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The topological K-theory of certain crystallographic groups

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Abstract. Let Γ be a semidirect product of the form $\mathbb{Z}^n \rtimes_\rho \mathbb{Z}/p$ where p is prime and the $\mathbb{Z}/p$-action ρ on $\mathbb{Z}^n$ is free away from the origin. We will compute the topological K-theory of the real and complex group C*-algebra of Γ and show that Γ satisfies the unstable Gromov–Lawson–Rosenberg Conjecture. On the way we will analyze the (co-)homology and the topological K-theory of the classifying spaces $B\Gamma$ and $\underline{B}\Gamma$. The latter is the quotient of the induced $\mathbb{Z}/p$-action on the torus T^n.

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Contents

0. Introduction

Let p be a prime. Let $\rho\colon \mathbb{Z}/p \to \mathrm{Aut}(\mathbb{Z}^n) = \mathrm{GL}(n,\mathbb{Z})$ be a group homomorphism. Throughout this paper we will assume:

Condition 0.1 (Free conjugation action). The induced action of $\mathbb{Z}/p$ on $\mathbb{Z}^n$ is free when restricted to $\mathbb{Z}^n - 0$.

Denote by

$$\Gamma = \mathbb{Z}^n \rtimes_\rho \mathbb{Z}/p \tag{0.2}$$

the associated semidirect product. Since Γ has a finitely generated, free abelian subgroup which is normal, maximal abelian, and has finite index, Γ is isomorphic to a crystallographic group. An example of such group Γ is given by $\mathbb{Z}^{p-1} \rtimes_\rho \mathbb{Z}/p$ where the action ρ is given by the regular representation $\mathbb{Z}[\mathbb{Z}/p]$ modulo the ideal generated by the norm element. When $n = 1$ and $p = 2$, Γ is the infinite dihedral group.

Let $B\Gamma := \Gamma\backslash E\Gamma$ be the classifying space of Γ. Denote by $\underline{E}\Gamma$ be the classifying space for proper group actions of Γ. Let $\underline{B}\Gamma = \Gamma\backslash\underline{E}\Gamma$. The space $\underline{B}\Gamma$ is the quotient of the torus T^n under the $\mathbb{Z}/p$-action associated to ρ. It is not a manifold, but an orbifold quotient.

To compute the K-theory of the C*-algebra, we will use the Baum–Connes Conjecture which predicts for a group G that the complex and real assembly maps

$$K_n^G(\underline{E}G) \xrightarrow{\cong} K_n(C_r^*(G)),$$
$$\mathrm{KO}_n^G(\underline{E}G) \xrightarrow{\cong} \mathrm{KO}_n(C_r^*(G;\mathbb{R}))$$

are bijective for $n \in \mathbb{Z}$. The point of the Baum–Connes Conjecture is that it identifies the very hard to compute topological K-theory of the group C*-algebra of G to the better accessible evaluation at $\underline{E}G$ of the equivariant homology theory given by equivariant topological K-theory. The Baum–Connes Conjecture has been proved for a large class of groups which includes crystallographic groups (and many more) in [19]. We will later use the composite maps, where in each case the second map is induction with the projection $\Gamma \to \{1\}$.

$$K_m(C_r^*(\Gamma)) \xleftarrow{\cong} K_n^\Gamma(\underline{E}\Gamma) \to K_m(\underline{B}\Gamma),$$
$$\mathrm{KO}_m(C_r^*(\Gamma;\mathbb{R})) \xleftarrow{\cong} \mathrm{KO}_m^\Gamma(\underline{E}\Gamma) \to \mathrm{KO}_m(\underline{B}\Gamma).$$

Next we describe the main results of this paper. We will show in Lemma 1.9 (i) that $k = n/(p-1)$ is an integer. Let $\mathcal{P}$ be the set of conjugacy classes $\{(P)\}$ of finite non-trivial subgroups of Γ.

Theorem 0.3 (Topological K-theory of the complex group C*-algebra). *Let $\Gamma = \mathbb{Z}^n \rtimes_\rho \mathbb{Z}/p$ be a group satisfying Condition 0.1.*

(i) *If $p = 2$,*

$$K_m(C_r^*(\Gamma)) \cong \begin{cases} \mathbb{Z}^{3\cdot 2^{n-1}}, & m \text{ even}, \\ 0, & m \text{ odd}. \end{cases}$$

If p is odd,

$$K_m(C_r^*(\Gamma)) \cong \begin{cases} \mathbb{Z}^{d_{\mathrm{ev}}}, & m \text{ even}, \\ \mathbb{Z}^{d_{\mathrm{odd}}}, & m \text{ odd}, \end{cases}$$

where

$$d_{\mathrm{ev}} = \frac{2^{(p-1)k} + p - 1}{2p} + \frac{(p-1)\cdot p^{k-1}}{2} + (p-1)\cdot p^k,$$
$$d_{\mathrm{odd}} = \frac{2^{(p-1)k} + p - 1}{2p} - \frac{(p-1)\cdot p^{k-1}}{2}.$$

In particular $K_m(C_r^*(\Gamma))$ is always a finitely generated free abelian group.

(ii) *There is an exact sequence*

$$0 \to \bigoplus_{(P)\in\mathcal{P}} \widetilde{R}_{\mathbb{C}}(P) \to K_0(C_r^*(\Gamma)) \to K_0(\underline{B}\Gamma) \to 0,$$

where $\widetilde{R}_{\mathbb{C}}(P)$ is the kernel of the map $R_{\mathbb{C}}(P) \to \mathbb{Z}$ sending the class $[V]$ of a complex P-representation V to $\dim_{\mathbb{C}}(\mathbb{C} \otimes_{\mathbb{C}P} V)$.

(iii) *The map*

$$K_1(C_r^*(\Gamma)) \xrightarrow{\cong} K_1(\underline{B}\Gamma)$$

is an isomorphism. Restricting to the subgroup $\mathbb{Z}^n$ of Γ induces an isomorphism

$$K_1(C_r^*(\Gamma)) \xrightarrow{\cong} K_1(C_r^*(\mathbb{Z}^n_\rho))^{\mathbb{Z}/p}.$$

Remark 0.4 (Twisted group algebras). The computation of Theorem 0.3 has already been carried out in the case $p = 2$ and in the case $n = 2$ and $p = 3$ in [17], Theorem 0.4, Example 3.7. In view of [17], Theorem 0.3, the computation presented in this paper yields also computations for the topological K-theory $K_*(C_r^*(\Gamma, \omega))$ of twisted group algebras for appropriate cocycles ω. One may investigate whether the whole program of [17] can be carried over to the more general situation considered in this paper.

Remark 0.5 (Computations by Cuntz and Li). Cuntz and Li [13] compute the K-theory of C*-algebras that are associated with rings of integers in number fields. They have to make the assumption that the algebraic number field contains only $\{\pm 1\}$ as roots of unity. This is related to our computation in the case $p = 2$. Our results, in particular, if we could handle instead of a prime p any natural number, may be useful to extend their program to the arbitrary case. However, the complexity we already encounter in the case of a prime p shows that this is a difficult task.

We are also interested in the slightly more difficult real case because of applications to the question whether a closed smooth spin manifold carries a Riemannian metric with positive scalar curvature (see Theorem 0.7). The numbers r_l appearing in the next theorem will be defined in (1.4) and analyzed in Section 1.3.

Theorem 0.6 (Topological K-theory of the real group C*-algebra). *Let p be an odd prime. Let $\Gamma = \mathbb{Z}^n \rtimes \mathbb{Z}/p$ be a group satisfying Condition 0.1. Then for all $m \in \mathbb{Z}$:*

(i)
$$\mathrm{KO}_m(C_r^*(\Gamma;\mathbb{R})) \cong \begin{cases} \mathbb{Z}^{p^k(p-1)/2} \oplus \left(\bigoplus_{l=0}^n \mathrm{KO}_{m-l}(*)^{r_l}\right), & m \text{ even},\\ \bigoplus_{l=0}^n \mathrm{KO}_{m-l}(*)^{r_l}, & m \text{ odd}.\end{cases}$$

(ii) *There is an exact sequence*
$$0 \to \bigoplus_{(P)\in\mathcal{P}} \widetilde{\mathrm{KO}}_{2m}^{\mathbb{Z}/p}(*) \to \mathrm{KO}_{2m}(C_r^*(\Gamma;\mathbb{R})) \to \mathrm{KO}_{2m}(\underline{B}\Gamma) \to 0,$$
where $\widetilde{\mathrm{KO}}_m^{\mathbb{Z}/p}(*) = \ker(\mathrm{KO}_m^{\mathbb{Z}/p}(*) \to \mathrm{KO}_m(*)) \cong \mathbb{Z}^{(p-1)/2}$. The exact sequence is split after inverting p.

(iii) *The map*
$$\mathrm{KO}_{2m+1}(C_r^*(\Gamma;\mathbb{R})) \xrightarrow{\cong} \mathrm{KO}_{2m+1}(\underline{B}\Gamma)$$
is an isomorphism. Restricting to the subgroup $\mathbb{Z}^n$ of Γ induces an isomorphism
$$\mathrm{KO}_{2m+1}(C_r^*(\Gamma;\mathbb{R})) \xrightarrow{\cong} \mathrm{KO}_{2m+1}(C_r^*(\mathbb{Z}^n_\rho;\mathbb{R}))^{\mathbb{Z}/p}.$$

If M is a closed spin manifold of dimension m with fundamental group G, one can define an invariant $\alpha(M) \in \mathrm{KO}_m(C_r^*(G;\mathbb{R}))$ as the index of a Dirac operator. If M admits a metric of positive scalar curvature, then $\alpha(M) = 0$. This theory and connections with the Gromov–Lawson–Rosenberg Conjecture will be reviewed in Section 12.1.

Theorem 0.7 ((Unstable) Gromov–Lawson–Rosenberg Conjecture). *Let p be an odd prime. Let M be a closed spin manifold of dimension $m \geq 5$ and fundamental group Γ as defined in (0.2). Then M admits a metric of positive scalar curvature if and only if $\alpha(M)$ is zero. Moreover if m is odd, then M admits a metric of positive scalar curvature if and only if the p-sheeted covering associated to the projection $\Gamma \to \mathbb{Z}/p$ does.*

Example 0.8. Here is an example where the last sentence of Theorem 0.7 applies. Choose an odd integer $k > 1$. Let M be a balanced product $S^k \times_\Gamma \mathbb{R}^n$ where Γ acts on the sphere via the projection $\Gamma \to \mathbb{Z}/p$ and a free action of $\mathbb{Z}/p$ on the sphere and Γ acts on $\mathbb{R}^n$ via its crystallographic action. Then its p-fold cover $S^k \times T^n$ admits a metric of positive scalar curvature since it is a spin boundary or since it is a product of a closed manifold with a closed Riemannian manifold with positive scalar curvature, and hence M admits a metric of positive scalar curvature.

Remark 0.9. Notice that Theorem 0.7 is not true for $\mathbb{Z}^4 \times \mathbb{Z}/3$ (see Schick [39]), whereas it is true for $\mathbb{Z}^4 \rtimes_\rho \mathbb{Z}/3$ for appropriate ρ by Theorem 0.7.

The computation of the topological K-theory of the reduced complex group C*-algebra $C_r^*(\Gamma)$ and of the reduced real group C*-algebra $C_r^*(\Gamma;\mathbb{R})$ will be done in a sequence of steps, passing in each step to a more difficult situation.

We will first compute the (co-)homology of $B\Gamma$ and $\underline{B}\Gamma$. A complete answer is given in Theorem 1.7 and Theorem 2.1.

Then we will analyze the complex and real topological K-cohomology and K-homology of $B\Gamma$ and $\underline{B}\Gamma$. A complete answer is given in Theorem 3.1, Theorem 4.1, Theorem 5.1 and Theorem 6.1 except for the exact structure of the p-torsion in $K^{2m+1}(\underline{B}\Gamma)$, $\mathrm{KO}^{2m+1}(\underline{B}\Gamma)$, $K_{2m}(\underline{B}\Gamma)$, and $\mathrm{KO}_{2m}(\underline{B}\Gamma)$.

In the third step we will compute the equivariant complex and real topological K-theory of $\underline{E}\Gamma$, and hence the K-theory of the complex and real C*-algebras of Γ. A complete answer is given in Theorem 0.3 and Theorem 0.6. It is rather surprising that we can give a complete answer although we do not know the full answer for $\underline{B}\Gamma$.

Finally we use the Baum–Connes Conjecture to prove Theorem 0.3 and Theorem 0.6 in Sections 11.

The proof of Theorem 0.7 will be presented in Section 12.

Although we are interested in the homological versions, it is important in each step to deal first with the cohomological versions as well since we will make use of the multiplicative structure and the Atiyah–Segal Completion Theorem.

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1. Group cohomology

In this section we compute the cohomology of $B\Gamma$ and $\underline{E}\Gamma$ for the group Γ defined in (0.2). It fits into a split exact sequence

$$1 \to \mathbb{Z}^n \xrightarrow{\iota} \Gamma \xrightarrow{\pi} \mathbb{Z}/p \to 1 \tag{1.1}$$

We write the group operation in $\mathbb{Z}/p$ and Γ multiplicatively and in $\mathbb{Z}^n$ additively. We fix a generator $t \in \mathbb{Z}/p$ and denote the value of $\rho(t)$ by $\rho\colon \mathbb{Z}^n \to \mathbb{Z}^n$. When wish to emphasize that $\mathbb{Z}^n$ is a $\mathbb{Z}[\mathbb{Z}/p]$-module, we denote it by $\mathbb{Z}^n_\rho$.

1.1. Statement of the computation of the cohomology

Notation 1.2 ($\underline{E}G$ and $\underline{B}G$). For a discrete group G we let $\underline{E}G$ denote the *classifying space for proper G-actions*. Let $\underline{B}G$ be the quotient space $G\backslash\underline{E}G$.

Recall that a model for the classifying space for proper G-actions is a G-CW-complex $\underline{E}G$ such that $\underline{E}G^H$ is contractible if $H \subset G$ is finite and empty otherwise. Two models are G-homotopy equivalent. There is a G-map $EG \to \underline{E}G$ which is unique up to G-homotopy. Hence there is a map $BG \to \underline{B}G$, unique up to homotopy. If G is torsion-free, then $EG = \underline{E}G$ and $BG = \underline{B}G$. For more information about $\underline{E}G$ we refer for instance to the survey article [30].

We will write $H^m(G)$ and $H_m(G)$ instead of $H^m(BG)$ and $H_m(BG)$.

Example 1.3 ($\underline{E}\Gamma$ and $\underline{B}\Gamma$). Since the group Γ is crystallographic and hence acts properly on $\mathbb{R}^n$ by smooth isometries, a model for $\underline{E}\Gamma$ is given by $\mathbb{R}^n$ with this Γ-action. In particular $\underline{B}\Gamma$ is a quotient of the n-torus T^n by a $\mathbb{Z}/p$-action.

The main result of this section is the computation of the group cohomology of $B\Gamma$ and $\underline{B}\Gamma$. Most of the calculation for $H^*(B\Gamma)$ has already been carried out by Adem [3] and later, with different methods, by Adem–Ge–Pan–Petrosyan [5]. The computation of $H^*(\underline{B}\Gamma)$ has recently and independently obtained by different methods by Adem–Duman–Gomez [4]. We include a complete proof since the techniques will be needed later when we compute topological K-theory.

Let

$$N = t^0 + t + \cdots + t^{p-1} \in \mathbb{Z}[\mathbb{Z}/p]$$

be the *norm element*. Denote by $I(\mathbb{Z}/p)$ the *augmentation ideal*, i.e., the kernel of the augmentation homomorphism $\mathbb{Z}[\mathbb{Z}/p] \to \mathbb{Z}$. Let $\zeta = e^{2\pi i/p} \in \mathbb{C}$ be a primitive p-th root of unity. We have isomorphisms of $\mathbb{Z}[\mathbb{Z}/p]$-modules

$$\mathbb{Z}[\mathbb{Z}/p]/N \cong \mathbb{Z}[\zeta] \cong I(\mathbb{Z}/p).$$

Define, for $m, j, k \in \mathbb{Z}_{\geq 0}$, natural numbers

$$r_m := \operatorname{rk}_{\mathbb{Z}}((\textstyle\bigwedge^m(\mathbb{Z}[\zeta]^k)^{\mathbb{Z}/p}), \tag{1.4}$$

$$a_j := |\{(\ell_1, \ldots, \ell_k) \in \mathbb{Z}^k \mid \ell_1 + \cdots + \ell_k = j,\ 0 \leq \ell_i \leq p-1\}|, \tag{1.5}$$

$$s_m := \sum_{j=0}^{m-1} a_j, \tag{1.6}$$

where here and in the sequel $\bigwedge^m$ means the m-th exterior power of a $\mathbb{Z}$-module. Notice that these numbers r_m, a_j and s_m depend on k, but we omit this from the notation since k will be determined by the equation $n = k(p-1)$ (see Lemma 1.9 (i)) and hence by Γ. Note that $r_0 = 1$, $r_1 = 0$, $a_0 = 1$, $a_1 = k$, $s_0 = 0$, $s_1 = 1$, and $s_2 = k + 1$. We will give more information about these numbers in Section 1.3.

Theorem 1.7 (Cohomology of $B\Gamma$ and $\underline{B}\Gamma$).

(i) *For $m \geq 0$,*

$$H^m(\Gamma) \cong \begin{cases} \mathbb{Z}^{r_m} \oplus (\mathbb{Z}/p)^{s_m}, & m \text{ even}, \\ \mathbb{Z}^{r_m}, & m \text{ odd}. \end{cases}$$

(ii) *For $m \geq 0$ the restriction map*

$$H^m(\Gamma) \to H^m(\mathbb{Z}^n_\rho)^{\mathbb{Z}/p}$$

is split surjective. The kernel is isomorphic to $(\mathbb{Z}/p)^{s_m}$ if m is even and 0 if m is odd.

(iii) *The map induced by the various inclusions*

$$\varphi^m \colon H^m(\Gamma) \to \bigoplus_{(P)\in\mathcal{P}} H^m(P)$$

is bijective for $m > n$.

(iv) *For $m \geq 0$,*

$$H^m(\underline{B}\Gamma) \cong \begin{cases} \mathbb{Z}^{r_m}, & m \text{ even}, \\ \mathbb{Z}^{r_m} \oplus (\mathbb{Z}/p)^{p^k - s_m}, & m \text{ odd},\ m \geq 3, \\ 0, & m = 1. \end{cases}$$

Remark 1.8 (Multiplicative structure). A transfer argument shows that the kernel of the restriction map $H^m(\Gamma) \to H^m(\mathbb{Z}^n)$ is p-torsion. Theorem 1.7 together with the exact sequence (1.14) implies that the map induced by the restrictions to the various subgroups

$$H^m(\Gamma) \to H^m(\mathbb{Z}^n) \oplus \bigoplus_{(P)\in\mathcal{P}} H^m(P)$$

is injective. The multiplicative structure of the target is obvious. This allows in principle to detect the multiplicative structure on $H^*(\Gamma)$.

1.2. Proof of Theorem 1.7. The proof of Theorem 1.7 needs some preparation.

Lemma 1.9. (i) *We have an isomorphism of $\mathbb{Z}[\mathbb{Z}/p]$-modules,*

$$\mathbb{Z}^n_\rho \cong I_1 \oplus \cdots \oplus I_k,$$

where the I_j are non-zero ideals of $\mathbb{Z}[\zeta]$.
We have

$$\mathbb{Z}^n_\rho \otimes \mathbb{Q} \cong \mathbb{Q}(\zeta)^k,$$
$$n = k(p-1).$$

(ii) *Each non-trivial finite subgroup P of Γ is isomorphic to $\mathbb{Z}/p$ and its Weyl group $W_\Gamma P := N_\Gamma P/P$ is trivial.*

(iii) *There are isomorphisms*

$$H^1(\mathbb{Z}/p; \mathbb{Z}^n_\rho) \xrightarrow{\cong} \operatorname{cok}(\rho - \mathrm{id}\colon \mathbb{Z}^n \to \mathbb{Z}^n) \cong (\mathbb{Z}/p)^k$$

and a bijection

$$\operatorname{cok}(\rho - \mathrm{id}\colon \mathbb{Z}^n \to \mathbb{Z}^n) \xrightarrow{\cong} \mathcal{P} := \{(P) \mid P \subset \Gamma,\ 1 < |P| < \infty\}.$$

If we fix an element $s \in \Gamma$ of order p, the bijection sends the element $\bar{u} \in \mathbb{Z}^n_\rho/(1-\rho)\mathbb{Z}^n_\rho$ to the subgroup of order p generated by us.

(iv) *We have $|\mathcal{P}| = p^k$.*

(v) *There is a bijection from the $\mathbb{Z}/p$-fixed set of the $\mathbb{Z}/p$-space $T^n_\rho := \mathbb{R}^n_\rho/\mathbb{Z}^n_\rho$ with $H^1(\mathbb{Z}/p;\mathbb{Z}^n_\rho)$. In particular $(T^n_\rho)^{\mathbb{Z}/p}$ consists of p^k points.*

(vi) $[\Gamma,\Gamma] = \operatorname{im}(\rho - \mathrm{id}\colon \mathbb{Z}^n \to \mathbb{Z}^n)$.

(vii) $\Gamma/[\Gamma,\Gamma] \cong \operatorname{cok}(\rho - \mathrm{id}\colon \mathbb{Z}^n \to \mathbb{Z}^n) \oplus \mathbb{Z}/p = (\mathbb{Z}/p)^{k+1}$.

Proof. (i): Let $u \in \mathbb{Z}^n_\rho$. Then $N \cdot u$ is fixed by the action of $t \in \mathbb{Z}/p$ and hence is zero by assumption. Thus $\mathbb{Z}^n_\rho$ is a finitely generated module over the Dedekind domain $\mathbb{Z}[\mathbb{Z}/p]/N = \mathbb{Z}[\zeta]$. Any finitely generated torsion-free module over a Dedekind domain is isomorphic to a direct sum of non-zero ideals (see [36], p. 11). Since $I_j \otimes \mathbb{Q} \cong \mathbb{Q}(\zeta)$, we see $\operatorname{rk}_{\mathbb{Z}}(I_j) = p-1$.

(ii): This is obvious.

(iii): Since the norm element N acts trivially on $\mathbb{Z}^n_\rho$, we get

$$\operatorname{cok}(\rho - \mathrm{id}\colon \mathbb{Z}^n \to \mathbb{Z}^n) = H^1(\mathbb{Z}/p;\mathbb{Z}^n_\rho).$$

We will show

$$H^1(\mathbb{Z}/p;\mathbb{Z}^n_\rho) \cong \widehat{H}^0(\mathbb{Z}/p;H^1(\mathbb{Z}^n_\rho)) \cong (\mathbb{Z}/p)^k$$

in Lemma 1.10 (i). One easily checks that the map $\operatorname{cok}(\rho - \mathrm{id}\colon \mathbb{Z}^n \to \mathbb{Z}^n) \to \mathcal{P}$ is bijective.

(iv): This follows from assertion (iii).

(v): Consider the short exact sequence of $\mathbb{Z}[\mathbb{Z}/p]$-modules

$$0 \to \mathbb{Z}^n_\rho \to \mathbb{R}^n_\rho \to T^n_\rho \to 0$$

Then the long exact cohomology sequence

$$(\mathbb{Z}^n_\rho)^{\mathbb{Z}/p} \to (\mathbb{R}^n_\rho)^{\mathbb{Z}/p} \to (T^n_\rho)^{\mathbb{Z}/p} \to H^1(\mathbb{Z}/p;\mathbb{Z}^n_\rho) \to H^1(\mathbb{Z}/p;\mathbb{R}^n_\rho)$$

is isomorphic to

$$0 \to 0 \to (T^n_\rho)^{\mathbb{Z}/p} \to (\mathbb{Z}/p)^k \to 0\,.$$

(vi): For $(i,p) = 1$ we have $(\zeta^i - 1)/(\zeta - 1) \in \mathbb{Z}[\zeta]^\times$ and hence we get $\ker(\rho - \mathrm{id}) = \ker(\rho^i - \mathrm{id}) = 0$ and $\operatorname{im}(\rho - \mathrm{id}) = \operatorname{im}(\rho^i - \mathrm{id})$. This implies

$$[\Gamma,\Gamma] = \operatorname{im}(\rho - \mathrm{id}\colon \mathbb{Z}^n \to \mathbb{Z}^n).$$

(vii): The isomorphism

$$\operatorname{cok}(\rho - \mathrm{id}\colon \mathbb{Z}^n \to \mathbb{Z}^n) \oplus \mathbb{Z}/p \xrightarrow{\cong} \Gamma/[\Gamma, \Gamma]$$

sends $(\bar{u}, \bar{i}) \mapsto \overline{us^i}$. □

Next will analyze the *Hochschild–Serre spectral sequence* (see [12], p. 171)

$$E_2^{i,j} = H^i(\mathbb{Z}/p; H^j(\mathbb{Z}^n_\rho)) \Rightarrow H^{i+j}(\Gamma)$$

of the extension (1.1). We say that a spectral sequence *collapses* if all differentials $d_r^{i,j}$ are trivial for $r \geq 2$ and all extension problems are trivial. The basic properties of the Tate cohomology $\widehat{H}^i(G; M)$ of a finite group G with coefficients in a $\mathbb{Z}[G]$-module M are reviewed in Appendix 12.2.

