The Picard group of a noncommutative algebraic torus

Yuri Berest, Ajay Ramadoss, and Xiang Tang

Abstract. Let $A_q := \mathbb{C}\langle x^{\pm 1}, y^{\pm 1} \rangle / (xy - qyx)$. Assuming that q is not a root of unity, we compute the Picard group $\mathrm{Pic}(A_q)$ of the algebra A_q, describe its action on the space $\mathcal{R}(A_q)$ of isomorphism classes of rank 1 projective modules and classify the algebras Morita equivalent to A_q. Our computations are based on a 'quantum' version of the Calogero–Moser correspondence relating projective A_q-modules to irreducible representations of the double affine Hecke algebras $\mathbb{H}_{t,q^{-1/2}}(S_n)$ at $t = 1$. We show that, under this correspondence, the action of $\mathrm{Pic}(A_q)$ on $\mathcal{R}(A_q)$ agrees with the action of $\mathrm{SL}_2(\mathbb{Z})$ on $\mathbb{H}_{t,q^{-1/2}}(S_n)$ constructed by Cherednik [C1], [C2]. We compare our results with the smooth and analytic cases. In particular, when $|q| \neq 1$, we find that $\mathrm{Pic}(A_q) \cong \mathrm{Auteq}(\mathcal{D}^b(X))/\mathbb{Z}$, where $\mathcal{D}^b(X)$ is the bounded derived category of coherent sheaves on the elliptic curve $X = \mathbb{C}^*/\mathbb{Z}$.

1. Introduction

Let $\mathbb{C}\langle x^{\pm 1}, y^{\pm 1} \rangle$ be the group algebra of the free group on two generators x and y, with coefficients in $\mathbb{C}$. We define $A_q := \mathbb{C}\langle x^{\pm 1}, y^{\pm 1} \rangle / (xy - qyx)$ for a fixed parameter $q \in \mathbb{C}$. Unless specified otherwise, we will assume in this paper that

$$q^n \neq 1 \quad \text{for all } n \in \mathbb{N}. \tag{1}$$

Under this condition, we will (a) classify finitely generated (right) projective modules of A_q, (b) compute the Picard group $\mathrm{Pic}(A_q)$ and describe its action on projective modules, and (c) classify the algebras Morita equivalent to A_q.

There are several reasons to clarify these questions. First, A_q may be thought of as a ring of functions on a noncommutative *algebraic* torus. Now, for a noncommutative *smooth* torus $\mathcal{A}_q$, the answers to (a), (b) and (c) are well known and well understood (mostly thanks to the work of Rieffel, see [R1], [R2], [R3], [K1]). Geometrically, the algebras $\{\mathcal{A}_q\}$ arise as deformations of the ring $\mathcal{C}^\infty(\mathbb{T})$ of smooth functions on the two-dimensional torus $\mathbb{T} = \mathbb{S}^1 \times \mathbb{S}^1$, and as such, these are fundamental examples of noncommutative differentiable manifolds in the sense of A. Connes [Co]. On the

other hand, algebraically, $\mathcal{A}_q$ is just a certain completion of A_q, and it is natural to ask how the projective modules, Picard groups, Morita equivalences, etc. behave under this completion. One might expect that A_q is 'too rigid' compared to $\mathcal{A}_q$, and the answers to the above questions are trivial. We will demonstrate that this is not the case, and although the properties of A_q and $\mathcal{A}_q$ are indeed very different, the answers to (a), (b) and (c) in the algebraic case are at least as meaningful and interesting as in the smooth case.

Second, A_q may be viewed as a 'quantum' (or multiplicative) Weyl algebra. Under the assumption (1), the ring-theoretic properties of A_q are indeed similar to those of the usual Weyl algebra $A_1 = \mathbb{C}\langle x, y \rangle / (xy - yx - 1)$ (see [J]). From this perspective, our answer to (a) should not be very surprising. It is known that the rank one projective modules of A_1 are isomorphic to ideals and as such, can be parametrized by certain smooth algebraic varieties $\mathcal{C}_n$ called the *Calogero–Moser spaces* (see [BW1], [BW2], [BC]). Similarly, the ideals of A_q are described by a certain 'quantum' version of the Calogero–Moser spaces $\mathcal{C}_n^q$ (see Theorem 1), and, in fact, a similar geometric description exists for more general classes of noncommutative algebras (see, e.g., [KKO], [NB], [NS]). What *is* surprising is the fact that the classification of ideals of A_q allows one to compute the Picard group $\mathrm{Pic}(A_q)$ of the category of all A_q-modules. By contrast, the Picard group $\mathrm{Pic}(A_1)$ of the Weyl algebra was computed by Stafford who showed that $\mathrm{Pic}(A_1) = \mathrm{Aut}(A_1)$, using reduction to fields of characteristic $p > 0$ (see [St]). A different proof of Stafford's result can be found in [CH]: it relies on Dixmier's classification [D] of maximal commutative subalgebras of A_1. Both proofs are fairly involved and indirect; in particular, they do not follow from the results of [BW1], [BW2], [BC]. To the best of our knowledge, the group $\mathrm{Pic}(A_q)$ has not appeared in the literature: we therefore consider its computation (Theorem 3) together with a related Morita classification (Theorem 5) as the main results of this paper.

Third, the quantum Calogero–Moser spaces $\mathcal{C}_n^q$ parametrizing the ideals of A_q can be defined as the spectra of spherical subalgebras of Cherednik's double affine Hecke algebra (DAHA) $\mathbb{H}_{1,q^{-1/2}}(S_n)$ (see [O]). Our results then imply that there is a natural bijection between the set of equivalence classes of irreducible representations of $\mathbb{H}_{1,q^{-1/2}}(S_n)$ amalgamated for all n and the set $\mathcal{R}(A_q)$ of isomorphism classes of ideals of A_q. In the rational case, the analogous bijection was first discovered in [EG], and its proof was given in [BCE]. In this paper, we will extend the construction of [BCE] to the quantum case (see Section 5). A new interesting observation is that, under the Calogero–Moser correspondence, the action of $\mathrm{Pic}(A_q)$ on $\mathcal{R}(A_q)$ corresponds to the action of $\mathrm{SL}_2(\mathbb{Z})$ on $\mathbb{H}_{1,q^{-1/2}}(S_n)$ constructed (and exploited in many applications) by Cherednik [C1], [C2].

Finally, A_q is a fundamental noncommutative algebra which plays a role in many areas of mathematics and physics. As a specific motivation to clarify questions (a), (b) and (c) for A_q, we mention a recent appearance of this algebra in knot theory: it is shown in [FGL] (see also [G]) that the classical invariants of knots – the so-called A-polynomials – can be naturally quantized, and the corresponding quantizations are given by certain *noncyclic* ideals of A_q. Comparing these ideals for different knots

is not an easy problem. Having a general classification and canonical forms for all ideals of A_q should be helpful in this context.

As another application, we should mention the link to integrable systems. The quantum Calogero–Moser spaces $\mathcal{C}_n^q$ first appeared in [FR] in connection with 2D Toda hierarchy and the Ruijsenaars–Schneider system; since then they were discussed in numerous papers on integrable systems. By analogy with the Weyl algebra (see [W1], [BW1]), the ideals of A_q are related to algebraic solutions of these systems, which in turn can be described in terms of q-version of the adelic Grassmannian of G. Wilson [W1], [W2] (see [CH1], [CN] and [BC1] for more details).

The paper is organized as follows. In Section 2, we define the Calogero–Moser spaces $\mathcal{C}_n^q$ and give our classification of ideals of A_q. In Section 3, we compute the Picard group $\mathrm{Pic}(A_q)$ and discuss some implications. In Section 4, we describe the action of $\mathrm{Pic}(A_q)$ on the Calogero–Moser spaces and classify the algebras Morita equivalent to A_q in terms of this action. In Section 5, we explain the relation between the DAHA and A_q and outline proofs of our main results. Finally, in Section 6, we compare the properties of A_q with the properties of noncommutative smooth tori $\mathcal{A}_q$.

2. The Calogero–Moser correspondence for A_q

Under the assumption (1), the algebra A_q is a simple Noetherian domain of homological dimension 1 (see [J], Theorem 2.1). In that case, it is known that every right projective module of rank ≥ 2 is free, while every rank one projective module is isomorphic to a right ideal of A_q (see [We], Theorem 1). Thus the problem of classifying projectives over A_q reduces to classifying the right ideals of A_q. Below, we will give an explicit classification of such ideals similar to the classification of ideals in the Weyl algebra A_1.

