Idempotent states on compact quantum groups and their classification on $U_q(2)$, $SU_q(2)$, and $SO_q(3)$

Uwe Franz*, Adam Skalski, and Reiji Tomatsu

Abstract. Unlike for locally compact groups, idempotent states on locally compact quantum groups do not necessarily arise as Haar states of compact quantum subgroups. We give a simple characterisation of those idempotent states on compact quantum groups that do arise as Haar states on quantum subgroups. We also show that all idempotent states on the quantum groups $U_q(2)$, $SU_q(2)$, and $SO_q(3)$ ($q \in (-1, 0) \cup (0, 1]$) arise in this manner and list the idempotent states on the compact quantum semigroups $U_0(2)$, $SU_0(2)$, and $SO_0(3)$. In the Appendix we provide a short new proof of the coamenability of deformations of classical compact Lie groups based on their representation theory.

Mathematics Subject Classification (2010). 17B37, 43A05, 46L65.
Keywords. Compact quantum group, quantum subgroup, idempotent state, Haar state.

1. Introduction

It is well known that if X is a locally compact topological semigroup, then the space of regular probability measures on X possesses a natural convolution product. Analogously, if A is the algebra of continuous functions on a compact quantum semigroup, i.e., a unital C*-algebra together with a coproduct (i.e., coassociative unital $*$-homomorphism) $\Delta \colon A \to A \otimes A$, then one can consider a natural associative convolution product on the state space of A,

$$\lambda \star \mu = (\lambda \otimes \mu) \circ \Delta, \quad \lambda, \mu \in S(A).$$

It is natural to ask whether one can characterise the states that satisfy the idempotent property

$$\mu \star \mu = \mu.$$

A particular and most important example of an idempotent state is the Haar state on a given compact quantum group in the sense of Woronowicz [Wor98]. More general idempotent states arise naturally in considerations of Cesàro limits of convolution

operators on the algebras of functions on compact quantum groups, cf. [FS08a]. They are also an important ingredient in the construction of quantum hypergroups [CV99] and occur as initial value φ_0 of convolution semigroups $(\varphi_t)_{t \geq 0}$ of states on quantum groups if one relaxes the initial condition $\varphi_0 = \varepsilon$, cf. [FSc00]. In the finite-dimensional case idempotent states are in one-to-one correspondence with quantum pre-subgroups studied in [BBS99]. They may also be used in the study of Hopf images of representations of the algebras of functions on compact quantum groups introduced in [BB10].

For classical compact groups, Kawada and Itô proved that all idempotent measures are induced by Haar measures of compact subgroups, see [KI40], Theorem 3. Later this result was extended to arbitrary locally compact topological groups, see [Hey77] and references therein. In [Pal96] Pal showed that this characterisation does not extend to quantum groups by giving an example of an idempotent state on the Kac–Paljutkin quantum group that cannot arise as the Haar state on a quantum subgroup. In [FS09a] the first two authors began a systematic study of idempotent states on compact quantum groups, later continued in [FS09b]. In particular we exhibited further examples of idempotent states on quantum groups that are not induced by Haar states of quantum subgroups. Such examples are actually quite frequent and may be thought of as the dual manifestation of the existence of non-normal subgroups. We also gave a characterisation of idempotent states on finite quantum groups in terms of quantum sub-hypergroups and established a one-to-one correspondence between idempotent states and quantum pre-subgroups of [BBS99].

In this work we continue the analysis started in [FS09a] and show a simple and easily verifiable characterisation of those idempotent states that arise as Haar states on quantum subgroups, so-called Haar idempotents. This, as in the finite-dimensional case, leads to natural examples of Haar idempotents on group C*-algebras of amenable discrete groups having non-normal subgroups. The second main result of the present paper is the classification of all idempotent states on the compact quantum groups $U_q(2)$, $SU_q(2)$, and $SO_q(3)$ for $q \in (-1, 1] \setminus \{0\}$. Surprisingly, it turns out that they are all induced by quantum subgroups, cf. Theorems 4.7, 5.1, and 5.2. As a byproduct we obtain the classification of quantum subgroups of the afore-mentioned quantum groups. In the case of $SU_q(2)$ and $SO_q(3)$ these have been known due to the work of Podleś [Pod95], but our approach provides a different proof.

For the value $q = 0$, the quantum cancellation properties fail and $U_0(2)$, $SU_0(2)$, and $SO_0(3)$ are no longer compact quantum groups. But they can still be considered as compact quantum semigroups so that, as explained above, their state spaces have natural convolution products. Using this we determine all idempotent states on $U_0(2)$, $SU_0(2)$, and $SO_0(3)$, see Theorems 6.3 and 6.5 and 6.6. It turns out that in spite of the fact that the underlying C*-algebras are isomorphic for all $q \in (-1, 1)$, in the case $q = 0$ there exist additional families of idempotent states which do not appear when $q \neq 0$. Interestingly, for $q = -1$ the Kac algebra $U_{-1}(2)$ admits non-Haar idempotent states. The full classification is however in this case more complicated and requires different techniques, and we refer to future work [FST11].

The detailed plan of the paper is as follows. In Section 2 we list the background results and definitions we need in the rest of the paper. In particular we recall the definitions of the quantum groups being the subject of the paper, discuss their representation theory and present explicit formulas for their Haar states. Although most of the facts presented in this section are well known, the details of the respective representation theories are somewhat scattered through literature (and in the case of $U_q(2)$ even not fully recorded), so we decided to describe them here in detail to facilitate a coherent use of terminology and notation in the remainder of the paper.

Section 3 introduces idempotent states on compact quantum groups, provides the characterisation of those idempotent states that arise as Haar states on quantum subgroups (extending the results for finite quantum groups given in [FS09a]) and briefly discusses the cocommutative situation under the coamenability assumption.

Section 4 contains main technical arguments of the paper and ends with the characterisation of all idempotent states on $U_q(2)$ for $q \in (-1, 1] \setminus \{0\}$.

In Section 5 we show how one can deduce the corresponding statements for $SU_q(2)$ and $SO_q(3)$. In view of Theorem 4.1 of [FS09b] this allows us immediately to conclude which of the coidalgebras of $C(SU_q(2))$ studied by the third named author in [Tom08] are *expected*, i.e., are images of a Haar state preserving conditional expectation (we refer to both of these papers for precise terminology).

Section 6 contains the classification of the idempotent states on compact quantum semigroups $U_0(2)$, $SU_0(2)$, and $SO_0(3)$.

Finally in Section 7, we use the result of [FS09a] showing that idempotent states on compact quantum groups are group-like projections in the dual quantum group, giving rise to algebraic quantum hypergroups by the construction presented in [LVD07], and discuss the quantum hypergroups associated to these group-like projections for the case of $SU_q(2)$. The Appendix contains a new short proof of coamenability of deformations of classical compact Lie groups based on the representation theory developed in [KS98].

2. Notation and preliminaries

The symbol $\otimes$ will denote the spatial tensor product of C*-algebras, $\odot$ will be reserved for the purely algebraic tensor product. We write $\mathbb{N}_0$ or $\mathbb{Z}_+$ for $\mathbb{N} \cup \{0\}$.

2.1. Compact quantum groups. The notion of compact quantum groups has been introduced in [Wor87a]. Here we adopt a rewording of Definition 2.1 in [Wor98].

Definition 2.1. A C*-*bialgebra* (the algebra of continuous functions on a compact quantum semigroup) is a pair (A, Δ), where A is a unital C*-algebra, $\Delta \colon A \to A \otimes A$ is a unital, $*$-homomorphic map which is coassociative

$$(\Delta \otimes \mathrm{id}_A) \circ \Delta = (\mathrm{id}_A \otimes \Delta) \circ \Delta.$$

If the quantum cancellation properties

$$\overline{\mathrm{Lin}}((1 \otimes \mathsf{A})\Delta(\mathsf{A})) = \overline{\mathrm{Lin}}((\mathsf{A} \otimes 1)\Delta(\mathsf{A})) = \mathsf{A} \otimes \mathsf{A},$$

are satisfied, then the pair (A, Δ) is called the algebra of continuous functions on a *compact quantum group*.

In quantum group theory it is quite common to write $\mathsf{A} = C(\mathbb{G})$, where $\mathbb{G}$ denotes the underlying compact quantum group; note however that the symbol $\mathbb{G}$ is only defined indirectly.

The map Δ is called the *coproduct* of A, it induces the convolution product

$$\lambda \star \mu := (\lambda \otimes \mu) \circ \Delta, \quad \lambda, \mu \in \mathsf{A}^*.$$

If $\mathbb{G}$ is a compact quantum group, then a unitary $U \in \mathrm{M}_n(C(\mathbb{G}))$ is called a *finite-dimensional unitary representation of* $\mathbb{G}$ if for all $i, j = 1, \ldots, n$ we have $\Delta(U_{ij}) = \sum_{k=1}^n U_{ik} \otimes U_{kj}$. It is said to be *irreducible* if the only matrices $T \in \mathrm{M}_n(\mathbb{C})$ with $TU = UT$ are multiples of the identity matrix.

Possibly the most important feature of compact quantum groups is the existence of the dense $*$-subalgebra $\mathcal{A} \subset C(\mathbb{G})$ (the algebra of matrix coefficients of irreducible unitary representations of $\mathbb{G}$), which is in fact a Hopf $*$-algebra, for example $\Delta \colon \mathcal{A} \to \mathcal{A} \odot \mathcal{A}$. This $*$-Hopf algebra is also denoted by $\mathcal{A} = \mathrm{Pol}(\mathbb{G})$, and treated as the analog of polynomial functions of $\mathbb{G}$.

Another fact of crucial importance is given by the following result, Theorem 2.3 of [Wor98].

Theorem 2.2. *Let $\mathbb{G}$ be a compact quantum group and let $\mathsf{A} = C(\mathbb{G})$. There exists a unique state $h \in \mathsf{A}^*$ (called the* Haar state *of $\mathbb{G}$) such that*

$$(h \otimes \mathrm{id}_{\mathsf{A}}) \circ \Delta(a) = (\mathrm{id}_{\mathsf{A}} \otimes h) \circ \Delta(a) = h(a)\, 1$$

for all $a \in \mathsf{A}$.

The algebra $C(\mathbb{G})$ is said to be in *reduced form* if the Haar state h is faithful. If this is not the case, pass to the quotient by the kernel of h. This procedure in particular does not influence the underlying Hopf $*$-algebra $\mathcal{A}$; in fact the reduced object may be viewed as the natural completion of $\mathcal{A}$ in the GNS representation with respect to h (as opposed for example to the universal completion of $\mathcal{A}$, for details see [BMT01]). In general the reduced and universal object need not coincide. This leads to certain technical complications which are not of essential importance in our context (for example, if a discrete group Γ is not amenable, the reduced C*-algebra of Γ is a proper quantum subgroup of the universal C*-algebra of Γ, even though they have 'identical' Haar states). To avoid such difficulties we focus on the class of coamenable compact quantum groups ([BMT01], see the Appendix to this paper for more information), for which the reduced and universal C*-algebraic

completions of $\mathcal{A}$ are naturally isomorphic. All deformations of classical compact Lie groups, in particular the quantum groups $U_q(2)$, $SU_q(2)$ and $SO_q(3)$ we consider in Sections 4 and 5, are known to be coamenable ([Ban99]). Banica's proof is based on the fusion rules appearing in the quantum group-theoretic representation theory; in the Appendix we give an alternative proof based on the C*-algebraic representation theory developed in [KS98].

The following definition was introduced by Podleś in the context of compact matrix pseudogroups (Definition 1.3 of [Pod95]).

Definition 2.3. A compact quantum group $\mathbb{G}'$ is said to be a *quantum subgroup* of a compact quantum group $\mathbb{G}$ if there exists a surjective compact quantum group morphism $j : C(\mathbb{G}) \to C(\mathbb{G}')$, i.e., a surjective unital $*$-homomorphism $j : C(\mathbb{G}) \to C(\mathbb{G}')$ such that

$$\Delta_{C(\mathbb{G}')} \circ j = (j \otimes j) \circ \Delta_{C(\mathbb{G})}.$$

Strictly speaking, one should consider the pairs $(\mathbb{G}', j)$ since $C(\mathbb{G})$ can contain several copies of $C(\mathbb{G}')$ with different morphisms (in the same way as a classical group can have different but isomorphic subgroups). We will not distinguish between $(\mathbb{G}_1, j_1)$ and $(\mathbb{G}_2, j_2)$ if there exists an isomorphism of quantum groups $\Theta : C(\mathbb{G}_1) \to C(\mathbb{G}_2)$ such that $\Theta \circ j_1 = j_2$. Note that such isomorphic pairs induce the same idempotent state $\phi = h_{\mathbb{G}_1} \circ j_1 = h_{\mathbb{G}_2} \circ j_2$, since uniqueness of the Haar states implies $h_{\mathbb{G}_1} = h_{\mathbb{G}_2} \circ \Theta$.

Any coamenable compact quantum group contains itself and the trivial compact quantum group $\{e\}$ as quantum subgroups; further these two quantum subgroups will be called trivial. If G is a compact group and a compact quantum group $\mathbb{G}$ contains G as a quantum subgroup, via a morphism $j : C(\mathbb{G}) \to C(G)$, we will simply say that G is a subgroup of $\mathbb{G}$.

Definition 2.4. A quantum subgroup $(\mathbb{G}', j)$ of a compact quantum group $\mathbb{G}$ is called *normal* if the images of the conditional expectations

$$E_{\mathbb{G}/\mathbb{G}'} = (\mathrm{id} \otimes (h_{\mathbb{G}'} \circ j)) \circ \Delta,$$
$$E_{\mathbb{G}'\backslash\mathbb{G}} = ((h_{\mathbb{G}'} \circ j) \otimes \mathrm{id}) \circ \Delta,$$

coincide, cf. [Wan08], Proposition 2.1 and Definition 2.2. In this case the quotient $C(\mathbb{G}/\mathbb{G}') := E_{\mathbb{G}/\mathbb{G}'}(C(\mathbb{G}))$ has a natural structure of the algebra of continuous functions on a compact quantum group.

2.2. q-numbers. Let $q \neq 1$. We will use the following notation for q-numbers,

$$(x; q)_n = (1 - x)(1 - qx) \dots (1 - q^{n-1}x),$$
$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{(q; q)_n}{(q; q)_k (q; q)_{n-k}},$$

for $n \in \mathbb{Z}_+$, $0 \leq k \leq n$, $x \in \mathbb{R}$.

