

On the spectral characterization of manifolds

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Abstract. We show that the first five of the axioms we had formulated on spectral triples suffice (in a slightly stronger form) to characterize the spectral triples associated to smooth compact manifolds. The algebra, which is assumed to be commutative, is shown to be isomorphic to the algebra of all smooth functions on a unique smooth oriented compact manifold, while the operator is shown to be of Dirac type and the metric to be Riemannian.

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1. Introduction

The problem of spectral characterization of manifolds was initially formulated as an open question in [12]. The issue is to show that under the simple conditions of [12] on a spectral triple $(\mathcal{A}, \mathcal{H}, D)$, with \mathcal{A} commutative, the algebra \mathcal{A} is the algebra $C^\infty(X)$ of smooth functions on a (unique) smooth compact manifold X . The five conditions ([12]), in dimension p , are:

- (1) The n -th characteristic value of the resolvent of D is $O(n^{-1/p})$.
- (2) $[[D, a], b] = 0$ for all $a, b \in \mathcal{A}$.
- (3) For any $a \in \mathcal{A}$ both a and $[D, a]$ belong to the domain of δ^m , for any integer m where δ is the derivation: $\delta(T) = [[D], T]$.
- (4) There exists a Hochschild cycle $c \in Z_p(\mathcal{A}, \mathcal{A})$ such that $\pi_D(c) = 1$ for p odd, while for p even, $\pi_D(c) = \gamma$ is a $\mathbb{Z}/2$ grading.
- (5) Viewed as an \mathcal{A} -module the space $\mathcal{H}_\infty = \bigcap \text{Dom } D^m$ is finite and projective. Moreover the following equality defines a hermitian structure $(\cdot | \cdot)$ on this module: $(\xi, a \eta) = f a(\xi | \eta) |D|^{-p}$ for all $a \in \mathcal{A}$, for all $\xi, \eta \in \mathcal{H}_\infty$.

The notations are recalled at the beginning of §2 below. The strategy of proof was outlined briefly in [12]. It consists in using the components a_α^j ($j > 0$) of the cycle $c = \sum a_\alpha^0 \otimes a_\alpha^1 \otimes \cdots \otimes a_\alpha^p$ as tentative local charts. There are three basic difficulties:

- a) Show that the spectrum X of \mathcal{A} is *large enough* so that the range of “local charts” a_α contains an open set in \mathbb{R}^p .
- b) Show that the joint spectral measure of the components a_α^j ($j > 0$) of a “local chart” is the Lebesgue measure.
- c) Apply the basic inequality ([9], [10], Proposition IV.3.14) giving an upper bound on the Voiculescu obstruction [26] and use [26], Theorem 4.5, to show that the “local charts” are locally injective.

In a recent paper [23], Rennie and Varilly considered the above challenging problem. The paper [23] is a courageous attempt which contains a number of interesting ideas and a useful smooth calculus but also, unfortunately, several gaps, each being enough to invalidate the proof of the claimed result.

I will show in this paper how to prove a), b), c). I have tried to be very careful and give detailed proofs. The way to prove a) uses a new ingredient: the Implicit Function Theorem (whose presence is not a real surprise). We shall first assume that continuous $*$ -derivations of \mathcal{A} exponentiate, i.e., are generators of one-parameter groups (of automorphisms of \mathcal{A}). Then most of the work, done in §§5, 6, is to show that this hypothesis can be removed. In this very technical part of the paper we show that enough self-adjoint derivations of \mathcal{A} exponentiate. We first prove in §5 that enough derivations are *dissipative* for the C^* -algebra norm. We then proceed in §6 and use the self-adjointness of D and the third condition (regularity) in the strong form, to show the surjectivity of the resolvent, and apply the Hille–Yosida Theorem to integrate these derivations into one-parameter groups of automorphisms of the C^* -algebra. We then show that they are continuous for the Sobolev norms and define automorphisms of \mathcal{A} .

To prove b) one needs a key result which is the analogue in our context of the quasi-invariance under diffeomorphisms of the smooth measure class on a manifold, whose replacement in our case is given by the Dixmier trace. This is shown in

Proposition 6.16 at the end of §6. We then prove in §7 the required absolute continuity of the spectral measure using a smearing argument. In §8 we show the required inequality between the multiplicity of the map s_α and the spectral multiplicity of the a_α^j .

To prove c) a new strategy is required. Roughly one needs to know that the multiplicity function of a tentative local coordinate system is locally bounded while the information one obtains just by applying the strategy outlined in [12] (and pursued in [23]) is that it is a lower semi-continuous¹ integrable function. Typical examples of Lebesgue negligible dense G_δ sets² show that, as such, the situation is hopeless. In order to solve this problem, one needs a local form of the basic inequality ([9], [10], Proposition IV.3.14) giving an upper bound on the Voiculescu obstruction. We prove this result in §9. This key result is combined with Voiculescu's Theorem (Theorem 4.5 of [26]) and with the initial implicit function technique to conclude the proof in §11. Our main result can be stated as follows (cf. Theorem 11.3):

Theorem 1.1. *Let $(\mathcal{A}, \mathcal{H}, D)$ be a spectral triple, with \mathcal{A} commutative, fulfilling the first five conditions of [12] (cf. §2) in a slightly stronger form, i.e., we assume that:*

- *The regularity holds for all \mathcal{A} -endomorphisms of $\bigcap \text{Dom } D^m$.*
- *The Hochschild cycle c is antisymmetric.*

Then there exists a compact oriented smooth manifold X such that \mathcal{A} is the algebra $C^\infty(X)$ of smooth functions on X .

Moreover every compact oriented smooth manifold appears in this spectral manner. Our next result is the following variant (Theorem 11.5):

Theorem 1.2. *Let $(\mathcal{A}, \mathcal{H}, D)$ be a spectral triple with \mathcal{A} commutative, fulfilling the first five conditions of [12] (cf. §2) with the cycle c antisymmetric. Assume that the multiplicity of the action of \mathcal{A}'' in \mathcal{H} is $2^{p/2}$. Then there exists a smooth oriented compact (spin^c) manifold X such that $\mathcal{A} = C^\infty(X)$.*

This multiplicity hypothesis is a weak form of the Poincaré duality condition 6 of [12] and thus the above theorem can be seen as the solution of the original problem formulated in [12] and gives a characterization of spin^c manifolds. It follows from [12] (cf. [16] for the proof) that the operator D is then a Dirac operator. The reality condition selects spin manifolds among spin^c , and the spectral action ([4]) selects the Levi-Civita connection.

Finally we make a few remarks in §12. The first describes a different perspective on our main result. As explained many times, it is only because one drops commutativity that variables with continuous range can coexist with infinitesimal variables

¹The inverse image of $]a, \infty]$ is open.

²Countable intersection of open sets.

(which only affect finitely many values larger than a given ε). In the classical formulation of variables, as maps from a set X to the real numbers, infinitesimal variables cannot coexist with continuous variables. The formalism of quantum mechanics and the uniqueness of the separable infinite dimensional Hilbert space cure this problem. Using this formalism, variables with continuous range (i.e., self-adjoint operators with continuous spectrum) coexist, in the same operator theoretic framework, with variables with countable range, such as the infinitesimal ones (i.e., compact operators). The only new fact is that they do not commute. The content of Theorem 1.2 can be expressed in a suggestive manner from this coexistence between the continuum and the discrete. We fix the integer p and $N = 2^{\lfloor p/2 \rfloor}$ where $\lfloor p/2 \rfloor$ is the integral part of $p/2$. The continuum will only be used through its “measure theoretic” content. This is captured by a commutative von Neumann algebra M and, provided there is no atomic part in M , this algebra is then unique (up to isomorphism). It is uniquely represented in Hilbert space \mathcal{H} (which we fix once and for all, as a universal stage) once the spectral multiplicity is fixed equal to N . Thus the pair (M, \mathcal{H}) is unique (up to isomorphism). Let us now consider (separately first) an infinitesimal ds , i.e., a self-adjoint compact operator in \mathcal{H} . Equivalently we can talk about its inverse D which is unbounded and self-adjoint. We assume that ds is an infinitesimal of finite order $\alpha = \frac{1}{p}$. The information contained in the operator ds is entirely captured by a list of real numbers, namely the eigenvalues of ds (with their multiplicity). This list determines uniquely (up to isomorphism) the pair (\mathcal{H}, D) . Theorem 1.2 can now be restated as the birth of a geometry from the coexistence of (M, \mathcal{H}) with (\mathcal{H}, D) . This coexistence is encoded by a unitary isomorphism F between the Hilbert space of the canonical pair (M, \mathcal{H}) and the Hilbert space of the canonical pair (\mathcal{H}, D) . Thus the full information on the geometric space is subdivided into two pieces:

- (1) the list of eigenvalues of D ,
- (2) the unitary F .

We point out in §12 the analogy between these parameters for geometry and the parameters of the Yukawa coupling of the Standard Model ([7]) which are encoded similarly by

- (1) the list of masses,
- (2) the CKM matrix C .

This analogy as well as the precise definition of the corresponding unitary invariant of Riemannian geometry will be dealt with in details in the companion paper [13].

The second remark recalls a result of M. Hilsum on finite propagation (cf. [18]). We then discuss briefly in §12 the variations dealing with real analytic manifolds, non-integral dimensions and the non-commutative case.

We end with two appendices. In the first, §13, we discuss equivalent formulations of the regularity condition. In the second, §14, we recall the basic properties of the Dixmier trace and its relation with the heat expansion.

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2. Preliminaries

Let us recall the conditions for *commutative* geometry as formulated in [12]. We shall only use the first five conditions.

We let $(\mathcal{A}, \mathcal{H}, D)$ be a spectral triple, thus \mathcal{H} is a Hilbert space, \mathcal{A} an involutive algebra represented in \mathcal{H} and D is a self-adjoint operator in \mathcal{H} . We assume that \mathcal{A} is *commutative*. We are given an integer p which controls the dimension of our space. The conditions are:

- 1) Dimension: *The n -th characteristic value of the resolvent of D is $O(n^{-1/p})$.*
- 2) Order one: $[[D, f], g] = 0$ for all $f, g \in \mathcal{A}$.
We let $\delta(T) = [|D|, T]$ be the commutator³ with the absolute value of D :
- 3) Regularity: *For any $a \in \mathcal{A}$ both a and $[D, a]$ belong to the domain of δ^m , for any integer m .*

We let π_D be the linear map given by

$$\pi_D(a^0 \otimes a^1 \otimes \cdots \otimes a^p) = a^0 [D, a^1] \dots [D, a^p] \quad \text{for all } a^j \in \mathcal{A}.$$

- 4) Orientability: *There exists a Hochschild cycle $c \in Z_p(\mathcal{A}, \mathcal{A})$ such that $\pi_D(c) = 1$ for p odd, while for p even, $\pi_D(c) = \gamma$ satisfies*

$$\gamma = \gamma^*, \quad \gamma^2 = 1, \quad \gamma D = -D\gamma.$$

- 5) Finiteness and absolute continuity: *Viewed as an \mathcal{A} -module the space $\mathcal{H}_\infty = \bigcap_m \text{Dom } D^m$ is finite and projective. Moreover the following equality defines a hermitian structure $(\cdot | \cdot)$ on this module,*

$$(\xi, a \eta) = \int a(\xi | \eta) |D|^{-p} \quad \text{for all } a \in \mathcal{A} \text{ and all } \xi, \eta \in \mathcal{H}_\infty. \quad (1)$$

In other words the module can be written as $\mathcal{H}_\infty = e\mathcal{A}^n$ with $e = e^* \in M_n(\mathcal{A})$ defining the Hermitian structure so that

$$(\xi | \eta) = \sum \xi_i^* \eta_i \in \mathcal{A} \quad \text{for all } \xi, \eta \in e\mathcal{A}^n.$$

It follows from condition 4) and from⁴ [10], Theorem 8, IV.2. γ , and [16] that the operators $a|D|^{-p}$, $a \in \mathcal{A}$, are *measurable* ([10], Definition 7, IV.2. β) so that the

³The domain of δ is the set of bounded operators T with $T \text{ Dom } |D| \subset \text{Dom } |D|$ and $\delta(T)$ bounded.

⁴We shall not use this result in an essential manner since one can just fix a choice of Dixmier trace Tr_ω throughout the proof.

coefficient $f a |D|^{-p}$ of the logarithmic divergence of their trace is unambiguously defined.

It follows from condition 5 that the algebra \mathcal{A} is uniquely determined inside its weak closure \mathcal{A}'' (which is also the bicommutant of \mathcal{A} in \mathcal{H}) by the equality

$$\mathcal{A} = \{T \in \mathcal{A}'' \mid T \in \bigcap_{m>0} \text{Dom } \delta^m\}.$$

This was stated without proof in [12] and we give the argument below:

Lemma 2.1. *The following conditions are equivalent for $T \in \mathcal{A}''$:*

- (1) $T \in \mathcal{A}$.
- (2) $[D, T]$ is bounded and both T and $[D, T]$ belong to the domain of δ^m , for any integer m .
- (3) T belongs to the domain of δ^m , for any integer m .
- (4) $T\mathcal{H}_\infty \subset \mathcal{H}_\infty$.

Proof. Let us assume the fourth property. Then T defines an endomorphism of the finite projective module $\mathcal{H}_\infty = e\mathcal{A}^n$ over \mathcal{A} . As any endomorphism T is of the form,

$$T = e[a_{ij}]e, \quad a_{ij} \in \mathcal{A}, \quad (2)$$

i.e., it is the compression of a matrix $a = [a_{ij}] \in M_n(\mathcal{A})$.

Let us show that since T belongs to the weak closure of \mathcal{A} one can choose $a_{ij} = x\delta_{ij}$ for some element x of \mathcal{A} . The norm closure A of \mathcal{A} in $\mathcal{L}(\mathcal{H})$ is a commutative C^* -algebra, $A = C(X)$ for some compact space X , and since \mathcal{A} is a subalgebra of $\mathcal{L}(\mathcal{H})$ it injects in A . The equality

$$\lambda(f) = \int f |D|^{-p} \quad \text{for all } f \in A \quad (3)$$

defines a positive measure λ on X . We let $\mathcal{E} = eA^n$ be the induced finite projective module over A , which is intrinsically defined as $\mathcal{E} = \mathcal{H}_\infty \otimes_{\mathcal{A}} A$. We let S be the hermitian vector bundle on X such that $\mathcal{E} = C(X, S)$. By the absolute continuity relation (1), the representation of $A = C(X)$ in \mathcal{H} is obtained from its action in $L^2(X, \lambda)$ by the tensor product

$$\mathcal{H} = \mathcal{E} \otimes_A L^2(X, \lambda) = eL^2(X, \lambda)^n = L^2(X, \lambda, S). \quad (4)$$

This shows that the weak closure $\mathcal{A}'' = A''$ of \mathcal{A} in \mathcal{H} is given by the diagonal action of $L^\infty(X, \lambda)$ in $eL^2(X, \lambda)^n$. Thus, since $T \in \mathcal{A}''$, there exists $f \in L^\infty(X, \lambda)$ such that $T = ef$. It follows that the matrix $ea e$ belongs to the center of $eM_n(\mathcal{A})e$. This center is $e(1 \otimes \mathcal{A})$ and thus T agrees with an element of \mathcal{A} , which proves the implication (4) \implies (1). To be more specific, and for later use, let us give a formula for an element $x \in \mathcal{A}$ such that $T = ex$ in terms of the matrix elements

$t_{ij} \in \mathcal{A}$ of $T = e[a_{ij}]e$. First the fact that T belongs to the center of the algebra $\mathcal{B} = eM_n(\mathcal{A})e$ of endomorphisms of \mathcal{H}_∞ can be seen directly since any such endomorphism S is automatically continuous in \mathcal{H} using (4). Thus since $T \in \mathcal{A}''$ one has $ST = TS$. Since e is a self-adjoint idempotent and \mathcal{A} injects in $C(X)$ the element $\tau = \text{Tr}(e) = \sum e_{jj} \in \mathcal{A}$ is determined by its image in A which is just the function $\chi \in X \mapsto \dim S_\chi \in \{0, 1, \dots, n\}$. This determines $n + 1$ self-adjoint idempotents $p_j \in A$ by

$$\tau = \text{Tr}(e) = \sum j p_j, \quad \sum p_j = 1. \quad (5)$$

To check that $p_j \in \mathcal{A}$ it is enough to show that $p_j = P_j(\tau)$ where P_j is a polynomial with

$$P_j(k) = 0 \quad \text{for all } k \neq j, \quad 0 \leq k \leq n, \quad P_j(j) = 1.$$

One then has the following formula⁵ for x :

$$x = \left(\sum t_{ii} \right) \sum_{j>0} \frac{1}{j} p_j \in \mathcal{A}. \quad (6)$$

As T belongs to the center of $eM_n(C(X))e$ one gets an equality $T = ef$ for $f \in C(X)$ and working at every point $\chi \in X$ one then shows that $T = ex$.

The implication (1) \implies (2) follows from the regularity, and (2) \implies (3) is immediate. To show the implication (3) \implies (4) one uses the definition of \mathcal{H}_∞ as the intersection of domains of powers of $|D|$ and the implication

$$T \in \text{Dom } \delta^m, \quad \xi \in \text{Dom } |D|^m \implies T\xi \in \text{Dom } |D|^m$$

with the formula

$$|D|^m T\xi = \sum_{k=0}^m \binom{m}{k} \delta^k(T) |D|^{m-k} \xi \quad \text{for all } \xi \in \text{Dom } |D|^m, \quad (7)$$

which is proved by induction on m . More precisely this gives an estimate of the norms but one has to care for the domains and proceed as follows. By definition any $T \in \text{Dom } \delta$ preserves the domain $\text{Dom } |D|$ thus one gets (7) for $m = 1$. Let now $T \in \text{Dom } \delta^2$, i.e., $T \in \text{Dom } \delta$ and $\delta(T) \in \text{Dom } \delta$. Let $\xi \in \text{Dom } |D|^2$. Then since $T \in \text{Dom } \delta$ and $|D|\xi \in \text{Dom } |D|$, one has $T|D|\xi \in \text{Dom } |D|$. One has $\delta(T)\xi = |D|T\xi - T|D|\xi$ where both terms make sense separately. Since $\delta(T) \in \text{Dom } \delta$ one has $\delta(T)\text{Dom } |D| \subset \text{Dom } |D|$. Thus $\delta(T)\xi \in \text{Dom } |D|$. Hence $|D|T\xi = \delta(T)\xi + T|D|\xi \in \text{Dom } |D|$ so that T preserves $\text{Dom } |D|^2$. Moreover one gets (7) for $m = 2$ as an equality valid on any vector $\xi \in \text{Dom } |D|^2$. One can now proceed by induction on m . We assume to have shown that

⁵Note that $p_0 = 0$ because of the faithfulness of the action of \mathcal{A} in Hilbert space together with condition 5).

- for $q \leq m$, $S \in \text{Dom } \delta^q \implies S \text{ Dom } |D|^q \subset \text{Dom } |D|^q$,
- (7) holds for all $n \leq m$.

For $T \in \text{Dom } \delta^{m+1}$ and $\xi \in \text{Dom } |D|^{m+1}$, one has $\xi \in \text{Dom } |D|^m$ and one can use the induction hypothesis to get

$$|D|^m T \xi = \sum_{k=0}^m \binom{m}{k} \delta^k(T) |D|^{m-k} \xi.$$

Let us show that $\delta^k(T) |D|^{m-k} \xi \in \text{Dom } |D|$. One has $\delta^k(T) \in \text{Dom } \delta^{m+1-k} \subset \text{Dom } \delta$ and $|D|^{m-k} \xi \in \text{Dom } |D|^{1+k} \subset \text{Dom } |D|$ which gives the result. Thus each term of the sum belongs to $\text{Dom } |D|$ and one has

$$|D|^{m+1} T \xi = \sum_{k=0}^m \binom{m}{k} |D| \delta^k(T) |D|^{m-k} \xi.$$

Moreover, as $\delta^k(T) \in \text{Dom } \delta$ and $|D|^{m-k} \xi \in \text{Dom } |D|$ one has

$$|D| \delta^k(T) |D|^{m-k} \xi = \delta^{k+1}(T) |D|^{m-k} \xi + \delta^k(T) |D|^{m-k+1} \xi,$$

which gives (7) for $n + 1$. □

This shows that the whole geometric data $(\mathcal{A}, \mathcal{H}, D)$ is in fact uniquely determined by the triple $(\mathcal{A}'', \mathcal{H}, D)$ where \mathcal{A}'' is a commutative von Neumann algebra.

This also shows that \mathcal{A} is a pre-C*-algebra, i.e., that it is stable under the holomorphic functional calculus in the C*-algebra norm closure of \mathcal{A} , $A = \overline{\mathcal{A}}$. Since we assumed that \mathcal{A} was commutative, so is A and by Gelfand's theorem $A = C(X)$ is the algebra of continuous complex valued functions on $X = \text{Spec}(A)$. We note finally that characters χ of \mathcal{A} are automatically self-adjoint: $\chi(a^*) = \overline{\chi(a)}$ since the spectrum of self-adjoint elements of \mathcal{A} is real. Also they are automatically continuous since the C*-norm is uniquely determined algebraically by

$$\|a\| = \sup\{|\lambda| \mid a^*a - \lambda^2 \notin \mathcal{A}^{-1}\},$$

thus they extend automatically to A by continuity so that

$$\text{Spec } A = \text{Spec } \mathcal{A}.$$

We shall now show that \mathcal{A} is a Frechet algebra, i.e., a complete locally convex algebra whose topology is defined by the submultiplicative norms,

$$p_k(xy) \leq p_k(x)p_k(y) \quad \text{for all } x, y \in \mathcal{A},$$

associated to the regularity condition, for instance by

$$p_k(x) = \|\rho_k(x)\|, \quad \rho_k(x) = \begin{pmatrix} x & \delta(x) & \dots & \delta^k(x)/k! \\ 0 & x & \dots & \dots \\ \dots & \dots & x & \delta(x) \\ 0 & \dots & 0 & x \end{pmatrix} \quad (8)$$

since ρ_k is a representation of \mathcal{A} .

Proposition 2.2. (1) *The unbounded derivation δ is a closed operator in $\mathcal{L}(\mathcal{H})$.*

(2) *The algebra \mathcal{A} endowed with the norms p_k is a Frechet algebra.*

(3) *The semi-norms $p_k([D, a]) = p'_k(a)$ are continuous.*

Proof. (1) Let $G(|D|)$ be the graph of $|D|$. The graph of δ is

$$G(\delta) = \{(T, S) \in \mathcal{L}(\mathcal{H})^2 \mid (T\xi, T\eta + S\xi) \in G(|D|) \text{ for all } (\xi, \eta) \in G(|D|)\}.$$

It is therefore closed.

(2) Let us show that \mathcal{A} is complete. Let $a_n \in \mathcal{A}$ be a sequence which is a Cauchy sequence in any of the norms p_k . Then $a_n \rightarrow T$ in norm, so that $T \in A \subset \mathcal{A}''$. Since δ is a closed operator one has $T \in \text{Dom } \delta$ and $\delta(a_n) \rightarrow \delta(T)$ in norm. By induction one gets, using the closedness of δ that $T \in \text{Dom } \delta^k$ and $\delta^k(T) = \lim \delta^k(a_n)$. Thus $T \in \bigcap \text{Dom } \delta^m$ and by Lemma 2.1, we get $T \in \mathcal{A}$. Furthermore we also have the norm convergence $\delta^k(T) = \lim \delta^k(a_n)$. This shows that the a_n converge to T in the topology of the norms p_k and hence that \mathcal{A} is a Frechet space.

(3) Let us show that if we adjoin the semi-norms p'_k to the topology of \mathcal{A} we still get a complete space. The argument of the proof of (1) only uses the closedness of the operator $|D|$ and thus we get in the same way that the derivation $T \rightarrow d(T) = [D, T]$ with domain $\text{Dom } d = \{T \in \mathcal{L}(\mathcal{H}) \mid T \text{ Dom } D \subset \text{Dom } D, \|[D, T]\| < \infty\}$ is closed for the norm topology of $\mathcal{L}(\mathcal{H})$. Thus the above proof of completeness applies. The result then follows from the Open Mapping Theorem ([25], Corollary 2.12) applied to the identity map from \mathcal{A} endowed with the topology of the p_k, p'_k to \mathcal{A} endowed with the topology of the p_k . \square

In fact Lemma 2.1 shows that one has Sobolev estimates, using finitely many generators η_μ of the \mathcal{A} -module \mathcal{H}_∞ to define the Sobolev norms on \mathcal{A} by

$$\|a\|_s^{\text{Sobolev}} = \left(\sum_{\mu} \|(1 + D^2)^{s/2} a \eta_\mu\|^2 \right)^{1/2} \quad \text{for all } a \in \mathcal{A}. \quad (9)$$

One has

Proposition 2.3. (1) *When endowed with the norms (9), \mathcal{A} is a Frechet separable nuclear space.*

(2) *One has Sobolev estimates of the form*

$$p_k(a) \leq c_k \|a\|_{s_k}^{\text{Sobolev}}, \quad p_k([D, a]) \leq c'_k \|a\|_{s'_k}^{\text{Sobolev}} \quad \text{for all } a \in \mathcal{A},$$

with $c_k < \infty, c'_k < \infty$ and suitable sequences $s_k > 0, s'_k > 0$.

(3) *The spectrum $X = \text{Spec}(\mathcal{A})$ is metrizable.*

(4) *Any $T \in \text{End}_{\mathcal{A}} \mathcal{H}_\infty$ is continuous in \mathcal{H}_∞ and extends continuously to a bounded operator in \mathcal{H} .*

(5) The algebraic isomorphism $\mathcal{H}_\infty = e\mathcal{A}^n$ is topological.

(6) The map $(a, \xi) \mapsto a\xi$ and the \mathcal{A} -valued inner product are jointly continuous $\mathcal{A} \times \mathcal{H}_\infty \rightarrow \mathcal{H}_\infty$ and $\mathcal{H}_\infty \times \mathcal{H}_\infty \rightarrow \mathcal{A}$.

Proof. (1) By construction the family (9) is an increasing sequence of norms. Let us show that \mathcal{A} is complete. Let a_n be a sequence of elements of \mathcal{A} such that the vectors $(1 + D^2)^{s/2} a_n \eta_\mu$ converge for all s (and all μ). We then obtain vectors

$$\zeta_\mu = \lim a_n \eta_\mu \in \mathcal{H}_\infty \quad \text{for all } \mu,$$

where the convergence holds in the topology of \mathcal{H}_∞ . Let then T be the operator given by

$$T\xi = \lim a_n \xi \quad \text{for all } \xi \in \mathcal{H}_\infty. \quad (10)$$

It is well defined since one can write $\xi = \sum b^\mu \eta_\mu$ with $b^\mu \in \mathcal{A}$, which gives $a_n \xi = \sum b^\mu a_n \eta_\mu$ which converges, in the topology of \mathcal{H}_∞ , to $\sum b^\mu \zeta_\mu$ since the b^μ are continuous linear maps on \mathcal{H}_∞ using (7) and regularity. Thus T is a linear map on \mathcal{H}_∞ and it commutes with \mathcal{A} , i.e., it is an endomorphism of this finite projective module. Thus T is of the form (2) and in particular it is bounded in \mathcal{H} . Also since endomorphisms of the finite projective module are automatically continuous in \mathcal{H} , they commute with T using (10). Thus the argument of Lemma 2.1 shows that $T \in \mathcal{A}$. Moreover, since the convergence (10) holds in the topology of \mathcal{H}_∞ , one has $a_n \rightarrow T$ in the Sobolev topology and \mathcal{A} is complete in that topology. Thus \mathcal{A} is a Frechet space. It is by construction a closed subspace of the sum of finitely many spaces \mathcal{H}_∞ each being a separable nuclear space (of sequences of rapid decay). Thus it is a separable nuclear space.

(2) The identity map from the Frechet algebra \mathcal{A} with the norms p_k to the Frechet space \mathcal{A} with the Sobolev topology is continuous (using (7)) and surjective. Hence the Open Mapping Theorem ([25], Corollary 2.12) asserts that it is an open mapping. This shows that the inverse map is continuous, which gives the required estimates for the norms p_k . The result for the semi-norms $p_k([D, a])$ follows from Proposition 2.2.

(3) Since \mathcal{A} is a Frechet separable nuclear space, there is a sequence $x_n \in \mathcal{A}$ which is dense in any of the continuous norms and in particular using (2) in the p_0 norm. This shows that the C^* -algebra A is norm separable and hence that its spectrum is metrizable.

(4) By hypothesis T being an endomorphism is of the form (2). Using the inclusion $\mathcal{A} \subset A = C(X)$ of \mathcal{A} in its norm closure, we can view T as an endomorphism of the induced C^* -module \mathcal{E} over A . By (4), any element of $\text{End}_A(\mathcal{E})$ defines a bounded operator in \mathcal{H} . This shows that the graph of the operator T in $\mathcal{H}_\infty \times \mathcal{H}_\infty$ is closed and hence by the closed graph theorem that T is continuous in the Frechet topology of \mathcal{H}_∞ .

(5) The product $\mathcal{A} \times \mathcal{A} \rightarrow \mathcal{A}$ is jointly continuous using the submultiplicative norms p_k of (8). This shows that $e\mathcal{A}^n$ is a closed subspace of \mathcal{A}^n and hence is

complete. Moreover the map $(a_j) \mapsto \sum a_j \xi_j$ for given $\xi_j \in \mathcal{H}_\infty$ is continuous from \mathcal{A}^n to \mathcal{H}_∞ using (7). Thus the Open Mapping Theorem gives the result.

(6) follows from (5) and the joint continuity of the product $\mathcal{A} \times \mathcal{A} \rightarrow \mathcal{A}$. \square

We end this section with the stability of \mathcal{A} under the smooth functional calculus as first shown in [2] (cf. also [23], Proposition 2.8). We repeat the proof for convenience.

Proposition 2.4. *Let $a_j = a_j^*$ be n self-adjoint elements of \mathcal{A} and $f : \mathbb{R}^n \mapsto \mathbb{C}$ be a smooth function defined on a neighborhood of the joint spectrum of the a_j . Then the element $f(a_1, \dots, a_n) \in A$ belongs to $\mathcal{A} \subset A$.*

Proof. Let us first show that for $a = a^* \in \mathcal{A}$ one has for any $k \in \mathbb{N}$,

$$\|\delta^k(e^{isa})\| = O(|s|^k), \quad |s| \rightarrow \infty. \quad (11)$$

For $k = 1$ one has

$$\delta(e^{isa}) = is \int_0^1 e^{itsa} \delta(a) e^{i(1-t)sa} dt,$$

which proves (11) for $k = 1$. In general one has, with $\beta_u(T) = e^{iusa} T e^{-iusa}$,

$$\frac{1}{n!} \delta^n(e^{isa}) e^{-isa} = \sum_{k_j > 0, \sum k_j = n} i^\ell s^\ell \int_{S_\ell} \beta_{u_1} \left(\frac{\delta^{k_1}(a)}{k_1!} \right) \dots \beta_{u_\ell} \left(\frac{\delta^{k_\ell}(a)}{k_\ell!} \right) du$$

where $S_\ell = \{(u_j) \mid 0 \leq u_1 \leq \dots \leq u_\ell \leq 1\}$ is the standard simplex. This gives (11). Now the joint spectrum $K \subset \mathbb{R}^n$ of the a_j is a compact subset and one can extend f to a smooth function with compact support $f \in C_c^\infty(\mathbb{R}^n)$. The element $f(a_1, \dots, a_n) \in A$ is then given by

$$f(a_1, \dots, a_n) = (2\pi)^{-n} \int \hat{f}(s_1, \dots, s_n) \prod e^{is_j a_j} \prod ds_j, \quad (12)$$

where \hat{f} is the Fourier transform of f and is a Schwartz function $\hat{f} \in \mathcal{S}(\mathbb{R}^n)$. By (11) the integral (12) is convergent in any of the norms p_k which define the topology of \mathcal{A} and one gets $f(a_1, \dots, a_n) \in \mathcal{A}$. \square

3. Openness Lemma

In this section, we use the standard Implicit Function Theorem for smooth maps $\mathbb{R}^p \rightarrow \mathbb{R}^p$ to obtain the openness of the tentative local charts. We formulate the lemma in a rather abstract manner below and use it concretely in §7 for the local charts.