For an integer $n \geq 1$, let $\widetilde{\mathcal{C}}_n^q$ denote the space of matrices

$$\{(X, Y, i, j) \mid X, Y \in \mathrm{GL}_n(\mathbb{C}), \ i \in \mathbb{C}^n, \ j \in \mathrm{Hom}(\mathbb{C}^n, \mathbb{C})\},$$

satisfying the equation

$$qXY - YX + ij = 0.$$

The group $\mathrm{GL}_n(\mathbb{C})$ acts on $\widetilde{\mathcal{C}}_n^q$ in the natural way:

$$(X, Y, i, j) \mapsto (gXg^{-1}, gYg^{-1}, gi, jg^{-1}), \quad g \in \mathrm{GL}_n(\mathbb{C}), \tag{2}$$

and this action is free for all n. We define the *n-th Calogero–Moser space* $\mathcal{C}_n^q$ to be the quotient variety $\widetilde{\mathcal{C}}_n^q / \mathrm{GL}_n(\mathbb{C})$ modulo (2). $\mathcal{C}_n^q$ are smooth irreducible affine symplectic varieties of (complex) dimension $2n$ (see [O]). We set $\mathcal{C}^q := \bigsqcup_{n \geq 0} \mathcal{C}_n^q$, assuming that $\mathcal{C}_0^q$ is a point.

In a slightly more invariant way, we may think of $\mathcal{C}^q$ as the space of (isomorphism classes of) triples (V, X, Y), where V is a finite-dimensional complex vector space, and X, Y are automorphisms of V satisfying the condition

$$\mathrm{rk}(qXYX^{-1}Y^{-1} - 1_V) = 1. \tag{3}$$

Now, for each $n \geq 0$, there is a natural action on $\mathcal{C}_n^q$ by the lattice $\mathbb{Z}^2$:

$$(V, X, Y) \mapsto (V, q^k X, q^m Y), \quad (k, m) \in \mathbb{Z}^2.$$

We write $\overline{\mathcal{C}}_n^q := \mathcal{C}_n^q / \mathbb{Z}^2$ for the corresponding quotient spaces, and set $\overline{\mathcal{C}}^q := \bigsqcup_{n \geq 0} \overline{\mathcal{C}}_n^q$.

Let $\mathcal{R}^q := \mathcal{R}(A_q)$ be the set of isomorphism classes of right ideals of A_q. Our first main result is the following

Theorem 1. *There is a natural bijection* $\omega \colon \overline{\mathcal{C}}^q \xrightarrow{\sim} \mathcal{R}^q$.

As in the Weyl algebra case, the bijection ω can be described quite explicitly; it is induced by the map $\mathcal{C}^q \to \mathcal{R}^q$, assigning to $(V, X, Y) \in \mathcal{C}^q$ the class of (isomorphic) fractional ideals

$$M_x = \det(X - x1_V) \cdot A_q + \kappa^{-1} \det(Y - y1_V) \cdot A_q, \tag{4}$$

$$M_y = \det(Y - y1_V) \cdot A_q + \kappa \det(X - x1_V) \cdot A_q, \tag{5}$$

where κ and κ^{-1} are given by the following elements in the quotient field of A_q:

$$\kappa = 1 + j(qY - y1_V)^{-1}(X - x1_V)^{-1}i, \tag{6}$$

$$\kappa^{-1} = 1 - j(X - qx1_V)^{-1}(Y - y1_V)^{-1}i.$$

Theorem 1 thus implies that the right ideals of A_q (and hence rank one projective A_q-modules) are classified by the conjugacy classes of pairs of matrices $(X, Y) \in \mathrm{GL}(V) \times \mathrm{GL}(V)$ satisfying (3). Furthermore, every ideal of A_q is isomorphic to one of the form (4)–(5), with pairs (X, Y) and (X', Y') corresponding to isomorphic ideals if and only if $(X', Y') \sim (q^k X, q^m Y)$ for some $k, m \in \mathbb{Z}$. As mentioned in the Introduction, this result is similar to that for the Weyl algebra A_1, and in the existing literature there are several different proofs (see [BW1], [BW2], [BC], [BCE]). Each of these proofs can be extended to the quantum case. In Section 5, we will outline a proof generalizing the arguments of [BC], [BCE]: the advantage of this approach is that it exhibits an interesting connection between A_q and the representation theory of double affine Hecke algebras $\mathbb{H}_{1,q^{-1/2}}(S_n)$, which may be of independent interest.

We now discuss some interesting implications of Theorem 1.

3. The Picard group of A_q

We begin by introducing notation. Being a Noetherian domain, the algebra A_q can be embedded into a quotient skew-field, which we denote by Q. The spaces of nonzero (Laurent) polynomials in x and y form two Ore subsets in A_q: we write $\mathbb{C}(x)[y^{\pm 1}] \subset Q$, resp. $\mathbb{C}(y)[x^{\pm 1}] \subset Q$, for the corresponding localizations. Every element $a \in \mathbb{C}(x)[y^{\pm 1}]$ can be uniquely written in the form $a = \sum_{m \le i \le n} a_i(x) y^i$, with $a_i \in \mathbb{C}(x)$ and $a_m, a_n \ne 0$. We call $a_n(x)$ the *leading coefficient* of a, and the difference $n - m$ the *degree* of a.

Lemma 1. *Every ideal M in A_q is isomorphic to a fractional ideal M_x satisfying*

(1) $M_x \subset \mathbb{C}(x)[y^{\pm 1}]$ *and* $M_x \cap \mathbb{C}[x^{\pm 1}] \ne \{0\}$;

(2) *the leading coefficients of all elements in M_x belong to $\mathbb{C}[x^{\pm 1}]$;*

(3) M_x *contains an element with a constant leading coefficient.*

If M_x and M_x' are two fractional ideals of A_q, both isomorphic to M, and satisfying (1)–(3), then there is a unit $u \in A_q$ such that $M_x' = u\, M_x$.

Proof. The proof is essentially the same as in the Weyl algebra case (see [BW2], Lemma 5.1). First, by [St], Lemma 4.2, every ideal class of A_q contains a representative M such that $M \cap \mathbb{C}[x^{\pm 1}] \ne \{0\}$. The leading coefficients of all the elements of M form an ideal in $\mathbb{C}[x^{\pm 1}]$; taking a generator $p(x) \in \mathbb{C}[x^{\pm 1}]$ of this ideal, we set $M_x := p^{-1} M$. It is easy to see that M_x thus defined satisfies the properties (1)–(3).

Now, if M_x' is another (fractional) ideal isomorphic to M, we have $M_x' = \gamma M_x$ for some $\gamma \in Q$. If both M_x and M_x' satisfy (1), then γ must be a unit in $\mathbb{C}(x)[y^{\pm 1}]$ and hence has the form $\gamma = f(x) y^k$, with $f(x) \in \mathbb{C}(x) \setminus \{0\}$ and $k \in \mathbb{Z}$. Property (2) forces $f(x)$ to be polynomial, i. e. $f(x) \in \mathbb{C}[x^{\pm 1}]$, and then (3) implies that $f(x) = \alpha\, x^m$ for some $\alpha \in \mathbb{C}^*$ and $m \in \mathbb{Z}$. Thus $\gamma = \alpha x^m y^k$ is a unit in A_q. $\qquad\square$

Given any (fractional) ideal $M \subset Q$ of A_q, its endomorphism ring is naturally identified with a subring of Q:

$$\mathrm{End}_A(M) = \{\gamma \in Q \mid \gamma M \subseteq M\}.$$

This yields a group embedding $\mathrm{Aut}_A(M) \hookrightarrow Q^*$, whose image we denote by $U(M)$. Thus $U(M) = \{\gamma \in Q^* \mid \gamma M = M\}$. In particular, if $M = A_q$, we get the group of units $U(A_q)$ of A_q canonically embedded in Q^*.

Lemma 2. *Let M_x be a fractional ideal of A_q satisfying the properties (1)–(3) of Lemma 1. Then $U(M_x) \subseteq U(A_q)$. Moreover, if $M_x' \cong M_x$ is another representative satisfying (1)–(3), we have $U(M_x') = U(M_x)$ in $U(A_q)$. Thus the subgroup $U(M_x) \subseteq U(A_q)$ depends only on the class of M_x in $\mathcal{R}^q$.*

Proof. The fact that $U(M_x) \subseteq U(A_q)$ is immediate from (the last statement of) Lemma 1. It implies that every element in $U(M_x)$ has the form $\gamma = \beta x^k y^m$, where $\beta \in \mathbb{C}^*$, $k, m \in \mathbb{Z}$. Using the commutation relations of A_q, it is easy to see that $u\gamma u^{-1} = \tilde{\beta} x^k y^m$ for any $u \in U(A_q)$. Hence $uU(M_x)u^{-1} = U(M_x)$, which proves the second claim of the lemma. $\square$

Remark. Reversing the roles of x and y in the above lemmas, we obtain another set of distinguished representatives $M_y \subset \mathbb{C}(y)[x^{\pm 1}]$ for any given class in $\mathcal{R}^q$. The ideals (4) and (5) appearing in Theorem 1 are examples of such representatives.

The next result is an important consequence of Theorem 1.

Theorem 2. *Let M be a noncyclic right ideal of A_q. Then* $\text{Aut}_A(M) \cong \mathbb{C}^*$.

Proof. First of all, note that $\mathbb{C}^* \subseteq U(M) \subset Q$ for any ideal M. Now suppose that $\text{Aut}_A(M) \ncong \mathbb{C}^*$. Choose a representative M_x in the isomorphism class of M satisfying the conditions (1)–(3) of Lemma 1. Since $\text{Aut}_A(M_x) \cong \text{Aut}_A(M) \ncong \mathbb{C}^*$, we have $\mathbb{C}^* \subsetneq U(M_x)$.