2.3. The Woronowicz quantum group $SU_q(2)$ [Wor87a], [Wor87b]. For $q \in \mathbb{R}$, we denote by $\mathrm{Pol}(SU_q(2))$ the $*$-bialgebra generated by α and γ, with the relations

$$\alpha\gamma = q\gamma\alpha, \quad \alpha\gamma^* = q\gamma^*\alpha, \quad \gamma^*\gamma = \gamma\gamma^*,$$
$$\gamma^*\gamma + \alpha^*\alpha = 1, \quad \alpha\alpha^* - \alpha^*\alpha = (1-q^2)\gamma^*\gamma,$$

and comultiplication and counit defined by setting

$$\Delta(\alpha) = \alpha \otimes \alpha - q\gamma^* \otimes \gamma, \quad \Delta(\gamma) = \gamma \otimes \alpha + \alpha^* \otimes \gamma,$$

and $\varepsilon(\alpha) = 1$, $\varepsilon(\gamma) = 0$. For $q \neq 0$, $\mathrm{Pol}(SU_q(2))$ admits an antipode. On the generators, it acts as

$$S(\alpha) = \alpha^* \quad \text{and} \quad S(\gamma) = -q\gamma.$$

Denote by $C(SU_q(2))$ the universal enveloping C*-algebra of $\mathrm{Pol}(SU_q(2))$. Then Δ extends uniquely to a non-degenerate coassociative homomorphism $\Delta \colon C(SU_q(2)) \to C(SU_q(2)) \otimes C(SU_q(2))$, and the pair $(C(SU_q(2)), \Delta)$ is a C*-bialgebra. For $q \neq 0$, $SU_q(2)$ is even a compact quantum group.

Observe that the mapping $\alpha \mapsto \alpha^*$ and $\gamma \mapsto q\gamma^*$ induces isomorphisms $\mathrm{Pol}(SU_{1/q}(2)) \to \mathrm{Pol}(SU_q(2))$, $C(SU_{1/q}(2)) \to C(SU_q(2))$. Therefore it is sufficient to consider $q \in [-1, 1]$.

2.3.1. Representation theory of $C(SU_q(2))$ [Wor87b], [VS88]. The C*-algebra $C(SU_q(2))$ has two families of irreducible representations. The first family consists of the one-dimensional representations ρ_θ, $0 \leq \theta < 2\pi$, given by

$$\rho_\theta(\alpha) = e^{i\theta}, \quad \rho_\theta(\gamma) = 0.$$

The other family consists of infinite-dimensional representations π_θ, $0 \leq \theta < 2\pi$, acting on a separable Hilbert space $\mathfrak{h}$ by

$$\pi_\theta(\alpha)e_n = \begin{cases} \sqrt{1-q^{2n}}\, e_{n-1} & \text{if } n > 0, \\ 0 & \text{if } n = 0, \end{cases}$$
$$\pi_\theta(\gamma)e_n = e^{i\theta}q^n e_n,$$

where $\{e_n \mid n \in \mathbb{N}_0\}$ is an orthonormal basis for $\mathfrak{h}$.

This list is complete, i.e., any irreducible representation of $C(SU_q(2))$ is unitarily equivalent to a representation in one of the two families above (Theorem 3.2 of [VS88]). It is known that the C*-algebra $C(SU_q(2))$ is of type I, therefore any representation can be written as a direct integral over the irreducible representations given above.

2.3.2. Quantum subgroups of $SU_q(2)$. Let Pol(U(1)) denote the *-algebra generated by one unitary w, $ww^* = w^*w = 1$. With $\Delta(w) = w \otimes w$, $\varepsilon(w) = 1$, $S(w) = w^*$, this becomes a *-Hopf algebra. Its enveloping C*-algebra $C(U(1))$ is the algebra of continuous functions on a compact group U(1). Note that the *-algebra homomorphism $\mathrm{Pol}(SU_q(2)) \to \mathrm{Pol}(U(1))$ defined by $\alpha \mapsto w$, $\gamma \mapsto 0$ extends to a surjective compact quantum group morphism $j \colon C(SU_q(2)) \to C(U(1))$, i.e., U(1) is a quantum subgroup of $SU_q(2)$. Furthermore, Podleś ([Pod95], Theorem 2.1) showed that, for $q \in (-1,0) \cup (0,1)$, U(1) and its closed subgroups are the only non-trivial quantum subgroups of $SU_q(2)$. This will also follow from the results in Section 4.

There exists a second morphism $j' \colon C(SU_q(2)) \to C(U(1))$, determined by $j' \colon \alpha \mapsto w^*$, $\gamma \mapsto 0$. But we do not need to distinguish between the pairs $(C(U(1)), j)$ and $(C(U(1)), j')$ since they are related by the automorphism Θ of $C(U(1))$ with $\Theta(w^k) = (w^*)^k$, $\Theta((w^*)^k) = w^k$, $k \in \mathbb{N}_0$.

2.3.3. Representations of $SU_q(2)$. Let $q \in (-1,0) \cup (0,1)$. We recall a few basic facts about the representations of $SU_q(2)$, for more details see [Wor87a], [Wor87b], [VS88], [MMN$^+$88], [Koo89]. For each non-negative half-integer $s \in \frac{1}{2}\mathbb{Z}_+$ there exists a $(2s+1)$-dimensional irreducible unitary representation $u^{(s)} = (u^{(s)}_{k\ell})_{-s \le k, \ell \le s}$ of $SU_q(2)$ which is unique up to unitary equivalence. Note that the indices k, ℓ run over the set $\{-s, -s+1, \ldots, s-1, s\}$, they are integers if $s \in \mathbb{Z}_+$ is integer, and half-integers if $s \in (\frac{1}{2}\mathbb{Z}_+) \backslash \mathbb{Z}_+$ is a half-integer. This convention is used throughout the paper.

The matrix coefficients $u^{(s)}_{k\ell}$, $s \in \frac{1}{2}\mathbb{Z}_+$, $-s \le k, \ell \le s$, span $\mathrm{Pol}(SU_q(2))$ and are linearly dense in $C(SU_q(2))$, which will be sufficient for our calculations. We have

$$u^{(0)} = (1), \qquad u^{(1/2)} = \begin{pmatrix} \alpha & -q\gamma^* \\ \gamma & \alpha^* \end{pmatrix},$$

$$u^{(1)} = \begin{pmatrix} \alpha^2 & -q\sqrt{1+q^2}\gamma^*\alpha & q^2(\gamma^*)^2 \\ \sqrt{1+q^2}\gamma\alpha & 1-(1+q^2)\gamma^*\gamma & -q\sqrt{1+q^2}\alpha^*\gamma^* \\ \gamma^2 & \sqrt{1+q^2}\alpha^*\gamma & (\alpha^*)^2 \end{pmatrix},$$

and the matrix coefficients of the higher-dimensional representations are of the form

$$u^{(s)}_{k\ell} = \begin{cases} \alpha^{-k-\ell} p^{(s)}_{k\ell} \gamma^{k-\ell} & \text{for } k+\ell \le 0, \ k \ge \ell, \\ \alpha^{-k-\ell} p^{(s)}_{k\ell} (\gamma^*)^{\ell-k} & \text{for } k+\ell \le 0, \ k \le \ell, \\ (\alpha^*)^{k+\ell} p^{(s)}_{k\ell} \gamma^{k-\ell} & \text{for } k+\ell \ge 0, \ k \ge \ell, \\ (\alpha^*)^{k+\ell} p^{(s)}_{k\ell} (\gamma^*)^{\ell-k} & \text{for } k+\ell \ge 0, \ k \le \ell, \end{cases}$$

where $p^{(s)}_{k\ell}$ is a polynomial in $\gamma^*\gamma$.

In particular, for s an integer, $u^{(s)}_{00} = p_s(\gamma^*\gamma; 1, 1; q^2)$ is the little q-Legendre

polynomial and $u_{s0}^{(s)} = \sqrt{\begin{bmatrix} 2s \\ s \end{bmatrix}_{q^2}} (\alpha^*)^s \gamma^s$.

If we define a $\mathbb{Z}$-grading on $\mathrm{Pol}(\mathrm{SU}_q(2))$ by

$$\deg \alpha = \deg \alpha^* = 0, \quad \deg \gamma = 1, \quad \deg \gamma^* = -1,$$

then we have

$$\deg u_{k\ell}^{(s)} = k - \ell.$$

With this grading, it is straight-forward to verify the formula

$$S^2(a) = q^{2 \deg a} a$$

for the square of the antipode on homogeneous elements.

2.3.4. The Haar state of $\mathrm{SU}_q(2)$. As stated in Theorem 2.2, there exists a unique invariant state h on the compact quantum group $\mathrm{SU}_q(2)$, called the *Haar state*. It was first computed by Woronowicz in [Wor87a]; it is the identity on the one-dimensional representation, and vanishes on the matrix coefficients of all other irreducible representations, i.e.,

$$h(u_{k\ell}^{(s)}) = \delta_{0s}$$

for $s \in \frac{1}{2}\mathbb{Z}_+$, $-s \le k, \ell \le s$. On polynomials $p(\gamma^*\gamma) \in \mathbb{C}[\gamma^*\gamma]$, it is equal to Jackson's q-integral ([Koo89]),

$$h(p(\gamma^*\gamma)) = (1 - q^2) \sum_{k=0}^{\infty} q^{2k} p(q^{2k}) =: \int_0^1 p(x) \, \mathrm{d}_{q^2} x.$$

2.4. The compact quantum group $\mathrm{SO}_q(3)$. A compact quantum group $\mathbb{G}'$ is called a *quotient group* of $\mathbb{G}$ if there exists an injective morphism of quantum groups $j : C(\mathbb{G}') \to C(\mathbb{G})$. The compact quantum group $\mathrm{SO}_q(3)$ can be defined as the quotient of $\mathrm{SU}_q(2)$ by the quantum subgroup $\mathbb{Z}_2$, cf. [Pod95]. $\mathrm{Pol}(\mathrm{SO}_q(3))$ is the subalgebra of $\mathrm{Pol}(\mathrm{SU}_q(2))$ spanned by the matrix coefficients of the unitary irreducible representations of $\mathrm{SU}_q(2)$ with integer label, and $C(\mathrm{SO}_q(3))$ is its norm closure. The Haar state on $\mathrm{SO}_q(3)$ is simply the restriction of the Haar state on $\mathrm{SU}_q(2)$.

Podleś [Pod95] showed that $\mathrm{SO}_q(3)$ and $\mathrm{SO}_{-q}(3)$ are isomorphic.

2.4.1. The semigroup case $q = 0$. We define $\mathrm{Pol}(\mathrm{SO}_0(3))$ to be the unital $*$-subalgebra of $\mathrm{Pol}(\mathrm{SU}_0(2))$ generated by $\alpha^2, \gamma^2, \gamma\alpha, \gamma^*\alpha$, and $\gamma^*\gamma$, i.e.,

$$\mathrm{Pol}(\mathrm{SO}_0(3)) = \mathrm{span}\{(\alpha^*)^r \gamma^k \alpha^s, (\alpha^*)^r (\gamma^*)^k \alpha^s \mid r, k, s \in \mathbb{Z}_+ \text{ with } r + k + s \text{ even}\}.$$

Since

$$\Delta((\alpha^*)^r \gamma^k \alpha^s) = \sum_{\kappa=0}^{k} (\alpha^*)^{r+k-\kappa} \gamma^\kappa \alpha^s \otimes (\alpha^*)^r \gamma^{k-\kappa} \alpha^{s+\kappa},$$

$$\Delta((\alpha^*)^r (\gamma^*)^k \alpha^s) = \sum_{\kappa=0}^{k} (\alpha^*)^r (\gamma^*)^\kappa \alpha^{s+k-\kappa} \otimes (\alpha^*)^{r+\kappa} (\gamma^*)^{k-\kappa} \alpha^s,$$

this is a sub $*$-Hopf algebra in $\mathrm{Pol}(SU_0(2))$. The C*-bialgebra $C(SO_0(3))$ is then defined as the norm closure of $\mathrm{Pol}(SO_0(3))$ in $C(SU_0(2))$. Note that C*-algebras $C(SU_q(2))$ are isomorphic for all $q \in (-1, 1)$ [Wor87b], Theorem A2.2.

2.4.2. The conditional expectation $E: C(SU_q(2)) \to C(SO_q(3))$. Looking at the defining relations of $SU_q(2)$, it is clear that $\vartheta: \alpha \to -\alpha$, $\gamma \to -\gamma$ extends to a unique $*$-algebra automorphism of $C(SU_q(2))$. Therefore $E = \frac{1}{2}(\mathrm{id} + \vartheta)$ defines a completely positive unital map from $C(SU_q(2))$ to itself. If $\phi_2 = h_{\mathbb{Z}_2} \circ j$ denotes the idempotent state on $SU_q(2)$ induced by the Haar measure of $\mathbb{Z}_2$, with $j: SU_q(2) \to C(\mathbb{Z}_2)$ the corresponding surjective morphism (see Definitions 3.1 and 3.2), then we can write E also as

$$E = (\mathrm{id} \otimes \phi_2) \circ \Delta.$$

Checking

$$E(u_{k\ell}^{(s)}) = \begin{cases} u_{k\ell}^{(s)} & \text{if } s \in \mathbb{Z}_+, \\ 0 & \text{else,} \end{cases}$$

we can show that the range of E is equal to $C(SO_q(3))$. Furthermore, E satisfies

$$\Delta \circ E = (\mathrm{id} \otimes E) \circ \Delta = (E \otimes \mathrm{id}) \circ \Delta = (E \otimes E) \circ \Delta.$$

2.4.3. Quantum subgroups of $SO_q(3)$. Here the restriction of the morphism $j: C(SU_q(2)) \to C(U(1))$ to $C(SO_q(3))$ is no longer surjective, its range is equal to the subalgebra $\{f \in C(U(1)) \mid f(z) = f(-z) \text{ for all } z \in U(1)\} = C(U(1)/\mathbb{Z}_2)$. Since $U(1)/\mathbb{Z}_2 \cong U(1)$, we see that $SO_q(3)$ contains $U(1) \cong SO(2)$ and its closed subgroups as quantum subgroups. Podleś [Pod95] showed that these are the only nontrivial quantum subgroups of $SO_q(3)$. Again this can be deduced from the results of Section 5.

2.5. The compact quantum group $U_q(2)$ [Koe91], [Wys04], [ZZ05]. Let $q \in \mathbb{R}$. Then $\mathrm{Pol}(U_q(2))$ is defined as the $*$-bialgebra generated by a, c, and v, with the relations

$$av = va, \quad cv = vc, \quad cc^* = c^*c,$$
$$ac = qca, \quad ac^* = qc^*a, \quad vv^* = v^*v = 1,$$
$$aa^* + q^2cc^* = 1 = a^*a + c^*c,$$
$$\Delta(a) = a \otimes a - qc^*v^* \otimes c, \quad \Delta(c) = c \otimes a + a \otimes c, \quad \Delta(v) = v \otimes v,$$

$$\varepsilon(a) = \varepsilon(v) = 1, \quad \varepsilon(c) = 0.$$

For $q \neq 0$, $\mathrm{Pol}(\mathrm{U}_q(2))$ admits an antipode, given by

$$S(a) = a^*, \quad S(v) = v^*, \quad S(c) = -qcv,$$

on the generators.

