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The odd-dimensional analogue of a theorem of Getzler and Wu

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Abstract. We prove an analogue for odd-dimensional manifolds with boundary, in the b-calculus setting, of the higher Atiyah–Patodi–Singer index theorem by Getzler and by Wu, and thus obtain a natural counterpart of the eta invariant for even-dimensional closed manifolds.

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Introduction

The goal of this paper is to prove an analogue for odd-dimensional manifolds with boundary of the higher Atiyah–Patodi–Singer index theorem of Getzler [Get93a] and Wu [Wu93]. For notational simplicity, we will restrict the discussion mainly to spin manifolds. However all results can be straightforwardly extended to general manifolds, with appropriate modification.

Suppose N is an odd-dimensional spin manifold with boundary and carries an exact b-metric [Mel93], cf. Section 1. For $g \in U_k(C^\infty(N))$ a unitary over N, let $\mathrm{Ch}_\bullet(g)$ (resp. $\mathrm{Ch}_\bullet^{\mathrm{dR}}(g)$) be the Chern character of g in entire cyclic homology of $C^\infty(N)$ (resp. de Rham cohomology of N). In the following, $⨍_N$ is the regularized integral on N with respect to its b-metric (see Section 1) and $\hat{A}(N)$ is the $\hat{A}$-genus form of N. Let D be the Dirac operator on N and ${}^\partial D$ be its restriction to the boundary ∂N. Denote the higher eta cochain of ${}^\partial D$ by $\eta^\bullet({}^\partial D)$, introduced by Wu [Wu93].

Theorem. *Let N be an odd-dimensional spin manifold with boundary. Endow N with an exact b-metric and let D be its associated Dirac operator. Assume ${}^\partial D$ is invertible. For $g \in U_k(C^\infty(N))$ a unitary over N, if $\|d\,{}^\partial g\| < \lambda$ where λ the lowest nonzero eigenvalue of $|{}^\partial D|$ and ${}^\partial g$ is the restriction of g to the boundary, then*

$$\mathrm{sf}(D, g^{-1}Dg) = ⨍_N \hat{A}(N) \wedge \mathrm{Ch}_\bullet^{\mathrm{dR}}(g) - \langle \eta^\bullet({}^\partial D), \mathrm{Ch}_\bullet({}^\partial g)\rangle. \tag{0.1}$$

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Here $\mathrm{sf}(D, g^{-1}Dg)$ is the spectral flow of the path $D_u = (1-u)D + ug^{-1}Dg$ with $u \in [0,1]$ (see Section 4). In order for $\mathrm{sf}(D, g^{-1}Dg)$ to be well defined, the infimum of the essential spectrum of $|D_u|$, denoted by $\inf \mathrm{spec}_{\mathrm{ess}}(|D_u|)$, has to be greater than zero for each u. The latter condition is fulfilled if and only if the restriction D_u to the boundary ∂N is invertible for each u. Thus the almost flatness condition $\|d\,{}^{\partial}g\| < \lambda$ ensures that $\mathrm{sf}(D, g^{-1}Dg)$ is well defined.

Let ${}^{\mathrm{b}}\mathrm{Ch}^{\bullet}(D_t)$ be the b-analogue of the odd Chern character by Jaffe–Lesniewski–Osterwalder [JLO88], cf. Section 2. The above theorem is proved by interpolating between the limit of ${}^{\mathrm{b}}\mathrm{Ch}^{\bullet}(D_t)$ as $t \to \infty$ and its limit as $t \to 0$, where $D_t = tD$. In fact, the limit at $t = \infty$ does not exist in general. However, when evaluated at $\mathrm{Ch}_{\bullet}(g)$ with g satisfying the almost flat condition above, the limit of ${}^{\mathrm{b}}\mathrm{Ch}^{\bullet}(D_t)$ as $t \to \infty$ does exist and gives the spectral flow $\mathrm{sf}(D, g^{-1}Dg)$. To prove this, i.e., the equality

$$\lim_{t\to\infty} \langle {}^{\mathrm{b}}\mathrm{Ch}^{\bullet}(D_t), \mathrm{Ch}_{\bullet}(g)\rangle = \mathrm{sf}(D, g^{-1}Dg), \tag{0.2}$$

we first show (see Proposition 4.6 below) that

$$\mathrm{sf}(D, g^{-1}Dg) = \lim_{\varepsilon\to\infty} \frac{\varepsilon}{\sqrt{\pi}} \int_0^1 {}^{\mathrm{b}}\mathrm{Tr}(\dot{D}_u e^{-\varepsilon^2 D_u^2})\, du. \tag{0.3}$$

This is a generalization to the b-calculus setting of Getzler's spectral flow formula for closed manifolds, cf. [Get93b], Corollary 2.7. Once we show eq. (0.3), the proof of eq. (0.2) reduces to

$$\lim_{t\to\infty} \frac{t}{\sqrt{\pi}} \int_0^1 {}^{\mathrm{b}}\mathrm{Tr}(\dot{D}_u e^{-t^2 D_u^2}) du = \lim_{t\to\infty} \langle {}^{\mathrm{b}}\mathrm{Ch}^{\bullet}(D_t), \mathrm{Ch}_{\bullet}(g)\rangle. \tag{0.4}$$

In turn, to verify this, we consider a multiparameter version of the Chern character $\mathrm{Ch}(\mathbb{A})$ of the superconnection $\mathbb{A}$ (see [Get93b], also Section 5 below, for the precise definition). Each side of eq. (0.4) corresponds to one term in the formula obtained by applying Stokes theorem to $\mathrm{Ch}(\mathbb{A}_t)$ for each fixed t. We then show the vanishing of the rest of the terms as $t \to \infty$, hence prove the validity of eq. (0.4), cf. Section 5. The rest of the proof proceeds along the lines of Getzler's even counterpart, cf. [Get93a].

Due to the fact that ${}^{\mathrm{b}}\mathrm{Tr}$ is not a trace, ${}^{\mathrm{b}}\mathrm{Ch}^{\bullet}(D_t)$ is not a closed cochain. The integral of its boundary from 0 to ∞ gives the odd eta cochain $\eta^{\bullet}({}^{\partial}D)$ on the right-hand side of (0.1).

As a corollary of the main theorem, by comparing eq. (0.1) with Dai–Zhang's Toeplitz index formula for odd-dimensional manifolds with boundary [DZ06], we show that if $\|d\,{}^{\partial}g\| < \lambda$, then

$$\langle \eta^{\bullet}({}^{\partial}D), \mathrm{Ch}_{\bullet}({}^{\partial}g)\rangle = -\bar{\eta}(\partial N, {}^{\partial}g) \mod \mathbb{Z},$$

where $\bar{\eta}(\partial N, {}^{\partial}g)$ is the (reduced) eta invariant of Dai–Zhang. This equality provides more evidence for the naturality of the Dai–Zhang eta invariant for even-dimensional closed manifolds.

An outline of this article is as follows. In Section 1, we recall some facts from b-calculus on manifolds with boundary and Chern characters in cyclic homology. In Section 2, we define a b-analogue of the JLO Chern character and prove its entireness. Then we state our main theorem (Theorem 3.1 below) in Section 3. We prove the main step of the proof to the main theorem in Section 4 and 5.

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1. Preliminaries

Throughout the paper, we denote by $\mathrm{Cl}_q(\mathbb{C})$ the complex Clifford algebra with odd-degree generators c_i, $1 \leq i \leq q$, and relations

$$c_i c_j + c_j c_i = -2\delta_{ij}.$$

This is a $\mathbb{Z}_2$-graded $*$-algebra with $c_i^* = -c_i$.

1.1. Manifolds with Boundary and b-metrics.

Let M be an odd-dimensional spin manifold with boundary. We fix a Riemannian metric, say w, and a spin structure on M. Furthermore, we assume the Riemannian metric is of product type near the boundary, that is, on $[0,\varepsilon)_x \times \partial M$ a collar neighborhood of ∂M, it takes the form

$$w = (dx)^2 + h,$$

where h is the Riemannian metric on ∂M. Denote by $\widehat{M}$ the manifold obtained by attaching an infinite cylinder $(-\infty, 0] \times \partial M$ to M along ∂M:

$$\widehat{M} = (-\infty, 0] \times \partial M \cup_{\partial M} M.$$

The Riemannian metric M extends naturally to a Riemannian metric on $\widehat{M}$, still denoted by w.

Notice that $(\widehat{M}, w)$ is isometric to a standard b-manifold, that is, a manifold with boundary carrying a b-metric. To see this, one performs the change of variable $x \mapsto r = e^x$ on the cylindrical end. This replaces $(-\infty, 0]_x \times \partial M$ by a compact cylinder $[0,1]_r \times \partial M$. Moreover, the metric w induces a metric on $N = [0,1] \times \partial M \cup_{\partial M} M$ under the coordinate change. In particular, the induced metric restricted on $[0,1]_r \times \partial M$ takes the form

$$\left(\frac{dr}{r}\right)^2 + h$$

which is an exact b-metric on N, cf. [Mel93] and [Loy05]. Unless otherwise specified, all b-metrics in this paper are assumed to be exact.

1.2. Clifford modules and Dirac operators. Consider $N = [0,1]_r \times \partial M \cup_{\partial M} M$ with an exact b-metric. The set of b-vector fields, that is, vector fields on N tangential to ∂N, is closed under Lie bracket. By the Swan–Serre Theorem, such vector fields are smooth sections of a vector bundle ${}^bTN$ over N, called the b-tangent bundle of N, cf. [Mel93], Lemma 2.5. We denote its dual bundle, the b-cotangent bundle, by ${}^bT^*N$.

By a Clifford module over N of degree q, we mean a $\mathbb{Z}_2$-graded Hermitian vector bundle $\mathcal{E}$ over N with commuting graded $*$-actions of the Clifford algebra $\mathrm{Cl}_q(\mathbb{C})$ and the Clifford bundle $\mathrm{Cl}({}^bT^*N)$, cf. [Get93a].

Let $\mathcal{S}$ be the spinor bundle over N and $\mathbb{C}^{1|1} = \mathbb{C}^+ \oplus \mathbb{C}^-$ be a $\mathbb{Z}_2$-graded two dimensional vector bundle. Then $\mathcal{S} \otimes \mathbb{C}^{1|1}$ is a Clifford module of degree 1 over N, where each $\omega \in \Gamma(N, \mathrm{Cl}({}^bT^*N))$ acts on $\mathcal{S} \otimes \mathbb{C}^{1|1}$ by $\left(\begin{smallmatrix} 0 & c(\omega) \\ c(\omega) & 0 \end{smallmatrix}\right)$ and the generator e_1 of $\mathrm{Cl}_1(\mathbb{C})$ acts on $\mathcal{S} \otimes \mathbb{C}^{1|1}$ by $\left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$.

1.3. b-norm. In this section, we introduce a b-norm on $C^\infty_{\exp}(\widehat{M})$. We shall use this b-norm to prove the entireness of the b-JLO Chern character in Section 2. Here $C^\infty_{\exp}(\widehat{M})$ is the space of smooth functions on $\widehat{M}$ which expands exponentially on the infinite cylinder $(-\infty, 0]_x \times \partial M$, cf. [Loy05]. A smooth function $f \in C^\infty(\widehat{M})$ expands exponentially on $(-\infty, 0]_x \times \partial M$ if

$$f(x,y) \sim \sum_{k=0}^{\infty} e^{kx} f_k(y)$$

for $(x,y) \in (-\infty, 0]_x \times \partial M$, where $f_k(y) \in C^\infty(\partial M)$ for each k. More precisely, we have

$$f(x,y) - \sum_{k=0}^{N-1} e^{kx} f_k(y) = e^{Nx} R_N(x,y),$$

where all derivatives of $R_N(x,y)$ in x and y are bounded.

Remark 1.1. Notice that $C^\infty_{\exp}(\widehat{M})$ becomes exactly $C^\infty(N)$ if one performs the change of variable $x \to e^x$ on the cylindrical end.

On $(-\infty, 0]_x \times \partial M$, for each $a \in C^\infty_{\exp}(\widehat{M})$, we have

$$a = a_c + e^x a_\infty,$$

where $a_c, a_\infty \in C^\infty_{\exp}(\widehat{M})$ and a_c is constant with respect to x. We define a norm on $C^\infty_{\exp}(\widehat{M})$ by

$${}^{\mathrm{b}}\|a\| := \|a\|_1 + 2\|a_\infty\|_1$$

where $\|a\|_1$ is the C^1-norm of a and $\|a_\infty\|_1$ is the C^1-norm of a_∞.

Lemma 1.2. ${}^{\mathrm{b}}\|\cdot\|$ *is a well-defined multiplicative norm.*

Proof. Note that $(a+b)_\infty = a_\infty + b_\infty$ and $(ab)_\infty = a_c b_\infty + a_\infty b_c + e^x a_\infty b_\infty$. So it is clear that

$$^{\mathrm{b}}\|\lambda a\| = |\lambda| \cdot {}^{\mathrm{b}}\|a\|, \quad {}^{\mathrm{b}}\|a+b\| \le {}^{\mathrm{b}}\|a\| + {}^{\mathrm{b}}\|b\|.$$

To prove the norm is multiplicative, we first notice that $\|a_c\| \le \|a\|$, $\|da_c\| \le \|da\|$ and

$$d(e^x a_\infty b_\infty) = (e^x dx) a_\infty b_\infty + e^x d(a_\infty b_\infty).$$

Thus we have

$$\begin{aligned} 2&\|a_c b_\infty + a_\infty b_c + e^x a_\infty b_\infty\|_1 \\ &= 2\|a_c b_\infty + a_\infty b_c + e^x a_\infty b_\infty\| + 2\|d(a_c b_\infty + a_\infty b_c + e^x a_\infty b_\infty)\| \\ &\quad + 2\|a\| \cdot \|db_\infty\| + 2\|da_\infty\| \cdot \|b\| \\ &\le 2\|a\|_1 \cdot \|b_\infty\|_1 + 2\|a_\infty\|_1 \cdot \|b\|_1 + 4\|a_\infty\|_1 \cdot \|b_\infty\|_1. \end{aligned}$$

By applying the inequality $\|ab\|_1 \le (\|a\| + \|da\|)(\|b\| + \|db\|)$, we obtain

$$^{\mathrm{b}}\|ab\| \le {}^{\mathrm{b}}\|a\| \cdot {}^{\mathrm{b}}\|b\|. \qquad \square$$

1.4. b-trace. For $f \in C^\infty_{\exp}(\widehat{M})$, we have

$$f = f_c + e^x f_\infty$$

on the cylindrical end $(-\infty, 0] \times \partial M$, where f_c is constant with respect to x.

