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Erratum to "C*-algebras associated with integral domains and crossed products by actions on adele spaces"

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1. Introduction

In [Cu-Li], we had computed the K-theory for C*-algebras associated with rings of integers in number fields. Unfortunately, there was a miscalculation in [Cu-Li], §6.4, case c), where the case of number fields with roots of unity $+1, -1$ and with an even strictly positive number of real places was treated (i.e., the case where $\#\{v_{\mathbb{R}}\} \geq 2$ even). In this case the final result for the K-theory of the ring C*-algebra $\mathfrak{A}[\mathcal{o}]$ of the ring of integers $\mathcal{o}$ of our number field should not be $K_*(\mathfrak{A}[\mathcal{o}]) \cong \Lambda^*(\Gamma) \oplus ((\mathbb{Z}/2\mathbb{Z}) \otimes_{\mathbb{Z}} \Lambda^*(\Gamma))$, but $K_*(\mathfrak{A}[\mathcal{o}]) \cong \Lambda^*(\Gamma)$. This means that the torsion-free part in §6.4, case c) of [Cu-Li] was determined correctly, but the torsion part was not computed correctly. The correct computation shows that the K-theory of the ring C*-algebra is torsion-free.

On the whole, the correct final result is the following (compare [Cu-Li], §6): Let K be a number field with roots of unity $\mu = \{\pm 1\}$. Choose a free abelian subgroup Γ of $K^\times$ such that $K^\times = \mu \times \Gamma$. We obtain for the K-theory of the ring C*-algebra $\mathfrak{A}[\mathcal{o}]$ attached to the ring of integers $\mathcal{o}$ of K:

$$K_*(\mathfrak{A}[\mathcal{o}]) \cong \begin{cases} K_0(C^*(\mu)) \otimes_{\mathbb{Z}} \Lambda^*(\Gamma) & \text{if } \#\{v_{\mathbb{R}}\} = 0, \\ \Lambda^*(\Gamma) & \text{if } \#\{v_{\mathbb{R}}\} \geq 1. \end{cases}$$

The distinction between the formulas in the two different cases corresponds to a natural identification on the level of generators. As abstract groups one obtains the same K-theory independently of the number of real embeddings.

2. The correct computation

Let us first of all explain what went wrong in our original computation in [Cu-Li], §6.4, case c): Let $\theta \in \mathrm{Aut}(C_0(\mathbb{R}))$ be the flip, i.e., $\theta(f)(x) = f(-x)$ for all $f \in C_0(\mathbb{R})$.

By equivariant Bott periodicity, we know that

$$K_i(C_0(\mathbb{R}^2) \rtimes_{\theta\otimes\theta} \mathbb{Z}/2\mathbb{Z}) \cong \begin{cases} \mathbb{Z}^2 & \text{if } i = 0, \\ \{0\} & \text{if } i = 1. \end{cases}$$

In the first part of the proof of Lemma 6.4 in [Cu-Li], we have claimed that the automorphism $\mathrm{id} \otimes \theta$ of $C_0(\mathbb{R}^2) \rtimes_{\theta\otimes\theta} \mathbb{Z}/2\mathbb{Z}$ acts as $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ in K-theory (in [Cu-Li], $\mathrm{id} \otimes \theta$ is denoted by $\hat{\beta}_{(1,-1)}$). This however cannot be true. The reason is that using the Pimsner–Voiculescu sequence, we would obtain as an immediate consequence that $K_0(C_0(\mathbb{R}^2) \rtimes_{\theta\otimes\theta} \mathbb{Z}/2\mathbb{Z} \rtimes_{\mathrm{id}\otimes\theta} \mathbb{Z}) \cong \mathbb{Z} \oplus (\mathbb{Z}/2\mathbb{Z})$. But as Lemma 2.1 below shows, the correct result is $K_0(C_0(\mathbb{R}^2) \rtimes_{\theta\otimes\theta} \mathbb{Z}/2\mathbb{Z} \rtimes_{\mathrm{id}\otimes\theta} \mathbb{Z}) \cong \mathbb{Z}$.

In the first part of the proof of Lemma 6.4 in [Cu-Li], we had considered the number field $K = \mathbb{Q}[\sqrt{2}]$ with ring of integers $\mathcal{o} = \mathbb{Z} + \mathbb{Z}\sqrt{2}$. The problem in our original computation was that we have assumed that in this particular case, the element $[u^1]_1 \times [u^{\sqrt{2}}]_1$ is part of a $\mathbb{Z}$-basis for $G_{\mathrm{inf}} \subseteq K_0(C^*(\mathcal{o} \rtimes \mu))$ (in the terminology of [Cu-Li], Lemma 6.1). But this is not the case, only up to finite index. This is why Lemma 6.4 in [Cu-Li] is false.

Here is now the correct computation:

Lemma 2.1. *$K_i(C_0(\mathbb{R}^2) \rtimes_{\theta\otimes\theta} \mathbb{Z}/2\mathbb{Z} \rtimes_{\mathrm{id}\otimes\theta} \mathbb{Z}) \cong \mathbb{Z}$ for $i = 0, 1$.*