Denote the universal enveloping C*-algebra of $\mathrm{Pol}(\mathrm{U}_q(2))$ by $C(\mathrm{U}_q(2))$. Then $\Delta \colon \mathrm{Pol}(\mathrm{U}_q(2)) \to \mathrm{Pol}(\mathrm{U}_q(2)) \odot \mathrm{Pol}(\mathrm{U}_q(2))$ extends uniquely to a non-degenerate coassociative homomorphism $\Delta \colon C(\mathrm{U}_q(2)) \to C(\mathrm{U}_q(2)) \otimes C(\mathrm{U}_q(2))$, and the pair $(C(\mathrm{U}_q(2)), \Delta)$ is a C*-bialgebra. For $q \neq 0$, $\mathrm{U}_q(2)$ is even a compact quantum group. It is again sufficient to consider $q \in [-1, 1]$ since $\mathrm{U}_q(2)$ and $\mathrm{U}_{1/q}(2)$ are isomorphic.

2.5.1. Quantum subgroups of $\mathrm{U}_q(2)$. The mapping $a \mapsto \alpha$, $c \mapsto \gamma$, $v \mapsto 1$ extends to a surjective compact quantum group morphism $C(\mathrm{U}_q(2)) \to C(\mathrm{SU}_q(2))$ and shows that $\mathrm{SU}_q(2)$ is a quantum subgroup of $\mathrm{U}_q(2)$. The C*-algebra $C(\mathrm{U}_q(2))$ is isomorphic to the tensor product of $C(\mathrm{SU}_q(2))$ and $C(\mathrm{U}(1))$. Moreover the compact quantum group, $\mathrm{U}_q(2)$ is equal to a twisted product of $\mathrm{SU}_q(2)$ and $\mathrm{U}(1)$, cf. [Wys04], written as $\mathrm{U}_q(2) = \mathrm{SU}_q(2) \ltimes_\sigma \mathrm{U}(1)$.

Another quantum subgroup of $\mathrm{U}_q(2)$ is the two-dimensional torus. Denote by $\mathrm{Pol}(\mathbb{T}^2)$ the *-Hopf algebra generated by two commuting unitaries, i.e., by w_1, w_2 with the relations

$$w_1 w_1^* = 1 = w_1^* w_1, \quad w_2 w_2^* = 1 = w_2^* w_2, \quad w_1 w_2 = w_2 w_1, \quad w_1 w_2^* = w_2^* w_1,$$
$$\Delta(w_1) = w_1 \otimes w_1, \quad \Delta(w_2) = w_2 \otimes w_2, \quad \varepsilon(w_1) = \varepsilon(w_2) = 1.$$

Then $C(\mathbb{T}^2)$ is the C*-enveloping algebra of $\mathrm{Pol}(\mathbb{T}^2)$. Then the mapping $a \mapsto w_1$, $c \mapsto 0$, $v \mapsto w_2$ extends to a unique surjective compact quantum group morphism $C(\mathrm{U}_q(2)) \to C(\mathbb{T}^2)$.

We will see that the twisted products $\mathrm{SU}_q(2) \ltimes_\sigma \mathbb{Z}_n$, $n \in \mathbb{N}$, the torus $\mathbb{T}^2$, and its closed subgroups are the only non-trivial quantum subgroups of $\mathrm{U}_q(2)$, cf. Corollary 4.8.

2.5.2. Representations of $\mathrm{U}_q(2)$. Unitary irreducible representations of $\mathrm{U}_q(2)$ can be obtained as tensor product of unitary irreducible representations of $\mathrm{SU}_q(2)$ with representations of $\mathrm{U}(1)$, cf. [Wys04]. In this way one obtains the following family of unitary irreducible representations of $\mathrm{U}_q(2)$,

$$v^{(s,p)} = (u_{k\ell}^{(s)} v^{p+s+\ell})_{-s \leq k, \ell \leq s}$$

for $p \in \mathbb{Z}$, $s \in \frac{1}{2}\mathbb{Z}_+$. The matrix coefficients of these representations clearly span $\mathrm{Pol}(\mathrm{U}_q(2))$. Therefore they are dense in $C(\mathrm{U}_q(2))$ and will be sufficient for the calculations in this paper.

Assume that $q \neq 0$. We want to compute the action of the square of the antipode on the matrix coefficients of the unitary irreducible representations defined above. Since we have $S^2(a) = a$, $S^2(c) = q^2c$, and $S^2(v) = v$, we get

$$S^2(u_{k\ell}^{(s)} v^{p+s+\ell}) = q^{2(k-\ell)} u_{k\ell}^{(s)} v^{p+s+\ell} \tag{2.1}$$

with $p \in \mathbb{Z}$, $s \in \frac{1}{2}\mathbb{Z}_+$, and $-s \leq k, \ell \leq s$.

2.5.3. The Haar state of $U_q(2)$. The Haar state h on $U_q(2)$ can be written as a tensor product of the Haar state on $SU_q(2)$ and the Haar state on $U(1)$. It acts on the matrix coefficients of the unitary irreducible representations given above by

$$h(u_{k\ell}^{(s)} v^{p+s+\ell}) = \delta_{0s}\delta_{0p}$$

for $s \in \frac{1}{2}\mathbb{Z}_+$, $-s \leq k, \ell \leq s$, $p \in \mathbb{Z}$.

2.6. Multiplicative domain of a completely positive unital map. The following result by Choi on multiplicative domains will be useful for us.

Theorem 2.5 ([Cho74], Theorem 3.1, [Pau02], Theorem 3.18). *Let $T : A \to B$ be a completely positive unital linear map between two C*-algebras A and B. Set*

$$D_T = \{a \in A \mid T(aa^*) = T(a)T(a^*), \ T(a^*a) = T(a^*)T(a)\}.$$

Then we have

$$T(ab) = T(a)T(b) \quad and \quad T(ba) = T(b)T(a)$$

for all $a \in D_T$ and $b \in A$.

3. Idempotent states on compact quantum groups

In this section we formally introduce the notion of idempotent states on a C*-bi-algebra, provide a characterisation of those idempotent states on compact quantum groups which arise as Haar states on quantum subgroups and discuss commutative and cocommutative cases.

Definition 3.1. Let (A, Δ) be a C*-bialgebra. A state $\phi \in A^*$ is called an *idempotent state* if

$$(\phi \otimes \phi) \circ \Delta = \phi,$$

i.e., if it is idempotent for the convolution product.

If $A = C(\mathbb{G})$ for a compact quantum group $\mathbb{G}$, we will also simply say that ϕ as above is an idempotent state on $\mathbb{G}$.

Definition 3.2. A state $\phi \in C(\mathbb{G})^*$ is called a *Haar state on a quantum subgroup of* $\mathbb{G}$ (or a *Haar idempotent*) if there exists a quantum subgroup $(\mathbb{G}', j)$ of $\mathbb{G}$ and $\phi = h_{\mathbb{G}'} \circ j$, where $h_{\mathbb{G}'}$ denotes the Haar state on $\mathbb{G}'$.

It is easy to check that each Haar state on a quantum subgroup of $\mathbb{G}$ is idempotent. It follows from the example of Pal in [Pal96] and our work in [FS09a] that not every idempotent state is a Haar idempotent. We have the following simple characterisation, extending Theorem 4.5 of [FS09a].

Theorem 3.3. *Let* $\mathbb{G}$ *be a compact quantum group, let* $\phi \in C(\mathbb{G})^*$ *be an idempotent state and let* $N_\phi = \{a \in C(\mathbb{G}) : \phi(a^*a) = 0\}$ *denote the null space of* ϕ. *Then* ϕ *is a Haar idempotent if and only if* N_ϕ *is a two-sided* (*equivalently, self-adjoint*) *ideal.*

Proof. It is an easy consequence of the Cauchy–Schwarz inequality that N_ϕ is a left ideal; thus it is a two-sided ideal if and only if it is self-adjoint.

Write $\mathsf{A} = C(\mathbb{G})$. Suppose first that ϕ is a Haar idempotent, i.e., there exists a compact quantum group $\mathbb{G}'$ and a surjective compact quantum group morphism $j : \mathsf{A} \to C(\mathbb{G}')$ such that $\phi = h_{\mathbb{G}'} \circ j$. Recall that we assumed $h_{\mathbb{G}'}$ to be faithful, so that $N_\phi = \{a \in \mathsf{A} \mid j(a^*a) = 0\} = \{a \in \mathsf{A} \mid j(a) = 0\}$, which is obviously self-adjoint.

Suppose then that N_ϕ is a two-sided self-adjoint ideal. Let $\mathsf{B} := \mathsf{A}/N_\phi$ and let $\pi_\phi : \mathsf{A} \to \mathsf{B}$ denote the canonical quotient map. We want to define the coproduct on B by the formula

$$\Delta_\mathsf{B} \circ \pi_\phi(a) = (\pi_\phi \otimes \pi_\phi) \circ \Delta(a) \in \mathsf{B} \otimes \mathsf{B}. \tag{3.1}$$

We need to check that it is well defined: to this end we employ a slightly modified idea from the proof of Theorem 2.1 of [BMT01]. A standard use of Cauchy–Schwarz inequality implies that $\phi|_{N_\phi} = 0$, so that there exists a faithful state $\psi \in \mathsf{B}^*$ such that $\psi \circ \pi_\phi = \phi$. Faithfulness of ψ implies that also the map $\mathrm{id}_\mathsf{B} \otimes \psi : \mathsf{B} \otimes \mathsf{B} \to \mathsf{B}$ is faithful (note that here faithfulness of a positive map T is understood in the usual sense, namely $Ta = 0$ and $a \geq 0$ imply $a = 0$) and thus also $\psi \otimes \psi \in (\mathsf{B} \otimes \mathsf{B})^*$ is faithful. Suppose then that $a \in N_\phi$. We have then

$$0 = \phi(a^*a) = (\phi \otimes \phi) \circ \Delta(a^*a) = (\psi \otimes \psi) \circ (\pi_\phi \otimes \pi_\phi)(\Delta(a^*a)),$$

so $(\pi_\phi \otimes \pi_\phi)\Delta(a^*a) = 0$ as well. The last statement implies that $(\pi_\phi \otimes \pi_\phi)\Delta(a) = 0$ and the validity of the definition given in the formula (3.1) is established. The fact that Δ_B is a coassociative unital $*$-homomorphism follows immediately from the analogous properties of Δ; similarly the cancellation properties of B follow from obvious equalities of the type

$$(\mathsf{B} \otimes 1_\mathsf{B})\Delta_\mathsf{B}(\mathsf{B}) = (\pi_\phi \otimes \pi_\phi)((\mathsf{A} \otimes 1_\mathsf{A})\Delta(\mathsf{A}))$$

and the cancellation properties of A. Thus $(\mathsf{B}, \Delta_\mathsf{B})$ is the algebra of continuous functions on a compact quantum group $\mathbb{G}'$ and it remains to check that ψ defined

above is actually the invariant state on B. This is however an immediate consequence of the observation

$$(\psi \otimes \psi) \circ \Delta_B \circ \pi_\phi = (\phi \otimes \phi) \circ \Delta = \phi = \psi \circ \pi_\phi,$$

so that ψ is an idempotent state and, as it is faithful, it has to coincide with the Haar state of $\mathbb{G}'$ ([Wor98]). $\qquad\square$

Note that in fact the theorem remains valid without the assumption of faithfulness of $h_{\mathbb{G}'}$. The proof of the 'only if' part remains the same, and the other implication follows from the modular properties of Haar states on not necessarily coamenable compact quantum groups implying that their null spaces are always self-adjoint.

The following proposition proved in [FS09a], Section 3, will be useful for the classification of idempotent states in the next two sections.

Proposition 3.4. *Let $\phi \in C(\mathbb{G})^*$ be an idempotent state. Then ϕ is invariant under the antipode, in the sense that $\phi(a) = \phi \circ S(a)$ for all $a \in \mathrm{Pol}(\mathbb{G})$.*

Note that the states invariant under the antipode are automatically invariant under the *scaling automorphism group* $\{\tau_t \mid t \in \mathbb{R}\}$ discussed in [Wor98]. This is the content of the next proposition.

Proposition 3.5. *Let $\omega \in C(\mathbb{G})^*$ satisfy the condition $\omega|_{\mathrm{Pol}(\mathbb{G})} = \omega \circ S|_{\mathrm{Pol}(\mathbb{G})}$. Then*

$$\omega \circ \tau_t = \omega, \quad t \in \mathbb{R}.$$

Proof. The assumed invariance of ω under the antipode is equivalent to the equality $\omega \circ \tau_{-i}|_{\mathrm{Pol}(\mathbb{G})} = \omega|_{\mathrm{Pol}(\mathbb{G})}$. The idea of the rest of the proof is based on the use of Woronowicz characters, as in the proof of Lemma 2.9 (2) in [Tom07]. Let $v \in B(\mathsf{H}_v) \odot \mathrm{Pol}(\mathbb{G})$ be an irreducible unitary representation of $\mathbb{G}$. We have

$$T_v := (\mathrm{id}_{B(\mathsf{H}_v)} \otimes \omega)(v) = (\mathrm{id}_{B(\mathsf{H}_v)} \otimes \omega \circ \tau_{-i})(v) = F_v(\mathrm{id}_{B(\mathsf{H}_v)} \otimes \omega)(v)(F_v)^{-1},$$

where $F_v = (\mathrm{id}_{B(\mathsf{H}_v)} \otimes f_1)(v)$ and f_1 is the Woronowicz character on $\mathrm{Pol}(\mathbb{G})$. The last formula implies that T_v commutes with F_v. Since F_v is a (strictly) positive operator on a finite-dimensional Hilbert space, we can consider the unitary group it generates and deduce immediately that for each $t \in \mathbb{R}$ we have $F_v^{it} T_v = T_v F_v^{it}$. But this implies that

$$(\mathrm{id}_{B(\mathsf{H}_v)} \otimes \omega)(v) = F_v^{it}(\mathrm{id}_{B(\mathsf{H}_v)} \otimes \omega)(v) F_v^{-it} = (\mathrm{id}_{B(\mathsf{H}_v)} \otimes \omega \circ \tau_t)(v).$$

The fact that $\mathrm{Pol}(\mathbb{G})$ is a linear span of the coefficients of the irreducible unitary representations of $\mathbb{G}$ shows that $\omega(a) = \omega(\tau_t(a))$ for each $a \in \mathrm{Pol}(\mathbb{G})$; density of $\mathrm{Pol}(\mathbb{G})$ in $C(\mathbb{G})$ ends the proof. $\qquad\square$

Corollary 3.6. *Let $\phi \in C(\mathbb{G})^*$ be an idempotent state. Then ϕ is preserved by the scaling automorphism group and the conditional expectation $\mathrm{E}_\phi := (\phi \otimes \mathrm{id}_{C(\mathbb{G})}) \circ \Delta$ associated to ϕ commutes with the modular group of the Haar state:*

$$\mathrm{E}_\phi \circ \sigma_t = \sigma_t \circ \mathrm{E}_\phi, \quad t \in \mathbb{R}.$$

Proof. It is an immediate consequence of the last lemma and the commutation relation $\Delta \circ \sigma_t = (\sigma_t \otimes \tau_{-t}) \circ \Delta$ $(t \in \mathbb{R})$. The fact that E_ϕ is a conditional expectation was established in [FS09b]. $\qquad\square$

3.1. Idempotent states on cocommutative compact quantum groups. Suppose now that a compact quantum group $\mathbb{G}$ is cocommutative, i.e., $\Delta = \tau \circ \Delta$, where $\tau \colon C(\mathbb{G}) \otimes C(\mathbb{G}) \to C(\mathbb{G}) \otimes C(\mathbb{G})$ denotes the usual tensor flip. It is easy to deduce from the general theory of duality for quantum groups ([KV00]) that $C(\mathbb{G})$ is isomorphic to the C*-algebra of a (classical) discrete group Γ, which should be thought of as the algebra of continuous functions on a quantum group dual $\widehat{\Gamma}$. Note that the notation $C^*(\Gamma) \approx C(\widehat{\Gamma})$, which can be considered as a definition of the compact quantum group $\widehat{\Gamma}$, is compatible with the usual Pontryagin duality for locally compact abelian groups. For the reasons mentioned in the previous section, in general, we need to distinguish between the reduced and the universal version of $C(\widehat{\Gamma})$; thus we restrict our attention to amenable Γ. The following generalises Theorem 6.2 of [FS09a] to the infinite-dimensional context.