As above and in [23], we let \mathcal{A} be a Frechet pre-C*-algebra. We recall, for involutive algebras, the reality condition which defines a *-derivation:

$$\delta_0(a^*) = \delta_0(a)^* \quad \text{for all } a \in \mathcal{A}.$$

We let $\text{Der } \mathcal{A}$ be the Lie algebra of continuous *-derivations of \mathcal{A} .

Definition 3.1. Let \mathcal{A} be a Frechet pre-C*-algebra. A continuous *-derivation $\delta_0 \in \text{Der } \mathcal{A}$ *exponentiates* iff one has a unique solution, depending continuously on $(t, a) \in \mathbb{R} \times \mathcal{A}$, of the differential equation

$$\partial_t y(t, a) = \delta_0(y(t, a)), \quad y(0, a) = a.$$

We say that \mathcal{A} is *exposable* when any continuous *-derivation $\delta_0 \in \text{Der } \mathcal{A}$ exponentiates.

We shall show in §§5, 6 that in our context enough derivations exponentiate but for clarity of the argument we shall first assume that the algebra \mathcal{A} is expable. We refer to [17], §I.3, for the discussion of differentiability in the context of Frechet spaces. We just recall that a map $y: F \rightarrow G$ of Frechet spaces is of class C^1 when the directional derivative

$$Dy(x, h) = \lim_{\varepsilon \rightarrow 0} \frac{1}{\varepsilon} (y(x + \varepsilon h) - y(x)) \quad (13)$$

exists and is a jointly continuous function of $(x, h) \in F \times F$. The map is of class C^n when the higher derivatives $D^k y(x, h_1, \dots, h_k)$ which are defined by iteration of (13) exist and are jointly continuous functions for $k \leq n$. The map is smooth (or of class C^∞) iff it is of class C^n for all n .

Proposition 3.2. *One has for any $a, b \in \mathcal{A}$,*

$$y(t, ab) = y(t, a)y(t, b), \quad y(t, a^*) = y(t, a)^*, \quad y(t, a + b) = y(t, a) + y(t, b), \quad (14)$$

$$y(t_1 + t_2, a) = y(t_1, y(t_2, a)), \quad y(t, \delta_0(a)) = \delta_0(y(t, a)). \quad (15)$$

Moreover $y(t, a)$ is a smooth function of (t, a) with n -th derivative given by

$$D^n y(t, a, s_1, h_1, \dots, s_n, h_n) = \delta_0^n(y(t, a)) \prod_j s_j + \sum_i \delta_0^{n-1} y(t, h_i) \prod_{j \neq i} s_j. \quad (16)$$

Proof. The equalities in (14) and (15) follow from the uniqueness of the solution. To prove (16) we consider the Frechet spaces $F = \mathbb{R} \times \mathcal{A}$ and $G = \mathcal{A}$ and compute the first derivative Dy . One has

$$y(t + \varepsilon s, a + \varepsilon h) - y(t, a) = y(t + \varepsilon s, a) - y(t, a) + \varepsilon y(t + \varepsilon s, h)$$

so that

$$Dy(t, a, s, h) = s\delta_0(y(t, a)) + y(t, h)$$

Since by (15) one has $\delta_0^k(y(t, a)) = y(t, \delta_0^k(a))$ for all k , one gets (16) by induction on n . \square

The Taylor expansion at (t, a) is thus of the form

$$y(t + s, a + h) \sim \sum (\delta_0^k(y(t, a))s^k + \delta_0^k(y(t, h))s^k)/k!$$

Lemma 3.3. *Let \mathcal{A} be commutative, and $a = (a^j)$ be p self-adjoint elements of \mathcal{A} . Let χ be a character of \mathcal{A} . Assume that there exists p derivations $\delta_j \in \text{Der } \mathcal{A}$ such that*

- *each δ_j exponentiates,*
- *the determinant of the matrix $\chi(\delta_j(a^k))$ does not vanish.*

Then the image under a of any neighborhood of χ in the spectrum $\text{Spec}(\mathcal{A})$ of \mathcal{A} contains a neighborhood of $a(\chi)$ in \mathbb{R}^p .

Proof. By hypothesis the derivations $\delta_j \in \text{Der } \mathcal{A}$ can be exponentiated to the corresponding one-parameter groups $F^j(t) \in \text{Aut}(\mathcal{A})$ of automorphisms of \mathcal{A} . Note that the flows F^j do not commute pairwise in general. We then define a map h from \mathbb{R}^p to $\text{Spec}(\mathcal{A})$ by

$$h = \chi \circ \sigma, \quad \sigma_{(t_1, \dots, t_p)} = F_{t_1}^1 \circ \dots \circ F_{t_p}^p,$$

which defines a character since $F^j(t) \in \text{Aut}(\mathcal{A})$ by (14). The map h is continuous since the topology of $\text{Spec}(\mathcal{A})$ is the weak topology and for any $a \in \mathcal{A}$ the map $(t_1, \dots, t_p) \in \mathbb{R}^p \mapsto \sigma_{(t_1, \dots, t_p)}(a)$ is continuous using Definition 3.1. The coordinates of the map $\phi = a \circ h$, from \mathbb{R}^p to \mathbb{R}^p , are given by

$$\phi^k(t_1, \dots, t_p) = h(t_1, \dots, t_p)(a^k) = \chi \circ F_{t_1}^1 \circ \dots \circ F_{t_p}^p(a^k).$$

The map

$$(t_1, \dots, t_p) \in \mathbb{R}^p \mapsto F_{t_1}^1 \circ \dots \circ F_{t_p}^p(a^k)$$

is a smooth map from \mathbb{R}^p to \mathcal{A} . Indeed the maps $(t, a) \mapsto F_t^j(a)$ are smooth, and compositions of smooth maps are smooth (cf. [17], Theorem 3.6.4), while the above map is the composition

$$\begin{aligned} & F^1 \circ (\text{Id} \times F^2) \circ \dots \circ (\text{Id}^{p-2} \times F^{p-1}) \circ (\text{Id}^{p-1} \times F^p(a^k)), \\ \mathbb{R}^p & \xrightarrow{F^p(a^k)} \mathbb{R}^{p-1} \times \mathcal{A} \xrightarrow{F^{p-1}} \mathbb{R}^{p-2} \times \mathcal{A} \rightarrow \dots \rightarrow \mathbb{R} \times \mathcal{A} \xrightarrow{F^1} \mathcal{A}. \end{aligned} \tag{17}$$

Thus the map $\phi = a \circ h$, obtained by composition with the character χ which is linear and continuous and hence smooth, is a smooth map from \mathbb{R}^p to \mathbb{R}^p . The image of $0 \in \mathbb{R}^p$ is $a(\chi)$. The partial derivatives at 0 are

$$(\partial_j \phi^k)(0) = \chi(\delta_j(a^k)),$$

thus we know from the hypothesis of the lemma that the Jacobian does not vanish at 0. It then follows from the Implicit Function Theorem that the mapping $\phi = a \circ h$

maps by a diffeomorphism a suitable neighborhood of 0 to a neighborhood of $a(\chi)$. In particular the image under a of a neighborhood W of χ contains the image under ϕ of $h^{-1}(W)$ which, since h is continuous, is a neighborhood of $0 \in \mathbb{R}^p$. This shows that the image under a of any neighborhood of χ in the spectrum $\text{Spec}(\mathcal{A})$ of \mathcal{A} contains a neighborhood of $a(\chi)$ in \mathbb{R}^p . \square

The above proof yields the following more precise statement:

Lemma 3.4. *Under the hypothesis of Lemma 3.3, there exists a smooth family $\sigma_t \in \text{Aut}(\mathcal{A})$, $t \in \mathbb{R}^p$, a neighborhood Z of χ in $X = \text{Spec}(\mathcal{A})$ and a neighborhood W of $0 \in \mathbb{R}^p$ such that, for any $\kappa \in Z$, the map $t \mapsto a(\kappa \circ \sigma_t)$ is a diffeomorphism, depending continuously on κ , of W with a neighborhood of $a(\kappa)$ in \mathbb{R}^p .*

Proof. Let as above

$$\sigma_{(t_1, \dots, t_p)} = F_{t_1}^1 \circ \dots \circ F_{t_p}^p.$$

The map which to $\kappa \in X$ associates the map ψ_κ from \mathbb{R}^p to \mathbb{R}^p given by $\psi_\kappa(t) = a(\kappa \circ \sigma_t)$ yields by restriction a continuous map $X \rightarrow C^\infty(K, \mathbb{R}^p)$ where K is a closed ball centered at $0 \in \mathbb{R}^p$. Indeed for each j the map $t \in K \mapsto \sigma_t(a^j) \in \mathcal{A}$ is smooth by (17) and thus its partial derivatives $\partial_t^\alpha \sigma_t(a^j)$ are elements of \mathcal{A} which depend continuously of t . One has

$$\partial_t^\alpha \psi_\kappa^j(t) = \kappa(\partial_t^\alpha \sigma_t(a^j)),$$

and thus the partial derivatives of $\psi_\kappa(t)$ are continuous functions of (κ, t) . Since the determinant of the jacobian $\chi(\delta_j(a^k))$ does not vanish, the result follows from the Implicit Function Theorem (see e.g. [17], Theorem 5.2.3). \square

4. Jacobian and openness of local charts

We first briefly recall the well-known properties of multiple commutators which we need later.

Definition 4.1. Let $T_j \in \mathcal{B}$ be elements of a noncommutative algebra \mathcal{B} , one lets

$$[T_1, T_2, \dots, T_n] = \sum_{\sigma} \varepsilon(\sigma) T_{\sigma(1)} T_{\sigma(2)} \dots T_{\sigma(n)}$$

where σ varies through all permutations of $\{1, \dots, n\}$ and $\varepsilon(\sigma)$ is its signature.

We mention the following general properties.

Proposition 4.2. *Let $T_j \in \mathcal{B}$ be elements of a noncommutative algebra \mathcal{B} .*

(a) For any permutation α of $\{1, \dots, n\}$, one has

$$[T_{\alpha(1)}, T_{\alpha(2)}, \dots, T_{\alpha(n)}] = \varepsilon(\alpha) [T_1, T_2, \dots, T_n].$$

(b) If two of the T_j are equal one has

$$[T_1, T_2, \dots, T_n] = 0.$$

(c) Let $\mathcal{A} \subset \mathcal{B}$ be a commutative subalgebra and $\mathcal{A}' \subset \mathcal{B}$ its relative commutant in \mathcal{B} . Let $a_k^j \in \mathcal{A}$, $\gamma_j \in \mathcal{A}'$. Then, with $T_k = \sum a_k^j \gamma_j$, one has

$$[T_1, T_2, \dots, T_n] = \text{Det}((a_k^j)) [\gamma_1, \gamma_2, \dots, \gamma_n]. \quad (18)$$

(d) The equality (18) extends to the case of a rectangular matrix $a_k^j \in \mathcal{A}$ as follows:

$$[T_1, T_2, \dots, T_n] = \sum_F \text{Det}((a_k^j(F))) [\gamma_1(F), \gamma_2(F), \dots, \gamma_n(F)], \quad (19)$$

where the sum is over all subsets $F \subset \{1, \dots, m\}$ with $\#F = n$, the matrix $a_k^j(F)$ is the restriction of a_k^j to $j \in F$ and the $\gamma_j(F)$ are the γ_i , $i \in F$, ordered with increasing index in F .

Proof. (a) This follows from $\varepsilon(\sigma \circ \alpha) = \varepsilon(\sigma)\varepsilon(\alpha)$.

(b) The permutation of the two indices is odd but does not affect the expression which must vanish.

(c) One has

$$[T_1, T_2, \dots, T_n] = \sum_{(j_k)} \prod_{k=1}^n a_k^{j_k} [\gamma_{j_1}, \gamma_{j_2}, \dots, \gamma_{j_n}] \quad (20)$$

where, a priori, the (j_k) is an arbitrary map from $\{1, \dots, n\}$ to $\{1, \dots, m\}$. By the second statement of the lemma, these terms vanish when two of the indices j_k are equal. Thus one can take the sum over permutations (j_k) and one can use the first statement of the lemma to rewrite the corresponding term as

$$[\gamma_{j_1}, \gamma_{j_2}, \dots, \gamma_{j_n}] = \varepsilon(j) [\gamma_1, \gamma_2, \dots, \gamma_n].$$

It follows that

$$[T_1, T_2, \dots, T_n] = \text{Det}((a_k^j)) [\gamma_1, \gamma_2, \dots, \gamma_n].$$

(d) One decomposes the sum (20) according to the range F of the injection j from $\{1, \dots, n\}$ to $\{1, \dots, m\}$. \square

Let us now go back to spectral triples $(\mathcal{A}, \mathcal{H}, D)$ fulfilling the five conditions of §2.

Lemma 4.3. *Let \mathcal{B} be the algebra of endomorphisms of \mathcal{H}_∞ . One has a finite decomposition*

$$[D, a] = \sum \delta_j(a) \gamma_j \quad \text{for all } a \in \mathcal{A}, \quad (21)$$

where $\gamma_j \in \mathcal{B}$ and the δ_j are derivations of the form

$$\delta_j(a) = i(\xi_j|[D, a]\xi_j) \quad \text{for all } a \in \mathcal{A}, \quad (22)$$

for some $\xi_j \in \mathcal{H}_\infty$.

Proof. First $[D, a]\mathcal{H}_\infty \subset \mathcal{H}_\infty$ using regularity and (7). Thus the order one condition shows that $[D, a] \in \mathcal{B}$. One has $\mathcal{H}_\infty = e\mathcal{A}^n$, $\mathcal{B} = eM_n(\mathcal{A})e$ for a self-adjoint idempotent $e \in M_n(\mathcal{A})$. Thus every element $T \in \mathcal{B}$ can be written uniquely, as any element of $M_n(\mathcal{A})$ in the form

$$T = \sum a_{k\ell} \varepsilon_{k\ell}, \quad a_{k\ell} \in \mathcal{A},$$

in terms of the matrix units ε_{ij} . The coefficients $a_{k\ell} \in \mathcal{A}$ are uniquely determined, using the elements $\eta_k = e\zeta_k \in \mathcal{H}_\infty$ where $\zeta_k \in \mathcal{A}^n$ is the element all of whose components vanish except the k -th one which is equal to 1. Using the \mathcal{A} -valued inner product, one has

$$a_{k\ell} = (\eta_k|T\eta_\ell) = L_{k\ell}(T) \quad \text{for all } k, \ell. \quad (23)$$

Moreover one has, since $T = eTe$ and the $a_{k\ell}$ commute with e ,

$$T = \sum a_{k\ell} e \varepsilon_{k\ell} e. \quad (24)$$

One has $L_{k\ell}(aT) = aL_{k\ell}(T)$ for any $a \in \mathcal{A}$. Applying this to $T = [D, b]$, the maps $a \mapsto L_{ij}([D, a])$ give derivations of \mathcal{A} . They are not self-adjoint but can be decomposed as linear combinations of self-adjoint derivations, which, using (24), gives the required formula (21). More precisely, the derivations δ_j can be written using the \mathcal{A} -valued inner product on \mathcal{H}_∞ in the form (22) for some $\xi_j \in \mathcal{H}_\infty$ (with i to ensure self-adjointness). Indeed one obtains (22) applying to (23) the polarization identity:

$$\begin{aligned} 2(\xi|T\eta) &= ((\xi + \eta)|T(\xi + \eta)) - (\xi|T\xi) - (\eta|T\eta) \\ &\quad - i(((\xi + i\eta)|T(\xi + i\eta)) - (\xi|T\xi) - (i\eta|Ti\eta)). \end{aligned} \quad (25)$$

In particular, using Proposition 2.3, the δ_j are continuous. \square

By hypothesis the cycle c is of the form

$$c = \sum_{\alpha} a_{\alpha}^0 \omega_{\alpha}, \quad \omega_{\alpha} = \sum_{\beta} \varepsilon(\beta) 1 \otimes a_{\alpha}^{\beta(1)} \otimes \cdots \otimes a_{\alpha}^{\beta(p)}, \quad (26)$$

where one can assume that the a_α^μ are self-adjoint for $\mu > 0$. We define the conditional expectation $E_{\mathcal{A}} : \text{End}_{\mathcal{A}}(\mathcal{H}_\infty) \rightarrow \mathcal{A}$, using the projections p_j of (5),

$$E_{\mathcal{A}}(T) = \sum_{j>0} \frac{1}{j} p_j \sum T_{kk} \quad \text{for all } T = (T_{k\ell}) \in eM_n(\mathcal{A})e, \quad (27)$$

using the identification $\mathcal{H}_\infty = e\mathcal{A}^n$. We obtain a self-adjoint $\rho_\alpha \in \mathcal{A}$ given⁶ by

$$\rho_\alpha = i^{\frac{p(p+1)}{2}} E_{\mathcal{A}}(\gamma \sum_{\beta} \varepsilon(\beta) [D, a_\alpha^{\beta(1)}] \dots [D, a_\alpha^{\beta(p)}]). \quad (28)$$

One lets

$$C_\alpha = \{x \in X \mid \rho_\alpha(x) = 0\}$$

and $U_\alpha = C_\alpha^c$ be its complement, i.e., the open set where ρ_α does not vanish.

Lemma 4.4. *The U_α form an open cover of $X = \text{Spec}(\mathcal{A})$.*

Each U_α is the disjoint union of the two open subsets U_α^\pm corresponding to the sign of ρ_α ,

$$\pm \rho_\alpha(x) > 0 \quad \text{for all } x \in U_\alpha^\pm.$$

Proof. It is enough to show that any $x \in X$ belongs to some U_α . One has $\pi_D(c) = \gamma$, so that by (26)

$$\gamma \sum_{\alpha} a_\alpha^0 \sum_{\beta} \varepsilon(\beta) [D, a_\alpha^{\beta(1)}] \dots [D, a_\alpha^{\beta(p)}] = 1.$$

By (28) and the conditional expectation module property $E_{\mathcal{A}}(aT) = aE_{\mathcal{A}}(T)$,

$$i^{-\frac{p(p+1)}{2}} \sum_{\alpha} a_\alpha^0 \rho_\alpha = 1$$

and $\rho_\alpha(x) \neq 0$ for some α . The second statement follows since ρ_α is a non-vanishing real valued function on U_α . \square

We let s_α be the natural continuous map from X to \mathbb{R}^p given by

$$\chi \in \text{Spec}(\mathcal{A}) \rightarrow (\chi(a_\alpha^j)) \in \mathbb{R}^p. \quad (29)$$

Lemma 4.5. *Assume that derivations of the form (22) exponentiate. Let $\chi \in U_\alpha$.*

- *There exists p derivations. $\delta_j \in \text{Der}(\mathcal{A})$ such that $\chi(\text{Det}((\delta_j(a_\alpha^k)))) \neq 0$.*
- *The map a_α from U_α to \mathbb{R}^p is open.*

⁶There is no γ in the odd case.

- *There exists a smooth family $\sigma_t \in \text{Aut}(\mathcal{A})$, $t \in \mathbb{R}^p$, a neighborhood Z of χ in $X = \text{Spec}(\mathcal{A})$ and a neighborhood W of $0 \in \mathbb{R}^p$ such that, for any $\kappa \in Z$, the map $t \mapsto s_\alpha(\kappa \circ \sigma_t)$ is a diffeomorphism, depending continuously on κ , of W with a neighborhood of $a(\kappa)$ in \mathbb{R}^p .*

Proof. We let, as above, \mathcal{B} be the algebra of endomorphisms of the \mathcal{A} -module \mathcal{H}_∞ . It contains $\mathcal{A} \subset \mathcal{B}$ as a subalgebra of its center. By Lemma 4.3, one has derivations $\delta_j \in \text{Der}(\mathcal{A})$ of the form (22) such that the formula (21) holds:

$$[D, a] = \sum_1^m \delta_j(a) \gamma_j \quad \text{for all } a \in \mathcal{A}.$$

By hypothesis we have $\rho_\alpha(\chi) \neq 0$. Thus, the following endomorphism of the \mathcal{A} -module \mathcal{H}_∞ does not vanish,

$$[[D, a_\alpha^1], [D, a_\alpha^2], \dots, [D, a_\alpha^p]](\chi) \neq 0 \quad \text{for all } \chi \in U_\alpha.$$

It thus follows, from (19) of Proposition 4.2, that for $\chi \in U_\alpha$ one can find p elements $\delta_j \in \text{Der}(\mathcal{A})$ among the above δ_j such that

$$\chi(\text{Det}((\delta_j(a_\alpha^k)))) \neq 0.$$

Now let $V \subset U_\alpha$ be open. To show that $s_\alpha(V)$ is open one needs to show that, for any $\chi \in V$, $s_\alpha(V)$ contains a neighborhood of $s_\alpha(\chi)$. But V is a neighborhood of χ in $\text{Spec}(\mathcal{A})$ and the hypothesis of Lemma 3.3 is fulfilled so that this lemma shows that $s_\alpha(V)$ contains a neighborhood of $s_\alpha(\chi)$. The third statement follows from Lemma 3.4. \square

5. Dissipative derivations

We assumed in the above discussion that the algebra \mathcal{A} is expable. It is of course desirable to remove this hypothesis, and this will be done in this section and the next one. We need a form of existence and uniqueness for solutions of linear differential equations with values in a Frechet space E . Simple examples show that in that generality one has neither existence nor uniqueness. For failure of existence just let $\mathcal{A} = C^\infty([0, 1])$, $\delta_0 = \partial_x$. For failure of uniqueness, let E be the space of sequences $x_n \in \mathbb{R}$ for $n \geq 1$ and take the shift operator S . Then the equation

$$\partial_t x = Sx, \quad (Sx)_n = x_{n+1},$$

has no uniqueness of solutions. Indeed we can take for $x_1(t)$ any smooth function which is flat at $t = 0$, i.e., $\partial_t^n x_1(0) = 0$, and then define by induction $x_{n+1}(t) =$

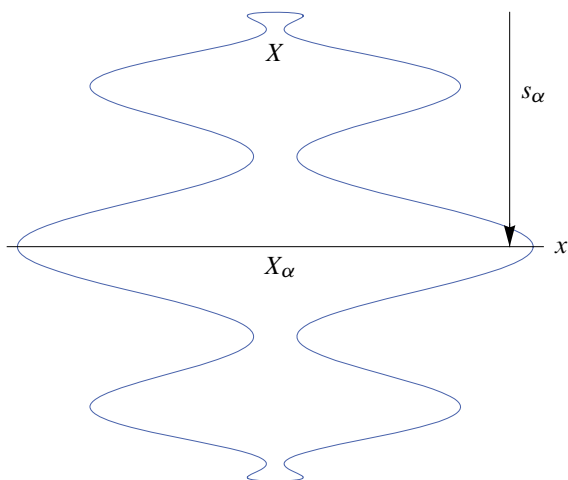


Figure 1. The map s_α from X to X_α .

$\partial_t x_n(t)$ so that $\partial_t x = Sx$ holds and the initial condition $x(0) = 0$ does not imply uniqueness.

In our case we need to know that any derivation $\delta \in \text{Der } \mathcal{A}$ can be exponentiated, i.e., that one has existence and uniqueness for the differential equation

$$\frac{dy(t)}{dt} = \delta(y(t)), \quad y(t) \in \mathcal{A}.$$

It is only the compactness of X that ensures this, and also the fact that one is dealing with a *real* vector field. This means that we first need to make sure that the derivation exponentiates at the level of the C^* -algebra as discussed in [3].

One step towards this would be to show directly the following corollary of expansibility:

Lemma 5.1. *Assume that the derivations of the form the form (22) exponentiate. Then, for any $h = h^* \in \mathcal{A}$, the commutator $[D, h]$ vanishes where⁷ h reaches its maximum. Conversely if this property holds the derivations $\pm\delta_j$ of the form (22) are dissipative (cf. [3], Definition 1.4.6), i.e.,*

$$\|x + \lambda\delta_j(x)\| \geq \|x\| \quad \text{for all } x \in \mathcal{A}, \lambda \in \mathbb{R}.$$

Proof. One has, by (21), $[D, h] = \sum \delta_j(h)\gamma_j$ where $\delta_j \in \text{Der } \mathcal{A}$. Thus it is enough to show that $\delta_j(h)(\chi) = 0$ where $h = h^* \in \mathcal{A}$ reaches its maximum at χ . This follows from the existence of $e^{t\delta_j} \in \text{Aut}(\mathcal{A})$ using the differentiable function $f(t) = \chi(e^{t\delta_j}(h))$ which has a maximum at $t = 0$ and hence vanishing derivative.

⁷This makes sense since $[D, h]$ commutes with h .

Conversely, the derivations δ_j are of the form (22), i.e., $\delta_j(h) = i(\xi|[D, h]\xi)$. Thus the vanishing of $[D, h](\chi)$, where $h = h^* \in \mathcal{A}$ reaches its maximum, ensures that $\delta_j(h)(\chi) = i(\xi(\chi), [D, h](\chi)\xi(\chi)) = 0$ also vanishes. Thus one has

$$\|h + \lambda\delta_j(h)\| \geq \|h\| \quad \text{for all } h = h^* \in \mathcal{A}, \lambda \in \mathbb{R},$$

since for a character χ of \mathcal{A} with $\chi(\pm h) = \|h\|$ one has $\chi(\pm(h + \lambda\delta_j(h))) = \|h\|$. In the complex case, i.e., for an arbitrary $x \in \mathcal{A}$, let ψ be a state on $A \supset \mathcal{A}$ such that $|\psi(x)| = \|x\|$. Replacing $x \mapsto ux$ for $u \in \mathbb{C}$, $|u| = 1$, one can assume that $\psi(x) > 0$. Then writing $x = h + ik$ with $h = h^*$ and $k = k^*$, one has $\psi(x) = \psi(h) = \|h\|$ so that $\psi(\delta_j(h)) = 0$ from the above discussion. Then one has, for $\lambda \in \mathbb{R}$,

$$\psi(x + \lambda\delta_j(x)) = \psi(h) + i\lambda\psi(\delta_j(k))$$

and $|\psi(x + \lambda\delta_j(x))| \geq \psi(h) = \|x\|$. □

Note that the commutativity of $[D, h]$ with h and the self-adjointness of D do not suffice to entail the conclusion of Lemma 5.1. This can be seen from the following spectral triple:

$$\mathcal{A} = C^\infty([0, 1]), \quad \mathcal{H} = L^2([0, 1]) \otimes \mathbb{C}^2, \quad D = \begin{pmatrix} 0 & \partial_x \\ -\partial_x & 0 \end{pmatrix} \quad (30)$$

with the boundary condition

$$\text{Dom } D = \left\{ \xi = \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} \mid \xi_1(0) = 0, \xi_2(1) = 0 \right\}. \quad (31)$$

For any $h \in \mathcal{A}$ one has $[D, h] = \partial_x h \gamma_1$,

$$\gamma_1 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

so that $[D, h]$ commutes with h . For $h(x) = x$ the maximum is at $x = 1$ and $[D, h]$ does not vanish there. This example shows that the hypothesis of expability of the algebra \mathcal{A} appears at first sight as essential. However, in this example, condition 5) fails since the boundary condition (31) does not yield a finite projective submodule of $C^\infty([0, 1]) \otimes \mathbb{C}^2$ over $\mathcal{A} = C^\infty([0, 1])$. Also $[D, h] = \partial_x h \gamma_1$ does not preserve the domain of D which is the same as the domain of $|D|$, thus *regularity fails*. Orientability also fails in this example. We shall now show that regularity allows in fact to obtain the required dissipativity.

Let us consider the one-parameter group of automorphisms of $\mathcal{L}(\mathcal{H})$ given by

$$\alpha_t(T) = e^{itD} T e^{-itD} \quad \text{for all } t \in \mathbb{R}.$$

Lemma 5.2. *Let T preserve $\text{Dom } D$ and $[D, T]$ be bounded. Then the function $t \mapsto \alpha_t(T)$ is norm continuous,*

$$\|\alpha_s(T) - \alpha_t(T)\| \leq |s - t| \| [D, T] \|,$$

and when $s \rightarrow 0$ the difference quotient

$$\frac{\alpha_s(T) - T}{s} = \frac{i}{s} \int_0^s \alpha_t([D, T]) dt$$

converges to $i[D, T]$ in the strong topology.

Proof. Let $\xi \in \text{Dom } D$. Then $\frac{1}{s}(e^{isD} - 1)\xi \rightarrow iD\xi$ (in norm) when $s \rightarrow 0$. Thus using

$$\frac{1}{s}(e^{isD}Te^{-isD} - T)\xi = \frac{1}{s}e^{isD}T(e^{-isD} - 1)\xi + \frac{1}{s}(e^{isD} - 1)T\xi$$

one gets (in norm)

$$\frac{1}{s}(e^{isD}Te^{-isD} - T)\xi \rightarrow i[D, T]\xi$$

when $s \rightarrow 0$. Thus $t \mapsto \alpha_t(T)\xi$ is of class C^1 . Its derivative is $t \mapsto i\alpha_t([D, T])\xi$. Thus

$$(\alpha_s(T) - \alpha_t(T))\xi = i \int_t^s \alpha_u([D, T])\xi du \quad (32)$$

holds for all $\xi \in \text{Dom } D$ and hence all $\xi \in \mathcal{H}$ since the map $u \mapsto \alpha_u([D, T])\xi$ is continuous, as follows from the continuity of $s \mapsto e^{isD}\eta$ for any $\eta \in \mathcal{H}$. Both statements follow. \square

We can now consider the C^* -algebra C generated by the $\alpha_s(h)$ for $h = h^* \in \mathcal{A}$ as above. It is norm separable and the $\alpha_s \in \text{Aut}(C)$ form a norm continuous one-parameter group. To try and prove that $[D, h]$ vanishes where $h = h^* \geq 0$ reaches its maximum, one considers a state ϕ on C such that $\phi(h) = \|h\|$. It is obtained by extension using the inclusion $C^*(h) \subset C$. The function

$$f(s) = \phi(\alpha_s(h))$$

is a Lipschitz function and reaches its maximum: $\|h\|$ at $s = 0$. Thus if one could assert that the derivative at $s = 0$ is given by $\phi([D, h])$, one would get the vanishing $\phi([D, h]) = 0$. The problem is that $\alpha_u([D, h])$ is not in general a norm continuous function of u and thus the differentiability only holds in the strong topology but not in the norm topology.

Things are easier with $|D|$ since the regularity conditions ensures that the map

$$t \rightarrow \gamma_t(a) = e^{it|D|}ae^{-it|D|} \quad (33)$$

is in fact of class C^∞ in the norm topology (cf. Lemma 13.3 of §13). Moreover the following lemma shows that it is enough to show the vanishing of $[[D^2, a], b]$ at χ for all $b \in \mathcal{A}$ to get the vanishing of $[D, a]$ at χ .

Lemma 5.3. *Let $h = h^* \in \mathcal{A}$ and $\chi \in \text{Spec}(\mathcal{A})$. If $[[D^2, h], h]$ vanishes at χ , then $[D, h]$ vanishes at χ .*

Proof. One has

$$[[D^2, h], h] = 2[D, h]^2 \quad (34)$$

using the order one condition. \square

Note that (34) shows that $[D, h]^2$ and hence $|[D, h]|$ only depends upon D^2 and hence $|D|$ and not upon the phase of the polar decomposition of D . This comes from the order one condition. Moreover one has the following vanishing of $|[D], h]$ where $h = h^* \in \mathcal{A}$ reaches its maximum.

