which is denoted by $\mathcal{A}^0(\Gamma)$.

Finally, we recall the definition of the Hopf algebra $\mathcal{H}_1$ that acts on the algebra $\mathcal{A}(\Gamma)$. This Hopf algebra $\mathcal{H}_1$ belongs to the family of algebras $\{\mathcal{H}_n\}_{n\geq 1}$ defined by Connes and Moscovici in [1], where it is interpreted as the "Hopf algebra of codimension one foliations". As an algebra, $\mathcal{H}_1$ is the universal enveloping algebra of the Lie algebra $\mathfrak{L}_1$ with generators $X, Y, \delta_n, n \geq 1$, satisfying the relations

$$[Y, X] = X, \quad [Y, \delta_n] = n\delta_n, \quad [X, \delta_n] = \delta_{n+1}, \quad [\delta_k, \delta_l] = 0 \quad \text{for all } k, l \in \mathbb{N}, \tag{2.2}$$

along with the coproducts

$$\begin{aligned}
\Delta(X) &= X \otimes 1 + 1 \otimes X + \delta_1 \otimes Y,\\
\Delta(Y) &= Y \otimes 1 + 1 \otimes Y,\\
\Delta(\delta_1) &= \delta_1 \otimes 1 + 1 \otimes \delta_1
\end{aligned}$$

and the antipode

$$S(Y) = -Y, \quad S(X) = -X + \delta_1 Y, \quad S(\delta_1) = -\delta_1.$$

For any $f \in \mathcal{M}_k$, the operator X is defined as

$$X(f) = \tfrac{1}{2\pi i}\big(\tfrac{d}{dz} f - (1/6)\tfrac{d}{dz}(\log \Delta) Y(f)\big),$$

where Y is the grading operator $Y(f) = \frac{k}{2} f$ and $\Delta(z)$ is the modular discriminant

$$\Delta(z) = (2\pi)^{12} q \prod_{n=1}^{\infty} (1-q^n)^{24}, \quad q = e^{2\pi i z},$$

which is a modular form of weight 12 and level $\Gamma(1) = \mathrm{SL}_2(\mathbb{Z})$. It may be checked that X defines an operator $X\colon \mathcal{M}_k \to \mathcal{M}_{k+2}$. We set

$$\widetilde{X}(f) = (2\pi i) \cdot X(f).$$

Moreover, the operator X (and hence $\widetilde{X}$) determines a derivation on $\mathcal{M}$. Now, given $F \in \mathcal{A}(\Gamma)$, $\mathcal{H}_1$ acts on $\mathcal{A}(\Gamma)$ as follows: for $\alpha \in \mathrm{GL}_2^+(\mathbb{Q})$ and $F \in \mathcal{A}(\Gamma)$ we have

$$X(F)_\alpha = X(F_\alpha), \quad Y(F)_\alpha = Y(F_\alpha), \quad \delta_1(F)_\alpha = \mu_\alpha \cdot F,$$

where $\mu_\alpha = (1/12\pi i)\frac{d}{dz}\big(\log \frac{\Delta|\alpha}{\Delta}\big)$. Note that μ_α measures the difference

$$\mu_\alpha \cdot Y(f) = X(f) - X(f|_k \alpha^{-1})|_{k+2}\alpha, \tag{2.3}$$

whence it follows directly that $\mu_\alpha = 0$ for all $\alpha \in \mathrm{SL}_2(\mathbb{Z})$. For the sake of convenience, set

$$\tilde{\mu}_\alpha = (2\pi i) \cdot \mu_\alpha \quad \text{for all } \alpha \in \mathrm{GL}_2^+(\mathbb{Q}).$$

2.2. Archimedean cohomology and the fibre at infinity. The cohomology of the "fibre at infinity" of an arithmetic variety has been studied by Consani in [4] by means of an Archimedean bicomplex with monodromy and Frobenius operators N and Φ, respectively. The fibre at infinity is a complex manifold and we shall deal with the case where it is a modular curve $X(N)$. The terms K_N^{**} of this bicomplex (K_N^{**}, d', d'') can be expressed as a direct sum of terms $K_N^{i,j} = \bigoplus_{i=0}^{\infty} K_N^{i,j,k}$, where the $K_N^{i,j,k}$ are modules of real differential forms twisted with an appropriate power of $(2\pi i)$, defined by (for any $i, j, k \in \mathbb{Z}$)

$$K_N^{i,j,k} = \begin{cases} \displaystyle\bigoplus_{\substack{a+b=j+1\\ a+b=j+1\\ |a-b|\le 2k-i}} \Omega^{a,b}_{X(N),\mathbb{R}}\big(\tfrac{1+j-i}{2}\big), & k \ge \max\{0, i\}, \\ 0 & \text{otherwise.} \end{cases} \tag{2.4}$$

Here $\Omega^{a,b}_{X(N),\mathbb{R}}$ is the abelian group of real differential forms of type $(a,b)+(b,a)$ on $X(N)$ and, for any $p \in \mathbb{Z}$, $\Omega^{a,b}_{X(N),\mathbb{R}}(p)$ refers to the p-th Tate twist of $\Omega^{a,b}_{X(N),\mathbb{R}}$, i.e., $\Omega^{a,b}_{X(N),\mathbb{R}}(p) = (2\pi i)^p \Omega^{a,b}_{X(N),\mathbb{R}}$. The differentials are as follows: given $\omega \in K_N^{i,j,k}$, we set

$$d' \colon K_N^{i,j,k} \to K_N^{i+1,j+1,k+1}, \quad \omega \mapsto (\partial + \bar{\partial})(\omega),$$
$$d'' \colon K_N^{i,j,k} \to K_N^{i+1,j+1,k}, \quad \omega \mapsto \sqrt{-1}(\bar{\partial} - \partial)(\omega).$$

The complex (K_N^{**}, d', d'') is equipped with a "monodromy operator" N and "a Frobenius operator Φ", defined (see [5], 3.2) by

$$N \colon K_N^{i,j,k} \to K_N^{i+2,j,k+1}, \quad \omega \mapsto (2\pi i)^{-1}\omega,$$
$$\Phi \colon K_N^{i,j,k} \to K_N^{i,j,k}, \quad \omega \mapsto \tfrac{1+j-i}{2}\omega.$$

We note that these operators satisfy the relation $[-\Phi, N] = N$. For more details on the complex (K_N^{**}, d', d''), see [6], [5], [4].

In Section 3, we shall tensor the objects $K_N^{i,j,k}$ with modular forms of appropriate weights to define modules $\mathcal{K}_N^{i,j,k}$ (see (3.1)) and consider the direct limit

$$\mathcal{K}^{i,j,k} = \varinjlim_N \mathcal{K}_N^{i,j,k}.$$

Then, by replacing the modular tower $\mathcal{M}$ in Definition 2.1 by the direct limit $\mathcal{K}^* = \bigoplus_{i,j,k\in\mathbb{Z}} \mathcal{K}^{i,j,k}$, we define an object $\mathbb{B}^*(\Gamma)$ that replaces the modular Hecke algebra $\mathcal{A}(\Gamma)$ in our theory.

Our basic motivation is to compare the relation $[-\Phi, N] = N$ between the monodromy and Frobenius operators on the Archimedean complex to the relation $[Y, X] = X$ (see (2.2)) between the generators of the Hopf algebra $\mathcal{H}_1$. As described in Section 2.1, the generators Y and X act as operators on the modular Hecke algebra $\mathcal{A}(\Gamma)$. Therefore, we shall describe an action of $\mathcal{H}_1$ on $\mathbb{B}^*(\Gamma)$ such that $Y \in \mathcal{H}_1$ acts as (the negative of) the Frobenius Φ on $\mathbb{B}^*(\Gamma)$ and $X \in \mathcal{H}_1$ acts as the monodromy N on $\mathbb{B}^*(\Gamma)$.

3. The Archimedean complex with Hopf algebra action

In this section we will maintain all the notation introduced in Section 2. Let $X(N)$, $N \geq 1$ denote the N-th modular curve. For any nonnegative integers a, b and any $r \in \mathbb{Z}$, we denote by $\Omega^{a,b}_{X(N),\mathbb{R}}$ the module of real differentials of type $(a,b)+(b,a)$ on $X(N)$ and for any $r \in \mathbb{Z}$, $\Omega^{a,b}_{X(N),\mathbb{R}}(r)$ refers to the r-th Tate twist of $\Omega^{a,b}_{X(N),\mathbb{R}}$, i.e., $\Omega^{a,b}_{X(N),\mathbb{R}}(r) = (2\pi i)^r \Omega^{a,b}_{X(N),\mathbb{R}}$. We set (for any $i, j, k \in \mathbb{Z}$)

$$\mathcal{K}_N^{i,j,k} = \Big(\bigoplus_{l \geq i-j-1} \mathcal{M}_l(\Gamma(N)) \Big) \otimes_{\mathbb{R}} \bigoplus_{\substack{a \leq b \\ a+b=j+1 \\ |a-b| \leq 2k-i}} \Omega^{a,b}_{X(N),\mathbb{R}}\big(\tfrac{1+j-i}{2}\big). \tag{3.1}$$