Theorem 3.7. *Let Γ be an amenable discrete group and $\mathsf{A} = C^*(\Gamma) \approx C(\widehat{\Gamma})$. There is a one-to-one correspondence between idempotent states on A and subgroups of Γ. An idempotent state $\phi \in \mathsf{A}^*$ is a Haar idempotent if and only if the corresponding subgroup of Γ is normal.*

Proof. The dual of A may be identified with the Fourier–Stieltjes algebra $B(\Gamma)$. The convolution of functionals in A^* then corresponds to the pointwise multiplication of functions in $B(\Gamma)$, and $\phi \in B(\Gamma)$ corresponds to a positive (respectively, unital) functional on A if and only if it is positive definite (respectively, $\phi(e) = 1$). This implies that $\phi \in B(\Gamma)$ corresponds to an idempotent state if and only if it is an indicator function (of a certain subset $S \subset \Gamma$) which is positive definite. It is a well-known fact that this happens if and only if S is a subgroup of Γ ([HR70], Corollary (32.7) and Example (34.3 a)). It remains to prove that if S is a subgroup of Γ then $\chi_S \in B(\Gamma)$ is a Haar state on a quantum subgroup of $\widehat{\Gamma}$ if and only if S is normal. For the 'if' direction assume that S is a normal subgroup and consider the C*-bialgebra $\mathsf{B} = C^*(\Gamma/S)$ (recall that quotients of amenable groups are amenable). Let $F(\Gamma)$ denote the dense *-subalgebra of A given by the functions $f = \sum_{\gamma \in \Gamma} \alpha_\gamma \lambda_\gamma$ ($\alpha_\gamma \in \mathbb{C}$, $\{\gamma \mid \alpha_\gamma \neq 0\}$ finite). Define $j \colon F(\Gamma) \to \mathsf{B}$ by

$$j(f) = \sum_{\gamma \in \Gamma} \alpha_\gamma \lambda_{[\gamma]},$$

where f is as above. So defined j is bounded: note that it is a restriction of the transpose of the map $T : \mathsf{B}^* \to \mathsf{A}^*$ given by

$$T(\phi)(\gamma) = \phi([\gamma]), \quad \phi \in B(\Gamma/S), \; \gamma \in \Gamma.$$

The map T is well defined as it maps positive definite functions into positive definite functions; these generate the relevant Fourier–Stieltjes algebras. Further the closed graph theorem allows to prove that T is bounded; therefore so is $T^* : \mathsf{A}^{**} \to \mathsf{B}^{**}$ and $j = T^*|_{F(\Gamma)}$. It is now easy to check that the extension of j to A is a surjective unital $*$-homomorphism (onto B). Since the invariant state on B is given by

$$h_\mathsf{B}\Big(\sum_{\kappa \in \Gamma/S} \alpha_\kappa \lambda_\kappa \Big) = \alpha_{[e]},$$

there is

$$h_\mathsf{B}(j(f)) = \sum_{\gamma \in S} \alpha_\gamma,$$

so that $h_\mathsf{B} \circ j$ corresponds via the identification of A^* and $B(\Gamma)$ exactly to the characteristic function of S.

The other direction follows exactly as in [FS09a]; we reproduce the argument for the sake of completeness. Suppose that S is a subgroup of Γ which is not normal and let $\gamma_0 \in \Gamma$, $s_0 \in S$ be such that $\gamma_0 s_0 \gamma_0^{-1} \notin S$. Denote by ϕ_S the state on A corresponding to the indicator function of S. Define $f \in \mathsf{A}$ by $f = \lambda_{\gamma_0 s_0} - \lambda_{\gamma_0}$. Then

$$f^* f = 2\lambda_e - \lambda_{s_0^{-1}} - \lambda_{s_0}, \quad f f^* = 2\lambda_e - \lambda_{\gamma_0 s_0^{-1} \gamma_0^{-1}} - \lambda_{\gamma_0 s_0 \gamma_0^{-1}}.$$

This implies that

$$\phi_S(f^* f) = 0, \quad \phi_S(f f^*) = 2,$$

so that $\ker \phi_S$ is not self-adjoint and ϕ_S cannot be a Haar idempotent. $\qquad\square$

Corollary 3.8. *Let $\mathbb{G}$ be a coamenable cocommutative compact quantum group. The following are equivalent:*

(1) *all idempotent states on $\mathbb{G}$ are Haar idempotents;*

(2) *$C(\mathbb{G}) \cong C^*(\Gamma)$ for an amenable Dedekind (i.e., containing no non-normal subgroups) discrete group Γ.*

Finite Dedekind groups have been first studied by Dedekind and then characterised by Baer in [Bae66]. They are also sometimes called *quasi-Hamiltonian* (and *Hamiltonian* groups are non-abelian Dedekind groups).

Remark 3.9. If Γ is a discrete abelian group, then there is a one-to-one correspondence between subgroups of $\widehat{\Gamma}$ (the Pontryagin dual of Γ) and subgroups of Γ: if

N is a subgroup of Γ, then $\widehat{\Gamma/N}$ is a subgroup of $\widehat{\Gamma}$. This is part of the general fact that the subgroup of a dual group corresponds to the dual of a quotient group. Theorem 3.7 and its proof show that the analogous statement remains true for not necessary abelian discrete groups Γ – quantum subgroups of $\widehat{\Gamma}$ are given by $\widehat{\Gamma/N}$, where N is a *normal* subgroup of Γ (normality is needed to make Γ/N a group). It is natural to ask whether one can construct the C*-algebra $C^*(\Gamma/S)$ and equip it with some extra algebraic structure if S is a non-normal subgroup of Γ. A partial answer to this question (for finite S) can be found in [DeVD10].

4. Idempotent states on $U_q(2)$ ($q \in (-1, 0) \cup (0, 1]$)

For $q = 1$, $C(U_q(2))$ is equal to the C*-algebra of continuous functions on the unitary group $U(2)$, and by Kawada and Itô's classical theorem all idempotent states on $C(U(2))$ come from Haar measures of compact subgroups of $U(2)$. In this section we shall classify the idempotent states on $C(U_q(2))$ for $-1 < q < 1$, $q \neq 0$. It turns out that they all correspond to Haar states of quantum subgroups of $U_q(2)$.

We begin with some preparatory lemmas.

Lemma 4.1. *Let $\phi \colon \mathrm{Pol}(U_q(2)) \to \mathbb{C}$ be an idempotent state. Then we have*

$$\phi(u_{k\ell}^{(s)} v^r) = 0 \quad \text{if } k \neq \ell,$$

and $\phi(u_{kk}^{(s)} v^r) \in \{0, 1\}$ for all $s \in \frac{1}{2}\mathbb{Z}_+$, $r \in \mathbb{Z}$, $-s \leq k, \ell \leq s$.

Proof. By Proposition 3.4, we have $\phi \circ S = \phi$ on $\mathrm{Pol}(U_q(2))$. Therefore, by equation (2.1),

$$\phi(u_{k\ell}^{(s)} v^r) = \phi \circ S^2(u_{k\ell}^{(s)} v^r) = q^{2(k-\ell)} \phi(u_{k\ell}^{(s)} v^r),$$

i.e., $\phi(u_{k\ell}^{(s)} v^r) = 0$ for $k \neq \ell$.

Define the matrices $M_{s,p}(\phi) \in M_{2s+1}(\mathbb{C})$ by

$$M_{s,p}(\phi) = (\phi(u_{k\ell}^{(s)} v^{p+s+\ell}))_{-s \leq k, \ell \leq s}.$$

Then $\phi = \phi \star \phi$ is equivalent to

$$M_{s,p}(\phi) = (M_{s,p}(\phi))^2$$

for all $s \in \frac{1}{2}\mathbb{Z}_+$, $p \in \mathbb{Z}$. Since we have already seen that these matrices are diagonal, it follows that the diagonal entries can take only the values 0 and 1. $\square$

Lemma 4.2. *If $\phi \colon \mathrm{Pol}(U_q(2)) \to \mathbb{C}$ is an idempotent state with $\phi(u_{00}^{(1)}) = 1$, then there exists an idempotent state $\tilde{\phi} \colon \mathrm{Pol}(\mathbb{T}^2) \to \mathbb{C}$ such that*

$$\phi = \tilde{\phi} \circ \pi_{\mathbb{T}^2}.$$

Proof. By the previous lemma, $\phi(c) = \phi(c^*) = 0$

We have $u_{00}^{(1)} = 1 - (1 + q^2)c^*c$, therefore $\phi(u_{00}^{(s)}) = 1$ is equivalent to $\phi(c^*c) = 0$. Then, by Theorem 2.5, $c, c^* \in D_\phi$, and ϕ vanishes on expressions of the form uc, cu, uc^*, c^*u with $u \in \mathrm{Pol}(U_q(2))$. But since $vc = cv$ and $u_{k\ell}^{(s)}c = q^{-(k-\ell)}cu_{k\ell}^{(s)}$, $u_{k\ell}^{(s)}c^* = q^{-(k-\ell)}c^*u_{k\ell}^{(s)}$ for $s \in \frac{1}{2}\mathbb{Z}_+$, $-s \le k, \ell \le s$, we can deduce that ϕ vanishes on the ideal

$$\mathcal{I}_c = \{u_1 c u_2, u_1 c^* u_2 \mid u_1, u_2 \in \mathrm{Pol}(U_q(2))\}$$

generated by c and c^*. It follows that we can divide out $\mathcal{I}_c$, i.e., there exists a unique state $\tilde{\phi}$ on $\mathrm{Pol}(U_q(2))/\mathcal{I}_c$ such that the diagram

$$\mathrm{Pol}(U_q(2)) \xrightarrow{\ \pi\ } \mathrm{Pol}(U_q(2))/\mathcal{I}_c$$

commutes.

But $\varepsilon(\mathcal{I}_c) = 0$,

$$\Delta(\mathcal{I}_c) \subseteq \mathcal{I}_c \odot \mathrm{Pol}(U_q(2)) + \mathrm{Pol}(U_q(2)) \odot \mathcal{I}_c,$$

and $S(\mathcal{I}_c) \subseteq \mathcal{I}_c$, i.e., $\mathcal{I}_c$ is also a Hopf $*$-ideal and $\mathrm{Pol}(U_q(2))/\mathcal{I}_c$ is a $*$-Hopf algebra. One easily verifies that $\mathrm{Pol}(U_q(2))/\mathcal{I}_c \cong \mathrm{Pol}(\mathbb{T}^2)$. Since $\pi_{\mathbb{T}^2} : \mathrm{Pol}(U_q(2)) \to \mathrm{Pol}(\mathbb{T}^2)$ is a surjective coalgebra morphism, its dual $\pi_{\mathbb{T}^2}^* : (\mathrm{Pol}(\mathbb{T}^2))^* \ni f \mapsto \pi_{\mathbb{T}^2}^*(f) = f \circ \pi_{\mathbb{T}^2} \in (\mathrm{Pol}(U_q(2)))^*$ is an injective algebra homomorphism, and $\tilde{\phi} = (\pi_{\mathbb{T}^2}^*)^{-1}(\phi)$ is again idempotent. $\qquad\square$

Lemma 4.3. *If $\phi: \mathrm{Pol}(U_q(2)) \to \mathbb{C}$ is an idempotent state with $\phi(u_{00}^{(1)}) = 0$, then $\phi(u_{00}^{(s)}) = 0$ for all integers $s \ge 1$, i.e., we have $\phi|_{\mathbb{C}[c^*c]} = h|_{\mathbb{C}[c^*c]}$.*

Proof. Recall that $u_{00}^{(1)} = 1 - (1 + q^2)c^*c$. Therefore $\phi(u_{00}^{(s)}) = 0$ is equivalent to $\phi(c^*c) = \frac{1}{1+q^2}$.

Assume that there exists an integer $s > 1$ with $\phi(u_{00}^{(s)}) = 1$. Then the Cauchy–Schwarz inequality implies that $\phi((u_{00}^{(s)})^*u_{00}^{(s)}) \ge 1$. The unitarity of the representation $v^{(s,p)}$ gives

$$1 = \sum_{k=-s}^{s} (u_{k0}^{(s)}v^{p+s})^*u_{k0}^{(s)}v^{p+s} = \sum_{k=-s}^{s} (u_{k0}^{(s)})^*u_{k0}^{(s)},$$

therefore

$$\phi\left(\sum_{\substack{k\in\{-s,\ldots,s\} \\ k\neq 0}} (u_{k0}^{(s)})^*u_{k0}^{(s)}\right) \le 0,$$

and in particular $\phi((u_{s0}^{(s)})^* u_{s0}^{(s)}) = 0$. We have

$$
\begin{aligned}
(u_s^{(s)})^* u_{s0}^{(s)} &= \begin{bmatrix} 2s \\ s \end{bmatrix}_{q^2} ((a^*)^s c^s)^* (a^*)^s c^s \\
&= \begin{bmatrix} c2s \\ s \end{bmatrix}_{q^2} (c^*)^s c^s a^s (a^*)^s \\
&= (c^*c)^s (1 - q^2 c^* c) \dots (1 - q^{2s} c^* c).
\end{aligned}
$$

By the representation theory of $C(\mathrm{SU}_q(2))$, c^*c is a positive self-adjoint contraction, with the spectrum $\sigma(c^*c) \subseteq \{q^{2n} \mid n \in \mathbb{Z}_+\} \cup \{0\}$, and therefore the product $(1 - q^2 c^* c) \dots (1 - q^{2s} c^* c)$ defines a strictly positive operator. Therefore $\phi((u_{s,0}^{(s)})^* u_{s,0}^{(s)}) = 0$ implies $\phi((c^*c)^s) = 0$, which is impossible if $\phi(c^*c) = \frac{1}{1+q^2} > 0$.