Lemma 1.10. (i) *We have*

$$\widehat{H}^i(\mathbb{Z}/p; H^j(\mathbb{Z}^n_\rho)) \cong \bigoplus_{\substack{\ell_1+\cdots+\ell_k=j \\ 0\leq \ell_q \leq p-1}} \widehat{H}^{i+j}(\mathbb{Z}/p; \mathbb{Z}) = \begin{cases} (\mathbb{Z}/p)^{a_j}, & i+j \text{ even}, \\ 0, & i+j \text{ odd}. \end{cases}$$

(ii) *The Hochschild–Serre spectral sequence associated to the extension* (1.1) *collapses.*

Proof. (i): There is a sequence of $\mathbb{Z}[\mathbb{Z}/p]$-isomorphisms

$$H^1(\mathbb{Z}^n_\rho) \cong \mathrm{Hom}_{\mathbb{Z}}(H_1(\mathbb{Z}^n_\rho), \mathbb{Z}) \cong \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}^n_\rho, \mathbb{Z}) \cong \mathbb{Z}^n_{\rho^*},$$

where $\rho(t)^*\colon \mathbb{Z}^n \to \mathbb{Z}^n$ for $t \in \mathbb{Z}/p$ is given by the transpose of the matrix describing $\rho(t)\colon \mathbb{Z}^n \to \mathbb{Z}^n$. The natural map given by the product in cohomology

$$\bigwedge^j H^1(\mathbb{Z}^n) \xrightarrow{\cong} H^j(\mathbb{Z}^n)$$

is bijective and hence is a $\mathbb{Z}[\mathbb{Z}/p]$-isomorphism by naturality. Thus we obtain a $\mathbb{Z}[\mathbb{Z}/p]$-isomorphism

$$H^j(\mathbb{Z}^n_\rho) \cong \bigwedge^j \mathbb{Z}^n_{\rho^*}.$$

Given a non-zero ideal $I \subset \mathbb{Z}[\zeta]$, There exists an isomorphism of $\mathbb{Z}_{(p)}[\zeta]$-modules

$$I \otimes \mathbb{Z}_{(p)} \xrightarrow{\cong} \mathbb{Z}[\zeta] \otimes_{\mathbb{Z}} \mathbb{Z}_{(p)} = \mathbb{Z}_{(p)}[\zeta].$$

This is true since $\mathbb{Z}_{(p)}[\zeta]$ is a discrete valuation ring, hence all ideals are principal. Since $\mathbb{Z}^n_{\rho^*}$ is isomorphic to a direct sum of ideals of $\mathbb{Z}[\zeta]$, we obtain for an appropriate natural number k isomorphisms of $\mathbb{Z}[\zeta] \otimes_{\mathbb{Z}} \mathbb{Z}_{(p)} = \mathbb{Z}_{(p)}[\zeta]$-modules

$$H^j(\mathbb{Z}^n_\rho) \otimes_{\mathbb{Z}} \mathbb{Z}_{(p)} \cong \bigwedge^j \mathbb{Z}^n_{\rho^*} \otimes_{\mathbb{Z}} \mathbb{Z}_{(p)} \cong \bigwedge^j \mathbb{Z}[\zeta]^k \otimes_{\mathbb{Z}} \mathbb{Z}_{(p)}.$$

For every $\mathbb{Z}[\mathbb{Z}/p]$-module M the obvious map

$$\widehat{H}^i(\mathbb{Z}/p; M) \to \widehat{H}^i(\mathbb{Z}/p; M \otimes_{\mathbb{Z}} \mathbb{Z}_{(p)})$$

is bijective. Hence we obtain an isomorphism

$$\widehat{H}^i(\mathbb{Z}/p; H^j(\mathbb{Z}^n_\rho)) \cong \widehat{H}^i(\mathbb{Z}/p; \textstyle\bigwedge^j \mathbb{Z}[\zeta]^k).$$

Since

$$\textstyle\bigwedge^*(\bigoplus_k \mathbb{Z}[\zeta])) = \bigotimes_k \bigwedge^*(\mathbb{Z}[\zeta])$$

and $\bigwedge^l(\mathbb{Z}[\zeta]) = 0$ for $l \geq p$, we get

$$\textstyle\bigwedge^j(\mathbb{Z}[\zeta]^k)) = \bigoplus_{\substack{\ell_1+\cdots+\ell_k=j\\ 0\leq \ell_q\leq p-1}} \bigwedge^{\ell_1}\mathbb{Z}[\zeta] \otimes \cdots \otimes \bigwedge^{\ell_k}\mathbb{Z}[\zeta].$$

Therefore we obtain an isomorphism

$$\widehat{H}^i(\mathbb{Z}/p; H^j(\mathbb{Z}^n_\rho)) \cong \bigoplus_{\substack{\ell_1+\cdots+\ell_k=j\\ 0\leq \ell_q\leq p-1}} \widehat{H}^i(\mathbb{Z}/p; \textstyle\bigwedge^{\ell_1}\mathbb{Z}[\zeta] \otimes \cdots \otimes \bigwedge^{\ell_k}\mathbb{Z}[\zeta])).$$

Hence it suffices to show

$$\widehat{H}^i(\mathbb{Z}/p; \textstyle\bigwedge^{\ell_1}\mathbb{Z}[\zeta] \otimes \cdots \otimes \bigwedge^{\ell_k}\mathbb{Z}[\zeta]) \cong \widehat{H}^{i+\sum_{a=1}^k l_a}(\mathbb{Z}/p;\mathbb{Z})$$

for $l_1,\ldots,l_k$ in $\{0,1,\ldots,p-1\}$. This will be done by induction over $j = \sum_{a=1}^k l_a$. The induction beginning $j=0$ is trivial, the induction step from $j-1$ to $j \geq 1$ done as follows. We can assume without loss of generality that $1 \leq l_1 \leq p-1$ otherwise permute the factors. There is an exact sequence of $\mathbb{Z}[\mathbb{Z}/p]$-modules

$$0 \to \mathbb{Z} \to \mathbb{Z}[\mathbb{Z}/p] \to \mathbb{Z}[\zeta] \to 0,$$

where $1 \in \mathbb{Z}$ maps to the norm element $N \in \mathbb{Z}[\mathbb{Z}/p]$. Since this exact sequence splits as an exact sequence of $\mathbb{Z}$-modules, it induces an exact sequence of $\mathbb{Z}[\mathbb{Z}/p]$-modules

$$1 \to \textstyle\bigwedge^{l_1-1}\mathbb{Z}[\zeta] \to \bigwedge^{l_1}\mathbb{Z}[\mathbb{Z}/p] \to \bigwedge^{l_1}\mathbb{Z}[\zeta] \to 1, \tag{1.11}$$

where the second map is induced by the epimorphism $\mathbb{Z}[\mathbb{Z}/p] \to \mathbb{Z}[\zeta]$ and the first sends $u_1 \wedge u_2 \wedge \cdots \wedge u_{l_1-1}$ to $u'_1 \wedge u'_2 \wedge \cdots \wedge u'_{l_1-1} \wedge N$, where $u'_b \in \mathbb{Z}[\mathbb{Z}/p]$ is any element whose image under the projection $\mathbb{Z}[\mathbb{Z}/p] \to \mathbb{Z}[\zeta]$ is u_b. This is independent of the choice of the u'_b's since two such choices differ by a multiple of the norm element $N \in \mathbb{Z}[\mathbb{Z}/p]$.

We next show that the middle term of (1.11) is a free $\mathbb{Z}[\mathbb{Z}/p]$-module when $1 \leq l_1 \leq p-1$. Since $\mathbb{Z}/p = \{t^0, t^1, \ldots, t^{p-1}\}$ is a $\mathbb{Z}$-basis for $\mathbb{Z}[\mathbb{Z}/p]$, we obtain a $\mathbb{Z}$-basis for $\bigwedge^{l_1}\mathbb{Z}[\mathbb{Z}/p]$ by

$$\{t^I \mid I \subset \mathbb{Z}/p,\ |I| = l_1\},$$

where $t^I = t^{i_1} \wedge t^{i_2} \wedge \cdots \wedge t^{i_{l_1}}$ for $I = \{i_1, i_2, \ldots, i_{l_1}\}$ with $1 \le i_1 < i_2 < \cdots < i_{l_1} \le p-1$. An element $s \in \mathbb{Z}/p$ acts on $\bigwedge^{l_1} \mathbb{Z}[\mathbb{Z}/p]$ by sending the basis element t^I to $\pm t^{s+I}$. The $\mathbb{Z}/p$ action on $\{I \subset \mathbb{Z}/p, |I| = l_1\}$ which sends I to $s + I$ for $s \in \mathbb{Z}/p$, is free. Indeed, for $s \in \mathbb{Z}/p - \{0\}$, the permutation of the p-element set $\mathbb{Z}/p$ given by $a \mapsto s + a$ cannot have any proper invariant sets since the permutation has order p and p is prime. This implies that the $\mathbb{Z}[\mathbb{Z}/p]$-module $\bigwedge^{l_1} \mathbb{Z}[\mathbb{Z}/p]$ is free.

We obtain from the exact sequence (1.11) an exact sequence of $\mathbb{Z}[\mathbb{Z}/p]$-modules with a free $\mathbb{Z}[\mathbb{Z}/p]$-module in the middle

$$\begin{aligned} 1 &\to \textstyle\bigwedge^{l_1-1} \mathbb{Z}[\zeta] \otimes \bigwedge^{\ell_2} \mathbb{Z}[\zeta] \otimes \cdots \otimes \bigwedge^{\ell_k} \mathbb{Z}[\zeta] \\ &\to \textstyle\bigwedge^{l_1} \mathbb{Z}[\mathbb{Z}/p] \otimes \bigwedge^{\ell_2} \mathbb{Z}[\zeta] \otimes \cdots \otimes \bigwedge^{\ell_k} \mathbb{Z}[\zeta] \\ &\to \textstyle\bigwedge^{l_1} \mathbb{Z}[\zeta] \otimes \bigwedge^{\ell_2} \mathbb{Z}[\zeta] \otimes \cdots \otimes \bigwedge^{\ell_k} \mathbb{Z}[\zeta] \to 1. \end{aligned}$$

Hence we obtain for $i \in \mathbb{Z}$ an isomorphism

$$\widehat{H}^i(\mathbb{Z}/p; \textstyle\bigwedge^{\ell_1} \mathbb{Z}[\zeta] \otimes \cdots \otimes \bigwedge^{\ell_k} \mathbb{Z}[\zeta]) \cong \widehat{H}^{i+1}(\mathbb{Z}/p; \bigwedge^{\ell_1 - 1} \mathbb{Z}[\zeta] \otimes \cdots \otimes \bigwedge^{\ell_k} \mathbb{Z}[\zeta]).$$

Now apply the induction hypothesis. This finishes the proof of assertion (i).

(ii): Next we want to show that the differentials $d_r^{i,j}$ are zero for all $r \ge 2$ and i, j. By the checkerboard pattern of the E_2-term it suffices to show for $r \ge 2$ and that the differentials $d_r^{0,j}$ are trivial for $r \ge 2$ and all odd $j \ge 1$. This is equivalent to show that for every odd $j \ge 1$ the edge homomorphism (see Proposition A.5)

$$\iota^j \colon H^j(\Gamma) \to H^j(\mathbb{Z}^n_\rho)^{\mathbb{Z}/p} = E_2^{0,j}$$

is surjective. But $\widehat{H}^0(\mathbb{Z}/p, H^j(\mathbb{Z}^n_\rho)) = 0$ by assertion (i), so the norm map $N = \iota^j \circ \mathrm{trf}^j \colon H^j(\mathbb{Z}^n_\rho)_{\mathbb{Z}/p} \to H^j(\mathbb{Z}^n_\rho)^{\mathbb{Z}/p}$ is surjective (see Theorem A.3), so ι^j is surjective.

It remains to show that all extensions are trivial. Since the composite

$$H^{i+j}(\Gamma) \xrightarrow{\iota^{i+j}} H^{i+j}(\mathbb{Z}^n_\rho) \xrightarrow{\mathrm{trf}^{i+j}} H^{i+j}(\Gamma)$$

is multiplication with p, the torsion in $H^{i+j}(\Gamma)$ has exponent p. Since $p \cdot E_\infty^{i,j} = p \cdot E_2^{i,j} = 0$ for $i > 0$, all extensions are trivial and

$$H^m\Gamma \cong \bigoplus_{i+j=m} E_\infty^{i,j} = \bigoplus_{i+j=m} E_2^{i,j}. \qquad \square$$

Proof of assertions (i) and (ii) of Theorem 1.7. These are direct consequences of Lemma 1.10. $\square$

Proof of assertion (iii) of Theorem 1.7. We obtain from [34], Corollary 2.11, together with Lemma 1.9 (ii) a cellular Γ-pushout

$$\begin{array}{ccc} \coprod_{(P)\in\mathcal{P}} \Gamma \times_P EP & \xrightarrow{i_0} & E\Gamma \\ {\scriptstyle \coprod_{(P)\in\mathcal{P}} \mathrm{pr}_P}\downarrow & & \downarrow{\scriptstyle f} \\ \coprod_{(P)\in\mathcal{P}} \Gamma/P & \xrightarrow{i_1} & \underline{E}\Gamma, \end{array} \tag{1.12}$$

where i_0 and i_1 are inclusions of Γ-CW-complexes, pr_P is the obvious Γ-equivariant projection and $\mathcal{P}$ is the set of conjugacy classes of subgroups of Γ of order p. Taking the quotient with respect to the Γ-action we obtain from (1.12) the cellular pushout

$$\begin{array}{ccc} \coprod_{(P)\in\mathcal{P}} BP & \xrightarrow{j_0} & B\Gamma \\ {\scriptstyle \coprod_{(P)\in\mathcal{P}} \overline{\mathrm{pr}}_P}\downarrow & & \downarrow{\scriptstyle \bar{f}} \\ \coprod_{(P)\in\mathcal{P}} * & \xrightarrow{j_1} & \underline{B}\Gamma \end{array} \tag{1.13}$$

where j_0 and j_1 are inclusions of CW-complexes, $\overline{\mathrm{pr}}_P$ is the obvious projection. It yields the following long exact sequence for $m \geq 0$

$$\begin{aligned} 0 \to H^{2m}(\underline{B}\Gamma) \xrightarrow{\bar{f}^*} H^{2m}(\Gamma) \xrightarrow{\varphi^{2m}} \bigoplus_{(P)\in\mathcal{P}} \widetilde{H}^{2m}(P) \\ \xrightarrow{\delta^{2m}} H^{2m+1}(\underline{B}\Gamma) \xrightarrow{\bar{f}^*} H^{2m+1}(\Gamma) \to 0, \end{aligned} \tag{1.14}$$

where φ^* is the map induced by the various inclusions $P \subset \Gamma$ for $(P) \in \mathcal{P}$.

Now assertion (iii) follows from (1.14) since there is a n-dimensional model for $\underline{B}\Gamma$. □

We still need to prove assertion (iv) of Theorem 1.7.

In order to compute $H^*(\underline{B}\Gamma)$, we need to compute the kernel and image of φ^{2m}.

Lemma 1.15. *Let $m \geq 1$.*

(i) *Let K^{2m} be the kernel of φ^{2m}. There is a short exact sequence*

$$0 \to K^{2m} \to H^{2m}(\mathbb{Z}^n_\rho)^{\mathbb{Z}/p} \to \widehat{H}^0(\mathbb{Z}/p; H^{2m}(\mathbb{Z}^n_\rho)) \to 0,$$

where the first non-trivial map is the restriction of $\iota^*\colon H^{2m}(\Gamma) \to H^{2m}(\mathbb{Z}^n_\rho)^{\mathbb{Z}/p}$ to K^{2m} and the second non-trivial map is given by the quotient map appearing in the definition of Tate cohomology. It follows that $K^{2m} \cong \mathbb{Z}^{r_m}$.

(ii) *The image of φ^{2m} is isomorphic to*

$$\ker(H^{2m}(\Gamma) \to H^{2m}(\mathbb{Z}^n_\rho)^{\mathbb{Z}/p}) \oplus \widehat{H}^0(\mathbb{Z}/p; H^{2m}(\mathbb{Z}^n_\rho)) \cong (\mathbb{Z}/p)^{s_{2m+1}}.$$

Proof. (i): Let $\beta \in H^2(\mathbb{Z}/p) \cong \mathbb{Z}/p$ be a generator. Let L^{2m} be the kernel of

$$- \cup \pi^*(\beta)^n \colon H^{2m}(\Gamma) \to H^{2m+2n}(\Gamma).$$

We first claim that $K^{2m} = L^{2m}$. Indeed, the following diagram commutes

$$\begin{tikzcd} H^{2m}(\Gamma) \arrow[r, "\varphi^{2m}"] \arrow[d, "-\cup\pi^*(\beta)^n"'] & \bigoplus_{(P)\in\mathcal{P}} H^{2m}(P) \arrow[d, "-\cup\beta^n"] \\ H^{2m+2n}(\Gamma) \arrow[r, "\varphi^{2m+2n}"] & \bigoplus_{(P)\in\mathcal{P}} H^{2m+2n}(P) \end{tikzcd}$$

Since $\dim(\underline{B}\Gamma) \le n$, we have $H^{i+2n}(\underline{B}\Gamma) = 0$ for $i \ge 1$. Hence the lower horizontal arrow is bijective by (1.14). The right vertical arrow is bijective. Thus $K^{2m} = L^{2m}$.

Recall that we have a descending filtration

$$H^{2m}(\Gamma) = F^{0,2m} \supset F^{1,2m-1} \supset \cdots \supset F^{2m,0} \supset F^{2m+1,-1} = 0$$

with $F^{r,2m-1}/F^{r+1,2m-r-1} \cong E_\infty^{r,2m-r}$. Recall that $E_2^{2,0} = H^2(\mathbb{Z}/p; H^0(\mathbb{Z}^n_\rho)) = H^2(\mathbb{Z}/p)$ so that we can think of β as an element in $E_2^{2,0}$. Recall that $E_2^{i,j} = E_\infty^{i,j}$ by Lemma 1.10 (ii). From the multiplicative structure of the spectral sequence we see that the image of the map

$$- \cup \pi^*(\beta)^n \colon H^{2m}(\Gamma) \to H^{2m+2n}(\Gamma)$$

lies in $F^{2n,2m}$ and the diagram

$$\begin{tikzcd} 0 \arrow[d] & 0 \arrow[d] \\ F^{1,2m-1} \arrow[r, "\cong", "-\cup\pi^*(\beta)^n"'] \arrow[d] & F^{2n+1,2m-1} \arrow[d] \\ H^{2m}(\Gamma) \arrow[r, "-\cup\pi^*(\beta)^n"'] \arrow[d] & F^{2n,2m} \arrow[d] \\ E_\infty^{0,2m} \arrow[r, "-\cup\beta^n"'] \arrow[d] & E_\infty^{2n,2m} \arrow[d] \\ 0 & 0 \end{tikzcd} \tag{1.16}$$

commutes, where the columns are exact. The upper horizontal arrow is bijective. Namely, one shows by induction over $r = -1, 0, 1, \ldots, 2m-1$ that the map

$$- \cup \pi^*(\beta)^n \colon F^{2m-r,r} \to F^{2m-r+2n,r}$$

is bijective. The induction beginning $r = -1$ is trivial since then both the source and the target are trivial, and the induction step from $r - 1$ to r follows from the five lemma and the fact that the map

$$\begin{aligned} - \cup \beta^n \colon E_\infty^{2m-r,r} &= H^{2m-r}(\mathbb{Z}/p; H^r(\mathbb{Z}^n_\rho)) \\ &\to E_\infty^{2m-r+2n,r} = H^{2m-r+2n}(\mathbb{Z}/p; H^r(\mathbb{Z}^n_\rho)) \end{aligned}$$

is bijective.

The bottom horizontal map in diagram (1.16) can be identified with the composition of the canonical quotient map

$$H^0(\mathbb{Z}/p; H^{2m}(\mathbb{Z}^n_\rho)) \to \widehat{H}^0(\mathbb{Z}/p; H^{2m}(\mathbb{Z}^n_\rho)).$$

with the isomorphism

$$- \cup \beta^n \colon \widehat{H}^0(\mathbb{Z}/p; H^{2m}(\mathbb{Z}^n_\rho)) \xrightarrow{\cong} \widehat{H}^{2n}(\mathbb{Z}/p; H^{2m}(\mathbb{Z}^n_\rho)).$$

So what do we know about diagram (1.16)? The top horizontal map is an isomorphism, the kernel of middle horizontal map is L^{2m}, and the bottom horizontal map is onto. We conclude from the snake lemma that the middle map is an epimorphism and that we have a short exact sequence

$$0 \to L^{2m} \to E_\infty^{0,2m} \to E_\infty^{2n,2m} \to 0.$$

The first non-trivial map is the composite of the inclusion $K^{2m} = L^{2m} \subset H^{2m}(\Gamma)$ with the epimorphism

$$H^{2m}(\Gamma) \to E_\infty^{0,2m} = H^{2m}(\mathbb{Z}^n_\rho)^{\mathbb{Z}/p}$$

induced by the inclusion $\iota\colon \mathbb{Z}^n \to \Gamma$. We have already identified the second non-trivial map (up to isomorphism) with the quotient map as desired. Hence the sequence in assertion (i) is exact. Since the middle term is isomorphic to $\mathbb{Z}^{r_m}$ and the right term is finite, K^{2m} is also isomorphic to $\mathbb{Z}^{r_m}$.

(ii): The exact sequence

$$0 \to \ker(H^{2m}(\Gamma) \to H^{2m}(\mathbb{Z}^n_\rho)^{\mathbb{Z}/p}) \to H^{2m}(\Gamma) \xrightarrow{\iota^{2m}} H^{2m}(\mathbb{Z}^n_\rho)^{\mathbb{Z}/p} \to 0$$

has the property that ι^{2m} restricted to K^{2m} is injective. Thus we can quotient by K^{2m} and $\iota^{2m}(K^{2m})$ in the middle and right-hand term respectively and maintain exactness. Hence we have the exact sequence

$$\begin{aligned} 0 \to \ker(H^{2m}(\Gamma) \to H^{2m}(\mathbb{Z}^n_\rho)^{\mathbb{Z}/p}) &\to H^{2m}(\Gamma)/K^{2m} \\ &\to \widehat{H}^0(\mathbb{Z}/p; H^{2m}(\mathbb{Z}^n_\rho)) \to 0. \end{aligned} \tag{1.17}$$

where we used assertion (i) to compute the right-hand term. We conclude from Lemma 1.10 that

$$\widehat{H}^0(\mathbb{Z}/p; H^{2m}(\mathbb{Z}^n_\rho)) \cong (\mathbb{Z}/p)^{a_{2m}}, \tag{1.18}$$

$$\ker(H^{2m}(\Gamma) \to H^{2m}(\mathbb{Z}^n_\rho)^{\mathbb{Z}/p}) \cong \bigoplus_{i=1}^{2m} E^{i,2m-i} \cong \bigoplus_{j=0}^{2m-1} (\mathbb{Z}/p)^{a_j}. \tag{1.19}$$

Since $H^{2m}(\Gamma)/K^{2m}$ is isomorphic to a subgroup of $\bigoplus_{(P)\in\mathcal{P}} \widetilde{H}^{2m}(P)$ by the long exact cohomology sequence (1.14) it is annihilated by multiplication with p. Hence the short exact sequence (1.17) splits and we conclude from (1.18) and (1.19) that

$$H^{2m}(\Gamma)/K^{2m} \cong \bigoplus_{j=0}^{2m} (\mathbb{Z}/p)^{a_j} \cong (\mathbb{Z}/p)^{s_{2m+1}}.$$

This finishes the proof of Lemma 1.15. □

We conclude from the exact sequence (1.14), Theorem 1.7 (i), Lemma 1.9 (iv), and Lemma 1.15

Corollary 1.20. *For $m \geq 1$ the long exact sequence* (1.14) *can be identified with*

$$0 \to \mathbb{Z}^{r_{2m}} \to \mathbb{Z}^{r_{2m}} \oplus (\mathbb{Z}/p)^{s_{2m}} \to (\mathbb{Z}/p)^{p^k}$$
$$\to \mathbb{Z}^{r_{2m+1}} \oplus (\mathbb{Z}/p)^{p^k - s_{2m+1}} \to \mathbb{Z}^{r_{2m+1}} \to 0.$$

Proof of assertion (iv) *of Theorem* 1.7. Obviously $H^0(\underline{B}\Gamma) \cong \mathbb{Z}$. Since $(\mathbb{Z}^n)^{\mathbb{Z}/p} = 0$ by assumption, we get $H^1(\Gamma) = 0$ from assertion (ii) of Theorem 1.7. We conclude $H^1(\underline{B}\Gamma) \cong 0$ from the long exact sequence (1.14). The values of $H^m(\underline{B}\Gamma)$ for $m \geq 2$ have already been determined in Corollary 1.20. Hence assertion (iv) of Theorem 1.7 follows. This finishes the proof of Theorem 1.7. □

1.3. On the numbers r_m. In this section we collect some basic information about the numbers r_m, a_j and s_m introduced in (1.4),(1.5), and (1.6).