By Theorem 1, we may assume that M_x has the form (4) and take M_y to be the second representative (5). The ideals M_x and M_y are related by $M_y = \kappa M_x$, where $\kappa = \kappa(x, y) \in Q$ is defined in (6). It follows that

$$\gamma \in U(M_x) \iff \kappa\gamma\kappa^{-1} \in U(M_y).$$

Now choose $\gamma \in U(M_x) \setminus \mathbb{C}^*$. Without loss of generality, we can write $\gamma = x^k y^l$ for some $(k, l) \neq (0, 0)$. Then $\kappa\gamma\kappa^{-1} = \alpha x^c y^d$ for some $(c, d) \neq (0, 0)$ and $\alpha \in \mathbb{C}^*$. Whence $\kappa\gamma^m\kappa^{-1} = (\alpha x^c y^d)^m$ for every $m \in \mathbb{Z}$. It follows that, for any $m \in \mathbb{Z}$, there exists $\alpha_m \in \mathbb{C}^*$ such that

$$\gamma^{-m}\kappa\gamma^m\kappa^{-1} = \alpha_m x^{m(c-k)} y^{m(d-l)}. \tag{7}$$

If $(k, l) \neq (c, d)$, we may choose m so that $\max\{m(c-k), m(d-l)\} > 0$. For such m, the equation (7) clearly does not hold. Therefore we have $(c, d) = (k, l)$. Further, by comparing the constant terms of the Laurent series expansions on both sides of (7) (for $m = 1$), we find that $\alpha_1 = 1$. Using the commutation relations in Q, we may therefore rewrite the equation (7) for $m = 1$ in the form

$$\kappa(x, y) = \kappa(q^l x, q^{-k} y). \tag{8}$$

Expanding now κ into the Laurent series

$$\kappa(x, y) = 1 + \sum_{s,r \geq 0} a_{sr} y^{-s-1} x^{-r-1}, \quad a_{sr} = q^s j Y^s X^r i,$$

and substituting it into (8), we get

$$a_{sr}(1 - q^{k(s+1)-l(r+1)}) = 0 \quad \text{for all } r, s \in \mathbb{Z}_+.$$

It follows that $a_{rs} = 0$ for all r, s such that $k(s + 1) \neq l(r + 1)$. Now, since $X \in \mathrm{GL}_n(\mathbb{C})$, the characteristic polynomial of X has a nonzero constant term. Hence $1 = \sum_{p=1}^{n} c_p X^p$ for some $c_p \in \mathbb{C}$. Suppose that $s, r \in \mathbb{Z}_+$ satisfy $k(s + 1) = l(r + 1)$. Then k and l are both nonzero. Since $k(s + 1) \neq l(r + p + 1)$ for any $p > 0$, we have

$$a_{sr} = q^s j Y^s X^r i = \sum_{p=1}^{n} c_p q^s j Y^s X^{r+p} i = \sum_{p=1}^{n} c_p a_{sr+p} = 0.$$

This shows that $a_{sr} = 0$ for all $s, r \in \mathbb{Z}_+$. Thus $\kappa = 1$ and $M_x = M_y = A_q$. It follows that $M \cong M_x$ is a cyclic ideal. $\qquad\square$

Now, using Theorem 1, we compute the Picard group of the algebra A_q. Recall that the elements of $\mathrm{Pic}(A_q)$ are the isomorphism classes of invertible bimodules of A_q, which are symmetric over $\mathbb{C}$. There is a natural group homomorphism $\Omega \colon \mathrm{Aut}(A_q) \to \mathrm{Pic}(A_q)$, taking $\sigma \in \mathrm{Aut}(A_q)$ to the class of the bimodule $(A_q)^{\sigma}$, which is isomorphic to A_q as a right module, with left action of A_q twisted by σ^{-1}. By [F], Theorem 1, the kernel of this homomorphism is precisely the group $\mathrm{Inn}(A_q)$ of inner automorphisms of A_q, while $\mathrm{Im}(\Omega)$ consists of those invertible A_q-bimodules that are cyclic as right modules. Since A_q is a domain, an invertible bimodule over A_q is just a right ideal M of A_q such that $\mathrm{End}_{A_q}(M) \cong A_q$ as $\mathbb{C}$-algebras. This last condition implies that $\mathrm{Aut}_{A_q}(M) \cong U(A_q)$, so by Theorem 2, M is indeed cyclic. Thus Ω is surjective, and we have

Proposition 1. *The canonical sequence of groups*

$$1 \to \mathrm{Inn}(A_q) \to \mathrm{Aut}(A_q) \xrightarrow{\Omega} \mathrm{Pic}(A_q) \to 1$$

is exact.

With Proposition 1, the problem of computing $\mathrm{Pic}(A_q)$ reduces to describing the automorphisms of A_q. Let $(\mathbb{C}^*)^2$ denote the direct product of multiplicative groups of $\mathbb{C}$. We define an action of $\mathrm{SL}_2(\mathbb{Z})$ on $(\mathbb{C}^*)^2$ by

$$g \colon (\alpha, \beta) \mapsto (\alpha^a \beta^b, \alpha^c \beta^d), \quad g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}), \tag{9}$$

and form the semidirect product $(\mathbb{C}^*)^2 \rtimes \mathrm{SL}_2(\mathbb{Z})$ relative to this action. The following result is probably well known (see, e.g., [AD1], [AD2]).

Lemma 3. $\mathrm{Aut}(A_q) \cong (\mathbb{C}^*)^2 \rtimes \mathrm{SL}_2(\mathbb{Z})$.

Proof. We first recall the well-known presentation of $\mathrm{SL}_2(\mathbb{Z})$ as a quotient of the braid group B_2:

$$\mathrm{SL}_2(\mathbb{Z}) = \langle g_1, g_2 \mid g_1 g_2 g_1 = g_2 g_1 g_2, (g_1 g_2)^6 = 1 \rangle. \tag{10}$$

The braid generators of g_1 and g_2 correspond under (10) to the matrices

$$g_1 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad g_2 = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}.$$

Now, using this presentation, it is easy to check that

$$g_1 \colon (x, y) \mapsto (yx, y), \quad g_2 \colon (x, y) \mapsto (x, yx^{-1}).$$

extends to a well-defined group homomorphism $\mathrm{SL}_2(\mathbb{Z}) \to \mathrm{Aut}(A_q)$ of the form

$$g \colon (x, \, y) \mapsto (\alpha_g \, y^b x^a, \beta_g \, y^d x^c),$$

where $g \in \mathrm{SL}_2(\mathbb{Z})$ as in (9) and $\alpha_g, \beta_g \in \mathbb{C}^*$ are some constants depending on g. On the other hand, there is an obvious homomorphism $(\mathbb{C}^*)^2 \to \mathrm{Aut}(A_q)$, mapping $(x, \, y) \mapsto (\alpha x, \beta y)$ for $(\alpha, \beta) \in (\mathbb{C}^*)^2$. These two homomorphisms fit together giving

$$(\mathbb{C}^*)^2 \rtimes \mathrm{SL}_2(\mathbb{Z}) \to \mathrm{Aut}(A_q), \tag{11}$$

which is easily seen to be injective.

On the other hand, any element $\sigma \in \mathrm{Aut}(A_q)$ takes units to units: in particular, it maps the generators (x, y) to elements of the form $(\alpha y^b x^a, \beta y^d x^c)$, with $\alpha, \beta \in c^*$ and $a, b, c, d \in \mathbb{Z}$. The invertibility of σ means that $ad - bc = \pm 1$ and the relation $\sigma(xyx^{-1}y^{-1}) = q$ ensures that $ad - bc = 1$. This proves that (11) is surjective. $\qquad\square$

Under the identification of Lemma 3, the inner automorphisms of A_q correspond to the elements $(q^n x, q^m y; 1) \in (\mathbb{C}^*)^2 \rtimes \mathrm{SL}_2(\mathbb{Z})$, with $n, m \in \mathbb{Z}$. Indeed, one can compute easily that the canonical projection $U(A_q) \twoheadrightarrow \mathrm{Inn}(A_q)$, $u \mapsto \mathrm{Ad}(u)$, is given on generators by $\mathrm{Ad}(u) \colon (x, y) \mapsto (q^{-b} x, q^a y)$, where $u = \alpha x^a y^b \in U(A_q)$. Thus, we arrive at the following theorem, which is the main result of this paper.

Theorem 3. *With identification of Lemma 3, the canonical map Ω induces an isomorphism of groups*

$$\mathrm{Pic}(A_q) \cong (\mathbb{C}^*/\mathbb{Z})^2 \rtimes \mathrm{SL}_2(\mathbb{Z}),$$

where $\mathbb{Z}$ is identified with the cyclic subgroup of $\mathbb{C}^*$ generated by q.