Therefore $\phi(u_{00}^{(s)}) = 0$ for all integers $s \geq 1$. $\qquad\square$

Lemma 4.4. *Let*

$$
\mathcal{A}_0 = \mathrm{span}\{u_{k\ell}^{(s)} v^r \mid s \in \tfrac{1}{2}\mathbb{Z}_+, \; s > 0, \; -s \leq k, \ell \leq s, \; r \in \mathbb{Z}\},
$$

i.e., $\mathcal{A}_0$ is the subspace spanned by the matrix coefficients of the unitary irreducible representations of dimension at least two.

Assume that $\phi|_{\mathbb{C}[c^*c]} = h|_{\mathbb{C}[c^*c]}$, i.e.,

$$
\phi(u_{00}^{(s)}) = \delta_{0s}
$$

for $s \in \mathbb{Z}_+$.

Then we have $\phi|_{\mathcal{A}_0} = h|_{\mathcal{A}_0}$, i.e.,

$$
\phi(u_{k\ell}^{(s)} v^r) = 0
$$

for all $r \in \mathbb{Z}$, $s \in \tfrac{1}{2}\mathbb{Z}_+$, $s > 0$, and $-s \leq k, \ell \leq s$.

Proof. By Lemma 4.1, we already know that $\phi(u_{k\ell}^{(s)} v^r) \in \{0, 1\}$, and $\phi(u_{k\ell}^{(s)} v^r) = 0$ for $k \neq \ell$. Assume that there exist $s \in \tfrac{1}{2}\mathbb{Z}_+$, $s > 0$, $-s \leq k \leq s$ and $r \in \mathbb{Z}$ such that

$$
\phi(u_{kk}^{(s)} v^r) = 1.
$$

We will show that this is impossible if ϕ agrees with the Haar state h on the subalgebra generated by c^*c.

By the Cauchy–Schwarz inequality, we have

$$
\phi((u_{kk}^{(s)})^* u_{kk}^{(s)}) \geq |\phi(u_{kk}^{(s)} v^r)|^2 = 1.
$$

Applying ϕ to

$$\sum_{\ell=-s}^{s} (u_{\ell k}^{(s)})^* u_{\ell k}^{(s)} = 1,$$

we can deduce

$$\sum_{\substack{\ell \in \{-s,\dots,s\} \\ \ell \neq k}} \phi((u_{\ell k}^{(s)})^* u_{\ell k}^{(s)}) = 0.$$

But this contradicts $\phi|_{\mathbb{C}[c^*c]} = h|_{\mathbb{C}[c^*c]}$ because

$$\sum_{\substack{\ell \in \{-s,\dots,s\} \\ \ell \neq k}} (u_{\ell k}^{(s)})^* u_{\ell k}^{(s)} = 1 - (u_{kk}^{(s)})^* u_{kk}^{(s)}$$

is a non-zero positive element in $\mathbb{C}[c^*c]$ and the Haar state is faithful. $\qquad\square$

Before we formulate the main theorem of this section we need two more remarks which will be used in the proof.

Remark 4.5. Since $C(\mathbb{T}^2)$ is commutative, by Kawada and Itô's theorem all idempotent states on $C(\mathbb{T}^2)$ are induced by Haar measures of compact subgroups of the two-dimensional torus $\mathbb{T}^2$.

Remark 4.6. As a compact quantum group $\mathbb{Z}_n$ is given by

$$\text{Pol}(\mathbb{Z}_n) = C(\mathbb{Z}_n) = \text{span}\{w_0, \dots, w_{n-1}\},$$

with $w_k w_\ell = w_{k+\ell \bmod n}$, $S(w_k) = w_{n-k} = (w_k)^*$, $\Delta(w_k) = w_k \otimes w_k$, and $\varepsilon(w_k) = 1$ for $k = 0, \dots, n-1$. The Haar state of $\mathbb{Z}_n$ is given by $h(w_k) = \delta_{0k}$. $C(\mathbb{Z}_n)$ can also be obtained from $\text{Pol}(U(1))$ by dividing out the Hopf $*$-ideal $\{u_1(w^n - 1)u_2; u_1, u_2 \in \text{Pol}(U(1))\}$.

Analogous to [Wys04], Section 4, one can define the twisted product $SU_q(2) \ltimes_\sigma \mathbb{Z}_n$. Alternatively, $\text{Pol}(SU_q(2) \ltimes_\sigma \mathbb{Z}_n)$ can be obtained from $U_q(2) \cong SU_q(2) \ltimes_\sigma U(1)$ by dividing out the Hopf ideal $\{u_1(v^n - 1)u_2 \mid u_1, u_2 \in \text{Pol}(U_q(2))\}$, and defining $C(SU_q(2) \ltimes_\sigma \mathbb{Z}_n)$ as its C*-completion. This construction shows that $SU_q(2) \ltimes_\sigma \mathbb{Z}_n$ is a quantum subgroup of $U_q(2) \cong SU_q(2) \ltimes_\sigma U(1)$. As in the case of $U_q(2)$, $C(SU_q(2) \ltimes_\sigma \mathbb{Z}_n) = C(SU_q(2)) \otimes C(\mathbb{Z}_n)$ and the Haar state of $SU_q(2) \ltimes_\sigma \mathbb{Z}_n$ is equal to the tensor product of the Haar states of $SU_q(2)$ and $\mathbb{Z}_n$.

We can now give a description of all idempotent states on $U_q(2)$. It turns out that they are all induced by Haar states of quantum subgroups of $U_q(2)$.

Theorem 4.7. *Let $q \in (-1, 0) \cup (0, 1)$. Then the following is a complete list of the idempotent states on the compact quantum group $U_q(2)$.*

(1) *The Haar state h of $U_q(2)$.*

(2) $\tilde{\phi} \circ \pi_{\mathbb{T}^2}$, *where $\pi_{\mathbb{T}^2}$ denotes the surjective quantum group morphism $\pi_{\mathbb{T}^2} \colon C(\mathrm{U}_q(2)) \to C(\mathbb{T}^2)$ and $\tilde{\phi}$ is an idempotent state on $C(\mathbb{T}^2)$. In particular if $\tilde{\varepsilon}$ denotes the counit of $C(\mathbb{T}^2)$, then $\tilde{\varepsilon} \circ \pi_{\mathbb{T}^2}$ is the counit of $\mathrm{U}_q(2)$.*

(3) *The states induced by the Haar states of the compact quantum subgroups $\mathrm{SU}_q(2) \ltimes_\sigma \mathbb{Z}_n$ of $\mathrm{U}_q(2) \cong \mathrm{SU}_q(2) \ltimes_\sigma \mathrm{U}(1)$, for $n \in \mathbb{N}$. The case of $n = 1$ corresponds to the Haar state on $\mathrm{SU}_q(2)$ viewed as a quantum subgroup of $\mathrm{U}_q(2)$.*

Proof. Let $\phi \colon C(\mathrm{U}_q(2)) \to \mathbb{C}$ be an idempotent state on $\mathrm{U}_q(2)$. Clearly ϕ is uniquely determined by its restriction to $\mathrm{Pol}(\mathrm{U}_q(2))$.

We distinguish two cases.

Case (i): $\phi(u_{00}^{(1)}) = 1$. In this case Lemma 4.2 shows that ϕ is induced by an idempotent state on the quantum subgroup $\mathbb{T}^2$ of $\mathrm{U}_q(2)$, i.e., $\phi = \tilde{\phi} \circ \pi_{\mathbb{T}^2}$ for some idempotent state $\tilde{\phi} \colon C(\mathbb{T}^2) \to \mathbb{C}$. This case includes the counit ε of $\mathrm{U}_q(2)$, it corresponds to the trivial subgroup $\{1\}$ of $\mathbb{T}^2$.

Case (ii): $\phi(u_{00}^{(1)}) = 0$. In this case Lemma 4.3 and Lemma 4.4 imply that ϕ agrees with the Haar state h on the subspace $\mathcal{A}_0$, i.e.,

$$\phi(u_{k\ell}^{(s)} v^r) = 0$$

for all $s \in \frac{1}{2}\mathbb{Z}_+$, $s > 0$, $-s \le k, \ell \le s$, and $r \in \mathbb{Z}$. It remains to determine ϕ on the $*$-subalgebra $\mathrm{alg}\{v, v^*\}$ generated by v since $\mathrm{Pol}(\mathrm{U}_q(2)) = \mathcal{A}_0 \oplus \mathrm{alg}\{v, v^*\}$ as a vector space. But this subalgebra is isomorphic to the $*$-Hopf algebra $\mathrm{Pol}(\mathrm{U}(1))$ of polynomials on the unit circle, and therefore $\phi|_{\mathrm{alg}\{v, v^*\}}$ has to be induced by the Haar measure of a compact subgroup of $\mathrm{U}(1)$. We have the following possibilities.

(1) $\phi|_{\mathrm{alg}\{v, v^*\}} = \varepsilon_{\mathrm{U}(1)}$, i.e., the restriction of ϕ to $\mathrm{alg}\{v, v^*\}$ is equal to the counit of $\mathrm{Pol}(\mathrm{U}(1))$. In this case we have

$$\phi(u_{k\ell}^{(s)} v^r) = \begin{cases} 1 & \text{if } s = k = \ell = 0 \text{ and } r \in \mathbb{Z}, \\ 0 & \text{else.} \end{cases}$$

This formula shows that $\phi = h_{\mathrm{SU}_q(2)} \circ \pi_{\mathrm{SU}_q(2)}$, where $\pi_{\mathrm{SU}_q(2)}$ is the quantum groups morphism from $C(\mathrm{U}_q(2))$ onto $C(\mathrm{SU}_q(2))$ and $h_{\mathrm{SU}_q(2)}$ denotes the Haar state of $\mathrm{SU}_q(2)$.

(2) $\phi|_{\mathrm{alg}\{v, v^*\}} = h_{\mathrm{U}(1)}$, i.e., the restriction of ϕ to $\mathrm{alg}\{v, v^*\}$ is equal to the Haar state of $\mathrm{Pol}(\mathrm{U}(1))$. In this case we have

$$\phi(u_{k\ell}^{(s)} v^r) = \begin{cases} 1 & \text{if } s = k = \ell = 0 \text{ and } r = 0, \\ 0 & \text{else.} \end{cases}$$

We see that ϕ is the Haar state h of $\mathrm{U}_q(2)$.

(3) $\phi|_{\text{alg}\{v,v^*\}}$ is the idempotent state on U(1) induced by the Haar measure of the subgroup $\mathbb{Z}_n \subseteq U(1)$ for some $n \in \mathbb{N}$, $n \geq 2$. In this case we have

$$\phi(u_{k\ell}^{(s)} v^r) = \begin{cases} 1 & \text{if } s = k = \ell = 0 \text{ and } r \equiv 0 \mod n, \\ 0 & \text{else.} \end{cases}$$

It follows that ϕ is induced by the Haar state of the quantum subgroup $SU_q(2) \ltimes_\sigma \mathbb{Z}_n$ of $U_q(2) \cong SU_q(2) \ltimes_\sigma U(1)$.

Conversely, all the states we have found are induced by Haar states on quantum subgroups of $U_q(2)$, therefore they are clearly idempotent. This can be also checked directly. □

We see that all idempotent states on $U_q(2)$ are induced from Haar states of quantum subgroups of $U_q(2)$. We can also deduce the complete list of quantum subgroups of $U_q(2)$.

Corollary 4.8. *Let $q \in (-1, 0) \cup (0, 1)$. Then the following is a complete list of the non-trivial quantum subgroups of $U_q(2)$.*

(1) *The two-dimensional torus and its closed subgroups.*

(2) *The compact quantum groups of the form $SU_q(2) \ltimes_\sigma \mathbb{Z}_n$, with $n \in \mathbb{N}$ (here the twisting is identical to the one appearing in the identification $U_q(2) \cong SU_q(2) \ltimes_\sigma \mathbb{T}$).*

5. Idempotent states on compact quantum groups $SU_q(2)$ and $SO_q(3)$ ($q \in (-1, 0) \cup (0, 1]$)

Let us first discuss the case $q = 1$. $C(SU_1(2))$ and $C(SO_1(3))$ are respectively the algebras of continuous functions on the groups $SU(2)$ and $SO(3)$. All idempotent states correspond to Haar measures on compact subgroups. The list of these subgroups can be found, e.g., in [Pod95].

Consider now the generic case $q \in (-1, 0) \cup (0, 1)$. Every idempotent state on $SU_q(2)$ induces an idempotent state on $U_q(2)$ since $SU_q(2)$ is a quantum subgroup of $U_q(2)$. This observation allows us to deduce all idempotent states on $SU_q(2)$ from Theorem 4.7. We omit the details and just state the result.

Theorem 5.1. *Let $q \in (-1, 0) \cup (0, 1)$. The Haar state, the counit, and the idempotent states induced by the quantum subgroups U(1) and $\mathbb{Z}_n$, $2 \leq n \leq \infty$, are the only idempotent states on $SU_q(2)$.*

Since the morphism $j : C(\mathrm{SU}_q(2)) \to C(\mathrm{U}(1))$ gives the diagonal matrices

$$(j(u_{k\ell}^{(s)}))_{-s \le k, \ell \le s} = \begin{pmatrix} z^{2s} & & & \\ & z^{2s-2} & & \\ & & \ddots & \\ & & & z^{-2s} \end{pmatrix},$$

we get

$$(h_{\mathrm{U}(1)} \circ j)(u_{k\ell}^{(s)}) = \begin{cases} 1 & \text{if } s \in \mathbb{Z}_+, \ k = \ell = 0, \\ 0 & \text{else;} \end{cases}$$

and, for $n \in \mathbb{N}$,

$$(h_{\mathbb{Z}_{2n}} \circ j)(u_{k\ell}^{(s)}) = \begin{cases} 1 & \text{if } s \in \mathbb{Z}_+, \ k = \ell, \ 2k \equiv 0 \mod 2n, \\ 0 & \text{else,} \end{cases} \tag{5.1}$$

$$(h_{\mathbb{Z}_{2n+1}} \circ j)(u_{k\ell}^{(s)}) = \begin{cases} 1 & \text{if } k = \ell, \ 2k \equiv 0 \mod 2n + 1, \\ 0 & \text{else.} \end{cases} \tag{5.2}$$

Consider now the idempotent states on $\mathrm{SO}_q(3)$. Since $C(\mathrm{SO}_q(3))$ is a subalgebra of $C(\mathrm{SU}_q(2))$ and since the inclusion map is a quantum group morphism, every idempotent state on $\mathrm{SU}_q(2)$ gives an idempotent state on $\mathrm{SO}_q(3)$ by restriction. We will show that all idempotent states on $\mathrm{SO}_q(3)$ arise in this way. It follows that all idempotent states on $\mathrm{SO}_q(3)$ are induced from Haar states of quantum subgroups.