Then, in the notation of Section 2.2, $\mathcal{K}_N^{i,j,k} = \bigoplus_{l\geq i-j-1} \mathcal{M}_l(\Gamma(N)) \otimes_{\mathbb{R}} K_N^{i,j,k}$. We define the two differentials on $\mathcal{K}_N^{i,j,k}$ as follows: given $f \otimes \omega \in \mathcal{K}_N^{i,j,k}$, with $f \in \bigoplus_{l\geq i-j-1} \mathcal{M}_l(\Gamma(N))$ and $\omega \in K_N^{i,j,k}$, we set

$$d'\colon \mathcal{K}_N^{i,j,k} \to \mathcal{K}_N^{i+1,j+1,k+1}, \quad (f\otimes\omega) \mapsto f \otimes (\partial + \bar{\partial})(\omega),$$
$$d''\colon \mathcal{K}_N^{i,j,k} \to \mathcal{K}_N^{i+1,j+1,k}, \quad (f\otimes\omega) \mapsto f \otimes \sqrt{-1}(\bar{\partial} - \partial)(\omega).$$

For any integers $N, N' \geq 1$, the projections $p\colon X(NN') \to X(N)$ induce morphisms (for each $l \geq i-j-1$)

$$\mathcal{M}_l(\Gamma(N)) \otimes \Omega^{a,b}_{X(N),\mathbb{R}}(\tfrac{1+j-i}{2}) \to \mathcal{M}_l(\Gamma(NN')) \otimes \Omega^{a,b}_{X(NN'),\mathbb{R}}(\tfrac{1+j-i}{2})$$

by tensoring pullback maps p^* on differential forms with the inclusions $\mathcal{M}_l(\Gamma(N)) \hookrightarrow \mathcal{M}_l(\Gamma(NN'))$. We define $\mathcal{K}^{i,j,k}$ to be the colimit

$$\mathcal{K}^{i,j,k} = \varinjlim_N \mathcal{K}_N^{i,j,k}$$

of this system and set $\mathcal{K}^{i,j} = \bigoplus_k \mathcal{K}^{i,j,k}$, $\mathcal{K}^* = \bigoplus_{i+j=*} \mathcal{K}^{i,j}$.

Remark 3.1. We have already noted that $\mathcal{K}_N^{i,j,k} = \bigoplus_{l\geq i-j-1} \mathcal{M}_l(\Gamma(N)) \otimes_{\mathbb{R}} K_N^{i,j,k}$. The term $\mathcal{K}_N^{i,j,k}$ is defined by tensoring the term $K_N^{i,j,k}$ in the Archimedean complex for the modular curve $X(\Gamma(N))$ with modular forms of level $\Gamma(N)$. Further, from (2.4), we know that the Tate twist appearing in the term $K_N^{i,j,k}$ of the Archimedean complex for the modular curve $X(N)$ is $(\frac{1+j-i}{2})$. We view the Tate twist as indicating the "weight" of the term $K_N^{i,j,k}$, which suggests that the term $K_N^{i,j,k}$ be tensored with modular forms of weight $-(1+j-i)$ and above to form the term $\mathcal{K}_N^{i,j,k} = \bigoplus_{l\geq i-j-1} \mathcal{M}_l(\Gamma(N)) \otimes_{\mathbb{R}} K_N^{i,j,k}$, thus forming an "enriched Archimedean complex".

We shall now define the module $\mathbb{B}^*(\Gamma)$ which replaces the modular Hecke algebra $\mathcal{A}(\Gamma)$ in our theory.

Definition 3.2. For a congruence subgroup $\Gamma \subset \mathrm{SL}_2(\mathbb{Z})$ and any integers $i, j, k \in \mathbb{Z}$, define $\mathbb{B}^{i,j,k}(\Gamma)$ to be the set of all functions of finite support

$$F\colon \Gamma\backslash \mathrm{GL}_2^+(\mathbb{Q}) \to \mathcal{K}^{i,j,k}$$

satisfying the following covariance condition: If $F_\alpha = \sum_{l=1}^m f_l \otimes \omega_l \in \mathcal{K}^{i,j,k}$, then

$$F_{\alpha\gamma} = \sum_{l=1}^m f_l|\gamma \otimes \omega_l \quad \text{for all } \gamma \in \Gamma,$$

where F_α denotes the function F evaluated on the coset $\Gamma\alpha$ for each $\alpha \in \mathrm{GL}_2^+(\mathbb{Q})$. Again, we can make $\mathbb{B}^{i,j,k}(\Gamma)$ into a complex by defining differentials

$$d'\colon \mathbb{B}^{i,j,k}(\Gamma) \to \mathbb{B}^{i+1,j+1,k+1}(\Gamma), \quad d'(F)_\alpha = d'(F_\alpha),$$
$$d''\colon \mathbb{B}^{i,j,k}(\Gamma) \to \mathbb{B}^{i+1,j+1,k}(\Gamma), \quad d''(F)_\alpha = d''(F_\alpha).$$

Define $\mathbb{B}^{i,j}(\Gamma) = \bigoplus_k \mathbb{B}^{i,j,k}(\Gamma)$ for each pair of integers i, j, and let $\mathbb{B}^*(\Gamma) = \bigoplus_{i+j=*} \mathbb{B}^{i,j}(\Gamma)$. This gives us a complex $(\mathbb{B}^*(\Gamma), d' + d'')$.

For the sake of simplicity, we shall frequently use the expression $f_\alpha \otimes \omega_\alpha$ to denote the sum $F_\alpha = \sum_{l=1}^k f_i \otimes \omega_i$ for any $F \in \mathbb{B}^*(\Gamma)$ and any $\alpha \in \mathrm{GL}_2^+(\mathbb{Q})$.

Our next step is to define an algebra $\mathcal{A}_T(\Gamma)$ which is a variant of the modular Hecke algebra $\mathcal{A}(\Gamma)$ of Connes and Moscovici [2]. We will show that $\mathbb{B}^*(\Gamma)$ is a module over $\mathcal{A}_T(\Gamma)$ and that $\mathcal{H}_1$ acts on both $\mathcal{A}_T(\Gamma)$ and $\mathbb{B}^*(\Gamma)$, and that the action is well behaved (or "flat") in a sense we will make precise in Definition 3.11.

Definition 3.3. Let $\Gamma \subset \mathrm{SL}_2(\mathbb{Z})$ be a congruence subgroup and let $\mathbb{R}[T]$ denote the polynomial ring in one variable over $\mathbb{R}$. Denote by $\mathcal{A}_T(\Gamma)$ the set of all functions of finite support

$$G\colon \Gamma\backslash G_2^+(\mathbb{Q}) \to \mathcal{M} \otimes_{\mathbb{R}} \mathbb{R}[T]$$

satisfying the following covariance condition: If $G_\alpha = \sum_{l=1}^m g_l \otimes \varepsilon_l \in \mathcal{M} \otimes_{\mathbb{R}} \mathbb{R}[T]$, then

$$G_{\alpha\gamma} = \sum_{l=1}^m g_l|\gamma \otimes \varepsilon_l$$

for any $\gamma \in \Gamma$. For simplicity, we will forgo the summation signs and write the sum $G_\alpha = \sum_{l=1}^m g_l \otimes \varepsilon_l$ simply as $g_\alpha \otimes \varepsilon_\alpha$. Also we define the submodule $\mathcal{A}_T^0(\Gamma)$ of all functions in $\mathcal{A}_T(\Gamma)$ whose values lie in the cuspidal part $\mathcal{M}^0 \otimes \mathbb{R}[T]$.

Remark 3.4. Although the algebra $\mathcal{A}_T(\Gamma)$ is isomorphic to $\mathcal{A}(\Gamma)[T]$, the polynomial ring in one variable over $\mathcal{A}(\Gamma)$, we maintain the separate notation $\mathcal{A}_T(\Gamma)$. This is done in order to avoid the following confusion: the action of the algebra $\mathcal{H}_1$ on $\mathcal{A}(\Gamma)$ extends naturally to the polynomial ring $\mathcal{A}(\Gamma)[T]$; for instance the action of $X \in \mathcal{H}_1$ extends to $\mathcal{A}(\Gamma)[T]$ since

$$X\Big(\sum_{k=1}^N G_k T^k\Big) = \sum_{k=1}^N X(G_k)T^k, \quad G_k \in \mathcal{A}(\Gamma),\ 1 \le k \le N. \tag{3.2}$$

However, the action (3.2) is not the action of $X \in \mathcal{H}_1$ that we intend to use (see (3.4)).

We will show that $\mathbb{B}^*(\Gamma)$ is a module over $\mathcal{A}_T(\Gamma)$, that $\mathcal{H}_1$ acts on both $\mathcal{A}_T(\Gamma)$ and $\mathbb{B}^*(\Gamma)$, and that the action is well behaved (or "flat") in the sense of (1.1).

