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A “natural” graded Hopf algebra and its graded Hopf-cyclic cohomology

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Abstract. We prove that the category of differential graded algebras is monoidally equivalent to the category of left graded comodule algebras over a certain graded Hopf algebra. After calculating the graded Hopf-cyclic cohomology of that graded Hopf algebra, we construct cyclic cocycles on any graded differential algebra with a closed graded trace by means of a characteristic homomorphism.

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Introduction

In [3], [4] Connes and Moscovici defined a cocyclic module associated with a Hopf algebra with a modular pair in involution. They also constructed a characteristic homomorphism from the Hopf-cyclic cohomology of a Hopf algebra to the usual cyclic cohomology of an algebra which is a module algebra over that Hopf algebra and is endowed with an α-invariant π-trace, where (α, π) is a modular pair in involution of that Hopf algebra.

We consider in this paper the cyclic cohomology in graded categories. The only difference occurring in the definition of graded (co)cyclic modules comes from the cyclic operator whose action is multiplied by a Koszul–Quillen sign, that is, whenever a flip is acted on two homogeneous elements a and b, (-1) raised to the power $\deg a \cdot \deg b$ is introduced. Then also the last face map is defined with the sign. This ‘twisted’ flip (flip with a sign) is in fact a special symmetric braiding. In [1], [6], [7] (co)cyclic modules in a symmetric braided tensor category are introduced, where the ‘twisted’ flip is replaced by a braid action.

Pareigis [10] constructed a “natural” non-commutative and non-cocommutative Hopf algebra such that the category of complexes and the category of comodules over that Hopf algebra are equivalent. We define a graded Hopf algebra $\mathscr{P}$ which has a similar categorial property, that is, the category of its left graded comodule algebras is monoidally equivalent to the category of differential graded algebras with differentials of degree 1. Differences occur, since here gradings are considered everywhere.

The graded Hopf-cyclic cohomology of $\mathcal{P}$ is computed and the cyclic cocycles on the graded Hopf algebra $\mathcal{P}$ are found. By using Connes–Moscovici's characteristic homomorphism and cyclic cocycles on $\mathcal{P}$, we obtain cyclic cocycles on any differential graded algebra with a closed graded trace.

We organize this paper as follows. In Section 1, we recall some definitions and properties of graded Hopf algebras, graded module algebras, and graded comodule algebras, etc. A graded Hopf pairing and some of its properties are given. In Section 2, we define the special graded Hopf algebra $\mathcal{P}$ and prove the equivalence of categories. In Section 3, we provide the characteristic homomorphism from the graded Hopf-cyclic cohomology of a graded Hopf algebra to the graded cyclic cohomology of a graded algebra which is a graded module algebra over that graded Hopf algebra and is endowed with a special trace. In Section 4, we compute the graded Hochschild cohomology and the graded Hopf-cyclic cohomology of $\mathcal{P}$. In Section 5, we use the characteristic homomorphism to provide cyclic cocycles on a differential graded algebra.

In this paper, let k be a field of characteristic 0. Every algebra here is an associative k-algebra with unit. And we use Sweedler's notation to express coproducts and comodule actions.

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1. Graded Hopf algebras and graded modules

In this section, in order to make precise our conventions, some definitions in graded categories are recalled, and some properties, which will be used later, are given without straightforward proofs.

Definition 1.1. A graded algebra A is called a *differential graded algebra* (denoted by DGA for short) if there exists a degree 1 differential $\partial\colon A_n \to A_{n+1}$ such that

$$\partial^2 = 0 \quad \text{and} \quad \partial(ab) = \partial(a)b + (-1)^{|a|} a\partial(b),$$

where a, b are homogeneous elements of A and $|a|$ is the degree of a.

Definition 1.2. Let (A, ∂) be a DGA. A k-linear form $f\colon A \to k$ is called a *graded trace* on A if $f(ab) = (-1)^{|a||b|} f(ba)$ for any homogeneous elements $a, b \in A$. Moreover, it is *closed* if $f(\partial a) = 0$ for all $a \in A$.

Definition 1.3. A graded algebra A (resp. graded coalgebra C) is called graded commutative (resp. graded cocommutative) if

$$aa' = (-1)^{|a||a'|}a'a,$$
$$\text{(resp. } \Delta(c) = \sum_{(c)} c_{(1)} \otimes c_{(2)} = \sum_{(c)} (-1)^{|c_{(1)}||c_{(2)}|} c_{(2)} \otimes c_{(1)})$$

for any homogeneous elements $a, a' \in A$ (resp. $c \in C$).

The tensor product of graded algebras (resp. DGAs) is also a graded algebra (resp. DGA) with the “twisted” multiplication defined as follows.

Proposition 1.4. (i) *If A and B are graded algebras, then $A \otimes B$ is also a graded algebra with multiplication defined by*

$$(a \otimes b) \cdot (a' \otimes b') = (-1)^{|b||a'|} aa' \otimes bb', \tag{1}$$

where $a, a' \in A$ and $b, b' \in B$ are homogeneous elements. We denote this graded algebra by $A \mathbin{\hat{\otimes}} B$.

(ii) *If (A, ∂_A) and (B, ∂_B) are DGAs, then so is $A \mathbin{\hat{\otimes}} B$ with the differential defined by*

$$\partial_{A \hat{\otimes} B}(a \otimes b) = \partial_A a \otimes b + (-1)^{|a|} a \otimes \partial_B b, \tag{2}$$

where $a \in A$ and $b \in B$ are homogeneous elements.

Due to this proposition, the category of DGAs becomes a tensor category. Precisely, let $\mathcal{D}$ be the category of DGAs whose morphisms are degree 0 algebra homomorphisms which are compatible with the differentials. The ground field k can be regarded as a DGA concentrated in degree 0 with the trivial differential, i.e., $\partial(k) = 0$.

Corollary 1.5. *The category $(\mathcal{D}, \hat{\otimes})$ is a tensor category with $\mathbf{1}_{\mathcal{D}} = k$. For A and B two objects of $\mathcal{D}$, the graded algebra $A \mathbin{\hat{\otimes}} B$ is also an object of $\mathcal{D}$ with the multiplication defined by (1) and the differential defined by (2).*

Definition 1.6. Let $\mathcal{H}$ be a graded Hopf algebra.

(i) A (left) $\mathcal{H}$-(co)module M is called a (left) *graded $\mathcal{H}$-(co)module* or a (left) *graded (co)module over $\mathcal{H}$* if $M = \oplus_{n \in \mathbb{Z}} M_n$ and the (co)module structure map is of degree 0.

(ii) An algebra A is called a *graded $\mathcal{H}$-module algebra* if A is a graded algebra and a graded $\mathcal{H}$-module sharing the same grading such that

$$h \cdot (ab) = \sum_{(h)} (-1)^{|a||h_{(2)}|} (h_{(1)} \cdot a)(h_{(2)} \cdot b) \quad \text{and} \quad h \cdot 1 = \varepsilon(h)$$

for any homogeneous elements $a, b \in A$ and $h \in \mathcal{H}$.