Lemma 5.4. *For any $h = h^* \in \mathcal{A}$, $h \geq 0$, reaching its maximum at $\chi \in \text{Spec}(\mathcal{A})$ and any sequence $b_n \in \mathcal{A}$, $\|b_n\| \leq 1$, with support tending to $\{\chi\}$, one has*

$$\|b_n^*|[D], h]b_n\| \rightarrow 0. \quad (35)$$

Proof. Let $\xi_n \in \mathcal{H}$ be unit vectors with support tending to $\{\chi\}$. Then consider any limit state on $\mathcal{L}(\mathcal{H})$:

$$\eta(T) = \lim_{\omega} \langle \xi_n, T \xi_n \rangle. \quad (36)$$

One has $\eta(h) = h(\chi)$ since h is a continuous function on $X = \text{Spec}(\mathcal{A})$. Thus $\eta(h) = \|h\|$. When applied to $|D|$ instead of D , Lemma 5.2 shows that both $\gamma_s(h)$ and $\gamma_s(\delta(h))$ are Lipschitz functions of s , while

$$\frac{\gamma_s(h) - h}{s} = \frac{1}{s} \int_0^s \gamma_t(i\delta(h)) dt \rightarrow i\delta(h)$$

so that $\gamma_s(h)$ is of class C^1 in norm. It follows that the function $s \mapsto \eta(\gamma_s(h))$ is of class C^1 . It is maximal for $s = 0$ and hence its derivative vanishes so that $\eta(\delta(h)) = 0$. Thus

$$\lim_{\omega} \langle \xi_n, \delta(h) \xi_n \rangle = 0 \quad (37)$$

and this continues to hold for any bounded sequence $\xi_n \in \mathcal{H}$ with support tending to $\{\chi\}$. Now let b_n be as in the lemma; then if (35) does not hold, one can find a subsequence n_k with $\|b_{n_k}^*|[D], h]b_{n_k}\| \geq \varepsilon > 0$ for all k . Using polarization (25), one gets unit vectors $\xi'_k \in \mathcal{H}$ such that

$$|\langle \xi'_k, b_{n_k}^*|[D], h]b_{n_k}\xi'_k \rangle| \geq \varepsilon' > 0,$$

which contradicts (37) for $\xi_k = b_{n_k}\xi'_k$. \square

We shall use the analogue in our context of the notion of symbol for pseudodifferential operators. The symbol of T can be viewed as a weak limit of the conjugate operators of the form

$$\tau^{-k} e^{i\tau\phi} T e^{-i\tau\phi}, \quad \tau \rightarrow \infty,$$

where the integer k is the order of T . For instance the symbol of D is given by $-i[D, \phi]$ since the order one condition gives

$$\tau^{-1} e^{i\tau\phi} D e^{-i\tau\phi} = -i[D, \phi] + \tau^{-1} D. \quad (38)$$

One expects the symbol of D^2 to be of the form

$$\lim_{\tau \rightarrow \infty} \frac{1}{\tau^2} e^{i\tau\phi} D^2 e^{-i\tau\phi} \xi = -[D, \phi]^2 \xi \quad \text{for all } \xi \in \text{Dom } D^2.$$

This is obtained by squaring (38), but one needs to know that $\text{Dom } D$ is invariant under $[D, \phi]$ to control the term $D[D, \phi]$. This is insured by regularity.

Remark 5.5. In the example (30) considered above, $[D, \phi]$ does not map $\text{Dom } D^2$ to $\text{Dom } D$ so that $D[D, \phi]\xi$ does not make sense in that case. In fact regularity fails, and $\text{Dom } D^2$ is not invariant under ϕ , unless $[D, \phi]$ vanishes on the boundary. To see this, note that the boundary condition for D^2 is

$$\xi = \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} \in \text{Dom } D^2 \iff \xi_1(0) = 0, \partial_x \xi_2(0) = 0, \xi_2(1) = 0, \partial_x \xi_1(1) = 0,$$

which contains the Neumann condition $\partial_x \xi_2(0) = 0$ while $\xi_2(0)$ is arbitrary. Thus $\partial_x \phi \xi_2(0) = \partial_x \phi(0) \xi_2(0)$ vanishes only when $\partial_x \phi(0) = 0$.

In the case of an operator of order 0 there is no power of τ and one deals with a bounded family so that one can expect the limit to be a weak limit. We need to guess the symbol of $[|D|, h]$. We expect that if we choose $\phi = h$, this symbol will just be $i[|D, h|] \in \text{End } S$. The symbol of $[D^2, h]$ is (using $[D^2, h] = D[D, h] + [D, h]D$)

$$\tau^{-1} e^{i\tau\phi} [D^2, h] e^{-i\tau\phi} = -i([D, \phi][D, h] + [D, h][D, \phi]) + \tau^{-1} [D^2, h].$$

To see why we should expect the symbol of $[|D|, h]$ for $\phi = h$ to just be $i[|D, h|]$ we have:

Lemma 5.6. *Assume that when $\tau \rightarrow \infty$ the following limit holds in the strong topology:*

$$\lim_{\tau \rightarrow \infty} e^{i\tau h} [|D|, h] e^{-i\tau h} = T.$$

Then one has, with strong convergence on $\text{Dom } D$,

$$\lim_{\tau \rightarrow \infty} \tau^{-1} e^{i\tau h} |D| e^{-i\tau h} = -iT.$$

Proof. One defines a one-parameter group β_u of automorphisms:

$$\beta_u(Y) = e^{iuh} Y e^{-iuh}.$$

One has, at the formal level, $\frac{d}{du}\beta_u(Y) = -i\beta_u([Y, h])$. Taking $Y = |D|$ one gets, with the notations of the lemma and using regularity,

$$e^{iuh}[|D|, h]e^{-iuh}\xi = i\frac{d}{du}\beta_u(|D|)\xi \quad \text{for all } \xi \in \text{Dom } |D|,$$

which gives

$$\int_0^\tau e^{iuh}[|D|, h]e^{-iuh}\xi du = i(\beta_\tau(|D|) - |D|)\xi \quad \text{for all } \xi \in \text{Dom } |D|. \quad (39)$$

Note that this equality continues to hold for any $\xi \in \mathcal{H}$ since $\beta_\tau(|D|) - |D|$ is a bounded operator. Now $e^{iuh}[|D|, h]e^{-iuh}$ is uniformly bounded and converges strongly by hypothesis to T . Thus one has, for the Cesàro mean,

$$\lim_{\tau \rightarrow \infty} \tau^{-1} \int_0^\tau e^{iuh}[|D|, h]e^{-iuh}\xi du = T\xi \quad \text{for all } \xi \in \mathcal{H},$$

which gives the result since one controls $\tau^{-1}|D|\xi \rightarrow 0$ for $\xi \in \text{Dom } D$. \square

Moreover we expect the symbol map to be a morphism so that the symbol of $|D|$ is given by the absolute value of the symbol of D , i.e., by $|[D, h]|$. In fact we do not need to prove the converse of Lemma 5.6 since we can use the regularization by Cesàro mean to compose the states η , with weak limits of $C_\tau(T)$,

$$C_\tau(T) = \tau^{-1} \int_0^\tau \beta_u(T) du. \quad (40)$$

Lemma 5.7. *With h as above one has*

- (1) C_τ is a completely positive map from $\mathcal{L}(\mathcal{H})$ to itself and $C_\tau(1) = 1$,
- (2) $C_\tau(aTb) = aC_\tau(T)b$ for all $a, b \in \mathcal{A}$,
- (3) $C_\tau([|D|, h]) = \frac{i}{\tau} (e^{i\tau h}|D|e^{-i\tau h} - |D|)$.

Proof. The first two statements follow from (40) using the commutativity of \mathcal{A} to get $\beta_u(aTb) = a\beta_u(T)b$. The last statement follows from (39). \square

We can then compose the vector states $\{\xi_n, \bullet\xi_n\}$ used in the construction of η (36) with C_{τ_n} to replace $[|D|, h]$ by $i|[D, h]|$.

Thus we need to determine the principal symbol of $|D|$. The intuitive idea is as follows: one has

$$\beta_\tau(D) = e^{i\tau h} D e^{-i\tau h} = D - i\tau[D, h] \quad (41)$$

since h commutes with $[D, h]$ so that $[D, f(h)] = f'(h)[D, h]$ for f smooth (cf. [18]). Thus, by homogeneity of the absolute value,

$$\frac{1}{\tau}\beta_\tau(|D|) = \frac{1}{\tau}e^{i\tau h}|D|e^{-i\tau h} = \left|\frac{D}{\tau} - i[D, h]\right| \quad \text{for all } \tau > 0.$$

We need the weak limit in \mathcal{H} for $\tau \rightarrow \infty$ of $\frac{1}{\tau}\beta_\tau(|D|)\xi$ for $\xi \in \text{Dom } D$. These vectors are bounded in norm as follows from

$$\|\frac{1}{\tau}\beta_\tau(|D|)\xi\| = \|\varepsilon D - i[D, h]\xi\| = \|(\varepsilon D - i[D, h])\xi\|, \quad \varepsilon = 1/\tau, \quad (42)$$

which is bounded since $\xi \in \text{Dom } D$ so that $\|\varepsilon D\xi\| \rightarrow 0$. Note also that $\frac{1}{\tau}\beta_\tau(|D|)$ is a positive operator so that any weak limit η of $\frac{1}{\tau}\beta_\tau(|D|)\xi$ fulfills $\langle \xi, \eta \rangle \geq 0$.

Let us now show how to use regularity to obtain the strong convergence of

$$\frac{1}{\tau}\beta_\tau(|D|)\xi = |\varepsilon D - i[D, h]\xi \quad (43)$$

when $\varepsilon \rightarrow 0$ and $\xi \in \text{Dom } D$. By (42) we can assume that $\xi \in \mathcal{H}_\infty$. We let $X(\varepsilon) = \varepsilon D - i[D, h]$. By (41) it is a self-adjoint operator with \mathcal{H}_∞ as a core since \mathcal{H}_∞ is invariant under $e^{i\tau h}$. The same holds for $|X(\varepsilon)|$.

Lemma 5.8. *One has, with $X(\varepsilon) = \varepsilon D - i[D, h]$,*

$$|X(\varepsilon)| = Y(\varepsilon) + f_0(X(\varepsilon)) \quad (44)$$

where

$$Y(\varepsilon)\xi = \frac{2}{\pi} \int_0^\infty \frac{X(\varepsilon)^2}{1 + u^2 + X(\varepsilon)^2} \xi \, du \quad \text{for all } \xi \in \text{Dom } D \quad (45)$$

and

$$f_0(x) = |x| - x^2(1 + x^2)^{-1/2} \quad \text{for all } x \in \mathbb{R}.$$

Proof. For any self-adjoint operator T one has $\|(1 + u^2 + T^2)^{-1}\| \leq (1 + u^2)^{-1}$ and the norm convergent expression

$$(1 + T^2)^{-1/2} = \frac{2}{\pi} \int_0^\infty \frac{1}{1 + u^2 + T^2} \, du,$$

which gives, for any $\xi \in \text{Dom } T$,

$$T^2(1 + T^2)^{-1/2}\xi = \frac{2}{\pi} \int_0^\infty \frac{T^2}{1 + u^2 + T^2} \xi \, du.$$

Note that the partial sums

$$\int_0^v \frac{T}{1 + u^2 + T^2} \, du$$

are uniformly bounded but do not converge in norm to $T(1 + T^2)^{-1/2}$ since the function $x(1 + x^2)^{-1/2}$ does not vanish at ∞ . Thus we get strong convergence on $\text{Dom } T$. This applies to $X(\varepsilon)$, which is, up to a scale factor, conjugate to D by an automorphism of $\text{Dom } D$ so that (45) holds with $Y(\varepsilon) = f(X(\varepsilon))$, $f(x) = x^2(1 + x^2)^{-1/2}$. Finally one has $f_0 \in C_0(\mathbb{R})$ and $f(x) + f_0(x) = |x|$. \square

For each $\lambda \geq 0$ we define a transformation on operators acting in \mathcal{H}_∞ by

$$\theta_\lambda(T) = (D^2 + \lambda)T(D^2 + \lambda)^{-1} \quad (46)$$

Lemma 5.9. *Let $h = h^* \in \mathcal{A}$. There exists $\lambda < \infty$ such that*

$$\|\theta_\lambda((1 + u^2 + X(0)^2)^{-1})\| \leq (\frac{1}{2} + u^2)^{-1} \quad \text{for all } u.$$

Proof. Let $\xi \in \mathcal{H}_\infty$ and let us give a lower bound for $\|\theta_\lambda(1 + u^2 + X(0)^2)\xi\|$. Using

$$\theta_\lambda(T) = T + [D^2, T](D^2 + \lambda)^{-1} \quad (47)$$

we get

$$\theta_\lambda(X(0)^2) = X(0)^2 - [D^2, [D, h]^2](D^2 + \lambda)^{-1}.$$

Now the regularity shows (cf. §13) that $[D^2, [D, h]^2](D^2 + \lambda)^{-1}$ is compact so that for $\lambda \rightarrow \infty$ its norm goes to 0 (in fact it is of the form $B_\lambda(D^2 + \lambda)^{-1/2}$ with the norm of B_λ bounded, so its norm decays like $\lambda^{-1/2}$). Thus we can choose λ large enough so that $Z = \theta_\lambda(X(0)^2) - X(0)^2$ fulfills $\|Z\| \leq 1/2$. We then get

$$\langle \xi, \theta_\lambda(1 + u^2 + X(0)^2)\xi \rangle \geq \langle \xi, 1 + u^2 + X(0)^2\xi \rangle - |\langle \xi, Z\xi \rangle| \geq (\frac{1}{2} + u^2)\|\xi\|^2$$

(using $X(0)^2 = -[D, h]^2 \geq 0$) so that

$$\|\theta_\lambda(1 + u^2 + X(0)^2)\xi\| \geq (\frac{1}{2} + u^2)\|\xi\| \quad \text{for all } \xi \in \mathcal{H}_\infty.$$

It remains to show that $\theta_\lambda(1 + u^2 + X(0)^2)$ is invertible as an operator acting in \mathcal{H}_∞ . Since $D^2 + \lambda$ is an automorphism of \mathcal{H}_∞ , it is enough to show that $1 + u^2 + X(0)^2$ is invertible as an operator acting in \mathcal{H}_∞ . One has $X(0)^2 = -[D, h]^2 \geq 0$ so that $1 + u^2 + X(0)^2$ is invertible as an operator in \mathcal{H} . Its invertibility in \mathcal{H}^∞ follows from the stability under smooth functional calculus (Proposition 2.4) of the algebra

$$\{T \in \mathcal{L}(\mathcal{H}) \mid T\mathcal{H}_\infty \subset \mathcal{H}_\infty, \|\delta^m(T)\| < \infty \text{ for all } m\}$$

and the fact that, by regularity, $[D, h]$ belongs to this algebra. \square

Lemma 5.10. *Let $h = h^* \in \mathcal{A}$. Then when $\varepsilon \rightarrow 0$,*

$$Y(\varepsilon)\xi \rightarrow Y(0)\xi \quad \text{for all } \xi \in \mathcal{H}_\infty.$$

Proof. One has for the action on \mathcal{H}_∞ ,

$$\begin{aligned} X(\varepsilon)^2 &= (\varepsilon D - i[D, h])^2 = \varepsilon^2 D^2 - i\varepsilon(D[D, h] + [D, h]D) - [D, h]^2 \\ &= \varepsilon^2 D^2 - i\varepsilon[D^2, h] - [D, h]^2. \end{aligned}$$

We first estimate, for $\xi \in \mathcal{H}_\infty$,

$$\eta(u, \varepsilon) = \left(\frac{X(\varepsilon)^2}{1 + u^2 + X(\varepsilon)^2} - \frac{X(0)^2}{1 + u^2 + X(0)^2} \right) \xi.$$

One has

$$\begin{aligned}
\eta(u, \varepsilon) &= (1 + u^2) \left(\frac{1}{1 + u^2 + X(0)^2} - \frac{1}{1 + u^2 + X(\varepsilon)^2} \right) \xi \\
&= \frac{1 + u^2}{1 + u^2 + X(\varepsilon)^2} (X(\varepsilon)^2 - X(0)^2) \frac{1}{1 + u^2 + X(0)^2} \xi \\
&= \frac{1 + u^2}{1 + u^2 + X(\varepsilon)^2} (\varepsilon^2 D^2 - i\varepsilon[D^2, h]) \frac{1}{1 + u^2 + X(0)^2} \xi \\
&= \frac{1 + u^2}{1 + u^2 + X(\varepsilon)^2} (\varepsilon^2 D^2 - i\varepsilon[D^2, h]) (D^2 + \lambda)^{-1} \\
&\quad \theta_\lambda ((1 + u^2 + X(0)^2)^{-1}) (D^2 + \lambda) \xi.
\end{aligned}$$

Now one has, using regularity,

$$\|(\varepsilon^2 D^2 - i\varepsilon[D^2, h]) (D^2 + \lambda)^{-1}\| = k(\varepsilon) = O(\varepsilon)$$

while, since $X(\varepsilon)$ is self-adjoint,

$$\left\| \frac{1 + u^2}{1 + u^2 + X(\varepsilon)^2} \right\| \leq 1.$$

Moreover $(D^2 + \lambda)\xi \in \mathcal{H}_\infty \subset \mathcal{H}$. By Lemma 5.9, for λ large enough, one thus gets

$$\|\theta_\lambda ((1 + u^2 + X(0)^2)^{-1}) (D^2 + \lambda)\xi\| \leq (\tfrac{1}{2} + u^2)^{-1} \|(D^2 + \lambda)\xi\| \quad \text{for all } u.$$

Thus after integrating in u we get the estimate

$$\|(Y(\varepsilon) - Y(0))\xi\| \leq \frac{2}{\pi} \int_0^\infty k(\varepsilon) (\tfrac{1}{2} + u^2)^{-1} \|(D^2 + \lambda)\xi\| du = O(\varepsilon),$$

which gives the required result. \square

It remains to estimate the continuity for $\varepsilon \rightarrow 0$ of $f_0(X(\varepsilon))\xi$. The above proof shows that for $g_a(x) = (a + x^2)^{-1}$ and any $a > 0$ one has the norm continuity of $g_a(X(\varepsilon))\xi$ when $\varepsilon \rightarrow 0$ (we showed convergence only for $\xi \in \mathcal{H}_\infty$, but it holds in general using the boundedness of the functions g_a). The even functions in $f \in C_0(\mathbb{R})$ for which

$$\|f(X(\varepsilon))\xi - f(X(0))\xi\| \rightarrow 0 \quad \text{for all } \xi \in \mathcal{H} \quad (48)$$

form a norm closed subalgebra of $C_0(\mathbb{R})^{\text{even}}$. This algebra contains the functions g_a , thus the Stone–Weierstrass Theorem shows that (48) holds for all $f \in C_0(\mathbb{R})^{\text{even}}$ and in particular for f_0 . We thus get:

Proposition 5.11. *Let $h = h^* \in \mathcal{A}$. Then one has, with norm convergence,*

$$\lim_{\tau \rightarrow \infty} \tau^{-1} e^{i\tau h} |D| e^{-i\tau h} \xi = |[D, h]| \xi \quad \text{for all } \xi \in \text{Dom } D.$$

Proof. By (43) we just need to show that $|X(\varepsilon)|\xi \rightarrow |X(0)|\xi$ when $\varepsilon \rightarrow 0$ for any $\xi \in \mathcal{H}_\infty$. By (44), $|X(\varepsilon)| = Y(\varepsilon) + f_0(X(\varepsilon))$. By Lemma 5.9 we have $Y(\varepsilon)\xi \rightarrow Y(0)\xi$ for $\xi \in \mathcal{H}_\infty$, and by the above discussion $f_0(X(\varepsilon))\xi$ is continuous at $\varepsilon = 0$. Thus we get the required result for $\xi \in \mathcal{H}_\infty$. The general case $\xi \in \text{Dom } D$ follows using (42). \square

Remark 5.12. Proposition 5.11 shows that, under the regularity hypothesis,

$$[[D, h], [D, a]] = 0 \quad \text{for all } h = h^*, a \in \mathcal{A}. \quad (49)$$

Indeed one has

$$[e^{i\tau h}|D|e^{-i\tau h}, [D, a]] = e^{i\tau h}[[D], [D, a]]e^{-i\tau h}$$

and the norm of $[[D], [D, a]]$ is finite so that $\tau^{-1}\|[e^{i\tau h}|D|e^{-i\tau h}, [D, a]]\| \rightarrow 0$ for $\tau \rightarrow \infty$. Thus one has

$$\lim_{\tau \rightarrow \infty} \tau^{-1} (e^{i\tau h}|D|e^{-i\tau h}[D, a]\xi - [D, a]e^{i\tau h}|D|e^{-i\tau h}\xi) = 0 \quad \text{for all } \xi \in \text{Dom } D,$$

and, since $[D, a]$ preserves $\text{Dom } D$,

$$[[D, h], [D, a]]\xi = 0 \quad \text{for all } \xi \in \text{Dom } D.$$

Note also that, by the same argument, under the *strong regularity* hypothesis of Definition 6.1 below, this shows that

$$[D, h]^2 \in \mathcal{A} \quad \text{for all } h = h^* \in \mathcal{A}.$$

Indeed $[[D, h]]$ then commutes with all endomorphisms of \mathcal{H}_∞ . Its square $[D, h]^2$, being itself an endomorphism, belongs to the center of $\text{End}_{\mathcal{A}}(\mathcal{H}_\infty)$ and is, by (6), an element of \mathcal{A} .

We can now show that regularity suffices to ensure the dissipative property of Lemma 5.1.

Theorem 5.13. *Let $(\mathcal{A}, \mathcal{H}, D)$ be a regular spectral triple with \mathcal{A} commutative fulfilling the order one condition. Then for any $h = h^* \in \mathcal{A}$, the commutator $[D, h]$ vanishes where h reaches its maximum, i.e., for any sequence $b_n \in \mathcal{A}$, $\|b_n\| \leq 1$, with support tending to $\{\chi\}$, where χ is a character such that $|\chi(h)|$ is maximum, one has*

$$\|[D, h]b_n\| \rightarrow 0.$$

Proof. By Proposition 5.11 combined with the third statement of Lemma 5.7 one has, first for $\xi \in \text{Dom } D$ and then by uniformity for all $\xi \in \mathcal{H}$,

$$\lim_{\tau \rightarrow \infty} C_\tau([D], h)\xi = \lim_{\tau \rightarrow \infty} \frac{i}{\tau} (e^{i\tau h}|D|e^{-i\tau h}\xi - |D|\xi) = i[[D, h]]\xi$$

and thus the $C_\tau([D], h)$ converge strongly to $i|[D], h|$ when $\tau \rightarrow \infty$. By the second statement of Lemma 5.7, one has

$$C_\tau(b_n^*[D], h)b_n = b_n^*C_\tau([D], h)b_n.$$

Thus fixing n and taking the limit for $\tau \rightarrow \infty$ one gets

$$C_\tau(b_n^*[D], h)b_n \rightarrow i b_n^*[D], h|b_n.$$

One thus gets

$$\|b_n^*[D], h|b_n\| \leq \|b_n^*[D], h|b_n\|.$$

But Lemma 5.4 shows that $\|b_n^*[D], h|b_n\| \rightarrow 0$ when $n \rightarrow \infty$, which gives the required result. Moreover, since $|[D], h|$ commutes with the b_n , this can be formulated by $\|[D], h|b_n\| \rightarrow 0$. \square

Corollary 5.14. *Let $(\mathcal{A}, \mathcal{H}, D)$ be a spectral triple with \mathcal{A} commutative fulfilling the five conditions of §2. The derivations $\pm\delta_j$ of Lemma 4.3 are dissipative.*

Proof. This follows from Theorem 5.13 and Lemma 5.1. \square

Corollary 5.15. *Let $h = h^* \in \mathcal{A}$. The principal symbol of the operator*

$$\text{Grad}(h) = [D^2, h]$$

vanishes where h reaches its maximum.

Proof. One has $[D^2, h] = D[D, h] + [D, h]D$ and since $[D, h]$ commutes with \mathcal{A} , one gets the principal symbol of $[D^2, h]$ from that of D , which gives

$$\lim_{\tau \rightarrow \infty} \frac{1}{\tau} e^{i\tau\phi} [D^2, h] e^{i\tau\phi} = -i([D, \phi][D, h] + [D, h][D, \phi]).$$

Thus the result follows from Theorem 5.13. \square

6. Self-adjointness and derivations

We now introduce a technical hypothesis which will play an important role.

Definition 6.1. A spectral triple is *strongly regular* when all endomorphisms of the \mathcal{A} -module \mathcal{H}_∞ are regular.

Our goal is to obtain self-adjoint operators from the operator D , in the form A^*DA where A is regular, i.e., belongs to the domains of δ^m for all m .

Lemma 6.2. *Let A be regular, then $A \text{ Dom } D \subset \text{Dom } D$ and the adjoint of $A^* D$ is the closure of the densely defined operator T*

$$\text{Dom } T = \text{Dom } D, \quad T\xi = D(A\xi) \quad \text{for all } \xi \in \text{Dom } D. \quad (50)$$

Proof. By regularity both A and A^* preserve the domain $\text{Dom } |D| = \text{Dom } D$ so that (50) makes sense. The domain of $A^* D$ is the domain of D . An $\eta \in \mathcal{H}$ belongs to the domain of the adjoint $S = (A^* D)^*$ when there exists a constant $C < \infty$ such that

$$|\langle A^* D\xi, \eta \rangle| \leq C \|\xi\| \quad \text{for all } \xi \in \text{Dom } D.$$

One has $\langle A^* D\xi, \eta \rangle = \langle D\xi, A\eta \rangle$ and, since D is self-adjoint, the above condition means that $A\eta \in \text{Dom } D$. Moreover one then has $S\eta = DA\eta$. In other words, $S = DA$ with domain

$$\text{Dom } S = \{\eta \mid A\eta \in \text{Dom } D\}, \quad S\xi = D(A\xi) \quad \text{for all } \xi \in \text{Dom } S.$$

To prove the lemma we need to show that S is the closure of the operator T of (50). Let $\eta \in \text{Dom } S$. We construct a sequence $\eta_n \in \text{Dom } D$ such that

$$\eta_n \rightarrow \eta, \quad DA\eta_n \rightarrow DA\eta.$$

In fact we let

$$\eta(\varepsilon) = (1 + \varepsilon|D|)^{-1}\eta \quad \text{for all } \varepsilon > 0.$$

It belongs to $\text{Dom } D$ by construction and $\eta(\varepsilon) \rightarrow \eta$ when $\varepsilon \rightarrow 0$. One has

$$DA\eta(\varepsilon) = D(1 + \varepsilon|D|)^{-1}A\eta + \varepsilon D(1 + \varepsilon|D|)^{-1}[|D|, A](1 + \varepsilon|D|)^{-1}\eta.$$

Since $A\eta \in \text{Dom } D$ one has $D(1 + \varepsilon|D|)^{-1}A\eta = (1 + \varepsilon|D|)^{-1}DA\eta \rightarrow DA\eta$. The remainder is of the form $B(\varepsilon)[|D|, A]\eta(\varepsilon)$, where $B(\varepsilon) = \varepsilon D(1 + \varepsilon|D|)^{-1}$ is of norm less than 1, $[|D|, A]$ is bounded and $\eta(\varepsilon) \rightarrow \eta$. Thus it behaves like $B(\varepsilon)[|D|, A]\eta$ and hence tends to 0 when $\varepsilon \rightarrow 0$ since $B(\varepsilon)\zeta \rightarrow 0$ for any $\zeta \in \mathcal{H}$. This shows that $DA\eta(\varepsilon) \rightarrow DA\eta$ and S is the closure of T . \square

Corollary 6.3. *Let $\varphi = \varphi^* \in \mathcal{A}$. Then the operator $H = \varphi D\varphi$ with domain $\text{Dom } D$ is essentially self-adjoint.*

Proof. One has $H = \varphi^2 D + \varphi[D, \varphi]$ on $\text{Dom } D$. The bounded perturbation $P = \varphi[D, \varphi]$ does not alter the domain of the adjoint H^* which is thus the same as the domain of H_0^* , $H_0 = \varphi^2 D$. By Lemma 6.2, the adjoint of H_0 is the closure of $D\varphi^2$ with domain $\text{Dom } D$. This is the same as the closure of $\varphi D\varphi + [D, \varphi]\varphi$ with domain $\text{Dom } D$. Since $[D, \varphi]\varphi$ is bounded, we thus get that the adjoint H_0^* of H_0 is the sum of the closure of $\varphi D\varphi$ with domain $\text{Dom } D$ with the bounded operator $[D, \varphi]\varphi$. Thus when adding $P^* = -[D, \varphi]\varphi$ to H_0^* , we obtain the closure of $\varphi D\varphi$ with domain $\text{Dom } D$, i.e., the operator H . \square

Lemma 6.4. *Let A be regular. Then A^*D is closable and*

- *for any ξ in the domain of the closure $\overline{A^*D}$ of A^*D , one has, for $\varepsilon > 0$,*

$$\begin{aligned} & (1 + \varepsilon|D|)^{-1} \overline{A^*D} \xi \\ &= A^*D(1 + \varepsilon|D|)^{-1} \xi - (1 + \varepsilon|D|)^{-1} [|D|, A^*] \varepsilon D (1 + \varepsilon|D|)^{-1} \xi, \end{aligned} \quad (51)$$

- *the domain of $\overline{A^*D}$ is the set of $\xi \in \mathcal{H}$ for which the $A^*D(1 + \varepsilon|D|)^{-1} \xi$ converge in norm for $\varepsilon \rightarrow 0$,*
- *the limit of the $A^*D(1 + \varepsilon|D|)^{-1} \xi$ gives $\overline{A^*D} \xi$.*

Proof. The operator A^*D is closable since its adjoint is densely defined by Lemma 6.2. The right-hand side of (51) is a bounded operator, thus it is enough to prove the equality for $\xi \in \text{Dom } D$ since $\overline{A^*D}$ is the closure of its restriction to $\text{Dom } D$. For $\xi \in \text{Dom } D$, (51) follows from

$$[(1 + \varepsilon|D|)^{-1}, A^*] = -(1 + \varepsilon|D|)^{-1} [\varepsilon|D|, A^*] (1 + \varepsilon|D|)^{-1}.$$

Let then ξ be in the domain of the closure $\overline{A^*D}$. By (51), $A^*D(1 + \varepsilon|D|)^{-1} \xi$ is the sum of $(1 + \varepsilon|D|)^{-1} \overline{A^*D} \xi \rightarrow \overline{A^*D} \xi$, and of $(1 + \varepsilon|D|)^{-1} [|D|, A^*] \varepsilon D (1 + \varepsilon|D|)^{-1} \xi$ which converges to 0 in norm since $(1 + \varepsilon|D|)^{-1} [|D|, A^*]$ is uniformly bounded while $\varepsilon D (1 + \varepsilon|D|)^{-1} \xi$ converges to 0 in norm. Thus $A^*D(1 + \varepsilon|D|)^{-1} \xi$ is convergent when $\varepsilon \rightarrow 0$. Conversely, if the $A^*D(1 + \varepsilon|D|)^{-1} \xi$ converge in norm for $\varepsilon \rightarrow 0$, then since $(1 + \varepsilon|D|)^{-1} \xi \rightarrow \xi$ and $(1 + \varepsilon|D|)^{-1} \xi \in \text{Dom } D$, one gets that ξ is in the domain of the closure $\overline{A^*D}$ of A^*D and that moreover the limit of the $A^*D(1 + \varepsilon|D|)^{-1} \xi$ gives $\overline{A^*D} \xi$. \square

Proposition 6.5. *Let A be regular then the operator $H = A^*DA$ with domain $\text{Dom } D$ is essentially self-adjoint. The domain of the closure of H is the set of $\xi \in \mathcal{H}$ for which the $A^*DA(1 + \varepsilon|D|)^{-1} \xi$ converge in norm for $\varepsilon \rightarrow 0$. The limit of the $A^*DA(1 + \varepsilon|D|)^{-1} \xi$ gives $\overline{H} \xi$.*

Proof. Let us first check that H is symmetric. One has for ξ and η in $\text{Dom } D$,

$$\langle H\xi, \eta \rangle = \langle A^*DA\xi, \eta \rangle = \langle DA\xi, A\eta \rangle = \langle A\xi, DA\eta \rangle = \langle \xi, A^*DA\eta \rangle = \langle \xi, H\eta \rangle.$$

Let us now show that H^* is the closure of H . Let $\eta \in \text{Dom } H^*$. Then there exists $C < \infty$ with

$$|\langle A^*DA\xi, \eta \rangle| \leq C \|\xi\| \quad \text{for all } \xi \in \text{Dom } D.$$

Since $\langle A^*DA\xi, \eta \rangle = \langle DA\xi, A\eta \rangle$, this means that $A\eta$ is in the domain of the adjoint of DA with domain $\text{Dom } D$, i.e.,

$$A\eta \in \text{Dom } T^*, \quad H^*\eta = T^*A\eta,$$

where T is defined in (50). By Lemma 6.2 the adjoint of A^*D is the closure of T : $(A^*D)^* = \overline{T}$. The adjoint T^* of T is the same as the adjoint of the closure \overline{T} , and is

the closure $\overline{A^*D} = (A^*D)^{**}$ of A^*D . Thus by Lemma 6.4 we have, since $A\eta$ is in the domain of $\overline{A^*D}$, the convergence of $A^*D(1 + \varepsilon|D|)^{-1}A\eta$ to $\overline{A^*D}A\eta = H^*\eta$. Moreover, as above, we have

$$\begin{aligned} A^*D(1 + \varepsilon|D|)^{-1}A\eta - A^*DA(1 + \varepsilon|D|)^{-1}\eta \\ = -A^*\varepsilon D(1 + \varepsilon|D|)^{-1}[|D|, A](1 + \varepsilon|D|)^{-1}\eta \end{aligned}$$

and the right-hand side converges to 0 in norm when $\varepsilon \rightarrow 0$. Thus we have shown that for any $\eta \in \text{Dom } H^*$ one gets the convergence of $A^*DA(1 + \varepsilon|D|)^{-1}\eta$ to $H^*\eta$. This shows, since $(1 + \varepsilon|D|)^{-1}\eta \in \text{Dom } H$, that H^* is the closure of H and hence that H is essentially self-adjoint. It also gives a characterization of the domain of the closure of H as required. \square

We now want to apply this result using endomorphisms of the \mathcal{A} -module \mathcal{H}_∞ which are of rank one, in order to obtain an operator on \mathcal{A} itself.