We end this section with a side observation. Assume that $|q| \neq 1$. Regarding $\mathbb{C}^*$ as complex analytic space, we can then identify the quotient $\mathbb{C}^*/\mathbb{Z}$ with a (smooth) elliptic curve X. Let $\mathcal{D}^b(X)$ denote the bounded derived category of coherent sheaves on X, and let $\mathrm{Auteq}\, \mathcal{D}^b(X)$ be the group of (exact) auto-equivalences of this category. Comparing our Theorem 3 with results of [Or], [ST], we get

Corollary 1. *There is a natural group isomorphism*

$$\gamma \colon \mathrm{Pic}(A_q) \xrightarrow{\sim} \mathrm{Auteq}(\mathcal{D}^b(X))/\mathbb{Z}, \tag{12}$$

where $\mathbb{Z} \subset \mathrm{Auteq}(\mathcal{D}^b(X))$ corresponds to the subgroup of translation functors on $\mathcal{D}^b(X)$.

We will briefly explain how to construct the isomorphism γ. Let $x_0 \in X$ be the point corresponding to the image of $1 \in \mathbb{C}^*$ under the canonical projection $\mathbb{C}^* \twoheadrightarrow X$. By [Or] (see also [ST]), the group $\mathrm{Auteq}\, \mathcal{D}^{\mathrm{b}}(X)$ is then generated by the functors

$$T_{\mathcal{O}}, \ T_{\mathcal{O}_{x_0}} \quad \text{and} \quad R_x, \ L_x \quad (x \in X), \tag{13}$$

where L_x is induced by tensoring with the line bundle $\mathcal{O}(x - x_0)$, R_x is the pull-back via the automorphism $X \to X, z \mapsto x \cdot z$, and $T_{\mathcal{E}}$ denotes the Fourier–Mukai transform with kernel

$$\mathrm{Cone}(\mathcal{E}^{\vee} \boxtimes \mathcal{E} \to \Delta_* \mathcal{O}_X), \quad \mathcal{E} \in \mathcal{D}^{\mathrm{b}}(X).$$

Now notice that there is the obvious homomorphism

$$\gamma_1 \colon X \times X \to \mathrm{Auteq}(\mathcal{D}^{\mathrm{b}}(X))/\mathbb{Z}, \quad (x, y) \mapsto L_x R_y. \tag{14}$$

More interestingly, using the known relations between the functors (13) (see [ST], Section 3d), one can easily check that $g_1 \mapsto T_{\mathcal{O}_{x_0}}$ and $g_2 \mapsto T_{\mathcal{O}}$ extend to a well-defined homomorphism

$$\gamma_2 \colon \mathrm{SL}_2(\mathbb{Z}) \to \mathrm{Auteq}(\mathcal{D}^{\mathrm{b}}(X))/\mathbb{Z}. \tag{15}$$

Another direct calculation shows that the maps (14) and (15) agree with each other extending to the semidirect product $(X \times X) \rtimes \mathrm{SL}_2(\mathbb{Z})$. By Theorem 3, this defines the desired isomorphism (12). It would be very interesting to find a conceptual explanation for this isomorphism (cf. [ST], Remark 1.5). The results of [BaEG] as well as recent papers [SV] and [Po] suggest that the existence of γ may not be a mere coincidence.

4. Morita classification

We begin by classifying the algebras A_q up to Morita equivalence within the family $\{A_q\}$ and then consider the general case.

Theorem 4. *Under the assumption* (1), *the algebras A_q and $A_{q'}$ are Morita equivalent if and only if they are isomorphic, i. e., if and only if $q' = q$ or $q' = q^{-1}$.*

Proof. The fact that $A_q \cong A_{q'} \iff q' = q^{\pm 1}$ follows easily from the defining relations of A_q and $A_{q'}$ (see [J], Theorem 1.3). Now assume that $A_{q'}$ and A_q are Morita equivalent. Then $A_{q'} \cong \mathrm{End}_{A_q}(M)$ for some f. g. projective right module M. Since $A_{q'}$ is a domain, M is isomorphic to a right ideal in A_q (see, e.g., [BEG], Lemma 3). Since $\mathrm{End}_{A_q}(M) \cong A_{q'}$, $\mathrm{Aut}_A(M) \not\cong \mathbb{C}^*$. By Theorem 2, $M \cong A_q$. Therefore, $A_{q'} \cong A_q$, which implies that $q' = q^{\pm 1}$. $\qquad\square$

Remark. Theorem 4 was first proven in [RS] by a different method. If $q = e^{2\pi i \theta}$ and $q' = e^{2\pi i \theta'}$, with $\theta, \theta' \in \mathbb{R} \backslash \mathbb{Q}$, then Theorem 4 says that A_θ and $A_{\theta'}$ are equivalent if and only if $\theta' \pm \theta \in \mathbb{Z}$. This result should be compared with a well-known Morita classification of smooth noncommutative tori $\mathcal{A}_\theta$ (see [R1]): in that case, $\mathcal{A}_\theta$ and $\mathcal{A}_{\theta'}$ are (strongly) Morita equivalent if and only if θ and θ' are in the same orbit of $\mathrm{GL}_2(\mathbb{Z})$ acting on $\mathbb{R} \backslash \mathbb{Q}$ by fractional linear transformations. Thus, unlike in the smooth case, there are no interesting Morita equivalences between the algebras A_q for different values of q. However, in the Morita class of each A_q there are many non-isomorphic algebras corresponding to different orbits of $\mathrm{Pic}(A_q)$ in $\mathcal{R}$. We will classify these algebras below, using Theorem 1.

We now describe an action of $\mathrm{Pic}(A_q)$ on the (reduced) Calogero–Moser spaces $\bar{\mathcal{C}}_n^q$. In Section 5, we will show that this action comes from a natural action of the braid group B_2 on the double affine Hecke algebra $\mathbb{H}_{1,\tau}(S_n)$ constructed by Cherednik in [C2]. We begin by defining an action of $\mathrm{Aut}(A_q)$ on $\mathcal{C}_n^q$. With the identification of Lemma 3, it suffices to construct two compatible group homomorphisms

$$f_1 \colon (\mathbb{C}^*)^2 \to \mathrm{Aut}(\mathcal{C}_n^q) \quad \text{and} \quad f_2 \colon \mathrm{SL}_2(\mathbb{Z}) \to \mathrm{Aut}(\mathcal{C}_n^q),$$

where $\mathrm{Aut}(\mathcal{C}_n^q)$ denotes the group of regular (algebraic) automorphisms of $\mathcal{C}_n^q$. First, we let

$$f_1(\alpha, \beta) \colon (X, Y) \mapsto (\alpha^{-1} X, \beta^{-1} Y), \quad (\alpha, \beta) \in (\mathbb{C}^*)^2.$$

Next, to define the second homomorphism we will use the presentation (10): on the braid generators, we define f_2 by

$$g_1 \colon (X, Y) \mapsto (Y^{-1} X, Y), \quad g_2 \colon (X, Y) \mapsto (X, YX).$$

A direct calculation then shows that $f_2(g_1 g_2 g_1) = f_2(g_2 g_1 g_2)$ and $f_2(g_1 g_2)^6 = 1$ in $\mathrm{Aut}(\mathcal{C}_n^q)$. Hence, this assignment extends to a well-defined homomorphism $f_2 \colon \mathrm{SL}_2(\mathbb{Z}) \to \mathrm{Aut}(\mathcal{C}_n^q)$. It is also easy to check that f_1 and f_2 are compatible in the sense that

$$f := (f_1, f_2) \colon (\mathbb{C}^*)^2 \rtimes \mathrm{SL}_2(\mathbb{Z}) \to \mathrm{Aut}(\mathcal{C}_n^q).$$

is a group homomorphism. Now, with the identification of Theorem 3, we see that f induces

$$\bar{f} \colon \mathrm{Pic}(A_q) \to \mathrm{Aut}(\bar{\mathcal{C}}_n^q), \tag{16}$$

which defines an action of $\mathrm{Pic}(A_q)$ on each of the spaces $\bar{\mathcal{C}}_n^q$ and hence on their disjoint union $\bar{\mathcal{C}}^q$. On the other hand, $\mathrm{Pic}(A_q)$ acts naturally on the space of ideal classes $\mathcal{R}^q = \mathcal{R}(A_q)$. With these actions, we have

Proposition 2. *The map* $\omega \colon \bar{\mathcal{C}}^q \to \mathcal{R}^q$ *of Theorem* 1 *is equivariant under* $\mathrm{Pic}(A_q)$.

We will prove Proposition 2 in Section 5. Its meaning becomes clear from the following theorem, which gives a geometric classification of algebras Morita equivalent to A_q.

Theorem 5. *There is a natural bijection between the orbits of* $\mathrm{Pic}(A_q)$ *in* $\overline{\mathcal{C}}^q$ *and the isomorphism classes of domains Morita equivalent to* A_q.