Consider the module $\mathcal{M} \otimes_{\mathbb{R}} \mathbb{R}[T]$ and the following maps: given $g \in \mathcal{M}$, $\varepsilon \in \mathbb{R}[T]$ and $\rho \in \mathrm{GL}_2^+(\mathbb{Q})$, we define functions

$$\begin{aligned} \psi_{g\otimes\varepsilon}\colon \mathcal{M} \otimes \mathbb{R}[T] &\to \mathcal{M} \otimes \mathbb{R}[T], \quad f' \otimes \varepsilon' \mapsto g \cdot f' \otimes \varepsilon\varepsilon', \\ T_\rho\colon \mathcal{M} \otimes \mathbb{R}[T] &\to \mathcal{M} \otimes \mathbb{R}[T], \quad f' \otimes \varepsilon' \mapsto f'|\rho \otimes \varepsilon', \end{aligned}$$

for any $f' \otimes \varepsilon' \in \mathcal{M} \otimes_{\mathbb{R}} \mathbb{R}[T]$. Let $G \in \mathcal{A}_T(\Gamma)$, with $G_\rho = g_\rho \otimes \varepsilon_\rho$ for each $\rho \in \mathrm{GL}_2^+(\mathbb{Q})$. If $g_\rho \otimes \varepsilon_\rho = G_\rho$ denotes the finite sum $\sum_{i=1}^k g_i \otimes \varepsilon_i$ in $\mathcal{M} \otimes_{\mathbb{R}} \mathbb{R}[T]$ with $g_i \in \mathcal{M}$, $\varepsilon_i \in \mathbb{R}[T]$, we use $\psi_{g_\rho\otimes\varepsilon_\rho}$ to denote the sum $\sum_{i=1}^k \psi_{g_i\otimes\varepsilon_i}$.

Proposition 3.5. *Let $\Gamma \subset \mathrm{SL}_2(\mathbb{Z})$ be a congruence subgroup. Then:*

(1) *$\mathcal{A}_T(\Gamma)$ is an associative algebra: Given $G, G' \in \mathcal{A}_T(\Gamma)$, with $G_\rho = g_\rho \otimes \varepsilon_\rho$ and $G'_\rho = g'_\rho \otimes \varepsilon'_\rho$ for each $\rho \in \mathrm{GL}_2^+(\mathbb{Q})$, the product structure is given by*

$$(G * G')_\alpha = \sum_{\beta\in\Gamma\backslash \mathrm{GL}_2^+(\mathbb{Q})} (g_\beta \cdot g'_{\alpha\beta^{-1}}|\beta) \otimes \varepsilon_\beta \varepsilon'_{\alpha\beta^{-1}}. \tag{3.3}$$

(2) *The Hopf algebra $\mathcal{H}_1$ acts on $\mathcal{A}_T(\Gamma)$ as follows:*

$$\begin{aligned} X(G)_\alpha &= \widetilde{X}(g_\alpha) \otimes T \cdot \varepsilon_\alpha, \\ Y(G)_\alpha &= Y(g_\alpha) \otimes \varepsilon_\alpha, \\ \delta_1(G)_\alpha &= \tilde{\mu}_\alpha \cdot g_\alpha \otimes T \cdot \varepsilon_\alpha. \end{aligned} \tag{3.4}$$

Moreover, the action of $\mathcal{H}_1$ on $\mathcal{A}_T(\Gamma)$ is flat in the sense that, given $G, G' \in \mathcal{A}_T(\Gamma)$ and $h \in \mathcal{H}_1$, with $\Delta(h) = \sum h_{(1)} \otimes h_{(2)}$, we have

$$h(G * G') = \sum h_{(1)}(G) * h_{(2)}(G').$$

Proof. (1) We choose any $G, G' \in \mathcal{A}_T(\Gamma)$, with $G_\rho = g_\rho \otimes \varepsilon_\rho$ and $G'_\rho = g'_\rho \otimes \varepsilon'_\rho$ for each $\rho \in \mathrm{GL}_2^+(\mathbb{Q})$. Then, from (3.3), we have

$$\begin{aligned} (G * G')_{\gamma\alpha} &= \sum_{\beta\in\Gamma\backslash \mathrm{GL}_2^+(\mathbb{Q})} (g_\beta \cdot g'_{\gamma\alpha\beta^{-1}}|\beta) \otimes \varepsilon_\beta \varepsilon'_{\gamma\alpha\beta^{-1}} \\ &= \sum_{\beta\in\Gamma\backslash \mathrm{GL}_2^+(\mathbb{Q})} \psi_{g_\beta\otimes\varepsilon_\beta} \circ T_\beta(g'_{\gamma\alpha\beta^{-1}} \otimes \varepsilon'_{\gamma\alpha\beta^{-1}}) \\ &= \sum_{\beta\in\Gamma\backslash \mathrm{GL}_2^+(\mathbb{Q})} \psi_{g_\beta\otimes\varepsilon_\beta} \circ T_\beta(g'_{\alpha\beta^{-1}} \otimes \varepsilon'_{\alpha\beta^{-1}}) \\ &= (G * G')_\alpha \end{aligned}$$

for any $\gamma \in \Gamma$. It follows that

$$\begin{aligned} (g_{\gamma\beta} \cdot g'_{\alpha\beta^{-1}\gamma^{-1}}|\gamma\beta) \otimes \varepsilon_{\gamma\beta}\varepsilon'_{\alpha\beta^{-1}\gamma^{-1}} &= \psi_{g_{\gamma\beta}\otimes\varepsilon_{\gamma\beta}} \circ T_{\gamma\beta}(g'_{\alpha\beta^{-1}\gamma^{-1}} \otimes \varepsilon'_{\alpha\beta^{-1}\gamma^{-1}}) \\ &= \psi_{g_\beta\otimes\varepsilon_\beta} \circ T_{\gamma\beta} \circ T_{\gamma^{-1}}(g'_{\alpha\beta^{-1}} \otimes \varepsilon'_{\alpha\beta^{-1}}) \\ &= (g_\beta \cdot g'_{\alpha\beta^{-1}}|\beta) \otimes \varepsilon_\beta\varepsilon'_{\alpha\beta^{-1}} \end{aligned}$$

for any $\beta \in \mathrm{GL}_2^+(\mathbb{Q})$ and $\gamma \in \Gamma$. Hence the expression for $(G * G')_\alpha$ in (3.3) is well defined and independent of the choice of coset representatives. Finally, we check the covariance condition:

$$\begin{aligned}
(G * G')_{\alpha\gamma} &= \sum_{\beta \in \Gamma \backslash \mathrm{GL}_2^+(\mathbb{Q})} (g_\beta \cdot g'_{\alpha\gamma\beta^{-1}}|\beta) \otimes \varepsilon_\beta \varepsilon'_{\alpha\gamma\beta^{-1}} \\
&= \sum_{\delta \in \Gamma \backslash \mathrm{GL}_2^+(\mathbb{Q})} (g_{\delta\gamma} \cdot g'_{\alpha\delta^{-1}}|\delta\gamma) \otimes \varepsilon_{\delta\gamma} \varepsilon'_{\alpha\delta^{-1}} \quad (\beta = \delta\gamma) \\
&= \sum_{\delta \in \Gamma \backslash \mathrm{GL}_2^+(\mathbb{Q})} \psi_{g_{\delta\gamma} \otimes \varepsilon_{\delta\gamma}} \circ T_{\delta\gamma}(g'_{\alpha\delta^{-1}} \otimes \varepsilon'_{\alpha\delta^{-1}}) \\
&= \sum_{\delta \in \Gamma \backslash \mathrm{GL}_2^+(\mathbb{Q})} (g_\delta \cdot g'_{\alpha\delta^{-1}})|\gamma \otimes \varepsilon_\delta \varepsilon'_{\alpha\delta^{-1}}.
\end{aligned}$$

(2) We check this on the generators. It is easy to check the Lie algebra relations $[Y, X] = X$, $[Y, \delta_1] = \delta_1$ and $[\delta_k, \delta_l] = 0 \ \forall\, k, l \in \mathbb{N}$ between the operators on $\mathcal{A}_T(\Gamma)$. We check the coproduct relations. We know that $\Delta X = X \otimes 1 + 1 \otimes X + \delta_1 \otimes Y$. Choose $G, G' \in \mathcal{A}_T(\Gamma)$. Then

$$\begin{aligned}
X(G * G')_\alpha &= \sum_{\beta \in \Gamma \backslash \mathrm{GL}_2^+(\mathbb{Q})} \tilde{X}(g_\beta \cdot g'_{\alpha\beta^{-1}}|\beta) \otimes T \cdot \varepsilon_\beta \varepsilon'_{\alpha\beta^{-1}} \\
&= \sum_{\beta \in \Gamma \backslash \mathrm{GL}_2^+(\mathbb{Q})} \tilde{X}(g_\beta) \cdot g'_{\alpha\beta^{-1}}|\beta \otimes T \cdot \varepsilon_\beta \varepsilon'_{\alpha\beta^{-1}} \\
&\quad + \sum_{\beta \in \Gamma \backslash \mathrm{GL}_2^+(\mathbb{Q})} g_\beta \cdot \tilde{X}(g'_{\alpha\beta^{-1}}|\beta) \otimes T \cdot \varepsilon_\beta \varepsilon'_{\alpha\beta^{-1}} \\
&= (X(G) * G')_\alpha + \sum_{\beta \in \Gamma \backslash \mathrm{GL}_2^+(\mathbb{Q})} (g_\beta \cdot \tilde{X}(g'_{\alpha\beta^{-1}})|\beta) \otimes T \cdot \varepsilon_\beta \varepsilon'_{\alpha\beta^{-1}} \\
&\quad - \sum_{\beta \in \Gamma \backslash \mathrm{GL}_2^+(\mathbb{Q})} (g_\beta (\tilde{\mu}_{\beta^{-1}} \cdot Y(g'_{\alpha\beta^{-1}}))|\beta) \otimes T \cdot \varepsilon_\beta \varepsilon'_{\alpha\beta^{-1}} \\
&= (X(G) * G')_\alpha + (G * X(G'))_\alpha \\
&\quad + \sum_{\beta \in \Gamma \backslash \mathrm{GL}_2^+(\mathbb{Q})} (\tilde{\mu}_\beta \cdot g_\beta) \cdot Y(g'_{\alpha\beta^{-1}})|\beta \otimes T \cdot \varepsilon_\beta \varepsilon'_{\alpha\beta^{-1}} \\
&\qquad\qquad (\text{as } \tilde{\mu}_{\beta^{-1}\beta} = 0 = \tilde{\mu}_{\beta^{-1}}|\beta + \tilde{\mu}_\beta) \\
&= (X(G) * G')_\alpha + (G * X(G'))_\alpha + (\delta_1(G) * Y(G'))_\alpha.
\end{aligned}$$