Definition 1.3. The regularized integral of $f \in C^\infty_{\exp}(\widehat{M})$ with respect to the b-metric is defined to be

$$⨍_{\widehat{M}} f \, d\mathrm{vol} := \int_M f|_M \, d\mathrm{vol} + \int_{(-\infty,0]\times\partial M} e^x f_\infty \, d\mathrm{vol}.$$

For $A \in {}^{\mathrm{b}}\Psi^{-\infty}(\widehat{M}, \mathcal{V})$, let K_A be its Schwartz kernel and $K_A|_\Delta$ the restriction of K_A to the diagonal $\Delta \subset \widehat{M} \times \widehat{M}$. Then the fiberwise trace of $K_A|_\Delta$, denoted by $\mathrm{tr}(K_A|_\Delta)$, is a function in $C^\infty_{\exp}(\widehat{M})$, cf. [Loy05]. We define the b-trace of $A \in {}^{\mathrm{b}}\Psi^{-\infty}(\widehat{M}, \mathcal{V})$ to be

$$^{\mathrm{b}}\mathrm{Tr}(A) := ⨍_{\widehat{M}} \mathrm{tr}(K_A|_\Delta) \, d\mathrm{vol}.$$

When $\mathcal{V}$ is $\mathbb{Z}_2$-graded, we define the b-supertrace of A by

$$^{\mathrm{b}}\mathrm{Str}(A) = ⨍_{\widehat{M}} \mathrm{str}(K_A|_\Delta) \, d\mathrm{vol},$$

where str is the fiberwise supertrace on $\mathrm{End}_{\mathbb{Z}_2}(\mathcal{V})$.

1.5. Cyclic homology and cyclic cohomology. For A an algebra over $\mathbb{C}$, let

$$C_n(A) = A \otimes (A/\mathbb{C})^{\otimes n}.$$

An element of $C_n(A)$ is denoted by $(a_0, a_1, \ldots, a_n)$. Sometimes we also write $(a_0, a_1, \ldots, a_n)_n$ to emphasise the degree of the element.

Definition 1.4.

$$b(a_0, \ldots, a_n) = \sum_{i=0}^{n} (-1)^i (a_0, \ldots, a_i a_{i+1}, \ldots, a_n),$$
$$B(a_0, \ldots, a_n) = \sum_{i=0}^{n} (-1)^{ni} (1, a_i, \ldots, a_n, a_0, \ldots, a_{i-1}).$$

Let $C_+(A) = \prod_k C_{2k}(A)$ and $C_-(A) = \prod_k C_{2k+1}(A)$, then we have the chain map

$$b + B \colon C_\pm(A) \to C_\mp(A).$$

The homology of this chain complex is called the periodic cyclic homology of A, denoted by $\mathrm{HP}_\pm(A)$.

When A is a Banach algebra, we use the inductive tensor product instead of the algebraic tensor product in the definition of $C_n(A)$. We denote the resulted space of continuous even (resp. odd) chains by $C_+^{\mathrm{top}}(A)$ (resp. $C_-^{\mathrm{top}}(A)$). Let us define

$$\|c_0 + c_1 + \cdots\|_\lambda = \sup_n \frac{\lambda^n}{\Gamma(n/2)} \|c_n\|.$$

Then an even chain $c_0 + c_2 + \cdots \in C_+^{\mathrm{top}}(A)$ is called entire if $\|c_0 + c_2 + \cdots\|_\lambda$ is finite for some $\lambda > 0$. Entire odd chains in $C_-^{\mathrm{top}}(A)$ are defined the same way. The space of even (resp. odd) entire chains will be denoted by $C_+^\omega(A)$ (resp. $C_-^\omega(A)$). It is easy to check that b and B are continuous maps from $C_\pm^\omega(A)$ to $C_\mp^\omega(A)$, hence $(C_\pm^\omega(A), b + B)$ is a well-defined chain complex. The resulting homology is called the entire cyclic homology of A, denoted $H_\pm^\omega(A)$.

Similarly, the entire cyclic cohomology of A, denoted by $H_\omega^\pm$, is defined to be the homology of the cochain complex $(C_\omega^\pm(A), b + B)$, where

$$C_\omega^\pm(A) := (C_\pm^\omega(A))^* = \text{the space of continuous linear functionals on } C_\pm^\omega(A),$$

with b and B being the obvious dual maps of those defined for cyclic homology.

1.6. Odd Chern character. For each Banach algebra A, we have the generalized trace map $\mathrm{Tr} : C_n(M_r(A)) \to C_n(A)$ with

$$\mathrm{Tr}(a_0, \ldots, a_n) = \sum_{0 \le i_0, \ldots, i_n \le r} ((a_0)_{i_0 i_1}, (a_1)_{i_1 i_2}, \ldots, (a_n)_{i_n i_0}).$$

It is easy to check that the generalized trace map induces a chain complex homomorphism $C^{\omega}_{\pm}(M_r(A)) \to C^{\omega}_{\pm}(A)$. For an invertible element $g \in \mathrm{GL}_r(A)$, its Chern character is defined to be

$$\mathrm{Ch}_{\bullet}(g) := \sum_{k=0}^{\infty} (-1)^k k!\, \mathrm{Tr}(g^{-1}, g, \ldots, g^{-1}, g)_{2k+1}.$$

We have $\mathrm{Ch}_{\bullet}(g) \in C^{\omega}_{-}(A)$ and $(b + B)\,\mathrm{Ch}_{\bullet}(g) = 0$. Moreover, let $h\colon [0,1] \to \mathrm{GL}_r(A)$ be a smooth path of invertible elements. Then we have (cf. [Get93b])

$$\frac{d}{dt}\,\mathrm{Ch}_{\bullet}(h_t) = (b + B)\widetilde{\mathrm{Ch}}_{\bullet}(h, t),$$

where the secondary Chern character $\widetilde{\mathrm{Ch}}_{\bullet}(h, t)$ of h is defined as

$$\widetilde{\mathrm{Ch}}_{\bullet}(h, t) = \mathrm{Tr}(h_t^{-1}\dot{h}_t) + \sum_{k=0}^{\infty} (-1)^{k+1} k! \sum_{j=0}^{k} \mathrm{Tr}(h_t^{-1} \otimes h_t)^{\otimes(j+1)} \otimes h_t^{-1}\dot{h}_t \otimes (h_t^{-1} \otimes h_t)^{\otimes(k-j)})_{2k+2}.$$

If we denote

$$\mathrm{Tch}_{\bullet}(h) = \int_0^1 \widetilde{\mathrm{Ch}}_{\bullet}(h, t)\,dt,$$

then

$$\mathrm{Ch}_{\bullet}(h_1) - \mathrm{Ch}_{\bullet}(h_0) = (b + B)\,\mathrm{Tch}_{\bullet}(h).$$

2. JLO Chern character in b-calculus

In this section, we shall define the b-JLO Chern character and prove its entireness. Let $\widehat{M}$ be as before and $\mathcal{S}$ be the spinor bundle over $\widehat{M}$. Then $\mathcal{S}_1 = \mathcal{S} \otimes \mathbb{C}^{1|1}$ is a Clifford module over $\widehat{M}$ of degree 1, where the generator e_1 of $\mathrm{Cl}_1(\mathbb{C})$ acts on $\mathcal{S}_1$ by $\left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$. Let D be the Dirac operator on $\widehat{M}$ and write

$$\mathfrak{D} = \begin{pmatrix} 0 & D \\ D & 0 \end{pmatrix} \in {}^{\mathrm{b}}\Psi^1(\widehat{M}; \mathcal{S}_1).$$

Notice that $\mathfrak{D}$ is odd and self-adjoint, and (graded) commutes with the action of $\mathrm{Cl}_1(\mathbb{C})$.

2.1. JLO Chern character in b-calculus.

For $A \in {}^{\mathrm{b}}\Psi^m(\widehat{M}; \mathcal{S}_1)$, we define

$${}^{\mathrm{b}}\mathrm{Str}_{(1)}(A) := \frac{1}{2\sqrt{\pi}}\,{}^{\mathrm{b}}\mathrm{Str}(e_1 A).$$

More generally, for $A \in {}^{\mathrm{b}}\Psi^m(\widehat{M};\mathcal{V})$ with $\mathcal{V}$ a Clifford module of degree q, we define

$$ {}^{\mathrm{b}}\mathrm{Str}_{(q)}(A) := \frac{1}{(4\pi)^{q/2}}\,{}^{\mathrm{b}}\mathrm{Str}(e_1 \dots e_q A), $$

where $\{e_1, \dots, e_q\}$ is a set of generators of $\mathrm{Cl}_q(\mathbb{C})$.

Definition 2.1. The b-JLO Chern character of $\mathfrak{D}$ is defined to be

$$ \begin{aligned} &{}^{\mathrm{b}}\mathrm{Ch}^n(\mathfrak{D})(a_0, \dots, a_n) \\ &\quad = {}^{\mathrm{b}}\langle a_0, [\mathfrak{D}, a_1], \dots, [\mathfrak{D}, a_n]\rangle \\ &\quad = \int_{\Delta^n} {}^{\mathrm{b}}\mathrm{Str}_{(1)}(a_0 e^{-\sigma_0 \mathfrak{D}^2}[\mathfrak{D}, a_1]e^{-\sigma_1 \mathfrak{D}^2} \dots [\mathfrak{D}, a_n]e^{-\sigma_n \mathfrak{D}^2})\, d\sigma, \end{aligned} $$

where $[-,-]$ stands for the graded commutator.

A straightforward calculation gives the following lemma.

Lemma 2.2. *We have*

$$ \begin{aligned} &{}^{\mathrm{b}}\mathrm{Ch}^{2k+1}(\mathfrak{D})(a_0, \dots, a_{2k+1}) \\ &\quad = \frac{1}{\sqrt{\pi}} \int_{\Delta^{2k+1}} {}^{\mathrm{b}}\mathrm{Tr}(a_0 e^{-\sigma_0 D^2}[D, a_1]e^{-\sigma_1 D^2} \dots [D, a_{2k+1}]e^{-\sigma_{2k+1} D^2}) d\sigma. \end{aligned} $$

We see that the definition of ${}^{\mathrm{b}}\mathrm{Ch}^{\bullet}(\mathfrak{D})$ is a natural generalization of the JLO odd Chern character (for Dirac operators on closed manifolds) to the b-calculus setting.

Lemma 2.3 ([Get93b], Lemma 4.4). *For $g \in U_r(C^\infty_{\exp}(\widehat{M}))$, we have*

$$ \langle {}^{\mathrm{b}}\mathrm{Ch}^{\bullet}(\mathfrak{D}), \textstyle\sum_{k=0}^\infty k!\,\mathrm{Str}(p, \dots, p)_{2k+1}\rangle = \langle {}^{\mathrm{b}}\mathrm{Ch}^{\bullet}(\mathfrak{D}), \mathrm{Ch}_{\bullet}(g^{-1}) - \mathrm{Ch}_{\bullet}(g)\rangle, $$

where $p = \left(\begin{smallmatrix} 0 & g^{-1} \\ g & 0 \end{smallmatrix}\right) \in C^\infty_{\exp}(\widehat{M}) \otimes \mathrm{End}(\mathbb{C}^{r|r})$ with $\mathbb{C}^{r|r} = (\mathbb{C}^r)^+ \oplus (\mathbb{C}^r)^-$ being $\mathbb{Z}_2$-graded.

Proof. Notice that

$$ [\mathfrak{D}, p] = \begin{pmatrix} 0 & [\mathfrak{D}, g^{-1}] \\ [\mathfrak{D}, g] & 0 \end{pmatrix} $$

and

$$ \begin{aligned} &[\mathfrak{D}, p]e^{-\sigma \mathfrak{D}^2}[\mathfrak{D}, p]e^{-\tau \mathfrak{D}^2} \\ &\quad = \begin{pmatrix} -[\mathfrak{D}, g^{-1}]e^{-\sigma \mathfrak{D}^2}[\mathfrak{D}, g]e^{-\tau \mathfrak{D}^2} & 0 \\ 0 & -[\mathfrak{D}, g]e^{-\sigma \mathfrak{D}^2}[\mathfrak{D}, g^{-1}]e^{-\tau \mathfrak{D}^2} \end{pmatrix}. \end{aligned} $$

It follows that

$$\begin{aligned}
{}^{\mathrm{b}}\langle p, [\mathfrak{D}, p], \dots, [\mathfrak{D}, p]\rangle &= (-1)^{(k+1)}\, {}^{\mathrm{b}}\langle g^{-1}, [\mathfrak{D}, g], \dots, [\mathfrak{D}, g^{-1}], [\mathfrak{D}, g]\rangle \\
&\quad - (-1)^{(k+1)}\, {}^{\mathrm{b}}\langle g, [\mathfrak{D}, g^{-1}], \dots, [\mathfrak{D}, g], [\mathfrak{D}, g^{-1}]\rangle \\
&= (-1)^{k\,\mathrm{b}}\langle g, [\mathfrak{D}, g^{-1}], \dots, [\mathfrak{D}, g], [\mathfrak{D}, g^{-1}]\rangle \\
&\quad - (-1)^{k\,\mathrm{b}}\langle g^{-1}, [\mathfrak{D}, g], \dots, [\mathfrak{D}, g^{-1}], [\mathfrak{D}, g]\rangle .
\end{aligned}$$

Hence the lemma follows. □

2.2. Entireness of the b-JLO Chern character. For $A \in {}^{\mathrm{b}}\Psi^{-\infty}(\widehat{M}, \mathcal{V})$, we let

$$\mathrm{Tr}^M(A) := \int_M \mathrm{tr}(K_A|_\Delta) \quad \text{and} \quad {}^{\mathrm{b}}\mathrm{Tr}^{\mathrm{end}}(A) := \fint_{(-\infty,0]\times\partial M} \mathrm{tr}(K_A|_\Delta).$$