Lemma 6.6. *Let $\xi, \eta \in \mathcal{H}_\infty$. Then the following gives an endomorphism of the \mathcal{A} -module \mathcal{H}_∞ :*

$$T_{\xi, \eta}(\zeta) = (\eta|\zeta)\xi \quad \text{for all } \zeta \in \mathcal{H}_\infty, \quad (52)$$

where $(\eta|\zeta)$ is the \mathcal{A} -valued inner product. One has

$$T_{a\xi, b\eta} = ab^*T_{\xi, \eta} \quad \text{for all } a, b \in \mathcal{A}, \quad T_{\xi, \eta}^* = T_{\eta, \xi}. \quad (53)$$

Proof. This follows from the \mathcal{A} -linearity of the inner product, which is linear in the second variable and antilinear in the first. The equality $T_{\xi, \eta}^* = T_{\eta, \xi}$ follows from

$$\begin{aligned} \langle T_{\eta, \xi}\alpha, \beta \rangle &= \langle (\xi|\alpha)\eta, \beta \rangle = \int (\xi|\alpha)^*(\eta|\beta) d\lambda, \\ \langle \alpha, T_{\xi, \eta}\beta \rangle &= \langle \alpha, (\eta|\beta)\xi \rangle = \int (\alpha|\xi)(\eta|\beta) d\lambda. \end{aligned} \quad \square$$

By Proposition 2.3 (4), the $T_{\xi, \eta}$ are bounded operators in \mathcal{H} . Let us now assume that all endomorphisms of the \mathcal{A} -module \mathcal{H}_∞ are regular as in Definition 6.1. We can then apply Proposition 6.5 and get that

$$D_{\xi, \eta} = T_{\eta, \xi} D T_{\xi, \eta} \quad (54)$$

defines an essentially self-adjoint operator with domain $\text{Dom } D$. We need to relate this operator with the derivation of \mathcal{A} given by (22), i.e.,

$$\delta_0(a) = i(\xi|[D, a]\xi) \quad \text{for all } a \in \mathcal{A}. \quad (55)$$

Lemma 6.7. *One has*

$$D_{\xi, \eta}\zeta = -i\delta_0((\eta|\zeta))\eta + (\xi|D\xi)T_{\eta, \eta}\zeta \quad \text{for all } \xi, \eta, \zeta \in \mathcal{H}_\infty. \quad (56)$$

The operator $V_\eta(a) = a\eta$, for all $a \in \mathcal{A}$, extends to a bounded linear map V_η from $L^2(X, d\lambda)$ to \mathcal{H} , and one has

$$V_\eta^* V_\eta = (\eta|\eta), \quad V_\eta V_\eta^* = T_{\eta, \eta}, \quad V_\eta^*(\zeta) = (\eta|\zeta) \quad \text{for all } \zeta \in \mathcal{H}_\infty.$$

Proof. One has $D_{\xi, \eta}\zeta = T_{\eta, \xi} D T_{\xi, \eta}\zeta = T_{\eta, \xi} D((\eta|\zeta)\xi)$. Thus using $(\xi, [D, a]\xi) = -i\delta_0(a)$ and $(\xi|aD\xi)\eta = (\xi|D\xi)a\eta$ for $a = (\eta|\zeta)$ one gets

$$D_{\xi, \eta}\zeta = (\xi|Da\xi)\eta = (\xi|[D, a]\xi)\eta + (\xi|D\xi)a\eta = -i\delta_0((\eta|\zeta))\eta + (\xi|D\xi)T_{\eta, \eta}\zeta,$$

which gives (56). To show that V_η is bounded, note that

$$\langle V_\eta(a), V_\eta(a) \rangle = \langle a\eta, a\eta \rangle = \int a^* a (\eta|\eta) |D|^{-p} = \int a^* a (\eta|\eta) d\lambda,$$

which also shows that $V_\eta^* V_\eta = (\eta|\eta)$. Let us check that $V_\eta^*(\zeta) = (\eta|\zeta)$. One has

$$\langle \zeta, V_\eta(a) \rangle = \langle \zeta, a\eta \rangle = \int a(\zeta|\eta) d\lambda = \int (\eta|\zeta)^* a d\lambda = \langle V_\eta^*(\zeta), a \rangle.$$

The equality $V_\eta V_\eta^* = T_{\eta, \eta}$ follows from (52). \square

The strategy now is to use the self-adjointness of $D_{\xi, \eta}$ and the fact that δ_0 can be compared to $iD_{\xi, \eta}$ plus a bounded perturbation to show that the resolvent problem $(1 + \varepsilon\delta_0)\xi = \eta$ can be solved first in L^2 . Then one wants to use the regularity to show that this problem can also be solved in the Sobolev spaces. Finally one wants to use the Sobolev estimates to show that it can be solved in the C^* norm. Then together with the results on dissipative derivations of §5 one gets the existence of the resolvent for the action on the C^* -algebra. One notes that it is enough to solve the resolvent problem for ε small enough. One then applies the Hille–Yosida Theorem.

More specifically we consider the equation

$$(1 + i\varepsilon(D_{\xi, \eta} - (\xi|D\xi)T_{\eta, \eta}))\zeta = a\eta, \quad (57)$$

where $a \in \mathcal{A}$ is given and ε can be taken as small as needed. Given a solution ζ of (57), one can under suitable regularity conditions on ζ take the inner product $(\eta|\zeta) = b$. One then has, at the formal level,

$$b + \varepsilon\delta_0(b)(\eta|\eta) = a(\eta|\eta). \quad (58)$$

We can assume that the support of ξ is small enough so that we can find η such that $(\eta|\eta) = 1$ in a neighborhood of the support K of ξ . Then by (55) one gets that since δ_0 vanishes outside K , one can replace $\delta_0(b)(\eta|\eta)$ in (58) by $\delta_0(b)$. Moreover one then gets

$$c + \varepsilon\delta_0(c) = a, \quad c = b + (1 - (\eta|\eta))a \quad (59)$$

since $(1 - (\eta|\eta))a$ belongs to the kernel of δ_0 because its support is disjoint from K . We need to know that $c \in A$, where $A = C(X)$ is the norm closure of \mathcal{A} , and in fact

also that $[D, c]$ is bounded, just to formulate the result. Thus we need to control the Sobolev norms of the solution of (57). To do that we use the transformation θ_λ of (46). One has, as in (47),

$$\theta_\lambda(T) = T + \mathcal{E}_\lambda(T), \quad \mathcal{E}_\lambda(T) = [D^2, T](D^2 + \lambda)^{-1}, \quad (60)$$

so that the binomial formula expresses $\theta_\lambda^N(T)$ in terms of the $\mathcal{E}_\lambda^k(T)$ for $k \leq N$ as

$$\theta_\lambda^N(T) = T + \sum_{k \geq 1} \binom{N}{k} \mathcal{E}_\lambda^k(T). \quad (61)$$

Note also that, for T regular, and on $\text{Dom } D$ one has

$$[D^2, T] = 2\delta(T)|D| + \delta^2(T), \quad (62)$$

as follows from $[D^2, T] = [|D|^2, T] = \delta(T)|D| + |D|\delta(T) = 2\delta(T)|D| + \delta^2(T)$.

Lemma 6.8. *Let T be regular.*

- (1) *The $\mathcal{E}_\lambda^k(T)$ are compact operators for $k > 0$ and converge in norm to 0 when $\lambda \rightarrow \infty$.*
- (2) *One has (with $\lambda \geq 1$)*

$$\begin{aligned} \|\mathcal{E}_\lambda(T)D\| &\leq 2\|\delta(T)\| + \|\delta^2(T)\|, \\ \|D\mathcal{E}_\lambda(T)\| &\leq 2\|\delta(T)\| + 3\|\delta^2(T)\| + \|\delta^3(T)\|. \end{aligned} \quad (63)$$

- (3) *For $k > 1$, the operators $D\mathcal{E}_\lambda^k(T)$ and $\mathcal{E}_\lambda^k(T)D$ are compact operators which converge in norm to 0 when $\lambda \rightarrow \infty$.*

Proof. (1) One has, using (60) and (62), that

$$\mathcal{E}_\lambda(T) = (2\delta(T)|D| + \delta^2(T))(D^2 + \lambda)^{-1}. \quad (64)$$

Thus the answer follows for $k = 1$ since both $(D^2 + \lambda)^{-1}$ and $|D|(D^2 + \lambda)^{-1}$ are compact operators which converge in norm to 0 when $\lambda \rightarrow \infty$. Since $\mathcal{E}_\lambda(T)$ is also regular, it follows also for $k > 1$.

- (2) The first inequality of (63) follows from (64). For the second, one has

$$[|D|, \mathcal{E}_\lambda(T)] = \mathcal{E}_\lambda(\delta(T)),$$

which gives (63) using

$$\|D\mathcal{E}_\lambda(T)\| \leq \|\mathcal{E}_\lambda(T)|D|\| + \|\mathcal{E}_\lambda(\delta(T))\|,$$

and the second inequality follows using the first and (64).

- (3) The statement is immediate for $\mathcal{E}_\lambda^k(T)D$ since $|D|(D^2 + \lambda)^{-1}$ is compact. For the second one uses $[|D|, \mathcal{E}_\lambda^k(T)] = \mathcal{E}_\lambda^k(\delta(T))$ as in the proof of (2). \square

Lemma 6.9. (1) For any integer $N \in \mathbb{N}$, there exists $\lambda < \infty$ and $\varepsilon_0 > 0$ such that the operator

$$\theta_\lambda^N (1 + i\varepsilon S_{\xi, \eta}), \quad S_{\xi, \eta} = D_{\xi, \eta} - (\xi | D\xi) T_{\eta, \eta}, \quad (65)$$

with domain $\text{Dom } D$ is closable and invertible in \mathcal{H} for any $\varepsilon \leq \varepsilon_0$, and the norm of its inverse fulfills

$$\|(\theta_\lambda^N (1 + i\varepsilon S_{\xi, \eta}))^{-1}\| \leq 1 + Nc_{\xi, \eta}\varepsilon, \quad (66)$$

where $c_{\xi, \eta} < \infty$ only depends on ξ and η .

(2) For any integer N there exists $\varepsilon_N > 0$ such that (57) can be solved in $\mathcal{H}_N = \text{Dom } |D|^N$.

Proof. (1) The operator $P = (\xi | D\xi) T_{\eta, \eta}$ is bounded and regular since it is an endomorphism of the \mathcal{A} -module \mathcal{H}_∞ . Thus it preserves the domain of $(D^2 + \lambda)^N$ and the $\mathcal{E}_\lambda^k(P)$ are compact operators for $k > 0$ and converge in norm to 0 when $\lambda \rightarrow \infty$ by Lemma 6.8. By (54), one has $D_{\xi, \eta} = T_{\eta, \xi} D T_{\xi, \eta}$, thus, by regularity of the $T_{\xi, \eta}$, the operator

$$\theta_\lambda^N (D_{\xi, \eta}) = (D^2 + \lambda)^N D_{\xi, \eta} (D^2 + \lambda)^{-N}$$

is well defined on $\text{Dom } D$. Moreover one has, by (61), and on $\text{Dom } D$,

$$\theta_\lambda^N (D_{\xi, \eta}) = \theta_\lambda^N (T_{\eta, \xi}) \theta_\lambda^N (D) \theta_\lambda^N (T_{\xi, \eta}) = \sum_{k, m} \binom{N}{k} \mathcal{E}_\lambda^k (T_{\eta, \xi}) D \binom{N}{m} \mathcal{E}_\lambda^m (T_{\xi, \eta})$$

so that one gets

$$\theta_\lambda^N (D_{\xi, \eta}) = D_{\xi, \eta} + N \mathcal{E}_\lambda (T_{\eta, \xi}) D T_{\xi, \eta} + N T_{\eta, \xi} D \mathcal{E}_\lambda (T_{\xi, \eta}) + Q(N, \lambda),$$

where the remainder $Q(N, \lambda)$ is a sum of terms proportional to $\mathcal{E}_\lambda^k (T_{\eta, \xi}) D \mathcal{E}_\lambda^m (T_{\xi, \eta})$ for $k + m > 1$. By Lemma 6.8 we get that $Q(N, \lambda)$ is a compact operator and $\|Q(N, \lambda)\| \rightarrow 0$ when $\lambda \rightarrow \infty$. Thus for $\lambda \rightarrow \infty$, we get the following estimate: there exists $C_{\xi, \eta} < \infty$ only depending on ξ and η such that

$$\liminf_{\lambda \rightarrow \infty} \|\theta_\lambda^N (D_{\xi, \eta}) - D_{\xi, \eta}\| \leq N C_{\xi, \eta}.$$

Since the $\mathcal{E}_\lambda^k(P)$ are compact operators for $k > 0$ and converge in norm to 0 when $\lambda \rightarrow \infty$, one gets similarly

$$\liminf_{\lambda \rightarrow \infty} \|\theta_\lambda^N (S_{\xi, \eta}) - S_{\xi, \eta}\| \leq N C_{\xi, \eta}.$$

Let λ be large enough so that $\|\theta_\lambda^N (S_{\xi, \eta}) - S_{\xi, \eta}\| \leq 2N C_{\xi, \eta}$. For ε small enough,

$$\|\theta_\lambda^N (1 + i\varepsilon S_{\xi, \eta}) - (1 + i\varepsilon D_{\xi, \eta})\| \leq 2N C_{\xi, \eta} \varepsilon + \varepsilon \|(\xi | D\xi) T_{\eta, \eta}\| < 1. \quad (67)$$

Since $D_{\xi, \eta}$ is essentially self-adjoint, the operator $K = 1 + i\varepsilon D_{\xi, \eta}$ is closable and invertible for any $\varepsilon > 0$ and the norm of its inverse is ≤ 1 . $\theta_\lambda^N(1 + i\varepsilon S_{\xi, \eta})$ is closable since it is a bounded perturbation $K - B$ of $K = 1 + i\varepsilon D_{\xi, \eta}$. Moreover by (67) it is invertible, and the norm of its inverse, which is given by the Neumann series $(K - B)^{-1} = \sum (K^{-1}B)^m K^{-1}$, fulfills (66). This proves the first statement.

(2) Let $\xi = a\eta \in \mathcal{H}_{2N}$, and consider $\xi' = (D^2 + \lambda)^N \xi \in \mathcal{H}$. Then, by the first statement, one can find a sequence $\zeta_n \in \text{Dom } D$, $\zeta_n \rightarrow \xi' \in \mathcal{H}$, such that

$$\theta_\lambda^N(1 + i\varepsilon(D_{\xi, \eta} - (\xi|D\xi)T_{\eta, \eta}))\zeta_n \rightarrow \xi',$$

where the convergence is in \mathcal{H} . Applying the bounded operator $(D^2 + \lambda)^{-N}$ gives

$$(1 + i\varepsilon(D_{\xi, \eta} - (\xi|D\xi)T_{\eta, \eta}))(D^2 + \lambda)^{-N}\zeta_n \rightarrow \xi.$$

One has $\zeta = (D^2 + \lambda)^{-N}\xi' \in \mathcal{H}_{2N} \subset \text{Dom } D$ and $(D^2 + \lambda)^{-N}\zeta_n \rightarrow (D^2 + \lambda)^{-N}\xi'$ in the topology of $\text{Dom } D$. Thus

$$(1 + i\varepsilon(D_{\xi, \eta} - (\xi|D\xi)T_{\eta, \eta}))\zeta = \xi,$$

and $\zeta \in \mathcal{H}_{2N}$ gives the required solution. \square

We now need to show that if $\eta \in \mathcal{H}_\infty$ and $\zeta \in \mathcal{H}_N$ for N large enough, the inner product $(\eta|\zeta)$ gives an element of $A = C(X)$ and in fact in the domain of δ^k . To see this we use Proposition 2.3. We recall that the Sobolev norms on \mathcal{A} are defined using generators η_μ of the \mathcal{A} -module \mathcal{H}_∞ by (9), i.e.,

$$\|a\|_s^{\text{sobolev}} = \left(\sum_\mu \|(1 + D^2)^{s/2} a \eta_\mu\|^2 \right)^{1/2} \quad \text{for all } a \in \mathcal{A}.$$

Thus when we want to control the Sobolev norms of $(\eta|\zeta)$, we need to control the norms

$$\|(1 + D^2)^{s/2}(\eta|\zeta)\eta_\mu\|.$$

The point then is that $(\eta, \zeta)\eta_\mu = T_{\eta_\mu, \eta}\zeta$ while the endomorphism $T_{\eta_\mu, \eta}$ is regular by hypothesis so that $\theta_\lambda^N(T_{\eta_\mu, \eta})$ is bounded and (with $\lambda = 1$) one gets:

Lemma 6.10. *Assuming strong regularity, one has, for $\eta \in \mathcal{H}_\infty$,*

$$\|(\eta|\zeta)\|_s^{\text{sobolev}} \leq C_s \|(1 + D^2)^{s/2}\zeta\|. \quad (68)$$

Proof. It is enough to prove the estimate when $s/2 = N$ is an integer. For each μ one has

$$\begin{aligned} \|(1 + D^2)^N(\eta|\zeta)\eta_\mu\| &= \|\theta_1^N(T_{\eta_\mu, \eta})(1 + D^2)^N\zeta\| \\ &\leq \|\theta_1^N(T_{\eta_\mu, \eta})\| \|(1 + D^2)^N\zeta\|. \end{aligned} \quad \square$$

Theorem 6.11. *Let $(\mathcal{A}, \mathcal{H}, D)$ be a strongly regular spectral triple with \mathcal{A} commutative, fulfilling the five conditions of §2. Then any derivation of \mathcal{A} of the form (22), i.e., $\delta_0(a) = i(\xi|[D, a]\xi)$ for all $a \in \mathcal{A}$, is closable for the C^* -norm of A , and its closure is the generator of a one-parameter group of automorphisms $U(t)$ of $A = C(X)$, $X = \text{Spec}(\mathcal{A})$.*

Proof. By Corollary 5.14 the derivation δ_0 , with domain $\mathcal{A} \subset A$, is dissipative for the C^* -norm of A . Thus it is closable ([3], Proposition 1.4.7) and we let $D(\delta_0)$ be the domain of its closure. To apply the Hille–Yosida–Lumer–Phillips Theorem we need to show that for sufficiently small ε one has

$$(1 + \varepsilon\delta_0)D(\delta_0) = A. \quad (69)$$

By Corollary 5.14, we have

$$\|(1 + \varepsilon\delta_0)(x)\| \geq \|x\| \quad \text{for all } x \in D(\delta_0).$$

Thus $(1 + \varepsilon\delta_0)D(\delta_0)$ is closed in norm and it is enough to show that $(1 + \varepsilon\delta_0)\mathcal{A}$ is norm dense in A . Let then $\eta \in \mathcal{H}_\infty$ be such that $(\eta|\eta) = 1$ in a neighborhood of the support of ξ (with $\delta_0(a) = i(\xi|[D, a]\xi)$). Let then $N \in \mathbb{N}$ be such that the Sobolev estimate holds (Proposition 2.3):

$$\|a\|_{C^*} \leq C \|a\|_N^{\text{sobolev}} \quad \text{for all } a \in \mathcal{A}. \quad (70)$$

Let $a \in \mathcal{A}$, one has $a\eta \in \mathcal{H}_\infty$. By Lemma 6.9 there exists $\varepsilon_{N+1} > 0$ such that for any $\varepsilon \leq \varepsilon_{N+1}$ one can find a solution in $\zeta \in \mathcal{H}_{N+1}$ of the equation (57). Since \mathcal{H}_∞ is dense in \mathcal{H}_{N+1} , we thus get a sequence $\zeta_n \in \mathcal{H}_\infty$ such that $\zeta_n \rightarrow \zeta$ in \mathcal{H}_{N+1} . The operator $S_{\xi, \eta} = D_{\xi, \eta} - (\xi|D\xi)T_{\eta, \eta}$ is continuous from \mathcal{H}_{N+1} to \mathcal{H}_N . One thus has, with convergence in \mathcal{H}_N ,

$$(1 + i\varepsilon S_{\xi, \eta})\zeta_n \rightarrow (1 + i\varepsilon S_{\xi, \eta})\zeta = a\eta.$$

Combining Lemma 6.10 with (70), one gets that the $b_n = (\eta|\zeta_n) \in \mathcal{A}$ converge in the C^* -norm $\|x\|$. Moreover, by (56) and (65),

$$S_{\xi, \eta}\zeta_n = -i\delta_0((\eta|\zeta_n))\eta, \quad (1 + i\varepsilon S_{\xi, \eta})\zeta_n = \zeta_n + \varepsilon\delta_0(b_n)\eta \rightarrow a\eta$$

with convergence in \mathcal{H}_N . Thus applying $(\eta|\cdot)$ and using (68) and (70),

$$b_n + \varepsilon\delta_0(b_n)(\eta|\eta) \rightarrow a(\eta|\eta)$$

in the C^* -norm, as in (58). Since $(\eta|\eta) = 1$ in a neighborhood of the support of ξ , one has $\delta_0(b_n)(\eta|\eta) = \delta_0(b_n)$. Moreover, one has $\delta_0((1 - (\eta|\eta))a) = 0$ since $(1 - (\eta|\eta))a$ vanishes in a neighborhood of the support of ξ . Thus we have the norm convergence

$$c_n + \varepsilon\delta_0(c_n) \rightarrow a, \quad c_n = b_n + (1 - (\eta|\eta))a,$$

and this shows that $(1 + \varepsilon\delta_0)\mathcal{A}$ is norm dense in A . Since $(1 + \varepsilon\delta_0)D(\delta_0)$ is norm closed, it is equal to A . Thus, we have shown that for sufficiently small ε one has (69). Thus the Hille–Yosida–Lumer–Phillips Theorem ([3], Theorem 1.5.2, [22], Theorem X.47 (a)) shows that δ_0 generates a contraction semi-group of A . Since the same holds for $-\delta_0$, one gets a one-parameter group of isometries $U(t) = e^{t\delta_0}$ of the C^* -algebra A . Moreover $U(t)(a)$ is a norm continuous function of t for fixed $a \in A$. Using the operators of the form

$$U(f) = \int f(t)U(t) dt: A \rightarrow A \quad (71)$$

for f such that the L^1 -norms of the derivatives $\|f^{(n)}\|_1$ fulfill $\sum \frac{t^n}{n!} \|f^{(n)}\|_1 < \infty$, one gets a dense domain of analytic elements and one checks that since δ_0 is a derivation on $D(\delta_0)$ the $U(t)$ are automorphisms of A . \square

It remains to show that the $U(t) \in \text{Aut}(A)$ respect the smoothness. Let us first show that we need only understand what happens to $U(t)(a)\eta$ as an element of \mathcal{H} because $U(t)$ is the identity in the complement of the support of ξ .

Lemma 6.12. *Let $x \in A$ have support disjoint from the support of ξ . Then $U(t)(x) = x$ for all $t \in \mathbb{R}$.*

Proof. We can assume that $x \in \mathcal{A}$. Let us show that $\delta_0(x) = 0$. There exists $\phi \in \mathcal{A}$ with $x = x\phi^2$ and $\phi\xi = 0$. One has $\delta_0(x) = i(\xi|[D, x]\xi)$ and $[D, x] = [D, x]\phi^2 + 2x[D, \phi]\phi$ so that $[D, x]\xi = 0$ and $\delta_0(x) = 0$. It follows that for f as in (71), one gets $\delta_0(U(f)(x)) = 0$ since $U(f)$ commutes with δ_0 . With f analytic for L^1 one gets that $U(f)(x)$ is an analytic element such that $\delta_0(U(f)(x)) = 0$ and hence $\delta_0^n(U(f)(x)) = 0$ for all $n \geq 1$. It follows that

$$U(t)(U(f)(x)) = \sum \frac{t^n}{n!} \delta_0^n(U(f)(x)) = U(f)(x) \quad \text{for all } t \in \mathbb{R}.$$

Thus since $U(f_n)(x) \rightarrow x$ in norm for a suitable sequence f_n , one gets $U(t)(x) = x$ for all $t \in \mathbb{R}$. \square

Lemma 6.13. *Let S be the closure of $S_{\xi, \eta} = D_{\xi, \eta} - (\xi|D\xi)T_{\eta, \eta}$ as an unbounded operator in \mathcal{H} . Then for any $a \in \text{Dom } \delta_0$ one has $a\eta \in \text{Dom } S$ and*

$$S(a\eta) = -i\delta_0(a)\eta. \quad (72)$$

For any $a \in A$ and $\varepsilon > 0$, let $b = (1 + \varepsilon\delta_0)^{-1}(a)$. Then $b\eta \in \text{Dom } S$ and

$$(1 + i\varepsilon S)(b\eta) = a\eta. \quad (73)$$

Proof. For $a_n \rightarrow a$ and $\delta_0(a_n) \rightarrow \delta_0(a)$ in norm one has $a_n\eta \rightarrow a\eta$ and $\delta_0(a_n)\eta \rightarrow \delta_0(a)\eta$ in \mathcal{H} . Thus, since S is closed and \mathcal{A} is a core for δ_0 , it is enough to prove (72) for $a \in \mathcal{A}$. In that case one gets

$$S(a\eta) = S_{\xi, \eta}(a\eta) = ((\xi|D(a(\eta|\eta)\xi)) - (\xi|D\xi)(\eta|a\eta))\eta.$$

One has $a(\eta|\eta)\xi = a\xi$ since $(\eta|\eta) = 1$ on the support of ξ . Similarly, $(\xi|D\xi)(\eta|a\eta) = (\xi|D\xi)a = (\xi|aD\xi)$. Thus we get

$$S(a\eta) = (\xi|[D, a]\xi) \quad \text{for all } a \in \mathcal{A},$$

which gives (72).

To prove (73), note that by Theorem 6.11 the resolvent $(1 + \varepsilon\delta_0)^{-1}$ exists for any $\varepsilon > 0$ and maps A to the domain of δ_0 . Thus applying the first part of the lemma to $b = (1 + \varepsilon\delta_0)^{-1}(a)$, one gets

$$(1 + i\varepsilon S)(b\eta) = b\eta + i\varepsilon(-i\delta_0(b)\eta) = ((1 + \varepsilon\delta_0)b)\eta = a\eta,$$

which gives (73). □

Lemma 6.14. *The one-parameter group $U(t) \in \text{Aut}(A)$ fulfills for each N an estimate of the form*

$$\|U(t)(a)\|_N^{\text{sobolev}} \leq c_1 e^{Nc_{\xi, \eta}|t|} \|a\|_N^{\text{sobolev}}. \quad (74)$$

Proof. We first use the Sobolev semi-norm given by

$$\|a\|_{N, \eta, \lambda}^{\text{sobolev}} = (D^2 + \lambda)^{N/2} a\eta,$$

with $\lambda > 0$ determined by Lemma 6.9. We let $\varepsilon_0 > 0$ be as in Lemma 6.9. By (73) one has for any $a \in A$

$$(1 + \varepsilon\delta_0)^{-1}(a)\eta = (1 + i\varepsilon S)^{-1}a\eta \quad \text{for all } \varepsilon \leq \varepsilon_0.$$

Assume that $\|a\|_{N, \eta, \lambda}^{\text{sobolev}} < \infty$. One then has

$$a\eta = (D^2 + \lambda)^{-N/2}\zeta, \quad \zeta = (D^2 + \lambda)^{N/2}a\eta \in \mathcal{H}.$$

By Lemma 6.9 one gets, for $\varepsilon \leq \varepsilon_0$, using (66),

$$\begin{aligned} \|(1 + \varepsilon\delta_0)^{-1}(a)\|_{N, \eta, \lambda}^{\text{sobolev}} &= \|(D^2 + \lambda)^{N/2}(1 + \varepsilon\delta_0)^{-1}(a)\eta\| \\ &= \|(D^2 + \lambda)^{N/2}(1 + i\varepsilon S)^{-1}a\eta\| \\ &= \|\theta_\lambda^{N/2}((1 + i\varepsilon S)^{-1})\zeta\| \leq (1 + Nc_{\xi, \eta}\varepsilon)\|\zeta\| \\ &= (1 + Nc_{\xi, \eta}\varepsilon)\|a\|_{N, \eta, \lambda}^{\text{sobolev}}. \end{aligned}$$

This shows that $\|(1 + \varepsilon\delta_0)^{-1}(a)\|_{N,\eta,\lambda}^{\text{sobolev}} < \infty$, and thus one can iterate and obtains

$$\|(1 + \varepsilon\delta_0)^{-m}(a)\|_{N,\eta,\lambda}^{\text{sobolev}} \leq (1 + Nc_{\xi,\eta}\varepsilon)^m \|a\|_{N,\eta,\lambda}^{\text{sobolev}} \quad \text{for all } \varepsilon \leq \varepsilon_0. \quad (75)$$

Now for $t > 0$ and with norm convergence in A one has

$$U(-t)a = \lim_{n \rightarrow \infty} (1 + \frac{t\delta_0}{n})^{-n}(a).$$

This shows that $U(-t)a$ is the norm limit of the sequence $a_n = (1 + \frac{t\delta_0}{n})^{-n}(a)$ and moreover one has, from (75), $\|a_n\|_{N,\eta,\lambda}^{\text{sobolev}} \leq (1 + Nc_{\xi,\eta}t/n)^n \|a\|_{N,\eta,\lambda}^{\text{sobolev}}$. Thus,

$$\limsup \|a_n\|_{N,\eta,\lambda}^{\text{sobolev}} \leq e^{Nc_{\xi,\eta}|t|} \|a\|_{N,\eta,\lambda}^{\text{sobolev}}. \quad (76)$$

Since $a_n \rightarrow b = U(-t)a$ in norm, one has $a_n\eta \rightarrow b\eta$ also in norm in \mathcal{H} . Since the operator $(D^2 + \lambda)^{N/2}$ is closed, and by (76) the $(D^2 + \lambda)^{N/2}a_n\eta$ are uniformly bounded, it follows that $b\eta \in \text{Dom}(D^2 + \lambda)^{N/2}$ and thus $\|U(-t)a\|_{N,\eta,\lambda}^{\text{sobolev}} < \infty$. More precisely we get

$$\|U(-t)a\|_{N,\eta,\lambda}^{\text{sobolev}} \leq e^{Nc_{\xi,\eta}|t|} \|a\|_{N,\eta,\lambda}^{\text{sobolev}}.$$

Now the semi-norm $\|a\|_{N,\eta,\lambda}^{\text{sobolev}}$ is not equivalent to the Sobolev norm, but the latter is equivalent to the sum

$$\|(D^2 + \lambda)^{N/2}a\eta\| + \sum \|(D^2 + \lambda)^{N/2}a\eta_\mu\|,$$

where one can choose the η_μ so that their supports are disjoint from the support of ξ . This can be seen using the strong regularity. It then follows from Lemma 6.12 that the semi-norm $\sum \|(D^2 + \lambda)^{N/2}a\eta_\mu\|$ is preserved by $U(t)$ since $U(t)(a)\eta_\mu = a\eta_\mu$ for all μ . Thus one obtains (74). \square

Theorem 6.15. *Let $(\mathcal{A}, \mathcal{H}, D)$ be a strongly regular spectral triple with \mathcal{A} commutative, fulfilling the five conditions of §2. Then any derivation of \mathcal{A} of the form (22), i.e., $\delta_0(a) = i(\xi[D, a]\xi)$ for all $a \in \mathcal{A}$, is the generator of a one-parameter group of automorphisms $\sigma_t \in \text{Aut}(\mathcal{A})$ such that*

- $\partial_t \sigma_t(a) = \delta_0(\sigma_t(a))$,
- the map $(t, a) \in \mathbb{R} \times \mathcal{A} \mapsto \sigma_t(a) \in \mathcal{A}$ is jointly continuous.