Further, we know that $\Delta Y = Y \otimes 1 + 1 \otimes Y$ and $\Delta\delta_1 = \delta_1 \otimes 1 + 1 \otimes \delta_1$. It can be checked that both Y and δ_1 are derivations on the algebra $\mathcal{A}_T(\Gamma)$, and so the action of $\mathcal{H}_1$ on $\mathcal{A}_T(\Gamma)$ is flat. □

Corollary 3.6. (1) *$\mathcal{A}_T^0(\Gamma)$ is an ideal in $\mathcal{A}_T(\Gamma)$, which we shall refer to as the cuspidal ideal.*

(2) *The cuspidal ideal $\mathcal{A}_T^0(\Gamma)$ is invariant under the action of $\mathcal{H}_1$.*

Proof. (1) follows directly from the definition of the product in (3.3).

(2) From [2], we know that $\widetilde{X}$ preserves cuspidal modular forms. From the expression (3.4) for the actions of the generators X, Y and δ_1 on $\mathcal{A}_T(\Gamma)$, it is now clear that $\mathcal{A}_T^0(\Gamma)$ is preserved by the action of $\mathcal{H}_1$. □

For any $\alpha \in \mathrm{GL}_2^+(\mathbb{Q})$ and the element $g \otimes T^l \in \mathcal{M} \otimes \mathbb{R}[T]$, we define functions

$$\begin{aligned} \phi_{g\otimes T^l} \colon \mathcal{K}^{i,j,k} &\to \mathcal{K}^{i+2l,j,k+l}, \quad f \otimes \omega \mapsto g \cdot f \otimes T^l \cdot \omega := f \otimes (2\pi i)^{-l}\omega, \\ T_\alpha \colon \mathcal{K}^{i,j,k} &\to \mathcal{K}^{i,j,k}, \qquad\quad f \otimes \omega \mapsto f|\alpha \otimes \omega, \end{aligned} \tag{3.5}$$

for any $f \otimes \omega \in \mathcal{K}^{i,j,k}$. Here it is understood that the image of $\phi_{g\otimes T^l}$ in (3.5) is zero if $\mathcal{K}^{i+2l,j,k+l} = 0$. Let $G \in \mathcal{A}_T(\Gamma)$, with $G_\rho = g_\rho \otimes \varepsilon_\rho$ for each $\rho \in \mathrm{GL}_2^+(\mathbb{Q})$. If $g_\rho \otimes \varepsilon_\rho = G_\rho$ denotes the finite sum $\sum_{i=1}^k g_i \otimes T^{l_i}$ in $\mathcal{M} \otimes_{\mathbb{R}} \mathbb{R}[T]$ with $g_i \in \mathcal{M}$, $l_i \in Z$, we use $\phi_{g_\rho\otimes\varepsilon_\rho}$ to denote the sum $\sum_{i=1}^k \phi_{g_i\otimes T^{l_i}}$.

Proposition 3.7. *Let Γ be a congruence subgroup of $\mathrm{SL}_2(\mathbb{Z})$ and let $G \in \mathcal{A}_T(\Gamma)$, $F \in \mathbb{B}^*(\Gamma)$. Let $G_\alpha = g_\alpha \otimes \varepsilon_\alpha$ for any $\alpha \in \mathrm{GL}_2^+(\mathbb{Q})$. Let $F \in \mathbb{B}^{i,j,k}(\Gamma)$ for some $i, j, k \in \mathbb{Z}$ and let $F_\alpha = f_\alpha \otimes \omega_\alpha$ for each $\alpha \in \mathrm{GL}_2^+(\mathbb{Q})$. Then we have a module action of $\mathcal{A}_T(\Gamma)$ on $\mathbb{B}^*(\Gamma)$ defined as*

$$\begin{aligned} (G * F)_\alpha &= \sum_{\beta\in\Gamma\backslash \mathrm{GL}_2^+(\mathbb{Q})} \phi_{g_\beta\otimes\varepsilon_\beta} \circ T_\beta(f_{\gamma\alpha\beta^{-1}} \otimes \omega_{\gamma\alpha\beta^{-1}}) \\ &= \sum_{\beta\in\Gamma\backslash \mathrm{GL}_2^+(\mathbb{Q})} (g_\beta \cdot f_{\alpha\beta^{-1}}|\beta) \otimes \varepsilon_\beta \cdot \omega_{\alpha\beta^{-1}}, \end{aligned} \tag{3.6}$$

where the right-hand side of (3.6) belongs to the direct sum $\bigoplus_{l=0}^\infty \mathbb{B}^{i+2l,j,k+l}(\Gamma)$. (It is understood that if any of the summands $\mathbb{B}^{i+2l,j,k+l}(\Gamma)$ vanishes, the corresponding term on the right-hand side is taken to be zero.)

Proof. To prove that the module action is well defined, we check that, for $\gamma \in \Gamma$,

$$\begin{aligned} (G * F)_{\gamma\alpha} &= \sum_{\beta\in\Gamma\backslash \mathrm{GL}_2^+(\mathbb{Q})} (g_\beta \cdot f_{\gamma\alpha\beta^{-1}}|\beta) \otimes \varepsilon_\beta \cdot \omega_{\gamma\alpha\beta^{-1}} \\ &= \sum_{\beta\in\Gamma\backslash \mathrm{GL}_2^+(\mathbb{Q})} \phi_{g_\beta\otimes\varepsilon_\beta} \circ T_\beta(f_{\gamma\alpha\beta^{-1}} \otimes \omega_{\gamma\alpha\beta^{-1}}) \\ &= \sum_{\beta\in\Gamma\backslash \mathrm{GL}_2^+(\mathbb{Q})} \phi_{g_\beta\otimes\varepsilon_\beta} \circ T_\beta(f_{\alpha\beta^{-1}} \otimes \omega_{\alpha\beta^{-1}}) \\ &= \sum_{\beta\in\Gamma\backslash \mathrm{GL}_2^+(\mathbb{Q})} g_\beta \cdot f_{\alpha\beta^{-1}}|\beta \otimes \varepsilon_\beta \cdot \omega_{\alpha\beta^{-1}} = (G * F)_\alpha. \end{aligned}$$

This action is also independent of the choice of coset representatives β, i.e.,

$$\begin{aligned}(g_{\gamma\beta}\cdot f_{\alpha\beta^{-1}\gamma^{-1}}|\gamma\beta)\otimes\varepsilon_{\gamma\beta}\cdot\omega_{\alpha\beta^{-1}\gamma^{-1}} &= (g_\beta\cdot f_{\alpha\beta^{-1}\gamma^{-1}}|\gamma\beta)\otimes\varepsilon_\beta\cdot\omega_{\alpha\beta^{-1}\gamma^{-1}}\\ &= \phi_{g_\beta\otimes\varepsilon_\beta}\circ T_{\gamma\beta}(f_{\alpha\beta^{-1}\gamma^{-1}}\otimes\omega_{\alpha\beta^{-1}\gamma^{-1}})\\ &= \phi_{g_\beta\otimes\varepsilon_\beta}\circ T_{\gamma\beta}\circ T_{\gamma^{-1}}(f_{\alpha\beta^{-1}}\otimes\omega_{\alpha\beta^{-1}})\\ &= \phi_{g_\beta\otimes\varepsilon_\beta}\circ T_{\beta}(f_{\alpha\beta^{-1}}\otimes\omega_{\alpha\beta^{-1}})\\ &= g_\beta\cdot f_{\alpha\beta^{-1}}|\beta\otimes\varepsilon_\beta\cdot\omega_{\alpha\beta^{-1}}.\end{aligned}$$

Finally, we check the covariance condition, for $\gamma\in\Gamma$,

$$\begin{aligned}(G*F)_{\alpha\gamma} &= \sum_{\beta\in\Gamma\backslash \mathrm{GL}_2^+(\mathbb{Q})}(g_\beta\cdot f_{\alpha\gamma\beta^{-1}}|\beta\otimes\varepsilon_\beta\cdot\omega_{\alpha\gamma\beta^{-1}})\\ &= \sum_{\delta\in\Gamma\backslash \mathrm{GL}_2^+(\mathbb{Q})}(g_{\delta\gamma}\cdot f_{\alpha\delta^{-1}}|\delta\gamma\otimes\varepsilon_{\delta\gamma}\cdot\omega_{\alpha\delta^{-1}})\quad(\beta=\delta\gamma)\\ &= \sum_{\delta\in\Gamma\backslash \mathrm{GL}_2^+(\mathbb{Q})}\phi_{g_{\delta\gamma}\otimes\varepsilon_{\delta\gamma}}(f_{\alpha\delta^{-1}}|\delta\gamma\otimes\omega_{\alpha\delta^{-1}})\\ &= \sum_{\delta\in\Gamma\backslash \mathrm{GL}_2^+(\mathbb{Q})}(g_\delta\cdot f_{\alpha\delta^{-1}}|\delta)|\gamma\otimes\varepsilon_\delta\cdot\omega_{\alpha\delta^{-1}}.\end{aligned}$$