Since $\mathbb{Z}^n$ acts freely on $\underline{E}\Gamma = \mathbb{R}^n$, we conclude from Lemma 1.9 (i) and Proposition A.4

$$\begin{aligned} r_m &= \operatorname{rk}_{\mathbb{Q}}((\Lambda^m_{\mathbb{Q}}(\mathbb{Q}(\zeta)^k)^{\mathbb{Z}/p})) = \operatorname{rk}_{\mathbb{Q}}(H^m(B\mathbb{Z}^n_\rho;\mathbb{Q})^{\mathbb{Z}/p}) \\ &= \operatorname{rk}_{\mathbb{Q}}(H^m(\underline{B}\Gamma;\mathbb{Q})) = \operatorname{rk}_{\mathbb{Q}}(H^m(\Gamma;\mathbb{Q})). \end{aligned}$$

Since Tate cohomology is rationally trivial, the norm map is a rational isomorphism, hence also

$$r_m = \operatorname{rk}_{\mathbb{Q}}(\Lambda^m_{\mathbb{Q}}(\mathbb{Q}[\zeta]^k) \otimes_{\mathbb{Q}[\mathbb{Z}/p]} \mathbb{Q}). \tag{1.21}$$

Lemma 1.22. (i) *We have $r_0 = 1$, $r_1 = 0$, $a_0 = 1$, $a_1 = k$, $s_0 = 0$, $s_1 = 1$, and $s_2 = k+1$. We get $r_m = 0$ for $m \geq n+1$ and $s_m = p^k$ for $m \geq n$.*

(ii) *If p is odd, we get*

$$\sum_{\substack{m \geq 0 \\ m \text{ even}}} r_m = \frac{2^{(p-1)k} + p - 1}{2p} + \frac{(p-1) \cdot p^{k-1}}{2},$$
$$\sum_{\substack{m \geq 0 \\ m \text{ odd}}} r_m = \frac{2^{(p-1)k} + p - 1}{2p} - \frac{(p-1) \cdot p^{k-1}}{2}.$$

If $p = 2$, we get

$$\sum_{\substack{m \geq 0 \\ m \text{ even}}} r_m = 2^{n-1}, \quad \sum_{\substack{m \geq 0 \\ m \text{ odd}}} r_m = 0.$$

(iii) *Suppose that $k = 1$. Then*

$$\begin{aligned}
r_m &= \tfrac{1}{p} \cdot \left(\tbinom{p-1}{m} + (-1)^m \cdot (p-1)\right) \quad \text{for } 0 \leq m \leq (p-1), \\
r_m &= 0 \quad \text{for } m \geq p, \\
a_m &= 1 \quad \text{for } 0 \leq m \leq p-1, \\
a_m &= 0 \quad \text{for } p \leq m, \\
s_m &= m \quad \text{for } 0 \leq m \leq p-1, \\
s_m &= p \quad \text{for } m \geq p.
\end{aligned}$$

Proof. In the proof below we write $\bigwedge^l V$ instead of $\bigwedge^l_{\mathbb{Q}} V$ for a $\mathbb{Q}$-vector space V.

(i): This follows directly from the definitions.

(ii): Suppose that $1 \leq l \leq p-1$. By rationalizing the exact sequence (1.11) we have the short exact sequence of $\mathbb{Q}[\mathbb{Z}/p]$-modules

$$0 \to \bigwedge^{l-1} \mathbb{Q}[\zeta] \to \bigwedge^l \mathbb{Q}[\mathbb{Z}/p] \to \bigwedge^l \mathbb{Q}[\zeta] \to 0.$$

Since $\bigwedge^l \mathbb{Z}[\mathbb{Z}/p]$ is finitely generated free as $\mathbb{Z}[\mathbb{Z}/p]$-module (see proof of Lemma 1.10 (i)), the following equation holds in the rational representation ring $R_{\mathbb{Q}}(\mathbb{Z}/p)$:

$$[\bigwedge^l \mathbb{Q}[\zeta]] + [\bigwedge^{l-1} \mathbb{Q}[\zeta]] = \frac{1}{p} \cdot \binom{p}{l} \cdot [\mathbb{Q}[\mathbb{Z}/p]].$$

One shows by induction over l for $0 \leq l \leq p-1$,

$$[\bigwedge^l (\mathbb{Q}[\zeta])] = (-1)^l \cdot [\mathbb{Q}] + \frac{1}{p}\left(\binom{p-1}{l} - (-1)^l\right) \cdot [\mathbb{Q}[\mathbb{Z}/p]]. \tag{1.23}$$

Since $\sum_{l=0}^{p-1} \binom{p-1}{l} = 2^{p-1}$, we get

$$\sum_{l=0}^{p-1} [\bigwedge^l \mathbb{Q}[\zeta]] = \begin{cases} [\mathbb{Q}] + \frac{2^{p-1}-1}{p} \cdot [\mathbb{Q}[\mathbb{Z}/p]] & \text{if } p \text{ is odd}, \\ [\mathbb{Q}[\mathbb{Z}/2]] & \text{if } p = 2. \end{cases} \tag{1.24}$$

Since

$$\bigwedge^*(\bigoplus_k \mathbb{Q}[\zeta]) = \bigotimes_k \bigwedge^*(\mathbb{Q}[\zeta])$$

and $\bigwedge^l(\mathbb{Q}[\zeta]) = 0$ for $l \geq p$, we get

$$[\bigwedge^j(\mathbb{Q}[\zeta]^k)] = \sum_{\substack{\ell_1+\cdots+\ell_k=j \\ 0\leq \ell_i \leq p-1}} \prod_{i=1}^k [\bigwedge^{\ell_i}(\mathbb{Q}[\zeta])]. \tag{1.25}$$

We conclude from (1.24) and (1.25) that

$$\begin{aligned}
\sum_{j\geq 0} [\bigwedge^j(\mathbb{Q}[\zeta]^k)] &= \sum_{j\geq 0} \Big(\sum_{\substack{\ell_1+\cdots+\ell_k=j \\ 0\leq \ell_i \leq p-1}} \prod_{i=1}^k [\bigwedge^{\ell_i}(\mathbb{Q}[\zeta])]\Big) \\
&= \sum_{\substack{l_1,l_2,\ldots,l_k \\ 0\leq \ell_q \leq p-1}} \prod_{i=1}^k [\bigwedge^{\ell_i}(\mathbb{Q}[\zeta])] \\
&= \prod_{i=1}^k \sum_{0\leq \ell_i \leq p-1} [\bigwedge^{\ell_i}(\mathbb{Q}[\zeta])] \\
&= \begin{cases} ([\mathbb{Q}] + \frac{2^{p-1}-1}{p} \cdot [\mathbb{Q}[\mathbb{Z}/p]])^k & \text{if } p \text{ is odd,} \\ [\mathbb{Q}[\mathbb{Z}/2]]^k & \text{if } p = 2. \end{cases}
\end{aligned}$$

Since $[\mathbb{Q}]$ is the multiplicative unit in $R_{\mathbb{Q}}(\mathbb{Z}/p)$, and $[\mathbb{Q}[\mathbb{Z}/p]]^i = p^{i-1} \cdot [\mathbb{Q}[\mathbb{Z}/p]]$, we obtain the following equality in $R_{\mathbb{Q}}(\mathbb{Z}/p)$ if p is odd:

$$\begin{aligned}
\sum_{j\geq 0} [\bigwedge^j(\mathbb{Q}[\zeta]^k)] &= \sum_{i=0}^k \binom{k}{i} \cdot \frac{(2^{p-1}-1)^i}{p^i} \cdot [\mathbb{Q}[\mathbb{Z}/p]]^i \cdot [\mathbb{Q}]^{k-i} \\
&= [\mathbb{Q}] + \frac{1}{p} \cdot (-1 + \sum_{i=0}^k \binom{k}{i} (2^{p-1}-1)^i) \cdot [\mathbb{Q}[\mathbb{Z}/p]] \\
&= [\mathbb{Q}] + \frac{1}{p} \cdot (-1 + 2^{(p-1)k}) \cdot [\mathbb{Q}[\mathbb{Z}/p]] \\
&= [\mathbb{Q}] + \frac{2^{(p-1)k}-1}{p} \cdot [\mathbb{Q}[\mathbb{Z}/p]].
\end{aligned} \tag{1.26}$$

If $p = 2$, we obtain

$$\sum_{j\geq 0} [\bigwedge^j(\mathbb{Q}[\zeta]^k)] = 2^{k-1} \cdot [\mathbb{Q}[\mathbb{Z}/2]].$$

There is a homomorphism of abelian groups

$$\Phi\colon R_{\mathbb{Q}}(\mathbb{Z}/p) \to \mathbb{Z}, \quad [V] \mapsto \mathrm{rk}_{\mathbb{Q}}(V \otimes_{\mathbb{Q}[\mathbb{Z}/p]} \mathbb{Q}).$$

By (1.21) it sends $\mathbb{Q}$, $\mathbb{Q}[\mathbb{Z}/p]$, and $[\bigwedge^m(\mathbb{Q}[\zeta]^k)]$ to 1, 1, and r_m respectively. Hence we conclude from (1.26)

$$\sum_{m \geq 0} r_m = \frac{2^{(p-1)k} - 1}{p} + 1 \quad \text{for } p \text{ odd}, \tag{1.27}$$

$$\sum_{m \geq 0} r_m = 2^{k-1} \quad \text{for } p = 2. \tag{1.28}$$

If X is a finite $\mathbb{Z}/p$-CW-complex with orbit space $\bar{X}$, then the Riemann–Hurwitz formula states that

$$\chi(\bar{X}) = \frac{1}{p}\chi(X) + \frac{p-1}{p}\chi(X^{\mathbb{Z}/p}).$$

One derives this formula by verifying it for both fixed and freely permuted cells. Applying Proposition A.4, the Riemann–Hurwitz formula, and Lemma 1.9 (v) to the $\mathbb{Z}/p$-action on the torus T^n, one sees

$$\sum_{m \geq 0} (-1)^m r_m = \chi((\mathbb{Z}/p)\backslash T^m) = 0 + (p-1)p^{k-1}. \tag{1.29}$$

We conclude from (1.27) and (1.29) if p is odd

$$\sum_{\substack{m \geq 0 \\ m \text{ even}}} r_m = \frac{2^{(p-1)k} + p - 1}{2p} + \frac{(p-1) \cdot p^{k-1}}{2},$$

$$\sum_{\substack{m \geq 0 \\ m \text{ odd}}} r_m = \frac{2^{(p-1)k} + p - 1}{2p} - \frac{(p-1) \cdot p^{k-1}}{2}.$$

If $p = 2$, we obtain from (1.28) and (1.29)

$$\sum_{\substack{m \geq 0 \\ m \text{ even}}} r_m = 2^{n-1}, \qquad \sum_{\substack{m \geq 0 \\ m \text{ odd}}} r_m = 0$$

since $n = k \cdot (p-1)$.

(iii): The first formula follows from (1.21) and applying the homomorphism Φ to (1.23). The rest of (iii) is clear from the definitions. $\square$

2. Group homology

Next we determine the group homology of the group Γ. Recall that, for a $\mathbb{Z}[G]$-module M, the *coinvariants* are $M_G = M \otimes_{\mathbb{Z}[G]} \mathbb{Z}$.

Theorem 2.1 (Homology of $B\Gamma$ and $\underline{B}\Gamma$).

(i) *For $m \geq 0$,*

$$H_m(\Gamma) \cong \begin{cases} \mathbb{Z}^{r_m} \oplus (\mathbb{Z}/p)^{s_{m+1}}, & m \text{ odd}, \\ \mathbb{Z}^{r_m}, & m \text{ even}. \end{cases}$$

(ii) *For $m \geq 0$, the inclusion map $\mathbb{Z}^n \to \Gamma$ induces an isomorphism*

$$H_{2m}(\mathbb{Z}^n_\rho)_{\mathbb{Z}/p} \xrightarrow{\cong} H_{2m}(\Gamma).$$

(iii) *The map induced by the various inclusions*

$$\varphi_m \colon \bigoplus_{(P)\in\mathcal{P}} H_m(P) \to H_m(\Gamma)$$

is bijective for $m > n$.

(iv) *For $m \geq 0$,*

$$H_m(\underline{B}\Gamma) \cong \begin{cases} \mathbb{Z}^{r_m}, & m \text{ odd}, \\ \mathbb{Z}^{r_m} \oplus (\mathbb{Z}/p)^{p^k - s_{m+1}}, & m \text{ even},\ m \geq 2, \\ \mathbb{Z}, & m = 0. \end{cases}$$

(v) *For $m \geq 1$ the long exact homology sequence associated to the pushout* (1.13)

$$0 \to H_{2m}(\Gamma) \to H_{2m}(\underline{B}\Gamma) \to \bigoplus_{(P)\in\mathcal{P}} H_{2m-1}(P)$$
$$\to H_{2m-1}(\Gamma) \to H_{2m-1}(\underline{B}\Gamma) \to 0$$

can be identified with

$$0 \to \mathbb{Z}^{r_{2m}} \to \mathbb{Z}^{r_{2m}} \oplus (\mathbb{Z}/p)^{p^k - s_{2m+1}}$$
$$\to (\mathbb{Z}/p)^{p^k} \to \mathbb{Z}^{r_{2m-1}} \oplus (\mathbb{Z}/p)^{s_{2m}} \to \mathbb{Z}^{r_{2m-1}} \to 0.$$

Proof. (i), (iii), (iv) and (v): Recall there is a exact sequence

$$0 \to \operatorname{Ext}^1_{\mathbb{Z}}(H^{n+1}(X), \mathbb{Z}) \to H_n(X) \to \operatorname{Hom}_{\mathbb{Z}}(H^n(X), \mathbb{Z}) \to 0$$

for every CW-complex X with finite skeleta, natural in X. This, Theorem 1.7 and Corollary 1.20 imply (i) (iv), and (v).

(ii): Here again we use the Hochschild–Serre spectral sequence

$$E^2_{i,j} = H_i(\mathbb{Z}/p; H_j(\mathbb{Z}^n_\rho)) \Rightarrow H_{i+j}(\Gamma).$$

Then the Universal Coefficient Theorem, Lemma A.1, and Lemma 1.10 (i) imply that

$$\widehat{H}^{i+1}(\mathbb{Z}/p; H_j(\mathbb{Z}^n_\rho)) \cong \widehat{H}^{i+1}(\mathbb{Z}/p; H^j(\mathbb{Z}^n_\rho)^*) \cong \widehat{H}^{-i-1}(\mathbb{Z}/p; H^j(\mathbb{Z}^n_\rho)) = 0$$

for $i + j$ even. Therefore, $E^2_{i,j} = 0$ when $i + j$ is even and $i > 0$. Because $\widehat{H}^{-1}(\mathbb{Z}/p; H_{2m}(\mathbb{Z}^n_\rho)) = 0$, the norm map

$$H_{2m}(\mathbb{Z}^n_\rho)_{\mathbb{Z}/p} \to H_{2m}(\mathbb{Z}^n_\rho)^{\mathbb{Z}/p}$$

is injective. Thus $E^2_{0,2m} = H_{2m}(\mathbb{Z}^n_\rho)_{\mathbb{Z}/p}$ is torsion-free. Since, for $i > 0$, $E^2_{i,j}$ is torsion,

$$H_{2m}(\mathbb{Z}^n_\rho)_{\mathbb{Z}/p} = E^2_{0,2m} = E^\infty_{0,2m} \xrightarrow{\cong} H_{2m}(\Gamma). \qquad \square$$

3. K-cohomology

Next we analyze the values of complex K-theory K^* on $B\Gamma$ and $\underline{B}\Gamma$. Recall that by Bott periodicity K^* is 2-periodic, $K^0(*) = \mathbb{Z}$, and $K^1(*) = 0$.

A rational computation of $K^*(BG) \otimes \mathbb{Q}$ has been given for groups G with a cocompact G-CW-model for $\underline{E}G$ in [31], Theorem 0.1, namely

$$\begin{aligned} &K^m(BG) \otimes \mathbb{Q} \\ &\xrightarrow{\cong} \Big(\prod_{l\in\mathbb{Z}} H^{2l+m}(BG;\mathbb{Q})\Big) \times \Big(\prod_{q \text{ prime}} \prod_{(g)\in \operatorname{con}_q(G)} \prod_{l\in\mathbb{Z}} H^{2l+m}(BC_G\langle g\rangle; \widehat{\mathbb{Q}_q})\Big), \end{aligned}$$

where $\operatorname{con}_q(G)$ is the set of conjugacy classes (g) of elements $g \in G$ of order q^d for some integer $d \geq 1$ and $C_G\langle g\rangle$ is the centralizer of the cyclic subgroup $\langle g\rangle$.

It gives in particular for $G = \Gamma$ because of Theorem 1.7 (ii) and (i) and Lemma 1.9:

$$\begin{aligned} K^0(B\Gamma) \otimes \mathbb{Q} &\cong \mathbb{Q}^{\sum_{l\in\mathbb{Z}} r_{2l}} \oplus (\widehat{\mathbb{Q}_p})^{(p-1)p^k}, \\ K^1(B\Gamma) \otimes \mathbb{Q} &\cong \mathbb{Q}^{\sum_{l\in\mathbb{Z}} r_{2l+1}}. \end{aligned}$$

Recall that we have computed $\sum_{l\in\mathbb{Z}} r_{2l}$ and $\sum_{l\in\mathbb{Z}} r_{2l+1}$ in Lemma 1.22 (ii).

We are interested in determining the integral structure, namely, we want to show

Theorem 3.1 (K-cohomology of $B\Gamma$ and $\underline{B}\Gamma$).

(i) *For* $m \in \mathbb{Z}$,

$$K^m(B\Gamma) \cong \begin{cases} \mathbb{Z}^{\sum_{l\in\mathbb{Z}} r_{2l}} \oplus (\widehat{\mathbb{Z}_p})^{(p-1)p^k}, & m \text{ even}, \\ \mathbb{Z}^{\sum_{l\in\mathbb{Z}} r_{2l+1}}, & m \text{ odd}. \end{cases}$$

Here $\widehat{\mathbb{Z}_p}$ *is the* p*-adic integers.*

(ii) *There is a split exact sequence of abelian groups*

$$0 \to (\widehat{\mathbb{Z}_p})^{(p-1)p^k} \to K^0(B\Gamma) \to K^0(B\mathbb{Z}^n_\rho)^{\mathbb{Z}/p} \to 0$$

and $K^0(B\mathbb{Z}^n_\rho)^{\mathbb{Z}/p} \cong \mathbb{Z}^{\sum_{l\in\mathbb{Z}} r_{2l}}$.

(iii) *Restricting to the subgroup* $\mathbb{Z}^n$ *of* Γ *induces an isomorphism*

$$K^1(B\Gamma) \xrightarrow{\cong} K^1(B\mathbb{Z}^n_\rho)^{\mathbb{Z}/p}$$

and $K^1(B\mathbb{Z}^n_\rho)^{\mathbb{Z}/p} \cong \mathbb{Z}^{\sum_{l\in\mathbb{Z}} r_{2l+1}}$.

(iv) *We have*

$$K^0(\underline{B}\Gamma) \cong \mathbb{Z}^{\sum_{l\in\mathbb{Z}} r_{2l}}.$$

(v) *We have*

$$K^1(\underline{B}\Gamma) \cong \mathbb{Z}^{\sum_{l\in\mathbb{Z}} r_{2l+1}} \oplus T^1$$

for a finite abelian p*-group* T^1 *for which there exists a filtration*

$$T^1 = T^1_1 \supset T^1_2 \supset \cdots \supset T^1_{[(n/2)+1]} = 0$$

such that

$$T^1_i / T^1_{i+1} = (\mathbb{Z}/p)^{t_i} \quad \textit{for } i = 1, 2, \ldots, [(n/2)+1]$$

for integers t_i *which satisfy* $0 \le t_i \le p^k - s_{2i+1}$.

(vi) *The map* $K^1(\underline{B}\Gamma) \to K^1(B\Gamma)$ *induces an isomorphism*

$$K^1(\underline{B}\Gamma)/p\textit{-torsion} \xrightarrow{\cong} K^1(B\Gamma).$$

Its kernel is isomorphic to T^1 *and is isomorphic to the cokernel of the map*

$$K^0(B\Gamma) \xrightarrow{\varphi^0} \bigoplus_{(P)\in\mathcal{P}} \widetilde{K}^0(BP).$$

The proof of Theorem 3.1 needs some preparation. We will use two spectral sequences. The *Atiyah–Hirzebruch spectral sequence* (see [43], Chapter 15) for topological K-theory

$$E_2^{i,j} = H^i(\underline{B}\Gamma; K^j(*)) \Rightarrow K^{i+j}(\underline{B}\Gamma)$$

converges since $\underline{B}\Gamma$ has a model which is a finite dimensional CW-complex. We also use the *Leray–Serre spectral sequence* (see [43, Chapter 15]) of the fibration $B\mathbb{Z}^n \to B\Gamma \to B\mathbb{Z}/p$. Recall that its E_2-term is $E_2^{i,j} = H^i(\mathbb{Z}/p; K^j(B\mathbb{Z}^n_\rho))$ and it converges to $K^{i+j}(B\Gamma)$. The Leray–Serre spectral sequence converges (with no $\lim^1$-term) by [32], Theorem 6.5.

Lemma 3.2. *In the Atiyah–Hirzebruch spectral sequence converging to* $K^*(\underline{B}\Gamma)$*,*

$$E_\infty^{i,j} \cong \begin{cases} \mathbb{Z}^{r_i}, & i \textit{ even}, \ j \textit{ even}, \\ \mathbb{Z}^{r_i} \oplus (\mathbb{Z}/p)^{t'_i}, & i \textit{ odd}, \ i \ge 3, \ j \textit{ even}, \\ 0, & i = 1, j \textit{ even}, \\ 0, & j \textit{ odd}, \end{cases}$$

where $0 \le t'_i \le p^k - s_i$.

Proof. Since $\underline{B}\Gamma$ has a finite CW-model, all differentials in the Atiyah–Hirzebruch spectral sequence converging to $K^*(\underline{B}\Gamma)$ are rationally trivial and there exists an N so that for all i, j, $E_N^{i,j} = E_\infty^{i,j}$. The E_2-term of the Atiyah–Hirzebruch spectral sequence converging to $K^*(\underline{B}\Gamma)$ is given by Theorem 1.7 (i):

$$E_2^{i,j} = H^i(\underline{B}\Gamma; K^j(*)) \cong \begin{cases} \mathbb{Z}^{r_i}, & i \text{ even}, \ j \text{ even}, \\ \mathbb{Z}^{r_i} \oplus (\mathbb{Z}/p)^{p^k - s_i}, & i \text{ odd}, \ i \geq 3, \ j \text{ even}, \\ 0, & i = 1, \ j \text{ even}, \\ 0, & j \text{ odd}. \end{cases}$$

A map with a torsion-free abelian group as target is already trivial, if it vanishes rationally. Now consider (i, j) such that it is not true that i is odd and j is even. Then one shows by induction over $r \geq 2$ that $E_r^{i,j}$ is zero for odd j and $\mathbb{Z}^{r_i}$ for even j, the differential ending at (i, j) in the E_r-term is trivial and the image of the differential starting at (i, j) is finite, and $E_r^{i,j}$ is an abelian subgroup of $E_{r+1}^{i,j}$ of finite index. Next consider (i, j) such that i is odd and j is even. Then one shows by induction over $r \geq 2$ that the image of the differential ending at (i, j) in the E^r-term lies in the torsion subgroup of $E_{r+1}^{i,j}$, the differential starting at (i, j) is trivial, the rank of $E_{r+1}^{i,j}$ is r_i and its torsion subgroup is isomorphic to $\mathbb{Z}/p^t$ for some t with $t \leq p^k - s_i$.

This finishes the proof of Lemma 3.2. □

Lemma 3.3. (i) *For every $m \in \mathbb{Z}$, there is an isomorphism of $\mathbb{Z}[\mathbb{Z}/p]$-modules*

$$K^m(B\mathbb{Z}^n_\rho) \cong \bigoplus_l H^{m+2l}(\mathbb{Z}^n_\rho);$$

in particular we get

$$K^m(B\mathbb{Z}^n_\rho)^{\mathbb{Z}/p} \cong \mathbb{Z}^{\sum_l r_{m+2l}}.$$

(ii)

$$\begin{aligned} \widehat{H}^i(\mathbb{Z}/p; K^j(B\mathbb{Z}^n_\rho)) &\cong \bigoplus_{l \in \mathbb{Z}} \widehat{H}^i(\mathbb{Z}/p; H^{j+2l}(\mathbb{Z}^n_\rho)) \\ &\cong \begin{cases} (\mathbb{Z}/p)^{\sum_{l \in \mathbb{Z}} a_{j+2l}}, & i + j \text{ even}, \\ 0, & i+j \text{ odd}. \end{cases} \end{aligned}$$

(iii) *All differentials in the Leray–Serre spectral sequence are trivial.*

Proof. (i): Since $K^*(*)$ is torsion-free, Lemma 3.4 below shows that the Chern character gives an isomorphism

$$\mathrm{ch}^m \colon K^m(T^n) \xrightarrow{\cong} \bigoplus_{i+j=m} H^i(T^n; K^j(*)) = \bigoplus_l H^{m+2l}(T^n).$$

Since T^n is a model for the $\mathbb{Z}/p$-space $B\mathbb{Z}^n_\rho$ and ch^m is natural with respect to self-maps of the torus, ch^m is an isomorphism of $\mathbb{Z}[\mathbb{Z}/p]$-modules.

Since $H^{m+2l}(\mathbb{Z}^n_\rho)^{\mathbb{Z}/p} \cong \mathbb{Z}^{r_{m+2l}}$ by Theorem 1.7 (ii) and (i), assertion (i) follows.

(ii): This follows from Lemma 1.10 (i) and assertion (i).

(iii): Next we want to show that the differentials $d_r^{i,j}$ are zero for all $r \geq 2$ and i, j. By the checkerboard pattern of the E_2-term it suffices to show for $r \geq 2$ that the differentials $d_r^{0,j}$ are trivial for $r \geq 2$ and all odd $j \geq 1$. This is equivalent to showing that for every odd $j \geq 1$ the edge homomorphism (see Proposition A.5)

$$\iota^j : K^j(B\Gamma) \to K^j(B\mathbb{Z}^n_\rho)^{\mathbb{Z}/p} = E_2^{0,j}$$

is surjective. To show this we use the transfer, whose properties are reviewed in Appendix 12.2. For j odd, $\widehat{H}^0(\mathbb{Z}/p, K^j(\mathbb{Z}^n_\rho)) = 0$ by assertion (ii). Thus the norm map $N = \iota^j \circ \mathrm{trf}^j$ is surjective, and so ι^j is surjective as desired. □

Let $\mathcal{H}_*$ be a generalized homology theory and $\mathcal{H}^*$ a generalized cohomology theory. Dold defined (see [16] and [27, Example 4.1]) Chern characters

$$\mathrm{ch}_m \colon \bigoplus_{i+j=m} H_i(X, Y; \mathcal{H}_j(*)) \to \mathcal{H}_m(X, Y) \otimes \mathbb{Q},$$
$$\mathrm{ch}^m \colon \mathcal{H}^m(X, Y) \to \bigoplus_{i+j=m} H^i(X, Y; \mathcal{H}^j(*)) \otimes \mathbb{Q}.$$

The homological Chern character is a natural transformations of homology theories defined on the category of CW-pairs. When $X = *$, then $\mathrm{ch}_m = i_{\mathbb{Q}} \colon \mathcal{H}_m(*) = \mathcal{H}_m(*) \otimes \mathbb{Z} \to \mathcal{H}_m(*) \otimes \mathbb{Q}$, after the obvious identification of the targets. Hence these Chern characters are rational isomorphisms for any CW-pair. In cohomology there are parallel statements after restricting oneself to the category of finite CW-pairs. (If the disjoint union axiom is fulfilled, finite dimensional suffices.)