Proof. The map $\omega : \overline{\mathcal{C}}^q \to \mathcal{R}^q$ assigns to a point in $\overline{\mathcal{C}}^q$ an isomorphism class $[M]$ of ideals in A_q. Choosing a representative M in such a class and taking its endomorphism ring $\mathrm{End}_{A_q}(M)$ yields a domain Morita equivalent to A_q. Now a well-known result in ring theory (see, e.g, [F], Theorem 1) says that the rings $\mathrm{End}_{A_q}(M)$ and $\mathrm{End}_{A_q}(M')$ are isomorphic iff the corresponding classes $[M]$ and $[M']$ are in the same orbit of $\mathrm{Pic}(A_q)$ in $\mathcal{R}_q$. Since, by the Morita Theorem, every domain equivalent to A_q is of the form $\mathrm{End}_{A_q}(M)$, Theorem 5 follows immediately from Theorem 1 and Proposition 2. □

Remark. If D is an algebra Morita equivalent to A_q, which is *not* a domain, then $D \cong \mathbb{M}_r(A_q)$ for some $r \geq 2$. This follows from the fact that all projective modules over A_q of rank $r \geq 2$ are free.

In general, the domains Morita equivalent to A_q seem to have a complicated structure: they are not easy to describe in terms of generators and relations, like A_q. However, their automorphism groups can be described geometrically, in terms of the action of $\mathrm{Pic}(A_q)$ on $\overline{\mathcal{C}}^q$. Precisely, we have the following consequence of Theorem 1 and Theorem 2.

Proposition 3. *Let M be a noncyclic right ideal of A_q, and let $E = \mathrm{End}_{A_q}(M)$ denote its endomorphism algebra. Then $\mathrm{Aut}_{\mathbb{C}}(E)$ is naturally isomorphic to the isotropy group of the point $\omega^{-1}[M]$, corresponding to the class of M under the Calogero–Moser bijection, for the action of $\mathrm{Pic}(A_q)$ on $\overline{\mathcal{C}}^q$ defined by* (16).

Proof. Put $A := A_q$. Then M is naturally a E-A-bimodule. Let $M^* := \mathrm{Hom}_A(M, A)$ be its dual which is an A-E-bimodule. By the Morita Theorem, we have $M^* \otimes_E M \cong A$ as A-bimodules and $M \otimes_A M^* \cong E$ as E-bimodules. Now consider the canonical sequence of groups

$$1 \to \mathrm{Inn}(E) \to \mathrm{Aut}(E) \xrightarrow{\Omega} \mathrm{Pic}(E) \xrightarrow{\Psi} \mathrm{Pic}(A),$$

where the last map is an isomorphism of groups given by $P \mapsto M^* \otimes_E P \otimes_E M$. The image of Ω consists of those invertible E-bimodules that are cyclic as right E-modules. Given such an E-bimodule L, we have

$$M \otimes_A M^* \otimes_E L \otimes_E M \cong E \otimes_E L \otimes_E M \cong L \otimes_E M \cong M$$

as right A-modules. Hence, $\Psi(L) = M^* \otimes_E L \otimes_E M$ is in the stabilizer of $[M]$ under the action of $\mathrm{Pic}(A)$ on $\mathcal{R}^q$. Conversely, if P is an invertible A-bimodule such that $M \otimes_A P \cong M$, then

$$M \otimes_A P \otimes_A M^* \cong M \otimes_A M^* \cong E$$

as right E-modules. Thus, $\Psi^{-1}(P) \cong M \otimes_A P \otimes_A M^*$ is in the image of Ω. It follows now from Theorem 1 that $\text{Im}(\Omega)$ is isomorphic to the stabilizer of $\omega^{-1}[M]$ under the $\text{Pic}(A)$-action on $\overline{\mathcal{C}}^q$. On the other hand, by Theorem 2, $\text{Inn}(E)$ is trivial. Hence $\text{Aut}(E) \cong \text{Im}(\Omega)$. This finishes the proof of the proposition. $\qquad\square$

5. Double affine Hecke algebras and the Calogero–Moser correspondence

In this section, we describe a relation between A_q and the double affine Hecke algebras $H_{q,n} := \mathbb{H}_{1,q^{-1/2}}(S_n)$. Our construction generalizes (and simplifies) the results of [BCE], where a similar relation between A_1 and the rational Cherednik algebra has been studied. A key role in this construction is played by a multiplicative version of the deformed preprojective algebra of a quiver introduced in [CBS]. We draw the reader's attention to the fact that we use a more general form of these algebras in which weights are assigned not only to the vertices but also to the edges of the quiver. To simplify exposition we omit most routine calculations, especially those ones parallel to [BCE]. However, we will give some details in the proof of Proposition 2, since the idea of this proof has not been used in the case of the Weyl algebra.

5.1. DAHA and Calogero–Moser spaces. We recall the presentation of the double affine Hecke algebra $H_{q,n}$ (see [C2]).

Generators:

$$X_1^{\pm 1}, X_2^{\pm 1}, \ldots, X_n^{\pm 1}; \quad T_1, T_2, \ldots, T_{n-1}; \quad \pi.$$

Relations:

$$
\begin{aligned}
&X_i X_j = X_j X_i, \quad T_i X_i T_i = X_{i+1}, \quad \pi X_i = X_{i+1}\pi \quad (1 \le i \le n-1),\\
&T_i T_{i+1} T_i = T_{i+1} T_i T_{i+1}, \quad \pi T_i = T_{i+1}\pi \quad (1 \le i \le n-2),\\
&T_i X_j = X_j T_i \quad (j - i \ne 0, 1), \quad [T_i, T_j] = 0 \quad (|i - j| > 1),\\
&\pi X_n = X_1 \pi, \quad \pi^n T_i = T_i \pi^n \quad (1 \le i \le n-1),\\
&(T_i - \tau)(T_i + \tau^{-1}) = 0, \quad \tau := q^{-1/2}.
\end{aligned}
$$

Further, following [C2], we introduce n pairwise commuting elements in $H_{q,n}$:

$$Y_i := T_i T_{i+1} \ldots T_{n-1} \pi^{-1} T_1^{-1} \ldots T_{i-1}^{-1}, \quad i = 1, 2, \ldots, n,$$

satisfying the relations

$$T_i Y_{i+1} T_i = Y_i \quad \text{and} \quad T_i Y_j = Y_j T_i, \quad j - i \ne 0, 1.$$

The algebra $H_{q,n}$ contains a copy of the finite Hecke algebra $\mathcal{H}_\tau(S_n)$ of the group S_n, which, in turn, contains the idempotents

$$\varepsilon := \frac{\sum_{w \in S_n} \tau^{l(w)} T_w}{\sum_{w \in S_n} \tau^{2l(w)}} \quad \text{and} \quad \varepsilon' := \frac{\sum_{w \in S_{n-1}} \tau^{l(w)} T_w}{\sum_{w \in S_{n-1}} \tau^{2l(w)}}. \tag{17}$$

In formula (17), S_{n-1} is regarded as the subgroup of S_n consisting of permutations that fix the first element in $\{1, 2, \ldots, n\}$; for $w \in S_n$, the element T_w denotes the product $T_{i_1} \ldots T_{i_{l(w)}}$, where $w = s_{i_1} \ldots s_{i_{l(w)}}$ is a reduced expression of w with $s_i := (i, i+1) \in S_n$.

The following proposition is analogous to [EG], Theorem 1.23 and Theorem 1.24, in the case of the rational Cherednik algebra.

Proposition 4 ([O], Theorem 5.1 and Theorem 6.1). (1) *The algebra $\varepsilon H_{q,n} \varepsilon$ is Morita equivalent to $H_{q,n}$, the equivalence $\mathrm{Mod}(H_{q,n}) \to \mathrm{Mod}(\varepsilon H_{q,n} \varepsilon)$ being the canonical functor $\varepsilon \colon M \mapsto \varepsilon M$.*

(2) $\varepsilon H_{q,n} \varepsilon$ is a commutative algebra isomorphic to the coordinate ring of $\mathcal{C}_n^q$.

Next we introduce a multiplicative version of the deformed preprojective algebra of a quiver Q, due to Crawley-Boevey and Shaw [CBS] (see also [vdB]). Our definition is slightly more general than that of [CBS] as we assign weights to both the vertices and the arrows of Q (see Remark below).