When $\mathcal{V}$ is a Clifford module of degree 1, we define

$$\mathrm{Str}^M_{(1)}(A) := \int_M \mathrm{str}_{(1)}(K_A|_\Delta) \quad \text{and} \quad {}^{\mathrm{b}}\mathrm{Str}^{\mathrm{end}}_{(1)}(A) := \fint_{(-\infty,0]\times\partial M} \mathrm{str}_{(1)}(K_A|_\Delta).$$

When $A|_{(-\infty,0]\times\partial M}$ is of trace class, we also write $\mathrm{Str}^{\mathrm{end}}_{(1)}(A)$ instead of ${}^{\mathrm{b}}\mathrm{Str}^{\mathrm{end}}_{(1)}(A)$. Now let us give an upper bound in terms of ${}^{\mathrm{b}}\|a_i\|$ for

$$\begin{aligned}
&\int_{\Delta^n} {}^{\mathrm{b}}\mathrm{Str}_{(1)}(a_0 e^{-\sigma_0\mathfrak{D}^2}[\mathfrak{D}, a_1]e^{-\sigma_1\mathfrak{D}^2} \dots [\mathfrak{D}, a_n]e^{-\sigma_n\mathfrak{D}^2})\, d\sigma \\
&\quad = \int_{\Delta^n} \mathrm{Str}^M_{(1)}(a_0 e^{-\sigma_0\mathfrak{D}^2}[\mathfrak{D}, a_1]e^{-\sigma_1\mathfrak{D}^2} \dots [\mathfrak{D}, a_n]e^{-\sigma_n\mathfrak{D}^2})\, d\sigma && (2.1)\\
&\qquad + \int_{\Delta^n} {}^{\mathrm{b}}\mathrm{Str}^{\mathrm{end}}_{(1)}(a_0 e^{-\sigma_0\mathfrak{D}^2}[\mathfrak{D}, a_1]e^{-\sigma_1\mathfrak{D}^2} \dots [\mathfrak{D}, a_n]e^{-\sigma_n\mathfrak{D}^2})\, d\sigma && (2.2)
\end{aligned}$$

For the first summand (2.1), by standard differential calculus on compact manifolds, one has

$$\left|\int_{\Delta^n} \mathrm{Str}^M_{(1)}(a_0 e^{-\sigma_0\mathfrak{D}^2} \dots [\mathfrak{D}, a_n]e^{-\sigma_n\mathfrak{D}^2}) d\sigma\right| \le \mathrm{Tr}^M(e^{-\mathfrak{D}^2})\, {}^{\mathrm{b}}\|a_0\|^{\mathrm{b}}\|a_1\| \dots {}^{\mathrm{b}}\|a_n\|,$$

cf. [GS89], Lemma 2.1.

For the second summand (2.2), first notice that on $(-\infty, 0] \times \partial M$,

$$[\mathfrak{D}, a] = c(da_c) + e^x[c(a_\infty dx) + c(da_\infty)] = C + e^x B,$$

where $C = c(da_c)$, $B = c(a_\infty dx) + c(da_\infty)$ and $c(-)$ stands for the Clifford multiplication. Similarly, we write

$$[\mathfrak{D}, a_i] = C_i + e^x B_i \quad \text{and} \quad a_0 = C_0 + e^x B_0,$$

where C_i is constant along the normal direction x. Notice that $\|B_i\| \le {}^{\mathrm{b}}\|a_i\|$ and $\|C_i\| \le {}^{\mathrm{b}}\|a_i\|$. Hence the term (2.2) is a sum of the following two types:

$$\text{(I)} \int_{\Delta^n} {}^{\mathrm{b}}\mathrm{Str}^{\mathrm{end}}_{(1)}(C_0 e^{-\sigma_0 \mathfrak{D}^2} \dots C_n e^{-\sigma_n \mathfrak{D}^2})\, d\sigma,$$

$$\text{(II)} \int_{\Delta^n} {}^{\mathrm{b}}\mathrm{Str}^{\mathrm{end}}_{(1)}(C_0 e^{-\sigma_0 \mathfrak{D}^2} \dots e^{-\sigma_i \mathfrak{D}^2} e^x B_i\, e^{-\sigma_{i+1} \mathfrak{D}^2} \dots C_n e^{-\sigma_n \mathfrak{D}^2})\, d\sigma.$$

Let us denote the Dirac operator on $\mathbb{R} \times \partial M$ by $D_{\mathbb{R}}$ and write $\mathfrak{D}_{\mathbb{R}} = \begin{pmatrix} 0 & D_{\mathbb{R}} \\ D_{\mathbb{R}} & 0 \end{pmatrix}$. By [LMP09], Proposition 3.1, $(e^{-\sigma \mathfrak{D}_{\mathbb{R}}^2} - e^{-\sigma \mathfrak{D}^2})|_{(-\infty,0]\times\partial M}$ is of trace class and there is a constant $\mathcal{K}_0$ such that

$$\left|\mathrm{Tr}(e^{-\sigma \mathfrak{D}_{\mathbb{R}}^2} - e^{-\sigma \mathfrak{D}^2})|_{(-\infty,0]\times\partial M}\right| \le \mathcal{K}_0 \quad \text{for all } 0 \le \sigma \le 1. \tag{2.3}$$

Type I. Since $\|e^{-\sigma \mathfrak{D}_{\mathbb{R}}^2}\| \le 1$ and $\|e^{-\sigma \mathfrak{D}^2}\| \le 1$, one has

$$\begin{aligned} &\left|{}^{\mathrm{b}}\mathrm{Str}^{\mathrm{end}}_{(1)}(C_0 e^{-\sigma_0 \mathfrak{D}^2} \dots C_n e^{-\sigma_n \mathfrak{D}^2}|_{(-\infty,0]\times\partial M})\right| \\ &\quad \le (n+1)\mathcal{K}_0 \prod_{i=0}^{n} \|C_i\| + \left|{}^{\mathrm{b}}\mathrm{Str}^{\mathrm{end}}_{(1)}(C_0 e^{-\sigma_0 \mathfrak{D}_{\mathbb{R}}^2} \dots C_n e^{-\sigma_n \mathfrak{D}_{\mathbb{R}}^2}|_{(-\infty,0]\times\partial M})\right| \\ &\quad = (n+1)\mathcal{K}_0 \prod_{i=0}^{n} \|C_i\|, \end{aligned}$$

where the last equality follows from the fact

$${}^{\mathrm{b}}\mathrm{Str}^{\mathrm{end}}_{(1)}(C_0 e^{-\sigma_0 \mathfrak{D}_{\mathbb{R}}^2} \dots C_n e^{-\sigma_n \mathfrak{D}_{\mathbb{R}}^2}|_{(-\infty,0]\times\partial M}) = 0$$

by the definition of the b-trace.

Type II. Due to the presence of the factor e^x,

$$C_0 e^{-\sigma_0 \mathfrak{D}^2} \dots e^{-\sigma_i \mathfrak{D}^2} e^x B_i\, e^{-\sigma_{i+1} \mathfrak{D}^2} \dots C_n e^{-\sigma_n \mathfrak{D}^2}$$

is of trace class.

Without loss of generality, it suffices to give an upper bound for

$$\mathrm{Str}^{\mathrm{end}}_{(1)}(e^x B_0 e^{-\sigma_0 \mathfrak{D}^2} A_1 e^{-\sigma_1 \mathfrak{D}^2} \dots A_n e^{-\sigma_n \mathfrak{D}^2}),$$

where $A_i = B_i$ or C_i as defined above for $1 \le i \le n$. First by the inequality (2.3),

$$\begin{aligned} &|\mathrm{Str}^{\mathrm{end}}_{(1)}(e^x B_0 e^{-\sigma_0 \mathfrak{D}^2} A_1 e^{-\sigma_1 \mathfrak{D}^2} \dots A_n e^{-\sigma_n \mathfrak{D}^2})| \\ &\qquad \le (n+1)\mathcal{K}_0 \|B_0\| \prod_{i=1}^{n} \|A_i\| \\ &\qquad\quad + |\mathrm{Str}^{\mathrm{end}}_{(1)}(e^x B_0 e^{-\sigma_0 \mathfrak{D}_{\mathbb{R}}^2} A_1 e^{-\sigma_1 \mathfrak{D}_{\mathbb{R}}^2} \dots A_n e^{-\sigma_n \mathfrak{D}_{\mathbb{R}}^2}|_{(-\infty,0]\times\partial M})|. \end{aligned}$$

Now we can rewrite

$$\begin{aligned} &e^x B_0 e^{-\sigma_0 \mathfrak{D}_{\mathbb{R}}^2} A_1 e^{-\sigma_1 \mathfrak{D}_{\mathbb{R}}^2} \dots A_n e^{-\sigma_n \mathfrak{D}_{\mathbb{R}}^2} \\ &\quad = (e^x B_0 e^{-\sigma_0 \mathfrak{D}_{\mathbb{R}}^2} e^{-\beta_1 x})(e^{\beta_1 x} A_1 e^{-\sigma_1 \mathfrak{D}_{\mathbb{R}}^2} e^{-\beta_2 x}) e^{\beta_2 x} \dots e^{-\beta_n x}(e^{\beta_n x} A_n e^{-\sigma_n \mathfrak{D}_{\mathbb{R}}^2}), \end{aligned}$$

with $1 > \beta_1 > \beta_2 > \cdots > \beta_n > 0$. By [LMP09], Proposition 3.7, there is a constant $\mathcal{K}'$ such that

$$\|e^{\beta_i x} A_i e^{-\sigma_i \mathfrak{D}_{\mathbb{R}}^2} e^{-\beta_{i+1} x}\|_{\sigma_i^{-1}} \leq \mathcal{K}' \|A_i\| (\beta_i - \beta_{i+1})^{-\sigma_i} (\sigma_i^{-\frac{\dim M + 1}{2}\sigma_i})$$

for all i. Notice that

$$\sigma^{-\frac{\dim M+1}{2}\sigma} = e^{-\frac{\dim M+1}{2}\sigma \ln(\sigma)}$$

is bounded on $[0,1]$. If we take $\beta_i = (n+1-i)/(n+1)$, then by the Hölder inequality one has

$$\begin{aligned} &\left|\mathrm{Str}_{(1)}^{\mathrm{end}}(e^x B_0 e^{-\sigma_0 \mathfrak{D}_{\mathbb{R}}^2} A_1 e^{-\sigma_1 \mathfrak{D}_{\mathbb{R}}^2} \ldots A_n e^{-\sigma_n \mathfrak{D}_{\mathbb{R}}^2}|_{(-\infty,0]\times\partial M})\right| \\ &\qquad \leq \mathcal{K}_1^{n+1}(n+1)\|B_0\| \prod_{i=1}^{n} \|A_i\| \end{aligned}$$

for some fixed constant $\mathcal{K}_1$.

Applying the estimates above, we have the following proposition.

Proposition 2.4. ${}^{\mathrm{b}}\mathrm{Ch}^{\bullet}(\mathfrak{D})$ *is an entire cyclic cochain.*

Proof. We have

$$\begin{aligned} &|{}^{\mathrm{b}}\mathrm{Ch}^n(\mathfrak{D})(a_0,\ldots,a_n)| \\ &\quad = |{}^{\mathrm{b}}\langle a_0, [\mathfrak{D}, a_1], \ldots, [\mathfrak{D}, a_n]\rangle| \\ &\quad = \left|\int_{\Delta^n} {}^{\mathrm{b}}\mathrm{Str}_{(1)}(a_0 e^{-\sigma_0 \mathfrak{D}^2}[\mathfrak{D}, a_1] e^{-\sigma_1 \mathfrak{D}^2} \ldots [\mathfrak{D}, a_n] e^{-\sigma_n \mathfrak{D}^2}) d\sigma\right| \\ &\quad \leq \frac{2^n (n+1)(\mathcal{K}_1^n + 2\mathcal{K}_0)}{n!} {}^{\mathrm{b}}\|a_0\| {}^{\mathrm{b}}\|a_1\| \ldots {}^{\mathrm{b}}\|a_n\|. \end{aligned}$$

It follows that ${}^{\mathrm{b}}\mathrm{Ch}^{\bullet}(\mathfrak{D})$ defines a continuous linear functional on $C_{-}^{\omega}(A)$, i.e., an entire cyclic cochain in $C_{\omega}^{-}(A)$. □

3. Odd APS index theorem for manifolds with boundary

In this section, we shall state and prove the main theorem of this paper. Let $\widehat{M}$ be an odd-dimensional spin b-manifold with a b-metric as before and D its associated Dirac operator. Recall that the spinor bundle $\mathcal{S}$ of $\widehat{M}$ naturally induces a Clifford module of degree 1, denoted by $\mathcal{S} \otimes \mathbb{C}^{1|1}$, where the generator e_1 of $\mathrm{Cl}_1(\mathbb{C})$ acts on $\mathcal{S} \otimes \mathbb{C}^{1|1}$ by $\left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$, where $\mathbb{C}^{1|1} = \mathbb{C}^+ \oplus \mathbb{C}^-$ is $\mathbb{Z}_2$-graded. We put

$$\mathfrak{D} = \begin{pmatrix} 0 & D \\ D & 0 \end{pmatrix} \quad \text{and} \quad \mathfrak{D}_t = t\mathfrak{D}.$$

We define

$$ {}^{\mathrm{b}}\mathrm{Ch}^n(\mathfrak{D},t)(a_0,a_1,\dots,a_n) := {}^{\mathrm{b}}\langle\!\langle a_0,[\mathfrak{D}_t,a_1],\dots,[\mathfrak{D}_t,a_n]\rangle\!\rangle $$

for $a_i \in C^\infty_{\exp}(\widehat{M})$. Here

$$ {}^{\mathrm{b}}\langle\!\langle A_0,A_1,\dots,A_n\rangle\!\rangle := \int_{\Delta^n} {}^{\mathrm{b}}\mathrm{Str}_{(1)}(A_0e^{-\sigma_0(d\mathfrak{D}_t+\mathfrak{D}_t^2)}\dots A_ne^{-\sigma_n(d\mathfrak{D}_t+\mathfrak{D}_t^2)})\,d\sigma, $$

with

$$ e^{-s(d\mathfrak{D}_t+\mathfrak{D}_t^2)} := \sum_{k=0}^{\infty}(-s)^k\int_{\Delta^k} e^{-\sigma_0 s\mathfrak{D}_t^2}d\mathfrak{D}_te^{-\sigma_1 s\mathfrak{D}_t^2}\dots d\mathfrak{D}_te^{-\sigma_k s\mathfrak{D}_t^2}\,d\sigma. $$