A CW-pair (X, Y) is *$\mathcal{H}_*$-Chern integral* if for all m,

$$i_{\mathbb{Q}} \colon \mathcal{H}_m(X, Y) \to \mathcal{H}_m(X, Y) \otimes \mathbb{Q}$$

is a monomorphism, and ch_m gives an isomorphism onto the image of $i_{\mathbb{Q}}$. There is a similar definition of *$\mathcal{H}^*$-Chern integral* for finite CW-pairs.

Lemma 3.4 (Chern character).

(i) *A point is $\mathcal{H}_*$-Chern integral if and only if $\mathcal{H}_*(*)$ is torsion-free.*

(ii) *If X is $\mathcal{H}_*$-Chern integral, then so is $X \times S^1$.*

Similar statements hold in cohomology.

Proof. (i): If a point is $\mathcal{H}_*$-Chern integral, then $\mathcal{H}_*(*) \to \mathcal{H}_*(*) \otimes \mathbb{Q}$ is injective, hence $\mathcal{H}_*(*)$ is torsion-free. If $\mathcal{H}_*(*)$ is torsion-free, then $i_{\mathbb{Q}}$ is injective. Since $\mathrm{ch}_m = i_{\mathbb{Q}}$, a point is $\mathcal{H}_*$-Chern integral.

(ii): Consider the following commutative diagram with split exact columns.

$$
\begin{tikzcd}
0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] \\
\bigoplus_{i+j=m} H_i(X \times D^1; \mathcal{H}_j(*)) \arrow[r, "\mathrm{ch}_m"] \arrow[d, shift right] & \mathcal{H}_m(X \times D^1) \otimes \mathbb{Q} \arrow[d, shift right] & \mathcal{H}_m(X \times D^1) \arrow[l, "i_{\mathbb{Q}}"'] \arrow[d, shift right] \\
\bigoplus_{i+j=m} H_i(X \times S^1; \mathcal{H}_j(*)) \arrow[r, "\mathrm{ch}_m"] \arrow[u, dashed, shift right] \arrow[d] & \mathcal{H}_m(X \times S^1) \otimes \mathbb{Q} \arrow[u, dashed, shift right] \arrow[d] & \mathcal{H}_m(X \times S^1) \arrow[l, "i_{\mathbb{Q}}"'] \arrow[u, dashed, shift right] \arrow[d] \\
\bigoplus_{i+j=m} H_i(X \times (S^1, D^1); \mathcal{H}_j(*)) \arrow[r, "\mathrm{ch}_m"] \arrow[d] & \mathcal{H}_m(X \times (S^1, D^1)) \otimes \mathbb{Q} \arrow[d] & \mathcal{H}_m(X \times (S^1, D^1)) \arrow[l, "i_{\mathbb{Q}}"'] \arrow[d] \\
0 & 0 & 0
\end{tikzcd}
$$

The columns come from the long exact sequence of a pair where D^1 is included in S^1 as the upper semicircle. The splitting maps are given by a constant map $S^1 \to D^1$. It is elementary to see that the bottom row is isomorphic to

$$\bigoplus_{i+j=m-1} H_i(X; \mathcal{H}_j(*)) \xrightarrow{\mathrm{ch}_{m-1}} \mathcal{H}_{m-1}(X) \otimes \mathbb{Q} \xleftarrow{i_{\mathbb{Q}}} \mathcal{H}_{m-1}(X).$$

Since X is $\mathcal{H}_*$-Chern integral, so are $X \times (D^1, S^1)$ and $X \times D^1$. It follows that $X \times S^1$ is $\mathcal{H}_*$-Chern integral as desired.

One may also argue by using the fact that stably $X \times S^1$ agrees with $X \vee S^1 \vee \Sigma X$ and the property $\mathcal{H}_*$-Chern integral is inherited by suspensions and wedges. $\square$

Proof of Theorem 3.1. (iv), (v): These assertions follow from the Atiyah–Hirzebruch spectral sequence converging to $K^*(\underline{B}\Gamma)$ using Lemma 3.2.

(ii), (iii), (vi): We first claim that for all $m \in \mathbb{Z}$, the inclusion $\iota \colon \mathbb{Z}^n \to \Gamma$ induces an epimorphism

$$\iota^m \colon K^m(B\Gamma) \to K^m(B\mathbb{Z}^n_\rho)^{\mathbb{Z}/p}$$

and $K^m(B\mathbb{Z}^n_\rho)^{\mathbb{Z}/p} \cong \mathbb{Z}^{\sum_{l \in \mathbb{Z}} r_{m+2l}}$. We will also show that for m odd, the map ι^m is an isomorphism. By Lemma 3.3 (iii), the Leray–Serre spectral sequence collapses, so $E_2^{0,m} = E_\infty^{0,m}$. Hence the edge homomorphism ι^m is onto (see Proposition A.5). The computation of $K^m(B\mathbb{Z}^n_\rho)$ is given in Lemma 3.3 (i). Now assume m is odd. For any $i > 0$, $E_2^{i,m-i} = 0$ by Lemma 3.3 (ii). Hence $H^m(B\Gamma) = E_\infty^{0,m}$, so the edge homomorphism is injective. We have now proved assertion (iii) of our theorem.

We next note that for all $m \in \mathbb{Z}$, the kernel and cokernel of the composite

$$K^m(\underline{B}\Gamma) \to K^m(B\Gamma) \to K^m(B\mathbb{Z}^n_\rho)^{\mathbb{Z}/p}$$

are finitely generated abelian p-groups. This follows from Proposition A.4 and the commutative diagram

$$\begin{tikzcd} B\mathbb{Z}^n_\rho = T^n \times S^\infty \arrow[r, "\simeq"] \arrow[d, "\pi"] & T^n = \mathbb{R}^n/\mathbb{Z}^n \arrow[d, "\underline{\pi}"] \\ B\Gamma = T^n \times_{\mathbb{Z}/p} S^\infty \arrow[r] & \underline{B}\Gamma = \mathbb{R}^n/\Gamma. \end{tikzcd}$$

By Lemma 1.9 (iv), the number of conjugacy classes of order p subgroups of Γ is p^k. By the Atiyah–Segal Completion Theorem (see [8]),

$$\widetilde{K}^m(B\mathbb{Z}/p) \cong \begin{cases} \mathbb{I}_{\mathbb{C}}(\mathbb{Z}/p) \otimes \widehat{\mathbb{Z}_p} \cong (\widehat{\mathbb{Z}_p})^{p-1}, & \text{if } m \text{ even,} \\ 0, & \text{if } m \text{ odd,} \end{cases}$$

where $\mathbb{I}_{\mathbb{C}}(\mathbb{Z}/p) \subset R_{\mathbb{C}}(\mathbb{Z}/p)$ is the augmentation ideal. Hence

$$\bigoplus_{(P)\in\mathcal{P}} \widetilde{K}^0(BP) \cong (\widehat{\mathbb{Z}_p})^{(p-1)p^k}.$$

We are now in a position to analyze the long exact sequence

$$0 \to K^0(\underline{B}\Gamma) \xrightarrow{\bar{f}^0} K^0(B\Gamma) \xrightarrow{\varphi^0} \bigoplus_{(P)\in\mathcal{P}} \widetilde{K}^0(BP) \xrightarrow{\delta^0} K^1(\underline{B}\Gamma) \xrightarrow{\bar{f}^1} K^1(B\Gamma) \to 0 \tag{3.5}$$

associated to the cellular pushout (1.13). We will work from right to left.

Since $K^1(B\Gamma) \cong K^1(B\mathbb{Z}^n_\rho)^{\mathbb{Z}/p}$ is torsion-free, it follows that the kernel of $\bar{f}^1$ equals T^1, the p-torsion subgroup of $K^1(\underline{B}\Gamma)$. By exactness of (3.5), T^1 also equals the cokernel of φ^0. This completes the proof of assertion (vi).

We showed above that $\ker \bar{f}^1 = \operatorname{im}\delta^0$ is a finite abelian p-group. It follows that $\ker\delta^0 = \operatorname{im}\varphi^0$ is also isomorphic to $(\widehat{\mathbb{Z}_p})^{(p-1)p^k}$ since any finite abelian p-group A is p-adically complete, and hence a $\widehat{\mathbb{Z}_p}$-module, a $\mathbb{Z}$-homomorphism from $(\widehat{\mathbb{Z}_p})^{(p-1)p^k} \to A$ is automatically a $\widehat{\mathbb{Z}_p}$-homomorphism and $\widehat{\mathbb{Z}_p}$ is a principal ideal domain.

Consider the commutative diagram with exact rows

$$\begin{tikzcd} 0 \arrow[r] & K^0(\underline{B}\Gamma) \arrow[r] \arrow[d, "\underline{\iota}^0"] & K^0(B\Gamma) \arrow[r] \arrow[d, "\iota^0"] & \operatorname{im}\varphi^0 \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & K^0(B\mathbb{Z}^n_\rho)^{\mathbb{Z}/p} \arrow[r] & K^0(B\mathbb{Z}^n_\rho)^{\mathbb{Z}/p} \arrow[r] & 0 \arrow[r] & 0. \end{tikzcd}$$

We have already seen that the middle vertical map is surjective with free abelian target, hence split surjective. Let K be the kernel of ι^0. Then by the snake lemma, there is a short exact sequence

$$0 \to K \to \operatorname{im}\varphi^0 \to \operatorname{coker}(\underline{\iota}^0) \to 0.$$

As we noted above, $\operatorname{im}\varphi^0 \cong (\widehat{\mathbb{Z}_p})^{(p-1)p^k}$ and $\operatorname{coker}(\iota^0)$ is a finite abelian p-group. Thus K is also isomorphic to $(\widehat{\mathbb{Z}_p})^{(p-1)p^k}$. This completes the proof of assertion (ii).

(i): This follows from assertions (ii) and (iii).

This finishes the proof of Theorem 3.1. □

4. K-homology

In this section we compute complex K-homology of $B\Gamma$ and $\underline{B}\Gamma$. Rationally this can be done using the Chern character of Dold [16] which gives for every CW-complex a natural isomorphism

$$\bigoplus_{l \in \mathbb{Z}} H_{m+2l}(X) \otimes \mathbb{Q} \xrightarrow{\cong} \mathrm{KO}_m(X) \otimes \mathbb{Q}.$$

In particular we get from Theorem 2.1 (i) and (iv)

$$K_m(B\Gamma) \otimes \mathbb{Q} \cong \mathbb{Q}^{\sum_{l\in\mathbb{Z}} r_{m+2l}}, \quad K_m(\underline{B}\Gamma) \otimes \mathbb{Q} \cong \mathbb{Q}^{\sum_{l\in\mathbb{Z}} r_{m+2l}}.$$

We are interested in determining the integral structure, namely, we want to show

Theorem 4.1 (K-homology of $B\Gamma$ and $\underline{B}\Gamma$).

(i) *For $m \in \mathbb{Z}$,*

$$K_m(B\Gamma) \cong \begin{cases} \mathbb{Z}^{\sum_{l\in\mathbb{Z}} r_{2l}}, & m \text{ even}, \\ \mathbb{Z}^{\sum_{l\in\mathbb{Z}} r_{2l+1}} \oplus (\mathbb{Z}/p^\infty)^{(p-1)p^k}, & m \text{ odd}. \end{cases}$$

Here $\mathbb{Z}/p^\infty = \operatorname{colim}_{n\to\infty} \mathbb{Z}/p^n \cong \mathbb{Z}[1/p]/\mathbb{Z}$.

(ii) *The inclusion map $\mathbb{Z}^n \to \Gamma$ induces an isomorphism*

$$K_0(B\mathbb{Z}^n_\rho)_{\mathbb{Z}/p} \xrightarrow{\cong} K_0(B\Gamma)$$

and $K_0(B\mathbb{Z}^n_\rho)_{\mathbb{Z}/p} \cong \mathbb{Z}^{\sum_{l\in\mathbb{Z}} r_{2l}}$.

(iii) *There is a split short exact sequence of abelian groups*

$$0 \to (\mathbb{Z}/p^\infty)^{(p-1)p^k} \to K_1(B\Gamma) \to K_1(\underline{B}\Gamma) \to 0.$$

(iv) *We have*

$$K_0(\underline{B}\Gamma) \cong \mathbb{Z}^{\sum_{l\in\mathbb{Z}} r_{2l}} \oplus T^1,$$

where T^1 is the finite abelian p-group appearing in Theorem 3.1 (v).

(v) *We have*

$$K_1(\underline{B}\Gamma) \cong \mathbb{Z}^{\sum_{l\in\mathbb{Z}} r_{2l+1}}.$$

(vi) *The group T^1 is isomorphic to a subgroup of the kernel of*

$$\bigoplus_{(P)\in\mathcal{P}} K_1(BP) \to K_1(B\Gamma).$$

Its proof needs some preparation.

Theorem 4.2 (Universal Coefficient Theorem for K-theory). *For any CW-complex X there is a short exact sequence*

$$0 \to \mathrm{Ext}_{\mathbb{Z}}(K_{*-1}(X), \mathbb{Z}) \to K^*(X) \to \mathrm{Hom}_{\mathbb{Z}}(K_*(X), \mathbb{Z}) \to 0.$$

If X is a finite CW-complex, there is also the K-homological version

$$0 \to \mathrm{Ext}_{\mathbb{Z}}(K^{*+1}(X), \mathbb{Z}) \to K_*(X) \to \mathrm{Hom}_{\mathbb{Z}}(K^*(X), \mathbb{Z}) \to 0.$$

Proof. A proof for the first short exact sequence can be found in [6] and [46], (3.1), the second sequence follows then from [1], Note 9 and 15. □

Proof of Theorem 4.1. (iv), (v): These assertions follow from Theorem 3.1 (iv) and (v) and Theorem 4.2 since there is a finite CW-model for $\underline{B}\Gamma$, namely $\Gamma\backslash\mathbb{R}^n$.

(iii): We will use Pontryagin duality for locally compact abelian groups. For such a group G, the *Pontryagin dual* $\widehat{G}$ is $\mathrm{Hom}(G, S^1)$, given the compact-open topology. A reference for the basic properties is [18]. These include: $\widehat{G}$ is also a locally compact abelian group. The natural map from G to its double dual is a isomorphism. G is discrete if and only if $\widehat{G}$ is compact. If $0 \to A \to B \to C \to 0$ is exact, then so is $0 \to \widehat{C} \to \widehat{B} \to \widehat{A} \to 0$. Our primary example of duality is

$$\widehat{\mathbb{Z}/p^\infty} \cong \widehat{\mathbb{Z}_p}.$$

Here $\mathbb{Z}/p^\infty$ is given the discrete topology and the p-adic integers $\widehat{\mathbb{Z}_p}$ are given the p-adic topology. This statement is included in [18], paragraph 25.2, but also follows from the following assertion proved in [25], 20.8, if $H_1 \to H_2 \to H_3 \to \cdots$ is a sequence of maps of locally compact abelian groups, then

$$\widehat{\operatorname{colim}_{n\to\infty} H_n} \cong \lim_{n\to\infty} \widehat{H}_n$$

We will now give the computation of $K_*(B\mathbb{Z}/p)$. The Atiyah–Hirzebruch Spectral Sequence shows that $\widetilde{K}_0(B\mathbb{Z}/p) = 0$. Vick [44] shows that $K_1(BG)$ is the Pontryagin dual of $\widetilde{K}_0(BG)$ for any finite group G. Applying these facts to $G = \mathbb{Z}/p$ we get (see also Knapp [22], Proposition 2.11)

$$K_m(B\mathbb{Z}/p) \cong \begin{cases} (\mathbb{Z}/p^\infty)^{p-1}, & \text{if } m \text{ is odd,} \\ \mathbb{Z}, & \text{if } m \text{ is even.} \end{cases}$$

Thus the long exact K-homology sequence associated to the cellular pushout (1.13) reduces to the exact sequence

$$0 \to K_0(B\Gamma) \xrightarrow{\bar{f}_0} K_0(\underline{B}\Gamma) \xrightarrow{\partial_0} \bigoplus_{(P)\in\mathcal{P}} K_1(BP) \xrightarrow{\varphi_0} K_1(B\Gamma) \xrightarrow{\bar{f}_1} K_1(\underline{B}\Gamma) \to 0. \tag{4.3}$$

Note that $\operatorname{im}\partial_0$ is a finite abelian p-group since it is a finitely generated subgroup of the p-torsion group

$$\bigoplus_{(P)\in\mathcal{P}} K_1(BP) \cong (\mathbb{Z}/p^\infty)^{(p-1)p^k}.$$

Dualizing the exact sequence

$$0 \to \operatorname{im}\partial_0 \to (\mathbb{Z}/p^\infty)^{(p-1)p^k} \to \operatorname{im}\varphi_0 \to 0,$$

we see that $\widehat{\operatorname{im}\varphi_0}$ has finite p-power index in $(\widehat{\mathbb{Z}_p})^{(p-1)p^k}$, hence is itself isomorphic to $(\widehat{\mathbb{Z}_p})^{(p-1)p^k}$ (compare the proof of Theorem 3.1 (iv) and (v)). Dualizing again, we see $\operatorname{im}\varphi_0 \cong (\mathbb{Z}/p^\infty)^{(p-1)p^k}$.

The map $\bar{f}_1$ is split surjective since its target is free abelian by assertion (v).

(ii): The Universal Coefficient Theorem in K-theory shows that $K^0(B\mathbb{Z}^n_\rho) \cong K_0(B\mathbb{Z}^n_\rho)^*$. In Lemma 3.3 we showed there is an isomorphism of $\mathbb{Z}[\mathbb{Z}/p]$-modules $K^0(B\mathbb{Z}^n_\rho) \cong \bigoplus_\ell H^{2\ell}(\mathbb{Z}^n_\rho)$. Now we proceed exactly as in the proof of Theorem 2.1 (ii), using the Leray–Serre spectral sequence

$$E^2_{i,j} = H_i(\mathbb{Z}/p; K_j(B\mathbb{Z}^n_\rho)) \Rightarrow K_{i+j}(B\Gamma).$$

One shows that $E^2_{0,2m} = K_{2m}(B\mathbb{Z}^n_\rho)_{\mathbb{Z}/p}$ is torsion-free, and for $i > 0$, $E^2_{i,j}$ has exponent p and vanishes if $i + j$ is even. Thus

$$K_{2m}(B\mathbb{Z}^n_\rho)_{\mathbb{Z}/p} = E^2_{0,2m} = E^\infty_{0,2m} \xrightarrow{\cong} K_{2m}(B\Gamma).$$

By Remark A.2 and the Universal Coefficient Theorem, $(K_{2m}(B\mathbb{Z}^n_\rho)_{\mathbb{Z}/p})^* \cong K^{2m}(B\mathbb{Z}^n_\rho)^{\mathbb{Z}/p}$, which is isomorphic to $\mathbb{Z}^{\sum_{l\in\mathbb{Z}} r_{2l}}$ by Lemma 3.3 (i).

(i): This follows from assertions (ii), (iii) and (v).

(vi): This follows from assertion (iv) and the exact sequence (4.3). This finishes the proof of Theorem 4.1. □

5. KO-cohomology

In this section we compute real K-cohomology KO^* of $B\Gamma$.

Recall that by Bott periodicity KO^* is 8-periodic, i.e., there is a natural isomorphism $\mathrm{KO}^m(X) \cong \mathrm{KO}^{m+8}(X)$ for every $m \in \mathbb{Z}$ and CW-complex X, and $\mathrm{KO}^{-m}(*)$

is given for $m = 0, 1, 2, \dots, 7$ by $\mathbb{Z}$, $\mathbb{Z}/2$, $\mathbb{Z}/2$, 0, $\mathbb{Z}$, 0, 0, 0. We will assume from now on that p is odd in order to avoid the extra difficulties arising from the fact that $\mathrm{KO}^m(*) \cong \mathbb{Z}/2$ for $m = 1, 2$.

Theorem 5.1 (KO-cohomology of $B\Gamma$ and $\underline{B}\Gamma$). *Let p be an odd prime and let m be any integer.*

(i)

$$\mathrm{KO}^m(B\Gamma) \cong \begin{cases} (\bigoplus_{l \in \mathbb{Z}} \mathrm{KO}^{m-l}(*)^{r_l}) \oplus (\widehat{\mathbb{Z}_p})^{p^k(p-1)/2}, & m \text{ even}, \\ \bigoplus_{l \in \mathbb{Z}} \mathrm{KO}^{m-l}(*)^{r_l}, & m \text{ odd}. \end{cases}$$

(ii) *There is a split exact sequence of abelian groups*

$$0 \to (\widehat{\mathbb{Z}_p})^{p^k(p-1)/2} \to \mathrm{KO}^{2m}(B\Gamma) \to \mathrm{KO}^{2m}(B\mathbb{Z}^n_\rho)^{\mathbb{Z}/p} \to 0,$$

and $\mathrm{KO}^{2m}(B\mathbb{Z}^n_\rho)^{\mathbb{Z}/p} \cong \bigoplus_{l \in \mathbb{Z}} \mathrm{KO}^{2m-l}(*)^{r_l}$.

(iii) *Restricting to the subgroup $\mathbb{Z}^n$ of Γ induces an isomorphism*

$$\mathrm{KO}^{2m+1}(B\Gamma) \xrightarrow{\cong} \mathrm{KO}^{2m+1}(B\mathbb{Z}^n_\rho)^{\mathbb{Z}/p}.$$

and $\mathrm{KO}^{2m+1}(B\mathbb{Z}^n_\rho)^{\mathbb{Z}/p} \cong \bigoplus_{l \in \mathbb{Z}} \mathrm{KO}^{2m+1-l}(*)^{r_l}$.

(iv) *We have*

$$\mathrm{KO}^{2m}(\underline{B}\Gamma) \cong \oplus_{l \in \mathbb{Z}} \mathrm{KO}^{2m-l}(*)^{r_l}.$$

(v) *We have*

$$\mathrm{KO}^{2m+1}(\underline{B}\Gamma) \cong (\bigoplus_{l \in \mathbb{Z}} \mathrm{KO}^{2m+1-l}(*)^{r_l}) \oplus \mathrm{TO}^{2m+1},$$

where TO^{2m+1} *is a finite abelian p-group for which there exists a filtration*

$$\mathrm{TO}^{2m+1} = \mathrm{TO}^{2m+1}_1 \supset \mathrm{TO}^{2m+1}_2 \supset \cdots \supset \mathrm{TO}^{2m+1}_{[(n+4-(-1)^m)/4]} = 0$$

such that $\mathrm{TO}^{2m+1}_i / \mathrm{TO}^{2m+1}_{i+1} = (\mathbb{Z}/p)^{\mathrm{to}_i}$ *holds for integers* to_i *which satisfy* $0 \le \mathrm{to}_i \le p^k - s_{4i+(-1)^m}$.

(vi) *The map* $\mathrm{KO}^{2m+1}(\underline{B}\Gamma) \to \mathrm{KO}^{2m+1}(B\Gamma)$ *induces an isomorphism*

$$\mathrm{KO}^{2m+1}(\underline{B}\Gamma)/p\text{-torsion} \xrightarrow{\cong} \mathrm{KO}^{2m+1}(B\Gamma).$$

Its kernel is isomorphic to TO^{2m+1} *and is isomorphic to the cokernel of the map*

$$\mathrm{KO}^{2m}(B\Gamma) \to \bigoplus_{(P) \in \mathcal{P}} \widetilde{\mathrm{KO}}^{2m}(BP).$$

Lemma 5.2. *Let p be an odd prime. In the Atiyah–Hirzebruch spectral sequence converging to $K^*(\underline{B}\Gamma)$ after localizing at p*

$$(E_\infty^{i,j})_{(p)} \cong \begin{cases} \mathbb{Z}_{(p)}^{r_i}, & i \text{ even},\ j \equiv 0 \bmod 4,\\ \mathbb{Z}_{(p)}^{r_i} \oplus (\mathbb{Z}/p)^{t_i'}, & i \text{ odd},\ i \geq 3,\ j \equiv 0 \bmod 4,\\ 0, & i = 1,\ j \equiv 0 \bmod 4,\\ 0, & j \not\equiv 0 \bmod 4,\end{cases}$$

where $0 \leq t_i' \leq p^k - s_i$.