Theorem 5.2. *Let $q \in (-1, 0) \cup (0, 1)$ and n an odd integer. Then the restrictions to $C(\mathrm{SO}_q(3))$ of the idempotent states $h_{\mathbb{Z}_n} \circ j$ and $h_{\mathbb{Z}_{2n}} \circ j$ coincide.*

Furthermore, the Haar state, the counit, and the states induced from the Haar states on the quantum subgroups $\mathrm{U}(1) \cong \mathrm{SO}(2)$ and its closed subgroups are the only idempotent states on $\mathrm{SO}_q(3)$.

Proof. The first statement follows from equations (5.1) and (5.2).

Let now ϕ be an idempotent state on $\mathrm{SO}_q(3)$. Denote by E the conditional expectation from $C(\mathrm{SU}_q(2))$ onto $C(\mathrm{SO}_q(3))$ introduced in Paragraph 2.4.2. Then $\hat{\phi} = \phi \circ \mathrm{E}$ defines an idempotent state on $\mathrm{SU}_q(2)$ such that $\phi = \hat{\phi}|_{C(\mathrm{SO}_q(3))}$. With this observation Theorem 5.2 follows immediately from Theorem 5.1. $\qquad \square$

Remark 5.3. This method applies to quotient quantum groups in general. If $(\mathbb{G}', j)$ is a normal quantum subgroup of $\mathbb{G}$ (see Definition 2.4), then all idempotent states on $C(\mathbb{G}/\mathbb{G}')$ arise as restrictions of idempotent states on $C(\mathbb{G})$.

As a corollary to Theorems 5.1 and 5.2, we recover Podleś' classification [Pod95] of the quantum subgroups of $\mathrm{SU}_q(2)$ and $\mathrm{SO}_q(3)$.

Corollary 5.4. *Let $q \in (-1, 0) \cup (0, 1)$. Then $\mathrm{U}(1) \cong \mathrm{SO}(2)$ and its closed subgroups are the only non-trivial quantum subgroups of both $\mathrm{SU}_q(2)$ and $\mathrm{SO}_q(3)$.*

6. Idempotent states on compact quantum semigroups $U_0(2)$, $SU_0(2)$ and $SO_0(3)$

In this section we compute all idempotent states on $U_0(2)$, $SU_0(2)$ and $SO_0(3)$. As in the cases $q \neq 0$ considered earlier, we begin with the C*-bialgebra $C(U_0(2))$. Again we first need some preparatory observations and lemmas.

Note that $\pi_{\mathbb{T}^2} \colon C(U_q(2)) \to C(\mathbb{T}^2)$ is a well-defined $*$-algebra and coalgebra morphism also for $q = 0$, so $\mathbb{T}^2$ and its compact subgroups induce idempotent states on $U_0(2)$.

For $q = 0$ the algebraic relations of a and c become

$$cc^* = c^*c, \quad aa^* = 1, \quad ac = ac^* = 0, \quad a^*a = 1 - c^*c.$$

Since a is a coisometry, we have a decreasing family of orthogonal projections $(a^*)^n a^n$, $n \in \mathbb{N}$, which are group-like, i.e., $\Delta((a^*)^n a^n) = (a^*)^n a^n \otimes (a^*)^n a^n$, and $c^*c = 1 - a^*a$ is also an orthogonal projection.

Denote by M the unital semigroup $U(1) \times (\mathbb{Z}_+ \cup \{\infty\})$ with the operation

$$(z_1, n_1) \cdot (z_2, n_2) = (z_1 z_2, \min(n_1, n_2))$$

for $z_1, z_2 \in U(1), n_1, n_2 \in \mathbb{Z}_+ \cup \{\infty\}$. This is an abelian semigroup with unit element $e_M = (1, \infty)$. Equip $\mathbb{Z}_+ \cup \{\infty\}$ with the topology in which a subset of $\mathbb{Z}_+ \cup \{\infty\}$ is open if and only if it is either an arbitrary subset of $\mathbb{Z}_+$ or the complement of a finite subset of $\mathbb{Z}_+$ (i.e., $\mathbb{Z}_+ \cup \{\infty\}$ is the one-point-compactification of $\mathbb{Z}_+$), and equip $M = U(1) \times (\mathbb{Z}_+ \cup \{\infty\})$ with the product topology.

The C*-bialgebra $C(M)$ will play an important role in this section.

Lemma 6.1. *A probability measure μ on M is idempotent if and only if there exists an $n \in \mathbb{Z}_+ \cup \{\infty\}$ and an idempotent probability ρ on $U(1)$ such that $\mu = \rho \otimes \delta_n$.*

Proof. Any probability on M can be expressed as a sum $\mu = \sum_{n=0}^{\infty} \rho_n \otimes \delta_n + \rho_\infty \otimes \delta_\infty$, where ρ_n, $n \in \mathbb{Z}_+ \cup \{\infty\}$ are uniquely determined positive measures on $U(1)$ with total mass $\sum_{n=0}^{\infty} \rho_n(U(1)) + \rho_\infty(U(1)) = 1$, and δ_n denotes the Dirac measure on $\mathbb{Z}_+ \cup \{\infty\}$, i.e.,

$$\delta_n(Q) = \begin{cases} 1 & \text{if } n \in Q, \\ 0 & \text{if } n \notin Q, \end{cases}$$

for $Q \subseteq \mathbb{Z}_+ \cup \{\infty\}$. We have

$$(\delta_n \star \delta_m)(Q) = (\delta_n \otimes \delta_m)(\{(k, \ell) \in (\mathbb{Z}_+ \cup \{\infty\})^2 \mid \min(k, \ell) \in Q\})$$
$$= \begin{cases} 1 & \text{if } \min(n, m) \in Q, \\ 0 & \text{if } \min(n, m) \notin Q, \end{cases}$$

i.e., $\delta_n \star \delta_m = \delta_{\min(n,m)}$. Therefore

$$\mu^{\star 2} = \rho_\infty^{\star 2} \otimes \delta_\infty + \sum_{n=0}^{\infty} \left(\rho_n \star \left(\rho_n + 2 \sum_{m=n+1}^{\infty} \rho_m + 2 \rho_\infty \right) \right) \otimes \delta_n.$$

Clearly, if ρ_∞ is an idempotent probability on U(1) and $\rho_n = 0$ for $n \in \mathbb{Z}_+$, then $\mu = \rho_\infty \otimes \delta_\infty$ is idempotent.

Assume now that $\rho_\infty(\mathrm{U}(1)) < 1$. Then there exists a unique $n \in \mathbb{Z}_+$ such that $\sum_{m=n+1}^\infty \rho_m(\mathrm{U}(1)) < 1$, $\sum_{m=n}^\infty \rho_m(\mathrm{U}(1)) = 1$ (i.e., n is the biggest integer m for which $\rho_{\geq m} = \rho_\infty + \sum_{k=m}^\infty \rho_k$ is a probability). Let $p = \rho_n(\mathrm{U}(1))$. If μ is idempotent, then we have

$$p = \mu(\mathrm{U}(1) \times \{n\}) = \mu^{\star 2}(\mathrm{U}(1) \times \{n\}) = (\rho_n \star (2\rho_{\geq n} - \rho_n))(\mathrm{U}(1)) = 2p - p^2.$$

Since $p = \rho_n(\mathrm{U}(1)) > 0$ by the choice of n, we get $p = 1$, i.e., $\rho_m = 0$ for $m \neq n$ and ρ_n is a probability. Then ρ_n has to be idempotent, and $\mu = \rho_n \otimes \delta_n$ is of the desired form.

Conversely, any probability of the form $\mu = \rho_n \otimes \delta_n$ with $n \in \mathbb{Z}_+ \cup \{\infty\}$ and ρ_n idempotent is idempotent. $\qquad\square$

For $k \in \mathbb{Z}$ and $n \in \mathbb{Z}_+$, define functions $\Theta_n^k \colon M \to \mathbb{C}$ by

$$\Theta_n^k(z, m) = \begin{cases} z^k & \text{if } m \geq n, \\ 0 & \text{if } m < n. \end{cases}$$

The span of these functions is dense in $C(M)$, and they satisfy

$$\Theta_n^k \Theta_m^\ell = \Theta_{\max(n,m)}^{k+\ell}, \quad (\Theta_n^k)^* = \Theta_n^{-k}, \quad \text{and} \quad \varepsilon(\Theta_n^k) = \Theta_n^k(e_M) = 1.$$

For their coproduct, we have

$$\Delta\Theta_n^k((z_1, m_1), (z_2, m_2)) = \Theta_n^k(z_1 z_2, \min(m_1, m_2))$$
$$= \begin{cases} (z_1 z_2)^k & \text{if } m_1, m_2 \geq n, \\ 0 & \text{else,} \end{cases}$$

i.e., $\Delta\Theta_n^k = \Theta_n^k \otimes \Theta_n^k$.

Proposition 6.2. *The semigroup M is a quantum quotient semigroup of $\mathrm{U}_0(2)$, in the sense that there exists an injective $*$-algebra homomorphism j from $\mathrm{Pol}(M) :=$ $\mathrm{span}\{\Theta_n^k \mid n \in \mathbb{Z}_+, \ k \in \mathbb{Z}\}$ to $\mathrm{Pol}(\mathrm{U}_0(2))$ such that*

$$\Delta_M \circ j = (j \otimes j) \circ \Delta.$$

Proof. For $n \in \mathbb{Z}_+$ and $k \in \mathbb{Z}$, define $E_n^k = (\alpha^*)^n \alpha^n v^k \in \mathrm{Pol}(\mathrm{U}_0(2))$. From the defining relations of $\mathrm{U}_0(2)$, one can check that the E_n^k satisfy the same $*$-algebraic and coalgebraic relations as the Θ_n^k, i.e., $j(\Theta_n^k) = E_n^k$ defines a $*$-bialgebra homomorphism $j \colon \mathrm{Pol}(M) \to \mathrm{Pol}(\mathrm{U}_0(2))$.

Let us show that j is injective. Assume there exists a non-zero function $f = \sum_{k,n} \lambda_{k,n} \Theta_n^k$ such that $j(f) = 0$. Let n_0 be the smallest integer for which there

exists a $k \in \mathbb{Z}$ such that $\lambda_{k,n} \neq 0$. Take the representation $\pi = \pi_0 \otimes \mathrm{id}_{L^2(\mathbb{T})}$ of $C(U_0(2)) \cong C(SU_0(2)) \otimes C(U(1))$ (recall that π_0 was defined in Section 2.3.1). Since $j(f) = 0$, the operator $\chi = \pi(j(f)) = \sum \lambda_{k,n} \pi_0((\alpha^*)^n \alpha^n) v^k$ has to vanish. Apply χ to $e_{n_0} \otimes 1$. Since $\pi_0((\alpha^*)^n \alpha^n) e_{n_0} = 0$ for $n > n_0$, we get $\chi(e_{n_0} \otimes 1) = \sum_k \lambda_{n_0,k} e_0 \otimes v^k \in \mathfrak{h} \otimes L^2(\mathbb{T})$, which implies $\lambda_{n_0,k} = 0$ for all k, a contradiction to the choice of n_0. $\qquad\square$

We can now give a description of all idempotent states on $U_0(2)$.

Theorem 6.3. *The following gives a complete list of the idempotent states on $U_0(2)$.*

(1) *The idempotent states induced by the Haar measures on the two-dimensional torus $\mathbb{T}^2$ and its closed subgroups. If ρ denotes the Haar measure of $\mathbb{T}^2$ or one of its closed subgroups, then the corresponding idempotent state ϕ_ρ is given by*

$$\phi_\rho((a^*)^r c^k a^s v^\ell) = \delta_{0k} \int_{\mathbb{T}^2} w_1^{s-r} w_2^\ell \, \mathrm{d}\rho(w_1, w_2)$$

for $n, m \in \mathbb{Z}_+$, $k, \ell \in \mathbb{Z}$. This includes the counit of $U_0(2)$, for the trivial subgroup $\{1\}$ of $\mathbb{T}^2$.

(2) *The family $\Psi_{n,m} = \psi_n \otimes \phi_m$, $n \in \mathbb{Z}_+$, $m \in \mathbb{N} \cup \{\infty\}$. Here ϕ_m is an idempotent state on $C(U(1))$, namely the Haar measure on $U(1)$ for $m = \infty$ and the idempotent state induced by the Haar measure of $\mathbb{Z}_m$ for $m \in \mathbb{N}$. And ψ_n, $n \in \mathbb{Z}_+$, is the idempotent state on $SU_0(2)$ defined by*

$$\psi_n((\alpha^*)^r \gamma^k \alpha^s) = \begin{cases} 1 & \text{if } r = s \leq n \text{ and } k = 0, \\ 0 & \text{else.} \end{cases}$$

Proof. Let $\phi \colon C(U_0(2)) \to \mathbb{C}$ be an idempotent state on $U_0(2)$. Then ϕ induces an idempotent state $\phi \circ j$ on $C(M)$. By Lemma 6.1, $\phi \circ j$ is integration against a probability measure of the form $\rho \otimes \delta_n$ with $n \in \mathbb{Z}_+ \cup \{\infty\}$ and ρ an idempotent measure on $U(1)$. This determines ϕ on the subalgebra generated by v and $(a^*)^r a^r$, $r \in \mathbb{N}$: we have

$$\phi((a^*)^r a^r v^k) = \begin{cases} \rho(v^k) & \text{if } r \leq n, \\ 0 & \text{else,} \end{cases}$$

for $k \in \mathbb{Z}$, $r \in \mathbb{Z}_+$.

Case (i): $n = \infty$. For $k > 0$ and any $r, s \in \mathbb{Z}_+$, $\ell \in \mathbb{Z}$, we have

$$\begin{aligned} |\phi((a^*)^r c^k a^s v^\ell)|^2 &\leq \phi((a^*)^r a^r) \phi((a^*)^s (c^*)^k c^k a^s) \\ &= \phi((a^*)^s a^s - (a^*)^{s+1} a^{s+1}) \\ &= 0, \end{aligned}$$

and therefore ϕ vanishes on the $*$-ideal $\mathcal{I}_c$ generated by c. As in the proof of Lemma 4.2, it follows that ϕ is induced by an idempotent state on $\mathrm{Pol}(\mathrm{U}_0(2))/\mathcal{I}_c \cong \mathrm{Pol}(\mathbb{T}^2)$, i.e., ϕ is of the form given in (1).