Notice that (cf. [Get93a], Lemma 2.5)

$$ {}^{\mathrm{b}}\langle\!\langle A_0,A_1,\dots,A_n\rangle\!\rangle = {}^{\mathrm{b}}\langle A_0,\dots,A_n\rangle - \sum_{i=0}^{n}{}^{\mathrm{b}}\langle A_0,\dots,A_i,dt\mathfrak{D},A_{i+1},\dots,A_n\rangle $$

Therefore

$$ \begin{aligned} &{}^{\mathrm{b}}\mathrm{Ch}^k(\mathfrak{D},t)(a_0,\dots,a_n)\\ &\quad= {}^{\mathrm{b}}\langle a_0,[\mathfrak{D}_t,a_1],\dots,[\mathfrak{D}_t,a_n]\rangle\\ &\qquad-\sum_{i=0}^{n}(-1)^i dt\,{}^{\mathrm{b}}\langle a_0,[\mathfrak{D}_t,a_1],\dots,[\mathfrak{D}_t,a_i],\mathfrak{D},[\mathfrak{D}_t,a_{i+1}],\dots,[\mathfrak{D}_t,a_n]\rangle. \end{aligned} $$

Recall that

$$ {}^{\mathrm{b}}\mathrm{Tr}[Q,K] = \frac{i}{2\pi}\int_{-\infty}^{\infty}{}^{\partial}\mathrm{Tr}\left(\frac{d\,\mathrm{I}(Q,\lambda)}{d\lambda}\,\mathrm{I}(K,\lambda)\right)d\lambda $$

if either Q or K is in $\Psi_b^{-\infty}(\widehat{M},\mathcal{V})$, where $\mathrm{I}(Q,\lambda)$ (resp. $\mathrm{I}(K,\lambda)$) is the indicial family of Q (resp. K), cf. [Loy05], Theorem 2.5. For the Dirac operator D above, we have

$$ I(D,\lambda) = {}^{\partial}D + i\lambda c(\nu), $$

where $\nu = dx$ is the normal cotangent vector and $c(\nu)$ is the Clifford multiplication of ν, cf. [Get93b], Proposition 5.4. In the following identities ([Get93a], Lemma 6.3), we assume that the indicial family $I(A_i,\lambda)$ of A_i is independent of λ and commutes with $\mathrm{Cl}_1(\mathbb{C})$ and $c(v)$. Let us denote the degree of A_i with respect to the $\mathbb{Z}_2$-grading by $|A_i|$.

(1) If we denote $\varepsilon_i = (|A_0|+\dots+|A_{i-1}|)(|A_i|+\dots+|A_n|)$, then

$$ {}^{\mathrm{b}}\langle\!\langle A_0,A_1,\dots,A_n\rangle\!\rangle = (-1)^{\varepsilon_i}\,{}^{\mathrm{b}}\langle\!\langle A_i,\dots,A_n,A_0,\dots,A_{i-1}\rangle\!\rangle. $$

(2) $\sum_{i=0}^{n}(-1)^{\varepsilon_i}\,{}^{\mathrm{b}}\langle\!\langle 1,A_i,\dots,A_n,A_0,\dots,A_{i-1}\rangle\!\rangle = {}^{\mathrm{b}}\langle\!\langle A_0,A_1,\dots,A_n\rangle\!\rangle.$

(3) We have

$$
\begin{aligned}
&{}^{\mathrm{b}}\langle\!\langle A_0, \dots, A_{i-1}, [d\mathfrak{D}_t + \mathfrak{D}_t^2, A_i], A_{i+1}, \dots, A_n \rangle\!\rangle \\
&\qquad = {}^{\mathrm{b}}\langle\!\langle A_0, \dots, A_{i-1}A_i, \dots, A_n \rangle\!\rangle - {}^{\mathrm{b}}\langle\!\langle A_0, \dots, A_i A_{i+1}, \dots, A_n \rangle\!\rangle.
\end{aligned}
$$

(4) Write $\delta_i = |A_0| + \cdots + |A_{i-1}|$. Then

$$
\begin{aligned}
&d\,{}^{\mathrm{b}}\langle\!\langle A_0, A_1, \dots, A_n \rangle\!\rangle - \langle\langle {}^{\partial}A_0, {}^{\partial}A_1, \dots, {}^{\partial}A_n \rangle\!\rangle \\
&\qquad = \sum_{i=0}^{n} (-1)^{\delta_i}\, {}^{\mathrm{b}}\langle\!\langle A_0, A_1, \dots, A_{i-1}, [d + \mathfrak{D}_t, A_i], A_{i+1}, \dots, A_n \rangle\!\rangle.
\end{aligned}
$$

Here we define

$$
\begin{aligned}
&{}^{\partial}\langle\!\langle {}^{\partial}A_0, {}^{\partial}A_1, \dots, {}^{\partial}A_n \rangle\!\rangle \\
&\qquad = \int_{\Delta^n} {}^{\mathrm{b}}\mathrm{Str}_{(2)}\big({}^{\partial}A_0 e^{-\sigma_0(d\,{}^{\partial}\mathfrak{D}_t + {}^{\partial}\mathfrak{D}_t^2)} \dots {}^{\partial}A_n e^{-\sigma_n(d\,{}^{\partial}\mathfrak{D}_t + {}^{\partial}\mathfrak{D}_t^2)}\big)\, d\sigma,
\end{aligned}
$$

where

$$
{}^{\mathrm{b}}\mathrm{Str}_{(2)}(B) = \frac{1}{4\pi}\, {}^{\mathrm{b}}\mathrm{Str}\,(e_1 e_2 B) \quad \text{with } e_1 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},\ e_2 = \begin{pmatrix} 0 & c(\nu) \\ c(\nu) & 0 \end{pmatrix}
$$

and

$$
{}^{\partial}\mathfrak{D}_t = t\,{}^{\partial}\mathfrak{D} \quad \text{with } {}^{\partial}\mathfrak{D} = \begin{pmatrix} 0 & {}^{\partial}D \\ {}^{\partial}D & 0 \end{pmatrix}.
$$

Now if we let $A_0 = a_0$ and $A_i = [\mathfrak{D}_t, a_i]$ for $1 \le i \le n$, where $a_i \in C^{\infty}_{\exp}(\hat{M})$, then it is easy to see that

$$
[d + \mathfrak{D}_t, A_0] = [d + \mathfrak{D}_t, a_0] = [\mathfrak{D}_t, a_0]
$$

and

$$
[d + \mathfrak{D}_t, A_i] = [d + \mathfrak{D}_t, [\mathfrak{D}_t, a_i]] = [d\mathfrak{D}_t + \mathfrak{D}_t^2, a_i] \quad \text{with } 1 \le i \le n.
$$

Therefore, by applying the identities above to $A_0 = a_0$ and $A_i = [\mathfrak{D}_t, a_i]$, we see that ([Get93a], Theorem 6.2)

$$
(d - b - B)\,{}^{\mathrm{b}}\mathrm{Ch}^{\bullet}(\mathfrak{D}, t) = \mathrm{Ch}^{\bullet}({}^{\partial}\mathfrak{D}, t), \tag{3.1}
$$

where

$$
\mathrm{Ch}^n({}^{\partial}\mathfrak{D}, t)({}^{\partial}a_0, {}^{\partial}a_1, \dots, {}^{\partial}a_n) = {}^{\partial}\langle\!\langle {}^{\partial}a_0, [{}^{\partial}\mathfrak{D}_t, {}^{\partial}a_1], \dots, [{}^{\partial}\mathfrak{D}_t, {}^{\partial}a_n] \rangle\!\rangle.
$$

Let $\alpha \in \Omega^*(0, \infty)$ be the differential form $\alpha = \langle {}^{\mathrm{b}}\mathrm{Ch}^{\bullet}(\mathfrak{D}, t), \mathrm{Ch}_{\bullet}(g) \rangle$. Then

$$
d\alpha = \langle \mathrm{Ch}^{\bullet}({}^{\partial}\mathfrak{D}, t), \mathrm{Ch}_{\bullet}({}^{\partial}g) \rangle.
$$

By the fundamental theorem of calculus,

$$\alpha(t_2) - \alpha(t_1) = \langle \int_{t_1}^{t_2} \mathrm{Ch}^\bullet({}^\partial\mathfrak{D}, t), \mathrm{Ch}_\bullet({}^\partial g) \rangle.$$

Therefore, if both of the following limits exist, then

$$\lim_{t\to\infty} \alpha(t) - \lim_{t\to 0} \alpha(t) = -\langle \eta^\bullet({}^\partial\mathfrak{D}), \mathrm{Ch}_\bullet({}^\partial g)\rangle.$$

Here $\eta^\bullet({}^\partial\mathfrak{D}) = \int_0^\infty \mathrm{Ch}^\bullet({}^\partial\mathfrak{D}, t)$, which is the higher eta cochain of Wu [Wu93], although our normalization factor is different from that of Wu. More explicitly,

$$\eta^{2k+1}({}^\partial\mathfrak{D})(a_0, a_1, \ldots, a_{2k+1}) = \frac{1}{2\pi} \sum_{i=0}^{2k+1} (-1)^i \int_0^\infty dt\, {}^\partial\langle a_0, [{}^\partial D_t, a_1], \ldots$$
$$\ldots, [{}^\partial D_t, a_i], {}^\partial D, [{}^\partial D_t, a_{i+1}], \ldots, [{}^\partial D_t, a_{2k+1}]\rangle_t$$

where

$${}^\partial\langle A_0, \ldots, A_n\rangle_t = \int_{\Delta^n} \mathrm{Str}(A_0 e^{-\sigma_0 {}^\partial D_t^2} \ldots A_n e^{-\sigma_n {}^\partial D_t^2})\, d\sigma.$$

Let us also write $\eta^\bullet({}^\partial D) = \eta^\bullet({}^\partial\mathfrak{D})$. This higher eta cochain $\eta^\bullet({}^\partial D)$ has a finite radius of convergence, cf. [Wu93], Proposition 1.5. In order for

$$\langle \eta^\bullet({}^\partial D), \mathrm{Ch}_\bullet({}^\partial g)\rangle$$

to converge, we shall make the following assumptions throughout the rest of this paper:

(1) *${}^\partial D$ is invertible and the lowest eigenvalue of $|{}^\partial D|$ is λ;*

(2) $\|\,[{}^\partial D, {}^\partial g]\,\| = \|d\, {}^\partial g\| < \lambda$.

Let us denote by

$$\mathrm{Ch}_\bullet^{\mathrm{dR}}(g) := \sum_{k=0}^\infty \frac{1}{(2\pi i)^{k+1}} \frac{k!}{(2k+1)!} \mathrm{tr}((g^{-1}dg)^{2k+1}) \in \Omega^*(\widehat{M})$$

the Chern character of g in the de Rham cohomology of $\widehat{M}$. Then we have the following main theorem of this paper.

Theorem 3.1. *Let $\widehat{M}$ be an odd-dimensional spin b-manifold with a b-metric and D its associated Dirac operator. Assume that ${}^\partial D$ is invertible. Let $g \in U_k(C^\infty(\widehat{M}))$ be a unitary over $\widehat{M}$. If $\|d\,{}^\partial g\| < \lambda$, where λ is the lowest nonzero eigenvalue of $|{}^\partial D|$, then*

$$\mathrm{sf}(D, g^{-1}Dg) = {}^b\!\!\int_{\widehat{M}} \hat{A}(\widehat{M}) \wedge \mathrm{Ch}_\bullet^{\mathrm{dR}}(g) - \langle \eta^\bullet({}^\partial D), \mathrm{Ch}_\bullet({}^\partial g)\rangle.$$

Here $\hat{A}(\widehat{M}) = \det\left(\frac{{}^b\nabla^2/4\pi i}{\sinh {}^b\nabla^2/4\pi i}\right)^{1/2}$, with ${}^b\nabla$ the Levi-Civita b-connection associated to the b-metric on $\widehat{M}$.

Proof. We need to identify the limits of $\alpha(t)$ for $t = \infty$ and $t = 0$. In the case of closed manifolds, the local formula for the limit of $\alpha(t)$ as $t \to 0$ follows from Getzler's asymptotic calculus, cf. [BF90], [CM90], [Get83]. A direct calculation in the b-calculus setting is carried out in [LMP09], Sections 5 and 6. In particular, we have

$$\lim_{t\to 0}\alpha(t) = \lim_{t\to 0}\langle {}^{\mathrm{b}}\mathrm{Ch}^{\bullet}(\mathfrak{D}), \mathrm{Ch}_{\bullet}(g)\rangle = {}^{\mathrm{b}}\!\!\int_{\widehat{M}} \hat{A}(\widehat{M}) \wedge \mathrm{Ch}_{\bullet}^{\mathrm{dR}}(g).$$

Now the theorem follows immediately once we have

$$\lim_{t\to\infty}\alpha(t) = \mathrm{sf}(D, g^{-1}Dg),$$

which we will prove in Proposition 5.8 below. □

Corollary 3.2. *With the same notation as above,*

$$\langle \eta^{\bullet}({}^{\partial}D), \mathrm{Ch}_{\bullet}({}^{\partial}g)\rangle = -\bar{\eta}(\partial\widehat{M}, {}^{\partial}g) \mod \mathbb{Z}.$$

Proof. Here $\bar{\eta}(\partial\widehat{M}, {}^{\partial}g)$ is the eta invariant of Dai and Zhang [DZ06]. Without loss of generality, we can assume that the unitary g is constant along the normal direction of the cylindrical end. In this case, we have

$${}^{\mathrm{b}}\!\!\int_{\widehat{M}} \hat{A}(\widehat{M}) \wedge \mathrm{Ch}_{\bullet}^{\mathrm{dR}}(g) = \int_M \hat{A}(M) \wedge \mathrm{Ch}_{\bullet}^{\mathrm{dR}}(g)$$

by definition of the regularized integral. Now comparing the above theorem with the Toeplitz index theorem on odd-dimensional manifolds by Dai and Zhang [DZ06], Theorem 2.3, we have

$$\langle \eta^{\bullet}({}^{\partial}D), \mathrm{Ch}_{\bullet}({}^{\partial}g)\rangle = -\bar{\eta}(\partial\widehat{M}, {}^{\partial}g) \mod \mathbb{Z}. \qquad \square$$

This equality provides more evidence for the naturality of the Dai–Zhang eta invariant for even-dimensional closed manifolds.