Proof. Because of Theorem 1.7 (i) the E_2-term of the spectral sequence converging to $K^*(B\Gamma)_{(p)}$ is given after localization at p by

$$\begin{aligned}(E_2^{i,j})_{(p)} &= H^i(\underline{B}\Gamma; \mathrm{KO}^j(*))_{(p)}\\ &\cong \begin{cases} \mathbb{Z}_{(p)}^{r_i}, & i \text{ even},\ j \equiv 0 \bmod 4,\\ \mathbb{Z}_{(p)}^{r_i} \oplus (\mathbb{Z}/p)^{p^k - s_i}, & i \text{ odd},\ i \geq 3,\ j \equiv 0 \bmod 4,\\ 0, & i = 1,\ j \equiv 0 \bmod 4,\\ 0, & j \not\equiv 0 \bmod 4.\end{cases}\end{aligned}$$

The rest of the proof is analogous to the proof of Lemma 3.2. □

Lemma 5.3. *Let p be an odd prime. For every $m \in \mathbb{Z}$, there are isomorphisms of $\mathbb{Z}[\mathbb{Z}/p]$-modules*

$$\mathrm{KO}^m(B\mathbb{Z}_\rho^n) \otimes \mathbb{Z}[1/2] \cong \bigoplus_{i \in \mathbb{Z}} H^i(B\mathbb{Z}_\rho^n) \otimes \mathrm{KO}^{m-i}(*) \otimes \mathbb{Z}[1/2],$$
$$\mathrm{KO}^m(B\mathbb{Z}_\rho^n) \otimes \mathbb{Z}[1/p] \cong \bigoplus_{i \in \mathbb{Z}} H^i(B\mathbb{Z}_\rho^n) \otimes \mathrm{KO}^{m-i}(*) \otimes \mathbb{Z}[1/p].$$

Proof. Since $\mathrm{KO}^*(X) \otimes \mathbb{Z}[1/2]$ is a generalized cohomology theory with torsion-free coefficients, the Chern character and Lemma 3.4 give the first isomorphism.

One proves that there are isomorphisms of abelian groups

$$\mathrm{KO}^m(B\mathbb{Z}_\rho^n) \cong \bigoplus_{i \in \mathbb{Z}} H^i(B\mathbb{Z}_\rho^n) \otimes \mathrm{KO}^{m-i}(*)$$

by induction on n using excision and the fact that $B\mathbb{Z}^n = S^1 \times B\mathbb{Z}^{n-1}$. It follows that the Atiyah–Hirzebruch spectral sequence $E_2^{i,j} = H^i(B\mathbb{Z}^n; K^j(*)[1/p]) \Rightarrow \mathrm{KO}^{i+j}(B\mathbb{Z}^n)[1/p]$ collapses. This spectral sequence is natural with respect to automorphisms of $\mathbb{Z}^n$. Hence we obtain a descending filtration by $\mathbb{Z}[1/p][\mathbb{Z}/p]$-modules

$$\mathrm{KO}^m(B\mathbb{Z}_\rho^n)[1/p] = F^{0,m} \supset F^{1,m-1} \supset F^{2,m-2} \supset \cdots \supset F^{m,0} \supset F^{m+1,-1} = 0$$

and exact sequences

$$0 \to F^{i+1,m-i-1} \to F^{i,m-i} \xrightarrow{\pi} H^i(\mathbb{Z}_\rho^n) \otimes K^{m-i}(*) \otimes \mathbb{Z}[1/p] \to 0.$$

It thus suffices to show that these exact sequences split over $\mathbb{Z}[1/p][\mathbb{Z}/p]$ for all i. If $m-i \equiv 3,5,6,7 \bmod 8$, this follows from the fact that $\mathrm{KO}^{m-i}(*) = 0$. If $m-i \equiv 0,4 \bmod 8$, then $K^{m-i}(*) \cong \mathbb{Z}$ and $H^i(\mathbb{Z}^n_\rho) \otimes K^{m-i}(*) \otimes \mathbb{Z}[1/p]$ is a finitely generated $\mathbb{Z}[1/p]$-torsion-free module over the ring $\mathbb{Z}[1/p][\mathbb{Z}/p]$ $\xrightarrow{\cong} \mathbb{Z}[1/p] \times \mathbb{Z}[1/p][\zeta]$, hence is projective. Finally, suppose $m-i \equiv 1,2 \bmod 8$. Since the Atiyah–Hirzebruch spectral sequence collapses, there is a homomorphism of abelian groups $s\colon H^i(\mathbb{Z}^n_\rho) \otimes K^{m-i}(*) \otimes \mathbb{Z}[1/p] \to F^{i,m-i}$ so that $\pi \circ s = \mathrm{id}$. Define

$$\tilde{s}\colon H^i(\mathbb{Z}^n_\rho) \otimes K^{m-i}(*) \otimes \mathbb{Z}[1/p] \to F^{i,m-i}, \quad x \mapsto \sum_{g \in \mathbb{Z}/p} g \cdot s(g^{-1}x).$$

Then $\tilde{s}$ is a homomorphism of $\mathbb{Z}[\mathbb{Z}/p]$-modules and $\pi \circ \tilde{s}$ is multiplication by p and hence is the identity since $K^{m-i}(*) \cong \mathbb{Z}/2$. $\square$

Lemma 5.4. *Let p be an odd prime.*

(i) *For every $m \in \mathbb{Z}$, there is an isomorphism of abelian groups*

$$\mathrm{KO}^m(B\mathbb{Z}^n_\rho)^{\mathbb{Z}/p} \cong \bigoplus_{l \in \mathbb{Z}} \mathrm{KO}^{m-l}(*)^{r_l}.$$

(ii)

$$\begin{aligned} \widehat{H}^i(\mathbb{Z}/p; \mathrm{KO}^j(B\mathbb{Z}^n_\rho)) &\cong \bigoplus_{l \in \mathbb{Z}} \widehat{H}^i(\mathbb{Z}/p; H^{j+4l}(\mathbb{Z}^n_\rho)) \\ &\cong \begin{cases} (\mathbb{Z}/p)^{\sum_{l \in \mathbb{Z}} a_{j+4l}}, & i+j \text{ even}, \\ 0, & i+j \text{ odd}. \end{cases} \end{aligned}$$

(iii) *All differentials in the Leray–Serre spectral sequence associated to the extension* (1.1) *converging to* $\mathrm{KO}^*(B\Gamma)$ *vanish.*

Proof. (i): It suffices to show the isomorphism exists after inverting 2 and after localizing at 2. Furthermore, if M is a $\mathbb{Z}[\mathbb{Z}/p]$-module, then $M^{\mathbb{Z}/p} \otimes \mathbb{Z}[1/2] \cong (M \otimes \mathbb{Z}[1/2])^{\mathbb{Z}/p}$ and $M^{\mathbb{Z}/p} \otimes \mathbb{Z}_{(2)} \cong (M \otimes \mathbb{Z}_{(2)})^{\mathbb{Z}/p}$ since localization is an exact functor. The assertion then follows from Lemma 5.3 and the definition of the numbers r_l.

(ii): Since $\mathbb{Z}[1/2] \subset \mathbb{Z}_{(p)}$, Lemma 5.3 implies that

$$\mathrm{KO}^j(B\mathbb{Z}^n_\rho) \otimes \mathbb{Z}_{(p)} \cong \bigoplus_{l \in \mathbb{Z}} H^{j+4l}(B\mathbb{Z}^n_\rho) \otimes \mathbb{Z}_{(p)}.$$

The first isomorphism in assertion (ii) then follows since localization is an exact functor and the Tate cohomology groups are p-torsion. The second isomorphism follows from Lemma 1.10 (i).

(iii): First note that the Leray–Serre spectral sequence converges with no $\lim^1$-term, see [32], Theorem 6.5.

It suffices to prove the differentials vanish after inverting p and after localizing at p. If we invert p, the claim follows from

$$E_2^{i,j}[1/p] = H^i(\mathbb{Z}/p; \mathrm{KO}^j(B\mathbb{Z}^n_\rho))[1/p] = 0 \quad \text{for } i \geq 1.$$

If we localize at p, the proof that the differentials vanish is identical to the proof of Lemma 3.3 (iii). □

Proof of Theorem 5.1. (iv): We first note that Proposition A.4 and Lemma 5.4 (i) imply that, for all $m \in \mathbb{Z}$, the kernel and cokernel of the composite

$$\mathrm{KO}^m(\underline{B}\Gamma) \to \mathrm{KO}^m(B\Gamma) \to \mathrm{KO}^m(B\mathbb{Z}^n_\rho)^{\mathbb{Z}/p} \cong \oplus_{l\in\mathbb{Z}} \mathrm{KO}^{m-l}(*)^{r_l} \tag{5.5}$$

are finitely generated p-groups. This implies that the desired isomorphism holds after inverting p. It holds at p by Lemma 5.2.

(iii): As in the proof of Theorem 3.1, one shows that the map

$$\iota^m\colon\ \mathrm{KO}^m(B\Gamma) \to \mathrm{KO}^m(B\mathbb{Z}^n_\rho)^{\mathbb{Z}/p}$$

is an isomorphism for m odd and an epimorphism for m even.

(v), (vi): Since p is odd, every non-trivial irreducible $\mathbb{Z}/p$-representation is of complex type. Hence we get from [40], Remark on p. 133 after Proposition 2.2, that $\mathrm{KO}^m_{\mathbb{Z}/p}(*) \cong \mathrm{KO}^m(*) \oplus K^m(*) \otimes \mathbb{I}_{\mathbb{R}}(\mathbb{Z}/p)$. The Atiyah–Segal Completion Theorem for KO^* (see [8]) implies

$$\widetilde{\mathrm{KO}}^m(B\mathbb{Z}/p) \cong \begin{cases} \mathbb{I}_{\mathbb{R}}(\mathbb{Z}/p) \otimes \widehat{\mathbb{Z}_p} \cong (\widehat{\mathbb{Z}_p})^{(p-1)/2}, & m \text{ even},\\ 0, & \text{otherwise}.\end{cases}$$

The cellular pushout (1.13) yields for $m \in \mathbb{Z}$ a long exact sequence

$$\begin{aligned} 0 \to \mathrm{KO}^{2m}(\underline{B}\Gamma) &\xrightarrow{\bar{f}^{2m}} \mathrm{KO}^{2m}(B\Gamma) \xrightarrow{\varphi^{2m}} \bigoplus_{(P)\in\mathcal{P}} \widetilde{\mathrm{KO}}^{2m}(BP)\\ &\xrightarrow{\delta^{2m}} \mathrm{KO}^{2m+1}(\underline{B}\Gamma) \xrightarrow{\bar{f}^{2m+1}} \mathrm{KO}^{2m+1}(B\Gamma) \to 0.\end{aligned}$$

Define TO^{2m+1} to be the kernel of the surjection $\bar{f}^{2m+1}$. Since $\bar{f}^{2m+1}$ is an isomorphism after inverting p by (5.5) and assertion (iii), TO^{2m+1} is p-torsion. We next claim $\bar{f}^{2m+1}$ is split. We only need verify this after localizing at p in which case it follows since $K^{2m+1}(B\Gamma) \otimes \mathbb{Z}_{(p)}$ is free over $\mathbb{Z}_{(p)}$ by assertion (iii) and Lemma 5.4 (i). Finally, the stated filtration of TO^{2m+1} is a consequence of Lemma 5.2. The completes the proof of assertion (v). Assertion (vi) is a consequence.

(ii): The proof of this is identical to that of Theorem 3.1 (ii); the only missing part is to show the epimorphism

$$\iota^{2m}\colon \mathrm{KO}^{2m}(B\Gamma) \to \mathrm{KO}^{2m}(B\mathbb{Z}^n_\rho)^{\mathbb{Z}/p}$$

is split. At p, this follows since $\mathrm{KO}^{2m}(B\mathbb{Z}^n_\rho)^{\mathbb{Z}/p} \otimes \mathbb{Z}_{(p)}$ is free over $\mathbb{Z}_{(p)}$. After inverting p, the splitting is provided by composing the inverse of the composite (5.5) with the map $\mathrm{KO}^{2m}(\underline{B}\Gamma)[1/p] \to \mathrm{KO}^{2m}(B\Gamma)[1/p]$. □

6. KO-homology

In this section we want to compute the real K-homology KO_* of $B\Gamma$ and $\underline{B}\Gamma$. Rationally this can be done using the Chern character of Dold [16]: for every CW-complex there is a natural isomorphism

$$\bigoplus_{l\in\mathbb{Z}} H_{m+4l}(X)\otimes\mathbb{Q} \xrightarrow{\cong} \mathrm{KO}_m(X)\otimes\mathbb{Q}.$$

In particular we get from Theorem 2.1 (i) and (iv):

$$\mathrm{KO}_m(B\Gamma)\otimes\mathbb{Q} \cong \mathbb{Q}^{\sum_{l\in\mathbb{Z}} r_{m+4l}},$$
$$\mathrm{KO}_m(\underline{B}\Gamma)\otimes\mathbb{Q} \cong \mathbb{Q}^{\sum_{l\in\mathbb{Z}} r_{m+4l}}.$$

We are interested in determining the integral structure.

Theorem 6.1 (KO-homology of $B\Gamma$ and $\underline{B}\Gamma$). *Let p be an odd prime and m be any integer.*

(i)
$$\mathrm{KO}_m(B\Gamma) \cong \begin{cases} \mathbb{Z}^{\sum_{l\in\mathbb{Z}} r_{2l}}, & m \text{ even},\\ \mathbb{Z}^{\sum_{l\in\mathbb{Z}} r_{2l+1}} \oplus (\mathbb{Z}/p^\infty)^{p^k(p-1)/2}, & m \text{ odd}.\end{cases}$$

(ii) *The inclusion map $\mathbb{Z}^n \to \Gamma$ induces an isomorphism*

$$\mathrm{KO}_{2m}(B\mathbb{Z}^n_\rho)_{\mathbb{Z}/p} \xrightarrow{\cong} \mathrm{KO}_{2m}(B\Gamma)$$

and $\mathrm{KO}_{2m}(B\mathbb{Z}^n_\rho)_{\mathbb{Z}/p} \cong \bigoplus_{l\in\mathbb{Z}} \mathrm{KO}_{2m-l}(*)^{r_l}$.

(iii) *There is a split short exact sequence of abelian groups*

$$0 \to (\mathbb{Z}/p^\infty)^{p^k(p-1)/2} \to \mathrm{KO}_{2m+1}(B\Gamma) \to \mathrm{KO}_{2m+1}(\underline{B}\Gamma) \to 0.$$

(iv) *We have*
$$\mathrm{KO}_{2m}(\underline{B}\Gamma) \cong \Big(\bigoplus_{l\in\mathbb{Z}} \mathrm{KO}_{2m-l}(*)^{r_l}\Big) \oplus \mathrm{TO}^{2m+5},$$

where TO^{2m+5} is the finite abelian p-group appearing in Theorem 5.1 (v).

(v) *We have*
$$\mathrm{KO}_{2m+1}(\underline{B}\Gamma) \cong \bigoplus_{l\in\mathbb{Z}} \mathrm{KO}_{2m+1-l}(*)^{r_l}.$$

(vi) *The group TO^{2m+5} is isomorphic to a subgroup of the kernel of*

$$\bigoplus_{(P)\in\mathcal{P}} \mathrm{KO}_{2m+1}(BP) \to \mathrm{KO}_{2m+1}(B\Gamma).$$

Theorem 6.2 (Universal Coefficient Theorem for KO-theory). *For any CW-complex X there is a short exact sequence*

$$0 \to \operatorname{Ext}_{\mathbb{Z}}(\mathrm{KO}_{n+3}(X), \mathbb{Z}) \to \mathrm{KO}^n(X) \to \operatorname{Hom}(\mathrm{KO}_{n+4}(X), \mathbb{Z}) \to 0.$$

If X is a finite CW-complex, there is a short exact sequence

$$0 \to \operatorname{Ext}_{\mathbb{Z}}(\mathrm{KO}^{n+5}(X), \mathbb{Z}) \to \mathrm{KO}_n(X) \to \operatorname{Hom}_{\mathbb{Z}}(\mathrm{KO}^{n+4}(X), \mathbb{Z}) \to 0.$$

Proof. A proof for the first short exact sequence can be found in [6] and [46], (3.1), the second sequence follows then from [1], Note 9 and 15. □

Proof of Theorem 6.1. (iv), (v): These assertions follow from Theorem 5.1 (iv) and (v), and Theorem 6.2.

(ii): There are natural transformations of cohomology theories $i^*\colon \mathrm{KO}^* \to K^*$ and $r^*\colon K^* \to \mathrm{KO}^*$, induced by sending a real representation V to its complexification $\mathbb{C} \otimes_{\mathbb{R}} V$ and a complex representation to its restriction as a real representation. The composite $r^* \circ i^*\colon \mathrm{KO}^* \to \mathrm{KO}^*$ is multiplication by two. Since the map

$$K_0(B\mathbb{Z}^n_\rho)_{\mathbb{Z}/p} \xrightarrow{\cong} K_0(B\Gamma).$$

is bijective by Theorem 4.1 (ii), the map

$$\mathrm{KO}_{2m}(B\mathbb{Z}^n_\rho)_{\mathbb{Z}/p} \xrightarrow{\cong} \mathrm{KO}_{2m}(B\Gamma)$$

is bijective after inverting 2. In order to show that it is itself bijective, it remains to show that it is bijective after inverting p. This follows from Proposition A.4.

Since we are dealing with KO-homology, the Atiyah–Hirzebruch spectral sequence converges also for the infinite-dimensional CW-complex $B\Gamma$. Because of the existence of Dold's Chern character, all its differentials vanish rationally. For $m \in \mathbb{Z}$ we have $H_{2m}(B\Gamma) \cong \mathbb{Z}^{r_{2m}}$ by Theorem 2.1. Hence we get for an odd prime p since $\mathrm{KO}_m(*)_{(p)}$ is $\mathbb{Z}_{(p)}$ for $m \equiv 0 \bmod 4$ and 0 otherwise

$$\mathrm{KO}_{2m}(B\Gamma)_{(p)} \cong (\mathbb{Z}_{(p)})^{\sum_{l \in \mathbb{Z}} r_{2m+4l}}.$$

We conclude that

$$\mathrm{KO}_{2m}(B\Gamma) \cong \bigoplus_{l \in \mathbb{Z}} \mathrm{KO}_{2m-l}(*)^{r_l}$$

holds after localizing at p. It remains to show that it holds after inverting p. This follows from Proposition A.4 and assertion (iv).

(iii) The Atiyah–Hirzebruch spectral sequence shows that $\widetilde{\mathrm{KO}}_{2m}(B\mathbb{Z}/p) = 0$ for all $m \in \mathbb{Z}$. The methods of [44] together with the Universal Coefficient Theorem for KO-theory show that $\widetilde{\mathrm{KO}}_{2m+3}(BG)$ is the Pontryagin dual of $\widetilde{\mathrm{KO}}^{2m}(BG)$ for any finite group G. Applying these facts to $G = \mathbb{Z}/p$ for an odd prime p, we see that

$$\widetilde{\mathrm{KO}}_m(B\mathbb{Z}/p) = \begin{cases} (\mathbb{Z}/p^\infty)^{(p-1)/2}, & m \text{ odd}, \\ 0, & m \text{ even}. \end{cases}$$

Thus the long exact KO-homology sequence associated to the cellular pushout (1.13) reduces to the exact sequence

$$\begin{aligned} 0 \to \mathrm{KO}_{2m}(B\Gamma) \xrightarrow{\bar{f}_{2m}} \mathrm{KO}_{2m}(\underline{B}\Gamma) \xrightarrow{\partial_{2m}} \bigoplus_{(P)\in\mathcal{P}} \mathrm{KO}_{2m-1}(BP) \\ \xrightarrow{\varphi_{2m-1}} \mathrm{KO}_{2m-1}(B\Gamma) \xrightarrow{\bar{f}_{2m-1}} \mathrm{KO}_{2m-1}(\underline{B}\Gamma) \to 0. \end{aligned} \tag{6.3}$$

Note $\operatorname{im}\partial_{2m}$ is a finite abelian p-group, since it is a finitely generated subgroup of the p-torsion group

$$\bigoplus_{(P)\in\mathcal{P}} \mathrm{KO}_{2m-1}(BP) \cong (\mathbb{Z}/p^{\infty})^{(p-1)p^k/2}.$$

Thus $\operatorname{im}\varphi_{2m-1} \cong (\mathbb{Z}/p^{\infty})^{(p-1)p^k/2}$ (compare with the proof of Theorem 3.1 (iii)). It remains to see that $\bar{f}_{2m-1}$ splits, which we verify at p and away from p. The target of $\bar{f}_{2m-1}$ is free after localizing at p by assertion (v), so it splits. After inverting p, the exact sequence 6.3 shows that $\bar{f}_{2m-1}[1/p]$ is an isomorphism.

(i): This follows from assertions (ii), (iii) and (v).

(vi): This follows from assertions (ii) and (iv) and the long exact sequence (6.3). This finishes the proof of Theorem 6.1. □

7. Equivariant K-cohomology

In the sequel an equivariant cohomology theory is to be understood in the sense of [29], Section 1. Equivariant topological complex K-theory $K^*_?$ is an example as shown in [29], Example 1.6, based on [32]. This applies also to equivariant topological real K-theory $\mathrm{KO}^*_?$.

Rationally one obtains

$$K^0_\Gamma(\underline{E}\Gamma)\otimes\mathbb{Q} \cong \mathbb{Q}^{(p-1)p^k+\sum_{l\in\mathbb{Z}} r_{2l}}, \quad K^1_\Gamma(\underline{E}\Gamma)\otimes\mathbb{Q} \cong \mathbb{Q}^{\sum_{l\in\mathbb{Z}} r_{2l+1}}$$

from [32], Theorem 5.5 and Lemma 5.6, using Theorem 1.7 (iv) and Lemma 1.9. We want to get an integral computation. Recall that we have computed $\sum_{l\in\mathbb{Z}} r_{2l}$ and $\sum_{l\in\mathbb{Z}} r_{2l+1}$ in Lemma 1.22 (ii).

Theorem 7.1 (Equivariant K-cohomology of $\underline{E}\Gamma$).

(i) *We have*

$$K^m_\Gamma(\underline{E}\Gamma) \cong \begin{cases} \mathbb{Z}^{(p-1)p^k+\sum_{l\in\mathbb{Z}} r_{2l}}, & m \text{ even},\\ \mathbb{Z}^{\sum_{l\in\mathbb{Z}} r_{2l+1}}, & m \text{ odd}.\end{cases}$$

(ii) *There is an exact sequence*

$$0 \to K^0(\underline{B}\Gamma) \to K^0_\Gamma(\underline{E}\Gamma) \to \bigoplus_{(P)\in\mathcal{P}} \mathbb{I}_{\mathbb{C}}(P) \to T^1 \to 0,$$

where T^1 is the finite abelian p-group appearing in Theorem 3.1 (v).

(iii) *The canonical maps*

$$K^1_\Gamma(\underline{E}\Gamma) \xrightarrow{\cong} K^1(B\Gamma), \quad K^1(B\Gamma) \xrightarrow{\cong} K^1(B\mathbb{Z}^n_\rho)^{\mathbb{Z}/p}$$

are isomorphisms.

In the sequel we will often use the following lemma.

Lemma 7.2. (i) *Let $\mathcal{H}^*_?$ be an equivariant cohomology theory in the sense of* [29], *Section* 1. *Then there is a long exact sequence*

$$\cdots \to \mathcal{H}^m(\underline{B}\Gamma) \xrightarrow{\mathrm{ind}_{\Gamma\to 1}} \mathcal{H}^m_\Gamma(\underline{E}\Gamma) \xrightarrow{\varphi^m} \bigoplus_{(P)\in\mathcal{P}} \bar{\mathcal{H}}^m_P(*)$$
$$\to \mathcal{H}^{m+1}(\underline{B}\Gamma) \xrightarrow{\mathrm{ind}_{\Gamma\to 1}} \mathcal{H}^{m+1}_\Gamma(\underline{E}\Gamma) \to \cdots,$$

where $\bar{\mathcal{H}}^m_P(*)$ is the cokernel of the induction map $\mathrm{ind}_{P\to 1}\colon \mathcal{H}^m(*) \to \mathcal{H}^m_P(*)$ *and the map φ^m is induced by the various inclusions $P \to \Gamma$.*
The map

$$\mathrm{ind}_{\Gamma\to 1}[1/p]\colon \mathcal{H}^m(\underline{B}\Gamma)[1/p] \to \mathcal{H}^m_\Gamma(\underline{E}\Gamma)[1/p]$$

is split injective.

(ii) *Let $\mathcal{H}^?_*$ be an equivariant homology theory in the sense of* [27], *Section* 1. *Then there is a long exact sequence*

$$\cdots \to \mathcal{H}^\Gamma_{m+1}(\underline{E}\Gamma) \xrightarrow{\mathrm{ind}_{\Gamma\to 1}} \mathcal{H}_{m+1}(\underline{B}\Gamma) \to \bigoplus_{(P)\in\mathcal{P}} \widetilde{\mathcal{H}}^P_m(*)$$
$$\xrightarrow{\varphi_m} \mathcal{H}^\Gamma_m(\underline{E}\Gamma) \xrightarrow{\mathrm{ind}_{\Gamma\to 1}} \mathcal{H}_m(\underline{B}\Gamma) \to \cdots,$$

where $\widetilde{\mathcal{H}}^P_m(*)$ is the kernel of the induction map $\mathrm{ind}_{P\to 1}\colon \mathcal{H}^P_m(*) \to \mathcal{H}_m(*)$ *and the map φ_m is induced by the various inclusions $P \to \Gamma$.*
The map

$$\mathrm{ind}_{\Gamma\to 1}[1/p]\colon \mathcal{H}^\Gamma_m(\underline{E}\Gamma)[1/p] \to \mathcal{H}_m(\underline{B}\Gamma)[1/p]$$

is split surjective.