Proof. By Lemma 6.14, the one-parameter group $U(t) \in \text{Aut}(A)$ preserves the sub-algebra $\mathcal{A} \subset A$. We let $\sigma_t \in \text{Aut}(\mathcal{A})$ be the corresponding automorphisms. For $a \in \mathcal{A}$ one has $a \in \text{Dom } \delta_0$ and thus

$$\sigma_t(a) - a = \int_0^t \sigma_u(\delta_0(a)) du,$$

where $\sigma_u(\delta_0(a))$ is a norm continuous function of u . By (74) applied to $\delta_0(a)$, this shows that $\|\sigma_t(a) - a\|_N^{\text{sobolev}} = O(|t|)$ when $t \rightarrow 0$. One has

$$\frac{1}{t}(\sigma_t(a) - a) - \delta_0(a) = \frac{1}{t} \int_0^t (\sigma_u(\delta_0(a)) - \delta_0(a)) du,$$

which, since $\|\sigma_u(\delta_0(a)) - \delta_0(a)\|_N^{\text{sobolev}} = O(|u|)$, gives

$$\|\frac{1}{t}(\sigma_t(a) - a) - \delta_0(a)\|_N^{\text{sobolev}} = O(|t|) \quad \text{for } t \rightarrow 0.$$

This shows that $\partial_t \sigma_t(a) = \delta_0(\sigma_t(a))$ in the Frechet space \mathcal{A} . Let us check the joint continuity of $(t, a) \mapsto \sigma_t(a)$. Let $(t_n, a_n) \rightarrow (t, a) \in \mathbb{R} \times \mathcal{A}$. One has $\sigma_{t_n}(a_n) - \sigma_t(a) = \sigma_{t_n}(a_n - a) + \sigma_{t_n}(a) - \sigma_t(a)$. The norm $\|\sigma_{t_n}(a) - \sigma_t(a)\|_N^{\text{sobolev}}$ converges to 0 by the above discussion. Moreover, Lemma 6.14 shows that one controls the Sobolev norms of $\sigma_{t_n}(a_n - a)$ by those of $(a_n - a)$, which gives the required continuity. \square

We can now also prove directly the absolute continuity of the transformed measure $\sigma_t^*(\lambda)$ with respect to λ .

Proposition 6.16. *Let $(\mathcal{A}, \mathcal{H}, D)$, δ_0 and σ_t be as in Theorem 6.15. Then for each $t \in \mathbb{R}$ the measure λ of (3) is strongly⁸ equivalent to its transform under σ_t .*

Proof. Let $\delta_0(a) = i(\xi|[D, a]\xi)$ for all $a \in \mathcal{A}$. By Lemma 6.12 the measure $\sigma_t^*(\lambda)$ given by $\sigma_t^*(\lambda)(f) = \lambda(\sigma_t(f))$ agrees with $\lambda(f)$ whenever the support of f is disjoint from the support of ξ . With $\eta \in \mathcal{H}_\infty$ as above one has $(\eta|\eta) = 1$ in a neighborhood V of the support of ξ . To obtain the required strong equivalence, it is enough to compare $\lambda(\sigma_t(f))$ and $\lambda(f)$ for f and $\sigma_t(f)$ with support contained in V . Using (1) one then has

$$\lambda(\sigma_t(f)) = \int \sigma_t(f) |D|^{-p} = \int \sigma_t(f)(\eta|\eta) |D|^{-p} = \langle \eta, \sigma_t(f)\eta \rangle.$$

Let, as above, S be the closure of $S_{\xi, \eta} = D_{\xi, \eta} - (\xi|D\xi)T_{\eta, \eta}$. It is by construction a bounded perturbation of the self-adjoint operator (closure of) $D_{\xi, \eta}$ and one can define e^{itS} for $t \in \mathbb{R}$ using the expansional formula ([1])

$$e^{A+B}e^{-A} = \left(\sum_n \int_{S_n} \alpha_{u_1}(B) \dots \alpha_{u_n}(B) du \right), \quad \alpha_u(B) = e^{uA} B e^{-uA}, \quad (77)$$

with $A = itD_{\xi, \eta}$ and $B = -it(\xi|D\xi)T_{\eta, \eta}$. Let us show that

$$\sigma_t(a)\eta = e^{itS}a\eta \quad \text{for all } a \in \mathcal{A}. \quad (78)$$

⁸ μ is strongly equivalent to ν iff there is $c > 0$ with $c\nu \leq \mu \leq c^{-1}\nu$.

By Theorem 6.15 and (72) the \mathcal{H} -valued function $t \mapsto \eta(t) = \sigma_t(a)\eta$ solves the differential equation

$$\frac{d\eta(t)}{dt} = iS\eta(t), \quad \eta(0) = a\eta, \quad \eta(t) \in \text{Dom } S \text{ for all } t \in \mathbb{R}.$$

This implies that $\frac{d}{dt}(e^{-itS}\eta(t)) = 0$ and thus $e^{-itS}\eta(t) = a\eta$, which proves (78). It follows from (78) that

$$\langle \eta, \sigma_t(a)\eta \rangle = \langle \eta, e^{itS}a\eta \rangle = \langle e^{-itS^*}\eta, a\eta \rangle. \quad (79)$$

Note that S is not self-adjoint in general because of the additional term $-(\xi|D\xi)T_{\eta,\eta}$. The difference $S - S^*$ is a bounded operator and an endomorphism of the \mathcal{A} -module \mathcal{H}_∞ given by

$$S - S^* = \rho T_{\eta,\eta}, \quad \rho = (\xi|D\xi)^* - (\xi|D\xi), \quad (80)$$

since $T_{\eta,\eta}$ is self-adjoint by (53). We can now write a formula for $e^{-itS^*}\eta$,

$$e^{-itS^*}\eta = \left(\sum_n i^n t^n \int_{S_n} \sigma_{-tu_1}(\rho) \dots \sigma_{-tu_n}(\rho) du \right) \eta \quad (81)$$

with $S_n = \{(u_j) \mid 0 \leq u_1 \leq \dots \leq u_n \leq 1\}$ the standard simplex. Indeed one has $-itS^* = -itS + P$ with $P = it\rho T_{\eta,\eta}$, which is bounded which allows one to use the expansional formula (77), with $A = -itS$, $B = P$. Now by (78) one has $e^{itS}\eta = \eta$ for all $t \in \mathbb{R}$ thus the left-hand side of (77) applied to η gives $e^{-itS^*}\eta$. Let us compute the right-hand side. We first show that

$$e^{isS} \rho T_{\eta,\eta} a \eta = \sigma_s(\rho a) \eta \quad \text{for all } a \in \mathcal{A}. \quad (82)$$

Indeed one has $T_{\eta,\eta} a \eta = (\eta|a\eta)\eta = a(\eta|\eta)\eta$ and since $(\eta|\eta) = 1$ on the support of ρ (using (80)), one gets that $\rho T_{\eta,\eta} a \eta = \rho a(\eta|\eta)\eta = \rho a \eta$. Thus (82) follows from (78). We then get

$$\begin{aligned} & \alpha_{u_1}(P) \dots \alpha_{u_n}(P) \eta \\ &= e^{-itu_1 S} P e^{-it(u_2-u_1)S} P \dots e^{-it(u_n-u_{n-1})S} P \eta \\ &= i^n t^n e^{-itu_1 S} \rho T_{\eta,\eta} e^{-it(u_2-u_1)S} \rho T_{\eta,\eta} \dots e^{-it(u_n-u_{n-1})S} \rho T_{\eta,\eta} \eta \\ &= i^n t^n \sigma_{-tu_1}(\rho \sigma_{-t(u_2-u_1)}(\rho \dots (\rho \sigma_{-t(u_n-u_{n-1})}(\rho)) \dots)) \eta, \end{aligned}$$

which yields (81) from (77). Now the series

$$h(t) = \sum_n i^n t^n \int_{S_n} \sigma_{-tu_1}(\rho) \dots \sigma_{-tu_n}(\rho) du \quad (83)$$

converges in the Frechet algebra \mathcal{A} since, for each k , the $p_k(\sigma_s(\rho))$ are uniformly bounded on compact sets of s , while the volume of the simplex S_n is $1/n!$. Thus $h(t) \in \mathcal{A}$, and combining (79) and (81) one has

$$\langle \eta, \sigma_t(f)\eta \rangle = \langle e^{-itS^*}\eta, f\eta \rangle = \langle h(t)\eta, f\eta \rangle = \langle \eta, \bar{h}(t)f\eta \rangle$$

so that we get, for all f with support in V ,

$$\lambda(\sigma_t(f)) = \lambda(\bar{h}(t)f). \quad (84)$$

Since, by construction, one has $h = 1$ outside the support of ξ , (using Lemma 6.12), equality (84) holds for all $f \in \mathcal{A}$. The norm continuity $\|h(t) - 1\| \rightarrow 0$ when $t \rightarrow 0$ (using (83)) then gives the required strong equivalence. \square

7. Absolute continuity

The following equality defines a positive measure λ on X :

$$\int a \, d\lambda = \int a |D|^{-p} \quad \text{for all } a \in C(X). \quad (85)$$

This measure is locally equivalent to the spectral measure of the representation of $A = C(X)$ in \mathcal{H} . More precisely:

Lemma 7.1. *For any open set $V \subset X$ the following two measures are strongly equivalent:*

- *The restriction $\lambda|_V$ to V of the measure λ of (85).*
- *The restriction to V of the spectral measure associated to a vector $\xi \in \mathcal{H}^\infty$ whose \mathcal{A} -valued inner product $\langle \xi, \xi \rangle$ is strictly positive on \bar{V} .*

Proof. By the condition of absolute continuity one has a relation of the form

$$\langle \xi, a\xi \rangle = \int a(\xi, \xi) |D|^{-p}$$

and since $\langle \xi, \xi \rangle \in \mathcal{A}$ is strictly positive on \bar{V} , one gets the strong equivalence between the restriction to V of the spectral measure associated to the vector $\xi \in \mathcal{H}^\infty$ and the measure $\lambda|_V$ of (85). \square

We let

$$B_\varepsilon = \{t \in \mathbb{R}^p \mid |t| < \varepsilon\}.$$

Given an automorphism $\sigma \in \text{Aut}(\mathcal{A})$ we use the covariant notation

$$\sigma(\kappa) = \kappa \circ \sigma^{-1} \quad \text{for all } \kappa \in \text{Spec}(\mathcal{A}) \quad (86)$$

and view σ as a homeomorphism of $X = \text{Spec}(\mathcal{A})$. We use the notations U_α, s_α of Lemma 4.4 and of (29).

Lemma 7.2. *Let $V \subset U_\alpha$ be an open set and $\chi \in V$. There exists a smooth family $\sigma_t \in \text{Aut}(\mathcal{A})$, $t \in \mathbb{R}^p$, a neighborhood Z of χ in V and $\varepsilon > 0$, $\varepsilon' > 0$ such that:*

- (1) For any $\kappa \in Z$, the map $t \mapsto s_\alpha(\sigma_t(\kappa)) = F(\kappa, t)$ is a diffeomorphism, depending continuously on κ , of B_ε with a neighborhood of $s_\alpha(\kappa)$ in \mathbb{R}^p and

$$s_\alpha(\kappa) + B_{\varepsilon'} \subset F(\kappa, B_{\varepsilon/2}) \quad \text{for all } \kappa \in \bar{Z}. \quad (87)$$

- (2) For any $t \in B_\varepsilon$ one has

$$\frac{1}{2}\lambda \leq \sigma_t(\lambda) \leq 2\lambda. \quad (88)$$

- (3) $Z_1 = \bigcap_{B_\varepsilon} \sigma_t Z$ is a neighborhood of χ .

- (4) $Z_2 = \bigcup_{B_\varepsilon} \sigma_t Z$ is contained in V .

Proof. Let $\sigma_t \in \text{Aut}(\mathcal{A})$, $t \in \mathbb{R}^p$, W and Z as in Lemma 4.5. We can replace the Z_0 of Lemma 4.5 by any neighborhood of χ contained in Z_0 and hence by a ball centered at χ and contained in $V \cap Z_0$. We use a metric d on X compatible with the topology (Proposition 2.3). Thus

$$Z = \{\kappa \in X \mid d(\kappa, \chi) < r\}$$

and we can take r small enough so that

$$\{\kappa \in X \mid d(\kappa, \chi) \leq 3/2 r\} \subset V. \quad (89)$$

The continuity of the map $(\kappa, t) \mapsto \sigma_t^{-1}(\kappa) = \kappa \circ \sigma_t$ yields $\varepsilon > 0$ with $B_\varepsilon \subset W$ and

$$d(\kappa, \sigma_t^{\pm 1}(\kappa)) \leq r/2 \quad \text{for all } \kappa \in X, t \in B_\varepsilon. \quad (90)$$

Then the first statement (1) follows from Lemma 4.5, with (87) coming from the continuity in κ . The second statement follows from (90) since for $d(\kappa, \chi) < r/2$ one gets $d(\chi, \sigma_t^{-1}(\kappa)) < r$ and $\sigma_t^{-1}(\kappa) \in Z$. Similarly the third statement follows from (90) and (89). Finally (88) follows from Proposition 6.16 for ε small. \square

Lemma 7.3. *Let $V \subset X$ be an open set with $\bar{V} \subset U_\alpha$ and λ_V (resp. $\lambda_{\bar{V}}$) be the spectral measure of the restriction to V (resp. \bar{V}) of the representation of $C(X)$ in \mathcal{H} . Then $s_\alpha(\lambda_V)$ is equivalent to the Lebesgue measure on $s_\alpha(V)$ and there exists $c < \infty$ such that*

$$\int_{\bar{V}} f \circ s_\alpha d\lambda_{\bar{V}} \leq c \int_{s_\alpha(\bar{V})} f(x) dx^p \quad \text{for all } f \in C_c^+(\mathbb{R}^p). \quad (91)$$

Proof. By Lemma 7.1, the spectral measure λ_V (resp. $\lambda_{\bar{V}}$) is equivalent to the measure λ of (85) restricted to V (resp. \bar{V}). We show that

- for any $\chi \in \bar{V}$ one can find a neighborhood Z_1 of χ in U_α such that $s_\alpha(\lambda|_{Z_1}) \leq c dx^p$ for some $c < \infty$,
- for any $\chi \in V$ one can find a neighborhood Z_2 of χ in V such that $ds_\alpha(\lambda|_{Z_2})/dx^p = \rho(x) > 0$ in a neighborhood of $s_\alpha(\chi)$.

Let $\chi \in \bar{V}$. We apply Lemma 7.2 relative to $V = U_\alpha$. We let $\sigma_t, Z, Z_j, \varepsilon$ and ε' be as in Lemma 7.2. We can assume that for $|t| < \varepsilon$ one has (88). Let then $h \in C_c^\infty(B_\varepsilon)$, $h(t) \in [0, 1]$, be equal to 1 on $B_{\varepsilon/2}$. By Lemma 7.2, for any $\kappa \in Z$, the map $t \mapsto F(\kappa, t) = s_\alpha(\sigma_t(\kappa))$ is a diffeomorphism F_κ of B_ε with a neighborhood of $s_\alpha(\kappa)$ in \mathbb{R}^p . It then follows that for fixed κ the image in \mathbb{R}^p of the measure $h(t)dt^p$ is a smooth multiple $g_\kappa(u)$ of the Lebesgue measure du^p ,

$$\int_{B_\varepsilon} f(F(\kappa, t))h(t) dt^p = \int_{\mathbb{R}^p} f(u)g_\kappa(u) du^p \quad \text{for all } f \in C_c(\mathbb{R}^p). \quad (92)$$

The function g_κ vanishes outside $F_\kappa(B_\varepsilon)$ and is given inside by

$$g_\kappa(u) = h(\psi(u))|d\psi(u)/du|,$$

where ψ is the inverse of the diffeomorphism F_κ and $d\psi(u)/du$ its Jacobian. The continuity of the map $\kappa \mapsto F_\kappa$ gives a uniform upper bound

$$g_\kappa(u) \leq c_1 \quad \text{for all } u \in \mathbb{R}^p, \kappa \in Z. \quad (93)$$

Since $h = 1$ on $B_{\varepsilon/2}$ and $s_\alpha(\kappa) + B_{\varepsilon'} \subset F_\kappa(B_{\varepsilon/2})$ by (87), one has $h(\psi(u)) = 1$ for $u \in s_\alpha(\kappa) + B_{\varepsilon'}$. The continuity of the map $\kappa \mapsto F_\kappa$ then yields $\varepsilon_1 > 0$ such that

$$g_\kappa(u) \geq \varepsilon_1 \quad \text{for all } u \in s_\alpha(\kappa) + B_{\varepsilon'} \text{ and all } \kappa \in Z. \quad (94)$$

We consider the image $d\nu$ under $(\kappa, t) \in Z \times B_\varepsilon \mapsto F(\kappa, t) \in \mathbb{R}^p$ of the finite positive measure $d\lambda(\kappa)h(t)dt^p$ on $Z \times B_\varepsilon$. It is given by

$$\int_{\mathbb{R}^p} f(x)d\nu(x) = \int_Z \int_{B_\varepsilon} f(F(\kappa, t))h(t)dt^p d\lambda(\kappa) \quad \text{for all } f \in C_c(\mathbb{R}^p),$$

and is equal, by (92), to

$$\int f(x) d\nu(x) = \iint f(u)g_\kappa(u) du^p d\lambda(\kappa) = \int f(u)\rho(u) du^p \quad (95)$$

where

$$\rho(u) = \int_Z g_\kappa(u)d\lambda(\kappa).$$

By (93) one has

$$\rho(u) \leq c_1\lambda(Z) < \infty \quad \text{for all } u \in \mathbb{R}^p. \quad (96)$$

Moreover (94) shows that

$$\rho(u) \geq \varepsilon_1\lambda(\{\kappa \in Z \mid |u - s_\alpha(\kappa)| < \varepsilon'\}).$$

We then have

$$\rho(u) > 0 \quad \text{for all } u \in s_\alpha(Z). \quad (97)$$

This strict positivity follows from the condition of absolute continuity which shows that the support of the measure λ is X . Indeed, for $u \in s_\alpha(Z)$, the open set $\{\kappa \in Z \mid |u - s_\alpha(\kappa)| < \varepsilon'\}$ is non-empty and it has strictly positive measure. This shows that the restriction of the measure ν to the open set $s_\alpha(Z)$ is equivalent to the Lebesgue measure.

We now use the quasi-invariance of $d\lambda$ given by (88) to compare $s_\alpha(\lambda|_{Z_j})$ with ν . Using (88) (for $\delta = 1/2$), one has $\frac{1}{2}d\lambda \leq d(\sigma_t(\lambda)) \leq 2d\lambda$ for $t \in B_\varepsilon$ so that, for any subset $E \subset X$ and any positive $f \in C_c^+(\mathbb{R}^p)$, one has, with 1_E the characteristic function of E ,

$$\frac{1}{2} \int (f \circ s_\alpha) 1_E d\lambda \leq \int (f \circ s_\alpha) 1_E d(\sigma_t(\lambda)) \leq 2 \int (f \circ s_\alpha) 1_E d\lambda.$$

The middle term is

$$\int (f \circ s_\alpha) 1_E d(\sigma_t(\lambda)) = \int (f \circ s_\alpha \circ \sigma_t)(1_E \circ \sigma_t) d\lambda,$$

and we thus get

$$\frac{1}{2} \int_E (f \circ s_\alpha) d\lambda \leq \int_{\sigma_t^{-1}E} (f \circ s_\alpha \circ \sigma_t) d\lambda \leq 2 \int_E (f \circ s_\alpha) d\lambda \quad \text{for all } f \in C_c^+(\mathbb{R}^p). \quad (98)$$

We let, as in Lemma 7.2,

$$Z_1 = \bigcap_{B_\varepsilon} \sigma_t Z, \quad Z_2 = \bigcup_{B_\varepsilon} \sigma_t Z.$$

One has $\sigma_t^{-1}(Z_1) \subset Z$ for $t \in B_\varepsilon$ so that, by the first inequality of (98) for $E = Z_1$,

$$\frac{1}{2} \int_{Z_1} (f \circ s_\alpha) d\lambda \leq \int_{\sigma_t^{-1}Z_1} (f \circ s_\alpha \circ \sigma_t) d\lambda \leq \int_Z (f \circ s_\alpha \circ \sigma_t) d\lambda$$

for all $t \in B_\varepsilon$, $f \in C_c^+(\mathbb{R}^p)$ so that, multiplying by $h(t)dt^p$ and integrating over $t \in B_\varepsilon$, we get $C < \infty$ with

$$\int_{Z_1} (f \circ s_\alpha) d\lambda \leq C \int_Z \int_{B_\varepsilon} f(s_\alpha(\sigma_t(\kappa))) h(t) dt^p d\lambda(\kappa) = C \int_{\mathbb{R}^p} f(x) d\nu(x),$$

where we used Fubini's theorem and the equality $s_\alpha(\sigma_t(\kappa)) = F(\kappa, t)$ for $\kappa \in Z$ and $t \in B_\varepsilon$. Thus, using (95) and (96),

$$\int_{Z_1} (f \circ s_\alpha) d\lambda \leq C \int_{\mathbb{R}^p} f(u) \rho(u) du^p \leq C' \int_{\mathbb{R}^p} f(u) du^p \quad \text{for all } f \in C_c^+(\mathbb{R}^p),$$

hence the image $s_\alpha(\lambda|_{Z_1})$ is absolutely continuous with respect to the Lebesgue measure and is majored by a constant multiple of Lebesgue measure. Thus, every

point of \bar{V} has a neighborhood Z_1 such that $s_\alpha(\lambda|_{Z_1}) \leq c_1 dx^p$. Covering the compact set \bar{V} by finitely many such Z_1 gives (91).

Let us now assume that $\chi \in V$. We can then assume by Lemma 7.2 that $Z_2 = \bigcup_{B_\varepsilon} \sigma_t Z$ is contained in V . One has $Z \subset \sigma_t^{-1}(Z_2)$ for $t \in B_\varepsilon$ so that, by the second inequality of (98) for $E = Z_2$,

$$\int_Z (f \circ s_\alpha \circ \sigma_t) d\lambda \leq \int_{\sigma_t^{-1}(Z_2)} (f \circ s_\alpha \circ \sigma_t) d\lambda \leq 2 \int_{Z_2} (f \circ s_\alpha) d\lambda \quad \text{for all } t \in B_\varepsilon,$$

thus, after integration over $t \in B_\varepsilon$,

$$C' \int_{\mathbb{R}^p} f(x) dv(x) = C' \int_Z \int_{B_\varepsilon} f(s_\alpha(\sigma_t(y))) h(t) dt^p d\lambda(y) \leq \int_{Z_2} (f \circ s_\alpha) d\lambda.$$

This shows, using (95), that

$$\int_{Z_2} (f \circ s_\alpha) d\lambda \geq C' \int_{\mathbb{R}^p} f(u) \rho(u) du^p \quad \text{for all } f \in C_c^+(\mathbb{R}^p).$$

By (97) one has $\rho(u) > 0$ for all $u \in s_\alpha(Z)$, thus $\rho(u) > 0$ in a neighborhood of $s_\alpha(\chi)$, in other words, $ds_\alpha(\lambda|_{Z_2})/dx^p = \rho_2(x) > 0$ in a neighborhood of $s_\alpha(\chi)$ as required. This shows that $s_\alpha(\lambda|_V)$ is equivalent to the Lebesgue measure on $s_\alpha(V)$. \square

8. Spectral multiplicity

We want to get an upper bound for the number of elements in the fiber of the map $s_\alpha: U_\alpha \rightarrow \mathbb{R}^p$. We shall first relate the multiplicity of the map s_α with the spectral multiplicity of the operators a_α^j in the Hilbert space \mathcal{H} . This is not automatic, indeed the first difficulty is that for an injective representation π of a C^* -algebra B with a subalgebra $A \subset B$, one can have the same double-commutants $\pi(A)'' = \pi(B)''$ even though $A \neq B$. Thus for instance one can take the subalgebra $C[0, 1] \subset C(K)$ where $K = \{0, 1, \dots, 9\}^{\mathbb{N}}$ is the Cantor set of the decimal digits and the inclusion is given by the decimal expansion. Both act in $L^2[0, 1]$ (by multiplication) and the spectral multiplicity of the function $x \in C[0, 1]$ is equal to one, but the number of elements in the fiber is equal to 2 for numbers of the form $k10^{-n}$. The point in this example is that the projection map $s: K \rightarrow [0, 1]$ is not an open mapping. Thus in particular the subset of K where the multiplicity of s is two is not an open subset of K (it is countable).

Lemma 8.1. *Let $s: X \rightarrow Y$ be a continuous open map of compact spaces. Then the function $n(y) = \#s^{-1}(y)$ is lower semi-continuous on Y .*

Proof. Assume that $n(y) \geq m$ and let us show that this inequality still holds in a neighborhood of y . Let $x_j \in X$ be m distinct points in $s^{-1}(y)$. One can then find disjoint open sets $V_j \ni x_j$ and let $W = \bigcap_j s(V_j)$ which is an open neighborhood of y . For any $z \in W$ the preimage $s^{-1}(z)$ contains at least m points. \square

Now let $s: X \rightarrow Y$ be a continuous open map of compact spaces. Let μ be a positive measure on X with support X and π the corresponding representation of $C(X)$ in $L^2(X, \mu)$. We want to compare the spectral multiplicity function $\Sigma(y)$ of the restriction of π to $C(Y)$ with $n(y) = \#s^{-1}(y)$. Let $\nu = s(\mu)$ be the image of the measure μ . One can disintegrate μ in the form

$$\mu = \int_Y \rho_y \, d\nu(y),$$

where the conditional measure ρ_y is supported by the closed subset $s^{-1}(y)$. The issue is what is the dimension of the Hilbert space $L^2(X, \rho_y)$. It might seem at first that if the support of the measure μ is X one should be able to conclude that the support of ρ_y is $s^{-1}(y)$ and obtain that the spectral multiplicity function $\Sigma(y)$ is larger than $n(y) = \#s^{-1}(y)$. However this fails as shown by the following example:

$$X = Y \times \{1, \dots, m\}, \quad s(y, k) = y,$$

and let μ_k be the measure on Y corresponding to the restriction of μ to $Y \times \{k\}$.

Lemma 8.2. *If the measures μ_k are mutually singular, then the spectral multiplicity function $\Sigma(y)$ is equal to 1 a.e.*

Proof. The representations of $C(Y)$ in $L^2(Y, \mu_k)$ are pairwise disjoint, and each is of multiplicity one. Thus the commutant of $C(Y)$ in the direct sum of these representations only contains block diagonal operators and is hence commutative so that the multiplicity is equal to one. \square

The above example gives the needed condition for the relation between $\Sigma(y)$ and $n(y)$, and one has:

Lemma 8.3. *Let X be a compact space and λ a finite positive measure on X , π the representation⁹ of $C(X)$ in $L^2(X, d\lambda)$. Let $a_j = a_j^* \in C(X)$ and let s be the map from X to \mathbb{R}^p with coordinates a_j . We let $U \subset X$ be an open set and ν a measure on \mathbb{R}^p , and assume that*

- *the restriction of s to U is an open mapping,*
- *for every open subset $V \subset U$ the image $s(\lambda|_V)$ is equivalent to the restriction of ν to $s(V)$.*

⁹By multiplication.

Let then $V \subset U$ be an open set and consider the operators $T_j = \pi(a_j)|_V$ obtained by restriction of the $\pi(a_j)$ to the subspace $L^2(V, d\lambda) \subset L^2(X, d\lambda)$. Then the joint spectral measure of the T_j is $\nu|_{s(V)}$ and the spectral multiplicity $\Sigma(y)$ fulfills

$$\Sigma(y) \geq n(y) = \#\{s^{-1}(y) \cap V\} \quad \text{for all } y \in s(V) \quad (99)$$

almost everywhere modulo ν .

Proof. Let $W = s(V)$, which is a bounded open set in \mathbb{R}^p . One can disintegrate $\lambda|_V$ in the form

$$\lambda|_V = \int_W \rho_y \, d\nu(y), \quad (100)$$

where the ρ_y are positive measures carried by $F_y = s^{-1}(y) \cap V$. Moreover the total mass of ρ_y is > 0 almost everywhere modulo ν for $y \in s(V)$. This follows from the assumed equivalence $s(\lambda|_V) \sim \nu|_{s(V)}$. One then has

$$L^2(V, d\lambda) = \int_W^\oplus L^2(F_y, \rho_y) \, d\nu(y).$$

For any $\xi \in L^2(V, d\lambda)$ and any $f \in C_c(\mathbb{R}^p)$ one has

$$\langle \xi, f((a_j))\xi \rangle = \int_W \int_{F_y} |\xi(x)|^2 d\rho_y f(y) \, d\nu(y),$$

which shows that the joint spectral measure of the a_j is absolutely continuous with respect to $\nu|_W$. Its equivalence with $\nu|_W$ follows from (100) taking $\xi(x) = 1$ and using the assumed equivalence of $s(\lambda|_V)$ with the restriction of ν to $s(V)$.