(iii) An algebra (A, ρ) is called a (left) *graded $\mathcal{H}$-comodule algebra* if A is a graded algebra and a (left) graded $\mathcal{H}$-comodule sharing the same grading, i.e., the comodule structure map $\rho(A_n) \subseteq \bigoplus_i \mathcal{H}_i \otimes A_{n-i}$, and for homogeneous elements $a, b \in A$, it satisfies that

$$\rho(ab) = \sum_{(a),(b)} (-1)^{|b_{(-1)}||a_{(0)}|} a_{(-1)} b_{(-1)} \otimes a_{(0)} \otimes b_{(0)}$$

and

$$\rho(1_A) = 1_{\mathcal{H}} \otimes 1_A,$$

where $\rho(a) = \sum_{(a)} a_{(-1)} \otimes a_{(0)}$ and $\rho(b) = \sum_{(b)} b_{(-1)} \otimes b_{(0)}$.

Example. The tensor algebra $T_k(\mathcal{H})$ of a graded Hopf algebra $\mathcal{H}$ is both a graded $\mathcal{H}$-module algebra and a graded $\mathcal{H}$-comodule algebra with the graded diagonal action

$$h \cdot (h^1 \otimes \ldots \otimes h^n) = \sum_{(h)} (-1)^{\sum_{i=1}^{n-1} (|h^1| + \cdots + |h^i|)|h_{(i+1)}|} h_{(1)} h^1 \otimes \ldots \otimes h_{(n)} h^n,$$

and the graded diagonal coaction

$$\rho(h^1 \otimes \ldots \otimes h^n) = \sum_{(h^i)} (-1)^{\sum_{i=1}^{n-1} (|h^1_{(0)}| + \cdots + |h^i_{(0)}|)|h^{i+1}_{(-1)}|} h^1_{(-1)} \ldots h^n_{(-1)} \otimes h^1_{(0)} \otimes \ldots \otimes h^n_{(0)}$$

for any homogeneous elements $h,\ h^i \in \mathcal{H}$.

For the tensor product of graded (co)modules (resp. graded (co)module algebras) we have the following proposition.

Proposition 1.7. *Let $\mathcal{H}$ be a graded Hopf algebra.*

(i) *If M and N are left graded $\mathcal{H}$-modules, then so is $M \otimes N$ with the module structure map*

$$h \cdot (m \otimes n) = \sum_{(h)} (-1)^{|m||h_{(2)}|} h_{(1)} \cdot m \otimes h_{(2)} \cdot n,$$

for homogeneous elements $h \in \mathcal{H}$, $m \in M$, and $n \in N$. Denote this graded module by $M \mathbin{\widehat{\otimes}} N$.

(ii) *Suppose that A and B are left graded $\mathcal{H}$-module algebras. If $\mathcal{H}$ is a graded cocommutative, then $A \mathbin{\widehat{\otimes}} B$ is also a left graded $\mathcal{H}$-module algebra.*

(iii) *If (M, ρ_M) and (N, ρ_N) are left graded $\mathcal{H}$-comodules, then so is $(M \otimes N, \rho)$ with the comodule structure map*

$$\rho(m \otimes n) = \sum_{(m),(n)} (-1)^{|m_{(0)}||n_{(-1)}|} m_{(-1)} n_{(-1)} \otimes m_{(0)} \otimes n_{(0)} \tag{3}$$

for homogeneous elements $m \in M$ and $n \in N$. Denote this graded comodule by $M \mathbin{\widehat{\otimes}} N$.

(iv) *Suppose that A and B are left graded $\mathcal{H}$-comodule algebras. If $\mathcal{H}$ is graded commutative, then $A \mathbin{\widehat{\otimes}} B$ is also a left graded $\mathcal{H}$-comodule algebra.*

Therefore under the conditions of (ii) (resp. (iv)) in the above proposition, the multiplication of a graded module (resp. comodule) algebra A is in fact a homomorphism of graded modules (resp. comodules) from $A \hat{\otimes} A$ to A.

Let $\mathcal{H}$ be the category of left graded comodule algebras over a graded commutative graded Hopf algebra $\mathscr{H}$ whose morphisms are degree 0 algebra homomorphisms and at the same time homomorphisms of $\mathscr{H}$-comodules. The ground field k can be regarded as the trivial graded $\mathscr{H}$-comodule algebra concentrated in degree 0 with the trivial graded comodule structure map, i.e., $\rho(1_k) = 1_{\mathscr{H}} \otimes 1_k$.

Corollary 1.8. *The category* $(\mathcal{H}, \hat{\otimes})$ *is a tensor category with* $\mathbf{1}_{\mathcal{H}} = k$. *For A and B two objects of $\mathcal{H}$, the graded algebra $A \hat{\otimes} B$ is also an object of $\mathcal{H}$ with the product defined by* (1) *and the comodule structure map defined by* (3).

For graded Hopf algebras we can also define a graded Hopf pairing.

Definition 1.9. Let A and B be two graded Hopf algebras. A *graded Hopf pairing* of A and B is a bilinear form $\langle -, - \rangle \colon A \times B \to k$ such that $\langle A_i, B_j \rangle = 0$ for $i + j \neq 0$ and

$$\begin{aligned} \langle a, b_1 b_2 \rangle &= \sum_{(a)} (-1)^{|b_1||b_2|} \langle a_{(1)}, b_1 \rangle \langle a_{(2)}, b_2 \rangle, \\ \langle a_1 a_2, b \rangle &= \sum_{(b)} (-1)^{|a_1||a_2|} \langle a_1, b_{(1)} \rangle \langle a_2, b_{(2)} \rangle, \\ \langle 1, b \rangle &= \varepsilon_B(b), \ \langle a, 1 \rangle = \varepsilon_A(a), \\ \langle S_A(a), b \rangle &= \langle a, S_B(b) \rangle \end{aligned} \tag{4}$$

for all homogeneous elements $a, a_1, a_2 \in A$ and $b, b_1, b_2 \in B$, where ε_A and ε_B are counits of A and B respectively, and S_A and S_B are antipodes of A and B respectively.

Remark 1.10. In fact, if A and B are two locally finite graded Hopf algebras, that is, $\dim A_n < \infty$ and $\dim B_n < \infty$ for each n, then the last equality in (4) is redundant, because of the uniqueness of S and ε.

Many graded Hopf algebras we shall consider will be defined by generators and relations. The following lemma, which is a graded version of Lemma 3.4 in [8], provides us with a method of constructing graded Hopf pairings.