It follows from (3.5) and (3.6) that the product lies entirely in the direct sum $\bigoplus_{l=0}^{\infty}\mathbb{B}^{i+2l,j,k+l}(\Gamma)$. □

Proposition 3.8. *The Hopf algebra $\mathcal{H}_1$ acts on the module $\mathbb{B}^*(\Gamma)$ as follows: Let $F\in\mathbb{B}^{i,j,k}(\Gamma)$ be such that $F_\alpha=f_\alpha\otimes\omega_\alpha$ for any $\alpha\in \mathrm{GL}_2^+(\mathbb{Q})$. Then*

$$\begin{aligned}&X:\mathbb{B}^{i,j,k}(\Gamma)\to\mathbb{B}^{i+2,j,k+1}(\Gamma), && X(F)_\alpha=\widetilde{X}(f_\alpha)\otimes(2\pi i)^{-1}\omega_\alpha,\\ &Y:\mathbb{B}^{i,j,k}(\Gamma)\to\mathbb{B}^{i,j,k}(\Gamma), && Y(F)_\alpha=Y(f_\alpha)\otimes\omega_\alpha,\\ &\delta_1:\mathbb{B}^{i,j,k}(\Gamma)\to\mathbb{B}^{i+2,j,k+1}(\Gamma), && \delta_1(F)_\alpha=\tilde{\mu}_\alpha\cdot f_\alpha\otimes(2\pi i)^{-1}\omega_\alpha.\end{aligned}\tag{3.7}$$

Proof. From (3.7) it follows that

$$\begin{aligned}YX(F)_\alpha &= Y(\widetilde{X}(f_\alpha))\otimes(2\pi i)^{-1}\omega_\alpha,\\ XY(F)_\alpha &= \widetilde{X}(Y(f_\alpha))\otimes(2\pi i)^{-1}\omega_\alpha.\end{aligned}\tag{3.8}$$

Since $[Y,\widetilde{X}]=\widetilde{X}$ on the modular tower $\mathcal{M}$, it follows from (3.8) that $[Y,X]=X$ as operators on $\mathbb{B}^*(\Gamma)$. Similarly, we can check that $[Y,\delta_1]=\delta_1$. The action of the operators δ_n for $n>1$ is determined by the relation $[X,\delta_n]=\delta_{n+1}$. Note that the relation

$$\delta_n(F_\alpha)=\sum\widetilde{X}^{n-1}(\tilde{\mu}_\alpha)\cdot f_\alpha\otimes(2\pi i)^{-n}\omega_\alpha\in\mathbb{B}^{i+2n,j,k+n}(\Gamma)\tag{3.9}$$

holds for $n = 1$. If (3.9) holds for n, then

$$\begin{aligned}\delta_{n+1}(F)_\alpha &= X\delta_n(F)_\alpha - \delta_n X(F)_\alpha \\ &= X(\widetilde{X}^{n-1}(\tilde{\mu}_\alpha)\cdot f_\alpha \otimes (2\pi i)^{-n}\omega_\alpha) \\ &\quad - (\widetilde{X}^{n-1}(\tilde{\mu}_\alpha)\cdot \widetilde{X}(f_\alpha)\otimes (2\pi i)^{-n-1}\omega_\alpha) \\ &= \widetilde{X}^n(\tilde{\mu}_\alpha)\cdot f_\alpha \otimes (2\pi i)^{-n-1}\omega_\alpha,\end{aligned}$$

which proves the result for all n by induction. From the expression (3.9), it is now obvious that $[\delta_k, \delta_l] = 0$ for all $k, l \in \mathbb{N}$. Hence $\mathbb{B}^*(\Gamma)$ carries an action of the Hopf algebra $\mathcal{H}_1$. □

Remark 3.9. Note that the operator X of Proposition 3.8 is a "composite" of the monodromy operator N on the Archimedean complex and the derivation X on modular forms. Further, we have explained in Remark 3.1 that the weight of the modular forms appearing in the expression $\mathcal{K}_N^{i,j,k} = \bigoplus_{l\geq i-j-1} \mathcal{M}_l(\Gamma(N))\otimes_{\mathbb{R}} K_N^{i,j,k}$ is related to the Tate twist $(\frac{1+j-i}{2})$ appearing in $K_N^{i,j,k}$. On the term $K_N^{i,j,k}$ of the Archimedean complex , the Frobenius operator Φ is defined to be $\Phi(x) = (\frac{1+j-i}{2})x$. Moreover, the grading operator on modular forms acts on the module $\mathcal{M}_{i-j-1}(\Gamma(N))$ (which appears in the first term of the direct sum $\mathcal{K}_N^{i,j,k} = \bigoplus_{l\geq i-j-1} \mathcal{M}_l(\Gamma(N))\otimes_{\mathbb{R}} K_N^{i,j,k}$) by multiplication with $-(\frac{1+j-i}{2})$. Hence, the definition of the operator Y reflects both the grading operator on modular forms and $-\Phi$ on the Archimedean complex. One can check that the operators Φ and N on the Archimedean complex satisfy the relation $[-\Phi, N] = N$, which leads to the comparison with the commutator relation $[Y, X] = X$ for operators X and Y on modular forms as explained above.

Corollary 3.10. *The action of the operators $X, Y, \delta_n \in \mathcal{H}_1$, $n \geq 1$, on $\mathbb{B}^*(\Gamma)$ commutes with the differentials d' and d''.*

Proof. For any $i, j, k \in \mathbb{Z}$, take $F \in \mathbb{B}^{i,j,k}(\Gamma)$ with $F_\alpha = f_\alpha \otimes \omega_\alpha$ for any $\alpha \in \mathrm{GL}_2^+(\mathbb{Q})$. Then

$$(d'X(F))_\alpha = d'(X(F)_\alpha) = \widetilde{X}(f_\alpha)\otimes (2\pi i)^{-1} d'(\omega_\alpha) = X(d'(F))_\alpha,$$

with both sides lying in $\mathbb{B}^{i+3,j+1,k+2}(\Gamma)$ and similarly for the differential d''. The same commutation relations hold for Y and δ_n, $n \geq 1$. □

Definition 3.11. Let M be a module over an algebra A. Suppose that $\mathcal{H}$ is a Hopf algebra acting on both A and M. Then the action of $\mathcal{H}$ on A is said to be flat if

$$h(a_1a_2) = \sum h_{(1)}(a_1)h_{(2)}(a_2), \quad \Delta(h) = \sum h_{(1)}\otimes h_{(2)}$$

for all $h \in \mathcal{H}$, $a_1, a_2 \in A$. The action of $\mathcal{H}$ on the system (A, M) is said to be flat if

$$h(am) = \sum h_{(1)}(a)h_{(2)}(m), \quad \Delta(h) = \sum h_{(1)}\otimes h_{(2)}$$

for all $h \in \mathcal{H}$, $a \in A$, $m \in M$.

In Proposition 3.8 we have already shown that the Hopf algebra $\mathcal{H}_1$ acts on $\mathbb{B}^*(\Gamma)$. We know from Proposition 3.5 that the Hopf algebra $\mathcal{H}_1$ has a flat action on the Hecke algebra $\mathcal{A}_T(\Gamma)$ and we proved in Proposition 3.7 that $\mathbb{B}^*(\Gamma)$ is a module over $\mathcal{A}_T(\Gamma)$. We will now show that the action of $\mathcal{H}_1$ on the system $(\mathcal{A}_T(\Gamma), \mathbb{B}^*(\Gamma))$ is flat, in the sense of Definition 3.11.

Proposition 3.12. *The action of $\mathcal{H}_1$ on the system $(\mathcal{A}_T(\Gamma), \mathbb{B}^*(\Gamma))$ is flat.*

Proof. For any $i, j, k \in \mathbb{Z}$, choose $F \in \mathbb{B}^{i,j,k}(\Gamma)$ and let $G \in \mathcal{A}_T(\Gamma)$. Let $F_\alpha = f_\alpha \otimes \omega_\alpha$ and $G_\alpha = g_\alpha \otimes \varepsilon_\alpha$ for each $\alpha \in \mathrm{GL}_2^+(\mathbb{Q})$. By definition,

$$X(G*F)_\alpha = \sum_{\beta\in\Gamma\backslash \mathrm{GL}_2^+(\mathbb{Q})} \widetilde{X}(g_\beta \cdot f_{\alpha\beta^{-1}}|\beta) \otimes (2\pi i)^{-1}\varepsilon_\beta \cdot \omega_{\alpha\beta^{-1}} \in \bigoplus_{l=1}^{\infty} \mathbb{B}^{i+2l,j,k+l}(\Gamma). \tag{3.10}$$