Case (ii): $n = 0$. Using again the Cauchy–Schwarz inequality, we get

$$|\phi((a^*)^r c^k a^0 v^\ell)|^2 \leq \phi((a^*)^r a^r)\phi((a^*)^s a^s - (a^*)^{s+1} a^{s+1}), \quad (6.1)$$

$$|\phi((a^*)^r c^k a^s v^\ell)|^2 \leq \phi((a^*)^r a^r - (a^*)^{r+1} a^{r+1})\phi((a^*)^s a^s), \quad (6.2)$$

$$|\phi((a^*)^r a^s v^\ell)|^2 \leq \phi((a^*)^r a^r)\phi((a^*)^s a^s) \quad (6.3)$$

for $k, r, s \in \mathbb{Z}_+, \ell \in \mathbb{Z}$. This shows that $\phi((a^*)^r c^k a^r v^\ell)$ vanishes, unless $r = s = 0$. But then $\phi = \phi \star \phi$ implies that

$$\phi(c^k v^\ell) = (\phi \otimes \phi)(\Delta(c^k v^\ell)) = \sum_{\kappa=0}^{k} \phi((a^*)^{k-\kappa} c^\kappa v^\ell)\phi(c^{k-\kappa} a^\kappa v^\ell) = 0$$

for $k > 0, s \in \mathbb{Z}$. By hermitianity $\phi((c^*)^k v^\ell) = 0$ for $k > 0$, and ϕ has the form given in (2) with $n = 0$.

Case (iii): $n \in \mathbb{N}$. We use once more the Cauchy–Schwarz inequality. For $k \neq 0$, (6.1) and (6.2) imply that $\phi((a^*)^r c^k a^s v^\ell)$ vanishes unless $r = s = n$. But then we can show that $\phi = \phi \star \phi$ implies $\phi((a^*)^n c^k a^n v^\ell) = 0$ in the same way as in the previous case.

For $k = 0$, we see from (6.3) that $\phi((a^*)^r a^s v^\ell)$ vanishes unless $r, s \leq n$. The elements $(a^*)^r a^s v^\ell$ are group-like, therefore $\phi((a^*)^r a^s v^\ell) \in \{0, 1\}$. If we can show $\phi((a^*)^r a^s v^\ell) \neq 1$ for $r \neq s$, we are done since then $\phi((a^*)^r c^k a^s v^\ell)$ is non-zero only if $r = s \leq n$. We get $\phi((a^*)^r a^s v^\ell) = \delta_{rs}\rho(v^\ell)$ for $r, s \leq n$, i.e., ϕ has the form given in (2).

We show $\phi((a^*)^r a^s v^\ell) \neq 1$ for $r \neq s$ by contradiction. Assume that there exists a triple (r_0, s_0, ℓ_0) such that $\phi((a^*)^{r_0} a^{s_0} v^{\ell_0}) = 1$ and choose such a triple with maximal r_0. Set

$$b = (a^*)^{r_0} a^{s_0} v^{\ell_0} + (a^*)^{s_0} a^{r_0} v^{-\ell_0} - 1.$$

Maximality of r_0 implies that $r_0 > s_0$ and

$$\phi((a^*)^{2r_0-s_0} a^{s_0} v^{2\ell_0}) = \phi((a^*)^{s_0} a^{2r_0-s_0} v^{-2\ell_0}) = 0,$$

therefore we get

$$\begin{aligned}
\phi(b^*b) &= \phi((a^*)^{r_0} a^{r_0}) + \phi((a^*)^{s_0} a^{s_0}) + 1 \\
&\quad + \phi((a^*)^{2r_0-s_0} a^{s_0} v^{2\ell_0}) + \phi((a^*)^{s_0} a^{2r_0-s_0} v^{-2\ell_0}) \\
&\quad - 2\phi((a^*)^{r_0} a^{s_0} v^{\ell_0}) - 2\phi((a^*)^{s_0} a^{r_0} v^{-\ell_0}) \\
&= -1,
\end{aligned}$$

which is clearly a contradiction to the positivity of ϕ.

Conversely, using the formulas

$$\Delta\big((a^*)^r c^k a^s v^\ell\big) = \sum_{\kappa=0}^{k} (a^*)^{r+k-\kappa} c^\kappa a^s v^\ell \otimes (a^*)^r c^{k-\kappa} a^{s+\kappa} v^\ell,$$

$$\Delta\big((a^*)^r (c^*)^k a^s v^\ell\big) = \sum_{\kappa=0}^{k} (a^*)^r (c^*)^\kappa a^{s+k-\kappa} v^\ell \otimes (a^*)^{r+\kappa} (c^*)^{k-\kappa} a^s v^\ell,$$

with $r, s, k \in \mathbb{Z}_+$, $\ell \in \mathbb{Z}$, for the coproduct in $\mathrm{Pol}(U_0(2))$, one can check that all states given in the theorem are indeed idempotent. $\qquad\square$

Remark 6.4. The state $\Psi_{0,\infty}$ introduced in the theorem above can be considered as the Haar state on $U_0(2)$ since it is invariant, i.e.,

$$\Psi_{0,\infty} \star f = f \star \Psi_{0,\infty} = f(1)\Psi_{0,\infty}$$

for any $f \in C(U_0(2))^*$. But $\Psi_{0,\infty}$ is not faithful. Its algebraic null space

$$N_{\Psi_{0,\infty}} = \{u \in \mathrm{Pol}(U_0(2)) \mid \Psi_{0,\infty}(u^*u) = 0\}$$
$$= \mathrm{span}\,\{(a^*)^k c^m a^\ell v^n \mid k \in \mathbb{Z}_+,\ m, n \in \mathbb{Z},\ \ell \geq 1\}$$

is a left ideal but not self-adjoint or two-sided. It is a subcoalgebra, i.e., we have

$$\Delta N_{\Psi_{0,\infty}} \subseteq N_{\Psi_{0,\infty}} \odot N_{\Psi_{0,\infty}},$$

but it is not a coideal since the counit does not vanish on $\mathcal{N}_{\Psi_{0,\infty}}$.

The complete description of the idempotent states on $SU_0(2)$ follows now directly from Theorem 6.3 and the comments before Theorem 4.7.

Theorem 6.5. *The following gives a complete list of the idempotent states on* $SU_0(2)$.

(1) *The family* ϕ_n, $n \in \mathbb{N} \cup \{\infty\}$ *where* ϕ_1 *is the counit,* ϕ_∞ *the idempotent induced by the Haar state of quantum subgroup* $U(1)$, *and* ϕ_n, $2 \leq n < \infty$, *denotes the idempotent state induced by the Haar state on the quantum subgroup* $\mathbb{Z}_n$.

(2) *The family* ψ_n, $n \in \mathbb{Z}_+$ *defined by*

$$\psi_n((\alpha^*)^r \gamma^k \alpha^s) = \begin{cases} 1 & \text{if } k = 0,\ r = s \leq n, \\ 0 & \text{else,} \end{cases}$$

for $r, s \in \mathbb{Z}_+$, $k \in \mathbb{Z}$, *with the convention* $\gamma^{-k} = (\gamma^*)^k$.

Similarly, using the conditional expectation introduced in Paragraph 2.4.2, we can derive a complete classification of the idempotent states on $SO_0(3)$. The proof is identical to the proof of Theorem 5.2 and therefore omitted.

Theorem 6.6. *All idempotent states on* $SO_0(3)$ *arise as restrictions of idempotent states on* $SU_0(2)$. *Moreover* $\phi_n|_{C(SO_0(3))} = \phi_{2n}|_{C(SO_0(3))}$ *for n an odd integer.*

7. The idempotent states on $SU_q(2)$ as elements of the dual and associated quantum hypergroups

Let $\mathbb{G}$ be a compact quantum group and $\mathcal{A} = \mathrm{Pol}(\mathbb{G})$ the corresponding Hopf $*$-algebra dense in $C(\mathbb{G})$. Then $\mathcal{A}$ is (the algebra of functions on) an *algebraic quantum group* in the sense of Van Daele ([VD98]), and so is its dual $\widehat{\mathcal{A}}$, given by the functionals of the form $h(\,\cdot\,a)$ with $a \in \mathcal{A}$. By [FS09a], Lemma 3.1, an idempotent state ϕ on $C(\mathbb{G})$ defines a group-like projection p_ϕ in the multiplier algebra $M(\widehat{\mathcal{A}})$ of the dual, and therefore, by [LVD07], Theorem 2.7, and [FS09a], Theorem 2.4, an *algebraic quantum hypergroup* $\widehat{\mathcal{A}}_{p_\phi}$. As an algebra, $\widehat{\mathcal{A}}_{p_\phi} = p_\phi \widehat{\mathcal{A}} p_\phi$, and the coproduct of $\widehat{\mathcal{A}}_{p_\phi}$ is given by

$$\widehat{\Delta}_{p_\phi} = (p_\phi \otimes p_\phi)\widehat{\Delta}(a)(p_\phi \otimes p_\phi)$$

for $a \in \widehat{\mathcal{A}}_{p_\phi}$, where $\widehat{\Delta}$ denotes the coproduct of $\widehat{\mathcal{A}}$.

Let $q \in (-1,0) \cup (0,1)$. In this section we will consider the case of the compact quantum group $SU_q(2)$ and describe the algebraic quantum hypergroups associated to its idempotent states. Note that in this case the dense Hopf $*$-algebra is $\mathcal{A} = \mathrm{Pol}(SU_q(2)) = \mathrm{span}\{u_{k\ell}^{(s)} \mid s \in \frac{1}{2}\mathbb{Z}_+, \, -s \le k, \ell \le s\}$. We will use the basis

$$e_{kl}^{(s)} = \frac{1 - q^{2(2s+1)}}{q^{2(s-k)}(1 - q^2)} \, h((u_{k\ell}^{(s)})^* \cdot)$$

for $\widehat{\mathcal{A}}$, which can be thought of as the algebra of trigonometric polynomials on the quantum group dual $\widehat{SU_q(2)}$. Using the orthogonality relation

$$h((u_{k\ell}^{(s)})^* u_{k'\ell'}^{(s')}) = \delta_{ss'}\delta_{kk'}\delta_{\ell\ell'} \, \frac{q^{2(s-k)}(1 - q^2)}{1 - q^{2(2s+1)}},$$

for $s, s' \in \frac{1}{2}\mathbb{Z}_+, \, -s \le k, \ell \le s, \, -s' \le k', \ell' \le s'$, cf. [Koo89], eq. (5.12), we can check that this basis is dual to the basis $\{u_{k\ell}^{(s)} \mid s \in \frac{1}{2}\mathbb{Z}_+, \, -s \le k, \ell \le s\}$ of $\mathrm{Pol}(SU_q(2))$. The algebraic quantum group $\widehat{\mathcal{A}} = \mathrm{Pol}(\widehat{SU_q(2)})$ is of discrete type and is equal to the algebraic direct sum

$$\widehat{\mathcal{A}} = \bigoplus_{s \in \frac{1}{2}\mathbb{Z}_+} M_{2s+1}.$$

The $e_{k\ell}^{(s)}$ form a basis of matrix units for $M_{2s+1} = \mathrm{span}\{e_{kl}^{(s)} \mid -s \le k, \ell \le s\}$.

The Haar state h and the counit ε give the elements $p_h = 1$ and $p_\varepsilon = e_{00}^{(0)}$ in $M(\widehat{\mathcal{A}})$, and the associated algebraic quantum hypergroups are $\widehat{\mathcal{A}}_{p_h} = \widehat{\mathcal{A}}$ and $\widehat{\mathcal{A}}_{p_\varepsilon} = \mathbb{C}$.

The remaining cases are more interesting.

7.1. The idempotent state $\phi_2 = h_{\mathbb{Z}_2} \circ j$ induced by the quantum subgroup $\mathbb{Z}_2$.
We have

$$p_{\phi_2} = \sum_{s=0}^{\infty} \sum_{k=-s}^{s} e_{kk}^{(s)} = \sum_{s=0}^{\infty} 1_{2s+1},$$

i.e., p_{ϕ_2} is the sum of the identity matrices from the odd-dimensional matrix algebras $M_{2s+1}, s \in \mathbb{Z}_+$. This projection is in the center of $M(\hat{\mathcal{A}})$, therefore $\hat{\mathcal{A}}_2 = p_{\phi_2} \hat{\mathcal{A}} p_{\phi_2}$ will be an algebraic quantum group. We get

$$\hat{\mathcal{A}}_2 = \bigoplus_{s \in \mathbb{Z}_+} M_{2s+1} = \mathrm{Pol}\,\widehat{\mathrm{SO}_q(3)},$$

i.e., $\hat{\mathcal{A}}_{p_{\phi_2}}$ is the algebra of trigonometric polynomials on the algebraic quantum group dual to $\mathrm{SO}_q(3)$. This is to be expected as $\mathbb{Z}_2$ is the only nontrivial normal quantum subgroup of $\mathrm{SU}_q(2)$ and $\mathrm{SO}_q(3)$ is the corresponding quotient quantum group.

7.2. The idempotent state $\phi_\infty = h_{\mathrm{U}(1)} \circ j$ induced by the quantum subgroup $\mathbf{U(1)}$. Here

$$p_{\phi_\infty} = \sum_{s=0}^{\infty} e_{00}^{(s)}$$

and this projection is not central. We get

$$\hat{\mathcal{A}}_{p_\infty} = \bigoplus_{s=0}^{\infty} \mathbb{C},$$

which is a commutative algebraic quantum hypergroup of discrete type. This is the dual of the hypergroup introduced in [Koo91], Section 7.

7.3. The idempotent states $\phi_n = h_{\mathbb{Z}_n} \circ j$, $3 \le n < \infty$. The remaining cases also give non-central projections,

$$p_{\phi_{2n}} = \sum_{s=0}^{\infty} \sum_{k=-\lceil \frac{s}{n} \rceil}^{\lceil \frac{s}{n} \rceil} e_{nk,nk}^{(s)}, \quad p_{\phi_{2n+1}} = \sum_{s \in \frac{1}{2}\mathbb{Z}_+} \sum_{\substack{-s \le k \le s \\ 2k \equiv 0 \bmod 2n+1}} e_{kk}^{(s)},$$

for $1 \le n < \infty$, cf. equations (5.1) and (5.2). They lead to non-commutative algebraic quantum hypergroups $\hat{\mathcal{A}}_n = p_{\phi_n} \hat{\mathcal{A}} p_{\phi_n}$,

$$\hat{\mathcal{A}}_{2n} = \bigoplus_{k=0}^{\infty} n\, M_{2k+1}, \quad \hat{\mathcal{A}}_{2n+1} = \bigoplus_{k=0}^{\infty} (2n+1)\, M_k,$$

of discrete type (in the formulas above $n\, M_{2k+1}$ denotes n direct copies of the matrix algebra M_{2k+1} and similarly $(2n+1)\, M_k$ denotes $2n+1$ direct copies of the matrix algebra M_k).