Lemma 1.11. *Let $\widetilde{A}$ (resp. $\widetilde{B}$) be a free graded algebra generated by homogeneous elements $a_1, \dots, a_p$ (resp. $b_1, \dots, b_q$) over k. Suppose that $\widetilde{A}$ (resp. $\widetilde{B}$) has a graded Hopf algebra structure such that each $\Delta(a_i)$ for $1 \leq i \leq p$ (resp. $\Delta(b_j)$ for $1 \leq j \leq q$) is a linear combination of tensors $a_r \otimes a_s$ (resp. of tensors $b_r \otimes b_s$). Given pq scalars $\lambda_{ij} \in k$ with $1 \leq i \leq p$ and $1 \leq j \leq q$, there is a unique graded Hopf pairing $\varphi \colon \widetilde{A} \otimes \widetilde{B} \to k$ such that $\varphi(a_i, b_j) = \lambda_{ij}$.*

Now A (resp. B) is the graded algebra obtained as the quotient of $\widetilde{A}$ (resp. $\widetilde{B}$) by the ideal generated by homogeneous elements $r_1, \dots, r_n \in \widetilde{A}$ (resp. $r'_1, \dots, r'_m \in \widetilde{B}$). Suppose also that the graded Hopf algebra structure of $\widetilde{A}$ (resp. $\widetilde{B}$) induces a graded Hopf algebra structure of A (resp. B). Then a graded Hopf pairing of $\widetilde{A}$ and $\widetilde{B}$ induces a graded Hopf pairing of A and B if and only if $\varphi(r_i, b_j) = 0$ for all $i = 1, \dots, n$ and $j = 1, \dots, q$, and $\varphi(a_i, r'_j) = 0$ for all $i = 1, \dots, p$ and $j = 1, \dots, m$.

Proposition 1.12. *Let A and B be two graded Hopf algebras endowed with a graded Hopf pairing $\langle -, - \rangle \colon A \times B \to k$, then a left graded B-comodule (algebra) M is also a right graded A-module (algebra) via $m.a = (-1)^{|a||m|}\langle a, m_{(-1)}\rangle m_{(0)}$.*

Furthermore, if A is graded commutative, then a left graded B-comodule algebra is also a left graded A-module algebra via $a.m = \langle a, m_{(-1)}\rangle m_{(0)}$.

2. Categorical equivalence

Pareigis, in [10], constructed a particular Hopf algebra and showed that the category of complexes is monoidally equivalent to the category of comodules over that Hopf algebra. Now we consider the graded case. We will define a graded Hopf algebra $\mathscr{P}$ which is graded commutative but not graded cocommutative.

Let $\mathscr{P}$ be a graded Hopf algebra generated by $\{s, t, t^{-1}\}$, where $|s| = -1$ and $|t| = 0$, with the relations

$$\begin{aligned} &s^2 = 0, && st = ts, && t^{-1}t = tt^{-1} = 1; \\ &\Delta(t) = t \otimes t, && \varepsilon(t) = 1, && S(t) = t^{-1}; \\ &\Delta(s) = s \otimes 1 + t^{-1} \otimes s, && \varepsilon(s) = 0, && S(s) = -st. \end{aligned}$$

Note that $\Delta(s^2) = s^2 \otimes 1 + st^{-1} \otimes s + (-1)t^{-1}s \otimes s + t^{-2} \otimes s^2 = 0$, so $\mathscr{P}$ is a well-defined graded Hopf algebra.

Let $\mathcal{P}$ be the category of left graded comodule algebras over $\mathscr{P}$ whose morphisms are degree 0 algebra homomorphisms and at the same time homomorphisms of $\mathscr{P}$-comodules. Recall that $\mathcal{D}$ defined in Section 1 is the category of differential graded algebras with a differential of degree 1 whose morphisms are degree 0 algebra homomorphisms which are compatible with the differentials.

Define a functor $F \colon \mathcal{D} \to \mathcal{P}$ by setting $F(A = \oplus A_n, \partial) = (A, \rho)$ for any object (A, ∂) of $\mathcal{D}$ such that

$$\rho(a) = t^n \otimes a + st^{n+1} \otimes \partial a \quad \text{for all } a \in A_n,$$

and setting $F(f) = f$ for any morphism f of $\mathcal{D}$.

It is easy to check that (A, ρ) is a left graded $\mathscr{P}$-comodule and that f is a morphism of $\mathcal{P}$. Since $\partial 1 = 0$, we get $\rho(1) = 1 \otimes 1$. For all $a \in A_n$ and $b \in A_m$, one has

$$\begin{aligned}\rho(ab) &= t^{n+m} \otimes ab + st^{n+m+1} \otimes \partial(ab) \\ &= t^{n+m} \otimes ab + st^{n+m+1} \otimes (\partial a)b + (-1)^n st^{n+m+1} \otimes a(\partial b) \\ &= (t^n \otimes a + st^{n+1} \otimes \partial a) \cdot (t^m \otimes b + st^{m+1} \otimes \partial b) \\ &= \rho(a) \cdot \rho(b),\end{aligned}$$

so the graded $\mathscr{P}$-comodule (A, ρ) is also a graded $\mathscr{P}$-comodule algebra.

For (A, ∂_A) and (B, ∂_B) two objects of $\mathcal{D}$, write $F(A, \partial_A) = (A, \rho_A)$, $F(B, \partial_B) = (B, \rho_B)$, and

$$F(A \mathbin{\hat{\otimes}} B, \partial_{A \hat{\otimes} B}) = (A \mathbin{\hat{\otimes}} B, \rho_{A \hat{\otimes} B}),$$

where the differential $\partial_{A \hat{\otimes} B}$ is defined by (2). We can check directly that $\rho_{A \hat{\otimes} B}$ defined from $\partial_{A \hat{\otimes} B}$ is equal to the graded comodule structure induced from the graded tensor product of ρ_A and ρ_B in (3). Thus

$$F(A \mathbin{\hat{\otimes}} B, \partial_{A \hat{\otimes} B}) = F(A, \partial_A) \mathbin{\hat{\otimes}} F(B, \partial_B),$$

the functor F is a functor of tensor categories.

Define another functor $G : \mathcal{P} \to \mathcal{D}$ by setting $G(A, \rho) = (\oplus A_n, \partial_\rho)$ for any object (A, ρ) of $\mathcal{P}$ such that

$$A_n = \{a \in A \mid \rho(a) = t^n \otimes a + st^{n+1} \otimes a',\ a' \in A\},$$

setting $\partial_\rho(a) = a'$ where $a \in A_n$ with $\rho(a) = t^n \otimes a + st^{n+1} \otimes a'$, and setting $G(g) = g$ for any morphism g of $\mathcal{P}$.

The graded Hopf algebra $\mathscr{P}$ has a k-basis $\{t^n, st^n, \text{ for all } n \in \mathbb{Z}\}$, so, for all $a \in A$, $\rho(a)$ can be written as

$$\rho(a) = \sum_n t^n \otimes a_n + \sum_n st^{n+1} \otimes a'_n, \tag{5}$$

where $a_n, a'_n \in A$.