Proof. The first step is the following simple observation:

$$\begin{aligned} & C_0(\mathbb{R}^2) \rtimes_{\theta\otimes\theta} \mathbb{Z}/2\mathbb{Z} \rtimes_{\mathrm{id}\otimes\theta} \mathbb{Z}/2\mathbb{Z} \\ &\quad \cong (C_0(\mathbb{R}) \otimes C_0(\mathbb{R})) \rtimes_{\theta\otimes\theta} \mathbb{Z}/2\mathbb{Z} \rtimes_{\mathrm{id}\otimes\theta} \mathbb{Z}/2\mathbb{Z} \\ &\quad \cong (C_0(\mathbb{R}) \otimes C_0(\mathbb{R})) \rtimes_{\theta\otimes\mathrm{id}} \mathbb{Z}/2\mathbb{Z} \rtimes_{\mathrm{id}\otimes\theta} \mathbb{Z}/2\mathbb{Z} \\ &\quad \cong ((C_0(\mathbb{R}) \rtimes_\theta \mathbb{Z}/2\mathbb{Z}) \otimes C_0(\mathbb{R})) \rtimes_{\mathrm{id}\otimes\theta} \mathbb{Z}/2\mathbb{Z} \\ &\quad \cong [C_0(\mathbb{R}) \rtimes_\theta \mathbb{Z}/2\mathbb{Z}] \otimes [C_0(\mathbb{R}) \rtimes_\theta \mathbb{Z}/2\mathbb{Z}]. \end{aligned} \tag{1}$$

To get from the second to the third line, we just made use of the automorphism $(\mathbb{Z}/2\mathbb{Z})^2 \cong (\mathbb{Z}/2\mathbb{Z})^2$ given by $t_1 \mapsto t_1 t_2$, $t_2 \mapsto t_2$. Here t_1 and t_2 are the generators of the two copies of $\mathbb{Z}/2\mathbb{Z}$.

Since $K_0(C_0(\mathbb{R}) \rtimes_\theta \mathbb{Z}/2\mathbb{Z}) \cong \mathbb{Z}$ and $K_1(C_0(\mathbb{R}) \rtimes_\theta \mathbb{Z}/2\mathbb{Z}) \cong \{0\}$ (see [Cu-Li], §3.3, Equation (12)), we deduce

$$K_i(C_0(\mathbb{R}^2) \rtimes_{\theta\otimes\theta} \mathbb{Z}/2\mathbb{Z} \rtimes_{\mathrm{id}\otimes\theta} \mathbb{Z}/2\mathbb{Z}) \cong \begin{cases} \mathbb{Z} & \text{if } i = 0, \\ \{0\} & \text{if } i = 1. \end{cases} \tag{2}$$

Now consider the automorphism $(\mathrm{id} \otimes \theta)\hat{}$ of $C_0(\mathbb{R}^2) \rtimes_{\theta\otimes\theta} \mathbb{Z}/2\mathbb{Z} \rtimes_{\mathrm{id}\otimes\theta} \mathbb{Z}/2\mathbb{Z}$ which is dual to the action of the second copy of $\mathbb{Z}/2\mathbb{Z}$. Under the isomorphism (1), $(\mathrm{id} \otimes \theta)\hat{}$ corresponds to the automorphism $\hat{\theta} \otimes \hat{\theta}$, where $\hat{\theta}$ is the automorphism on

$C_0(\mathbb{R}) \rtimes_\theta \mathbb{Z}/2\mathbb{Z}$ dual to θ. Since $\hat{\theta}$ is either id or $-$id on $K_0(C_0(\mathbb{R}) \rtimes_\theta \mathbb{Z}/2\mathbb{Z}) \cong \mathbb{Z}$, we conclude that

$$((\mathrm{id} \otimes \theta)\hat{\ })_* = \mathrm{id} \quad \text{on } K_0(C_0(\mathbb{R}^2) \rtimes_{\theta \otimes \theta} \mathbb{Z}/2\mathbb{Z} \rtimes_{\mathrm{id} \otimes \theta} \mathbb{Z}/2\mathbb{Z}) \cong \mathbb{Z}. \tag{3}$$

Plugging (2) and (3) into the exact sequence from [Bla], Theorem 10.7.1, which connects the K-theory of the crossed products by $\mathbb{Z}$ and by $\mathbb{Z}/2$ induced by id $\otimes\, \theta$ respectively, we obtain

$$K_i(C_0(\mathbb{R}^2) \rtimes_{\theta \otimes \theta} \mathbb{Z}/2\mathbb{Z} \rtimes_{\mathrm{id} \otimes \theta} \mathbb{Z}) \cong \mathbb{Z} \quad \text{for } i = 0, 1. \qquad \square$$

With this lemma, the computation of the K-theory of the ring C*-algebras follows the same line of arguments as in [Cu-Li]. Let us explain this briefly using the same notations as in the introduction and as in [Cu-Li], §6.4, case c). Combining Equation (4) in [Cu-Li] with Corollary 4.2 of [Cu-Li] and using a refined version of Lemma 6.3 in [Cu-Li], it is straightforward to see that the K-theory of $\mathfrak{A}[\mathcal{o}]$ coincides with the K-theory of $C_0(\mathbb{A}_\infty) \rtimes K^\times$. As in [Cu-Li], §6.4, case c), let $K^\times = \mu \times \Gamma$ and choose a $\mathbb{Z}$-basis $\{p, p_1, p_2, \dots\}$ of Γ, with $p \in \mathbb{Z}_{>0}$. We can arrange that $\#\{v_\mathbb{R} : v_\mathbb{R}(p_1) < 0\}$ is odd and $\#\{v_\mathbb{R} : v_\mathbb{R}(p_i) < 0\}$ is even for all $i > 1$. Let $\Gamma_m = \langle p, \dots, p_m \rangle$ and $\Gamma'_m = \langle p, p_2 \dots, p_m \rangle$. An iterative application of the Pimsner–Voiculescu sequence gives

$$K_*(C_0(\mathbb{A}_\infty) \rtimes (\mu \times \Gamma_m)) \cong \Lambda^*(\Gamma'_m)$$

and thus

$$K_*(\mathfrak{A}[\mathcal{o}]) \cong \Lambda^*(\Gamma).$$

References

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