Note that since the quantum subgroups considered in the last two sections are not normal, the objects we obtain have only the quantum hypergroup structure (and can be informally thought of as duals of quantum hypergroups obtained via the double coset construction, [CV99]).

Appendix

The goal of the appendix is to provide an alternative proof of coamenability of the deformations of classical compact Lie groups. To facilitate the discussion, for a compact quantum group $\mathbb{G}$ we use $\widehat{\mathbb{G}}$ to denote the dual of $\mathbb{G}$ (which is a discrete quantum group), $C(\mathbb{G})_{\mathrm{red}}$ and $C(\mathbb{G})$ to denote respectively the reduced and universal C*-algebras associated with $\mathbb{G}$, and $L^{\infty}(\mathbb{G})$ to denote the corresponding von Neumann algebra (we refer for example to [Tom07] for precise definitions). Note that contrary to the main body of the paper we do not assume that the Haar state on $\mathbb{G}$ is faithful so that $\mathbb{G}$ need not be in reduced form. We adopt the following definition ([BMT02], [Tom06]).

Definition A.1. A compact quantum group $\mathbb{G}$ is said to be *coamenable* if the dual quantum group $\widehat{\mathbb{G}}$ is amenable, that is, $L^{\infty}(\widehat{\mathbb{G}})$ has an invariant mean.

The following result gives a useful criterion to check coamenability:

Theorem A.2 ([BMT02], Theorem 4.7, [Tom06], Corollary 3.7, Theorem 3.8). *A compact quantum group $\mathbb{G}$ is coamenable if and only if there exists a counit on $C(\mathbb{G})_{\mathrm{red}}$ if and only if there exists a $*$-homomorphism from $C(\mathbb{G})_{\mathrm{red}}$ onto $\mathbb{C}$.*

The second equivalence is fairly easy to show, in the first the forward implication was established in [BMT02] and the backward implication in [Tom06].

Let $\mathbb{G}$ be a classical compact Lie group and $\mathbb{G}_q$ the q-deformation with the parameter $-1 < q < 1$, $q \neq 0$ (see [KS98]). The function algebra $C(\mathbb{G}_q)$ is the universal C*-algebra generated by certain polynomial elements. The Haar state is denoted by h.

The following theorem was proved by T. Banica ([Ban99], Corollary 6.2). We present another proof using Korogodski–Soibelman's results on the representation theory of $C(\mathbb{G}_q)$.

Theorem A.3. *The quantum group $\mathbb{G}_q$ is coamenable.*

Proof. Let us introduce the left ideal $N_h := \{a \in C(\mathbb{G}_q) \mid h(a^*a) = 0\}$, which is in fact an ideal of $C(\mathbb{G}_q)$. The reduced version $C(\mathbb{G}_q)_{\mathrm{red}}$ is defined as the quotient $C(\mathbb{G}_q)/N_h$. By Theorem A.2 to show that $\mathbb{G}_q$ is coamenable it suffices to show that the C*-algebra $C(\mathbb{G}_q)_{\mathrm{red}}$ has a character.

Consider an irreducible representation $\pi\colon C(\mathbb{G}_q)_{\mathrm{red}} \to B(H_\pi)$. Composing this map with the canonical surjection $\rho\colon C(\mathbb{G}_q) \to C(\mathbb{G}_q)_{\mathrm{red}}$, we get an irreducible representation $\pi \circ \rho$ of $C(\mathbb{G}_q)$. Thanks to [KS98], Theorem 6.2.7 (3), §3, we may assume that $\pi \circ \rho$ is of the form

$$\pi \circ \rho = (\pi_{s_{i_1}} \otimes \cdots \otimes \pi_{s_{i_k}} \otimes \pi_t) \circ \Delta^{(k)}$$

or

$$\pi \circ \rho = \pi_t,$$

where $s_{i_1} \ldots s_{i_k}$ is the reduced decomposition in the Weyl group of $\mathbb{G}$, and $t \in T$, the maximal torus of $\mathbb{G}$. In the latter case π is a one-dimensional representation. In the former case, we remark that the counit of $C(\mathbb{G}_q)$ factors through $\operatorname{im} \pi_{s_i}$ for every i, that is, there exists $\eta_i \colon \operatorname{im} \pi_{s_i} \to \mathbb{C}$ such that $\eta_i \circ \pi_{s_i} = \varepsilon$ (see the argument in [Tom07], p. 294).

Then we introduce a representation $\tilde{\pi} := (\eta_{i_1} \otimes \cdots \otimes \eta_{i_k} \otimes \operatorname{id}) \circ \pi$ of $C(\mathbb{G}_q)_{\mathrm{red}}$, which is well defined and one-dimensional. Indeed,

$$\tilde{\pi} \circ \rho = (\varepsilon \otimes \cdots \otimes \varepsilon \otimes \pi_t) \circ \Delta^{(k)} = \pi_t.$$

Thus we have proved in each case the existence of a one-dimensional representation of the C*-algebra $C(\mathbb{G}_q)_{\mathrm{red}}$, and $\mathbb{G}_q$ is coamenable. $\square$

Note added in proof. Several results of Section 3 of this paper have recently been generalised to idempotent states and contractive idempotent functionals on the algebras of functions on locally compact quantum groups, see [SS12], [NSSS12].

References

[BBS99] S. Baaj, E. Blanchard, and G. Skandalis, Unitaires multiplicatifs en dimension finie et leurs sous-objets. *Ann. Inst. Fourier (Grenoble)* **49** (1999), 1305–1344. Zbl 0938.46050 MR 1703090

[Bae66] R. Baer, Characterization of finite Dedekind groups. *Nagoya Math. J.* **27** (1966), 21–41. Zbl 0145.02801 MR 0199269

[Ban99] T. Banica, Representations of compact quantum groups and subfactors. *J. Reine Angew. Math.* **509** (1999), 167–198. Zbl 0957.46038 MR 1679171

[BB10] T. Banica and J. Bichon, Hopf images and inner faithful representations. *Glasgow Math. J.* **52** (2010), 677–703. Zbl 1243.16033 MR 2679923

[BMT01] E. Bédos, G. J. Murphy, and L. Tuset, Co-amenability of compact quantum groups. *J. Geom. Phys.* **40** (2001), 130–153. Zbl 1011.46056 MR 1862084

[BMT02] E. Bédos, G. J. Murphy, and L. Tuset, Amenability and coamenability of algebraic quantum groups. *Int. J. Math. Math. Sci.* **31** (2002), 577–601. Zbl 1022.46043 MR 1931751

[CV99]   Yu. A. Chapovsky and L. I. Vainerman, Compact quantum hypergroups. *J. Operator Theory* **41** (1999), 261–289. Zbl 0987.81039 MR 1681575

[Cho74] M. D. Choi, A Schwarz inequality for positive linear maps on C^*-algebras. *Illinois J. Math.* **18** (1974), 565–574. Zbl 0293.46043 MR 0355615

[DeVD10] L. Delvaux and A. Van Daele, Algebraic quantum hypergroups II. Constructions and examples. *Internat. J. Math.* **22** (2011), 407–434. Zbl 1215.43003 MR 2782694

[FSc00] U. Franz and M. Schürmann, Lévy processes on quantum hypergroups. In *Infinite dimensional harmonic analysis* (Kyoto, 1999), D. & M. Gräbner-Verlag, Altendorf/Bamberg 2000, 93–114. Zbl 0988.43007 MR 1851916

[FS08a] U. Franz and A. Skalski, On ergodic properties of convolution operators associated with compact quantum groups. *Colloq. Math.* **113** (2008), 13–23. Zbl 1157.46036 MR 2399662

[FS09b] U. Franz and A. Skalski, A new characterisation of idempotent states on finite and compact quantum groups. *C. R. Math. Acad. Sci. Paris* **347** (2009), 991–996. Zbl 1213.46067 MR 2549133

[FS09a] U. Franz and A. Skalski, On idempotent states on quantum groups. *J. Algebra* **322** (2009), 1774–1802. Zbl 1176.43005 MR 2543634

[FST11] U. Franz, A. Skalski and R. Tomatsu, Idempotent states on the Kac algebra $U_{-1}(2)$. In preparation.

[HR70]   E. Hewitt and K. A. Ross, *Abstract harmonic analysis*. Vol. II: Structure and analysis for compact groups. Analysis on locally compact Abelian groups. Grundlehren Math. Wiss. 152, Springer-Verlag, Berlin 1970. Zbl 0213.40103 MR 0262773

[Hey77] H. Heyer, *Probability measures on locally compact groups*. Ergeb. Math. Grenzgeb. 94, Springer-Verlag, Berlin 1977. Zbl 0376.60002 MR 0501241

[KI40]   Y. Kawada and K. Itô, On the probability distribution on a compact group. I. *Proc. Phys.-Math. Soc. Japan* (3) **22** (1940), 977–998. JFM 66.0544.04 Zbl 0026.13801 MR 0003462

[Koe91] H. T. Koelink, On ∗-representations of the Hopf ∗-algebra associated with the quantum group $U_q(n)$. *Compositio Math.* **77** (1991), 199–231. Zbl 0721.17014 MR 1091898

[Koo89] T. H. Koornwinder, Orthogonal polynomials in connection with quantum groups. In *Orthogonal polynomials* (Columbus, OH, 1989), NATO Adv. Sci. Inst. Ser. C Math. Phys. Sci. 294, Kluwer Acad. Publ., Dordrecht 1990, 257–292. Zbl 0697.42019 MR 1100297

[Koo91] T. H. Koornwinder, Positive convolution structures associated with quantum groups. In *Probability measures on groups, X* (Oberwolfach, 1990), Plenum, New York 1991, 249–268. Zbl 0823.60007 MR 1178988

[KS98]   L. I. Korogodski and Y. S. Soibelman, *Algebras of functions on quantum groups: Part I*. Math. Surveys Monogr. 56, Amer. Math. Soc., Providence, RI 1998. Zbl 0923.17017 MR 1614943

[KV00]   J. Kustermans and S. Vaes, Locally compact quantum groups. *Ann. Sci. École Norm. Sup.* (4) **33** (2000), 837–934. Zbl 1034.46508 MR 1832993

[LVD07]   M. B. Landstad and A. Van Daele, Compact and discrete subgroups of algebraic quantum groups I. Preprint 2007. arXiv:math/0702458

[MMN$^+$88] T. Masuda, K. Mimachi, Y. Nakagami, M. Noumi, and K. Ueno, Representations of quantum groups and a q-analogue of orthogonal polynomials. *C. R. Acad. Sci. Paris Sér. I Math.* **307** (1988), 559–564. Zbl 0658.22010 MR 967361

[NSSS12] M. Neufang, P. Salmi, A. Skalski, and N. Spronk, Contractive idempotents on locally compact quantum groups. Oberwolfach Preprint 2012-19, available at http://www.mfo.de/scientific-programme/publications/owp/2012/OWP2012_19.pdf.

[Pal96]   A. Pal, A counterexample on idempotent states on a compact quantum group. *Lett. Math. Phys.* **37** (1996), 75–77. Zbl 0849.17010 MR 1392148

[Pau02]   V. Paulsen, *Completely bounded maps and operator algebras*. Cambridge Stud. Adv. Math. 78, Cambridge University Press, Cambridge 2002. Zbl 1029.47003 MR 1976867

[Pod95]   P. Podleś, Symmetries of quantum spaces. Subgroups and quotient spaces of quantum $SU(2)$ and $SO(3)$ groups. *Comm. Math. Phys.* **170** (1995), 1–20. Zbl 0853.46074 MR 1331688

[SS12]   P. Salmi and A. Skalski, Idempotent states on locally compact quantum groups. emph*Quart. J. Math.* **63** (2012), no. 4, 1009–1032. Zbl 06117815

[Tom06]   R. Tomatsu, Amenable discrete quantum groups. *J. Math. Soc. Japan* **58** (2006), 949–964. Zbl 1129.46061 MR 2276175

[Tom07]   R. Tomatsu, A characterization of right coideals of quotient type and its application to classification of Poisson boundaries. *Comm. Math. Phys.* **275** (2007), 271–296. Zbl 1130.46042 MR 2335776

[Tom08]   R. Tomatsu, Compact quantum ergodic systems. *J. Funct. Anal.* **254** (2008), 1–83. Zbl 1137.46041 MR 2375065

[VD98]   A. Van Daele, An algebraic framework for group duality. *Adv. Math.* **140** (1998), 323–366. Zbl 0933.16043 MR 1658585

[VS88]   L. L. Vaksman and Ya. S. Soĭbel′man, Algebra of functions on the quantum group SU(2). *Funktsional. Anal. i Prilozhen.* **22** (1988), 1–14; English transl. *Funkt. Anal. Appl.* **22** (1988), 170–181. Zbl 0679.43006 MR 961757

[Wan08]   S. Wang, Simple compact quantum groups I. *J. Funct. Anal.* **256** (2009), 3313–3341. Zbl 1176.46063 MR 2504527

[Wor87a] S. L. Woronowicz, Compact matrix pseudogroups. *Comm. Math. Phys.* **111** (1987), 613–665. Zbl 0627.58034 MR 901157

[Wor87b] S. L. Woronowicz, Twisted $SU(2)$ group. An example of a non-commutative differential calculus. *Publ. Res. Inst. Math. Sci.* **23** (1987), 117–181. Zbl 0676.46050 MR 890482

[Wor98]   S. L. Woronowicz, Compact quantum groups. In *Symétries quantiques/Quantum symmetries* (Les Houches, 1995), North-Holland, Amsterdam 1998, 845–884. Zbl 0997.46045 MR 1616348

[Wys04]   J. Wysoczański, Twisted product structure and representation theory of the quantum group $U_q(2)$. *Rep. Math. Phys.* **54** (2004), 327–347. Zbl 1135.46310 MR 2115743

[ZZ05]   X. Zhang and E. Y. Zhao, The compact quantum group $U_q(2)$ I. *Linear Algebra Appl.* **408** (2005), 244–258. Zbl 1093.46042 MR 2166867

U. Franz, Département de mathématiques de Besançon, Université de Franche-Comté, 16, route de Gray, 25 030 Besançon cedex, France

E-mail: uwe.franz@univ-fcomte.fr

A. Skalski, Institute of Mathematics, Polish Academy of Sciences, ul. Śniadeckich 8, 00-956 Warszawa, Poland

E-mail: a.skalski@impan.pl

R. Tomatsu, Department of Mathematics, Hokkaido University, Kita 10, Nishi 8, Kita-Ku, Sapporo, Hokkaido, 060-0810, Japan

E-mail: tomatsu@math.sci.hokudai.ac.jp