Since $(\varepsilon \otimes \mathrm{id})\rho = \mathrm{id}_A$ and $(\mathrm{id} \otimes \rho)\rho = (\Delta \otimes \mathrm{id})\rho$, from (5) we obtain that $a = \sum a_n$, $\rho(a_n) = t^n \otimes a_n + st^{n+1} \otimes a'_n$, and $\rho(a'_n) = t^{n+1} \otimes a'_n$. So $A = \oplus A_n$ and $\partial_\rho^2(a_n) = \partial_\rho(a'_n) = 0$ for all $a_n \in A_n$.

Let (A, ρ) be a graded $\mathscr{P}$-comodule algebra. Then, for any $a \in A_n$ and $b \in A_m$, one has

$$\rho(ab) = \rho(a) \cdot \rho(b) = t^{n+m} \otimes ab + st^{n+m+1} \otimes a'b + (-1)^n st^{n+m+1} \otimes ab'.$$

Thus, $ab \in A_{n+m}$ and $\partial_\rho(ab) = a'b + (-1)^{|a|}ab' = (\partial_\rho a)b + (-1)^{|a|}a(\partial_\rho b)$. Hence $G(A, \rho)$ is a DGA. It is clear that G maps a morphism of $\mathcal{P}$ to a morphism of $\mathcal{D}$. Due to Propositions 1.4 and 1.7, the functor G is a functor of tensor categories.

It is clear that $FG = \mathrm{id}_{\mathcal{P}}$ and $GF = \mathrm{id}_{\mathcal{D}}$. Therefore we have:

Theorem 2.1. *The category of left graded $\mathscr{P}$-comodule algebras and the category of differential graded algebras with degree* 1 *differentials are monoidally equivalent.*

3. Cocyclic modules in graded categories

The notion of cocyclic modules was first defined by Connes in [2]. A cocyclic module is a covariant functor from the cyclic category Λ to the category of k-modules. Therefore, to give a cocyclic module, it is necessary and sufficient to give a sequence of k-modules $C^0, C^1, \ldots$ together with homomorphisms of k-modules $d_i : C^{n-1} \to C^n$ and $s_i : C^{n+1} \to C^n$, $0 \le i \le n$ for each $n \in \mathbb{N}$, such that

$$\begin{aligned} d_j d_i &= d_i d_{j-1} \quad \text{if } i < j, \\ s_j s_i &= s_i s_{j+1} \quad \text{if } i \le j, \\ s_j d_i &= \begin{cases} d_i s_{j-1} & \text{if } i < j, \\ \mathrm{id} & \text{if } i = j, j+1, \\ d_{i-1} s_j & \text{if } i > j+1, \end{cases} \end{aligned} \tag{6}$$

and

$$\begin{aligned} &\tau_n^{n+1} = \mathrm{id}, \\ &\tau_n d_i = d_{i-1}\tau_{n-1}, \ \tau_n s_i = s_{i-1}\tau_{n-1}, \quad 1 \le i \le n. \end{aligned} \tag{7}$$

The maps d_i's are called *face* maps, s_i's are called *degeneracy* maps, and τ_n's are called *cyclic* operators. Usually we write τ instead of τ_n.

In this paper we will consider several concrete cocyclic modules in graded categories.

Proposition 3.1. *Let A be a graded algebra. For each $n \ge 0$, set $C^n_G(A) = \mathrm{Hom}(A^{\otimes n+1}, k)$. For all $f \in C^n_G(A)$ and homogeneous elements a_i's in A, define the k-linear homomorphisms*

$$\begin{aligned} d_i(f)(a_0, \ldots, a_n, a_{n+1}) &= f(a_0, \ldots, a_i a_{i+1}, \ldots, a_{n+1}), \quad 0 \le i \le n, \\ d_{n+1}(f)(a_0, \ldots, a_n, a_{n+1}) &= (-1)^{|a_{n+1}|(|a_0|+\cdots+|a_n|)} f(a_{n+1}a_0, \ldots, a_n), \\ s_j(f)(a_0, \ldots, a_{n-1}) &= f(a_0, \ldots, a_j, 1, a_{j+1}, \ldots, a_{n-1}), \quad 0 \le j \le n-1, \\ \tau(f)(a_0, \ldots, a_n) &= (-1)^{|a_n|(|a_0|+\cdots+|a_{n-1}|)} f(a_n, a_0, \ldots, a_{n-1}). \end{aligned}$$

The homomorphisms d_i, s_i, τ satisfy the relations (6) *and* (7)*, so $(C^\bullet_G(A), d_i, s_i, \tau)$ defined above is a cocyclic module of the graded algebra A.*

In fact, the cyclic operator τ is only multiplied by a sign, and the last face map is just a composition of the first face map and the cyclic operator. Note that all morphisms d_i, s_j, and τ preserve the degree. Denote the Hochschild (resp. cyclic) cohomology of a graded algebra A by $\mathrm{HH}^*_G(A)$ (resp. $\mathrm{HC}^*_G(A)$).

Let $\mathcal{H}$ be a graded Hopf algebra.

Definition 3.2. A *character* of $\mathcal{H}$ is an algebra homomorphism of degree 0 from $\mathcal{H}$ to k. Let π be a group-like element of $\mathcal{H}$. A pair (α, π) is called a *graded modular* *pair* of $\mathcal{H}$ if α is a character of $\mathcal{H}$ and $\alpha(\pi) = 1$. Moreover, the graded modular pair (α, π) is called a *graded modular pair in involution* of $\mathcal{H}$ if

$$(\pi^{-1} \cdot S_\alpha)^2 = \mathrm{id},$$

where $S_\alpha(h) = (\alpha * S)(h) = \alpha(h_{(1)})S(h_{(2)})$ and $(\pi^{-1} \cdot S_\alpha)(h) = \pi^{-1} S_\alpha(h)$ for all $h \in \mathcal{H}$.

Remark 3.3. For a graded Hopf algebra $\mathcal{H}$, every group-like element is in $\mathcal{H}_0$, and every character of $\mathcal{H}$ maps $\mathcal{H}_n$ to 0 unless $n = 0$.

Lemma 3.4. *All the graded modular pairs in involution of $\mathcal{P}$ are of the form*

$$\{(\varepsilon, t^r), \textit{ for all } r \in \mathbb{Z}\}.$$

Proof. It is clear that all the group-like elements of $\mathcal{P}$ are of the form $\{t^r, \text{ for all } r \in \mathbb{Z}\}$. Assume that (α, t^r) is a graded modular pair in involution of $\mathcal{P}$. Because $(t^{-r} S_\alpha)^2(st^n) = \alpha(t^{-1-r})st^n$ according to the definition of a graded modular pair in involution, we have $\alpha(t^{-1-r}) = 1$ and $\alpha(t^r) = 1$. So $\alpha(t) = 1$. And $\alpha(s) = 0$ is clear as the degree of s is -1. Therefore α is just the counit of $\mathcal{P}$. □

Definition 3.5. Let (α, π) be a graded modular pair in involution of a graded Hopf algebra $\mathcal{H}$, and let A be a graded $\mathcal{H}$-module algebra. A k-linear map $\mathsf{T}\colon A \to k$ is called a *graded π-trace* if

$$\mathsf{T}(ab) = (-1)^{|a||b|}\mathsf{T}(b(\pi \cdot a))$$

for any homogeneous elements $a, b \in A$. Moreover, the graded π-trace T is called *α-invariant* if

$$\mathsf{T}((h \cdot a)b) = (-1)^{|h||a|}\mathsf{T}(a(S_\alpha(h) \cdot b)) \tag{8}$$

for any homogeneous elements $a, b \in A$ and $h \in \mathcal{H}$.