Since $\widetilde{X}$ is a derivation on $\mathcal{M}$, the right-hand side of (3.10) is equal to

$$\begin{aligned}
&\sum_{\beta\in\Gamma\backslash \mathrm{GL}_2^+(\mathbb{Q})} \widetilde{X}(g_\beta) \cdot f_{\alpha\beta^{-1}}|\beta \otimes (2\pi i)^{-1}\varepsilon_\beta \cdot \omega_{\alpha\beta^{-1}} \\
&\qquad + \sum_{\beta\in\Gamma\backslash \mathrm{GL}_2^+(\mathbb{Q})} (g_\beta \cdot \widetilde{X}(f_{\alpha\beta^{-1}}|\beta)) \otimes (2\pi i)^{-1}\varepsilon_\beta \cdot \omega_{\alpha\beta^{-1}} \\
&= (X(G) * F)_\alpha + \sum_{\beta\in\Gamma\backslash \mathrm{GL}_2^+(\mathbb{Q})} (g_\beta \cdot \widetilde{X}(f_{\alpha\beta^{-1}})|\beta) \otimes (2\pi i)^{-1}\varepsilon_\beta \cdot \omega_{\alpha\beta^{-1}} \\
&\qquad - \sum_{\beta\in\Gamma\backslash \mathrm{GL}_2^+(\mathbb{Q})} (g_\beta \cdot (\tilde{\mu}_{\beta^{-1}} \cdot Y(f_{\alpha\beta^{-1}}))|\beta) \otimes (2\pi i)^{-1}\varepsilon_\beta \cdot \omega_{\alpha\beta^{-1}} \\
&= (X(G) * F)_\alpha + (G * X(F))_\alpha \\
&\qquad + \sum_{\beta\in\Gamma\backslash \mathrm{GL}_2^+(\mathbb{Q})} ((\tilde{\mu}_\beta \cdot g_\beta) \cdot Y(f_{\alpha\beta^{-1}})|\beta) \otimes (2\pi i)^{-1}\varepsilon_\beta \cdot \omega_{\alpha\beta^{-1}} \\
&= (X(G) * F)_\alpha + (G * X(F))_\alpha + (\delta_1(G) * Y(F))_\alpha.
\end{aligned}$$

The result follows easily for the coproducts $\Delta(Y) = Y \otimes 1 + 1 \otimes Y$ and $\Delta(\delta_1) = \delta_1 \otimes 1 + 1 \otimes \delta_1$. □

4. Rankin–Cohen brackets and the restricted modular Hecke algebra

In this section we show that, by "restricting" the expression (3.3) for the product on $\mathcal{A}_T(\Gamma)$ as defined in Proposition 3.5 to coset representatives in $\mathrm{SL}_2(\mathbb{Z})$ instead of in $\mathrm{GL}_2^+(\mathbb{Q})$, we can define a "restricted algebra" $\mathcal{A}_T^r(\Gamma)$. We show that $\mathcal{A}_T^r(\Gamma)$ carries a flat action of the Hopf algebra $\mathcal{H}_1$ such that the action of each $\delta_n \in \mathcal{H}_1$, $n \geq 1$, is zero. We can make $\mathbb{B}^*(\Gamma)$ into a module over $\mathcal{A}_T^r(\Gamma)$ and with this module structure, we will refer to $\mathbb{B}^*(\Gamma)$ as $\mathbb{B}_r^*(\Gamma)$. Further, we show that $\mathcal{H}_1$ has a flat action on the system $(\mathcal{A}^r_T(\Gamma), \mathbb{B}^*_r(\Gamma))$. We conclude by constructing Rankin–Cohen brackets of all orders on $\mathbb{B}^*_r(\Gamma)$.

Proposition 4.1. (1) *Let $F, F' \in \mathcal{A}_T(\Gamma)$. Suppose that $F_\alpha = f_\alpha \otimes \varepsilon_\alpha$, $F'_\alpha = f'_\alpha \otimes \varepsilon'_\alpha$ for each $\alpha \in \mathrm{GL}_2^+(\mathbb{Q})$. Then $\mathcal{A}_T(\Gamma)$ becomes an algebra with the product*

$$(F * F')_\alpha = \sum_{\beta \in \Gamma \backslash \mathrm{SL}_2(\mathbb{Z})} (f_\beta \cdot f'_{\alpha\beta^{-1}} | \beta) \otimes \varepsilon_\beta \varepsilon'_{\alpha\beta^{-1}}. \tag{4.1}$$

Whenever we use the product of (4.1), we will refer to the algebra $\mathcal{A}_T(\Gamma)$ as $\mathcal{A}^r_T(\Gamma)$.

(2) *The Hopf algebra $\mathcal{H}_1$ has a flat action on $\mathcal{A}^r_T(\Gamma)$ defined by (for all $\alpha \in \mathrm{GL}_2^+(\mathbb{Q})$)*

$$X(F)_\alpha = \widetilde{X}(f_\alpha) \otimes T \cdot \varepsilon_\alpha, \quad Y(F)_\alpha = Y(f_\alpha) \otimes \varepsilon_\alpha, \quad \delta_1(F)_\alpha = 0.$$

Proof. (1) follows in the exact same manner as the proof of Proposition 3.5 (1).

To prove (2), we note that for any $\beta \in \mathrm{SL}_2(\mathbb{Z})$, $\mu_{\beta^{-1}} = 0$ and hence it follows from (2.3) that $X(f|\beta) = X(f)|\beta$ for any $f \in \mathcal{M}$. We know that $\Delta(X) = X \otimes 1 + 1 \otimes X + \delta_1 \otimes Y$. We check that

$$\begin{aligned}
X(F * F')_\alpha &= \sum_{\beta \in \Gamma \backslash \mathrm{SL}_2(\mathbb{Z})} \widetilde{X}(f_\beta \cdot f'_{\alpha\beta^{-1}} | \beta) \otimes T \cdot \varepsilon_\beta \varepsilon'_{\alpha\beta^{-1}} \\
&= \sum_{\beta \in \Gamma \backslash \mathrm{SL}_2(\mathbb{Z})} \widetilde{X}(f_\beta) \cdot f'_{\alpha\beta^{-1}} \otimes T \cdot \varepsilon_\beta \varepsilon'_{\alpha\beta^{-1}} \\
&\quad + \sum_{\beta \in \Gamma \backslash \mathrm{SL}_2(\mathbb{Z})} f_\beta \cdot \widetilde{X}(f'_{\alpha\beta^{-1}} | \beta) \otimes T \cdot \varepsilon_\beta \varepsilon'_{\alpha\beta^{-1}} \\
&= \sum_{\beta \in \Gamma \backslash \mathrm{SL}_2(\mathbb{Z})} \widetilde{X}(f_\beta) \cdot f'_{\alpha\beta^{-1}} \otimes T \cdot \varepsilon_\beta \varepsilon'_{\alpha\beta^{-1}} \\
&\quad + \sum_{\beta \in \Gamma \backslash \mathrm{SL}_2(\mathbb{Z})} f_\beta \cdot \widetilde{X}(f'_{\alpha\beta^{-1}}) | \beta \otimes T \cdot \varepsilon_\beta \varepsilon'_{\alpha\beta^{-1}} \\
&= (X(F) * F')_\alpha + (F * X(F'))_\alpha \\
&= (X(F) * F')_\alpha + (F * X(F'))_\alpha + (\delta_1(F) * Y(F'))_\alpha,
\end{aligned}$$

where the last equality follows from the fact that the action of δ_1 on $\mathcal{A}_T(\Gamma)$ has been defined to be zero. We can also check directly that Y is a derivation on $\mathcal{A}^r_T(\Gamma)$. Since $\Delta(Y) = Y \otimes 1 + 1 \otimes Y$ and the action of δ_1 (and hence that of any $\delta_n = [X, \delta_{n-1}]$, $n > 1$) on $\mathcal{A}^r_T(\Gamma)$ is zero, it follows that $\mathcal{H}_1$ has a flat action on $\mathcal{A}^r_T(\Gamma)$. $\square$

Proposition 4.2. (1) *Let $G \in \mathcal{A}^r_T(\Gamma)$ and $F \in \mathbb{B}^*(\Gamma)$. Let $G_\alpha = g_\alpha \otimes \varepsilon_\alpha$ and $F_\alpha = f_\alpha \otimes \omega_\alpha$ for each $\alpha \in \mathrm{GL}_2^+(\mathbb{Q})$. We set*

$$\begin{aligned}
(G * F)_\alpha &= \sum_{\beta \in \Gamma \backslash \mathrm{SL}_2(\mathbb{Z})} \phi_{g_\beta \otimes \varepsilon_\beta} \circ T_\beta (f_{\gamma\alpha\beta^{-1}} \otimes \omega_{\gamma\alpha\beta^{-1}}) \\
&= \sum_{\beta \in \Gamma \backslash \mathrm{SL}_2(\mathbb{Z})} (g_\beta \cdot f_{\alpha\beta^{-1}} | \beta) \otimes \varepsilon_\beta \cdot \omega_{\alpha\beta^{-1}},
\end{aligned} \tag{4.2}$$

where the functions $\phi_{g_\beta \otimes \varepsilon_\beta}$ *and* T_β *are as in* (3.5)*. This makes* $\mathbb{B}^*(\Gamma)$ *a module over* $\mathcal{A}^r_T(\Gamma)$*. With the module action of* (4.2)*, we will refer to* $\mathbb{B}^*(\Gamma)$ *as* $\mathbb{B}^*_r(\Gamma)$*.*

(2) *Given* $i, j, k \in \mathbb{Z}$*, for* $F \in \mathbb{B}^{i,j,k}_r(\Gamma) \subset \mathbb{B}^*_r(\Gamma)$*, define an action of* $\mathcal{H}_1$ *on* $\mathbb{B}^*_r(\Gamma)$ *by (for all* $\alpha \in \mathrm{GL}_2^+(\mathbb{Q})$*)*

$$X(F)_\alpha = \widetilde{X}(f_\alpha) \otimes (2\pi i)^{-1}\omega_\alpha, \quad Y(F)_\alpha = Y(f_\alpha) \otimes \omega_\alpha, \quad \delta_1(F)_\alpha = 0,$$

with the right-hand side lying in the direct sum $\bigoplus_{l=0}^{\infty} \mathbb{B}^{i+2l,j,k+l}_r(\Gamma)$*. This defines a flat action of* $\mathcal{H}_1$ *on the system* $(\mathcal{A}^r_T(\Gamma), \mathbb{B}^*_r(\Gamma))$*.*

Proof. The proof of (1) is analogous to that of Proposition 4.1 (1).