4. Spectral flow

In this section, we proceed to explain the notion of spectral flow and prove an analogue of Getzler's formula for spectral flow (cf. [Get93b], Corollary 2.7) in the b-calculus setting.

Following Booss-Bavnbek, Lesch and Phillips [BBLP05], we define the notion of spectral flow as follows.

Definition 4.1. Let $T_u \colon \mathcal{H} \to \mathcal{H}$ for $u \in [0,1]$ be a continuous path of (possibly unbounded) self-adjoint Fredholm operators. Then its spectral flow, denoted by $\mathrm{sf}(T_u)_{0\le u\le 1}$, is defined by

$$\mathrm{sf}(T_u)_{0\le u\le 1} = \mathrm{wind}(\kappa(T_u)_{0\le u\le 1}),$$

where $\kappa(T) = (T-i)(T+i)^{-1}$ is the Cayley transform of T and wind $(\kappa(T_u)_{0\leq u\leq 1})$ is the winding number of the path $\kappa(T_u)_{0\leq u\leq 1}$ (see also [KL04], Section 6). We also write $\mathrm{sf}(T_0, T_1)$ for the spectral flow if it is clear what the path is from the context.

Actually, in this paper, where we are concerned with smooth paths of self-adjoint Fredholm operators, we use the following equivalent working definition of the spectral flow (cf. [BBLP05, Section 2.2]). Let $T_u\colon \mathcal{H} \to \mathcal{H}$ for $u \in [0,1]$ be a smooth path of (possibly unbounded) self-adjoint Fredholm operators. For a fixed $u_0 \in [0,1]$, there exists $(a,b) \subset [0,1]$ such that

(1) $u_0 \in (a,b)$ (unless $u_0 = 0$ or 1, in which case $u_0 = a = 0$ if $u_0 = 0$, and $u_0 = b = 1$ if $u_0 = 1$);

(2) $\dim\ker(T_u) \leq \dim\ker(T_{u_0})$ for all $u \in (a,b)$.

By shrinking the neighborhood (a,b) if necessary, we can assume that the essential spectrum of $|T_u|$ for $u \in (a,b)$ is bounded below uniformly by λ_0 and the spectrum of T_u in $(-\lambda_0, \lambda_0)$ consists of discrete eigenvalues. We can further assume T_u has the same number of eigenvalues (counted with multiplicities) in $(-\lambda_0, \lambda_0)$, for all $u \in (a,b)$. By perturbation theory of linear operators (cf. [Kat95], II.6, V.4.3, VII.3), there are smooth functions β_k on (a,b) such that $\{\beta_k(u)\}_k$ gives a complete set of eigenvalues of T_u in $(-\lambda_0, \lambda_0)$. Let n_b (resp. n_a) be the number of nonnegative eigenvalues of T_b (resp. T_a) in $(-\lambda_0, \lambda)$. Then we define the spectral flow of $(T_u)_{a\leq u\leq b}$ to be

$$\mathrm{sf}(T_u)_{a\leq u\leq b} := (n_b - n_a). \tag{4.1}$$

We call an interval (a,b) as above together with $u_0 \in (a,b)$ a pointed gap interval. It is easy to see that the formula (4.1) is additive with respect to disjoint pointed gap intervals. Let us cover $[0,1]$ by finitely many intervals, say $[a_i, b_i]_{0\leq i\leq n}$ such that each (a_i, b_i) is a pointed gap interval, with $b_i = a_{i+1}$, $u_0 = a_0 = 0$, $u_n = b_n = 1$ and $u_j \in (a_j, b_j)$ for $1 \leq j \leq n-1$. Then we define

$$\mathrm{sf}(T_u)_{0\leq u\leq 1} := \sum_{j=0}^{n} \mathrm{sf}(T_u)_{a_j\leq u\leq b_j}.$$

By additivity of formula (4.1), we see that $\mathrm{sf}(T_u)_{0\leq u\leq 1}$ so defined is independent of the choice of pointed gap intervals.

Let $\widehat{M}$ be an odd-dimensional spin b-manifold with a b-metric as before and D its associated Dirac operator. Let $D_u = (1-u)D + ug^{-1}Dg$. Since by assumption $\|[{}^{\partial}D, {}^{\partial}g]\| < \lambda$, we see that ${}^{\partial}D_u$ is invertible for all $u \in [0,1]$. It follows that $\inf \mathrm{spec}_{\mathrm{ess}}(|D_u|) > 0$ for all $u \in [0,1]$. Thus $\{D_u\}_{0\leq u\leq 1}$ is an analytic family of self-adjoint Fredholm operators.

Following from the discussion above, we see that for each fixed $u_0 \in [0,1]$, there exists $(a,b) \subset [0,1]$ and $\lambda_0 > 0$ such that the spectrum of D_u in $(-\lambda_0, \lambda_0)$ consists of discrete eigenvalues for all $u \in (a,b)$. Moreover, we can assume D_u has the same number of eigenvalues in $(-\lambda_0, \lambda_0)$ for all $u \in (a, b)$. We put

$$\begin{aligned} A_u &:= D_u P_u, \\ B_u &:= D_u(I - P_u) + P_u, \\ C_u &:= D_u(I - P_u), \end{aligned} \tag{4.2}$$

where P_u is the spectral projection of D_u on $(-\lambda_0, \lambda_0)$. Let β_k be the smooth functions on (a, b) such that $\{\beta_k(u)\}_k$ gives the complete set of eigenvalues (counted with multiplicities) of A_u. Since $\{D_u\}_{0\le u\le 1}$ is an analytic family of operators, β_k is an analytic function of $u \in (a, b)$. It follows that for each k, β_k either has only finitely many isolated zeroes or is itself constantly zero. Hence by shrinking (a, b) as much as needed, we can assume that β_k either is a constant zero function or has only one zero in (a, b). In the latter case, by shrinking (a, b) again if necessary, we can assume that the isolated zeros can only occur at u_0. Moreover, for each $u \in (a, b)$, there is a set of orthonormal eigenvectors $\{\phi_k(u)\}_{1\le k\le m}$ such that $A_u\phi_k(u) = \beta_k(u)\phi_k(u)$ and the vector-valued function ϕ_k is analytic with respect to u for each $1 \le k \le m$.

Following Getzler [Get93b], we define the truncated eta invariant of D to be

$$\eta_\varepsilon(D) := \frac{1}{\sqrt{\pi}} \int_\varepsilon^\infty {}^{\mathrm{b}}\mathrm{Tr}(De^{-sD^2}) s^{-1/2} ds = \frac{2}{\sqrt{\pi}} \int_\varepsilon^\infty {}^{\mathrm{b}}\mathrm{Tr}(De^{-t^2D^2}) dt$$

and the reduced (truncated) eta invariant of D to be

$$\xi_\varepsilon(D) = \frac{\eta_\varepsilon(D) + \dim\ker(D)}{2}.$$

The following lemma is a natural extension of [Get93b], Proposition 2.5, to the b-calculus setting.

Lemma 4.2. *We have*

$$\frac{d\eta_\varepsilon(B_u)}{du} = -\frac{2\varepsilon}{\sqrt{\pi}} {}^{\mathrm{b}}\mathrm{Tr}(\dot{B}_u e^{-\varepsilon^2 B_u^2}) + E_\varepsilon(u),$$

where $E_\varepsilon(u)$ is defined by

$$\begin{aligned} E_\varepsilon(u) = &-\frac{2}{\sqrt{\pi}} \int_\varepsilon^\infty \int_0^1 t^{2\mathrm{b}}\mathrm{Tr}[e^{-st^2B_u^2} B_u^2, \dot{B}_u e^{-(1-s)t^2B_u^2}] ds dt \\ &- \frac{2}{\sqrt{\pi}} \int_\varepsilon^\infty \int_0^1 t^{2\mathrm{b}}\mathrm{Tr}[e^{-st^2B_u^2} B_u, \dot{B}_u B_u e^{-(1-s)t^2B_u^2}] ds dt. \end{aligned}$$

Proof. Using Duhamel's principle, we have

$$\begin{aligned}
&\frac{d}{du}\eta_\varepsilon(B_u)\\
&\quad=\frac{2}{\sqrt{\pi}}\int_\varepsilon^\infty {}^{\mathrm{b}}\mathrm{Tr}(\dot{B}_u e^{-t^2B_u^2})\,dt\\
&\qquad-\frac{2}{\sqrt{\pi}}\int_\varepsilon^\infty\int_0^1 {}^{\mathrm{b}}\mathrm{Tr}(B_u e^{-st^2B_u^2}t^2(B_u\dot{B}_u+\dot{B}_uB_u)e^{-(1-s)t^2B_u^2})\,dsdt\\
&\quad=\frac{2}{\sqrt{\pi}}\int_\varepsilon^\infty {}^{\mathrm{b}}\mathrm{Tr}(\dot{B}_u e^{-t^2B_u^2})dt-\frac{4}{\sqrt{\pi}}\int_\varepsilon^\infty {}^{\mathrm{b}}\mathrm{Tr}(t^2\dot{B}_uB_u^2e^{-t^2B_u^2})\,dt+E_\varepsilon(u).
\end{aligned}$$

Integration by parts shows that

$$\begin{aligned}
&\int_\varepsilon^\infty {}^{\mathrm{b}}\mathrm{Tr}(t^2\dot{B}_uB_u^2e^{-t^2B_u^2})\,dt\\
&\qquad=-\frac{1}{2}\int_\varepsilon^\infty t\frac{d}{dt}{}^{\mathrm{b}}\mathrm{Tr}(\dot{B}_u e^{-t^2B_u^2})dt\\
&\qquad=-\frac{1}{2}{}^{\mathrm{b}}\mathrm{Tr}(\dot{B}_u e^{-t^2B_u^2})t\Big|_{t=\varepsilon}^{t=\infty}+\frac{1}{2}\int_\varepsilon^\infty {}^{\mathrm{b}}\mathrm{Tr}(\dot{B}_u e^{-t^2B_u^2})\,dt.
\end{aligned}$$

Since B_u is invertible, ${}^{\mathrm{b}}\mathrm{Tr}(t\dot{B}_u e^{-t^2B_u^2})$ goes to 0 as $t\to\infty$. It follows that

$$\frac{d}{du}\eta_\varepsilon(B_u)=-\frac{2\varepsilon}{\sqrt{\pi}}{}^{\mathrm{b}}\mathrm{Tr}(\dot{B}_u e^{-\varepsilon^2B_u^2})+E_\varepsilon(u).\qquad\square$$

Corollary 4.3. *For $u\in(a,b)$,*

$$\frac{d\eta_\varepsilon(C_u)}{du}=-\frac{2\varepsilon}{\sqrt{\pi}}{}^{\mathrm{b}}\mathrm{Tr}(\dot{C}_u e^{-\varepsilon^2C_u^2})+E_\varepsilon(u)$$

Proof. By definition, we have

$$\eta_\varepsilon(C_u)=\eta_\varepsilon(B_u)-K\int_\varepsilon^\infty e^{-t^2}dt,$$

where $K=\mathrm{rank}(P_u)$ is independent of $u\in(a,b)$. Thus $\frac{d}{du}\eta_\varepsilon(C_u)=\frac{d}{du}\eta_\varepsilon(B_u)$. Notice that

$$\begin{aligned}
&{}^{\mathrm{b}}\mathrm{Tr}(\dot{B}_u e^{-\varepsilon^2B_u^2})\\
&\quad={}^{\mathrm{b}}\mathrm{Tr}(\dot{C}_u e^{-\varepsilon^2C_u^2})+{}^{\mathrm{b}}\mathrm{Tr}(\dot{P}_u e^{-\varepsilon^2P_u^2})+{}^{\mathrm{b}}\mathrm{Tr}(\dot{C}_u e^{-\varepsilon^2P_u^2})+{}^{\mathrm{b}}\mathrm{Tr}(\dot{P}_u e^{-\varepsilon^2C_u^2})\\
&\quad={}^{\mathrm{b}}\mathrm{Tr}(\dot{C}_u e^{-\varepsilon^2C_u^2})+\mathrm{Tr}(\dot{P}_u e^{-\varepsilon^2P_u^2})+\mathrm{Tr}(\dot{C}_u e^{-\varepsilon^2P_u^2})+\mathrm{Tr}(\dot{P}_u e^{-\varepsilon^2C_u^2})
\end{aligned}$$

since $\dot{P}_u e^{-\varepsilon^2 P_u^2}$, $\dot{C}_u e^{-\varepsilon^2 P_u^2}$ and $\dot{P}_u e^{-\varepsilon^2 C_u^2}$ are all trace class operators. In fact, since P_u is a projection and the rank of P_u remains constant for each $u \in (a,b)$, using Duhamel's formula we have

$$\operatorname{Tr}(\dot{P}_u e^{-\varepsilon^2 P_u^2}) = \frac{1}{2}\operatorname{Tr}((\dot{P}_u P_u + P_u \dot{P}_u)e^{-\varepsilon^2 P_u^2}) = \frac{-1}{2\varepsilon^2}\frac{d}{du}\operatorname{Tr}(e^{-\varepsilon^2 P_u^2}) = 0.$$

By the very definition of C_u (see formula (4.2) above), we have $P_u C_u = C_u P_u = 0$. In particular, $\operatorname{Tr}(C_u e^{-\varepsilon^2 P_u^2}) \equiv 0$. Therefore,

$$\begin{aligned} 0 = \frac{d}{du}\operatorname{Tr}(C_u e^{-\varepsilon^2 P_u^2}) &= \operatorname{Tr}(\dot{C}_u e^{-\varepsilon^2 P_u^2}) + \operatorname{Tr}(C_u e^{-\varepsilon^2 P_u^2}(P_u \dot{P}_u + \dot{P}_u P_u)) \\ &= \operatorname{Tr}(\dot{C}_u e^{-\varepsilon^2 P_u^2}). \end{aligned}$$