Remark 3.6. If $b = 1$ in (8), then we get

$$\mathsf{T}(h \cdot a) = \alpha(h)\mathsf{T}(a). \tag{9}$$

In fact if $\mathcal{H}$ is a usual Hopf algebra, that is, $\mathcal{H} = \mathcal{H}_0$, then (8) and (9) are equivalent ([4]).

Remark 3.7. For homogeneous elements $h_1, h_2 \in \mathcal{H}$, if (8) holds for h_1 and h_2 respectively, then it also holds for h_1h_2. Therefore it suffices to check (8) on the homogeneous generators of a graded Hopf algebra.

The cocyclic module of a graded Hopf algebra with a graded modular pair in involution is defined by Khalkhali and Pourkia in [7]. We restate it as follows:

Proposition 3.8. *Let $\mathcal{H}$ be a graded Hopf algebra with a graded modular pair in involution (α,π). For each $n\in\mathbb{N}$ define $C^n_{G(\alpha,\pi)}(\mathcal{H})=\mathcal{H}^{\otimes n}$ and define k-linear homomorphisms*

$$\begin{aligned}
d_0(h^1,\dots,h^n) &= (1,h^1,\dots,h^n),\\
d_i(h^1,\dots,h^n) &= (h^1,\dots,\Delta(h^i),\dots,h^n) \quad \text{for } 1\le i\le n,\\
d_{n+1}(h^1,\dots,h^n) &= (h^1,\dots,h^n,\pi),\\
s_j(h^1,\dots,h^n) &= (h^1,\dots,\varepsilon(h^{j+1}),\dots,h^n) \quad \text{for } 0\le j<n,\\
\tau(h^1,\dots,h^n) &= S_\alpha(h^1)\cdot(h^2,\dots,h^n,\pi)\\
&= \beta_1\beta_2(S(h^1_{(n)})\cdot h^2,\dots,S(h^1_{(2)})\cdot h^n, S_\alpha(h^1_{(1)})\cdot\pi),\\
d_0(1_k) = 1_{\mathcal{H}},\quad d_1(1_k) &= \pi,\quad \tau(1_k)=1_k,\quad \tau(h)=S_\alpha(h)\cdot\pi,
\end{aligned}$$

where the h^i's are homogeneous elements of $\mathcal{H}$ and

$$\begin{aligned}
\beta_1 &= \prod_{i=1}^{n-1}(-1)^{(|h^1_{(1)}|+\cdots+|h^1_{(i)}|)|h^1_{(i+1)}|},\\
\beta_2 &= \prod_{j=1}^{n-1}(-1)^{|h^1_{(j)}|(|h^2|+|h^3|+\cdots+|h^{n-j+1}|)}.
\end{aligned}$$

The homomorphisms d_i, s_i, τ satisfy the relations (6) and (7), so $(C^\bullet_G(\mathcal{H}),d_i,s_i,\tau)$ defined above is a cocyclic module of the graded Hopf algebra $\mathcal{H}$.

The cyclic cohomology of this cocyclic module is called the *Hopf-cyclic cohomology* of the graded Hopf algebra $\mathcal{H}$ and denoted by $\mathrm{HC}^*_{G(\alpha,\pi)}(\mathcal{H})$. Denote the Hochschild cohomology of this cosimplicial module by $\mathrm{HH}^*_\pi(\mathcal{H})$, as $\mathrm{HH}^*_\pi(\mathcal{H})$ is independent of the character α. Since all group-like elements are of degree 0, the last face map defined in Proposition 3.8 has no additional sign.

Theorem 3.9. *Let $\mathcal{H}$ be a graded Hopf algebra with (α,π) its graded modular pair in involution. Let A be a graded $\mathcal{H}$-module algebra with a graded α-invariant π-trace T. Then the k-linear map $\gamma\colon C^n_{G(\alpha,\pi)}(\mathcal{H})\to C^n_G(A)$ defined below is a morphism of cocyclic modules, that is, for $n>0$,*

$$\gamma(h^1,\dots,h^n)(a_0,\dots,a_n)=(-1)^{\sum_{i=1}^n|h^i|(|a_0|+\cdots+|a_{i-1}|)}\mathsf{T}(a_0(h^1\cdot a_1)\dots(h^n\cdot a_n)),$$

and for $n=0$,

$$\gamma(1_k)(a_0)=\mathsf{T}(a_0).$$

Hence γ induces a characteristic homomorphism of cyclic cohomologies for each n:

$$\gamma^*\colon\ \mathrm{HC}^n_{G(\alpha,\pi)}(\mathcal{H})\to\mathrm{HC}^n_G(A).$$

Proof. We need to prove that γ commutes with d_i, s_j and τ. For homogeneous elements h^i's $\in H$ and a_j's $\in A$,

$$\begin{aligned}
\gamma(d_0(h^1,\dots,h^n))&(a_0,a_1,\dots,a_{n+1})\\
&= \gamma(1,h^1,\dots,h^n)(a_0,a_1,\dots,a_{n+1})\\
&= (-1)^{\sum_{i=1}^n |h^i|(|a_0|+\dots+|a_i|)}\mathsf{T}(a_0a_1(h^1\cdot a_2)\dots(h^n\cdot a_{n+1})),\\
d_0(\gamma(h^1,\dots,h^n))&(a_0,a_1,\dots,a_{n+1})\\
&= \gamma(h^1,\dots,h^n)(a_0a_1,a_2,\dots,a_{n+1})\\
&= (-1)^{\sum_{i=1}^n |h^i|(|a_0|+\dots+|a_i|)}\mathsf{T}(a_0a_1(h^1\cdot a_2)\dots(h^n\cdot a_{n+1}));
\end{aligned}$$

for $1\le i\le n$,

$$\begin{aligned}
\gamma(d_i(h^1,\dots,h^n))&(a_0,a_1,\dots,a_{n+1})\\
&= \gamma(h^1,\dots,h^i_{(1)},h^i_{(2)},\dots,h^n))(a_0,a_1,\dots,a_{n+1})\\
&= (-1)^{\sum_{j=1}^i |h^j|(|a_0|+\dots+|a_{j-1}|)+\sum_{l=i+1}^n |h^l|(|a_0|+\dots+|a_l|)+|h^i_{(2)}||a_i|}\\
&\qquad \mathsf{T}(a_0(h^1\cdot a_1)\dots(h^i_{(1)}\cdot a_i)(h^i_{(2)}\cdot a_{i+1})\dots(h^n\cdot a_{n+1}))\\
&= (-1)^{\sum_{j=1}^i |h^j|(|a_0|+\dots+|a_{j-1}|)+\sum_{l=i+1}^n |h^l|(|a_0|+\dots+|a_l|)}\\
&\qquad \mathsf{T}(a_0(h^1\cdot a_1)\dots(h^i\cdot(a_ia_{i+1}))\dots(h^n\cdot a_{n+1}))\\
&= d_i(\gamma(h^1,\dots,h^n))(a_0,a_1,\dots,a_{n+1});
\end{aligned}$$