(2) also follows just as in the proof of Proposition 4.1 (2), using again the fact that $\widetilde{X}(f|\beta) = \widetilde{X}(f)|\beta$ for any $\beta \in \mathrm{SL}_2(\mathbb{Z})$ and any $f \in \mathcal{M}$. □

It follows from Proposition 4.1 and Proposition 4.2 that the operators $\delta_n \in \mathcal{H}_1$, $n \geq 1$, vanish in the action of $\mathcal{H}_1$ on the algebra $\mathcal{A}^r_T(\Gamma)$ as well as on the pair $(\mathcal{A}^r_T(\Gamma), \mathbb{B}^*_r(\Gamma))$. Consider, therefore, the smaller Hopf algebra $\mathfrak{h}_1$, which is the universal enveloping algebra of the Lie algebra $\mathfrak{l}_1$ with two generators X and Y satisfying the relation

$$[Y, X] = X.$$

The Lie algebra $\mathfrak{l}_1$ has been treated extensively in literature (see, for instance, [7]). It is well known that, up to isomorphism, $\mathfrak{l}_1$ is the only non abelian Lie algebra of dimension 2 over $\mathbb{C}$. A basis $\{e_1, e_2\}$ for the Lie algebra $\mathfrak{l}_1$ can be given in terms of the matrices

$$e_1 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad e_2 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix},$$

which then satisfy $[e_2, e_1] = e_2 e_1 - e_1 e_2 = e_1$. The universal enveloping algebra $\mathfrak{h}_1 = \mathcal{U}(\mathfrak{l}_1)$ is obtained from $\mathcal{H}_1$ by setting the operators $\delta_n \in \mathcal{H}_1$, $n \geq 1$, to zero. Hence, Proposition 4.1 shows that there is a flat action of $\mathfrak{h}_1$ on the algebra $\mathcal{A}^r_T(\Gamma)$, and Proposition 4.2 shows that there is a flat action of $\mathfrak{h}_1$ on the pair $(\mathcal{A}^r_T(\Gamma), \mathbb{B}^*_r(\Gamma))$.

From (2.1) in Section 2.1, we know that for any congruence subgroup Γ and elements $F^1, F^1 \in \mathcal{A}(\Gamma)$, the product on the modular Hecke algebra $\mathcal{A}(\Gamma)$ is given by

$$(F^1 * F^2)_\alpha = \sum_{\beta \in \Gamma \backslash \mathrm{GL}_2^+(\mathbb{Q})} F^1_\beta \cdot F^2_{\alpha\beta^{-1}}|\beta \quad \text{for all } F^1, F^2 \in \mathcal{A}(\Gamma). \tag{4.3}$$

The sum in (4.3) is taken over all right cosets of Γ in $\mathrm{GL}_2^+(\mathbb{Q})$. By restricting only to those right cosets of Γ that lie in $\mathrm{SL}_2(\mathbb{Z})$, we can define a product on $\mathcal{A}(\Gamma)$ as

$$(F^1 * F^2)_\alpha = \sum_{\beta \in \Gamma \backslash \mathrm{SL}_2(\mathbb{Z})} F^1_\beta \cdot F^2_{\alpha\beta^{-1}}|\beta \quad \text{for all } F^1, F^2 \in \mathcal{A}(\Gamma). \tag{4.4}$$

Whenever we use the product of (4.4), we shall refer to the algebra $\mathcal{A}(\Gamma)$ as $\mathcal{A}^r(\Gamma)$.

Our final aim is to define "Rankin–Cohen brackets" RC_n of any order $n \geq 1$ on $\mathbb{B}_r^*(\Gamma)$:

$$RC_n\colon \mathbb{B}_r^*(\Gamma) \otimes \mathbb{B}_r^*(\Gamma) \to \mathcal{A}^r(\Gamma)(1),$$

where $\mathcal{A}^r(\Gamma)(1) = \mathcal{A}^r(\Gamma)(2\pi i)$. The construction of the Rankin–Cohen bracket will combine the Rankin–Cohen brackets of [3], 1.5, with the pairing on the Archimedean complex used by Consani [4]. We start with the definition of the first Rankin–Cohen bracket. If f and g are modular forms of weight k and l respectively, the first Rankin–Cohen bracket can be expressed as

$$RC_1(f, g) = X(f)Y(g) - Y(f)X(g).$$

In [2], Connes and Moscovici have shown that the extension of the first Rankin–Cohen bracket to the modular Hecke algebra $\mathcal{A}(\Gamma)$ is defined by the generator of the transverse fundamental class $[F] \in HC^2(\mathcal{H}_1)$, which is a class in Hopf cyclic cohomology. Here, the class F is given by (see [3], 0.3)

$$F = X \otimes Y - Y \otimes X - \delta_1 Y \otimes Y. \tag{4.5}$$

However, on the algebra $\mathcal{A}_T^r(\Gamma)$, the action of δ_1 is zero, and hence the expression (4.5) for the first Rankin–Cohen bracket reduces to $X \otimes Y - Y \otimes X$. It follows that, for F_1, F_2 in $\mathcal{A}_T^r(\Gamma)$, the natural extension of the first Rankin–Cohen bracket is given by

$$RC_1(F_1, F_2) = X(F_1) * Y(F_2) - Y(F_1) * X(F_2). \tag{4.6}$$

In [4], 4.6, Consani has defined a pairing on the terms of the Archimedean complex taking values in $\mathbb{R}(1)$. We will now generalize this pairing to define a "Rankin–Cohen bracket" on $\mathbb{B}_r^*(\Gamma)$ taking values in the twisted module $\mathcal{A}^r(\Gamma)(1)$.

For any $m \in \mathbb{Z}$, let

$$\epsilon(m) = (-1)^{\frac{m(m+1)}{2}},$$

and, for a differential form ω of type (a, b), we set

$$C(\omega) = (\sqrt{-1})^{a-b}.$$

Combining [4], 4.6, with (4.5), we have a pairing

$$RC_{1,0}\colon \mathbb{B}_r^{-i-2,-j,k-1}(\Gamma) \otimes \mathbb{B}_r^{i,j,k+i}(\Gamma) \to \mathcal{A}^r(\Gamma)(1),$$

which is defined as follows. Let $F \in \mathbb{B}_r^{-i-2,-j,k-1}(\Gamma)$ and $F' \in \mathbb{B}_r^{i,j,k+i}(\Gamma)$ with $F_\rho = f_\rho \otimes \omega_\rho$ and $F'_\rho = f'_\rho \otimes \omega'_\rho$ for any $\rho \in G_2^+(\mathbb{Q})$. Then, for any $\alpha \in \mathrm{GL}_2^+(\mathbb{Q})$, the pairing is defined by (compare (4.6)):

$$\begin{aligned} &RC_{1,0}(F, F')_\alpha \\ &\quad = -\epsilon(1-j)(-1)^{k-1}(2\pi i)^{-2} \sum_{\beta \in \Gamma\backslash \mathrm{SL}_2(\mathbb{Z})} (\widetilde{X}(f_\beta) 2Y(f'_{\alpha\beta^{-1}})|\beta) \cdot \int \omega_\beta \wedge C\omega'_{\alpha\beta^{-1}} \\ &\qquad + \epsilon(1-j)(-1)^{k-1}(2\pi i)^{-2} \sum_{\beta \in \Gamma\backslash \mathrm{SL}_2(\mathbb{Z})} (2Y(f_\beta)\widetilde{X}(f_{\alpha\beta^{-1}})|\beta) \cdot \int \omega_\beta \wedge C\omega'_{\alpha\beta^{-1}}. \end{aligned}$$