Proof. (i): From the cellular Γ-pushout (1.12) we obtain a long exact sequence

$$\begin{aligned}\cdots \to \mathcal{H}^m_\Gamma(\underline{E}\Gamma) \to \mathcal{H}^m_\Gamma(E\Gamma) \oplus \bigoplus_{(P)\in\mathcal{P}} \mathcal{H}^m_\Gamma(\Gamma/P) \to \bigoplus_{(P)\in\mathcal{P}} \mathcal{H}^m_\Gamma(\Gamma\times_P EP)\\ \to \mathcal{H}^{m+1}_\Gamma(\underline{E}\Gamma) \to \mathcal{H}^{m+1}_\Gamma(E\Gamma) \oplus \bigoplus_{(P)\in\mathcal{P}} \mathcal{H}^{m+1}_\Gamma(\Gamma/P) \to \cdots.\end{aligned} \tag{7.3}$$

From the cellular pushout (1.13) we obtain the long exact sequence

$$\begin{aligned}\cdots \to \mathcal{H}^m(\underline{B}\Gamma) \to \mathcal{H}^m(B\Gamma) \oplus \bigoplus_{(P)\in\mathcal{P}} \mathcal{H}^m(*) \to \bigoplus_{(P)\in\mathcal{P}} \mathcal{H}^m(BP) \\ \to \mathcal{H}^{m+1}(\underline{B}\Gamma) \to \mathcal{H}^{m+1}(B\Gamma) \oplus \bigoplus_{(P)\in\mathcal{P}} \mathcal{H}^{m+1}(*) \to \cdots .\end{aligned} \tag{7.4}$$

Induction with the group homomorphism $\Gamma \to 1$ yields a map from the long exact sequence (7.4) to the long exact sequence (7.3). Recall that the induction homomorphism $\mathcal{H}^m(\Gamma\backslash X) \to \mathcal{H}^m_\Gamma(X)$ is an isomorphism if Γ acts freely on the proper Γ-CW-complex X. Therefore the maps

$$\bigoplus_{(P)\in\mathcal{P}} \mathcal{H}^m(BP) \xrightarrow{\cong} \bigoplus_{(P)\in\mathcal{P}} \mathcal{H}^m_\Gamma(\Gamma \times_P EP),$$
$$\mathcal{H}^m(B\Gamma) \xrightarrow{\cong} \mathcal{H}^m_\Gamma(E\Gamma)$$

are bijective. Hence one can splice the long exact sequences (7.3) and (7.4) together to obtain the desired long exact sequence, after noting the commutative diagram

$$\begin{tikzcd} \mathcal{H}^m_\Gamma(\Gamma/P) & \mathcal{H}^m(*) \arrow[l, "\mathrm{ind}_{\Gamma\to 1}"'] \\ \mathcal{H}^m_P(*) \arrow[u, "\mathrm{ind}_{P\to\Gamma}", "\cong"'] & \mathcal{H}^m(*). \arrow[l, "\mathrm{ind}_{P\to 1}"'] \arrow[u, "="'] \end{tikzcd}$$

We have the following commutative diagram, where the vertical arrow are given by induction with the group homomorphism $\Gamma \to 1$:

$$\begin{tikzcd} \mathcal{H}^m(\underline{B}\Gamma) \arrow[r, tail] \arrow[d] & \mathcal{H}^m(B\mathbb{Z}^n) \arrow[d, "\cong"] \\ \mathcal{H}^m_\Gamma(\underline{E}\Gamma) \arrow[r] & \mathcal{H}^m_\Gamma(\Gamma \times_{\mathbb{Z}^n} E\mathbb{Z}^n). \end{tikzcd}$$

The upper horizontal arrow is split injective after inverting p by Proposition A.4. The right vertical arrow is bijective since Γ acts freely on $\Gamma \times_{\mathbb{Z}^n} E\mathbb{Z}^n$. Hence $\mathcal{H}^m(\underline{B}\Gamma) \to \mathcal{H}^m_\Gamma(\underline{E}\Gamma)$ is injective after inverting p.

(ii) The proof is analogous to the one of assertion (i). This finishes the proof of Lemma 7.2. □

Proof of Theorem 7.1. Recall that $K^0_\Gamma(\Gamma/P) \cong R_{\mathbb{C}}(P)$ and $K^1_\Gamma(\Gamma/P) \cong 0$. Hence we obtain from Lemma 7.2 (i) the long exact sequence

$$0 \to K^0(\underline{B}\Gamma) \to K^0_\Gamma(\underline{E}\Gamma) \to \bigoplus_{(P)\in\mathcal{P}} \overline{R}_{\mathbb{C}}(P) \to K^1(\underline{B}\Gamma) \to K^1_\Gamma(\underline{E}\Gamma) \to 0, \tag{7.5}$$

where $\overline{R}_{\mathbb{C}}(P)$ is the cokernel of the homomorphism $R_{\mathbb{C}}(1) \to R_{\mathbb{C}}(P)$ given by restriction with $P \to 1$. Notice that the composite of the augmentation ideal

$\mathbb{I}_{\mathbb{C}}(P) \to R_{\mathbb{C}}(P)$ with the projection $R_{\mathbb{C}}(P) \to \bar{R}_{\mathbb{C}}(P)$ is an isomorphism of finitely generated free abelian groups

$$\mathbb{I}_{\mathbb{C}}(P) \xrightarrow{\cong} \bar{R}_{\mathbb{C}}(P) \tag{7.6}$$

and that $\mathbb{I}_{\mathbb{C}}(P)$ is isomorphic to $\mathbb{Z}^{p-1}$.

(iii): It was already shown in Theorem 3.1 (iii) that the map $K^1(B\Gamma) \xrightarrow{\cong} K^1(B\mathbb{Z}_p^n)^{\mathbb{Z}/p}$ is bijective and that $K^1(B\Gamma) \cong \mathbb{Z}^{\sum_{l\in\mathbb{Z}} r_{2l+1}}$. Hence it remains to prove that the composite

$$K^1_\Gamma(\underline{E}\Gamma) \to K^1_\Gamma(E\Gamma) \xrightarrow{\cong} K^1(B\Gamma)$$

is bijective. We obtain from (3.5) and (7.5) the following commutative diagram with exact rows

$$\begin{tikzcd}
\bigoplus_{(P)\in\mathcal{P}} \bar{R}_{\mathbb{C}}(P) \arrow[r] \arrow[d] & K^1(\underline{B}\Gamma) \arrow[r] \arrow[d, "\mathrm{id}"'] & K^1_\Gamma(\underline{E}\Gamma) \arrow[r] \arrow[d] & 0 \\
\bigoplus_{(P)\in\mathcal{P}} \widetilde{K}^0(BP) \arrow[r] & K^1(\underline{B}\Gamma) \arrow[r] & K^1(B\Gamma) \arrow[r] & 0.
\end{tikzcd}$$

By the five lemma it suffices to show that the map

$$\ker(K^1(\underline{B}\Gamma) \to K^1_\Gamma(\underline{E}\Gamma)) \to \ker(K^1(\underline{B}\Gamma) \to K^1(B\Gamma))$$

is surjective. We conclude from Theorem 3.1 (vi) that the kernel of $K^1(\underline{B}\Gamma) \to K^1(B\Gamma)$ is the finite abelian p-group T^1 appearing in Theorem 3.1 (v). Hence it remains to show for every integer $l > 0$ that the obvious composite

$$\bigoplus_{(P)\in\mathcal{P}} R_{\mathbb{C}}(P) \to \bigoplus_{(P)\in\mathcal{P}} K^0(BP) \to \Big(\bigoplus_{(P)\in\mathcal{P}} K^0(BP)\Big)/p^l \cdot \Big(\bigoplus_{(P)\in\mathcal{P}} K^0(BP)\Big)$$

is surjective. By the Atiyah–Segal Completion Theorem the map $R_{\mathbb{C}}(P) \to K^0(BP)$ can be identified with the map

$$\mathrm{id} \oplus i : \mathbb{Z} \oplus I(\mathbb{Z}/p) \to \mathbb{Z} \oplus \big(I(\mathbb{Z}/p) \otimes \widehat{\mathbb{Z}_p}\big)$$

Hence it suffices to show that the composite

$$\mathbb{Z} \to \widehat{\mathbb{Z}_p} \to \widehat{\mathbb{Z}_p}/p^l\widehat{\mathbb{Z}_p}$$

is surjective. This is true since the latter map can be identified with the canonical epimorphism $\mathbb{Z} \to \mathbb{Z}/p^l$.

(ii): This follows from Theorem 3.1 (vi), the long exact sequence (7.5), the isomorphism (7.6) and assertion (iii).

(i): We have shown that $K^0(\underline{B}\Gamma) \cong \mathbb{Z}^{\sum_{l\in\mathbb{Z}} r_{2l}}$ in Theorem 3.1 (iv). We have $\mathbb{I}(\mathbb{Z}/p) \cong \mathbb{Z}^{(p-1)/2}$. The order of $\mathcal{P}$ is p^k by Lemma 1.9 (iv). Hence we conclude from assertion (ii) that

$$K^0_\Gamma(\underline{E}\Gamma) \cong \mathbb{Z}^{(p-1)p^k + \sum_{l\in\mathbb{Z}} r_{2l}}.$$

The computation of $K^1_\Gamma(\underline{E}\Gamma)$ follows from Theorem 3.1 (iii) and assertion (iii). $\square$

Remark 7.7 (Geometric interpretation of T^1). The exact sequence appearing in Theorem 7.1 (ii) has the following interpretation in terms of equivariant vector bundles. Since Γ is a crystallographic group, Γ acts properly on $\mathbb{R}^n$ such that this action reduced to $\mathbb{Z}^n$ is the free standard action and $\mathbb{R}^n$ is a model for $\underline{E}\Gamma$. Hence the quotient of $\mathbb{Z}^n\backslash\mathbb{R}^n$ is the standard n-torus T^n together with a $\mathbb{Z}/p$-action. There is a bijection

$$\mathcal{P} \xrightarrow{\cong} (T^n)^{\mathbb{Z}/p}$$

coming from the fact that $(\mathbb{R}^n)^P$ consists of exactly one point for $(P) \in \mathcal{P}$. In particular $(T^n)^{\mathbb{Z}/p}$ consists of p^k points (see Lemma 1.9 (v).) Hence for any complex $\mathbb{Z}/p$-vector bundle ξ we obtain a collection of complex $\mathbb{Z}/p$-representations $\{\xi_x \mid x \in (T^n)^{\mathbb{Z}/p}\}$ satisfying $\dim_{\mathbb{C}}(\xi_x) = \dim_{\mathbb{C}}(\xi_y) = \dim(\xi)$ for $x, y \in (T^n)^{\mathbb{Z}/p}$. This yields a map

$$\beta\colon K^0_{\mathbb{Z}/p}(T^n) \to \bigoplus_{P \in (P)} I_{\mathbb{C}}(P)$$

sending the class of a $\mathbb{Z}/p$-vector bundle ξ to the collection $\{[\xi_x] - \dim(\xi)\cdot[\mathbb{C}] \mid x \in (T^n)^{\mathbb{Z}/p}\}$. Let

$$\alpha\colon K^0\big((\mathbb{Z}/p)\backslash T^n\big) \to K^0_{\mathbb{Z}/p}(T^n)$$

be the homomorphism coming from the pullback construction associated to the projection $T^n \to (\mathbb{Z}/p)\backslash T^n$. We obtain the exact sequence

$$0 \to K^0((\mathbb{Z}/p)\backslash T^n) \xrightarrow{\alpha} K^0_{\mathbb{Z}/p}(T^n) \xrightarrow{\beta} \bigoplus_{(P)\in\mathcal{P}} I_{\mathbb{C}}(P) \to T^1 \to 0,$$

which can be identified with exact sequence of Theorem 7.1 (ii).

Thus the group T^1 is related to (stable version of) the question when a collection of $\mathbb{Z}/p$-representations $\{V_x \mid x \in (T^n)^{\mathbb{Z}/p}\}$ with $\dim_{\mathbb{C}}(V_x) = \dim_{\mathbb{C}}(V_y)$ for $x, y \in (T^n)^{\mathbb{Z}/p}$ can be realized as the fibers of a $\mathbb{Z}/p$-vector bundle ξ over T^n at the points in $(T^n)^{\mathbb{Z}/p}$.

Moreover, a $\mathbb{Z}/p$-vector bundle over T^n is stably isomorphic to the pullback of a vector bundle over $(\mathbb{Z}/p)\backslash T^n$ if and only if for every $x \in (T^n)^{\mathbb{Z}/p}$ the $\mathbb{Z}/p$-representation ξ_x has trivial $\mathbb{Z}/p$-action.

8. Equivariant K-homology

In the sequel equivariant homology theory is to be understood in the sense of [27], Section 1. Equivariant topological complex K-homology $K^?_*$ is an example (see [14], [33], Section 6). The construction there yields the same for proper G-CW-complexes as the construction due to Kasparov [21]. It is two-periodic. For finite groups G the group $K^G_m(*)$ is $R_{\mathbb{C}}(G)$ for even m and trivial for odd m.

We obtain from [28], Theorem 0.7, using Lemma 1.9 an isomorphism

$$K_m(\underline{B}\Gamma)\big[\tfrac{1}{p}\big] \oplus \bigoplus_{(P)\in\mathcal{P}} K_m(*) \otimes I_{\mathbb{C}}(P)\big[\tfrac{1}{p}\big] \cong K^{\Gamma}_m(\underline{E}\Gamma)\big[\tfrac{1}{p}\big]$$

and hence from Theorem 4.1

$$K_0^\Gamma(\underline{E}\Gamma)\big[\tfrac{1}{p}\big] \cong (\mathbb{Z}[1/p])^{(p-1)p^k+\sum_l r_{2l}}, \quad K_1^\Gamma(\underline{E}\Gamma)\big[\tfrac{1}{p}\big] \cong (\mathbb{Z}[1/p])^{\sum_l r_{2l+1}}.$$

We want to get an integral computation.

Theorem 8.1 (Equivariant K-homology of $\underline{E}\Gamma$).

(i) *We have*

$$K_m^\Gamma(\underline{E}\Gamma) \cong \begin{cases} \mathbb{Z}^{(p-1)p^k+\sum_{l\in\mathbb{Z}} r_{2l}}, & m \text{ even},\\ \mathbb{Z}^{\sum_{l\in\mathbb{Z}} r_{2l+1}}, & m \text{ odd}. \end{cases}$$

(ii) *There is a natural isomorphism*

$$K_m^\Gamma(\underline{E}\Gamma) \xrightarrow{\cong} \operatorname{Hom}_{\mathbb{Z}}(K_\Gamma^m(\underline{E}\Gamma), \mathbb{Z}).$$

(iii) *The map $K_1^\Gamma(\underline{E}\Gamma) \to K_1(\underline{B}\Gamma)$ is an isomorphism. There is an exact sequence*

$$0 \to \bigoplus_{(P)\in\mathcal{P}} \widetilde{R}_{\mathbb{C}}(P) \to K_0^\Gamma(\underline{E}\Gamma) \to K_0(\underline{B}\Gamma) \to 0,$$

where $\widetilde{R}_{\mathbb{C}}(P)$ is the kernel of the map $R_{\mathbb{C}}(P) \to R_{\mathbb{C}}(1)$ which sends $[V]$ to $[\mathbb{C} \otimes_{\mathbb{C}P} V]$. It splits after inverting p.

Its proof needs some preparation.

Lemma 8.2. *Let G be a finite group. Then there is an isomorphism of $R_{\mathbb{C}}(G)$-modules*

$$R_{\mathbb{C}}(G) \xrightarrow{\cong} \operatorname{Hom}_{\mathbb{Z}}(R_{\mathbb{C}}(G), \mathbb{Z})$$

which sends $[V]$ to the homomorphism $R_{\mathbb{C}}(G) \to \mathbb{Z}$, $[W] \mapsto \dim_{\mathbb{C}}(\operatorname{Hom}_{\mathbb{C}G}(V, W))$. Here $R_{\mathbb{C}}(G)$ acts on $\operatorname{Hom}_{\mathbb{Z}}(R_{\mathbb{C}}(G); \mathbb{Z})$ by $([V]\cdot\phi)([W]) := \phi([V^*]\cdot[W])$.

In particular we get for any $R_{\mathbb{C}}(G)$-module M a natural isomorphism of $R_{\mathbb{C}}(G)$-modules

$$\operatorname{Ext}^i_{R_{\mathbb{C}}(G)}(M, R_{\mathbb{C}}(G)) \xrightarrow{\cong} \operatorname{Ext}^1_{\mathbb{Z}}(M, \mathbb{Z}) \quad \textit{for } i \geq 0.$$

Proof. See [35], 2.5 and 2.10. □

Theorem 8.3 (Universal coefficient theorem for equivariant K-theory). *Let G be a finite group and X be a finite G-CW-complex. Then there are for $n \in \mathbb{Z}$ natural exact sequences of $R_{\mathbb{C}}(G)$-modules*

$$0 \to \operatorname{Ext}_{R_{\mathbb{C}}(G)}(K^G_{n-1}(X), R_{\mathbb{C}}(G)) \to K^n_G(X) \to \operatorname{Hom}_{R_{\mathbb{C}}(G)}(K^G_n(X), R_{\mathbb{C}}(G)) \to 0$$

and

$$0 \to \operatorname{Ext}_{R_{\mathbb{C}}(G)}(K^{n+1}_G(X), R_{\mathbb{C}}(G)) \to K^G_n(X) \to \operatorname{Hom}_{R_{\mathbb{C}}(G)}(K^n_G(X), R_{\mathbb{C}}(G)) \to 0.$$

Proof. The first sequence is proved in [10]. The second sequence follows from the first by equivariant S-duality (see [35], [45]). □

Proof of Theorem 8.1. (ii): Since $\mathbb{Z}^n$ acts freely on $\underline{E}\Gamma$, induction with $\Gamma \to \mathbb{Z}/p$ induces isomorphisms

$$K_n^\Gamma(\underline{E}\Gamma) \xrightarrow{\cong} K_n^{\mathbb{Z}/p}(\mathbb{Z}^n\backslash\underline{E}\Gamma), \quad K^n_{\mathbb{Z}/p}(\mathbb{Z}^n\backslash\underline{E}\Gamma) \xrightarrow{\cong} K^n_\Gamma(\underline{E}\Gamma).$$

Since $\mathbb{Z}^n\backslash\underline{E}\Gamma$ is a finite $\mathbb{Z}/p$-CW-complex, we obtain from Lemma 8.2 and Theorem 8.3 the exact sequence of $R_\mathbb{C}(\mathbb{Z}/p)$-modules

$$\begin{aligned} 0 \to \operatorname{Ext}^1_\mathbb{Z}(K^{n+1}_{\mathbb{Z}/p}(\mathbb{Z}^n\backslash\underline{E}\Gamma), \mathbb{Z}) \to K_n^{\mathbb{Z}/p}(\mathbb{Z}^n\backslash\underline{E}\Gamma) \\ \to \operatorname{Hom}_\mathbb{Z}(K^n_{\mathbb{Z}/p}(\mathbb{Z}^n\backslash\underline{E}\Gamma), \mathbb{Z}) \to 0. \end{aligned}$$

(Another construction of the sequence above is given in [20].) Hence we get an exact sequence of $R_\mathbb{C}(\mathbb{Z}/p)$-modules (see also [35], Proposition 2.8)

$$0 \to \operatorname{Ext}^1_\mathbb{Z}\big(K^{n+1}_\Gamma(\underline{E}\Gamma), \mathbb{Z}\big) \to K_n^\Gamma(\underline{E}\Gamma) \to \operatorname{Hom}_\mathbb{Z}\big(K^n_\Gamma(\underline{E}\Gamma), \mathbb{Z}\big) \to 0.$$

Since $K^{n+1}_\Gamma(\underline{E}\Gamma)$ is a finitely generated free abelian group for all $n \in \mathbb{Z}$ by Theorem 7.1, we obtain for $n \in \mathbb{Z}$ an isomorphism of $R_\mathbb{C}(\mathbb{Z}/p)$-modules

$$K_n^\Gamma(\underline{E}\Gamma) \xrightarrow{\cong} \operatorname{Hom}_\mathbb{Z}\big(K^n_\Gamma(\underline{E}\Gamma), \mathbb{Z}\big).$$

(i): Apply Theorem 7.1 (i) and assertion (ii) to get the concrete identification of $K_n^\Gamma(\underline{E}\Gamma)$.

(iii): From Lemma 7.2 (ii) we obtain a long exact sequence

$$0 \to K_1^\Gamma(\underline{E}\Gamma) \to K_1(\underline{B}\Gamma) \to \bigoplus_{(P)\in\mathcal{P}} \widetilde{K}_0^{\mathbb{Z}/p}(*) \to K_0^\Gamma(\underline{E}\Gamma) \to K_0(\underline{B}\Gamma) \to 0.$$

where $\widetilde{K}_0^{\mathbb{Z}/p}(*)$ is the kernel of the map $K_0^{\mathbb{Z}/p}(*) \to K_0(*)$ coming from induction with $\mathbb{Z}/p \to 1$. Since $K_1^\Gamma(\underline{E}\Gamma)$ and $K_1(\underline{B}\Gamma)$ are finitely generated free abelian groups of the same rank by assertion (i) and Theorem 4.1 (v) and $\bigoplus_{(P)\in\mathcal{P}} \widetilde{K}_0^{\mathbb{Z}/p}(*)$ is torsion-free, the map $K_1^\Gamma(\underline{E}\Gamma) \to K_1(\underline{B}\Gamma)$ is bijective and we get a short exact sequence

$$0 \to \bigoplus_{(P)\in\mathcal{P}} \widetilde{K}_0^{\mathbb{Z}/p}(*) \to K_0^\Gamma(\underline{E}\Gamma) \to K_0(\underline{B}\Gamma) \to 0.$$ □

9. Equivariant KO-cohomology

Recall that equivariant topological real KO-theory $\mathrm{KO}^*_?$ is an equivariant cohomology theory in the sense of [29], Section 1. It is 8-periodic. Recall also that equivariant topological real K-homology $\mathrm{KO}^?_*$ is an equivariant homology theory in the sense of [27], Section 1. It is 8-periodic.

We first give some information about $\mathrm{KO}_m^G(*)$ and $\mathrm{KO}_G^m(*)$ for finite G. We have $\mathrm{KO}_m^G(*) = \mathrm{KO}_G^{-m}(*)$ If G is a finite group, then we get

$$\mathrm{KO}_G^{-m}(*) \cong \mathrm{KO}_m^G(*) \cong K_m(\mathbb{R}G)$$

for $m \in \mathbb{Z}$, where $K_m(\mathbb{R}G)$ is the topological K-theory of the real group C*-algebra $\mathbb{R}G$. Let $\{V_i \mid i = 0, 1, 2, \ldots, r\}$ be a complete set of representatives for the $\mathbb{R}G$-isomorphism classes of irreducible real G-representations. By Schur's Lemma the endomorphism ring $D_i = \mathrm{End}_{\mathbb{R}G}(V_i)$ is a skew-field over $\mathbb{R}$ and hence isomorphic to $\mathbb{R}$, $\mathbb{C}$ or $\mathbb{H}$. There are positive integers k_i for $i \in \{0, 1, \ldots, r\}$ such that we obtain a splitting

$$\mathbb{R}G \cong \prod_{i=0}^{r} M_{k_i}(D_i).$$

Since topological K-theory is compatible with products, by Morita equivalence we obtain for $m \in \mathbb{Z}$ an isomorphism

$$K_m(\mathbb{R}G) \cong \prod_{i=1}^{r} K_m(D_i).$$

The real K-theory of the building blocks are given by $\mathrm{KO}_m(\mathbb{R}) = \mathrm{KO}_m(*)$, $\mathrm{KO}_m(\mathbb{C}) = K_m(*)$, and $\mathrm{KO}_m(\mathbb{H}) = \mathrm{KO}_{m+4}(*)$. If $G = \mathbb{Z}/p$ for an odd prime p and we take for V_0 the trivial real $\mathbb{Z}/p$-representation $\mathbb{R}$, then $r = (p-1)/2$, $D_0 = \mathbb{R}$ and $D_i = \mathbb{C}$ for $i \in \{1, 2, \ldots (p-1)/2\}$. This implies that

$$\mathrm{KO}_m^{\mathbb{Z}/p}(*) \cong \mathrm{KO}_m(*) \oplus K_m(*)^{(p-1)/2}, \tag{9.1}$$

$$\mathrm{KO}^m_{\mathbb{Z}/p}(*) \cong \mathrm{KO}_{-m}(*) \oplus K_{-m}(*)^{(p-1)/2}. \tag{9.2}$$

Let $\widetilde{\mathrm{KO}}_m^{\mathbb{Z}/p}(*)$ be the kernel of the map $\mathrm{KO}_m^{\mathbb{Z}/p}(*) \to \mathrm{KO}_m(*)$ given by induction with $\mathbb{Z}/p \to 1$. This corresponds under the isomorphism (9.1) to the obvious projection of $\mathrm{KO}_m(*) \oplus K_m(*)^{(p-1)/2}$ onto $\mathrm{KO}_m(*)$. Let $\overline{\mathrm{KO}}^m_{\mathbb{Z}/p}(*)$ be the cokernel of the map $\mathrm{KO}^m(*) \to \mathrm{KO}^m_{\mathbb{Z}/p}(*)$ given by induction with $\mathbb{Z}/p \to 1$. This corresponds under the isomorphism (9.2) to the obvious inclusion of $\mathrm{KO}_{-m}(*)$ into $\mathrm{KO}_{-m}(*) \oplus \mathrm{KO}_{-m}(*)^{(p-1)/2}$. Hence we get

$$\widetilde{\mathrm{KO}}_m^{\mathbb{Z}/p}(*) \cong K_m(*)^{(p-1)/2}, \quad \overline{\mathrm{KO}}^m_{\mathbb{Z}/p}(*) \cong K_{-m}(*)^{(p-1)/2}.$$

This implies that

$$\widetilde{\mathrm{KO}}_m^{\mathbb{Z}/p}(*) \cong \overline{\mathrm{KO}}^m_{\mathbb{Z}/p}(*) \cong \begin{cases} \mathbb{Z}^{(p-1)/2}, & m \text{ even}, \\ 0, & m \text{ odd}. \end{cases} \tag{9.3}$$

We conclude from [29], Theorem 5.2, using Lemma 1.9 (i) for $m \in \mathbb{Z}$ that

$$\mathrm{KO}^{2m}_{\Gamma}(\underline{E}\Gamma) \otimes \mathbb{Q} \cong \mathbb{Q}^{p^k(p-1)/2 + \sum_{l \in \mathbb{Z}} r_{2m+4l}},$$

$$\mathrm{KO}^{2m+1}_{\Gamma}(\underline{E}\Gamma) \otimes \mathbb{Q} \cong \mathbb{Q}^{\sum_{l \in \mathbb{Z}} r_{2m+1+4l}}.$$

Again we seek an integral computation.