for $0\le j\le n-1$, since $\varepsilon(h^{j+1})=0$ unless $|h^{j+1}|=0$,

$$\begin{aligned}
\gamma(s_j(h^1,\dots,h^n))&(a_0,a_1,\dots,a_{n-1})\\
&= \varepsilon(h^{j+1})\gamma(h^1,\dots,h^j,h^{j+2},\dots,h^n)(a_0,a_1,\dots,a_{n-1})\\
&= (-1)^{\sum_{m=1}^j |h^m|(|a_0|+\dots+|a_{m-1}|)+\sum_{l=j+2}^n |h^l|(|a_0|+\dots+|a_{l-2}|)}\varepsilon(h^{j+1})\\
&\qquad \mathsf{T}(a_0(h^1\cdot a_1)\dots(h^j\cdot a_j)(h^{j+2}\cdot a_{j+1})\dots(h^n\cdot a_{n-1}))\\
&= (-1)^{\sum_{m=1}^{j+1} |h^m|(|a_0|+\dots+|a_{m-1}|)+\sum_{l=j+2}^n |h^l|(|a_0|+\dots+|a_{l-2}|)}\\
&\qquad \mathsf{T}(a_0(h^1\cdot a_1)\dots(h^j\cdot a_j)(h^{j+1}\cdot 1)(h^{j+2}\cdot a_{j+1})\dots(h^n\cdot a_{n-1}))\\
&= \gamma(h^1,\dots,h^n)(a_0,\dots,a_j,1,a_{j+1},\dots,a_{n-1})\\
&= s_j(\gamma(h^1,\dots,h^n))(a_0,a_1,\dots,a_{n-1}).
\end{aligned}$$

Note that

$$\begin{aligned}
\gamma(h\cdot(h^1,\dots,h^n))&(a_0,a_1,\dots,a_n)\\
&= (-1)^{|h||a_0|+\sum_{i=1}^n |h^i|(|a_0|+\dots+|a_{i-1}|)}\mathsf{T}(a_0(h\cdot((h^1\cdot a_1)\dots(h^n\cdot a_n)))).
\end{aligned}$$

So we have

$$
\begin{aligned}
&\gamma(\tau(h^1,\dots,h^n))(a_0,a_1,\dots,a_n)\\
&\quad = \gamma(S_\alpha(h^1)\cdot(h^2,\dots,h^n,\pi))(a_0,a_1,\dots,a_n)\\
&\quad = (-1)^{|h^1||a_0|+\sum_{i=2}^n |h^i|(|a_0|+\cdots+|a_{i-2}|)}\\
&\qquad\quad \cdot \mathsf{T}(a_0(S_\alpha(h^1)\cdot((h^2\cdot a_1)\dots(h^n\cdot a_{n-1})(\pi\cdot a_n))))\\
&\quad = (-1)^{\sum_{i=2}^n |h^i|(|a_0|+\cdots+|a_{i-2}|)}\mathsf{T}((h^1\cdot a_0)(h^2\cdot a_1)\dots(h^n\cdot a_{n-1})(\pi\cdot a_n))\\
&\quad = (-1)^{\sum_{i=2}^n |h^i|(|a_0|+\cdots+|a_{i-2}|)+|a_n|(|a_0|+\cdots+|a_{n-1}|+|h^1|+\cdots+|h^n|)}\\
&\qquad\quad \cdot \mathsf{T}(a_n(h^1\cdot a_0)(h^2\cdot a_1)\dots(h^n\cdot a_{n-1}))\\
&\quad = (-1)^{|a_n|(|a_0|+\cdots+|a_{n-1}|)}\gamma(h^1,\dots,h^n)(a_n,a_0,\dots,a_{n-1})\\
&\quad = \tau(\gamma(h^1,\dots,h^n))(a_0,a_1,\dots,a_n).
\end{aligned}
$$

Since $d_{n+1} = \tau d_0$, we get $\gamma d_{n+1} = d_{n+1}\gamma$. □

4. Hochschild and Hopf-cyclic cohomology of the graded Hopf algebra $\mathcal{P}$

We would like to compute the Hochschild cohomology of the graded Hopf algebra $\mathcal{P}$ endowed with a graded modular pair in involution (ε, t^r) of $\mathcal{P}$ using the method introduced in [5].

For a graded Hopf algebra $\mathcal{H}$ with a graded modular pair in involution (α,π), denote by $(C^\bullet(\mathcal{H}), d_\pi)$ the Hochschild cochain complex associated with the cocyclic module of $\mathcal{H}$ defined in Proposition 3.8, that is,

$$k \to \mathcal{H} \to \mathcal{H}^{\otimes 2} \to \cdots \to \mathcal{H}^{\otimes n} \xrightarrow{d_\pi} \mathcal{H}^{\otimes(n+1)} \to \cdots,$$

$$
\begin{aligned}
d_\pi(a^1,\dots,a^n) &= (1,a^1,\dots,a^n) + \sum_{i=1}^n (-1)^i (a^1,\dots,a^i_{(1)},a^i_{(2)},\dots,a^n)\\
&\quad + (-1)^{n+1}(a^1,\dots,a^n,\pi) \quad \text{for } n>0 \text{ and for all } a^i\in\mathcal{H},\\
d_\pi(1_k) &= 1_{\mathcal{H}} - \pi.
\end{aligned}
$$

Define another differential d'_π on $C^\bullet(\mathcal{H})$ by removing the first face map, that is,

$$
\begin{aligned}
d'_\pi(a^1,\dots,a^n) &= \sum_{i=1}^n (-1)^{i-1}(a^1,\dots,a^i_{(1)},a^i_{(2)},\dots,a^n)\\
&\quad + (-1)^n(a^1,\dots,a^n,\pi) \quad \text{for } n>0 \text{ and for all } a^i\in\mathcal{H},\\
d'_\pi(1_k) &= \pi.
\end{aligned}
$$

The complex $(C^\bullet(\mathcal{H}), d'_\pi)$ is acyclic as we can define a contraction $\tilde{\sigma}\colon C^n(\mathcal{H}) \to C^{n-1}(\mathcal{H})$ by

$$\tilde{\sigma}(a^1,\dots,a^n) = \varepsilon(a^1)(a^2,\dots,a^n) \quad \text{for } n>0 \text{ and for all } a^i\in\mathcal{H}$$

and

$$\tilde{\sigma}(1_k) = 0,$$

which satisfies $\tilde{\sigma} d'_\pi + d'_\pi \tilde{\sigma} = \mathrm{id}$.