Let us prove (99). Let $y \in W$ with $n(y) = \#\{s^{-1}(y) \cap V\} \geq m > 0$. Let $x_j \in V$ be m distinct points in $s^{-1}(y) \cap V$. One can then find disjoint open sets $B_j \ni x_j$ and let $Z = \bigcap_j s(B_j)$, which is an open neighborhood of y . For any $z \in Z$ the preimage $s^{-1}(z) \cap V$ contains at least m points since it contains at least one in each B_j . Moreover one has $s(s^{-1}(Z) \cap B_j) = Z$ for all j . Let λ_j be the restriction of λ to $V_j = s^{-1}(Z) \cap B_j$. From the first part of the lemma, for each j the spectral measure of the action of the a_j in $L^2(V_j, d\lambda_j)$ is the restriction $\nu|_{s(V_j)} = \nu_Z$. The action of the a_j in $L^2(V, d\lambda_j)$ contains the direct sum of the actions in the $L^2(V_j, d\lambda_j)$ and hence a copy of the action of the coordinates y_j in

$$\bigoplus_1^m L^2(Z, \nu_Z),$$

which shows that the spectral multiplicity fulfills $\Sigma(z) \geq m$ a.e. on the neighborhood Z of y . This shows that any element y in the open set $U_m = \{y \in U \mid n(y) \geq m\}$ admits an open neighborhood Z_y where $\Sigma(z) \geq m$ holds a.e. Since U_m is σ -compact, it follows that $\Sigma(y) \geq m$ almost everywhere modulo ν on U_m so that (99) holds. \square

Remark 8.4. With the hypothesis of Lemma 8.3, let E be a complex hermitian vector bundle over X with non-zero fiber dimension everywhere. Then the inequality $\Sigma_E(y) \geq n(y)$ holds, where Σ_E is the spectral multiplicity of the T_j acting on $L^2(X, d\lambda, E)$. This follows since, at the measurable level, one can find a nowhere vanishing section of E , which shows that the representation π_E of $C(X)$ in $L^2(X, d\lambda, E)$ contains the representation π of $C(X)$ in $L^2(X, d\lambda)$. Since π_E is contained in the sum of N copies of π , one obtains the conclusion.

Theorem 8.5. *Let $V \subset U_\alpha$ be an open set and let $a_\alpha^j|_V$ be the restriction of $a_\alpha^j \in \mathcal{A}$ to the range $1_V \mathcal{H} \subset \mathcal{H}$. Then*

- *the joint spectral measure of the $a_\alpha^j|_V$ is the Lebesgue measure on $s_\alpha(V)$,*
- *the spectral multiplicity $m_{ac}(y)$ fulfills*

$$m_{ac}(y) \geq n(y) = \#\{s_\alpha^{-1}(y) \cap V\} \quad \text{for all } y \in s_\alpha(V)$$

almost everywhere modulo the Lebesgue measure.

Proof. By Lemma 7.3, the hypothesis of Lemma 8.3 are fulfilled by the compact space X with measure λ , the open set U_α , the measure $dv = dx^p$ and the elements a_α^j . Thus the result follows from Lemma 8.3 and Remark 8.4. \square

9. Local form of the $\mathcal{L}^{(p,1)}$ estimate

We fix $p \in [1, \infty[$. Our goal is to control the size of the Lebesgue multiplicity $m_{ac}(y)$ which appears in Theorem 8.5. The idea here is to use a local form of the $\mathcal{L}^{(p,1)}$ estimate of [10], Proposition IV.3.14, with the right-hand side of the inequality now involving a closed subset $K \subset X$, by

$$\lambda(K) = \inf_{b \in \mathcal{A}^+, b|_K = 1_K} \int b|D|^{-p}.$$

It relies on the estimate given in [9] and on the crucial results of Voiculescu ([26], [27], [28]). The norm $\|T\|_{(p,1)}$ is defined¹⁰ for a compact operator T with characteristic values $\mu_n(T)$ in decreasing order by (cf. [26], p. 5),

$$\|T\|_{(p,1)} = \sum_1^\infty n^{-1+1/p} \mu_n. \quad (101)$$

In order to get an upper bound on $\|T\|_{(p,1)}$ for T an operator of finite rank, we can use an inequality of the form

$$\|T\|_{(p,1)} \leq C_p (\text{Rank } T)^{1/p} \|T\|_\infty, \quad (102)$$

¹⁰For $p = 1$ it agrees with the \mathcal{L}^1 -norm.

which follows using the characteristic values $\mu_n(T)$ from

$$\|T\|_{(p,1)} = \sum_{n=1}^N n^{-1+1/p} \mu_n \leq \|T\|_\infty \sum_{n=1}^N n^{-1+1/p} \leq C_p N^{1/p} \|T\|_\infty$$

where $N = \text{Rank } T$. Note also that the $\mathcal{L}^{(p,1)}$ norm fulfills

$$\|ATB\|_{(p,1)} \leq \|A\|_\infty \|T\|_{(p,1)} \|B\|_\infty. \quad (103)$$

Let D be a self-adjoint unbounded operator such that its resolvent is an infinitesimal of order $1/p$, i.e., is such that the characteristic values fulfill $\mu_n(|D|^{-1}) = O(n^{-1/p})$. We let for any $\lambda > 0$,

$$P(\lambda) = \mathbf{1}_{[0,\lambda]}(|D|), \quad \alpha(\lambda) = \text{Tr } P(\lambda). \quad (104)$$

By construction $\alpha(\lambda)$ is a non decreasing integer valued function. The hypothesis $\mu_n(|D|^{-1}) = O(n^{-1/p})$ implies that $\mu_n(|D|^{-1}) < C n^{-1/p}$ for some $C < \infty$, and it follows that $\alpha(C^{-1}n^{1/p}) < n$ since the n -th eigenvalue of $|D|$ in increasing order is $> C^{-1}n^{1/p}$. Thus, using for n the smallest integer above $C^p \lambda^p$, we get

$$\alpha(\lambda) \leq C^p \lambda^p \quad \text{for all } \lambda > 0. \quad (105)$$

Let us show

Lemma 9.1. *Let $f \in C_c^\infty(\mathbb{R})$. Then there is a finite constant C_f such that*

$$\|[f(\varepsilon D), a]\|_\infty \leq C_f \varepsilon \|[D, a]\| \quad \text{for all } a \in \mathcal{A}. \quad (106)$$

Under the hypothesis of Theorem 8.5, one has

$$\liminf \lambda^{-p} \alpha(\lambda) > 0. \quad (107)$$

Proof. One has

$$[e^{is\varepsilon D}, a] = is\varepsilon \int_0^1 e^{ius\varepsilon D} [D, a] e^{i(1-u)s\varepsilon D} du, \quad (108)$$

which gives (106) using the finiteness of $\int |s \hat{f}(s)| ds$ and

$$[f(\varepsilon D), a] = (2\pi)^{-1} \int \hat{f}(s) [e^{is\varepsilon D}, a] ds. \quad (109)$$

Assume that (107) does not hold. Then let $\lambda_n \rightarrow \infty$ be such that $\lim \lambda_n^{-p} \alpha(\lambda_n) = 0$. Let $f \in C_c^\infty(\mathbb{R})$ be an (even) cutoff function vanishing outside $[-1, 1]$. For $\varepsilon_n = \lambda_n^{-1}$ one has

$$\text{Rank } f(\varepsilon_n D) \leq \alpha(\lambda_n), \quad \text{Rank } [f(\varepsilon_n D), a] \leq 2\alpha(\lambda_n)$$

so that by (102) one gets, using (106),

$$\|[f(\varepsilon_n D), a]\|_{(p,1)} \leq C_p (2\alpha(\lambda_n))^{1/p} C \varepsilon_n \|[D, a]\|$$

and since $\lim_{n \rightarrow \infty} \lambda_n^{-p} \alpha(\lambda_n) = 0$,

$$\lim_{n \rightarrow \infty} \|[f(\varepsilon_n D), a]\|_{(p,1)} = 0.$$

The Voiculescu obstruction relative to an ideal J of compact operators is given by

$$k_J(\{a_j\}) = \liminf_{A \in \mathcal{R}_1^+, A \uparrow 1} \max \| [A, a_j] \|_J,$$

where \mathcal{R}_1^+ is the partially ordered set of positive, finite rank operators of norm less than one, in \mathcal{H} . We take $A_n = f(\varepsilon_n D)$. It is by construction an element of \mathcal{R}_1^+ . Moreover since $f(\varepsilon_n D) \rightarrow 1$ strongly in \mathcal{H} , this shows that for the ideal $J = \mathcal{L}^{(p,1)}$ one gets $k_J(\{a_j\}) = 0$. This contradicts the existence, shown in Theorem 8.5, of p self-adjoint elements a_j of \mathcal{A} whose joint spectral measure is the Lebesgue measure, using Theorem 4.5 of [26], which gives the equality, valid for p self-adjoint operators,

$$k_J(\{a_j\})^p = \gamma_p \int_{\mathbb{R}^p} m(y) d^p y, \quad (110)$$

where the function $m(y)$ is the multiplicity of the Lebesgue spectrum. \square

The rank of the operator $T = [f(\varepsilon D), a]$ is controlled by twice the rank of $f(\varepsilon D)$. We take f compactly supported and thus $f \leq g$, where g is equal to one on the support of f yields an inequality of the form

$$\text{Rank } f(\varepsilon D) \leq \text{Tr}(g(\varepsilon D)).$$

One has by Corollary 14.10 (Appendix 2) an estimate of the form

$$\liminf \varepsilon^p \text{Tr}(g(\varepsilon D)) \leq c_g \int |D|^{-p}.$$

This gives

$$\liminf \varepsilon^p \text{Rank } f(\varepsilon D) \leq c_g \int |D|^{-p} \quad (111)$$

and:

Lemma 9.2. *Let $f \in C_c^\infty(\mathbb{R})$, then there is a finite constant c_f such that*

$$\liminf \|[f(\varepsilon D), a]\|_{(p,1)} \leq c_f \left(\int |D|^{-p} \right)^{1/p} \|[D, a]\| \quad \text{for all } a \in \mathcal{A}. \quad (112)$$

Proof. Using (111) and $\text{Rank}[f(\varepsilon D), a] \leq 2 \text{Rank } f(\varepsilon D)$, one obtains a sequence $\varepsilon_q \rightarrow 0$ with

$$\text{Rank } T_q \leq 3\varepsilon_q^{-p} c_g \int |D|^{-p}, \quad T_q = [f(\varepsilon_q D), a].$$

Using (102) and (106) then gives

$$\|T_q\|_{(p,1)} \leq C_p (\text{Rank } T_q)^{1/p} \|T_q\|_\infty \leq C_p \left(3\varepsilon_q^{-p} c_g \int |D|^{-p} \right)^{1/p} C_f \varepsilon_q \| [D, a] \|,$$

which is the required estimate since $(\varepsilon_q^{-p})^{1/p} \varepsilon_q = 1$. \square

We now let $K \subset X$ be a compact subset and we want to localize the estimate (112) to K , i.e., to the range of K in \mathcal{H} .

Lemma 9.3.¹¹ *Let $h \in C_c^\infty(\mathbb{R})$ be an (even) cutoff function and $f = h^2$. Then*

$$\| [f(\varepsilon|D|), a] - \frac{1}{2}\varepsilon(f'(\varepsilon|D|)\delta(a) + \delta(a)f'(\varepsilon|D|)) \|_{(p,1)} = O(\varepsilon), \quad (113)$$

where $\delta(a) = [|D|, a]$ and one assumes that $a \in \bigcap_{j=1}^2 \text{Dom } \delta^j$.

Proof. First one has (cf. Corollary 10.16 of [16])

$$\| [h(\varepsilon|D|), a] - \varepsilon h'(\varepsilon|D|)\delta(a) \| \leq C_2 \varepsilon^2 \|\delta^2(a)\|, \quad (114)$$

with a similar estimate using $\varepsilon\delta(a)h'(\varepsilon|D|)$. Indeed, using (109) with $|D|$ instead of D , one gets

$$[h(\varepsilon|D|), a] = (2\pi)^{-1} \int \hat{h}(s) [e^{is\varepsilon|D|}, a] ds$$

so that by (108)

$$[h(\varepsilon|D|), a] = (2\pi)^{-1} \int \hat{h}(s) i s \varepsilon \int_0^1 e^{ius\varepsilon|D|} [|D|, a] e^{i(1-u)s\varepsilon|D|} du ds,$$

and since by (108) one has

$$\| [[|D|, a], e^{i(1-u)s\varepsilon|D|}] \| \leq |s\varepsilon| \|\delta^2(a)\|,$$

one gets

$$\| [h(\varepsilon|D|), a] - \varepsilon h'(\varepsilon|D|)\delta(a) \| \leq C_2 \varepsilon^2 \|\delta^2(a)\|, \quad C_2 = (2\pi)^{-1} \int s^2 |\hat{h}(s)| ds.$$

We follow the proof of Lemma 10.29 in [16]. One has

$$[f(\varepsilon|D|), a] - \frac{1}{2}\varepsilon(f'(\varepsilon|D|)\delta(a) + \delta(a)f'(\varepsilon|D|)) = A_\varepsilon B_\varepsilon + C_\varepsilon A_\varepsilon,$$

¹¹This is Lemma 10.29 in [16], but the proof given there is not correct, so we give the full details here.

where $A_\varepsilon = h(\varepsilon|D|)$, $B_\varepsilon = [h(\varepsilon|D|), a] - \varepsilon h'(\varepsilon|D|)\delta(a)$ and $C_\varepsilon = [h(\varepsilon|D|), a] - \varepsilon \delta(a)h'(\varepsilon|D|)$. By (114) one has $\|B_\varepsilon\| = O(\varepsilon^2)$, $\|C_\varepsilon\| = O(\varepsilon^2)$, while A_ε is uniformly bounded with $\text{Rank } A_\varepsilon = O(\varepsilon^{-p})$. Thus by (102) one has $\|A_\varepsilon\|_{(p,1)} = O(\varepsilon^{-1})$. Thus we get the required estimate using (103). \square

We then let $K \subset X$ be a compact subset as above, and consider the operators

$$R_\varepsilon = 1_K f(\varepsilon|D|) 1_K. \quad (115)$$

We let $b \in \mathcal{A}$ be equal to 1 on K , i.e., such that $b 1_K = 1_K$. One then has:

Lemma 9.4.

$$\|[R_\varepsilon, a] - \frac{1}{2}\varepsilon(1_K f'(\varepsilon|D|)b\delta(a)1_K + 1_K \delta(a)bf'(\varepsilon|D|)1_K)\|_{(p,1)} = O(\varepsilon).$$

Proof. One has

$$[R_\varepsilon, a] = 1_K [f(\varepsilon|D|), a] 1_K$$

since a commutes with 1_K . Thus multiplying on both sides by 1_K in (113), one gets (using (103))

$$\|[R_\varepsilon, a] - \frac{1}{2}\varepsilon(1_K f'(\varepsilon|D|)\delta(a)1_K + 1_K \delta(a)f'(\varepsilon|D|)1_K)\|_{(p,1)} = O(\varepsilon). \quad (116)$$

Lemma 9.1 and (105) show, using (102), that one has a uniform upper bound

$$\|[f'(\varepsilon|D|), b]\|_{(p,1)} \leq C \|[D|, b]\|$$

since f' has compact support. Thus in (116) one can replace $1_K f'(\varepsilon|D|) = 1_K bf'(\varepsilon|D|)$ by $1_K f'(\varepsilon|D|)b$, without affecting the behavior in $O(\varepsilon)$. The same applies to the other term. \square

We recall the interpolation inequality used in [10], §IV.2.8, but stated without proof there.

Lemma 9.5. *There exists for $1 \leq p < \infty$, a constant c_p such that, for $S \in \mathcal{L}^1$,*

$$\|S\|_{(p,1)} \leq c_p \|S\|_1^{1/p} \|S\|_\infty^{1-1/p}. \quad (117)$$

Proof. The inequality holds as an equality for $p = 1$ with $c_1 = 1$, thus we can assume that $p > 1$. We use the fact that $\mathcal{L}^{(p,1)}$ is obtained by real interpolation of index $(\theta, 1)$ for $\theta = \frac{1}{p}$ from the Banach spaces $Y_0 = \mathcal{K}$ and $Y_1 = \mathcal{L}^1$. The functoriality of the interpolation gives an inequality of the form

$$\|T(x)\|_{(Y_0, Y_1)_{(\theta, q)}} \leq M_0^{1-\theta} M_1^\theta \|x\|_{(X_0, X_1)_{(\theta, q)}}$$

for any linear operator from $X_0 + X_1$ to $Y_0 + Y_1$ such that

$$\|Tx\|_{Y_i} \leq M_i \|x\|_{X_i} \quad \text{for all } x \in X_i, \quad i = 0, 1.$$

We can take $X_0 = X_1 = \mathbb{C}$ and let T be such that $T(1) = S$. Then $M_0 = \|S\|_\infty$, $M_1 = \|S\|_1$ and the norm $\|x\|_{(X_0, X_1)_{(\theta, 1)}}$ is finite and non-zero. \square

Remark 9.6. In order not to depend on interpolation theory we give a direct proof of (117). We assume that $p > 1$. First, for $p > 1$ an equivalent norm on $\mathcal{L}^{(p,1)}$ is

$$\|T\|_{(p,1)'} = (1 - \theta) \sum N^{\theta-2} \sigma_N(T), \quad \theta = \frac{1}{p}, \quad (118)$$

where $\sigma_N(T)$ is the sum of the first N characteristic values. The equivalence of the norms (118) and (101) follows from $\mu_N \leq \sigma_N/N$ one way. For the other way, one applies Fubini to the double sum

$$\sum_n \sum_{m \geq n} \mu_n m^{\theta-2} = \sum_m \sum_{n \leq m} \mu_n m^{\theta-2}.$$

Now to estimate (118) assuming $\|T\|_\infty \leq 1$ and $\|T\|_1 = \rho \geq 1$, one splits the sum as follows:

$$\sum_1^\infty N^{\theta-2} \sigma_N(T) = \sum_{N < \rho} N^{\theta-2} \sigma_N(T) + \sum_{N \geq \rho} N^{\theta-2} \sigma_N(T).$$

Using $\|T\|_\infty \leq 1$ gives $\sigma_N(T) \leq N$ and one bounds the first sum as

$$\sum_{N < \rho} N^{\theta-2} N \sim C_\theta \rho^\theta.$$

Using $\|T\|_1 = \rho \geq 1$ gives $\sigma_N(T) \leq \rho$, and one bounds the second sum by

$$\sum_{N \geq \rho} N^{\theta-2} \rho \sim C'_\theta \rho^\theta,$$

which gives the required inequality (117).

Lemma 9.7. *There exists a constant $C_f \leq \infty$ such that for $b = b^* \in \mathcal{A}$, $b \geq 0$,*

$$\liminf \varepsilon^p \|bf'(\varepsilon|D|)b\|_1 \leq C_f \int b^2 |D|^{-p}.$$

Proof. Note that by construction of f as a cutoff function, its derivative $f' \leq 0$ on $[0, \infty[$. Let $h = -f' \in C_c^\infty(\mathbb{R})$ so that $h \geq 0$. One then has $bh(\varepsilon|D|)b \geq 0$ and

$$\|bh(\varepsilon|D|)b\|_1 = \text{Tr}(bh(\varepsilon|D|)b) = \text{Tr}(b^2 h(\varepsilon|D|)),$$

and the result follows from Corollary 14.10 (Appendix 2), which gives

$$\liminf \varepsilon^p \text{Tr}(b^2 h(\varepsilon|D|)) \leq \mu \int b^2 |D|^{-p},$$

where $\mu = p \int_0^\infty u^{p-1} h(u) du$. □

Lemma 9.8. *There exists a constant $C'_f \leq \infty$ such that, for $b = b^* \in \mathcal{A}$, $0 \leq b \leq 1$,*

$$\liminf \|\varepsilon b f'(\varepsilon |D|) b\|_{(p,1)} \leq C'_f \left(\int b^2 |D|^{-p} \right)^{1/p}. \quad (119)$$

Proof. By Lemma 9.7 one has, once b is fixed, a sequence $\varepsilon_q \rightarrow 0$ such that

$$\|b f'(\varepsilon_q |D|) b\|_1 \leq 2C_f \varepsilon_q^{-p} \int b^2 |D|^{-p}.$$

Also since f' is bounded one has

$$\|b f'(\varepsilon_q |D|) b\|_\infty \leq B = \|f'\|_\infty < \infty.$$

Thus it follows from (117) that

$$\|b f'(\varepsilon_q |D|) b\|_{(p,1)} \leq c_p \left(2C_f \varepsilon_q^{-p} \int b^2 |D|^{-p} \right)^{1/p} B^{1-1/p}.$$

After multiplication by ε_q one gets the required estimate. \square

Theorem 9.9. *There exists a finite constant κ_p such that for any operators $a_j \in \mathcal{A}$ and compact subset $K \subset X$ one has, with $J = \mathcal{L}^{(p,1)}$, the inequality*

$$k_J(\{a_j 1_K\}) \leq \kappa_p \max \|\delta(a_j)\|_\infty (\lambda(K))^{1/p},$$

where one lets¹²

$$\lambda(K) = \inf_{b \in \mathcal{A}^+, b|_K = 1_K} \int b |D|^{-p}.$$

Proof. By definition one has

$$k_J(\{a_j 1_K\}) = \liminf_{A \in \mathcal{R}_1^+, A \uparrow 1} \max \| [A, a_j 1_K] \|_J,$$

where \mathcal{R}_1^+ is the partially ordered set of positive, finite rank operators of norm less than one in $1_K \mathcal{H}$. We take $R_\varepsilon = 1_K f(\varepsilon |D|) 1_K$ as in (115). It is by construction an element of \mathcal{R}_1^+ . Moreover since $f(\varepsilon |D|) \rightarrow 1$ strongly in \mathcal{H} , one gets that $R_\varepsilon \rightarrow 1$ strongly in $1_K \mathcal{H}$. By Lemma 9.4 one has

$$\|[R_\varepsilon, a] - \frac{1}{2} \varepsilon (1_K f'(\varepsilon |D|) b \delta(a) 1_K + 1_K \delta(a) b f'(\varepsilon |D|) 1_K)\|_{(p,1)} = O(\varepsilon).$$

Using

$$1_K f'(\varepsilon |D|) b \delta(a) 1_K = 1_K b f'(\varepsilon |D|) b \delta(a) 1_K$$

¹²This is the natural extension of λ given by the Riesz Representation Theorem [24].

and (103) for $A = 1_K$, $T = bf'(\varepsilon|D|)b$, $B = \delta(a)1_K$ and similarly for the other term, one gets an estimate of the form

$$\|[R_\varepsilon, a]\|_{(p,1)} \leq O(\varepsilon) + \|\varepsilon bf'(\varepsilon|D|)b\|_{(p,1)} \|\delta(a)\|_\infty.$$

Thus, by Lemma 9.8 one gets that, for any $b \in \mathcal{A}^+$ equal to 1 on K , there exists a sequence $\varepsilon_q \rightarrow 0$ such that

$$\|\varepsilon_q bf'(\varepsilon_q|D|)b\|_{(p,1)} \leq 2C'_f \left(\int b^2 |D|^{-p} \right)^{1/p},$$

which gives, for q large enough,

$$\|[R_{\varepsilon_q}, a]\|_{(p,1)} \leq 2C'_f \left(\int b^2 |D|^{-p} \right)^{1/p} \|\delta(a)\|_\infty$$

for any $a \in \mathcal{A}$ and hence

$$\liminf \max \|[R_\varepsilon, a_j 1_K]\|_J \leq 2C'_f \max \|\delta(a_j)\|_\infty \left(\int b^2 |D|^{-p} \right)^{1/p}.$$

After varying b one obtains the required estimate. \square

Remark 9.10. a) One may worry that Voiculescu's definition of k_J involves the ordered set \mathcal{R}_1^+ while all we got was $R_\varepsilon \rightarrow 1$ strongly in $1_K \mathcal{H}$. Thus let us briefly mention how to get the $A \uparrow 1$ from R_ε by a small modification. Given a finite dimensional subspace of $1_K \mathcal{H}$, one lets P be the corresponding finite rank projection, with fixed rank N . One needs to construct $A \in \mathcal{R}_1^+$, $A \geq P$, with a control on $\max \|[A, a_j 1_K]\|_J$. One takes

$$A_\varepsilon = P + (1 - P)R_\varepsilon(1 - P),$$

so that $A \geq P$ by construction. Moreover one has

$$R_\varepsilon - A_\varepsilon = P(R_\varepsilon - 1)P + PR_\varepsilon(1 - P) + (1 - P)R_\varepsilon P. \quad (120)$$

Moreover by the strong convergence $R_\varepsilon \rightarrow 1$, one has

$$\|P(R_\varepsilon - 1)\|_\infty = \|(R_\varepsilon - 1)P\|_\infty \rightarrow 0$$

so that all three terms in the rhs of (120) converge to 0 in norm and hence in the J norm since their rank is less than N so that one can use (102). Thus one has

$$\|R_\varepsilon - A_\varepsilon\|_J \rightarrow 0$$

and one controls $\max \|[A, a_j 1_K]\|_J$ from $\max \|[R_\varepsilon, a_j 1_K]\|_J$.

b) It might seem possible at first sight to tensor the spectral triple $(\mathcal{A}, \mathcal{H}, D)$ by $(\mathbb{C}, \mathcal{H}', D')$, with the spectrum of D' growing fast enough so that the product triple¹³

$$(\mathcal{A} \otimes \mathbb{C}, \mathcal{H} \otimes \mathcal{H}', D'' = D \otimes 1 + \gamma \otimes D')$$

would still be of dimension p , i.e., such that the characteristic values of the inverse of D'' are $O(n^{-1/p})$. Let us show that this is only possible if the dimension of \mathcal{H}' is finite. Indeed the eigenvalues of $(D \otimes 1 + \gamma \otimes D')^2 = D^2 \otimes 1 + 1 \otimes D'^2$ are the independent sums of the eigenvalues of D^2 and of D'^2 . Thus having infinitely many eigenvalues of D'^2 contradicts the two inequalities

$$\alpha(\lambda) \geq c\lambda^p, \quad \alpha''(\lambda) \leq C''\lambda^p$$

for the counting functions $\alpha(\lambda) = \text{Tr}(1_{[0,\lambda]}(|D|))$, $\alpha''(\lambda) = \text{Tr}(1_{[0,\lambda]}(|D''|))$ since they yield

$$\dim(\mathcal{H}') \leq C''/c.$$

c) The constant C'_f in (119) is given, up to a function of p alone, by

$$C'_f = \left(\int_0^\infty u^{p-1} h(u) du \right)^{1/p} \|h\|_\infty^{1-1/p}, \quad h = -f' \geq 0,$$

and one needs to check that there is a lower bound to C'_f independent of the choice of the cutoff function f . Since $f(0) = 1$, the only information is about $\int_0^\infty h(u) du = 1$ and thus one needs to show a general inequality of the form

$$\int_0^\infty h(u) du \leq c(p) \left(p \int_0^\infty u^{p-1} h(u) du \right)^{1/p} \|h\|_\infty^{1-1/p}. \quad (121)$$

To prove this one lets $g(u) = h(u^{1/p})$ so that

$$p \int_0^\infty u^{p-1} h(u) du = \int_0^\infty g(v) dv, \quad \int_0^\infty h(u) du = \frac{1}{p} \int_0^\infty v^{1/p-1} g(v) dv,$$

and one uses the same argument as in Remark 9.6. First, with $k(u) = \int_0^u g(v) dv$,

$$\int_0^\infty v^{1/p-1} g(v) dv = \left(1 - \frac{1}{p}\right) \int_0^\infty v^{1/p-2} k(v) dv.$$

Next, assuming $\|h\|_\infty = 1$, one has $g(v) \leq 1$ for all $v > 0$ and thus, with $\rho = \int_0^\infty g(v) dv$,

$$\int_0^\infty v^{1/p-2} k(v) dv \leq \int_0^\rho v^{1/p-2} k(v) dv + \int_\rho^\infty v^{1/p-2} k(v) dv$$

¹³In the even case.

so that, since $k(v) \leq v$ and $k(v) \leq \rho$, one gets

$$\int_0^\infty v^{1/p-2} k(v) dv \leq \int_0^\rho v^{1/p-1} dv + \int_\rho^\infty v^{1/p-2} \rho dv = c_p \rho^{1/p}$$

with $c_p = p + (1 - \frac{1}{p})^{-1}$ which gives (121) with $c(p) = 1$.

We can now combine this with Theorem 4.5 of [26], which gives the equality, valid for p self-adjoint operators h_j ,

$$k_J(\{h_j\})^p = \gamma_p \int_{\mathbb{R}^p} m(y) d^p y. \quad (122)$$

Corollary 9.11. *Let $a_j = a_j^* \in \mathcal{A}$ be p self-adjoint elements. Then for any compact subset $K \subset X$ one has*

$$\int_{\mathbb{R}^p} m_{\text{ac}}^K(y) d^p y \leq \kappa'_p \max \|\delta(a_j)\|_\infty^p \lambda(K),$$

where the constant κ'_p only depends on p , and the function $m_{\text{ac}}^K(y)$ is the multiplicity of the Lebesgue spectrum of the restriction of the a_j to $1_K(\mathcal{H})$.

10. Local bound on $\#(s_\alpha^{-1}(x) \cap V)$

Let $V \subset U_\alpha$ be an open set with $\bar{V} \subset U_\alpha$.

Lemma 10.1. *There exists $C < \infty$ such that the spectral multiplicity $m_{\text{ac}}^V(x)$ on the absolutely continuous joint spectrum of the restriction $a_\alpha^j|_V$ of the a_α^j , to $1_V \mathcal{H}$ fulfills*

$$m_{\text{ac}}^V(x) \leq C, \quad \text{a.e. on } W = s_\alpha(V).$$

Proof. By Theorem 8.5, the joint spectral measure of the $a_\alpha^j|_V$ is the Lebesgue measure on $s_\alpha(V)$. Let $E \subset W$ be a compact subset, and $K = s_\alpha^{-1}(E) \cap \bar{V}$. Then Corollary 9.11 gives an inequality of the form

$$\int_{\mathbb{R}^p} m_{\text{ac}}^K(y) d^p y \leq \kappa' \lambda(K). \quad (123)$$

One has

$$m_{\text{ac}}^V(y) \leq m_{\text{ac}}^K(y) \quad \text{for all } y \in E \quad (124)$$

since one has a direct sum decomposition

$$1_V \mathcal{H} = 1_{s_\alpha^{-1}(E) \cap V} \mathcal{H} \oplus 1_{s_\alpha^{-1}(E^c) \cap V} \mathcal{H}$$

where the representation in the second term in the right-hand-side does not contribute to the multiplicity in E . Indeed, with $E^c = \bigcup E_n$ and E_n compact disjoint from E , the joint spectrum of $a_\alpha^j|_{s_\alpha^{-1}(E_n) \cap V}$ is contained in E_n and disjoint from E . Moreover the representation in the first term is dominated by the representation in $1_K \mathcal{H}$ since $s_\alpha^{-1}(E) \cap V \subset K = s_\alpha^{-1}(E) \cap \bar{V}$.

By (91) one has an inequality

$$\int_{\bar{V}} f \circ s_\alpha d\lambda_{\bar{V}} \leq c \int_{s_\alpha(\bar{V})} f(x) dx^p \leq c \int_{\mathbb{R}^p} f(x) dx^p \quad \text{for all } f \in C_c^+(\mathbb{R}^p),$$

which shows, taking $1_E = \inf f_n$ as an infimum of continuous functions $f_n \in C_c^+(\mathbb{R}^p)$, that

$$\lambda(K) = \int_{\bar{V}} 1_E \circ s_\alpha d\lambda_{\bar{V}} \leq \int_{\bar{V}} f_n \circ s_\alpha d\lambda_{\bar{V}} \leq c \int_{\mathbb{R}^p} f_n(x) dx^p \rightarrow c \int_E dx^p.$$

Thus, using (124) and (123),

$$\int_E m_{\text{ac}}^V(x) d^p x \leq \int_E m_{\text{ac}}^K(x) d^p x \leq \kappa' \lambda(K) \leq c\kappa' \int_E dx^p,$$

and there exists a constant $C = c\kappa'$ such that, for any compact $E \subset W$,

$$\int_E m_{\text{ac}}^V(x) d^p x \leq C \int_E d^p x,$$

which gives the inequality, valid almost everywhere,

$$m_{\text{ac}}^V(x) \leq C. \quad \square$$

Lemma 10.2. *Let V be as above. Then there exists $m < \infty$ such that*

$$\#\{s_\alpha^{-1}(x) \cap V\} \leq m \quad \text{for all } x \in W = s_\alpha(V).$$

Proof. By Theorem 8.5, one has, almost everywhere,

$$m_{\text{ac}}^V(y) \geq n(y) = \#\{s_\alpha^{-1}(y) \cap V\} \quad \text{for all } y \in s_\alpha(V),$$

so that the result follows from Lemma 10.1 and the semi-continuity of $n(y)$, which shows that an almost everywhere inequality remains valid everywhere. \square

Lemma 10.3. *Let $V \subset U_\alpha$ be an open set with $\bar{V} \subset U_\alpha$. There exists a dense open subset $Y \subset s_\alpha(V)$ such that every point of $s_\alpha^{-1}(Y) \cap V$ has a neighborhood N in X such that the restriction of s_α to N is a homeomorphism with an open set of \mathbb{R}^p .*

Proof. Let $W = s_\alpha(V)$ and

$$m_1 = \sup_{x \in W} \#(s_\alpha^{-1}(x) \cap V),$$

which is finite (and non-zero) by Lemma 10.2. Let

$$W_1 = \{x \in W \mid \#(s_\alpha^{-1}(x) \cap V) = m_1\}.$$

This is by Lemma 8.1 an open subset of W . Moreover for $x \in W_1$ one can find m_1 disjoint open neighborhoods V_j of the preimages x_j of x such that all V_j surject on the same neighborhood U of x in W . It follows that the restriction of s_α to each of the V_j is a bijection onto U and hence an isomorphism of a neighborhood of x_j with an open set in \mathbb{R}^p given by the a_α^j .