Theorem 9.4 (Equivariant KO-cohomology). *Let p be an odd prime and let m be any integer.*

(i)
$$\mathrm{KO}^m_\Gamma(\underline{E}\Gamma) \cong \begin{cases} \mathbb{Z}^{p^k(p-1)/2} \oplus \bigoplus_{i\in\mathbb{Z}} \mathrm{KO}^{m-i}(*)^{r_i}, & m \text{ even}, \\ \bigoplus_{i\in\mathbb{Z}} \mathrm{KO}^{m-i}(*)^{r_i}, & m \text{ odd}. \end{cases}$$

(ii) *If TO^{2m+1} is the finite abelian p-group appearing in Theorem 5.1 (v), then there is an exact sequence*
$$0 \to \mathrm{KO}^{2m}(\underline{B}\Gamma) \to \mathrm{KO}^{2m}_\Gamma(\underline{E}\Gamma) \to \bigoplus_{(P)\in\mathcal{P}} \overline{\mathrm{KO}}^{2m}_{\mathbb{Z}/p}(*) \to \mathrm{TO}^{2m+1} \to 0.$$

(iii) *The canonical maps*
$$\mathrm{KO}^{2m+1}_\Gamma(\underline{E}\Gamma) \xrightarrow{\cong} \mathrm{KO}^{2m+1}(B\Gamma),$$
$$\mathrm{KO}^{2m+1}(B\Gamma) \xrightarrow{\cong} \mathrm{KO}^{2m+1}(B\mathbb{Z}^n_\rho)^{\mathbb{Z}/p}$$

are isomorphisms.

Proof. (iii): Lemma 7.2 (i) together with (9.3) implies that there is a long exact sequence
$$\begin{aligned} 0 \to \mathrm{KO}^{2m}(\underline{B}\Gamma) \to \mathrm{KO}^{2m}_\Gamma(\underline{E}\Gamma) \to \bigoplus_{(P)\in\mathcal{P}} \overline{\mathrm{KO}}^{2m}_{\mathbb{Z}/p}(*) \\ \to \mathrm{KO}^{2m+1}(\underline{B}\Gamma) \to \mathrm{KO}^{2m+1}_\Gamma(\underline{E}\Gamma) \to 0, \end{aligned} \tag{9.5}$$

and that the kernel of the epimorphism $\mathrm{KO}^{2m+1}(\underline{B}\Gamma) \to \mathrm{KO}^{2m+1}_\Gamma(\underline{E}\Gamma)$ is a finite abelian p-group.

For $m \in \mathbb{Z}$ the composite
$$\mathrm{KO}^{2m+1}(\underline{B}\Gamma) \xrightarrow{\alpha} \mathrm{KO}^{2m+1}_\Gamma(\underline{E}\Gamma) \xrightarrow{\beta} \mathrm{KO}^{2m+1}(B\Gamma)$$

is surjective and has a finite abelian p-group as kernel by Theorem 5.1 (vi). Hence the map β is surjective for all $m \in \mathbb{Z}$. Since α is surjective by (9.5), the map $\ker(\beta\circ\alpha) \to \ker(\beta)$ is surjective and hence the kernel of β is a finite abelian p-group.

The following diagram commutes:

$$\begin{tikzcd} \mathrm{KO}^{2m+1}_\Gamma(\underline{E}\Gamma) \arrow[r] \arrow[d] \arrow[rr, bend left, "2\cdot\mathrm{id}"] & K^{2m+1}_\Gamma(\underline{E}\Gamma) \arrow[r] \arrow[d, "\cong"] & \mathrm{KO}^{2m+1}_\Gamma(\underline{E}\Gamma) \arrow[d] \\ \mathrm{KO}^{2m+1}(B\Gamma) \arrow[r] \arrow[rr, bend right, "2\cdot\mathrm{id}"'] & K^{2m+1}(B\Gamma) \arrow[r] & \mathrm{KO}^{2m+1}(B\Gamma). \end{tikzcd}$$

Here the left horizontal maps are given by induction with $\mathbb{R} \to \mathbb{C}$, the right horizontal maps by restriction with $\mathbb{R} \to \mathbb{C}$ and the middle vertical arrow is an isomorphism by Theorem 7.1. Hence the kernel of the epimorphism $\mathrm{KO}^{2m+1}_\Gamma(\underline{E}\Gamma) \to \mathrm{KO}^{2m+1}(B\Gamma)$ is an abelian group of exponent 2. We have already shown that its kernel is a finite abelian p-group. Since p is odd, we conclude that

$$\mathrm{KO}^{2m+1}_\Gamma(\underline{E}\Gamma) \xrightarrow{\cong} \mathrm{KO}^{2m+1}(B\Gamma)$$

is an isomorphism.

The bijectivity of $\mathrm{KO}^{2m+1}(B\Gamma) \xrightarrow{\cong} \mathrm{KO}^{2m+1}(B\mathbb{Z}^n_\rho)^{\mathbb{Z}/p}$ has already been proved in Theorem 5.1 (iii).

(i): Since kernel of the epimorphism $\mathrm{KO}^{2m+1}(\underline{B}\Gamma) \to \mathrm{KO}^{2m+1}_\Gamma(\underline{E}\Gamma)$ is a finite abelian p-group and $\bigoplus_{(P)\in\mathcal{P}} \overline{\mathrm{KO}}^{2m}_{\mathbb{Z}/p}(*)$ is isomorphic to $\mathbb{Z}^{p^k(p-1)/2}$ by Lemma 1.9 (iv) and by (9.3), we conclude from the exact sequence (9.5) that

$$\mathrm{KO}^{2m}_\Gamma(\underline{E}\Gamma) \cong \mathrm{KO}^{2m}(\underline{B}\Gamma) \oplus \mathbb{Z}^{p^k(p-1)/2}.$$

Since we have already computed $\mathrm{KO}^{2m}(\underline{B}\Gamma)$ and $\mathrm{KO}^{2m+1}(B\Gamma)$ in Theorem 5.1, assertion (i) follows using assertion (iii).

(ii): The kernel of the epimorphism $\mathrm{KO}^{2m+1}(\underline{B}\Gamma) \to \mathrm{KO}^{2m+1}(B\Gamma)$ is isomorphic to TO^{2m+1} by Theorem 5.1 (v) and (vi). Since $\mathrm{KO}^{2m+1}_\Gamma(\underline{E}\Gamma) \xrightarrow{\cong} \mathrm{KO}^{2m+1}(B\Gamma)$ is bijective by assertion (iii), the claim follows from the long exact sequence (9.5). $\square$

10. Equivariant KO-homology

We obtain from [28], Theorem 0.7, using Lemma 1.9 isomorphisms

$$\mathrm{KO}^\Gamma_{2m}(\underline{E}\Gamma) \otimes \mathbb{Q} \cong \mathbb{Q}^{p^k(p-1)/2+\sum_{l\in\mathbb{Z}} r_{4l+2m}},$$
$$\mathrm{KO}^\Gamma_{2m+1}(\underline{E}\Gamma) \otimes \mathbb{Q} \cong \mathbb{Q}^{\sum_{l\in\mathbb{Z}} r_{4l+2m+1}}.$$

We want to get an integral computation.

Theorem 10.1 (Equivariant KO-homology). *Let p be an odd prime and m be any integer.*

(i)
$$\mathrm{KO}^\Gamma_m(\underline{E}\Gamma) \cong \begin{cases} \mathbb{Z}^{p^k(p-1)/2} \oplus (\bigoplus_{i=0}^n \mathrm{KO}_{m-i}(*)^{r_i}), & m \text{ even}, \\ \bigoplus_{i=0}^n \mathrm{KO}_{m-i}(*)^{r_i}, & m \text{ odd}. \end{cases}$$

(ii) *For $m \in \mathbb{Z}$ the map $\mathrm{KO}^\Gamma_{2m+1}(\underline{E}\Gamma) \xrightarrow{\cong} \mathrm{KO}_{2m+1}(\underline{B}\Gamma)$ is an isomorphism.*

(iii) *There is a short exact sequence*

$$0 \to \bigoplus_{(P)\in\mathcal{P}} \widetilde{\mathrm{KO}}_{2m}^{\mathbb{Z}/p}(*) \to \mathrm{KO}_{2m}^{\Gamma}(\underline{E}\Gamma) \to \mathrm{KO}_{2m}(\underline{B}\Gamma) \to 0,$$

where $\widetilde{\mathrm{KO}}_{2m}^{\mathbb{Z}/p}(*)$ *is the kernel of the map* $KO_{2m}^{\mathbb{Z}/p}(*) \to \mathrm{KO}_{2m}(*)$ *coming from induction with* $\mathbb{Z}/p \to 1$. *It splits after inverting* p.

Proof. Lemma 7.2 (ii) implies that there is an long exact sequence

$$\begin{aligned} 0 \to \mathrm{KO}_{2m+1}^{\Gamma}(\underline{E}\Gamma) \to \mathrm{KO}_{2m+1}(\underline{B}\Gamma) \to \bigoplus_{(P)\in\mathcal{P}} \widetilde{\mathrm{KO}}_{2m}^{\mathbb{Z}/p}(*) \\ \to \mathrm{KO}_{2m}^{\Gamma}(\underline{E}\Gamma) \to \mathrm{KO}_{2m}(\underline{B}\Gamma) \to 0. \end{aligned} \tag{10.2}$$

and that the map

$$\mathrm{KO}_i^{\Gamma}(\underline{E}\Gamma)[1/p] \to \mathrm{KO}_i(\underline{B}\Gamma)[1/p]$$

is split surjective for $i \in \mathbb{Z}$. The cokernel of $\mathrm{KO}_{2m+1}^{\Gamma}(\underline{E}\Gamma) \to \mathrm{KO}_{2m+1}(\underline{B}\Gamma)$ is a finite abelian p-group. Since $\widetilde{\mathrm{KO}}_{2m}^{\mathbb{Z}/p}(*)$ is a finitely generated free abelian group by (9.3), the long exact sequence (10.2) reduces to an isomorphism

$$\mathrm{KO}_{2m+1}^{\Gamma}(\underline{E}\Gamma) \xrightarrow{\cong} \mathrm{KO}_{2m+1}(\underline{B}\Gamma)$$

and a short exact sequence

$$0 \to \bigoplus_{(P)\in\mathcal{P}} \widetilde{\mathrm{KO}}_{2m}^{\mathbb{Z}/p}(*) \to \mathrm{KO}_{2m}^{\Gamma}(\underline{E}\Gamma) \to \mathrm{KO}_{2m}(\underline{B}\Gamma) \to 0, \tag{10.3}$$

which splits after inverting p. We have proven assertions (ii) and (iii).

Since the composite

$$\mathrm{KO}_i^{\Gamma}(\underline{E}\Gamma) \to K_i^{\Gamma}(\underline{E}\Gamma) \to \mathrm{KO}_i^{\Gamma}(\underline{E}\Gamma)$$

is multiplication with 2 and $K_i^{\Gamma}(\underline{E}\Gamma)$ is a finitely generated free abelian group by Theorem 8.1, the torsion subgroup of the finitely generated abelian group $\mathrm{KO}_i^{\Gamma}(\underline{E}\Gamma)$ is annihilated by 2 for $i \in \mathbb{Z}$. Since, by Theorem 6.1 (iv),

$$\bigoplus_{(P)\in\mathcal{P}} \widetilde{\mathrm{KO}}_{2m}^{\mathbb{Z}/p}(*) \cong \mathbb{Z}^{p^k(p-1)/2},$$

$$\mathrm{KO}_{2m}(\underline{B}\Gamma) \cong \Big(\bigoplus_{i=0}^{n} \mathrm{KO}_{2m-i}(*)^{r_i}\Big) \oplus \mathrm{TO}^{2m+5}$$

for a finite abelian p-group TO^{2m+5} and the torsion in $\bigoplus_{i=0}^{n} \mathrm{KO}_{m-i}(*)^{r_i}$ is annihilated by multiplication with 2, we get from (10.3) an isomorphism of abelian groups

$$\mathrm{KO}_{2m}^{\Gamma}(\underline{E}\Gamma) \cong \mathbb{Z}^{p^k(p-1)/2} \oplus \Big(\bigoplus_{i=0}^{n} \mathrm{KO}_{2m-i}(*)^{r_i}\Big).$$

This is the even case of assertion (i). The odd case of assertion (i) follows from assertion (ii) and Theorem 6.1 (v). □

11. Topological K-theory of the group C^*-algebra

In this section we compute the topological K-theory $K_n(C_r^*(\Gamma))$ of the complex reduced group C*-algebra $C_r^*(\Gamma)$ and the topological K-theory $\mathrm{KO}_n(C_r^*(\Gamma;\mathbb{R})) := K_n(C_r^*(\Gamma;\mathbb{R}))$ of the real reduced group C*-algebra $C_r^*(\Gamma;\mathbb{R})$.

The Baum–Connes Conjecture (see [9], Conjecture 3.15 on p. 254) predicts for a group G that the complex and the real assembly maps

$$K_n^G(\underline{E}G) \xrightarrow{\cong} K_n(C_r^*(G)), \tag{11.1}$$

$$\mathrm{KO}_n^G(\underline{E}G) \xrightarrow{\cong} \mathrm{KO}_n(C_r^*(G;\mathbb{R})) \tag{11.2}$$

are bijective for $n \in \mathbb{Z}$. It has been proved for $G = \Gamma$ (and many more groups) in [19].

11.1. The complex case.

We begin with the complex case.

Proof of Theorem 0.3. Because of the isomorphism (11.1) all claims follow from Lemma 1.9 (i), Lemma 1.22 (ii) and Theorem 8.1 except the statement that

$$K_1(C_r^*(\Gamma)) \xrightarrow{\cong} K_1(C_r^*(\mathbb{Z}^n_\rho))^{\mathbb{Z}/p}$$

is bijective. Induction with $\iota\colon \mathbb{Z}^n \to \Gamma$ yields a homomorphism

$$K_1(C_r^*(\mathbb{Z}^n)) \to K_1(C_r^*(\Gamma))$$

and restriction with ι yields a homomorphism

$$K_1(C_r^*(\Gamma)) \to K_1(C_r^*(\mathbb{Z}^n)).$$

Since an inner automorphism of Γ induces the identity on $K_1(C_r^*(\Gamma))$, these homomorphisms induce homomorphisms

$$\iota_*\colon K_1(C_r^*(\mathbb{Z}^n_\rho)_{\mathbb{Z}/p} \to K_1(C_r^*(\Gamma)), \quad \iota^*\colon K_1(C_r^*(\Gamma)) \to K_1(C_r^*(\mathbb{Z}^n_\rho))^{\mathbb{Z}/p}.$$

By the double coset formula the composite $\iota^* \circ \iota_*$ is the norm map

$$N\colon K_1(C_r^*(\mathbb{Z}^n_\rho))_{\mathbb{Z}/p} \to K_1(C_r^*(\mathbb{Z}^n_\rho))^{\mathbb{Z}/p}.$$

The cokernel of the norm map is $\widehat{H}^0(\mathbb{Z}/p; K_1(C_r^*(\mathbb{Z}^n_\rho))$. Note that

$$\begin{aligned}
\widehat{H}^0(\mathbb{Z}/p; K_1(C_r^*(\mathbb{Z}^n_\rho)) &\cong \widehat{H}^0(\mathbb{Z}/p; K_1(B\mathbb{Z}^n_\rho)) && \text{(the BC Conjecture for } \mathbb{Z}^n)\\
&\cong \widehat{H}^0(\mathbb{Z}/p; K^1(B\mathbb{Z}^n_\rho)^*) && \text{(the UCT for K-theory 4.2)}\\
&\cong \widehat{H}^{-1}(\mathbb{Z}/p; K^1(B\mathbb{Z}^n_\rho)) && \text{(Lemma A.1 proven below)}\\
&= 0 && \text{(Lemma 3.3 (ii)).}
\end{aligned}$$

This implies that the norm map N and hence $\iota^*\colon K_1(C_r^*(\Gamma)) \xrightarrow{\cong} K_1(C_r^*(\mathbb{Z}^n_\rho))^{\mathbb{Z}/p}$ are surjective. Since source and target of ι^* are finitely generated free abelian groups of the same rank by assertion (i) and Lemma 3.3 (i), ι^* is an isomorphism. □

11.2. The real case. Next we treat the real case.

Proof of Theorem 0.6. Because of the isomorphisms (9.3) and (11.2) all claims follow from Theorem 10.1 except the claim that

$$\mathrm{KO}_{2m+1}(C_r^*(\Gamma;\mathbb{R})) \xrightarrow{\cong} \mathrm{KO}_{2m+1}(C_r^*(\mathbb{Z}^n_\rho;\mathbb{R}))^{\mathbb{Z}/p}$$

is bijective. As we have natural transformations of cohomology theories $i^*\colon \mathrm{KO}_* \to K_*$ and $r^*\colon K_* \to \mathrm{KO}_*$ with $r^* \circ i^* = 2 \cdot \mathrm{id}$, Theorem 0.3 (iii) implies that the map is bijective after inverting 2. Since p is odd, it remains to show that it is bijective after inverting p. Because of the bijectivity of $\mathrm{KO}_{2m+1}(C_r^*(\Gamma;\mathbb{R})) \xrightarrow{\cong} \mathrm{KO}_{2m+1}(\underline{B}\Gamma)$, the fact that $\mathrm{KO}_{2m+1}(B\mathbb{Z}^n_\rho)_{\mathbb{Z}/p} \to \mathrm{KO}_{2m+1}(\underline{B}\Gamma)$ is bijective after inverting p (use Proposition A.4), the fact that norm map is always bijective after inverting p, and the isomorphism (11.2) for $\mathbb{Z}^n$, the claim holds. □

12. The group Γ satisfies the (unstable) Gromov–Lawson–Rosenberg Conjecture

In this section we give the proof of Theorem 0.7, after first providing some background.

12.1. The Gromov–Lawson–Rosenberg Conjecture. For a closed, spin manifold M of dimension m with fundamental group G, one can define an invariant

$$\alpha(M) \in \mathrm{KO}_m(C_r^*(G);\mathbb{R}),$$

which vanishes if M admits a metric of positive scalar curvature (see [37]). The (*unstable*) *Gromov–Lawson–Rosenberg Conjecture* for a group G states that if $\alpha(M) = 0$ and $\dim M \geq 5$, then M admits a metric of positive scalar curvature. The (unstable) Gromov–Lawson–Rosenberg Conjecture is known to be valid for some fundamental groups, for example, the trivial group (see [41]), for finite groups with periodic cohomology (see [11] and [23]), some torsion-free infinite groups, for example, when G is a fundamental group of a complete Riemannian manifold of non-positive sectional curvature (see [37]), and some infinite groups with torsion, for example, cocompact Fuchsian groups (see [15]), but not in general – there is a counterexample when $G = \mathbb{Z}^4 \times \mathbb{Z}/3$ due to Schick [39].

There is a weaker version of the conjecture which may be valid for all groups. Suppose that B^8 is a "Bott manifold", that is, a simply-connected spin 8-manifold with $\widehat{A}$-genus equal to one. We say that a manifold M *stably admits a metric of positive scalar curvature* if $M \times (B^8)^j$ admits a metric of positive scalar curvature for some $j \geq 0$. The *stable Gromov–Lawson–Rosenberg Conjecture* formulated by Rosenberg–Stolz [38] states that, for a closed spin manifold M with fundamental group G, M stably admits a metric of positive scalar curvature if and only if $\alpha(M) = 0$. Since the Baum–Connes Conjecture implies the stable Gromov–Lawson–Rosenberg Conjecture (see [42], Theorem 3.10, for an outline of the proof) and Γ satisfies the Baum–Connes Conjecture, we know already that Γ satisfies the stable Gromov–Lawson–Rosenberg Conjecture.

There are two definitions of the invariant α, one topological and one analytic. Let $\mathbf{KO}$ be the periodic spectrum underlying real K-theory, and let $\boldsymbol{p}\colon \mathbf{ko} \to \mathbf{KO}$ be the 0-connective cover, that is, it induces an isomorphism on π_i for $i \geq 0$ and $\pi_i(\mathbf{ko}) = 0$ for i negative. Then the topological definition of $\alpha(M)$ is the image of the class $[f_M\colon M \to BG]$ where f_M induces the identity on the fundamental group under the composite

$$\Omega_m^{\mathrm{Spin}}(BG) \xrightarrow{D} \mathrm{ko}_m(BG) \xrightarrow{p_{BG}} \mathrm{KO}_m(BG) \xrightarrow{A} \mathrm{KO}_m(C_r^*(G)),$$

where D is the ko-orientation of spin bordism, p_{BG} is the canonical map from connective to the periodic K-theory, and A is the assembly map. The analytic definition of $\alpha(M)$ is the index of the Dirac operator. These two definitions agree (see [37]). Furthermore if M has positive scalar curvature, then the Bochner–Lichnerowicz–Weitzenböck formula shows that the index is zero so that $\alpha(M) = 0$.

Finally, we mention one more result in our quick review, and that is the generalization of the Gromov–Lawson surgery theorem of due to Jung and Stolz [38], 3.7.

Proposition 12.1. *Let M be a connected closed spin manifold with fundamental group G and dimension $m \geq 5$. Let $[f\colon N \to BG] \in \Omega_m^{\mathrm{Spin}}(BG)$. (Note that N need not have fundamental group G.) If $D[f_M\colon M \to BG] = D[f\colon N \to BG] \in \mathrm{ko}_m(BG)$ and N admits a metric of positive scalar curvature, then so does M.*

12.2. The proof of Theorem 0.7. The proof of Theorem 0.7 needs some preparation.

Lemma 12.2. *Let p be an odd prime. Then the map*

$$\widetilde{D}\colon \widetilde{\Omega}_m^{\mathrm{Spin}}(B\mathbb{Z}/p) \to \widetilde{\mathrm{ko}}_m(B\mathbb{Z}/p)$$

is surjective for all $m \geq 0$.

Proof. If M is a $\mathbb{Z}[\mathbb{Z}/p]$-module, then $H_i(\mathbb{Z}/p; M)[1/p] = 0$ for $i \geq 1$ and hence the canonical maps

$$H_i(B\mathbb{Z}/p; M) \xrightarrow{\cong} H_i(B\mathbb{Z}/p; M)_{(p)} \xrightarrow{\cong} H_i(B\mathbb{Z}/p; M_{(p)})$$

are bijective for $i \geq 1$. We conclude from the Atiyah–Hirzebruch spectral sequences that the vertical maps in the commutative diagram

$$\begin{tikzcd}
\widetilde{\Omega}_m^{\mathrm{Spin}}(B\mathbb{Z}/p) \arrow[r, "\widetilde{D}"] \arrow[d, "\cong"] & \widetilde{\mathrm{ko}}_m(B\mathbb{Z}/p) \arrow[d, "\cong"] \\
\widetilde{\Omega}_m^{\mathrm{Spin}}(B\mathbb{Z}/p)_{(p)} \arrow[r, "\widetilde{D}_{(p)}"] & \widetilde{\mathrm{ko}}_m(B\mathbb{Z}/p)_{(p)}
\end{tikzcd}$$

are bijective for $m \geq 0$. Hence it suffices to prove the surjectivity of the lower horizontal map. Since p is odd, $\Omega_j^{\mathrm{Spin}}(*)_{(p)}$ is zero for $j \not\equiv 0 \bmod 4$ and $\Omega_j^{\mathrm{Spin}}(*)_{(p)}$ is a finitely generated free $\mathbb{Z}_{(p)}$-module for $j \equiv 0 \bmod 4$ (see [7]). The same is true for $\mathrm{ko}_j(*)_{(p)}$ by Bott periodicity. Hence there are no differentials in Atiyah–Hirzebruch spectral sequences converging to $\widetilde{\Omega}_{i+j}^{\mathrm{Spin}}(B\mathbb{Z}/p)_{(p)}$ and $\widetilde{\mathrm{ko}}_{i+j}(B\mathbb{Z}/p)_{(p)}$ and we get for the E^∞-terms

$$\begin{aligned} E^\infty_{i,j}(\widetilde{\Omega}_{i+j}^{\mathrm{Spin}}(B\mathbb{Z}/p)_{(p)}) &\cong \widetilde{H}_i(\mathbb{Z}/p) \otimes \Omega_j^{\mathrm{Spin}}(*)_{(p)}, \\ E^\infty_{i,j}(\widetilde{\mathrm{ko}}_{i+j}(B\mathbb{Z}/p)_{(p)}) &\cong \widetilde{H}_i(\mathbb{Z}/p) \otimes \mathrm{ko}_j(*)_{(p)}. \end{aligned}$$

It suffices to show that the map on the E^∞-terms is surjective for all i, j. Hence it is enough to show that the map

$$D_{(p)} \colon \Omega_j^{\mathrm{Spin}}(*)_{(p)} \to \mathrm{ko}_j(*)_{(p)}$$

is surjective for all j. Since $\mathrm{ko}_*(*)_{(p)}$ is a polynomial algebra on a single generator in dimension 4, it suffices to prove $D_{(p)}$ is onto when $j = 4$. In this case both $\Omega_4^{\mathrm{Spin}}(*)$ and $\mathrm{ko}_4(*)$ are infinite cyclic with the former generated by a spin manifold of signature 16, for example the Kummer surface K. The $\widehat{A}$-genus of K is 2 and the index of the real Dirac operator is $\widehat{A}(K)/2$ (see [24], Theorem II.7.10). Hence $D \colon \Omega_4^{\mathrm{Spin}}(*) \to \mathrm{ko}_4(*)$ is an isomorphism. □

Theorem 12.3 (ko-homology). *Let p be an odd prime and let m be any integer.*

(i)

$$\mathrm{ko}_m(B\Gamma) \cong \begin{cases} \bigoplus_{i=0}^n \mathrm{ko}_{m-i}(*)^{r_i}, & m \text{ even}, \\ \mathrm{to}_m(B\Gamma) \oplus (\bigoplus_{i=0}^n \mathrm{ko}_{m-i}(*)^{r_i}), & m \text{ odd}, \end{cases}$$

where $\mathrm{to}_m(B\Gamma)$ is a finite abelian p-group defined for m odd.