Similarly, $\operatorname{Tr}(\dot{P}_u e^{-\varepsilon^2 C_u^2}) = 0$. We conclude that

$$ {}^{\mathrm{b}}\operatorname{Tr}(\dot{B}_u e^{-\varepsilon^2 B_u^2}) = {}^{\mathrm{b}}\operatorname{Tr}(\dot{C}_u e^{-\varepsilon^2 C_u^2}).$$

Hence the corollary follows. □

Lemma 4.4. *For $\tau \in (a,b)$ and $\tau \neq u_0$, we have*

$$\frac{d}{du}\eta_\varepsilon(A_u)\Big|_{u=\tau} = -\frac{2\varepsilon}{\sqrt{\pi}}\operatorname{Tr}(\dot{A}_u e^{-\varepsilon^2 A_u^2})\Big|_{u=\tau}.$$

Proof. Notice that

$$\begin{aligned} \eta_\varepsilon(A_u) &= \sum_k \frac{2}{\sqrt{\pi}}\int_\varepsilon^\infty \beta_k(u) e^{-t^2\beta_k^2(u)}\,dt \\ &= \sum_k \frac{2}{\sqrt{\pi}}\int_\varepsilon^\infty \langle A_u e^{-t^2 A_u^2}\phi_k(u),\ \phi_k(u)\rangle\,dt. \end{aligned}$$

If $\beta_k(\tau) \neq 0$, then the same argument from Lemma 4.2 shows that

$$\frac{d}{du}\int_\varepsilon^\infty \langle A_u e^{-t^2 A_u^2}\phi_k(u), \phi_k(u)\rangle\,dt\Big|_{u=\tau} = \varepsilon\langle \dot{A}_u e^{-\varepsilon^2 A_u^2}\phi_k(u), \phi_k(u)\rangle|_{u=\tau}.$$

If $\beta_k(\tau) = 0$, then $\beta_k \equiv 0$ on (a,b) by our choice of the interval (a,b). In particular, $A_u\phi_k(u) = 0$ for all $u \in (a,b)$. Then

$$\begin{aligned} &\frac{d}{du}\langle A_u\phi_k(u), \phi_k(u)\rangle \\ &\quad = \langle \dot{A}_u\phi_k(u), \phi_k(u)\rangle + \langle A_u\dot{\phi}_k(u), \phi_k(u)\rangle + \langle A_u\phi_k(u), \dot{\phi}_k(u)\rangle \\ &\quad = \langle \dot{A}_u\phi_k(u), \phi_k(u)\rangle. \end{aligned}$$

It follows that $\langle \dot{A}_u \phi_k(u), \phi_k(u)\rangle = 0$. Hence

$$\langle \dot{A}_u e^{-\varepsilon^2 A_u^2} \phi_k(u), \phi_k(u)\rangle = \langle \dot{A}_u \phi_k(u), \phi_k(u)\rangle = 0$$

for all $u \in (a,b)$. This finishes the proof. □

Corollary 4.5. *For $u \in (a,b)$ and $u \neq u_0$,*

$$\frac{d}{du}\eta_\varepsilon(D_u)du = -\frac{2\varepsilon}{\sqrt{\pi}}{}^{\mathrm{b}}\mathrm{Tr}(\dot{D}_u e^{-\varepsilon^2 D_u^2}) + E_\varepsilon(u).$$

Proof. Notice that

$$\eta_\varepsilon(D_u) = \eta_\varepsilon(A_u) + \eta_\varepsilon(C_u)$$

and

$${}^{\mathrm{b}}\mathrm{Tr}(\dot{A}_u e^{-\varepsilon^2 C_u^2}) = {}^{\mathrm{b}}\mathrm{Tr}(\dot{C}_u e^{-\varepsilon^2 A_u^2}) = 0.$$

The corollary follows from the above lemmas. □

If we denote by Q^+ the cardinality of the set $\{\beta_k \mid \beta_k(u_0) = 0 \text{ and } \beta_k(a) > 0\}$ and by Q^- the cardinality of the set $\{\beta_k \mid \beta_k(u_0) = 0 \text{ and } \beta_k(a) < 0\}$, then

$$\dim\ker D_{u_0} = \dim\ker D_u + Q^+ + Q^- \quad \text{for } u \in (a, u_0). \tag{4.3}$$

Since

$$\lim_{\lambda\to 0^\pm} \frac{2}{\sqrt{\pi}} \int_\varepsilon^\infty \lambda e^{-t^2\lambda^2} dt = \pm 1,$$

it follows that

$$\lim_{u\to u_0^-} \eta_\varepsilon(D_u) = \eta_\varepsilon(D_{u_0}) + Q^+ - Q^-. \tag{4.4}$$

Recall that, by definition, $\mathrm{sf}(D_a, D_{u_0}) = Q^-$ and

$$\xi_\varepsilon(D_{u_0}) = \frac{\eta_\varepsilon(D_{u_0}) + \dim\ker(D_{u_0})}{2}.$$

Therefore, the difference of equation (4.3) and equation (4.4) gives

$$\mathrm{sf}(D_a, D_{u_0}) = \xi_\varepsilon(D_{u_0}) - \lim_{u\to u_0^-} \xi_\varepsilon(D_u).$$

Similarly, $\mathrm{sf}(D_{u_0}, D_b) = \lim_{u\to u_0^+} \xi_\varepsilon(D_u) - \xi_\varepsilon(D_{u_0})$. Thus we have

$$\mathrm{sf}(D_a, D_b) = \lim_{u\to u_0^+} \xi_\varepsilon(D_u) - \lim_{u\to u_0^-} \xi_\varepsilon(D_u).$$

With the above results combined, we have the following proposition.

Proposition 4.6.

$$\mathrm{sf}(D, g^{-1}Dg) = \lim_{\varepsilon\to\infty} \frac{\varepsilon}{\sqrt{\pi}} \int_0^1 {}^{\mathrm{b}}\mathrm{Tr}(\dot{D}_u e^{-\varepsilon^2 D_u^2})\, du.$$

Proof. Let us cover [0, 1] by finitely many pointed gap intervals $[a_i, b_i]$, $0 \le i \le n$, with $u_i \in [a_i, b_i]$ such that $b_i = a_{i+1}$ with $u_0 = a_0 = 0$, $u_n = b_n = 1$ and $u_j \in (a_j, b_j)$ for $1 \le j \le n-1$. Then

$$\begin{aligned}\mathrm{sf}(D_0, D_1) &= \xi_\varepsilon(D_1) - \xi_\varepsilon(D_0) + \sum_i \lim_{u\to u_i^+} \xi_\varepsilon(D_u) - \lim_{u\to u_i^-} \xi_\varepsilon(D_u)\\ &= \xi_\varepsilon(D_1) - \xi_\varepsilon(D_0) - \frac{1}{2}\int_0^1 \frac{d}{du}\eta_\varepsilon(D_u)\, du\\ &= \xi_\varepsilon(D_1) - \xi_\varepsilon(D_0) + \frac{\varepsilon}{\sqrt{\pi}}\int_0^1 {}^{\mathrm{b}}\mathrm{Tr}(\dot{D}_u e^{-\varepsilon^2 D_u^2}) du - \frac{1}{2}\int_0^1 E_\varepsilon(u)\, du.\end{aligned}$$

Notice that $\xi_\varepsilon(g^{-1}Dg) = \xi_\varepsilon(D)$ and $\int_0^1 E_\varepsilon(u)du$ vanishes when $\varepsilon \to \infty$, hence the proposition follows. □

5. Large time limit

In this section, we prove the equality

$$\mathrm{sf}(D, g^{-1}Dg) = \lim_{t\to\infty} \langle {}^{\mathrm{b}}\mathrm{Ch}^\bullet(t\mathfrak{D}), \mathrm{Ch}_\bullet(g)\rangle.$$

This is the last step remaining to prove Theorem 3.1. We follow rather closely Getzler's proof for closed manifolds [Get93b].

Recall that we have

$$\mathfrak{D} = \begin{pmatrix} 0 & D \\ D & 0 \end{pmatrix} \in {}^{\mathrm{b}}\Psi^1(\widehat{M}; \mathcal{S}_1), \quad {}^{\partial}\mathfrak{D} = \begin{pmatrix} 0 & {}^{\partial}D \\ {}^{\partial}D & 0 \end{pmatrix} \in {}^{\mathrm{b}}\Psi^1(\partial M; \mathcal{S}_1)$$

and

$$p = \begin{pmatrix} 0 & g^{-1} \\ g & 0 \end{pmatrix} \in C^\infty_{\exp}(\widehat{M}) \otimes \mathrm{End}(\mathbb{C}^{r|r}),$$

with $\mathbb{C}^{r|r} = (\mathbb{C}^r)^+ \oplus (\mathbb{C}^r)^-$ being $\mathbb{Z}_2$-graded. Let us put

$$\mathfrak{D}_u = (1-u)\mathfrak{D} - up\mathfrak{D}p \in {}^{\mathrm{b}}\Psi^1(\widehat{M}; \mathcal{S}_1 \otimes_s \mathbb{C}^{r|r})$$

for $u \in [0, 1]$, where $\mathcal{S}_1 \otimes_s \mathbb{C}^{r|r}$ is the super-tensor product of $\mathcal{S}_1$ and $\mathbb{C}^{r|r}$. We see immediately that

$$\begin{aligned}\mathfrak{D}_u &= \begin{pmatrix} (1-u)\mathfrak{D} + ug^{-1}\mathfrak{D}g & 0 \\ 0 & (1-u)\mathfrak{D} + ug\mathfrak{D}g^{-1} \end{pmatrix}\\ &= \begin{pmatrix} \mathfrak{D} + ug^{-1}[\mathfrak{D}, g] & 0 \\ 0 & \mathfrak{D} + ug[\mathfrak{D}, g^{-1}] \end{pmatrix}.\end{aligned}$$

We write $\mathfrak{D}_{u,s} = \mathfrak{D}_u + sp$ (resp. ${}^{\partial}\mathfrak{D}_{u,s} = {}^{\partial}\mathfrak{D}_u + sp$), where $(u,s) \in [0,1]\times(-\infty,0]$. Consider the superconnections $\mathbb{A} = d + \mathfrak{D}_{u,s}$ and ${}^{\partial}\mathbb{A} = d + {}^{\partial}\mathfrak{D}_{u,s}$, where d is the standard de Rham differential on the parameter space $[0,1]\times(-\infty,0]$. We have

$$\mathbb{A}^2 = \mathfrak{D}_u^2 + s[\mathfrak{D}_u, p] + s^2 + du\,\dot{\mathfrak{D}}_u + dsp,$$
$${}^{\partial}\mathbb{A}^2 = {}^{\partial}\mathfrak{D}_u^2 + s[{}^{\partial}\mathfrak{D}_u, {}^{\partial}p] + s^2 + du\,{}^{\partial}\dot{\mathfrak{D}}_u + ds\,{}^{\partial}p.$$

Recall that the indicial family $I(D,\lambda)$ of D is

$$I(D,\lambda) = {}^{\partial}D + i\lambda c(\nu),$$

where $\nu = dx$ is the normal cotangent vector and $c(\nu)$ is the Clifford multiplication of ν, cf. [Get93b], Proposition 5.4. Therefore, we have

$$\mathrm{I}(\mathfrak{D}_u,\lambda) = {}^{\partial}\mathfrak{D}_u + i\lambda c(\nu),$$
$$\mathrm{I}(\mathfrak{D}_{u,s},\lambda) = {}^{\partial}\mathfrak{D}_u + i\lambda c(\nu) + s\,{}^{\partial}p,$$
$$\mathrm{I}(d\,\mathfrak{D}_{u,s},\lambda) = du\,{}^{\partial}\dot{\mathfrak{D}}_u + ds\,{}^{\partial}p,$$
$$\mathrm{I}(\mathfrak{D}_{u,s}^2,\lambda) = {}^{\partial}\mathfrak{D}_u^2 + \lambda^2 + s[{}^{\partial}\mathfrak{D}_u, {}^{\partial}p] + s^2.$$

Consider the Chern character of $\mathbb{A}$ defined by

$$\mathrm{Ch}(\mathbb{A}) := {}^{\mathrm{b}}\mathrm{Str}_{(1)}(e^{-\mathbb{A}^2}).$$

Denote Γ_u the contour $\{u\}\times[0,\infty)$ and γ_s the contour $[0,1]\times\{s\}$. By Stoke's theorem, we have

$$\int_{\Gamma_1}\mathrm{Ch}(\mathbb{A}) - \int_{\Gamma_0}\mathrm{Ch}(\mathbb{A}) + \int_{\gamma_0}\mathrm{Ch}(\mathbb{A}) - \lim_{s\to\infty}\int_{\gamma_s}\mathrm{Ch}(\mathbb{A}) = \int_{[0,1]\times[0,\infty)} d\,\mathrm{Ch}(\mathbb{A}). \tag{5.1}$$

5.1. Technical lemmas. In this section let us prove several technical lemmas. Notice that by definition, we have

$$\begin{aligned}\mathrm{Ch}(\mathbb{A}) = &-du\int_0^1 {}^{\mathrm{b}}\mathrm{Str}_{(1)}(e^{-\sigma(\mathfrak{D}_u^2+s[\mathfrak{D}_u,p]+s^2)}\dot{\mathfrak{D}}_u e^{-(1-\sigma)(\mathfrak{D}_u^2+s[\mathfrak{D}_u,p]+s^2)})\,d\sigma\\ &- ds\int_0^1 {}^{\mathrm{b}}\mathrm{Str}_{(1)}(e^{-\sigma(\mathfrak{D}_u^2+s[\mathfrak{D}_u,p]+s^2)}\,p\,e^{-(1-\sigma)(\mathfrak{D}_u^2+s[\mathfrak{D}_u,p]+s^2)})\,d\sigma\end{aligned}$$

Lemma 5.1. *We have*

$$\int_{\gamma_0}\mathrm{Ch}(\mathbb{A}) = -\int_0^1 {}^{\mathrm{b}}\mathrm{Str}_{(1)}(\dot{\mathfrak{D}}_u e^{-\mathfrak{D}_u^2})\,du.$$