It can be that W_1 is not dense in W , but then we just take the complement of its closure, $W^1 = W \setminus \overline{W_1}$, and let

$$m_2 = \sup_{x \in W^1} \#(s_\alpha^{-1}(x) \cap V),$$

which is $< m_1$ by construction. One then defines

$$W_2 = \{x \in W^1 \mid \#(s_\alpha^{-1}(x) \cap V) = m_2\},$$

which is by Lemma 8.1 an open subset of W^1 . The same argument as above shows that the subset $Z = W_1 \cup W_2$ fulfills the condition of the lemma. One proceeds in the same way and gets, by induction, a sequence W_k , with $Y = \bigcup W_j$ fulfilling the condition of the lemma. Since the sequence m_j is strictly decreasing, one gets that the process stops and Y is dense in W . \square

11. Reconstruction Theorem

We shall now use Lemma 10.3 together with the ability to move around in X by automorphisms of \mathcal{A} to prove the following key lemma:

Lemma 11.1. *For every point $\chi \in X$ there exists p real elements $x^\mu \in \mathcal{A}$ and a smooth family $\tau_t \in \text{Aut}(\mathcal{A})$, $t \in \mathbb{R}^p$, $\tau_0 = \text{Id}$, such that:*

- *The x^μ give a homeomorphism of a neighborhood of χ with an open set in \mathbb{R}^p .*
- *The map $t \mapsto h(t) = \chi \circ \tau_t$ is a homeomorphism of a neighborhood of 0 in \mathbb{R}^p with a neighborhood of χ .*
- *The map $x \circ h$ is a local diffeomorphism.*

Proof. Let $\chi \in X$. By Lemma 4.4, there exists α such that $\chi \in U_\alpha$. By Lemma 4.5, there exists a smooth family $\sigma_t \in \text{Aut}(\mathcal{A})$, $t \in \mathbb{R}^p$, a neighborhood Z of χ in

$X = \text{Spec}(\mathcal{A})$ and a neighborhood W of $0 \in \mathbb{R}^p$ such that, for any $\kappa \in Z$, the map $t \mapsto s_\alpha(\kappa \circ \sigma_t)$ is a diffeomorphism, depending continuously on κ , of W with a neighborhood of $s_\alpha(\kappa)$ in \mathbb{R}^p .

We now take for V a ball

$$V = B_r = \{y \in X \mid d(\chi, y) < r\} \subset U_\alpha, \quad \bar{V} \subset U_\alpha.$$

We apply Lemma 10.3 to $V = B_r$ and let Y be a dense open subset $Y \subset s_\alpha(V)$ such that every point of $s_\alpha^{-1}(Y) \cap V$ has a neighborhood N in X such that the restriction of s_α to N is a homeomorphism with an open set of \mathbb{R}^p . Since Y is dense in $s_\alpha(V)$ and by Lemma 4.5 the image of W by $t \mapsto \psi(t) = s_\alpha(\chi \circ \sigma_t)$ is an open neighborhood of $s_\alpha(\chi)$, one can choose a $u_0 \in W$ such that $\chi \circ \sigma_{u_0} \in V$ and $\psi(u_0) = s_\alpha(\chi \circ \sigma_{u_0}) \in Y$. One has $\kappa = \chi \circ \sigma_{u_0} \in s_\alpha^{-1}(Y) \cap V$. Thus by Lemma 10.3 there exists a neighborhood N of κ such that the restriction of s_α to N is an isomorphism with an open set of \mathbb{R}^p . Thus the a_α^μ are good local coordinates near κ . The automorphism $\sigma_{u_0} \in \text{Aut}(\mathcal{A})$ is such that

$$\kappa = \chi \circ \sigma_{u_0}, \quad \chi = \sigma_{u_0}(\kappa).$$

Recall that we use the covariant notation (86). We take

$$x^\mu = \sigma_{u_0}(a_\alpha^\mu)$$

as local coordinates near χ . The corresponding map x from $X = \text{Spec}(\mathcal{A})$ to \mathbb{R}^p is given by

$$\zeta \in X \mapsto \zeta(x^\mu) = \zeta(\sigma_{u_0}(a_\alpha^\mu)) = s_\alpha(\zeta \circ \sigma_{u_0}) = s_\alpha \circ \sigma_{u_0}^{-1}(\zeta).$$

Thus $x = s_\alpha \circ \sigma_{u_0}^{-1}$ and, since σ_{u_0} is a homeomorphism of X , $x = s_\alpha \circ \sigma_{u_0}^{-1}$ is a homeomorphism of the neighborhood $\sigma_{u_0}(N)$ of χ with an open set of \mathbb{R}^p . Thus the x^μ are good local coordinates at χ . Then let $\tau_t \in \text{Aut}(\mathcal{A})$ be given by

$$\tau_t = \sigma_{u_0+t} \circ \sigma_{u_0}^{-1}$$

so that $\tau_t \circ \sigma_{u_0} = \sigma_{u_0+t}$. One has

$$\chi \circ \tau_t(x^\mu) = \chi \circ \tau_t(\sigma_{u_0}(a_\alpha^\mu)) = \chi \circ \sigma_{u_0+t}(a_\alpha^\mu) = s_\alpha^\mu(\chi \circ \sigma_{u_0+t}) = \psi^\mu(u_0 + t).$$

Now the map h is given by $t \mapsto h(t) = \chi \circ \tau_t$, thus one has

$$x \circ h(t) = \psi(u_0 + t).$$

This shows that the map $x \circ h$ is a diffeomorphism from $W_1 = W - u_0$ (which is a neighborhood of $t = 0 \in \mathbb{R}^p$ since $u_0 \in W$) with an open set of \mathbb{R}^p . On W_1 , the map h is injective since $x \circ h$ is injective. Thus h is a homeomorphism with its range. One has $h(0) = \chi$, h is continuous, thus $W_2 = h^{-1}(\sigma_{u_0}(N)) \cap W_1$ is an open set containing 0 and $W_2' = x \circ h(W_2)$ is an open set in \mathbb{R}^p . The map x is a homeomorphism

of $\sigma_{u_0}(N)$ with an open set in \mathbb{R}^p and $x \circ h$ is a homeomorphism of W_1 with an open set in \mathbb{R}^p . One has $h(W_2) \subset \sigma_{u_0}(N)$. Thus $h(W_2) = x^{-1}(W'_2) \cap \sigma_{u_0}(N)$ is open in $\sigma_{u_0}(N)$ and since it contains $h(0) = \chi$, we get that h is a homeomorphism of a neighborhood of 0 in \mathbb{R}^p with a neighborhood of χ . Moreover, as we have seen above, the map $x \circ h$ is a diffeomorphism. \square

Lemma 11.2. *The algebra \mathcal{A} is locally the algebra of restrictions of smooth functions on \mathbb{R}^p to a bounded open set of \mathbb{R}^p .*

Proof. Let $\chi \in X$. By Lemma 11.1, we can assume that some $x^\mu \in \mathcal{A}$ give a homeomorphism of a neighborhood U of χ with a bounded open set $x(U) \subset \mathbb{R}^p$. By the smooth functional calculus the algebra $C_c^\infty(x(U))$ is contained in \mathcal{A} using the morphism $f \in C_c^\infty(x(U)) \mapsto f(x^\mu) \in \mathcal{A}$. Moreover for any $\kappa \in U$ one has $\kappa(f(x^\mu)) = f(\kappa(x^\mu))$ so that the function $f \circ x$ coincides on U with the element $f(x^\mu) \in \mathcal{A}$. Taking a smaller neighborhood V of χ with compact closure in U one gets that the algebra $C^\infty(\mathbb{R}^p)|_{x(V)}$ of restrictions to $x(V)$ of smooth functions on \mathbb{R}^p is contained in the algebra of restrictions to V of elements of \mathcal{A} , using x to identify V with the open set $x(V) \subset \mathbb{R}^p$. We need to show that any element of \mathcal{A} restricts to a smooth function on V , using the local coordinates x to define smoothness. For this we use (Lemma 11.1) the existence of a smooth family $\tau: \mathbb{R}^p \rightarrow \text{Aut}(\mathcal{A})$ such that $x \circ \tau$ is a local diffeomorphism around χ . Thus given $b \in \mathcal{A}$, to show that the restriction of b to V is smooth, it is enough to show that $\tau_t(b)$ evaluated at χ is a smooth function of t . This follows from the smoothness of the family τ_t . \square

Theorem 11.3. *Let $(\mathcal{A}, \mathcal{H}, D)$ be a strongly regular spectral triple fulfilling the five conditions of §2 (cf. [12]) with c antisymmetric. Then there exists an oriented smooth compact manifold X such that $\mathcal{A} = C^\infty(X)$.*

Proof. We let $X = \text{Spec}(\mathcal{A})$ be the spectrum of \mathcal{A} or equivalently of the norm closure A . By construction it is a compact space. By Lemma 11.1, for every point $\chi \in X$ there exists a neighborhood U of χ and p real elements $x^\mu \in \mathcal{A}$ which give a local homeomorphism ϕ of a neighborhood V of x with an open set in \mathbb{R}^p . Moreover by Lemma 11.2 one has

$$f \in \mathcal{A}|_V \quad \text{if and only if} \quad f \circ \phi^{-1} \in C^\infty(\mathbb{R}^p)|_{\phi(V)}.$$

This shows that on the intersection of such domains of local charts, the change of chart is of class C^∞ . We can thus, using compactness, take a finite cover and this endows X with a structure of p -dimensional smooth manifold. Lemma 11.2 shows that any $a \in \mathcal{A}$ restricts to a smooth function in each local chart and thus $\mathcal{A} \subset C^\infty(X)$. Moreover given $f \in C^\infty(X)$ there exists for each V_j in the finite open cover of X an $a_j \in \mathcal{A}$ with $f|_{V_j} = a_j|_{V_j}$. Then the existence of partitions of unity (Lemma 2.10 of [23]),

$$\psi_j \in \mathcal{A}, \quad \sum \psi_j = 1, \quad \text{Support } \psi_j \subset V_j,$$

shows that f agrees with $\sum \psi_j a_j \in \mathcal{A}$. We have shown that there exists a smooth compact manifold X such that $\mathcal{A} = C^\infty(X)$. The cycle c gives a nowhere vanishing section of the real exterior power $\wedge^p(TX)$ and hence shows that the manifold X is oriented. \square

We thus obtain the following characterization of the algebras $C^\infty(X)$:

Theorem 11.4. *An involutive algebra \mathcal{A} is the algebra of smooth functions on an oriented smooth compact manifold if and only if it admits a faithful¹⁴ representation in a pair (\mathcal{H}, D) fulfilling the five conditions of §2 (cf. [12]) with the cycle antisymmetric and the strong regularity.*

Proof. The direct implication follows from Theorem 11.3. Conversely, given an oriented smooth compact manifold X of dimension p , one can take the representation in $\mathcal{H} = L^2(X, \wedge_{\mathbb{C}}^*)$ the Hilbert space of square integrable differential forms with complex coefficients, and use the choice of a Riemannian metric to get the signature operator $D = d + d^*$ with the $\mathbb{Z}/2$ -grading γ in the even case coming from the Clifford multiplication by the volume form as in [19], Chapter 5. In the odd case one uses the Clifford multiplication γ by the volume form to reduce the Hilbert space \mathcal{H} to the subspace given by $\gamma\xi = \xi$. More specifically we consider the faithful representation of the Clifford algebra $\text{Cliff } T_x^*(X)$ in $\wedge^* T_x^*(X)$ given by the symbol of D , i.e.,

$$v \cdot \xi = v \wedge \xi - i_v \xi \quad \text{for all } v \in T_x^*(X), \xi \in \wedge^* T_x^*(X), \quad (125)$$

where i_v is the contraction by v . This gives (cf. [19] Proposition 3.9) a canonical isomorphism of vector spaces $\text{Cliff } T_x^*(X) \sim \wedge^* T_x^*(X)$. We let ω be the section of $\wedge^p T^*X$ given at each point by $\omega = e_1 \wedge \cdots \wedge e_p$, where e_1, \dots, e_p is any positively oriented orthonormal basis. In the Clifford algebra $\text{Cliff } T_x^*(X)$ one has $\omega^2 = (-1)^{\frac{p(p+1)}{2}}$ (cf. [19], (5.26)) and one defines

$$\gamma \xi = i^{-\frac{p(p+1)}{2}} \omega \xi \quad \text{for all } \xi \in \text{Cliff } T_x^*(X) \otimes \mathbb{C},$$

where the product $\omega \xi$ is the left Clifford multiplication by ω . By [19], (5.35), this left multiplication is related to the Hodge star operation by

$$\omega \xi = (-1)^{k(p-k) + \frac{k(k+1)}{2}} \star \xi \quad \text{for all } \xi \in \wedge^k.$$

With these notations one has $d^* = (-1)^{(p+1)} \gamma d \gamma$ (cf. [19], (5.10)), which shows that D commutes with γ when p is odd and anticommutes with γ when p is even.

To check the orientability condition 4), one uses (in both cases of the Dirac operator or the signature operator) local coordinates x^μ and the equalities

$$[D, f] = \sum \gamma^\mu \partial_\mu f, \quad \{\gamma^\mu, \gamma^\nu\} = -2g^{\mu\nu},$$

¹⁴I.e., with trivial kernel.

where the γ^μ correspond to the action of dx^μ through the representation of the Clifford algebra (given by (125) for the signature operator). One then has, for the multiple commutator,

$$[\gamma^1, \gamma^2, \dots, \gamma^p] = p! i^{\frac{p(p+1)}{2}} (\sqrt{g})^{-1} \gamma,$$

where \sqrt{g} is the square root of the determinant of the matrix $g_{\mu\nu}$ and $\gamma = \gamma^*$, $\gamma^2 = 1$ is the grading in the even case and is just 1 in the odd case.¹⁵ Thus in these local coordinates x^μ the cycle associated to the volume form

$$c = \frac{1}{p!} \sum_{\sigma} \varepsilon(\sigma) \sqrt{g} \otimes x^{\sigma(1)} \otimes \dots \otimes x^{\sigma(p)}$$

fulfills, locally, condition 4), up to the power of i , $i^{\frac{p(p+1)}{2}}$. Using a partition of unity gives the global form of c which is just the Hochschild cycle representing the global volume form.

The condition of strong regularity is checked using §13. One applies Lemma 13.2 to obtain the strong regularity since we take for D an elliptic differential operator of order one on a smooth compact manifold and the principal symbol of D^2 is a scalar multiple of the identity. This ensures that for any differential operator P of order m the symbol of order $m + 2$ of $[D^2, P]$ vanishes as it is given by the commutator of the principal symbols of order m and 2. Thus one gets that for any differential operator T of order 0, the operators $\delta_1^m(T)$ are of the form $P(1 + D^2)^{-m/2}$ where P is a differential operator of order m . Thus the theory of elliptic operators (cf. [15] Lemma 1.3.4 and 1.3.5) shows that they are bounded. This applies for D the Dirac operator or the signature operator, thus one gets the strong regularity in this case. \square

Theorem 11.5. *Let $(\mathcal{A}, \mathcal{H}, D)$ be a spectral triple with \mathcal{A} commutative, fulfilling the five conditions of §2 with the cycle c antisymmetric. Assume that the multiplicity¹⁶ of the action of \mathcal{A}'' in \mathcal{H} is $2^{p/2}$. Then there exists a smooth oriented compact ($spin^c$) manifold X such that $\mathcal{A} = C^\infty(X)$.*

Proof. We need to show that we can dispense with the hypothesis of strong regularity in Theorem 11.3. Indeed by the first part of Remark 5.12, we get

$$[[D, a][D, b] + [D, b][D, a], [D, c]] = 0 \quad \text{for all } a, b, c \in \mathcal{A} \quad (126)$$

since this is implied by the commutation (49) of $[[D, h]]$ with $[D, c]$. Thus if we work at a point $\chi \in \text{Spec}(\mathcal{A})$ and let S_χ be the fiber at $\chi \in X$ of the finite projective module \mathcal{H}_∞ and $M_\chi \subset \text{End } S_\chi$ be the subalgebra generated by the $[D, a]$ for $a \in \mathcal{A}$, it follows from (126) that

$$[D, a][D, b] + [D, b][D, a] \in Z(M_\chi) \quad \text{for all } a, b \in \mathcal{A}, \quad (127)$$

¹⁵Since we reduced the Hilbert space \mathcal{H} to the subspace given by $\gamma\xi = \xi$.

¹⁶We restrict ourselves to the even case.

where $Z(M_\chi)$ is the center of M_χ . Let e be a minimal projection in the center of M_χ . The equality $\pi_D(c) = \gamma$ shows that, at the point χ ,

$$\gamma e = \sum_{\alpha} e a_{\alpha}^0 [[D, a_{\alpha}^1], [D, a_{\alpha}^2], \dots, [D, a_{\alpha}^p]] \neq 0$$

so that the dimension of the space $T_e^*(\chi) = \{e[D, a] \mid a \in \mathcal{A}\}_{\chi}$ is at least equal to p . In fact, more precisely, for some α , the multiple commutator

$$[e[D, a_{\alpha}^1], e[D, a_{\alpha}^2], \dots, e[D, a_{\alpha}^p]] = e[[D, a_{\alpha}^1], [D, a_{\alpha}^2], \dots, [D, a_{\alpha}^p]] \neq 0$$

which can hold only if the $e[D, a_{\alpha}^j]$ are linearly independent. By (127) and the minimality of e , the following equality defines a positive quadratic form Q on the self-adjoint part of $T_e^*(\chi)$:

$$Q(e[D, a])e = (e[D, a])^2 \quad \text{for all } a \in \mathcal{A}.$$

It is non-degenerate since when $e[D, a]$ is self-adjoint, $(e[D, a])^2 = 0$ implies $e[D, a] = 0$. Let then C_Q be the Clifford algebra associated to the quadratic form Q on the self-adjoint part of $T_e^*(\chi)$. The latter has real dimension $\geq p$ and the relations (127) show that the map $e[D, a] \mapsto e[D, a]$ gives a representation of C_Q in the complex vector space eS_χ . Thus this shows that the dimension of eS_χ is then at least equal to $2^{p/2}$. The hypothesis of the theorem on the multiplicity of the action of \mathcal{A}'' in \mathcal{H} shows, using the condition of absolute continuity, that the fiber dimension of S is $2^{p/2}$. This shows that $e = 1$ and also, since the complexification of the algebra C_Q is an $N \times N$ matrix algebra for $N \geq 2^{p/2}$, that $M_\chi = \text{End } S_\chi$ for every $\chi \in X$. It also shows that the dimension of $T^*(\chi)$ is equal to p and that on U_α the $[D, a_{\alpha}^j]$ form a basis of $T^*(\chi)$. Consider then the monomials

$$\mu_F = [[D, a_{\alpha}^{j_1}], [D, a_{\alpha}^{j_2}], \dots, [D, a_{\alpha}^{j_k}]],$$

where $F = \{j_1 < j_2 < \dots < j_k\}$ is a subset with k elements of $\{1, 2, \dots, p\}$. For every $\chi \in U_\alpha$ the μ_F form a basis of $M_\chi = \text{End } S_\chi$. Thus any element T of $M_\chi = \text{End } S_\chi$ can be uniquely written in the form

$$T = \sum a_F \mu_F. \tag{128}$$

The coefficients a_F can be computed using the normalized trace on $\text{End } S_\chi$, the μ_F and the element T . Thus using the conditional expectation $E_{\mathcal{A}}$ of (27) one gets, for any endomorphism T of \mathcal{H}_∞ with support in U_α , that (128) holds with coefficients $a_F \in \mathcal{A}$. This shows that any endomorphism T of \mathcal{H}_∞ is a polynomial in the $[D, a]$ with coefficients in \mathcal{A} and it follows that it is automatically regular. Thus the strong regularity holds and we can apply Theorem 11.3. To see that X is a spin^c manifold one uses [12] (see [16] for the detailed proof). \square

12. Final remarks

12.1. The role of D . By Lemma 2.1, the spectral triple $(\mathcal{A}, \mathcal{H}, D)$ is entirely determined by $(\mathcal{A}'', \mathcal{H}, D)$ where $M = \mathcal{A}''$ is the commutative von Neumann algebra weak closure of \mathcal{A} . It follows in particular that, except for the dimension N of the bundle S which we may assume, for simplicity, to be constant and equal to $2^{p/2}$, there is no information in the pair $(\mathcal{A}'', \mathcal{H})$: they are all pairwise isomorphic. Similarly the only invariant of the pair (\mathcal{H}, D) is the spectrum of D , i.e., a list of real numbers with multiplicity. By [21] this spectrum does not suffice to reconstruct the geometry, and it is natural to wonder what additional invariant is required to do so. As we shall briefly explain it is the relative position of M and of the self-adjoint operator D which selects one geometric space, and it is worthwhile to look at the conditions from this point of view. The analogue in our context of the geodesic flow is the following one-parameter group,

$$\gamma_t(T) = e^{it|D|} T e^{-it|D|} \quad \text{for all } T \in \mathcal{L}(\mathcal{H}). \quad (129)$$

We say¹⁷ that an operator $T \in \mathcal{L}(\mathcal{H})$ is of class C^∞ when the map from \mathbb{R} to $\mathcal{L}(\mathcal{H})$ given by $t \mapsto \gamma_t(T)$ is of class C^∞ (for the norm topology of $\mathcal{L}(\mathcal{H})$) and we denote by $C^\infty(\mathcal{H}, D)$ this subalgebra of $\mathcal{L}(\mathcal{H})$. This algebra only depends upon (\mathcal{H}, D) and does not yet measure the compatibility of (M, \mathcal{H}) and (\mathcal{H}, D) . This is measured by the *weak density* in M of

$$C^\infty(M, \mathcal{H}, D) = \{T \in M \cap C^\infty(\mathcal{H}, D) \mid [D, T] \in M' \cap C^\infty(\mathcal{H}, D)\}, \quad (130)$$

where M' is the commutant of M . One checks that $\mathcal{A} = C^\infty(M, \mathcal{H}, D)$ is a subalgebra of M and its size measures the compatibility of (M, \mathcal{H}) and (\mathcal{H}, D) .

We now come to two equations which assert that $\mathcal{A} = C^\infty(M, \mathcal{H}, D)$ is large enough, so that we have maximal compatibility. One checks that $\mathcal{H}_\infty = \cap \text{Dom } D^m$ is automatically a module over \mathcal{A} (for the obvious action). The first equation requires that this module is *finite and projective* and that it admits a hermitian structure $(\cdot | \cdot)$ (necessarily unique) such that

$$\langle \xi, a \eta \rangle = \int (\xi | \eta) a |ds|^p \quad \text{for all } a \in \mathcal{A} \text{ and all } \xi, \eta \in \mathcal{H}_\infty, \quad (131)$$

where \int is the noncommutative integral given by the Dixmier trace.

The second equation means that we can find an element c of the tensor power $\mathcal{A}^{\otimes n}$, $n = p + 1$, totally antisymmetric in its last p -entries, and such that

$$c(D) = 1, \text{ where } (a_0 \otimes \cdots \otimes a_p)(D) = a_0 [D, a_1] \cdots [D, a_p] \text{ for all } a_j \in \mathcal{A}. \quad (132)$$

(This assumes p odd, in the even case one requires that for some c as above $c(D) = \gamma$ fulfills $\gamma = \gamma^*$, $\gamma^2 = 1$, $\gamma D = -D \gamma$.) We can now restate Theorem 11.5 as:

¹⁷Cf. Lemma 13.3 of §13.

Theorem 12.1. *Let (M, \mathcal{H}, D) fulfill (130), (131) and (132), and $N = 2^{p/2}$. Then there exists a unique smooth compact oriented spin^c Riemannian manifold (X, g) such that the triple (M, \mathcal{H}, D) is given by*

- $M = L^\infty(X, dv)$ where dv is the Riemannian volume form,
- $\mathcal{H} = L^2(X, S)$ where S is the spinor bundle,
- D is a Dirac operator associated to the Riemannian metric g .

Proof. We let $\mathcal{A} = C^\infty(M, \mathcal{H}, D)$. By the weak density in M of (130), we know that the multiplicity of the action of $\mathcal{A}'' = M$ in \mathcal{H} is $N = 2^{p/2}$. By construction, the triple $(\mathcal{A}, \mathcal{H}, D)$ fulfills the first three conditions. The fourth and fifth follow from (131) and (132). Thus by Theorem 11.5 we get $\mathcal{A} = C^\infty(X)$ for a smooth oriented compact spin^c manifold X . The conclusion then follows from [12] (see [16] for the detailed proof). Note that there is no uniqueness of D since we only know its principal symbol. This is discussed in [12] and [16]. \square

A striking feature of the above formulation is that the full information on the geometric space is subdivided into two pieces:

- (1) the list of eigenvalues of D ,
- (2) the unitary relation F between the Hilbert space of the canonical pair (M, \mathcal{H}) and the Hilbert space of the canonical pair (\mathcal{H}, D) .

Of course the conceptual meaning of the unitary F is the Fourier transform, but this second piece of data is now playing a role entirely similar to that of the CKM matrix in the Standard Model [7]. Moreover, in the latter, the information about the Yukawa coupling of the Higgs fields with the fermions (quarks and leptons) is organised in a completely similar manner, namely 1) the masses of the particles, 2) the CKM (and PMNS) matrix. At the conceptual level, such matrices describe the relative position of two different bases in the same Hilbert space. They are encoded by a double coset space closely related to Shimura varieties ([7]). These points deserve further investigations and will be pursued in a forthcoming paper.

12.2. Finite propagation. One can use in the above context a result of Hilsum [18] to obtain:

Lemma 12.2. *The support of the kernel $k_t(x, y)$ of the operator e^{itD} is contained in*

$$\{(x, y) \in X^2 \mid d(x, y) \leq |t|\},$$

where the distance d is defined by

$$d(x, y) = \sup |h(x) - h(y)|, \quad \|[D, h]\| \leq 1.$$

Proof. Let $(x, y) \in X^2$ with $d(x, y) > |t|$. There exists $h = h^*$ in \mathcal{A} such that $\|[D, h]\| \leq 1$ and $h(y) - h(x) > |t|$. Also h and $[D, h]$ commute by the order one condition. Thus by Lemma 1.10 of [18], one has $b < h(y)$, $a > h(x)$ such that

$$(h - b)_+ e^{-itD} (h - a)_- = 0$$

so that $k_t(x, y) = 0$. □

12.3. Immersion versus embedding. The proof of Theorem 11.3 shows that, with the cycle c given by (26), the map ψ from X to \mathbb{R}^N given by the components a_α^j for $j \geq 1$ is an *immersion*. It is not however an embedding in general even if one includes the components a_α^0 . To see this consider open balls $B \subset \mathbb{R}^p$ and $B_1 \subset B$ such that for some translation v the ball $B_2 = B_1 + v$ is disjoint from B_1 and contained in B . Then let x^μ be the coordinates in \mathbb{R}^p and $a_1^j \in C_c^\infty(B)$ be such that

$$a_1^j(x) = x^j \text{ for all } x \in B_1, \quad a_1^j(x) = x^j - v^j \text{ for all } x \in B_2.$$

Let N be a neighborhood of the complement of $B_1 \cup B_2$ in B . Let then $a_2^j(x) = b(x)x^j$, where $b(x) = 1$ for all $x \in N$ and vanishes in an open set of the form $B'_1 \cup B'_2$ where the $B'_j \subset B_j$ are smaller concentric balls. Let a_α^0 be a partition of unity in B for the covering by $B_1 \cup B_2$ and N . Then let c be the antisymmetrization of

$$\sum_1^2 a_\alpha^0 \otimes a_\alpha^1 \otimes \cdots \otimes a_\alpha^p.$$

For $x \in B'_1$ all the a_2^j vanish, including a_2^0 , and $a_1^0 = 1$ so that the following equality shows that the map ψ is not injective:

$$a_1^j(x + v) = a_1^j(x) \quad \text{for all } x \in B'_1.$$

12.4. The antisymmetry condition. We have used throughout the stronger form of condition 4) where the Hochschild cycle $c \in H_p(\mathcal{A}, \mathcal{A})$ is assumed to be totally antisymmetric in its last p -entries. It is unclear whether one can relax the antisymmetry condition on c . It is not true in general for commutative algebras that any Hochschild class can be represented in this way, but this is the case for $\mathcal{A} = C^\infty(X)$. In general, one has a natural projection on the antisymmetric chains, given by the antisymmetrisation map P . It is defined by the equality

$$P(a_0 \otimes a_1 \otimes \cdots \otimes a_p) = \frac{1}{p!} \sum_\beta \varepsilon(\beta) a_0 \otimes a_{\beta(1)} \otimes \cdots \otimes a_{\beta(p)}$$

Its range is contained in $Z_p(\mathcal{A}, \mathcal{A})$ since \mathcal{A} is commutative and any antisymmetric chain is a cycle ([20], Proposition 1.3.5). It is not obvious that P maps Hochschild

boundaries to Hochschild boundaries. This follows from the equality

$$P = \frac{1}{p!} \varepsilon_p \circ \pi_p,$$

where one lets $\Omega_K^p = \bigwedge_{\mathcal{A}}^p \Omega_K^1$ be the \mathcal{A} -module of Kähler p -forms (cf. [20], 1.3.11) and

$$\varepsilon_p: \Omega_K^p \rightarrow H_p(\mathcal{A}, \mathcal{A}), \quad \pi_p: H_p(\mathcal{A}, \mathcal{A}) \rightarrow \Omega_K^p$$

are defined in [20], Propositions 1.3.12 and 1.3.15. They are given by

$$\pi_k(a_0 \otimes a_1 \otimes \cdots \otimes a_k) = a_0 da_1 \wedge \cdots \wedge da_k$$

and

$$\varepsilon_k(a_0 da_1 \wedge \cdots \wedge da_k) = \sum \varepsilon(\sigma) a_0 \otimes a_{\sigma(1)} \otimes \cdots \otimes a_{\sigma(k)}.$$

12.5. Strong regularity. The hypothesis of strong regularity is, in general, stronger than regularity. Indeed the operation of direct sum $(\mathcal{A}, \mathcal{H}_1 \oplus \mathcal{H}_2, D_1 \oplus D_2)$ of two spectral triples for the same algebra \mathcal{A} preserves regularity but not, in general, strong regularity.