(ii) *The inclusion map $\mathbb{Z}^n \to \Gamma$ induces an isomorphism*

$$\mathrm{ko}_{2m}(B\mathbb{Z}^n_\rho)_{\mathbb{Z}/p} \xrightarrow{\cong} \mathrm{ko}_{2m}(B\Gamma)$$

and $\mathrm{ko}_{2m}(B\mathbb{Z}^n_\rho)_{\mathbb{Z}/p} \cong \bigoplus_{i=0}^n \mathrm{ko}_{2m-i}(*)^{r_i}$.

(iii) *There is a long exact sequence*

$$0 \to \mathrm{ko}_{2m}(B\Gamma) \xrightarrow{\bar{f}_{2m}} \mathrm{ko}_{2m}(\underline{B}\Gamma) \xrightarrow{\partial_{2m}} \bigoplus_{(P)\in\mathcal{P}} \widetilde{\mathrm{ko}}_{2m-1}(BP)$$
$$\xrightarrow{\varphi_{2m-1}} \mathrm{ko}_{2m-1}(B\Gamma) \xrightarrow{\bar{f}_{2m-1}} \mathrm{ko}_{2m-1}(\underline{B}\Gamma) \to 0.$$

Hence $\mathrm{ko}_m(B\Gamma)[1/p] \to \mathrm{ko}_m(\underline{B}\Gamma)[1/p]$ is an isomorphism for $m \in \mathbb{Z}$.

(iv) *We have*

$$\mathrm{ko}_{2m+1}(\underline{B}\Gamma) \cong \bigoplus_{i=0}^{2m+1} \mathrm{ko}_{2m+1-i}(*)^{r_i}.$$

(v) *Let* $\mathrm{to}_{2m}(\underline{B}\Gamma) = \operatorname{im}\partial_{2m}$ *and* $\mathrm{to}_{2m-1}(B\Gamma) = \operatorname{im}\varphi_{2m-1}$. *These are finite abelian* p*-groups. There is an exact sequence*

$$0 \to \mathrm{ko}_{2m}(B\Gamma) \to \mathrm{ko}_{2m}(\underline{B}\Gamma) \to \mathrm{to}_{2m}(\underline{B}\Gamma) \to 0$$

and an isomorphism

$$\mathrm{ko}_{2m+1}(B\Gamma) \cong \mathrm{to}_{2m+1}(B\Gamma) \oplus \bigoplus_{i=0}^{n} \mathrm{ko}_{2m+1-i}(*)^{r_i}.$$

Proof. (iii): The Atiyah–Hirzebruch spectral sequence implies that $\widetilde{\mathrm{ko}}_{2m}(B\mathbb{Z}/p)$ vanishes and that $\widetilde{\mathrm{ko}}_{2m+1}(B\mathbb{Z}/p)$ is a finite abelian p-group. Now the claim follows from the long exact sequence associated to the cellular pushout (1.13).

(ii): The proof is similar to that of Theorem 2.1 (ii). We analyze the Leray–Serre spectral sequence associated to the extension (1.1)

$$E^2_{i,j} = H_i(\mathbb{Z}/p; \mathrm{ko}_j(B\mathbb{Z}^n_\rho)) \Rightarrow \mathrm{ko}_{i+j}(B\Gamma).$$

One can show analogously to the proof of Lemma 5.3 that there are isomorphisms of $\mathbb{Z}[\mathbb{Z}/p]$-modules

$$\mathrm{ko}_j(B\mathbb{Z}^n_\rho) \otimes \mathbb{Z}[1/2] \cong \bigoplus_{l=0}^{n} H_l(\mathbb{Z}^n_\rho) \otimes \mathrm{ko}_{j-l}(*) \otimes \mathbb{Z}[1/2], \tag{12.4}$$

$$\mathrm{ko}_j(B\mathbb{Z}^n_\rho) \otimes \mathbb{Z}_{(2)} \cong \bigoplus_{l=0}^{n} H_l(\mathbb{Z}^n_\rho) \otimes \mathrm{ko}_{j-l}(*) \otimes \mathbb{Z}_{(2)}. \tag{12.5}$$

Since $\mathrm{ko}_m(*)_{(p)}$ is $\mathbb{Z}_{(p)}$ when m is divisible by 4 and vanishes otherwise,

$$\widehat{H}^{i+1}(\mathbb{Z}/p; \mathrm{ko}_j(B\mathbb{Z}^n_\rho)) \cong \bigoplus_{\ell} \widehat{H}^{i+1}(\mathbb{Z}/p; H_{j-4\ell}(\mathbb{Z}^n_\rho)).$$

This fact, the Universal Coefficient Theorem, Lemma A.1, and Lemma 1.10 (i) imply that $\widehat{H}^{i+1}(\mathbb{Z}/p; \mathrm{ko}_j(B\mathbb{Z}^n_\rho)) = 0$ when $i + j$ is even.

Thus $E^2_{0,2m} = \mathrm{ko}_{2m}(B\mathbb{Z}^n_\rho)_{\mathbb{Z}/p}$ maps injectively to $\mathrm{ko}_{2m}(B\mathbb{Z}^n_\rho)^{\mathbb{Z}/p}$ and hence is p-torsion-free, and for $i > 0$, $E^2_{i,j}$ has exponent p and vanishes if $i + j$ is even. Thus

$$\mathrm{ko}_{2m}(B\mathbb{Z}^n_\rho)_{\mathbb{Z}/p} \cong E^2_{0,2m} = E^\infty_{0,2m} \xrightarrow{\cong} \mathrm{ko}_{2m}(B\Gamma).$$

By(12.4), (12.5) and Theorem 2.1 (i), (ii),

$$\mathrm{ko}_{2m}(B\mathbb{Z}^n_\rho)_{\mathbb{Z}/p} \cong \bigoplus_{l=0}^{n} H_l(\mathbb{Z}^n_\rho)_{\mathbb{Z}/p} \otimes \mathrm{ko}_{2m-l}(*) \cong \textstyle\bigoplus_{l=0}^{n} \mathrm{ko}_{2m-l}(*)^{r_l}.$$

(iv): We will compute the group $\mathrm{ko}_{2m+1}(\underline{B}\Gamma)$ after localizing at p and after inverting p. We will begin with localizing at p. We use the Atiyah–Hirzebruch spectral sequence

$$E^2_{i,j} = H_i(\underline{B}\Gamma; \mathrm{ko}_j(*)_{(p)}) \Rightarrow \mathrm{ko}_{i+j}(\underline{B}\Gamma)_{(p)}$$

for the generalized homology theory $\mathrm{ko}_m(-)_{(p)}$. Note also that when i is odd, Theorem 2.1 (iv) states that $H_i(\underline{B}\Gamma) \cong \mathbb{Z}^{r_i}$. In particular, when $i+j$ is odd, $E^2_{i,j}$ is finitely generated free over $\mathbb{Z}_{(p)}$. Since the differentials in the Atiyah–Hirzebruch spectral sequence are rationally trivial, $E^\infty_{i,j} \subset E^2_{i,j}$ and has finite p-power index whenever $i+j$ is odd. Hence

$$\mathrm{ko}_{2m+1}(\underline{B}\Gamma)_{(p)} \cong \bigoplus_i E^\infty_{i,2m+1-i} \cong \bigoplus_i (\mathrm{ko}_{2m+1-i}(*)^{r_i})_{(p)}.$$

Now we invert p. For any integer $j \geq 0$,

$$\begin{aligned}
\mathrm{ko}_j(\underline{B}\Gamma)[1/p] &\xleftarrow{\cong} \mathrm{ko}_j(B\mathbb{Z}^n_\rho)_{\mathbb{Z}/p}[1/p] && \text{(Proposition A.4)}\\
&\cong \bigoplus_i H_i(B\mathbb{Z}^n_\rho)_{\mathbb{Z}/p} \otimes \mathrm{ko}_{j-i}(*)[1/p] && \text{(isomorphisms (12.4), (12.5))}\\
&\cong \bigoplus_i (\mathrm{ko}_{j-i}(*)^{r_i})[1/p]. && \text{(Theorem 2.1 (i), (ii))}
\end{aligned}$$

(v): The group $\mathrm{to}_{2m}(\underline{B}\Gamma)$ is a subgroup and the group $\mathrm{to}_{2m-1}(B\Gamma)$ is a quotient group of the finite abelian p-group $\widetilde{\mathrm{ko}}_{2m}(B\mathbb{Z}/p)$, hence are finite abelian p-groups themselves. To complete the proof of assertion (v), by assertions (iii) and (iv) we only need prove that $\bar{f}_{2m+1}$ is a split surjection. This follows since $\mathrm{ko}_{2m+1}(\underline{B}\Gamma)_{(p)}$ is free over $\mathbb{Z}_{(p)}$ and $\bar{f}_{2m+1} \otimes \mathrm{id}_{\mathbb{Z}[1/p]}$ is an isomorphism.

(i) This follows from assertions (ii) and (v). $\square$

Now we are ready to prove Theorem 0.7.

Proof of Theorem 0.7. Let M be a closed m-dimensional manifold with $m \geq 5$ and fundamental group $\pi_1(M) \cong \Gamma$. Suppose that $\alpha(M) = 0$. We have to show that M carries a metric with positive scalar curvature.

The following commutative diagram with exact rows is key to the proof.

$$\begin{tikzcd}
\bigoplus_{(P)\in\mathcal{P}} \widetilde{\mathrm{ko}}_m(BP) \arrow[r] & \mathrm{ko}_m(B\Gamma) \arrow[r, "\beta"] \arrow[d, "A\circ p_{B\Gamma}"'] & \mathrm{ko}_m(\underline{B}\Gamma) \arrow[d, "p_{\underline{B}\Gamma}"] \\
& \mathrm{KO}_m(C^*_r(\Gamma;\mathbb{R})) \arrow[r] & \mathrm{KO}_m(\underline{B}\Gamma)
\end{tikzcd}$$

Here the bottom map is the composite of the inverse of the Baum–Connes map $\mathrm{KO}^\Gamma_m(\underline{E}\Gamma) \to \mathrm{KO}^\Gamma_m(C^*_r(\Gamma;\mathbb{R}))$ (which is an isomorphism by [19]) and the map $\mathrm{KO}^\Gamma_m(\underline{E}\Gamma) \to \mathrm{KO}_m(\underline{B}\Gamma)$ coming from induction with $\Gamma \to 1$. The top row is exact by Theorem 12.3 (iii). The square commutes since the map $p_{\underline{B}\Gamma} \circ \beta$ equals the composite

$$\mathrm{ko}_m(B\Gamma) \to \mathrm{KO}_m(B\Gamma) = \mathrm{KO}^\Gamma_m(E\Gamma) \to \mathrm{KO}^\Gamma_m(\underline{E}\Gamma) \to \mathrm{KO}_m(B\Gamma).$$

Since by assumption $\alpha(M) = 0$, the image of $D[f_M : M \to B\Gamma] \in \mathrm{ko}_m(B\Gamma)$ under the composite $p_{\underline{B}\Gamma} \circ \beta$ is zero, where $f_M : M \to B\Gamma$ is the classifying map of M associated to $\pi_1(M) \cong \Gamma$.

Next we show that the map $p_{\underline{B}\Gamma}[1/p]$ is injective. Because of Proposition A.4, it suffices to show $\mathrm{ko}_m(B\mathbb{Z}^n_\rho)_{\mathbb{Z}/p}[1/p] \to \mathrm{KO}_m(B\mathbb{Z}^n_\rho)_{\mathbb{Z}/p}[1/p]$ is injective. Since p divides the order of $\mathbb{Z}/p$ it suffices to show that $\mathrm{ko}_m(B\mathbb{Z}^n) \to \mathrm{KO}_m(B\mathbb{Z}^n)$ is injective. This follows from the commutative square

$$\begin{tikzcd}
\bigoplus_{l=0}^n (\mathrm{ko}_{m-l}(*))^{\binom{n}{l}} \arrow[r, "\cong"] \arrow[d, "\bigoplus_{l=0}^n (p_*)^{\binom{n}{l}}"'] & \mathrm{ko}_m(B\mathbb{Z}^n) \arrow[d, "p_{B\mathbb{Z}^n}"] \\
\bigoplus_{l=0}^n (\mathrm{KO}_{m-l}(*))^{\binom{n}{l}} \arrow[r, "\cong"] & \mathrm{KO}_m(B\mathbb{Z}^n)
\end{tikzcd}$$

since $p_*\colon \mathrm{ko}_m(*) \to \mathrm{KO}_m(*)$ is injective for all $m \in \mathbb{Z}$. This finishes the proof that the kernel of the map $p_{\underline{B}\Gamma}$ consists of p-torsion. Hence $\beta(D[f_M : M \to B\Gamma]) \in \mathrm{ko}_m(\underline{B}\Gamma)$ is p-torsion.

Now we can finish the proof in the case that m is even. Then the map β is injective and its domain is a finitely generated abelian group without p-torsion by Theorem 12.3 (ii) and (iii). Hence $D[f_M : M \to B\Gamma] \in \mathrm{ko}_m(B\Gamma)$ is trivial and we conclude from Proposition 12.1 that M carries a metric with positive scalar curvature.

Hence we will now assume that m is odd. Then the target of β is a finitely generated abelian group without p-torsion by Theorem 12.3 (iv). Hence the image of $D[f_M : M \to B\Gamma] \in \mathrm{ko}_m(B\Gamma)$ under β is zero. We conclude from Theorem 12.3 (iii) that there is an element

$$(x_P)_{(P)\in\mathcal{P}} \in \bigoplus_{(P)\in\mathcal{P}} \widetilde{\mathrm{ko}}_m(BP)$$

which is mapped under $\bigoplus_{(P)\in\mathcal{P}} \widetilde{\mathrm{ko}}_m(BP) \to \mathrm{ko}_m(B\Gamma)$ to $D[f_M : M \to B\Gamma]$. Combining this with Lemma 12.2 yields elements $[N_P \to BP] \in \widetilde{\Omega}^{\mathrm{Spin}}_m(B\mathbb{Z}/p)$ such that the image of $[N_P \to BP]_{(P)\in\mathcal{P}}$ under the composite

$$\bigoplus_{(P)\in\mathcal{P}} \widetilde{\Omega}^{\mathrm{Spin}}_m(BP) \to \Omega^{\mathrm{Spin}}_m(B\Gamma) \xrightarrow{D} \mathrm{ko}_m(B\Gamma)$$

agrees with $D[f_M : M \to B\Gamma]$. By surgery we can arrange that the map $N_P \to BP$ is 2-connected and in particular a classifying map for N_P. Since m is odd, $\widetilde{\mathrm{KO}}_m(C^*_r(P;\mathbb{R})) = 0$ (see the beginning of Section 9). Hence since the Gromov–Lawson–Rosenberg conjecture holds for manifolds whose fundamental group is odd-order cyclic [23], each N_P admits a metric of positive scalar curvature. Recall that

$$D[f_M : M \to B\Gamma] = D[(\amalg_{P\in(\mathcal{P})} N_P) \to (\amalg_{P\in(\mathcal{P})} BP) \to B\Gamma] \in \mathrm{ko}_m(B\Gamma).$$

Hence, by Proposition 12.1, M admits a metric of positive scalar curvature.

Now we just need to show that the last sentence of Theorem 0.7 is valid.

Let M be a closed spin manifold with odd dimension $m \geq 5$ and fundamental group Γ. Suppose that its p-cover $\widehat{M}$ associated with the subgroup $\iota\colon \mathbb{Z}^n \to \Gamma$

admits a metric of positive scalar curvature. Then $0 = \alpha(\widehat{M}) = \iota^*\alpha(M) \in \mathrm{KO}_m(C_r^*(\mathbb{Z}^n;\mathbb{R}))$. Hence by Theorem 0.6 (iii), $\alpha(M) = 0$. Hence by our argument above, M admits a metric of positive scalar curvature. □

Appendix

Tate cohomology, duality, and transfers

Here we collect facts concerning duality in Tate cohomology, transfers in generalized (co)-homology theories, and edge homomorphisms in the Leray–Serre spectral sequence.

Recall that $\widehat{H}^*(G;M)$ denotes the *Tate cohomology* (see [12], VI.4) of a finite group G with coefficients in a $\mathbb{Z}[G]$-module M, that $\widehat{H}^i(G;M) = H^i(G;M)$ for $i \geq 1$, that $\widehat{H}^i(G;M) = H_{-i-1}(G;M)$ for $i \leq -2$, and that there is an exact sequence

$$0 \to \widehat{H}^{-1}(G;M) \to M_G \xrightarrow{N} M^G \to \widehat{H}^0(G;M) \to 0.$$

Here M^G are the *invariants* of M, $M_G = M \otimes_{\mathbb{Z}G} \mathbb{Z} = M/\langle gm - m\rangle_{g\in G, m\in M}$ are the *coinvariants* of M, and $N[m] = \sum_{g\in G} gm$ is the *norm map*. Note $M^G = H^0(G;M)$ and $M_G = H_0(G;M)$.

For a abelian group M, define the *dual* $M^* = \mathrm{Hom}_{\mathbb{Z}}(M,\mathbb{Z})$ and the *torsion dual* $M^\wedge = \mathrm{Hom}_{\mathbb{Z}}(M,\mathbb{Q}/\mathbb{Z})$. Note that if M is a finitely generated free abelian group (respectively a finite abelian group) then there is a non-canonical isomorphism $M \cong M^*$ (respectively $M \cong M^\wedge$). If M is a left $\mathbb{Z}G$-module, give M^* and $M^\wedge$ the structure of left $\mathbb{Z}G$-modules by defining $(g\varphi)(m) := \varphi(g^{-1}m)$ for $g \in G$ and $m \in M$.

Lemma A.1 (Tate duality). *Let G be a finite group and M be a finitely generated $\mathbb{Z}G$-module which contains no p-torsion for all primes p dividing the order of G. Then for all integers i there is an isomorphism of abelian groups*

$$\widehat{H}^i(G;M) \cong \widehat{H}^{-i}(G;M^*).$$

Hence for all integers $i > 0$,

$$H^{i+1}(G;M) \cong H_i(G;M^*).$$

Proof. The Tate cohomology group $\widehat{H}^i(G;M)$ is a finitely generated group of exponent $|G|$, hence is a finite abelian group. Thus there is a non-canonical isomorphism of abelian groups $\widehat{H}^i(G;M) \cong \widehat{H}^i(G;M)^\wedge$. Duality in Tate cohomology shows that

$$\widehat{H}^i(G;M)^\wedge \cong \widehat{H}^{-i-1}(G;M^\wedge)$$

(see [12], VI.7.3; duality holds for any $\mathbb{Z}G$-module). Let FM be M modulo its torsion subgroup. Then $(FM)^* \to M^*$ and $(FM)^\wedge \otimes \mathbb{Z}_{(|G|)} \to M^\wedge \otimes \mathbb{Z}_{(|G|)}$ are isomorphisms and

$$0 \to \operatorname{Hom}_{\mathbb{Z}}(FM, \mathbb{Z}) \to \operatorname{Hom}_{\mathbb{Z}}(FM, \mathbb{Q}) \to \operatorname{Hom}_{\mathbb{Z}}(FM, \mathbb{Q}/\mathbb{Z}) \to 0$$

is a short exact sequence. Thus

$$\widehat{H}^{-i-1}(G; M^\wedge) \cong \widehat{H}^{-i-1}(G; (FM)^\wedge) \cong \widehat{H}^{-i}(G; (FM)^*) \cong \widehat{H}^{-i}(G; M^*),$$

as desired. □

Remark A.2. Here is a related remark. Let $G = \langle g \rangle$ be a finite cyclic group and M be a $\mathbb{Z}G$-module. Then by dualizing the exact sequence

$$M \xrightarrow{g-1} M \to M_G \to 0$$

one obtains the exact sequence

$$0 \to (M_G)^* \to M^* \xrightarrow{g^{-1}-1} M^*.$$

Hence $(M_G)^* \cong (M^*)^G$.

Let $\pi\colon E \to B$ be a regular G-cover of CW-complexes. Let $\mathcal{H}_*$ a generalized homology theory and $\mathcal{H}^*$ a generalized cohomology theory. There are transfer maps trf_* and trf^* switching the domain and range of π_* and π^*. Their definition is given in [2], Chapter 4, when B is finite and in [26], Chapter IV, §3, in general. All four maps are G-equivariant with respect to the induced G-action on $\mathcal{H}_*(E)$ and the trivial G-action on $\mathcal{H}_*(B)$ and $\mathcal{H}^*(B)$. Hence we have maps

$$\begin{aligned}
\pi_*\colon \mathcal{H}_*(E)_G &\to \mathcal{H}_*(B),\\
\operatorname{trf}_*\colon \mathcal{H}_*(B) &\to \mathcal{H}_*(E)^G,\\
\pi^*\colon \mathcal{H}^*(B) &\to \mathcal{H}^*(E)^G,\\
\operatorname{trf}_*\colon \mathcal{H}^*(E)_G &\to \mathcal{H}^*(B).
\end{aligned}$$

The basic theorem connecting the two is this special case of the double coset formula [26], Corollary 6.4, p. 206.

Theorem A.3. *Both* $\operatorname{trf}_* \circ \pi_*$ *and* $\pi^* \circ \operatorname{trf}^*$ *are given by the norm map, i.e., multiplication by* $\sum_{g \in G} g$.

For ordinary (co)homology theory, $\pi_* \circ \operatorname{trf}_*$ and $\operatorname{trf}^* \circ \pi^*$ are both multiplication by $q = |G|$. This has the consequence that π_* and π_* are isomorphisms after inverting q. These last composite formulae are no longer true for generalized (co)homology theories, but one can say something.

A generalized homology theory is $1/q$-local if $\mathcal{H}_*(X) \otimes \mathbb{Z} \to \mathcal{H}_*(X) \otimes \mathbb{Z}[1/q]$ is an isomorphism for all X and m. For example, for any generalized homology theory, $\mathcal{H}_*(X) \otimes \mathbb{Z}[1/q]$ is a $1/q$-local generalized homology theory. There is an analogous definition and remark for generalized cohomology theories.

Proposition A.4. *Let G be a finite group of order q. Let $\mathcal{H}_*$ and $\mathcal{H}^*$ be $1/q$-local (co)homology theories. Let X be a G-CW-complex and $\pi\colon X \to \bar{X}$ the quotient map.*

(i) $\pi_m\colon \mathcal{H}_m(X)_G \xrightarrow{\cong} \mathcal{H}_m(\bar{X})$ *is an isomorphism for all $m \in \mathbb{Z}$.*

(ii) *If X is a finite CW-complex, then $\pi^m\colon \mathcal{H}^m(\bar{X}) \xrightarrow{\cong} \mathcal{H}^m(X)^G$ is an isomorphism for all $m \in \mathbb{Z}$.*

Proof. We give the argument only for homology, the one for cohomology is analogous.

Given a G-CW-complex X, we obtain a natural map

$$j_*\colon \mathcal{H}_*(X)_G \to \mathcal{H}_*(G\backslash X).$$

Since the functor sending a $\mathbb{Z}[1/q][G]$-module M to M_G is an exact functor, the assignment sending a G-CW-complex X to $\mathcal{H}_*(X)_G$ and to $\mathcal{H}_*(G\backslash X)$ are G-homology theories and j_* is a natural transformation of G-homology theories. One easily checks that j_* is a bijection when X is G/H for any subgroup $H \subset G$. A Mayer–Vietoris argument implies that j_* is a bijection for any finite G-CW-complex, and, since homology commutes with colimits, j_* is a bijection for any G-CW-complex. □

Atiyah's computation of $K^0(B\mathbb{Z}/p)$ shows that a finiteness hypothesis is necessary for a generalized cohomology theory.

At several places in this paper we use a property of edge homomorphisms in spectral sequences and we review this now. Let $\mathcal{H}_*$ and $\mathcal{H}^*$ be (co)homology theories. Let $F \to E \to B$ be a fibration. Assume that B is path-connected with fundamental group G. There are Leray–Serre spectral sequences

$$\begin{aligned} E^2_{i,j} &= H_i(B; \mathcal{H}_j(F)) \Rightarrow \mathcal{H}_{i+j}(E),\\ E_2^{i,j} &= H^i(B; \mathcal{H}^j(F)) \Rightarrow \mathcal{H}^{i+j}(E). \end{aligned}$$

These spectral sequences have coefficients twisted by the action of G on the (co)homology of the fiber, in particular

$$\begin{aligned} E^2_{0,j} &\cong H_0(G; \mathcal{H}_j(F)) = \mathcal{H}_j(F)_G,\\ E_2^{0,j} &\cong H^0(G; \mathcal{H}^j(F)) = \mathcal{H}^j(F)^G. \end{aligned}$$

The spectral sequences give maps

$$\begin{aligned} H_j(F)_G \cong E^2_{0,j} \twoheadrightarrow E^\infty_{0,j} \rightarrowtail \mathcal{H}_j(E),&\\ \mathcal{H}^j(E) \twoheadrightarrow E_\infty^{0,j} \rightarrowtail E_2^{0,j} \cong H^j(F)^G;& \end{aligned}$$

the composites are called the *edge homomorphisms*.

The proof of the proposition below follows the proof in the untwisted case [43], p. 354.

Proposition A.5 (Edge homomorphisms). *The edge homomorphisms*

$$\mathcal{H}_j(F)_G \to \mathcal{H}_j(E),$$
$$\mathcal{H}^j(E) \to \mathcal{H}^j(F)^G$$

equal the maps on (*co*)*homology induced by the inclusion of the fiber* $F \to E$.

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