Proof. Since

$$\begin{aligned}&{}^{\mathrm{b}}\mathrm{Str}_{(1)}[e^{-\sigma\mathfrak{D}_u^2},\dot{\mathfrak{D}}_u e^{-(1-\sigma)\mathfrak{D}_u^2}]\\&\quad=\frac{-1}{2\pi i}\int_{-\infty}^{\infty}\mathrm{Str}_{(1)}\Big(\frac{d(e^{-\sigma({}^{\partial}\mathfrak{D}^2+\lambda^2)})}{d\lambda}{}^{\partial}\dot{\mathfrak{D}}_u e^{-(1-\sigma)({}^{\partial}\mathfrak{D}^2+\lambda^2)}\Big)\,d\lambda=0,\end{aligned}$$

it follows that

$$\begin{aligned}\int_{\Gamma_0}\mathrm{Ch}(\mathbb{A})&=-\int_0^1 du\Big(\int_0^1{}^{\mathrm{b}}\mathrm{Str}_{(1)}(e^{-\sigma\mathfrak{D}_u^2}\dot{\mathfrak{D}}_u e^{-(1-\sigma)\mathfrak{D}_u^2})\,d\sigma\Big)\\&=-\int_0^1{}^{\mathrm{b}}\mathrm{Str}_{(1)}(\dot{\mathfrak{D}}_u e^{-\mathfrak{D}_u^2})\,du.\end{aligned}$$

□

Lemma 5.2. *We have*

$$\lim_{s\to\infty}\int_{\gamma_s}\mathrm{Ch}(\mathbb{A})=0.$$

Proof. First notice that a similar argument as that in Lemma 5.1 shows that

$$\begin{aligned}&\int_0^1{}^{\mathrm{b}}\mathrm{Str}_{(1)}(e^{-\sigma(\mathfrak{D}_u^2+s[\mathfrak{D}_u,p]+s^2)}p\,e^{-(1-\sigma)(\mathfrak{D}_u^2+s[\mathfrak{D}_u,p]+s^2)})\,d\sigma\\&\qquad={}^{\mathrm{b}}\mathrm{Str}_{(1)}(p\,e^{-\mathfrak{D}_u^2-s[\mathfrak{D}_u,p]-s^2}).\end{aligned}$$

Using Duhamel's principle, we have

$$\begin{aligned}&{}^{\mathrm{b}}\mathrm{Str}_{(1)}(p\,e^{-\mathfrak{D}_u^2-s[\mathfrak{D}_u,p]-s^2})\\&\quad=\sum_{n=0}^{\infty}e^{-s^2}(-s)^n\int_{\Delta^n}{}^{\mathrm{b}}\mathrm{Str}_{(1)}(pe^{-\sigma_0\mathfrak{D}_u^2}[\mathfrak{D}_u,p]e^{-\sigma_1\mathfrak{D}_u^2}\dots[\mathfrak{D}_u,p]e^{-\sigma_n\mathfrak{D}_u^2})\,d\sigma.\end{aligned}$$

The estimates in Section 2.2 show that

$$\begin{aligned}&\Big|\int_{\Delta^n}{}^{\mathrm{b}}\mathrm{Str}_{(1)}(pe^{-\sigma_0\mathfrak{D}_u^2}[\mathfrak{D}_u,p]e^{-\sigma_1\mathfrak{D}_u^2}\dots[\mathfrak{D}_u,p]e^{-\sigma_n\mathfrak{D}_u^2})d\sigma\Big|\\&\qquad\le 2^n(n+1)\frac{\mathcal{K}_1^n+2\mathcal{K}_0}{n!}\,{}^{\mathrm{b}}\|p\|^{n+2}\end{aligned}$$

for some constants $\mathcal{K}_0$ and $\mathcal{K}_1$. In fact $\mathcal{K}_0$ and $\mathcal{K}_1$ can be chosen independent of u since there is constant $\mathcal{C}$ such that

$$\big|\mathrm{Tr}(e^{-\sigma\mathfrak{D}_{\mathbb{R}}^2}-e^{-\sigma\mathfrak{D}_u^2})|_{(-\infty,0]\times\partial M}\big|\le\mathcal{C}$$

for all $\sigma,u\in[0,1]$ (cf. [LMP09], Proposition 3.1). Hence

$$\big|{}^{\mathrm{b}}\mathrm{Str}_{(1)}(p\,e^{-\mathfrak{D}_u^2-s[\mathfrak{D}_u,p]-s^2})\big|\le\mathcal{K}'e^{-s^2+2\mathcal{K}\,{}^{\mathrm{b}}\|p\|s}$$

for some constants $\mathcal{K}$ and $\mathcal{K}'$. Therefore $\int_{\gamma_s}\mathrm{Ch}(\mathbb{A})=O(e^{-s^2/2})$ as $s\to\infty$, hence the lemma follows. □

Lemma 5.3. *We have*

$$\int_{\Gamma_0} \mathrm{Ch}(\mathbb{A}) = -\int_{\Gamma_1} \mathrm{Ch}(\mathbb{A}) = \frac{1}{2}\langle {}^{\mathrm{b}}\mathrm{Ch}^{\bullet}(\mathfrak{D}), \textstyle\sum_{k=0}^{\infty} k!\,\mathrm{Str}(p,\dots,p)_{2k+1}\rangle.$$

Proof. When $u = 0$, we have $\mathbb{A}^2 = \mathfrak{D}^2 + s[\mathfrak{D}, p] + s^2 + dsp$. Using Duhamel's principle, we see that

$$\begin{aligned}
&\mathrm{Ch}(\mathbb{A})|_{\Gamma_0}\\
&= \sum_{n=0}^{\infty}(-s)^{2k+1}e^{-s^2}\int_{\Delta^n} {}^{\mathrm{b}}\mathrm{Str}_{(1)}(e^{-\sigma_0\mathfrak{D}^2}[\mathfrak{D}, p]e^{-\sigma_1\mathfrak{D}^2}\dots\\
&\qquad\qquad e^{-\sigma_i\mathfrak{D}^2}(-dsp)e^{-\sigma_{i+1}\mathfrak{D}^2}\dots[\mathfrak{D}, p]e^{-\sigma_n\mathfrak{D}^2})\,d\sigma \qquad (5.2)\\
&= \sum_{k=0}^{\infty} s^{2k+1}e^{-s^2}ds\sum_{i=0}^{2k+1} {}^{\mathrm{b}}\langle 1, [\mathfrak{D}, p],\dots,[\mathfrak{D}, p], p, [\mathfrak{D}, p],\dots,[\mathfrak{D}, p]\rangle\\
&= \sum_{k=0}^{\infty} s^{2k+1}e^{-s^2}\,ds\,{}^{\mathrm{b}}\langle p, [\mathfrak{D}, p],\dots,[\mathfrak{D}, p]\rangle_{2k+1},
\end{aligned}$$

where we have used the fact (cf. [Get93a], Lemma 6.3 (2))

$$\sum_{i=0}^{2k+1} {}^{\mathrm{b}}\langle 1, [\mathfrak{D}, p],\dots,[\mathfrak{D}, p], p, [\mathfrak{D}, p],\dots,[\mathfrak{D}, p]\rangle = {}^{\mathrm{b}}\langle p, [\mathfrak{D}, p],\dots,[\mathfrak{D}, p]\rangle_{2k+1}.$$

It follows that

$$\int_{\Gamma_0} \mathrm{Ch}(\mathbb{A}) = \frac{1}{2}\langle {}^{\mathrm{b}}\mathrm{Ch}^{\bullet}(\mathfrak{D}), \textstyle\sum_{k=0}^{\infty} k!\,\mathrm{Str}(p,\dots,p)_{2k+1}\rangle.$$

When $u = 1$, we have $\mathbb{A}^2 = \mathfrak{D}_1^2 + s[\mathfrak{D}_1, p] + s^2 + dsp$. Since $\mathfrak{D}_1 = -p\mathfrak{D}p$, we see that

$$[\mathfrak{D}_1, p] = -[\mathfrak{D}, p] \quad \text{and} \quad e^{-\mathfrak{D}_1^2} = pe^{-\mathfrak{D}^2}p.$$

Furthermore, we notice that $p[\mathfrak{D}, p]p = [\mathfrak{D}, p]$. Combining these with a calculation similar to that in equation (5.2) with $\mathfrak{D}$ replaced by $\mathfrak{D}_1$, we see that

$$\int_{\Gamma_1} \mathrm{Ch}(\mathbb{A}) = -\frac{1}{2}\langle {}^{\mathrm{b}}\mathrm{Ch}^{\bullet}(\mathfrak{D}_1), \textstyle\sum_{k=0}^{\infty} k!\,\mathrm{Str}(p,\dots,p)_{2k+1}\rangle \qquad \square$$

Lemma 5.4. *We have*

$$d\,\mathrm{Ch}(\mathbb{A}) = -{}^{\mathrm{b}}\mathrm{Str}[\mathfrak{D}_{u,s}, e^{-\mathbb{A}^2}] = \mathrm{Str}_{(2)}(e^{-\partial\mathbb{A}^2}).$$

Proof. Since $[\mathbb{A}, e^{-\mathbb{A}^2}] = 0$, we have

$$d\,{}^{\mathrm{b}}\mathrm{Str}_{(1)}(e^{-\mathbb{A}^2}) = {}^{\mathrm{b}}\mathrm{Str}_{(1)}[d, e^{-\mathbb{A}^2}] = -{}^{\mathrm{b}}\mathrm{Str}_{(1)}[\mathfrak{D}_{u,s}, e^{-\mathbb{A}^2}].$$

It follows that

$$\begin{aligned} d\,\mathrm{Ch}(\mathbb{A}) &= \frac{1}{2\pi i}\int_{-\infty}^{\infty} \mathrm{Str}_{(1)}\left(\frac{d\,\mathrm{I}(\mathfrak{D}_{u,s},\lambda)}{d\lambda}\,\mathrm{I}(e^{-\mathbb{A}^2},\lambda)\right)d\lambda \\ &= \frac{1}{2\pi i}\int_{-\infty}^{\infty} \mathrm{Str}_{(1)}(ic(\nu)\,\mathrm{I}(e^{-\mathbb{A}^2},\lambda))\,d\lambda \\ &= \frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty} \mathrm{Str}_{(2)}(\mathrm{I}(e^{-\mathbb{A}^2},\lambda))d\lambda \\ &= \frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty} e^{-\lambda^2}d\lambda\,\mathrm{Str}_{(2)}(e^{-{}^{\partial}\mathbb{A}^2}) \\ &= \mathrm{Str}_{(2)}(e^{-{}^{\partial}\mathbb{A}^2}). \end{aligned}$$

□

By Duhamel's principle, the 2-form components in $\mathrm{Str}_{(2)}(e^{-{}^{\partial}\mathbb{A}^2})$ can be expanded as

$$\begin{aligned} &\sum_{k=2}^{\infty}\sum_{1\le i<j\le k}(-s)^{k-2}e^{-s^2}\langle 1,[{}^{\partial}\mathfrak{D}_u,p],\ldots,[{}^{\partial}\mathfrak{D}_u,p],\underbrace{p}_{i-\text{th}},[{}^{\partial}\mathfrak{D}_u,p],\ldots \\ &\qquad \ldots,[{}^{\partial}\mathfrak{D}_u,p],\underbrace{{}^{\partial}\dot{\mathfrak{D}}_u}_{j\text{-th}},[{}^{\partial}\mathfrak{D}_u,p],\ldots,[{}^{\partial}\mathfrak{D}_u,p]\rangle_u\,\mathrm{d}u\mathrm{d}s \\ &-\sum_{k=2}^{\infty}\sum_{1\le i<j\le k}(-s)^{k-2}e^{-s^2}\langle 1,[{}^{\partial}\mathfrak{D}_u,p],\ldots,[{}^{\partial}\mathfrak{D}_u,p],\underbrace{{}^{\partial}\dot{\mathfrak{D}}_u}_{i-\text{th}},[{}^{\partial}\mathfrak{D}_u,p],\ldots \\ &\qquad \ldots,[{}^{\partial}\mathfrak{D}_u,p],\underbrace{p}_{j\text{-th}},[{}^{\partial}\mathfrak{D}_u,p],\ldots,[{}^{\partial}\mathfrak{D}_u,p]\rangle_u\,\mathrm{d}u\mathrm{d}s, \end{aligned} \tag{5.3}$$

where

$$\langle A_0,\ldots,A_n\rangle_u = \int_{\Delta^n} {}^{\mathrm{b}}\mathrm{Str}_{(2)}(A_0e^{-\sigma_0{}^{\partial}\mathfrak{D}_u^2}A_1e^{-\sigma_1{}^{\partial}\mathfrak{D}_u^2}\ldots A_ne^{-\sigma_n{}^{\partial}\mathfrak{D}_u^2})\,d\sigma.$$

Recall that (cf. [GS89], Lemma 2.2)

$$\begin{aligned} &\langle A_0,\ldots,A_n\rangle_u \\ &\qquad = \sum_{i=0}^{n}(-1)^{(|A_0|+\cdots+|A_{i-1}|)(|A_i|+\cdots+|A_n|)}\langle 1,A_i,\ldots,A_n,A_0,\ldots,A_{i-1}\rangle_u. \end{aligned}$$

Since ${}^{\partial}\mathfrak{D}_u$, ${}^{\partial}\dot{\mathfrak{D}}_u$ and p are of odd degree and $[{}^{\partial}\mathfrak{D}_u,p]$ is of even degree, one has

$$\begin{aligned} (5.3) = \sum_{k=2}^{\infty}\sum_{i=1}^{k-1}(-s)^{k-2}e^{-s^2}\langle p,[{}^{\partial}\mathfrak{D}_u,p],\ldots,[{}^{\partial}\mathfrak{D}_u,p],\underbrace{{}^{\partial}\dot{\mathfrak{D}}_u}_{i\text{-th}},[{}^{\partial}\mathfrak{D}_u,p],\ldots \\ \ldots,[{}^{\partial}\mathfrak{D}_u,p]\rangle_u\,\mathrm{d}u\mathrm{d}s. \end{aligned}$$

Let us define

$$\widetilde{\mathrm{Ch}}^n({}^\partial\mathfrak{D}_u, V)(a_0,\dots,a_n) = \iota(V)\langle a_0, [{}^\partial\mathfrak{D}_u, a_1],\dots,[{}^\partial\mathfrak{D}_u, a_n]\rangle_u,$$

where

$$\iota(V)\langle A_0,\dots,A_n\rangle_u := \sum_{0\le i\le n}(-1)^{|V|(|A_0|+\cdots+|A_i|)}\langle A_0,\dots,A_i,V,A_{i+1},\dots,A_n\rangle_u.$$

Then the calculation above shows that

$$\mathrm{Str}_{(2)}(e^{-{}^\partial\mathbb{A}^2}) = -\sum_{k=0}^{\infty}(-s)^k e^{-s^2}\langle\widetilde{\mathrm{Ch}}^k({}^\partial\mathfrak{D}_u, {}^\partial\dot{\mathfrak{D}}_u), \mathrm{Str}(p,\dots,p)_k\rangle_u\,\mathrm{d}u\mathrm{d}s.$$

We summarize this in the following lemma.