5.2. The (generalized) multiplicative preprojective algebra. Let Q be a quiver with vertex set I. Let $\bar{Q}$ be the double of Q obtained by adjoining a reverse arrow a^* to each arrow $a \in Q$. As in [CBS], we extend $a \mapsto a^*$ to an involution on $\bar{Q}$ by letting $(a^*)^* = a$, and define the function $\epsilon \colon \bar{Q} \to \{\pm 1\}$ by $\epsilon(a) = 1$ if $a \in Q$ and $\epsilon(a) = -1$ if $a^* \in Q$. Next we choose two sets of parameters (weights): $\{q_v\}_{v \in I}$ and $\{\hbar_a\}_{a \in \bar{Q}}$ with the assumption that $\hbar_{a^*} = \hbar_a$ for all $a \in Q$. The *multiplicative preprojective algebra* $\Lambda^{q,\hbar}(Q)$ is now defined by the algebra homomorphism $\mathbb{C}\bar{Q} \to \Lambda^{q,\hbar}(Q)$, which is *universal* among all algebra homomorphisms $\mathbb{C}\bar{Q} \to R$ satisfying the properties

$$aa^* + \hbar_a \text{ is a unit in } R \text{ for all } a \in \bar{Q},$$
$$\prod_{a \in \bar{Q}} (aa^* + \hbar_a)^{\epsilon(a)} = \sum_{v \in I} q_v e_v \quad \text{in } R.$$

Remark. The original definition of multiplicative preprojective algebras (see [CBS], Definition 1.2) corresponds to the choice $\hbar_a = 1$ for all $a \in \bar{Q}$. To see why we need an extension of this definition consider a quiver Q which consists of a single vertex v and a single loop a. Choosing then $q_v = q$ and $\hbar_a = 0$, we get the algebra $\Lambda^{q,\hbar}(Q)$ isomorphic to A_q (with $a \leftrightarrow x$ and $a^* \leftrightarrow y$). On the other hand, Example 1.3 in [CBS] shows that if $q_v = q$ and $\hbar_a = 1$, then $\Lambda^{q,\hbar}(Q) \cong \mathbb{C}\langle x, y, (1+xy)^{-1} \rangle / (xy - qyx - 1)$, which is a different quantized version of the first Weyl algebra, not isomorphic to A_q (see, e.g., [AD2]).

Now, as in [BCE], Sect. 2.2, we consider the *framed* one-loop quiver $Q = Q_\infty$ with two vertices, $I = \{0, \infty\}$, and two arrows $i \colon 0 \to \infty$ and $X \colon 0 \to 0$. Write $j := i^*$ and $Y := X^*$ for the reverse arrows in $\bar{Q}$. For the vertex and arrow weights, we take $(q_0, q_\infty) = (q, q^{-n})$ and $(\hbar_X, \hbar_i) = (0, 1)$, respectively. The

corresponding algebra $\Lambda := \Lambda^{q,\hbar}(Q)$ can then be identified with the quotient of $\mathbb{C}\bar{Q}\langle U, V \rangle := \mathbb{C}\bar{Q} * \mathbb{C}\langle U, V \rangle$ modulo the relations

$$U = eUe, \quad V = eVe, \quad XU = UX = e, \quad YV = VY = e,$$
$$XY - qYX - qYXij = 0, \quad ji = (q^{-n} - 1)e_\infty,$$

where e and e_∞ are the idempotents corresponding to the vertices 0 and ∞, respectively. The following lemma clarifies the relation between this algebra and A_q: it is a "multiplicative" analogue of [BCE], Lemma 3.

Lemma 4. *A_q is isomorphic to the quotient of Λ by the ideal generated by e_∞.*

In fact, the required isomorphism is induced by $\mathbb{C}\langle x^{\pm 1}, y^{\pm 1} \rangle \to \Lambda/\langle e_\infty \rangle$, $x \mapsto X$, $y \mapsto Y$.

Next, we explain the relation between Λ and the Cherednik algebra $H := H_{q,n}$. To this end, we consider the left projective H-module $P := H\varepsilon' \oplus H\varepsilon$, where ε and ε' are the idempotents defined in (17). The endomorphism ring of P can be identified with a matrix algebra:

$$\text{End}_H(P) = \begin{pmatrix} \varepsilon' H \varepsilon' & \varepsilon' H \varepsilon \\ \varepsilon H \varepsilon' & \varepsilon H \varepsilon \end{pmatrix}.$$

Using this identification, we can define an algebra map $\Theta \colon \mathbb{C}\bar{Q}\langle U, V \rangle \to \text{End}_H(P)^\circ$ by

$$X \mapsto \begin{pmatrix} X_1\varepsilon' & 0 \\ 0 & 0 \end{pmatrix}, \quad Y \mapsto \begin{pmatrix} Y_1\varepsilon' & 0 \\ 0 & 0 \end{pmatrix}, \quad i \mapsto \begin{pmatrix} 0 & 0 \\ \varepsilon & 0 \end{pmatrix}, \quad j \mapsto \begin{pmatrix} 0 & \varepsilon \\ 0 & 0 \end{pmatrix}(q^{-n}-1).$$

The following proposition shows that Θ is a multiplicative analogue of the map Θ^{quiver} constructed in [EGGO] (see loc. cit., (1.6.3)).

Proposition 5. *The map Θ induces an algebra homomorphism $\Lambda \to \text{End}_H(P)^\circ$.*

Proof. The key relation to verify is $\Theta(UVXY - qe - qvw) = 0$; all the other relations are immediate. First, using the fact that $\varepsilon \varepsilon' = \varepsilon' \varepsilon = \varepsilon$, we find

$$\Theta(UVXY - qe - qvw) = \begin{pmatrix} Y_1 X_1 Y_1^{-1} X_1^{-1}\varepsilon' - q\varepsilon' + q(1-q^{-n})\varepsilon & 0 \\ 0 & 0 \end{pmatrix}.$$

By [O], Lemma 4.1, $X_1 Y_1 X_1^{-1} Y_1^{-1} = T_1^{-1} T_2^{-1} \ldots T_{n-1}^{-2} T_{n-2}^{-1} \ldots T_1^{-1}$ in H. Hence $X_1 Y_1 X_1^{-1} Y_1^{-1}$ is an element of the finite Hecke algebra $\mathcal{H}_\tau(S_n) \subset H$. Now consider $\mathcal{H}_\tau(S_n)$ as a right $\mathcal{H}_\tau(S_{n-1})$-module. Its subspace of invariants $V := \mathcal{H}_\tau(S_n)\varepsilon' \subset \mathcal{H}_\tau(S_n)$ is then a left $\mathcal{H}_\tau(S_n)$-module. As shown in [O], Section 4.2, this last module admits a direct decomposition $V = \mathcal{H}_\tau(S_n)\varepsilon \oplus W$, where W has dimension $n - 1$. Moreover, the left multiplication by $Y_1 X_1 Y_1^{-1} X_1^{-1}$ acts on $\mathcal{H}_\tau(S_n)\varepsilon$ as a scalar operator with eigenvalue q^{1-n} and on W with eigenvalue q. The operator $q + q(q^{-n}-1)\varepsilon$ obviously acts on V in the same way, so $Y_1 X_1 Y_1^{-1} X_1^{-1}\varepsilon' - q\varepsilon' + q(1-q^{-n})\varepsilon = 0$, which proves the desired identity. $\square$

5.3. The Calogero–Moser functor. One way to interpret Proposition 5 is to say that P is an H-Λ-bimodule, with right Λ-module structure defined by Θ. In combination with Lemma 4, this allows us to define the functor

$$\mathrm{CM}_n : \mathcal{D}^{\mathrm{b}}(\mathrm{Mod}\, H) \to \mathcal{D}^{\mathrm{b}}(\mathrm{Mod}\, \Lambda) \to \mathcal{D}^{\mathrm{b}}(\mathrm{Mod}\, A_q), \quad V \mapsto (V \otimes_H P) \otimes_{\Lambda}^L A_q.$$

Note that P is a projective module on the left, so tensoring with P over H is an exact functor.

In view of Proposition 4, Theorem 1 of Section 2 is a consequence of the following result.

Theorem 6. (1) *The functor* CM_n *transforms the simple H-modules* (*viewed as 0-complexes in $\mathcal{D}^{\mathrm{b}}(\mathrm{Mod}\, H)$*) *to rank* 1 *projective A_q-modules* (*located in homological degree* -1).

(2) *Two simple H-modules* $V = (V; X_i, T_j, \pi)$ *and* $V' = (V'; X_i', T_j', \pi')$ *correspond to isomorphic A_q-modules if and only if there is* $(k, m) \in \mathbb{Z}^2$ *such that*

$$X_i' = q^k X_i, \quad T_j' = T_j, \quad \pi' = q^m \pi.$$

(*We call such H-modules equivalent.*)

(3) *For every rank* 1 *projective A_q-module, there is a unique* $n \in \mathbb{N}$ *and a simple module V over $H_{q,n}$ such that* $\mathrm{CM}_n(V) \cong M[1]$ *in* $\mathcal{D}^{\mathrm{b}}(\mathrm{Mod}\, A_q)$.

Thus the Calogero–Moser map ω of Theorem 1 is induced by the functors CM_n 'amalgamated' over all n. The proof of Theorem 6 is analogous to [BCE], Theorem 3: it is based on [BC], Theorem 3, and the following key lemma.

To simplify the notation we set $R := \mathbb{C}\langle x^{\pm 1}, y^{\pm 1}\rangle$ and denote by α the (surjective) algebra homomorphism $R \to e\Lambda e$ taking x, x^{-1}, y, y^{-1} to X, U, Y, V, respectively. Using this homomorphism, we define the linear map

$$\chi : R \to \varepsilon H\varepsilon, \quad r \mapsto \Theta(i\alpha(r)j).$$

Lemma 5. *Regarding P as a right Λ-module, we have:*

(1) $\mathrm{Tor}_k^{\Lambda}(P, A_q) = 0$ *for all* $k \neq 1$.