The integral in the expression above is well defined on the direct limit of the modular curves $X(N)$, $N \geq 1$ since the integral of a top dimensional differential form is left unchanged by pullback maps. We can also define a pairing

$$RC_{1,1}\colon \mathbb{B}_r^{-i,-j,k}(\Gamma) \otimes \mathbb{B}_r^{i-2,j,k+i-1}(\Gamma) \to \mathcal{A}^r(\Gamma)(1)$$

as follows: given $F \in \mathbb{B}_r^{-i,-j,k}(\Gamma)$ and $F' \in \mathbb{B}_r^{i-2,j,k+i-1}(\Gamma)$ with $F_\rho = f_\rho \otimes \omega_\rho$ and $F'_\rho = f'_\rho \otimes \omega'_\rho$ for any $\rho \in G_2^+(\mathbb{Q})$, we have, for each $\alpha \in \mathrm{GL}_2^+(\mathbb{Q})$,

$$\begin{aligned} &RC_{1,1}(F,F')_\alpha \\ &\quad = -\epsilon(1-j)(-1)^k(2\pi i)^{-2} \sum_{\beta\in\Gamma\backslash \mathrm{SL}_2(\mathbb{Z})} (\widetilde{X}(f_\beta)2Y(f'_{\alpha\beta^{-1}})|\beta)\cdot \int \omega_\beta \wedge C\omega'_{\alpha\beta^{-1}} \\ &\qquad + \epsilon(1-j)(-1)^k(2\pi i)^{-2} \sum_{\beta\in\Gamma\backslash \mathrm{SL}_2(\mathbb{Z})} (2Y(f_\beta)\widetilde{X}(f_{\alpha\beta^{-1}})|\beta)\cdot \int \omega_\beta \wedge C\omega'_{\alpha\beta^{-1}}. \end{aligned}$$

Extending the pairings $RC_{1,0}$ and $RC_{1,1}$ by zero, we have a first Rankin–Cohen bracket

$$\begin{aligned} RC_1\colon (\mathbb{B}_r^{-i-2,-j,k-1}(\Gamma) \oplus \mathbb{B}_r^{-i,-j,k}(\Gamma)) \otimes (\mathbb{B}_r^{i-2,j,k+i-1}(\Gamma) \oplus \mathbb{B}_r^{i,j,k+i}(\Gamma)) \\ \to \mathcal{A}^r(\Gamma)(1). \end{aligned}$$

In general, for the n-th Rankin–Cohen bracket, we will have $n+1$ distinct pairings $(p = 0,1,2,\ldots,n)$

$$RC_{n,p}\colon \mathbb{B}_r^{-i-2(n-p),-j,k-(n-p)}(\Gamma) \otimes \mathbb{B}_r^{i-2p,j,k+i-p}(\Gamma) \to \mathcal{A}^r(\Gamma)(1)$$

and we will extend by zero to define the brackets:

$$RC_n\colon \bigoplus_{p=0}^{n} \mathbb{B}_r^{-i-2p,-j,k-p}(\Gamma) \otimes \bigoplus_{p=0}^{n} \mathbb{B}_r^{i-2p,j,k+i-p}(\Gamma) \to \mathcal{A}^r(\Gamma)(1). \tag{4.7}$$

The n-th Rankin–Cohen bracket RC_n is defined as follows: choose any $p \in \{0,1,2,\ldots,n\}$, and let $F \in \mathbb{B}_r^{-i-2(n-p),-j,k-(n-p)}(\Gamma)$, $F' \in \mathbb{B}_r^{i-2p,j,k+i-p}(\Gamma)$ such that $F_\rho = f_\rho \otimes \omega_\rho$, $F'_\rho = f'_\rho \otimes \omega'_\rho$ for any $\rho \in \mathrm{GL}_2^+(\mathbb{Q})$. Then, for any $\alpha \in \mathrm{GL}_2^+(\mathbb{Q})$, we define the Rankin–Cohen brackets (compare [3], 1.5)

$$\begin{aligned} &RC_{n,p}(F,F')_\alpha \\ &\quad = \sum_{\beta\in\Gamma\backslash \mathrm{SL}_2(\mathbb{Z})} \sum_{l=0}^{n} \epsilon(1-j)(-1)^{k-n+p}(2\pi i)^{-n-1} \\ &\qquad \left(\left(\frac{(-\widetilde{X})^l}{l!}(2Y+l)_{n-l}(f_\beta)\right)\cdot\left(\frac{\widetilde{X}^{n-l}}{(n-l)!}(2Y+n-l)_l(f'_{\alpha\beta^{-1}})\right)\Big|\beta\right) \\ &\qquad \cdot\left(\int \omega_\beta \wedge C\omega'_{\alpha\beta^{-1}}\right), \end{aligned} \tag{4.8}$$

where $(2Y+k)_l = (2Y+k)(2Y+k+1)\dots(2Y+k+l-1)$ for any integers k, l. Extending by zeroes, we can define the Rankin–Cohen brackets RC_n for all n.

The Rankin–Cohen brackets defined in (4.7) can be related directly to the classical Rankin–Cohen brackets on modular forms. Let $f(z)$ and $g(z)$ be two given modular forms of level $\Gamma' \subseteq \mathrm{SL}_2(\mathbb{Z})$ and weights k and l respectively. Using the normalization in Zagier [8], let us denote by D the differential operator $D := (2\pi i)^{-1}\frac{d}{dz}$. Then the n-th Rankin–Cohen bracket of f and g can be expressed as

$$[f,g]_n = \sum_{r+s=n} (-1)^r \binom{n+k-1}{r}\binom{n+l-1}{s} D^r(f) D^s(g).$$

Then $[f,g]_n$ is a modular form of weight $k+l+2n$. We can express the Rankin–Cohen brackets defined in (4.7) more succinctly as follows.

Proposition 4.3. *Let $i, j, k \in \mathbb{Z}$. For any given n, choose some $p \in \{0,1,2,\dots,n\}$ and let $F \in \mathbb{B}_r^{-i-2(n-p),-j,k-(n-p)}(\Gamma)$, $F' \in \mathbb{B}_r^{i-2p,j,k+i-p}(\Gamma)$ such that $F_\rho = f_\rho \otimes \omega_\rho$, $F'_\rho = f'_\rho \otimes \omega'_\rho$ for any $\rho \in \mathrm{GL}_2^+(\mathbb{Q})$. Then, for any $\alpha \in \mathrm{GL}_2^+(\mathbb{Q})$, the formula (4.8) defining the n-th Rankin–Cohen brackets may be expressed as*

$$\begin{aligned} &RC_{n,p}(F,F')_\alpha \\ &\quad = \sum_{\beta\in\Gamma\backslash \mathrm{SL}_2(\mathbb{Z})} \epsilon(1-j)(-1)^{k-n+p}(2\pi i)^{-1}[f_\beta, f'_{\alpha\beta^{-1}}|\beta]_n \cdot \Big(\int \omega_\beta \wedge C\omega'_{\alpha\beta^{-1}}\Big). \end{aligned}$$

Proof. From (4.8), we know that

$$\begin{aligned} &RC_{n,p}(F,F')_\alpha \\ &\quad = \sum_{\beta\in\Gamma\backslash \mathrm{SL}_2(\mathbb{Z})} \sum_{l=0}^{n} \epsilon(1-j)(-1)^{k-n+p}(2\pi i)^{-n-1} \\ &\quad\quad \bigg(\Big(\frac{(-\widetilde{X})^l}{l!}(2Y+l)_{n-l}(f_\beta)\Big)\cdot\Big(\frac{\widetilde{X}^{n-l}}{(n-l)!}(2Y+n-l)_l(f'_{\alpha\beta^{-1}})\Big)\Big|\beta\bigg) \\ &\quad\quad \cdot \Big(\int \omega_\beta \wedge C\omega'_{\alpha\beta^{-1}}\Big). \end{aligned} \tag{4.9}$$

Since each coset representative β in (4.9) lies in $\mathrm{SL}_2(\mathbb{Z})$, we have $\widetilde{X}(g|\beta) = \widetilde{X}(g)|\beta$ for any element g in the modular tower $\mathcal{M}$. Further, we know that $(2\pi i)^{-1}\widetilde{X} = X$

and $Y(g|\beta) = Y(g)|\beta$ for any $g \in \mathcal{M}$. Therefore, we can rewrite (4.9) as

$$\begin{aligned} &RC_{n,p}(F,F')_\alpha \\ &\quad = \sum_{\beta\in\Gamma\backslash \mathrm{SL}_2(\mathbb{Z})} \epsilon(1-j)(-1)^{k-n+p}(2\pi i)^{-1} \\ &\qquad \Big(\sum_{l=0}^{n}\Big(\Big(\frac{(-X)^l}{l!}(2Y+l)_{n-l}(f_\beta)\Big)\cdot\Big(\frac{X^{n-l}}{(n-l)!}(2Y+n-l)_l(f'_{\alpha\beta^{-1}}|\beta)\Big)\Big)\Big) \\ &\qquad \cdot\Big(\int \omega_\beta\wedge C\omega'_{\alpha\beta^{-1}}\Big). \end{aligned} \tag{4.10}$$

On the other hand, we know from [3] that for any $g, h \in \mathcal{M}$, the Rankin–Cohen brackets of order n may be recovered from the action of the operators X and Y as

$$[g,h|\beta]_n = \sum_{l=0}^{n}\Big(\Big(\frac{(-X)^l}{l!}(2Y+l)_{n-l}(g)\Big)\cdot\Big(\frac{X^{n-l}}{(n-l)!}(2Y+n-l)_l(h|\beta)\Big)\Big).$$

Hence the expression in (4.10) may be rewritten as

$$\begin{aligned} &RC_{n,p}(F,F')_\alpha \\ &\quad = \sum_{\beta\in\Gamma\backslash \mathrm{SL}_2(\mathbb{Z})} \epsilon(1-j)(-1)^{k-n+p}(2\pi i)^{-1}[f_\beta, f'_{\alpha\beta^{-1}}|\beta]_n\cdot\Big(\int \omega_\beta\wedge C\omega'_{\alpha\beta^{-1}}\Big). \end{aligned}$$

□

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