Define a third differential $d_{(\pi_1,\pi_2)}$ on $C^\bullet(\mathcal{H})$ for two group-like elements π_1 and π_2 of $\mathcal{H}$ by

$$\begin{aligned} d_{(\pi_1,\pi_2)}(a^1,\ldots,a^n) = (\pi_1,a^1,\ldots,a^n) + \sum_{i=1}^{n} (-1)^i (a^1,\ldots,a^i_{(1)},a^i_{(2)},\ldots,a^n) \\ + (-1)^{n+1}(a^1,\ldots,a^n,\pi_2) \quad \text{for } n > 0 \text{ and for all } a^i \in \mathcal{H}, \\ d_{(\pi_1,\pi_2)}(1_k) = \pi_1 - \pi_2. \end{aligned}$$

Then we have

Lemma 4.1. *The cochain complexes $(C^\bullet(\mathcal{H}), d_{(\pi_1,\pi_2)})$ and $(C^\bullet(\mathcal{H}), d_\pi)$ are isomorphic where $\pi = \pi_1^{-1}\pi_2$.*

Indeed, we can construct an isomorphism of complexes Θ from $(C^\bullet(\mathcal{H}), d_{(\pi_1,\pi_2)})$ to $(C^\bullet(\mathcal{H}), d_\pi)$, that is, for each $n \geq 1$ put

$$\Theta(a^1,\ldots,a^n) = (\pi_1^{-1}a^1,\ldots,\pi_1^{-1}a^n) \quad \text{for all } a^i \in \mathcal{H},$$

and for $n = 0$ set $\Theta(1_k) = 1_k$.

Proposition 4.2. *For any integer r, if $n \neq r$, then $\mathrm{HH}^n_{t^r}(\mathcal{P}) = 0$ with $n \geq 0$.*

Proof. We proceed by induction on n. It is easy to see that $\mathrm{HH}^0_{t^r}(\mathcal{P}) = 0$ for all $r \neq 0$. For $n > 0$ with $n+1 \neq r$, let $z \in C^{n+1}(\mathcal{P})$ be an arbitrary Hochschild cocycle, and write $[z] \in \mathrm{HH}^{n+1}_{t^r}(\mathcal{P})$.

Let ϕ_m be the projection from $\mathcal{P}^{\otimes m}$ to its subspace $st \otimes \mathcal{P}^{\otimes(m-1)}$ for each $m > 0$, that is, for all $a^i \in \mathcal{P}$ define

$$\phi_m(a^1,a^2,\ldots,a^m) = \begin{cases} (a^1,a^2,\ldots,a^m) & \text{if } a^1 \in \mathrm{span}_k\{st\}, \\ 0 & \text{otherwise,} \end{cases}$$

and $\phi_0(1_k) = 0$. Then $\phi_\bullet$ becomes an endomorphism of the Hochschild cochain complex $(C^\bullet(\mathcal{P}), d_{t^r})$.

Let ξ_m be the projection from $\mathcal{P}^{\otimes m}$ to its subspace $k \otimes \mathcal{P}^{\otimes(m-1)} \cong \mathcal{P}^{\otimes(m-1)}$ for each $m > 0$, that is, for all $a^i \in \mathcal{P}$ define

$$\xi_m(a^1,a^2,\ldots,a^m) = \begin{cases} a^1(a^2,\ldots,a^m) & \text{if } a^1 \in k, \\ 0 & \text{otherwise.} \end{cases}$$

One can check directly that $d_{t^r}\xi_m + \xi_{m+1}d_{t^r} = \mathrm{id} - \phi_m$ for all $m > 0$. Hence every m-cocycle ($m > 0$) in the Hochschild cochain complex $(C^\bullet(\mathcal{P}), d_{t^r})$ can be represented by an element in the vector space $st \otimes \mathcal{P}^{\otimes(m-1)}$.

Therefore we might write as well $z = st \otimes u$ for $u \in \mathcal{P}^{\otimes n}$. Since $0 = d_{t^r}(st \otimes u) = -(st \otimes d_{(t,t^r)}u)$, the element u is an n-cocycle in the cochain complex $(C^\bullet(\mathcal{P}), d_{(t,t^r)})$. So $\Theta(u)$ is an n-cocycle in the Hochschild cochain complex $(C^\bullet(\mathcal{P}), d_{t^{r-1}})$ by Lemma 4.1. As $n \neq r-1$, by induction $\Theta(u)$ must be a coboundary. Thus there exists an element $w \in \mathcal{P}^{\otimes(n-1)}$ such that $d_{(t,t^r)}w = u$. Then the cocycle $z = d_{t^r}(-st \otimes w)$ is also a coboundary. The proof is finished. □

When $n = r$, we can not only calculate the Hochschild cohomology $\mathrm{HH}^r_{t^r}(\mathcal{P})$ but also find the representative cocycle of $\mathrm{HH}^r_{t^r}(\mathcal{P})$.

Proposition 4.3. *We have* $\mathrm{HH}^0_{t^0=1}(\mathcal{P}) = k$ *and* $\mathrm{HH}^r_{t^r}(\mathcal{P}) = k[(st, st^2, \ldots, st^r)] \cong k$ *for all* $r > 0$.

Proof. It is clear that $\mathrm{HH}^0_{t^0=1}(\mathcal{P}) = k$. For $r > 0$, use induction on r. Note that every Hochschild r-cocycle can be represented by an element in the vector space $st \otimes \mathcal{P}^{\otimes(r-1)}$ due to the proof of the above proposition. When $r = 1$, it is easy to see that the element st is a cocycle but not a coboundary, so $\mathrm{HH}^1_t(\mathcal{P}) = k[st] \neq 0$. Let $st \otimes u \in \mathcal{P}^{\otimes(r+1)}$ be a Hochschild $(r+1)$-cocycle, i.e., $[st \otimes u] \in \mathrm{HH}^{r+1}_{t^{r+1}}(\mathcal{P})$. Then $\Theta(u)$ is an r-cocycle in the Hochschild cochain complex $(C^\bullet(\mathcal{P}), d_{t^r})$, and by induction $\Theta(u) \in \mathrm{span}_k\{(st, st^2, \ldots, st^r)\}$. Hence $u \in \mathrm{span}_k\{(st^2, st^3, \ldots, st^{r+1})\}$, every $(r+1)$-cocycle can be represented by elements in $k[(st, st^2, \ldots, st^{r+1})]$. Now if $(st, st^2, \ldots, st^{r+1})$ is a coboundary, then there exists an element $w \in \mathcal{P}^{\otimes r}$ satisfying $d_{t^r}w = (st, st^2, \ldots, st^{r+1})$. However the degree of w is at least $-r$ and the differential $d_{t^{r+1}}$ preserves the degree, which is a contradiction since the degree of $(st, st^2, \ldots, st^{r+1})$ is $-r-1$. So $[(st, st^2, \ldots, st^{r+1})] \neq 0$. This completes the proof. □

Connes' periodicity exact sequence can be used here to calculate the graded Hopf-cyclic cohomology of $\mathcal{P}$. Therefore we get

Proposition 4.4. 1) *If* $r < 0$, *then* $\mathrm{HC}^n_{G(\varepsilon,t^r)}(\mathcal{P}) = 0$.