(2) *For $k = 1$, there is an isomorphism of right A_q-modules*

$$\mathrm{Tor}_1^{\Lambda}(P, A_q) \cong \mathrm{Ker}\left(\frac{H\varepsilon \otimes R}{H(\varepsilon \otimes rw - \chi(r) \otimes 1)R} \xrightarrow{\mu} H\varepsilon'\right), \quad (18)$$

where $w = q^{-1}(x^{-1}y^{-1}xy) - 1$.

The notation of Lemma 5 needs some explanation. The multiplication-action map μ in (18) is induced by $\mu(a\varepsilon \otimes r) = a\varepsilon\pi(r)$, where $a\varepsilon$ is viewed as an element in the direct summand of P (note that $a\varepsilon = a\varepsilon\varepsilon'$). The A_q-module structure on the right hand side of (18) descends then from the natural right R-module structure on $\mathrm{Ker}\, \mu$.

Proof. (1) By Proposition 4 (2), $\varepsilon H \varepsilon \cong \mathcal{O}(\mathcal{C}_n^q)$. Write $\mathcal{O} := \mathcal{O}(\mathcal{C}_n^q)$. Then, for every maximal ideal $\mathfrak{m} \subset \mathcal{O}$, $(\mathcal{O}/\mathfrak{m}) \otimes_{\mathcal{O}} (\varepsilon P)$ is a right Λ-module of dimension n over $\mathbb{C}$. It follows that $(\mathcal{O}/\mathfrak{m}) \otimes_{\mathcal{O}} (\varepsilon P) \otimes_{\Lambda} A_q$ is a finite dimensional right A_q-module and hence zero. Since $\varepsilon P \otimes_{\Lambda} A_q$ is a coherent $\mathcal{O}$-module, this implies that $\varepsilon P \otimes_{\Lambda} A_q = 0$ by Nakayama's Lemma. But then, by Morita equivalence of Proposition 4 (1), $P \otimes_{\Lambda} A_q = 0$. To see that the second and higher Tor's vanish we observe that the natural map $\Lambda e_{\infty} \Lambda \to \Lambda$ provides a left (and right) projective resolution of A_q, so A_q has projective dimension 1 as a left Λ-module.

The proof of part (2) is similar to the proof of Theorem 1 in [BCE]. We leave it to the reader. $\qquad \square$

5.4. Equivariance. We now turn to the proof of Proposition 2. Recall that $\mathrm{SL}_2(\mathbb{Z})$ is a quotient of the braid group B_2. Let $\phi \colon B_2 \to \mathrm{SL}_2(\mathbb{Z})$ denote the corresponding projection

$$g_1 \mapsto \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad g_2 \mapsto \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}.$$

The $\mathrm{SL}_2(\mathbb{Z})$-action on $(\mathbb{C}^*)^2$ defined by (9) induces (via ϕ) a B_2-action on $(\mathbb{C}^*)^2$. We write $G := (\mathbb{C}^*)^2 \rtimes B_2$ for the corresponding semidirect product of groups.

Lemma 6 ([C2]). *The following assignment extends to a well-defined group homomorphism* $\Phi_H \colon G \to \mathrm{Aut}_{\mathbb{C}}(H_{q,n})$:

$$(\alpha, \beta; \mathrm{id}) \mapsto \nu_{\alpha,\beta} := (X_i \mapsto \alpha X_i, \ Y_i \mapsto \beta Y_i, \ T_i \mapsto T_i),$$
$$(1, 1; g_1) \mapsto \tau := (X_i \mapsto Y_i X_i, \ Y_i \mapsto Y_i, \ T_i \mapsto T_i),$$
$$(1, 1; g_2) \mapsto \theta := (X_i \mapsto X_i, \ Y_i \mapsto X_i^{-1} Y_i, \ T_i \mapsto T_i).$$

Proof. It is obvious that $\nu_{\alpha,\beta} \in \mathrm{Aut}_{\mathbb{C}}(H_{q,n})$. The fact that θ and τ are also automorphisms, satisfying the braid relation $\theta \tau \theta = \tau \theta \tau$, is part of [C2], Theorem 4.3. (In [C2], θ^{-1} and τ are denoted by τ_+ and τ_-, respectively). The following calculation now completes the proof:

$$(1, 1; g_1)(\alpha, \beta; \mathrm{id})(X_i, Y_i, T_i) = (\alpha \beta Y_i X_i, \beta Y_i, T_i)$$
$$= (\alpha \beta, \beta; \mathrm{id})(1, 1; g_1)(X_i, Y_i, T_i)$$
$$= (g_1(\alpha, \beta); g_1)(X_i, Y_i, T_i),$$

$$(1, 1; g_2)(\alpha, \beta; \mathrm{id})(X_i, Y_i, T_i) = (\alpha X_i, \alpha^{-1} \beta X_i^{-1} Y_i, T_i)$$
$$= (\alpha, \alpha^{-1} \beta; \mathrm{id})(1, 1; g_2)(X_i, Y_i, T_i)$$
$$= (g_2(\alpha, \beta); g_2)(X_i, Y_i, T_i). \qquad \square$$

Note that any automorphism of H in the image of Φ_H fixes the generators $T_1, \dots, T_{n-1}$. It follows that the G-action on H defined by Ψ_H induces a G-action on the spherical algebra $\varepsilon H \varepsilon$ and the left $\varepsilon H \varepsilon$-module $\varepsilon P = \varepsilon H \varepsilon' \oplus \varepsilon H \varepsilon$. In other words,

we have group homomorphisms $\Phi_{\varepsilon H \varepsilon} \colon G \to \operatorname{Aut}_{\mathbb{C}}(\varepsilon H \varepsilon)$ and $\Phi_P \colon G \to \operatorname{Aut}_{\mathbb{C}}(\varepsilon P)$. In addition, we have

Lemma 7. *The following assignment extends to a well-defined group homomorphism* $\Phi_\Lambda \colon G \to \operatorname{Aut}_{\mathbb{C}}(\Lambda)$:

$$(1, 1; g_1) \mapsto (X \mapsto XY, \; Y \mapsto Y, \; j \mapsto Y^{-1} j, \; i \mapsto i\, Y),$$
$$(1, 1; g_2) \mapsto (X \mapsto X, \; Y \mapsto YX^{-1}, \; j \mapsto X\, j, \; i \mapsto i\, X^{-1}),$$
$$(\alpha, \beta; \operatorname{id}) \mapsto (X \mapsto \alpha X, \; Y \mapsto \beta Y, \; j \mapsto j, \; i \mapsto i).$$

Proof. A direct calculation similar to that of Lemma 6. We leave details to the reader.
□

Note that the above action of G on Λ preserves the idempotents e and e_∞ and hence restricts to the subalgebra $e\Lambda e \subset \Lambda$. By Proposition 5, εP is an $\varepsilon H \varepsilon$-$\Lambda$-bimodule. The subspace $\varepsilon Pe := \varepsilon P \otimes_\Lambda \Lambda e \subseteq \varepsilon P$ is preserved by any automorphism in $\operatorname{Aut}_{\mathbb{C}}(\varepsilon P)$, which is in the image of Φ_P. As a result, we have an action of G on εPe. Now form the semidirect product $[\varepsilon H \varepsilon \otimes (e\Lambda e)^\circ] \rtimes G$, with G acting diagonally on $\varepsilon H \varepsilon$ and $e\Lambda e$ as in Lemma 6 and Lemma 7.

Proposition 6. *The action of G on εPe defined above makes it a $[\varepsilon H \varepsilon \otimes (e\Lambda e)^\circ] \rtimes G$-module (equivalently, εPe is a G-equivariant $\varepsilon H \varepsilon$-$e\Lambda e$-bimodule).*

Proof. We need to verify that, for all $h \in \varepsilon H \varepsilon$, $m \in \varepsilon Pe$, $x \in e\Lambda e$ and $\sigma \in G$,

$$(\sigma \cdot h)(\sigma.m)(\Theta(\sigma \cdot x)) = \sigma \cdot (hm\Theta(x)). \tag{19}$$

Note that εPe may be identified with the direct summand $\varepsilon H \varepsilon'$ of εP. The G-action on $\varepsilon H \varepsilon'$ as well as that on $\varepsilon H \varepsilon$ are obtained by restricting Φ_H. It follows that $\sigma(h \cdot m) = (\sigma \cdot h)(\sigma \cdot m)$ for all $\sigma \in G$, $h \in \varepsilon H \varepsilon$ and $m \in \varepsilon Pe$. Therefore, it suffices to verify (19) with $h = 1$. For this, it suffices to show that $\Theta \colon (e\Lambda e)^\circ \to \varepsilon' H \varepsilon'$ is a G-module homomorphism. The following computation completes this verification (remember that we are working in the opposite algebra $(e\Lambda e)^\circ$).