Proposition 12.3. *Assuming regularity the subalgebra $Z_D(\mathcal{A})$ of $\text{End}_{\mathcal{A}}(\mathcal{H}_{\infty})$ generated by \mathcal{A} and the $[D, b][D, c] + [D, c][D, b]$ for $b, c \in \mathcal{A}$ is a commutative algebra containing \mathcal{A} and commuting with $[D, a]$ for all $a \in \mathcal{A}$.*

Proof. This follows from Remark 5.12 since (49) shows that $[D, b]^2$ commutes with $[D, a]$ for all $a \in \mathcal{A}$. \square

The understanding of the general situation when one does not assume strong regularity should be an interesting problem since the inclusion $\mathcal{A} \subset Z_D(\mathcal{A})$ should correspond to a finite “ramified cover” of the corresponding spectra, with $Y = \text{Spec } Z_D(\mathcal{A})$ covering $X = \text{Spec } \mathcal{A}$. It is easy to construct examples where Y has singularities. It is not clear that, assuming the first five conditions, the space X is always smooth. Similarly it is unclear what happens if one relaxes the regularity condition to the Lipschitz regularity, since we made heavy use of at least $C^{1+\varepsilon}$ -regularity in the above proofs. Finally it would be interesting also to investigate the meaning of *real analyticity* of the space X in terms of the real analyticity of the geodesic flow (129).

12.6. The noncommutative case. Among the five conditions of §2 the conditions 1), 3) and 5) make no use of the commutativity of the algebra \mathcal{A} and they extend as such to the noncommutative case. We refer to [12] for the extension of conditions 2) and 4) to the noncommutative case. The extension of the order one condition involves a new key ingredient which is an antilinear unitary operator J in \mathcal{H} which encodes the nuance between spin and spin^c. It turns out to be an incarnation not only of the

charge conjugation in physics terms and of the needed “real structure” to refine the K-theoretic meaning of the spectral triple from ordinary K -homology to KO -homology but, at a deeper level, of the Tomita operator which plays in the noncommutative case the role of a substitute for commutativity. All this plays an important role in the noncommutative geometry understanding of the standard model [7], [6], [5]. The extension of the “orientability” condition 4) exists and it certainly holds e.g. for noncommutative tori ([12]) but it is not fully satisfactory yet and its clarification should be considered as an open question.

13. Appendix 1: Regularity

The condition of regularity is not easy to check for smooth manifolds since it involves the module of the operator D . We give below the equivalent formulation in terms of D^2 (cf. [14]).

We deal with operators T which act on $\mathcal{H}_\infty = \cap \text{Dom } D^n$. We say that T is *bounded* when

$$\|T\| = \sup\{\|T\xi\| \mid \xi \in \mathcal{H}_\infty, \|\xi\| \leq 1\} \quad (133)$$

is finite. We still denote by T the unique continuous extension to a bounded operator in \mathcal{H} . By self-adjointness of D the domain \mathcal{H}_∞ is a core for powers of D or of $|D|$. The derivation $\delta(T) = [|D|, T]$ is defined algebraically as an operator in \mathcal{H}_∞ . The relation with the commutator in \mathcal{H} is given as follows.

Lemma 13.1. *Assume that both T and $[|D|, T]$ are bounded (as in (133)). Then T preserves $\text{Dom } |D| = \text{Dom } D$ and the bounded extension of $[|D|, T]$ coincides with the commutator $|D|T - T|D|$ on $\text{Dom } |D|$.*

Proof. Let $\xi \in \text{Dom } |D|$. There exists a sequence $\xi_n \in \mathcal{H}_\infty$ with $\xi_n \rightarrow \xi$ and $|D|\xi_n \rightarrow |D|\xi$. Since T is bounded the sequences $T\xi_n$ and $T|D|\xi_n$ are convergent and converge to $T\xi$ and $T|D|\xi$. Since $[|D|, T]$ is bounded, the sequence $(|D|T - T|D|)\xi_n$ converges. Thus $|D|T\xi_n$ converges, and as $|D|$ is closed one gets that $T\xi$ is in the domain of $|D|$. Thus $\text{Dom } |D|$ is invariant under T . Moreover one has $|D|T\xi = (|D|T - T|D|)\xi + T|D|\xi$. \square

In other words, saying that both T and $[|D|, T]$ are bounded is equivalent, for operators acting in \mathcal{H}_∞ to $T \in \text{Dom } \delta$ and moreover $\delta(T)$ is then the bounded extension of $|D|T - T|D|$.

We introduce the following variant of δ , defined on operators T acting in \mathcal{H}_∞ ,

$$\delta_1(T) = [D^2, T](1 + D^2)^{-1/2}.$$

Lemma 13.2. *Let T acting in \mathcal{H}_∞ be bounded.*

(1) *If $\delta_1(T)$ and $\delta_1^2(T)$ are bounded so is $\delta(T)$.*

(2) The $\delta_1^n(T)$ are bounded for all n iff so are the $\delta^n(T)$.

Proof. (1) The module $|D|$ is given by the following integral, which makes sense when applied to any $\xi \in \text{Dom } D$, which we omit for simplicity

$$|D| = \frac{2}{\pi} \int_0^\infty \frac{D^2}{D^2 + u^2} du. \quad (134)$$

To avoid dealing with the kernel of D we use the decomposition $\delta = \delta' + \delta_0$, where the derivations δ' and δ_0 commute, and δ_0 is bounded,

$$\begin{aligned} \delta'(T) &= [Q, T], \quad Q = D^2(1 + D^2)^{-1/2}, \\ \delta_0(T) &= [f_0(D), T], \quad f_0(x) = |x| - x^2(1 + x^2)^{-1/2} \text{ for all } x \in \mathbb{R}. \end{aligned}$$

One has $f \in C_0(\mathbb{R})$ and the derivation δ_0 is bounded, in fact $\|\delta_0\| \leq 1$ since $\|f_0\|_\infty < 1/2$. One has

$$Q = \frac{2}{\pi} \int_0^\infty \frac{D^2}{D^2 + 1 + u^2} du.$$

Thus

$$\begin{aligned} \delta'(T) &= [Q, T] = \frac{2}{\pi} \int_0^\infty \left[\frac{D^2}{D^2 + 1 + u^2}, T \right] du, \quad (135) \\ &= \left[\frac{D^2}{D^2 + 1 + u^2}, T \right] \\ &= - \left[\frac{1 + u^2}{D^2 + 1 + u^2}, T \right] \\ &= (1 + u^2) \frac{1}{D^2 + 1 + u^2} [D^2, T] \frac{1}{D^2 + 1 + u^2} \\ &= [D^2, T] \frac{1 + u^2}{(D^2 + 1 + u^2)^2} + (1 + u^2) \left[\frac{1}{D^2 + 1 + u^2}, [D^2, T] \right] \frac{1}{D^2 + 1 + u^2}. \end{aligned}$$

Thus using

$$\left[\frac{1}{D^2 + 1 + u^2}, [D^2, T] \right] = - \frac{1}{D^2 + 1 + u^2} [D^2, [D^2, T]] \frac{1}{D^2 + 1 + u^2}$$

we get

$$\begin{aligned} & \left[\frac{D^2}{D^2 + 1 + u^2}, T \right] \\ &= [D^2, T] \frac{1 + u^2}{(D^2 + 1 + u^2)^2} - \frac{1}{D^2 + 1 + u^2} [D^2, [D^2, T]] \frac{1 + u^2}{(D^2 + 1 + u^2)^2}. \end{aligned}$$

Thus combining with (135) one gets

$$\begin{aligned} \delta'(T) &= \frac{1}{2}[D^2, T](1 + D^2)^{-1/2} + \frac{1}{2}[D^2, T](1 + D^2)^{-3/2} \\ &\quad - \frac{2}{\pi} \int_0^\infty \frac{1}{D^2 + 1 + u^2} [D^2, [D^2, T]] \frac{1 + u^2}{(D^2 + 1 + u^2)^2} du, \end{aligned}$$

where we used

$$\frac{2}{\pi} \int_0^\infty \frac{u^2}{(D^2 + 1 + u^2)^2} du = \frac{1}{2}(1 + D^2)^{-1/2}$$

and

$$\frac{2}{\pi} \int_0^\infty \frac{1}{(D^2 + 1 + u^2)^2} du = \frac{1}{2}(1 + D^2)^{-3/2}.$$

Now one has $[D^2, [D^2, T]] = \delta_1^2(T)(1 + D^2)$ and

$$\left\| \frac{(1 + u^2)(1 + D^2)}{(D^2 + 1 + u^2)^2} \right\| \leq 1$$

so that

$$\left\| \frac{1}{D^2 + 1 + u^2} [D^2, [D^2, T]] \frac{1 + u^2}{(D^2 + 1 + u^2)^2} \right\| \leq \left\| \frac{1}{D^2 + 1 + u^2} \right\| \|\delta_1^2(T)\|$$

and one gets

$$\|\delta'(T)\| \leq \|\delta_1(T)\| + \|\delta_1^2(T)\|.$$

Now if both $\delta_1(T)$ and $\delta_1^2(T)$ are bounded, we get that $\delta'(T)$ is bounded, and since $\delta = \delta' + \delta_0$ with δ_0 bounded, we get that $\delta(T)$ is bounded, with

$$\|\delta(T)\| \leq \sum_0^2 \|\delta_1^j(T)\|. \quad (136)$$

(2) The operations δ and δ_1 commute since $|D|$ commutes with D^2 . Let us assume that the $\delta_1^n(T)$ are bounded. We have seen that $\delta(T)$ is bounded. To show that $\delta^2(T)$ is bounded it is enough to show that $\delta_1^m(\delta(T))$ are bounded for $m = 1, 2$. But $\delta_1^m(\delta(T)) = \delta(\delta_1^m(T))$, which is bounded since the $\delta_1^n(\delta_1^m(T))$ are bounded for $n \leq 2, m \leq 2$. More generally let us show by induction on n an inequality of the form

$$\|\delta^n(T)\| \leq \sum_0^{2n} c_{n,k} \|\delta_1^k(T)\|.$$

To get it for $n + 1$, assuming it for n , one uses (136), which gives

$$\|\delta(\delta^n(T))\| \leq \sum_0^2 \|\delta_1^j(\delta^n(T))\| = \sum_0^2 \|\delta^n(\delta_1^j(T))\| \leq \sum_0^2 \sum_0^{2n} c_{n,k} \|\delta_1^k(\delta_1^j(T))\|.$$

Thus we obtain by induction that $\delta^n(T)$ is bounded.

Conversely, the boundedness of the $\delta^n(T)$ implies that of the $\delta_1^n(T)$. Indeed the boundedness of the $\delta^n(T)$ is equivalent to the boundedness of the $\delta''^n(T)$, where $\delta''(T) = [(1 + D^2)^{1/2}, T]$, since $|D| - (1 + D^2)^{1/2}$ is bounded and commutes with $|D|$. Moreover the square of the operation

$$T \mapsto (1 + D^2)^{1/2}T(1 + D^2)^{-1/2} = T + \delta''(T)(1 + D^2)^{-1/2}$$

is

$$T \mapsto (1 + D^2)T(1 + D^2)^{-1} = T + [D^2, T](1 + D^2)^{-1},$$

which gives

$$[D^2, T](1 + D^2)^{-1} = 2\delta''(T)(1 + D^2)^{-1/2} + \delta''^2(T)(1 + D^2)^{-1}$$

so that

$$\delta_1(T) = 2\delta''(T) + \delta''^2(T)(1 + D^2)^{-1/2},$$

and one can proceed as above to get the boundedness of the $\delta_1^n(T)$. \square

Finally we relate the regularity condition with the smoothness of the geodesic flow $t \rightarrow \gamma_t(T) = e^{it|D|}Te^{-it|D|}$ of (33).

Lemma 13.3. *Let $T \in \mathcal{L}(\mathcal{H})$. Then the following conditions are equivalent:*

- (1) $T \in \cap_m \text{Dom } \delta^m$.
- (2) $t \rightarrow \gamma_t(T)$ is of class C^∞ in the norm topology.

Proof. Let us show that (1) implies (2). By (7), T preserves \mathcal{H}_∞ . We write the Taylor formula with remainder

$$f(t) = f(0) + tf'(0) + \cdots + \frac{t^n}{n!}f^{(n)}(0) + \frac{t^{n+1}}{n!} \int_0^1 (1-u)^n f^{(n+1)}(tu) du$$

for the function $f(t) = e^{it|D|}Te^{-it|D|}\xi$ with $\xi \in \mathcal{H}_\infty$. Since T preserves \mathcal{H}_∞ , this function is of class C^∞ . One gets

$$\begin{aligned} & \gamma_t(T)\xi \\ &= T\xi + it\delta(T)\xi + \cdots + \frac{i^n t^n}{n!}\delta^n(T)\xi + \frac{i^{n+1}t^{n+1}}{n!} \int_0^1 (1-u)^n \gamma_{tu}(\delta^{(n+1)}(T))\xi du \end{aligned}$$

since $f^{(k)}(s) = \gamma_s(\delta^{(k)}(T))\xi$ by induction on k . This shows that $t \rightarrow \gamma_t(T)$ is of class C^∞ in the norm topology, since the norm of the remainder is $O(t^{n+1})$.

Let us show that (2) implies (1). It is enough to show that if $T \in \mathcal{L}(\mathcal{H})$ and the following limit exists in norm $\lim_{t \rightarrow 0} \frac{1}{t}(\gamma_t(T) - T)$, then $T \in \text{Dom } \delta$ and the limit is $i\delta(T)$. One has, for $\xi \in \mathcal{H}$,

$$\xi \in \text{Dom } |D| \quad \text{if and only if there exists} \quad \lim_{t \rightarrow 0} \frac{1}{t}(e^{it|D|}\xi - \xi),$$

where the limit is supposed to exist in norm. Assuming that for some bounded operator $Y \in \mathcal{L}(\mathcal{H})$ one has $\lim_{t \rightarrow 0} \|\frac{1}{t}(\gamma_t(T) - T) - Y\| = 0$, one gets, for any $\xi \in \text{Dom } |D|$, $\frac{1}{t}(e^{it|D|}T\xi - T\xi) \rightarrow iT|D|\xi + Y\xi$. This shows that $T\xi \in \text{Dom } |D|$ and that $i|D|T\xi = iT|D|\xi + Y\xi$, which gives the required equality. \square

14. Appendix 2: The Dixmier trace and the heat expansion

We first recall the basic properties of the Dixmier trace. Recall that the characteristic value $\mu_n(T)$ of a compact operator T is the n -th eigenvalue of $|T|$ arranged in decreasing order and is equal to

$$\inf\{\|T|_{E^\perp}\| \mid \dim E = n - 1\}.$$

Definition 14.1. We define the Weyl norms by

$$\sigma_N(T) = \sum_1^N \mu_n(T).$$

The fact that they are norms and in particular fulfill

$$\sigma_N(T_1 + T_2) \leq \sigma_N(T_1) + \sigma_N(T_2)$$

follows from the next statement in which we use the same notation for a subspace $E \subset \mathcal{H}$ and the orthogonal projection on that subspace.

Proposition 14.2. *One has*

$$\sigma_N(T) = \sup\{\|TE\|_1 \mid \dim E = N\}.$$

Let T be a positive operator, then

$$\sigma_N(T) = \sup\{\text{Tr}(TE) \mid \dim E = N\}. \tag{137}$$

We use the following notation for refined limiting processes.

Definition 14.3. With the Cesàro mean M defined by

$$M(f)(\lambda) = \frac{1}{\log \lambda} \int_1^\lambda f(u) \frac{du}{u} \tag{138}$$

and $h(\lambda)$ a bounded function of $\lambda > 0$, ω a linear form on $C_b(\mathbb{R}_+^*)$ which is positive, $\omega(1) = 1$, and vanishes on $C_0(\mathbb{R}_+^*)$, and ϕ a homeomorphism of \mathbb{R}_+^* , we define

$$\text{Lim}_{\phi(\lambda) \rightarrow \omega}^k h(\lambda) = \omega(M^k(g)), \quad g(\lambda) = h(\phi^{-1}(\lambda)),$$

where the upper index k indicates that we iterate the Cesàro mean k -times.

We write Lim_ω as an abbreviation for Lim_ω^1 , and when we apply it to a sequence $(\alpha_N)_{N \in \mathbb{N}}$ we mean that the sequence has been extended to a function using

$$f_\alpha(\lambda) = \alpha_N \quad \text{for } \lambda \in]N - 1, N].$$

Also we consider the two-sided ideal containing compact operators of order one,

$$\mathcal{L}^{(1,\infty)}(\mathcal{H}) = \{T \in \mathcal{K} \mid \sigma_N(T) = O(\log N)\}.$$

Definition 14.4. For $T \geq 0$, $T \in \mathcal{L}^{(1,\infty)}(\mathcal{H})$, we set

$$\text{Tr}_\omega(T) = \text{Lim}_\omega \frac{1}{\log N} \sum_{n=1}^N \mu_n(T).$$

The basic properties of the Dixmier trace Tr_ω are summarized in the following ([10], Proposition 3, IV.2.β):

Proposition 14.5. Tr_ω extends uniquely by linearity to the entire ideal $\mathcal{L}^{(1,\infty)}(\mathcal{H})$ and has the following properties:

- (a) If $T \geq 0$, then $\text{Tr}_\omega(T) \geq 0$.
- (b) If S is any bounded operator and $T \in \mathcal{L}^{(1,\infty)}(\mathcal{H})$, then $\text{Tr}_\omega(ST) = \text{Tr}_\omega(TS)$.
- (c) $\text{Tr}_\omega(T)$ is independent of the choice of the inner product on \mathcal{H} , i.e., it depends only on the Hilbert space \mathcal{H} as a topological vector space.
- (d) Tr_ω vanishes on the ideal $\mathcal{L}_0^{(1,\infty)}(\mathcal{H})$, which is the closure, for the $\|\cdot\|_{1,\infty}$ -norm, of the ideal of finite-rank operators.

We fix $p \in [1, \infty]$. Let D be a self-adjoint unbounded operator such that its resolvent is an infinitesimal of order $1/p$, i.e., such¹⁸ that $\mu_n(D^{-1}) = O(n^{-1/p})$. We shall compare $\text{Tr}_\omega(T|D|^{-p})$ and $\lim \varepsilon^p \text{Tr}(f(\varepsilon D)T)$ for suitable even test functions f . We let E_N be the spectral projection¹⁹ on the first N -eigenvectors of $|D|$ so that $\dim E_N = N$, $E_N < E_{N+1}$ and

$$\text{Tr}(E_N|D|^{-p}) = \sigma_N(|D|^{-p}).$$

Lemma 14.6. For any bounded operator $T \in \mathcal{L}(\mathcal{H})$ one has

$$\text{Lim}_\omega \frac{1}{\log N} \text{Tr}(E_N|D|^{-p}T) = \text{Tr}_\omega(T|D|^{-p}). \quad (139)$$

¹⁸We replace D by a non-zero constant on its kernel so that D^{-1} makes sense.

¹⁹This is ambiguous when there is spectral multiplicity.

Proof. The hypothesis on D shows that $\text{Tr}(E_N|D|^{-p}) = O(\log N)$. Moreover, by construction of the Dixmier trace, one has

$$\text{Lim}_\omega \frac{1}{\log N} \text{Tr}(E_N|D|^{-p}) = \text{Lim}_\omega \frac{1}{\log N} \sigma_N(|D|^{-p}) = \text{Tr}_\omega(|D|^{-p}). \quad (140)$$

Let $\phi(T)$ be the left-hand side of (139). It makes sense since

$$|\text{Tr}(E_N|D|^{-p}T)| \leq \text{Tr}(E_N|D|^{-p})\|T\| = O(\log N)$$

so that the sequence $\frac{1}{\log N} \text{Tr}(E_N|D|^{-p}T)$ is bounded. The functional ϕ on $\mathcal{L}(\mathcal{H})$ is linear and positive (the trace of the product of the two positive operators $E_N|D|^{-p}$ and T is positive). Let $\psi(T)$ be the right-hand side of (139). Proposition 14.5 shows that, since $|D|^{-p} \in \mathcal{L}^{(1,\infty)}(\mathcal{H})$, the functional ψ is a positive linear functional on $\mathcal{L}(\mathcal{H})$. One uses Proposition 14.5 (b) to check the positivity, using for $T \geq 0$,

$$\text{Tr}_\omega(T|D|^{-p}) = \text{Tr}_\omega(T^{1/2}|D|^{-p}T^{1/2}) \geq 0.$$

Let us show that for any $T \geq 0$ one has $\phi(T) \leq \psi(T)$. One has

$$\sigma_N(T^{1/2}|D|^{-p}T^{1/2}) = \sigma_N(|D|^{-p/2}T|D|^{-p/2})$$

using $A = |D|^{-p/2}T^{1/2}$ in

$$\mu_n(A^*A) = \mu_n(AA^*) \quad \text{for all } A \in \mathcal{K}, n \in \mathbb{N}.$$

Thus one gets

$$\psi(T) = \text{Tr}_\omega(T|D|^{-p}) = \text{Lim}_\omega \frac{1}{\log N} \sigma_N(|D|^{-p/2}T|D|^{-p/2}).$$

By (137), one has

$$\begin{aligned} \sigma_N(|D|^{-p/2}T|D|^{-p/2}) &= \sup\{\text{Tr}(|D|^{-p/2}T|D|^{-p/2}E) \mid \dim E = N\} \\ &\geq \text{Tr}(|D|^{-p/2}T|D|^{-p/2}E_N) = \text{Tr}(E_N|D|^{-p}T) \end{aligned}$$

since E_N and $|D|^{-p/2}$ commute. Thus $\sigma_N(|D|^{-p/2}T|D|^{-p/2}) \geq \text{Tr}(E_N|D|^{-p}T)$ and after dividing by $\log N$ and applying Lim_ω to both sides one gets the inequality $\phi(T) \leq \psi(T)$. But, by (140), $\phi(1) = \text{Tr}_\omega(|D|^{-p}) = \psi(1)$, and thus the positive functional $\theta = \psi - \phi$ is equal to 0 by the Schwarz inequality $|\theta(T)|^2 \leq \theta(T^*T)\theta(1)$. \square

With $|D|$ as above, we let as in (104), for any $\lambda > 0$,

$$P(\lambda) = \mathbf{1}_{[0,\lambda]}(|D|), \quad \alpha(\lambda) = \text{Tr } P(\lambda).$$

Lemma 14.7. *Assume that*

$$\liminf \lambda^{-p} \alpha(\lambda) > 0. \quad (141)$$

Then, for any bounded operator $T \in \mathcal{L}(\mathcal{H})$, one has

$${}_p \text{Lim}_\omega \frac{1}{\log N} \text{Tr}(E_N |D|^{-p} T) = \text{Lim}_{\lambda^p \rightarrow \omega} \frac{1}{\log \lambda} \text{Tr}(P(\lambda) |D|^{-p} T). \quad (142)$$

Proof. We can assume by linearity that $T \geq 0$. We have (using (105)) constants $c_1 > 0$ and $c_2 < \infty$ such that

$$c_1 \lambda^p \leq \alpha(\lambda) \leq c_2 \lambda^p. \quad (143)$$

We let

$$f(N) = \text{Tr}(E_N |D|^{-p} T), \quad g(\lambda) = \text{Tr}(P(\lambda) |D|^{-p} T).$$

Since $\dim P(\lambda) \leq N$ implies $P(\lambda) \leq E_N$, we get, using $P(\lambda) |D|^{-p} \leq E_N |D|^{-p}$,

$$f(N) \geq g(\lambda) \quad \text{for all } \lambda, c_2 \lambda^p \leq N. \quad (144)$$

Similarly, since $\dim P(\lambda) \geq N$ implies $P(\lambda) \geq E_N$, we get

$$f(N) \leq g(\lambda) \quad \text{for all } \lambda, c_1 \lambda^p \geq N. \quad (145)$$

We extend $f(N)$ to positive real values of N as a non-decreasing step function. The arbitrariness of the extension is irrelevant since $f(N+1) - f(N) \rightarrow 0$ when $N \rightarrow \infty$ and we are interested in $\text{Lim}_\omega \frac{1}{\log N} f(N)$, which is insensitive to bounded perturbations of f . By construction, the Cesàro mean satisfies the following scale invariance, for bounded functions f ,

$$|M(\theta_\mu(f))(\lambda) - M(f)(\lambda)| \rightarrow 0 \quad \text{as } \lambda \rightarrow \infty, \quad (146)$$

where $\mu > 0$ and $\theta_\mu(f)(\lambda) = f(\mu^{-1}\lambda)$ for all $\lambda \in \mathbb{R}_+^*$. It follows from (144) and (145) that $f(c_1 N) \leq g(N^{1/p}) \leq f(c_2 N)$ up to $o(N)$ and for any positive real N . Thus the scale invariance of the Cesàro mean (146) together with $\log N / \log cN \rightarrow 1$ gives

$$M\left(\frac{1}{\log N} f(N)\right) - M\left(\frac{1}{\log N} g(N^{1/p})\right) \rightarrow 0$$

so that

$$\text{Lim}_\omega \frac{1}{\log N} f(N) = \frac{1}{p} \text{Lim}_\omega \frac{1}{\log N^{1/p}} g(N^{1/p}),$$

and the required equality (142) follows from Definition 14.3. \square

Corollary 14.8. *Assuming (141), one has*

$${}_p \text{Tr}_\omega(T |D|^{-p}) = \text{Lim}_{\lambda^p \rightarrow \omega} \frac{1}{\log \lambda} \text{Tr}(P(\lambda) |D|^{-p} T) \quad \text{for all } T \in \mathcal{L}(\mathcal{H}). \quad (147)$$

Proof. This follows from Lemmas 14.6 and 14.7. \square

Theorem 14.9. *Assume that (141) holds. Suppose that $f \in C_c([0, \infty[)$ and let $\rho = p \int_0^\infty u^{p-1} f(u) du$. Then for any bounded operator $T \in \mathcal{L}(\mathcal{H})$ one has*

$$\text{Lim}_{\varepsilon \rightarrow \omega}^2 \varepsilon^p \text{Tr}(f(\varepsilon|D|)T) = \rho \text{Tr}_\omega(T|D|^{-p}). \quad (148)$$

Proof. Let $g(u) = u^p f(u)$ viewed as an integrable function on the multiplicative group \mathbb{R}_+^* , endowed with its normalized Haar measure $d^*u = \frac{du}{u}$. We can assume that $T \geq 0$. We consider the positive measure on \mathbb{R}_+^* given by $d\beta(\lambda)$ where

$$\beta(\lambda) = \text{Tr}(P(\lambda)|D|^{-p}T), \quad (149)$$

which is a non-decreasing step function of λ . The measure $d\beta$ is a positive linear combination of Dirac masses, $d\beta = \sum \alpha_n \delta_{\lambda_n}$. One has

$$\begin{aligned} d\beta(\lambda) &= \text{Tr}(dP(\lambda)|D|^{-p}T) = \lambda^{-p} \text{Tr}(dP(\lambda)T), \\ \varepsilon^p \text{Tr}(f(\varepsilon|D|)T) &= \varepsilon^p \int f(\varepsilon\lambda) \text{Tr}(dP(\lambda)T) = \int \varepsilon^p \lambda^p f(\varepsilon\lambda) d\beta(\lambda) \end{aligned}$$

so that

$$\varepsilon^p \text{Tr}(f(\varepsilon|D|)T) = \int g(\varepsilon\lambda) d\beta(\lambda). \quad (150)$$

The convolution of the measure $d\beta$ with the function $\tilde{g}(u) = g(u^{-1})$ makes sense, since both have support in an interval $[u_0, \infty[$ with $u_0 > 0$, and gives the function

$$(\tilde{g} \star d\beta)(u) = \int g(u^{-1}\lambda) d\beta(\lambda).$$

Thus, with $h(\varepsilon) = \varepsilon^p \text{Tr}(f(\varepsilon|D|)T)$, one gets using (150),

$$h(u^{-1}) = (\tilde{g} \star d\beta)(u). \quad (151)$$

The convolution of the measures $\tilde{g}(u)d^*u$ and $d\beta$ is absolutely continuous with respect to d^*u and is given, with $\theta_u(v) = uv$ for all $u, v > 0$, by

$$(\tilde{g} \star d\beta)d^*u = \int \tilde{g}(u)\theta_u(d\beta) d^*u. \quad (152)$$

We extend the definition of the Cesàro mean (138) to measures μ by

$$M(\mu)(\lambda) = \frac{1}{\log \lambda} \int_1^\lambda d\mu,$$

so that

$$M(\mu)(\lambda) = M(h)(\lambda) \quad \text{for } \mu = hd^*u. \quad (153)$$

One has $\beta(\lambda) = O(\log \lambda)$ since

$$\beta(\lambda) \leq \text{Tr}(P(\lambda)|D|^{-p})\|T\| \leq \int_0^\lambda u^{-p} d\alpha(u)\|T\|,$$

while $\alpha(u) = 0$ near 0, and $u^{-p}\alpha(u)$ is bounded by (143). This gives after integrating by parts

$$\int_0^\lambda u^{-p} d\alpha(u) = \lambda^{-p}\alpha(\lambda) + \int_0^\lambda p u^{-p-1}\alpha(u) du \leq c_2(1 + p \log \lambda) + c'.$$

Moreover for $v > 1$ one gets, by the above integration by parts,

$$\beta(v\lambda) - \beta(\lambda) \leq \|T\| \int_\lambda^{v\lambda} u^{-p} d\alpha(u) \leq \|T\|c_2(1 + \log v).$$

One has

$$M(d\beta)(\lambda) = \frac{1}{\log \lambda} \int_1^\lambda d\beta = \frac{1}{\log \lambda} (\beta(\lambda) - \beta(1)). \quad (154)$$

Thus one has constants a and b such that

$$|M(\theta_u(d\beta))(\lambda) - M(d\beta)(\lambda)| \leq (a + b|\log u|)(\log \lambda)^{-1}$$

for any u . Thus since $\tilde{g}(u)$ and $|\log u|\tilde{g}(u)$ are integrable,

$$M\left(\int \tilde{g}(u)\theta_u(d\beta) d^*u\right) - M(d\beta) \int \tilde{g}(u) d^*u \rightarrow 0.$$

Equivalently, using (151), (152), (153) and $\int \tilde{g}(u) d^*u = \int g(u) d^*u$,

$$M(\tilde{h}) - M(d\beta) \int g(u) d^*u \rightarrow 0, \quad \tilde{h}(u) = h(u^{-1}).$$

Now by (154) and (147) one has

$$\text{Lim}_{\lambda^p \rightarrow \omega} M(d\beta)(\lambda) = p \text{Tr}_\omega(T|D|^{-p}).$$

Thus we finally get

$$(p \int g(u) d^*u) \text{Tr}_\omega(T|D|^{-p}) = \text{Lim}_{\lambda^p \rightarrow \omega} M(\tilde{h})(\lambda).$$

The right-hand side is given, by definition, by

$$\text{Lim}_{\lambda^p \rightarrow \omega} M(\tilde{h})(\lambda) = \omega(M(k)(u)), \quad k(u) = M(\tilde{h})(u^{1/p}).$$

Thus we still need to compare $k(u) = M(\tilde{h})(u^{1/p})$ with $k_1(u) = M(\tilde{h}(\lambda^{1/p}))(u)$, but a simple computation shows that $k(u) = k_1(u)$. \square

Corollary 14.10. *Assume that (141) holds. Let $f \in C_c([0, \infty]^+)$ be a positive function. Let $\rho = p \int_0^\infty u^{p-1} f(u) du$. One has, when $\varepsilon \rightarrow 0$,*

$$\liminf \varepsilon^p \operatorname{Tr}(f(\varepsilon|D|)T) \leq \rho \operatorname{Tr}_\omega(T|D|^{-p}). \quad (155)$$

Proof. Let $\delta = \liminf \varepsilon^p \operatorname{Tr}(f(\varepsilon|D|)T)$. Then for any $c < 1$ one has $h(\varepsilon) = \varepsilon^p \operatorname{Tr}(f(\varepsilon|D|)T) \geq c\delta$ for $\varepsilon \leq \varepsilon_c > 0$. It follows that $\operatorname{Lim}_{\varepsilon \rightarrow \omega}^2 h(\varepsilon) \geq c\delta$. Thus by (148) one has $c\delta \leq \rho \operatorname{Tr}_\omega(T|D|^{-p})$ and obtains (155). \square

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