Lemma 5.5. *We have*

$$\int_{[0,1]\times[0,\infty)}\mathrm{Str}_{(2)}(e^{-{}^\partial\mathbb{A}^2}) = \frac{1}{2}\langle\int_0^1\widetilde{\mathrm{Ch}}^\bullet({}^\partial\mathfrak{D}_u, {}^\partial\dot{\mathfrak{D}}_u)\mathrm{d}u, \sum_{k=0}^{\infty}k!\,\mathrm{Str}(p,\dots,p)_{2k+1}\rangle.$$

Proof. Notice that $\mathrm{Str}(p,\dots,p)_k = 0$ for k even, and

$$\int_0^\infty s^{2k+1}e^{-s^2}\mathrm{d}s = \frac{k!}{2}.$$

□

5.2. Large time limit. Recall that ${}^\partial D$ is invertible and $g\in U_k(C^\infty(N))$ is a unitary such that $\|\,[{}^\partial D, {}^\partial g]\,\| < \lambda$ with λ the lowest nonzero eigenvalue of $|{}^\partial D|$. In the following, we also write g for ${}^\partial g$ if there is no confusion. Notice that

$$\|\,[{}^\partial D, g^{-1}]\,\| = \|-g^{-1}[{}^\partial D, g]g^{-1}\| \le \|\,[{}^\partial D, g]\,\|,$$

and similarly $\|\,[{}^\partial D, g]\,\| \le \|\,[{}^\partial D, g^{-1}]\,\|$. Hence $\|\,[{}^\partial D, g^{-1}]\,\| = \|\,[{}^\partial D, g]\,\|$.

Lemma 5.6. *Let $\mathbb{A}_t = d + t\mathfrak{D}_{u,s}$. Then*

$$\lim_{t\to\infty}\int_{[0,1]\times[0,\infty)} d\,\mathrm{Ch}(\mathbb{A}_t) = 0.$$

Proof. Notice that

$$|{}^\partial D + ug^{-1}[{}^\partial D, g]| \ge |{}^\partial D| - u\|g^{-1}[{}^\partial D, g]\,\| \ge \lambda - u\|\,[{}^\partial D, g]\,\|,$$
$$|{}^\partial D + ug[{}^\partial D, g^{-1}]| \ge \lambda - u\|\,[{}^\partial D, g]\,\|.$$

When $u = 1$, ${}^\partial D + ug^{-1}[{}^\partial D, g] = g^{-1}\,{}^\partial D g$. The lowest eigenvalue of $|g^{-1}\,{}^\partial D g|$ is also λ since g is a unitary. Therefore, a similar argument as above shows that

$$|{}^\partial D + ug^{-1}[{}^\partial D, g]| \ge \lambda - (1-u)\|\,[{}^\partial D, g]\,\|,$$
$$|{}^\partial D + ug[{}^\partial D, g^{-1}]| \ge \lambda - (1-u)\|\,[{}^\partial D, g]\,\|.$$

Thus $|{}^{\partial}\mathfrak{D}_u|$ is bounded below by

$$\lambda_u := \max\{\lambda - u\|\,[{}^{\partial}D, g]\,\|,\ \lambda - (1-u)\|\,[{}^{\partial}D, g]\,\|\}.$$

Then there exists a constant C such that

$$\operatorname{Tr}(e^{-t^2\,{}^{\partial}\mathfrak{D}_u^2}) \leq Ce^{-t^2\lambda_u^2}$$

for all $t \geq 1$. where we may take $C = \sup_{u\in[0,1]} e^{\lambda_u^2}\operatorname{Tr}(e^{-{}^{\partial}\mathfrak{D}_u^2})$, cf. [GS89], Theorem C. One also notices that $\|\,[{}^{\partial}\mathfrak{D}_u, p]\,\| = (1-2u)\|\,[{}^{\partial}\mathfrak{D}, p]\,\| < \lambda_u$. Therefore we have

$$\begin{aligned}
&|t^{2k+2}\langle p, [{}^{\partial}\mathfrak{D}_u, p], \ldots, [{}^{\partial}\mathfrak{D}_u, p], {}^{\partial}\dot{\mathfrak{D}}_u, [{}^{\partial}\mathfrak{D}_u, p], \ldots, [{}^{\partial}\mathfrak{D}_u, p]\rangle_{u,2k+1}| \\
&\quad \leq \frac{1}{(2k)!}t^{2k+2}\operatorname{Tr}(e^{-t^2\,{}^{\partial}\mathfrak{D}_u^2})\|p\|\cdot\|\,[{}^{\partial}\mathfrak{D}_u, p]\,\|^{2k+1}\|{}^{\partial}\dot{\mathfrak{D}}_u\| \\
&\quad \leq \frac{C}{(2k)!}t^{2k+2}e^{-t^2\lambda_u^2}\|\,[{}^{\partial}\mathfrak{D}_u, p]\,\|^{2k+1}\|p[{}^{\partial}\mathfrak{D}, p]\,\|.
\end{aligned}$$

Hence

$$\begin{aligned}
\left|\int_{[0,1]\times[0,\infty)} d\operatorname{Ch}(\mathbb{A}_t)\right| &= \left|\int_{[0,1]\times[0,\infty)} \operatorname{Str}_{(2)}(e^{-{}^{\partial}\mathbb{A}_t^2})\right| \\
&= |\langle \int_0^1 \widetilde{\operatorname{Ch}}^\bullet(t\,{}^{\partial}\mathfrak{D}_u, t\,{}^{\partial}\dot{\mathfrak{D}}_u)du, \sum_{k=0}^{\infty} k!\operatorname{Str}(p, \ldots, p)_{2k+1}\rangle| \\
&\leq t\cdot\|p[{}^{\partial}\mathfrak{D}, p]\,\| \int_0^1 \sum_{k=1}^{\infty} \frac{C}{k!}e^{-t^2\lambda_u^2}(\|\,[{}^{\partial}\mathfrak{D}_u, p]\,\|\cdot t)^{2k+1}du \\
&\leq t^2C' \int_0^1 e^{(\|\,[{}^{\partial}\mathfrak{D}_u, p]\,\|^2-\lambda_u^2)t^2}du,
\end{aligned}$$

where the last term goes to 0 when $t \to \infty$. This finishes the proof. □

Lemma 5.7. *If $g \in U_k(C^\infty(N))$ is a unitary such that $\|\,[{}^{\partial}D, {}^{\partial}g]\,\| < \lambda$ with λ the lowest nonzero eigenvalue of $|{}^{\partial}D|$, then*

$$\lim_{t\to\infty}\langle {}^{\mathrm{b}}\operatorname{Ch}^\bullet(t\mathfrak{D}), \operatorname{Ch}_\bullet(g)\rangle = -\lim_{t\to\infty}\langle {}^{\mathrm{b}}\operatorname{Ch}^\bullet(t\mathfrak{D}), \operatorname{Ch}_\bullet(g^{-1})\rangle.$$

Proof. Recall equation (3.1):

$$(d - b - B)^{\mathrm{b}}\operatorname{Ch}^\bullet(\mathfrak{D}, t) = \operatorname{Ch}^\bullet({}^{\partial}\mathfrak{D}, t).$$

Since $\operatorname{Ch}_\bullet(g) + \operatorname{Ch}_\bullet(g^{-1}) - 2\operatorname{Ch}_\bullet(1) = (b + B)\operatorname{Tch}_\bullet(h)$, where h is a smooth path of unitaries connecting $\left(\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right)$ and $\left(\begin{smallmatrix}g&0\\0&g^{-1}\end{smallmatrix}\right)$ (cf. Section 1.6), we have

$$\begin{aligned}\lim_{t\to\infty}&\langle {}^{\mathrm{b}}\mathrm{Ch}^{\bullet}(t\mathfrak{D}),\mathrm{Ch}_{\bullet}(g)+\mathrm{Ch}_{\bullet}(g^{-1})\rangle\\ &=\lim_{t\to\infty}\langle {}^{\mathrm{b}}\mathrm{Ch}^{\bullet}(t\mathfrak{D}),\mathrm{Ch}_{\bullet}(g)+\mathrm{Ch}_{\bullet}(g^{-1})-2\,\mathrm{Ch}_{\bullet}(1)\rangle\\ &=\lim_{t\to\infty}\langle (b+B)\,{}^{\mathrm{b}}\mathrm{Ch}^{\bullet}(t\mathfrak{D}),\mathrm{Tch}_{\bullet}(h)\rangle\\ &=-\lim_{t\to\infty}\langle \mathrm{Ch}^{\bullet}(t\,{}^{\partial}\mathfrak{D}),\mathrm{Tch}_{\bullet}(h)\rangle.\end{aligned}$$

Now a similar argument (without the presence of the parameter u) as that in Lemma 5.6 above shows that

$$\lim_{t\to\infty}\langle \mathrm{Ch}^{\bullet}(t\,{}^{\partial}\mathfrak{D}),\mathrm{Tch}_{\bullet}(h)\rangle=0.$$

Hence the lemma follows. □

Proposition 5.8. *We have*

$$\mathrm{sf}(D,g^{-1}Dg)=\lim_{t\to\infty}\langle {}^{\mathrm{b}}\mathrm{Ch}^{\bullet}(t\mathfrak{D}),\mathrm{Ch}_{\bullet}(g)\rangle.$$

Proof. Notice that

$$\begin{aligned}\mathfrak{D}_u&=\begin{pmatrix}\mathfrak{D}+ug^{-1}[\mathfrak{D},g] & 0\\ 0 & \mathfrak{D}+ug[\mathfrak{D},g^{-1}]\end{pmatrix}\\ &=\begin{pmatrix}0 & D+ug^{-1}[D,g] & & \\ D+ug^{-1}[D,g] & 0 & & \\ & & 0 & D+ug[D,g^{-1}]\\ & & D+ug[D,g^{-1}] & 0\end{pmatrix}.\end{aligned}$$

Hence

$$\begin{aligned}\int_0^1 {}^{\mathrm{b}}\mathrm{Str}_{(1)}(\dot{\mathfrak{D}}_u e^{-\mathfrak{D}_u^2})\,du&=\frac{1}{\sqrt{\pi}}\int_0^1 {}^{\mathrm{b}}\mathrm{Tr}(g^{-1}[D,g]e^{-(D+ug^{-1}[D,g])^2})\,du\\ &\quad-\frac{1}{\sqrt{\pi}}\int_0^1 {}^{\mathrm{b}}\mathrm{Tr}(g[D,g^{-1}]e^{-(D+ug[D,g^{-1}])^2})\,du.\end{aligned}$$

It follows from Theorem 4.6 that

$$\begin{aligned}\lim_{t\to\infty}\int_0^1 {}^{\mathrm{b}}\mathrm{Str}_{(1)}(t\dot{\mathfrak{D}}_u e^{-t^2\mathfrak{D}_u^2})\,du&=\mathrm{sf}(D,g^{-1}Dg)-\mathrm{sf}(D,gDg^{-1})\\ &=2\,\mathrm{sf}(D,g^{-1}gD)\end{aligned}$$

since $\mathrm{sf}(D,gDg^{-1})=-\,\mathrm{sf}(D,g^{-1}Dg)$.

Now by applying Lemma 5.2, 5.3 and 5.1 to equation (5.1), we have

$$-\langle {}^{\mathrm{b}}\mathrm{Ch}^{\bullet}(t\mathfrak{D}), \sum_{k=0}^{\infty} k!\,\mathrm{Str}(p,\dots,p)_{2k+1}\rangle - \int_0^1 {}^{\mathrm{b}}\mathrm{Str}_{(1)}(t\dot{\mathfrak{D}}_u e^{-t^2\mathfrak{D}_u^2})\,du$$
$$= \int_{[0,1]\times[0,\infty)} d\,\mathrm{Ch}(\mathbb{A}_t).$$

Therefore,

$$\lim_{t\to\infty}\int_0^1 {}^{\mathrm{b}}\mathrm{Str}_{(1)}(t\dot{\mathfrak{D}}_u e^{-t^2\mathfrak{D}_u^2})\,du$$
$$= \lim_{t\to\infty}\langle {}^{\mathrm{b}}\mathrm{Ch}^{\bullet}(t\mathfrak{D}), \mathrm{Ch}_{\bullet}(g) - \mathrm{Ch}_{\bullet}(g^{-1})\rangle - \lim_{t\to\infty}\int_{[0,1]\times[0,\infty)} d\,\mathrm{Ch}(\mathbb{A}_t).$$

It follows from Lemma 5.6 that

$$2\,\mathrm{sf}(D, g^{-1}Dg) = \lim_{t\to\infty}\langle {}^{\mathrm{b}}\mathrm{Ch}^{\bullet}(t\mathfrak{D}), \mathrm{Ch}_{\bullet}(g) - \mathrm{Ch}_{\bullet}(g^{-1})\rangle.$$

Now applying Lemma 5.7, we have

$$\mathrm{sf}(D, g^{-1}Dg) = \lim_{t\to\infty}\langle {}^{\mathrm{b}}\mathrm{Ch}^{\bullet}(t\mathfrak{D}), \mathrm{Ch}_{\bullet}(g)\rangle. \qquad \square$$

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