$$\Theta((1, 1; g_1)(X)) = \Theta(YX) = Y_1 X_1 \varepsilon' = (1, 1; g_1)\, \Theta(X),$$
$$\Theta((1, 1; g_1)(Y)) = \Theta(Y) = Y_1 \varepsilon' = (1, 1; g_1)\, \Theta(Y),$$
$$\Theta((1, 1; g_2)(X)) = \Theta(X) = X_1 \varepsilon' = (1, 1; g_2)\, \Theta(X),$$
$$\Theta((1, 1; g_2)(Y)) = \Theta(YX^{-1}) = Y_1 X_1^{-1} \varepsilon' = (1, 1; g_2)\, \Theta(Y),$$
$$\Theta((\alpha, \beta; \operatorname{id})(X)) = \Theta(\alpha X) = \alpha X_1 \varepsilon' = (\alpha, \beta; \operatorname{id})\, \Theta(X),$$
$$\Theta((\alpha, \beta; \operatorname{id})(Y)) = \Theta(\beta Y) = \beta Y_1 \varepsilon' = (\alpha, \beta; \operatorname{id})\, \Theta(Y). \qquad \square$$

Proof of Proposition 2. To simplify the notation, we set $\Gamma := \varepsilon H \varepsilon \otimes (e \Lambda e)^\circ$ and $\Gamma_A := \varepsilon H \varepsilon \otimes A_q^\circ$. Note that G acts naturally on A_q. Further, the algebra homomorphism $e \Lambda e \to A_q$ given by Lemma 4 is G-equivariant. Hence, the induced homomorphism $\Gamma \to \Gamma_A$ is G-equivariant. Now, for any G-equivariant Γ-module M, we have an isomorphism Γ_A-modules

$$M \otimes_{e \Lambda e} A_q \cong \Gamma_A \otimes_\Gamma M \cong (\Gamma_A \rtimes G) \otimes_{\Gamma \rtimes G} M.$$

It follows that $M \otimes_{e \Lambda e}^L A_q \cong (\Gamma_A \rtimes G) \otimes_{\Gamma \rtimes G}^L M$ in the derived category of $\varepsilon H \varepsilon$-$A_q$-bimodules. Hence $M \otimes_{e \Lambda e}^L A_q$ is a G-equivariant $\varepsilon H \varepsilon$-$A_q$-bimodule. By Proposition 6, the $\varepsilon H \varepsilon$-$A_q$-bimodule $\tilde{P} := \varepsilon P e \otimes^L A_q$ is G-equivariant. Identify $\varepsilon H \varepsilon \cong \mathcal{O}(\mathcal{C}_n^q)$ as in Proposition 4, and let $\mu \colon \mathcal{O} \otimes \tilde{P} \otimes A_q \to \tilde{P}$ denote the structure map of the bimodule $\tilde{P}$ with this identification. Now, for any $\sigma \in G$ and for any maximal ideal $\mathfrak{m} \subset \mathcal{O}$, the commutative diagram

$$
\begin{array}{ccc}
\mathcal{O} \otimes \tilde{P} \otimes A_q & \xrightarrow{\mu} & \tilde{P} \\
{\scriptstyle \sigma \otimes \sigma \otimes \sigma} \downarrow & & \downarrow {\scriptstyle \sigma} \\
\mathcal{O} \otimes \tilde{P} \otimes A_q & \xrightarrow{\mu} & \tilde{P}
\end{array}
$$

induces

$$
\begin{array}{ccc}
(\mathcal{O}/\mathfrak{m}) \otimes [(\mathcal{O}/\mathfrak{m}) \otimes_\mathcal{O} \tilde{P}] \otimes A_q & \xrightarrow{\mu} & (\mathcal{O}/\mathfrak{m}) \otimes_\mathcal{O} \tilde{P} \\
{\scriptstyle \sigma \otimes \sigma \otimes \sigma} \downarrow & & \downarrow {\scriptstyle \sigma} \\
(\mathcal{O}/\mathfrak{m})_\sigma \otimes [(\mathcal{O}/\mathfrak{m})_\sigma \otimes_\mathcal{O} \tilde{P}] \otimes A_q & \xrightarrow{\mu} & (\mathcal{O}/\mathfrak{m})_\sigma \otimes_\mathcal{O} \tilde{P},
\end{array}
$$

where $(\mathcal{O}/\mathfrak{m})_\sigma$ denotes the twisting by σ of the $\mathcal{O}$-module $\mathcal{O}/\mathfrak{m} \cong \mathbb{C}$. It follows that the map $\omega_n \colon \mathcal{C}_n^q \to \mathcal{R}(A_q)$ is G-equivariant. Proposition 2 follows once we note that, by Theorem 3, $\mathrm{Pic}(A_q) \cong G/\mathbb{Z}^2$, and the G-action on $\mathcal{C}_n^q$ descends to the $\mathrm{Pic}(A_q)$-action on $\overline{\mathcal{C}}_n^q$. $\qquad\square$

6. Appendix: Comparison with noncommutative tori in the smooth case

In the following table, we compare the properties of algebraic and smooth noncommutative tori. In the algebraic case, most ring-theoretic and homological properties follow from the fact that A_q is a simple hereditary domain (see [J]); the classification of projectives and Morita classification are results of this paper, and the computation of Hochschild and cyclic homology can be found in [Wa]. In the smooth case, the description of projective modules and Morita classification can be found in [R2], the Picard group of $\mathcal{A}_q$ is computed in [K1], and results on Hochschild, cyclic homology, and homological dimension follow from [Co].

Algebraic vs smooth noncommutative tori

Properties	Algebraic torus A_q ($q^n \neq 1, n \in \mathbb{Z}$)	Smooth torus $\mathcal{A}_\theta$ ($q = e^{2\pi i \theta}, \theta \in \mathbb{R} \setminus \mathbb{Q}$)
Cancelation	Any f.g. projective module of rank ≥ 2 is free.	Stably isomorphic f.g. projective modules are isomorphic.
Projective modules	Any f.g. projective module is either free or isomorphic to a right ideal.	Any f.g. projective module is isomorphic to a standard module introduced by Connes.
Grothendieck group	$\mathbb{Z}$	$\mathbb{Z}^2$
Isomorphism classes	A_q is isomorphic to $A_{q'}$ if and only if $q' = q^{\pm 1}$.	$\mathcal{A}_\theta$ is isomorphic to $\mathcal{A}_{\theta'}$ if and only if $\theta \pm \theta' \in \mathbb{Z}$.
Morita classes	A_q is Morita equivalent to $A_{q'}$ if and only if $q' = q^{\pm 1}$. A unital algebra is Morita equivalent to A_q if and only if A is isomorphic to $\mathbb{M}_n(A_q)$ for some $n \geq 2$ or $\mathrm{End}_{A_q}(M)$ for some right ideal M of A_q.	A unital C^*-algebra is Morita equivalent to $\mathcal{A}_\theta$ if and only if it is isomorphic $\mathbb{M}_n(\mathcal{A}_{\theta'})$ for some $n \geq 2$, and $\theta' = \frac{a\theta + b}{c\theta + d}$, where $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{GL}(2, \mathbb{Z})$.
Outer automorphism group	$(\mathbb{C}^*/\mathbb{Z})^2 \rtimes \mathrm{SL}(2, \mathbb{Z})$	$(\mathbb{C}^*/\mathbb{Z})^2 \rtimes \mathrm{SL}(2, \mathbb{Z})$
Picard group	$(\mathbb{C}^*/\mathbb{Z})^2 \rtimes \mathrm{SL}(2, \mathbb{Z})$	If θ is not quadratic, $\mathrm{Pic}(\mathcal{A}_\theta)$ is isomorphic to $\mathrm{Out}(\mathcal{A}_\theta)$; if θ is quadratic, $\mathrm{Pic}(\mathcal{A}_\theta)$ is isomorphic to $(\mathrm{Out}\,\mathcal{A}_\theta) \rtimes \mathbb{Z}$.
Hochschild homology	$\mathrm{HH}_n(A_q) = \begin{cases} \mathbb{C}, & n = 0, 2, \\ \mathbb{C}^2, & n = 1, \\ 0, & n > 2. \end{cases}$	If θ satisfies a Diophantine condition, $\mathrm{HH}_n(\mathcal{A}_\theta) = \begin{cases} \mathbb{C}, & n = 0, 2, \\ \mathbb{C}^2, & n = 1, \\ 0, & n > 2. \end{cases}$ If θ does not satisfy a Diophantine condition, $\mathrm{HH}_0(\mathcal{A}_\theta)$ and $\mathrm{HH}_1(\mathcal{A}_\theta)$ are infinite-dimensional.
Cyclic homology	$\mathrm{HP}_{\mathrm{odd}}(A_q) = \mathrm{HP}_{\mathrm{ev}}(A_q) = \mathbb{C}^2$	$\mathrm{HP}_{\mathrm{odd}}(\mathcal{A}_\theta) = \mathrm{HP}_{\mathrm{ev}}(\mathcal{A}_\theta) = \mathbb{C}^2$
Homological dimension	1	2

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Yu. Berest, Department of Mathematics, Cornell University, Ithaca, NY 14853-4201,
U.S.A.

E-mail: berest@math.cornell.edu

A. Ramadoss, Departement Mathematik, ETH Zürich, 8092 Zürich, Switzerland

E-mail: ajay.ramadoss@math.ethz.ch

X. Tang, Department of Mathematics, Washington University, St. Louis, MO 63139, U.S.A.

E-mail: xtang@math.wustl.edu