2) *If* $r \geq 0$, *then* $\mathrm{HC}^n_{G(\varepsilon,t^r)}(\mathcal{P}) \cong \begin{cases} k & \text{for } n = r + 2m \text{ and all } m \in \mathbb{N}, \\ 0 & \text{otherwise.} \end{cases}$

Corollary 4.5. *For* $r > 0$, *the element* $(st, st^2, \ldots, st^r)$ *is a cyclic* r*-cocycle of* $C^\bullet_{G(\varepsilon,t^r)}(\mathcal{P})$.

Indeed we can check directly that

$$\begin{aligned}\tau(st, st^2, \ldots, st^r) &= (-1)^{r-1}(S(t)st^2, S(t)st^3, \ldots, S(t)st^r, S(st)t^r) \\ &= (-1)^r(st, st^2, \ldots, st^{r-1}, st^r).\end{aligned}$$

5. Application

Let $\mathcal{P}'$ be the graded Hopf algebra equipped with the same generators and relations as $\mathcal{P}$, but endowed with the opposite grading, that is, the element s is of degree -1 in $\mathcal{P}$ and degree 1 in $\mathcal{P}'$. The Hochschild cohomology and the Hopf-cyclic cohomology of a graded Hopf algebra are only concerned with $\mathbb{Z}_2$-grading, so $\mathcal{P}$ and $\mathcal{P}'$ have the same graded modular pairs in involution, the same cyclic cocycles, the same Hochschild cohomology, and the same Hopf-cyclic cohomology.

Proposition 5.1. *There exists a unique nontrivial graded Hopf pairing $\langle -, - \rangle$ of $\mathcal{P}'$ and $\mathcal{P}$ satisfying*

$$\langle s, s \rangle = \lambda_1 \quad \textit{and} \quad \langle t, t \rangle = \lambda_2$$

if and only if either $\lambda_1 = 0$ and $\lambda_2 \neq 0$, or $\lambda_1 \neq 0$ and $\lambda_2 = 1$.

Proof. In order to construct a graded Hopf pairing, we must have $\lambda_2 \neq 0$ and

$$\langle st - ts, s \rangle = \lambda_1 - \lambda_2^{-1} \lambda_1 = 0.$$

Conversely, the graded Hopf pairing is uniquely determined by λ_1 and λ_2, that is, for all $m, n \in \mathbb{Z}$,

$$\langle t^n, t^m \rangle = \lambda_2^{nm}, \quad \langle st^n, st^m \rangle = \lambda_1 \lambda_2^{nm}, \quad \langle t^n, st^m \rangle = \langle st^n, t^m \rangle = 0.$$

One can check directly that it is a graded Hopf paring of $\mathcal{P}'$ and $\mathcal{P}$. □

By Theorem 2.1 a DGA (A, ∂) is also a left graded $\mathcal{P}$-comodule algebra. Since $\mathcal{P}'$ is graded commutative, by Proposition 1.12, the DGA (A, ∂) is also a left graded $\mathcal{P}'$-module algebra. Let $\lambda_1 = \lambda_2 = 1$. Then the left action of $\mathcal{P}'$ on A is given by

$$t \cdot a = a \quad \text{and} \quad s \cdot a = \partial(a) \quad \text{for all } a \in A. \tag{10}$$

As (ε, t^r) is a graded modular pair in involution of $\mathcal{P}'$, by Definition 3.5, a k-linear map $\mathsf{T}\colon A \to k$ is a graded ε-invariant t^r-trace on A if the following three identities hold for any homogeneous elements $a, b \in A$:

$$\begin{aligned} \mathsf{T}(ab) &= (-1)^{|a||b|}\mathsf{T}(b(t^r \cdot a)), \\ \mathsf{T}((t \cdot a)b) &= \mathsf{T}(a(S(t) \cdot b)), \\ \mathsf{T}((s \cdot a)b) &= (-1)^{|a|}\mathsf{T}(a(S(s) \cdot b)). \end{aligned}$$

Due to (10) the above three identities are equivalent to the following two identities:

$$\begin{aligned} \mathsf{T}(ab) &= (-1)^{|a||b|}\mathsf{T}(ba), \\ \mathsf{T}((\partial a)b) &= (-1)^{|a|+1}\mathsf{T}(a(\partial b)). \end{aligned} \tag{11}$$

Since $\mathsf{T}(s \cdot (ab)) = \mathsf{T}(\partial(ab)) = \mathsf{T}((\partial a)b) + (-1)^{|a|}\mathsf{T}(a(\partial b))$, the identities (11) needed by a graded ε-invariant t^r-trace T are equivalent to the requirements of a closed graded trace on A.

Corollary 5.2. *Let (A, ∂) be a DGA which is also a graded $\mathcal{P}'$-module algebra via (10). Then a k-linear map T is a closed graded trace on A if and only if it is a graded ε-invariant t^r-trace on A.*

Applying Theorem 3.9 to $\mathcal{P}'$, we obtain the following corollary.

Corollary 5.3. *Let (A, ∂) be a DGA with a closed graded trace T. Then we have a characteristic homomorphism $\gamma_{\mathsf{T}}^{*}\colon \mathrm{HC}^r_{G(\varepsilon,t^r)}(\mathcal{P}') \to \mathrm{HC}^r_G(A)$ for all $r \geq 0$, which is induced from $\gamma_{\mathsf{T}}\colon C^r_{G(\varepsilon,t^r)}(\mathcal{P}') \to C^r_G(A)$. In particular, for $r > 0$,*

$$\gamma_{\mathsf{T}}(st, \dots, st^r)(a_0, \dots, a_r) = (-1)^{\sum_{i=0}^{r-1} |a_0|+\dots+|a_i|}\mathsf{T}(a_0(\partial a_1)\dots(\partial a_r)),$$

and for $r = 0$, $\gamma_{\mathsf{T}}(1_k)(a_0) = \mathsf{T}(a_0)$, where the a_i's are homogeneous elements of A.

Thus, if r is an even (resp. odd) positive integer, then

$$(-1)^{|a_1|+|a_3|+\dots+|a_{r-1}|}\mathsf{T}(a_0(\partial a_1)\dots(\partial a_r))$$
$$(resp. (-1)^{|a_0|+|a_2|+\dots+|a_{r-1}|}\mathsf{T}(a_0(\partial a_1)\dots(\partial a_r)))$$

defines a cyclic r-cocycle on the graded algebra A. If $r = 0$, then the closed graded trace T itself is a cyclic 0-cocycle on the graded algebra A.

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